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Abstract: In this research paper, we apply the idea of bipolar neutrosophic sets to incidence graphs. We present some notions, including bipolar neutrosophic incidence graphs, bipolar neutrosophic incidence cycle and bipolar neutrosophic incidence tree. We define strong path, strength and incidence strength of strongest path in bipolar neutrosophic incidence graphs. We investigate some properties of bipolar neutrosophic incidence graphs. We also describe an application of bipolar neutrosophic incidence graphs.

Keywords: Bipolar neutrosophic incidence graphs; Bipolar neutrosophic incidence cycle; Bipolar neutrosophic incidence tree.

1 Introduction

Graph theory is a mathematical structure which is used to represent a relationship between objects. It has been very successful in engineering and natural sciences. Sometimes, in many cases, graph theoretical concepts may be imprecise. To handle such cases, in 1975, Rosenfeld [1] gave the idea of fuzzy graphs. He considered fuzzy relations and proposed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. Bhutani and Rosenfeld [2] studied the strong edges in fuzzy graphs. By applying bipolar fuzzy sets [3] to graphs, Akram [4] introduced the notion of bipolar fuzzy graphs. He described the different methods to construct the bipolar fuzzy graphs and discussed the some of their properties. Broumi et al [5] introduced the single-valued neutrosophic graphs by applying the concept of single-valued neutrosophic sets to graphs. Later on, Akram and Sarwar [6] studied the novel multiple criteria decision making methods based on bipolar neutrosophic sets and bipolar neutrosophic graphs. They developed the independent and dominating sets of bipolar

neutrosophic graphs. Ishfaq et al [13, 14] introduced the rough neutrosophic digraphs and their applications. Later Akram et al [15] introduced the decision making approach based on neutrsophic rough information.
Dinesh [7, 8] studied the graph structures and introduced the fuzzy incidence graphs. Fuzzy incidence graphs not only give the limitation of the relation between elements contained in a set, but also give the influence or impact of an element to its relation pair. Fuzzy incidence graphs play an important role to interconnect the networks. Incidence relations have significant parts in human and natural made networks, including pipe, road, power and interconnection networks. Later Mathew and Mordeson [9] introduced the connectivity concepts in fuzzy incidence graphs and also introduced fuzzy influence graphs [10]. In this paper, we apply the idea of bipolar neutrosophic sets to incidence graphs and introduce a new concept, namely bipolar neutrosophic incidence graphs.
Some of essential preliminaries from [7] and [11] are given below:
Let V^* be a non-empty set. Then $G^* = (V^*, E^*, I^*)$ is an incidence graph, where E^* is a subset of $V^* \times V^*$ and I^* is a subset of $V^* \times E^*$.
A fuzzy incidence graph on an incidence graph $G^* = (V^*, E^*, I^*)$ is an ordered triplet $G' = (\mu', \lambda', \psi')$, where μ' is a fuzzy set on V^*, λ' is a fuzzy relation on V^* and ψ' is a fuzzy set on $V^* \times E^*$ such that

$$\psi'(y, yz) \leq \mu'(y) \wedge \lambda'(yz), \ \ \forall y, z \in V^*.$$

A bipolar neutrosophic set on a non-empty set V^* is an object having the form

$$B = \{(b, T_Y^+(b), I_Y^+(b), F_Y^+(b), T_Y^-(b), I_Y^-(b), F_Y^-(b)) : b \in V^*\},$$

where, $T_b^+, I_b^+, F_b^+ : V^* \longrightarrow [0,1]$ and $T_b^-, I_b^-, F_b^- : V^* \longrightarrow [-1,0]$.
For other notations and applications, readers are referred to [15-21].

2 Bipolar Neutrosophic Incidence Graphs

Definition 2.1. A *bipolar neutrosophic incidence graphs* (BNIG) on an incidence graph $G^* = (V^*, E^*, I^*)$ is an ordered triplet $G = (X, Y, Z)$, where

(1) X is a bipolar neutrosophic set on V^*.

(2) Y is a bipolar neutrosophic relation on V^*.

(3) Z is a bipolar neutrosophic set on $V^* \times E^*$ such that

$$\begin{aligned}
T_Z^+(x, xy) &\leq \min\{T_X^+(x), T_Y^+(xy)\}, \quad & T_Z^-(x, xy) &\geq \max\{T_X^-(x), T_Y^-(xy)\},\\
I_Z^+(x, xy) &\leq \min\{I_X^+(x), I_Y^+(xy)\}, \quad & I_Z^-(x, xy) &\geq \max\{I_X^-(x), I_Y^-(xy)\},\\
F_Z^+(x, xy) &\geq \max\{F_X^+(x), F_Y^+(xy)\}, \quad & F_Z^-(x, xy) &\leq \min\{F_X^-(x), F_Y^-(xy)\}, \forall x, y \in V^*.
\end{aligned}$$

Example 2.2. Let $G^* = (V^*, E^*, I^*)$ be an incidence graph, as shown in Fig. 1, where $V^* = \{w, x, y, z\}$, $E^* = \{wx, xy, yz, zw\}$ and $I^* = \{(w, wx), (x, wx), (x, xy), (y, xy), (y, yz), (z, yz), (z, zw), (w, zw)\}$. Let X be a bipolar neutrosophic set on V^* given as

$$\begin{aligned}
X = \{&(w,\ 0.2,\ 0.4,\ 0.7, -0.1, -0.2, -0.4), (x,\ 0.3,\ 0.5,\ 0.9, -0.1, -0.6, -0.7),\\
&(y,\ 0.4,\ 0.6,\ 0.9, -0.1, -0.2, -0.8), (z,\ 0.5,\ 0.6,\ 0.8, -0.2, -0.8, -0.6)\}.
\end{aligned}$$

Let Y be a bipolar neutrosophic relation on V^* given as

$$Y = \{(wx,\ 0.1,\ 0.2,\ 0.8, -0.1, -0.2, -0.9), (xy,\ 0.2,\ 0.4,\ 0.7, -0.2, -0.3, -0.9),$$
$$(yz,\ 0.1,\ 0.2,\ 0.8, -0.1, -0.2, -0.9), (zw,\ 0.2,\ 0.3,\ 0.6, -0.1, -0.2, -0.7)\}.$$

Let Z be a bipolar neutrosophic set on $V^* \times E^*$ given as

$$Z = \{((w,wx), 0.1,\ 0.1,\ 0.8, -0.2, -0.2, -0.9), ((x, wx), 0.1,\ 0.2,\ 0.8, -0.2, -0.3,\ -0.9),$$
$$((x,\ xy), 0.2,\ 0.3,\ 0.8, -0.2, -0.4, -0.9), ((y,\ xy), 0.1,\ 0.1,\ 0.8, -0.2, -0.2,\ -0.9),$$
$$((y,\ yz), 0.1,\ 0.2,\ 0.7, -0.2, -0.3, -0.9), ((z,\ yz), 0.1,\ 0.2,\ 0.7, -0.2, -0.3,\ -0.7),$$
$$((z, zw), 0.1,\ 0.1,\ 0.8, -0.2, -0.2, -0.9), ((w,zw), 0.2,\ 0.3,\ 0.5, -0.3, -0.3, -0.8)\}.$$

Then $G = (X, Y, Z)$ is a BNIG as shown in Fig. 2.

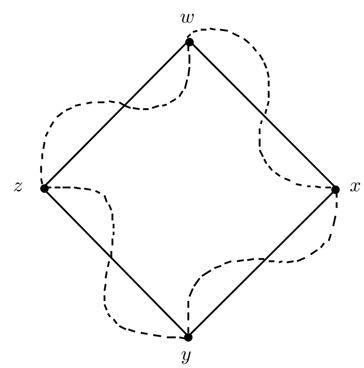

Figure 1: $G^* = (V^*, E^*, I^*)$

Definition 2.3. Let $G = (X, Y, Z)$ be a BNIG of G^*. Then *support* of $G = (X, Y, Z)$ is denoted by supp(G)=(supp(X),supp(Y),supp(Z)) such that

$$\text{supp}(X) = \{x \in X | T^+_X(x) > 0,\ I^+_X(x) > 0,\ F^+_X(x) > 0,$$
$$T^-_X(x) < 0,\ I^-_X(x) < 0,\ F^-_X(x) < 0\},$$

$$\text{supp}(Y) = \{xy \in Y | T^+_Y(xy) > 0,\ I^+_Y(xy) > 0,\ F^+_Y(xy) > 0,$$
$$T^-_Y(xy) < 0,\ I^-_Y(xy) < 0,\ F^-_Y(xy) < 0\},$$

$$\text{supp}(Z) = \{(x, xy) \in Z | T^+_Z(x, xy) > 0,\ I^+_Z(x, xy) > 0,\ F^+_Z(x, xy) > 0,$$
$$T^-_Z(x, xy) < 0,\ I^-_Z(x, xy) < 0,\ F^-_Z(x, xy) < 0\}.$$

Definition 2.4. A sequence
$x_0, (x_0, x_0x_1), x_0x_1, (x_1, x_0x_1), x_1, \cdots, x_{n-1}, (x_{n-1}, x_{n-1}x_n), x_{n-1}x_n, (x_n, x_{n-1}x_n), x_n$
of vertices, edges and pairs in BNIG G is called *walk*.

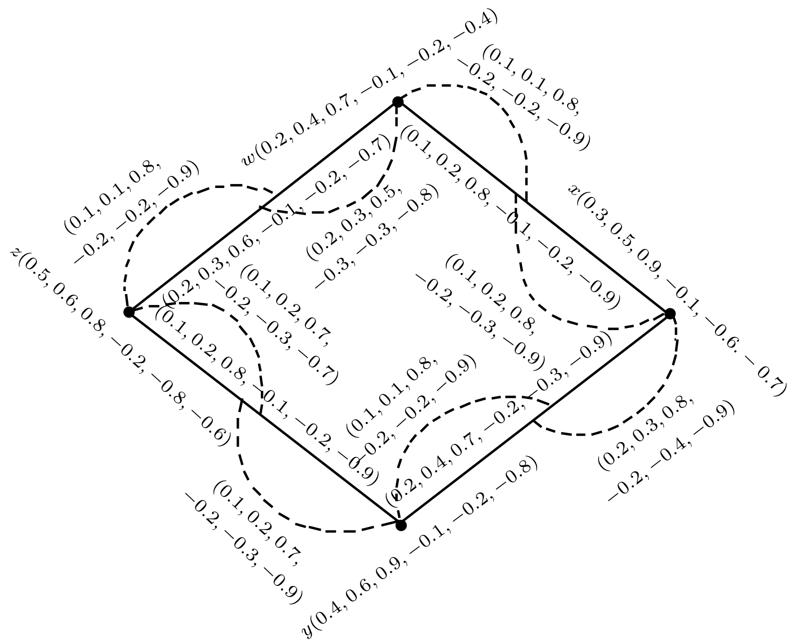

Figure 2: BNIG $G = (X, Y, Z)$

If $x_0 = x_n$, it is a *close walk*.
If edges are distinct, it is a *trail*.
If pairs are distinct, it is an *incidence trail*.
If vertices are distinct, it is a *path*.
If pairs are distinct, it is an *incidence path*.

Example 2.5. In a BNIG $G = (X, Y, Z)$ as shown in Fig.2,
$w, (w, wx), wx, (x, wx), x, (x, xy), xy, (y, xy), y, (y, yz), yz, (z, yz), z, (z, zw), zw, (w, zw), w, (w, wx), wx, (x, wx), x$ ia a walk. It is not a path, trail and an incidence trail.

$$w, (w, wx), wx, (x, wx), x, (x, xy), xy, (y, xy), y, (y, yz), yz, (z, yz), z$$

is a path, trail and an incidence trail.

Definition 2.6. The BNIG $G = (X, Y, Z)$ is a *cycle* if and only if supp(G)=(supp(X),supp(Y),supp(Z)) is a cycle.

Example 2.7. In a BNIG $G = (X, Y, Z)$ as shown in Fig.2, consider a walk

$$w, (w, wx), wx, (x, wx), x, (x, xy), xy, (y, xy), y, (y, yz), yz, (z, yz), z, (z, zw), zw, (w, zw), w.$$

which is a cycle. So $G = (X, Y, Z)$ is a cycle.

Definition 2.8. The BNIG $G = (X, Y, Z)$ is a *bipolar neutrosophic cycle* if and only if

$$\text{supp}(G) = (\text{supp}(X), \text{supp}(Y), \text{supp}(Z))$$

is a cycle and there exist at least two $xy \in$ supp(Y) such that

$$
\begin{aligned}
T_Y^+(xy) &= \min\{T_Y^+(uv)\ |uv \in \text{supp}(Y)\},\\
I_Y^+(xy) &= \min\{I_Y^+(uv)\ \ |uv \in \text{supp}(Y)\},\\
F_Y^+(xy) &= \max\{F_Y^+(uv)|uv \in \text{supp}(Y)\},\\
T_Y^-(xy) &= \max\{T_Y^-(uv)|uv \in \text{supp}(Y)\},\\
I_Y^-(xy) &= \max\{I_Y^-(uv)\ |uv \in \text{supp}(Y)\},\\
F_Y^-(xy) &= \min\{F_Y^-(uv)\ |uv \in \text{supp}(Y)\}.
\end{aligned}
$$

Example 2.9. In a BNIG $G = (X, Y, Z)$ as shown in Fig.2,
we have

$$
\begin{aligned}
T_Y^+(wx) &= \quad 0.1 = \min\{\ T_Y^+(wx),\ T_Y^+(xy),\ T_Y^+(yz),\ T_Y^+(zw)\},\\
I_Y^+(wx) &= \quad 0.2 = \min\{\ I_Y^+(wx),\ I_Y^+(xy),\ \ I_Y^+(yz),\ I_Y^+(zw)\},\\
F_Y^+(wx) &= \quad 0.8 = \max\{F_Y^+(wx),\ F_Y^+(xy),\ F_Y^+(yz),\ F_Y^+(zw)\},\\
T_Y^-(wx) &= -0.1 = \max\{T_Y^-(wx),\ T_Y^-(xy),\ T_Y^-(yz),\ T_Y^-(zw)\},\\
I_Y^-(wx) &= -0.2 = \max\{\ I_Y^-(wx),\ I_Y^-(xy),\ \ I_Y^-(yz),\ I_Y^-(zw)\},\\
F_Y^-(wx) &= -0.9 = \min\{\ F_Y^-(wx),\ F_Y^-(xy),\ F_Y^-(yz),\ F_Y^-(zw)\}.
\end{aligned}
$$

Also

$$
\begin{aligned}
T_Y^+(yz) &= \quad 0.1 = \min\{\ T_Y^+(wx),\ T_Y^+(xy),\ T_Y^+(yz),\ T_Y^+(zw)\},\\
I_Y^+(yz) &= \quad 0.2 = \min\{\ I_Y^+(wx),\ I_Y^+(xy),\ \ I_Y^+(yz),\ I_Y^+(zw)\},\\
F_Y^+(yz) &= \quad 0.8 = \max\{F_Y^+(wx),\ F_Y^+(xy),\ F_Y^+(yz),\ F_Y^+(zw)\},\\
T_Y^-(yz) &= -0.1 = \max\{T_Y^-(wx),\ T_Y^-(xy),\ T_Y^-(yz),\ T_Y^-(zw)\},\\
I_Y^-(yz) &= -0.2 = \max\{\ I_Y^-(wx),\ I_Y^-(xy),\ \ I_Y^-(yz),\ I_Y^-(zw)\},\\
F_Y^-(yz) &= -0.9 = \min\{\ F_Y^-(wx),\ F_Y^-(xy),\ F_Y^-(yz),\ F_Y^-(zw)\}.
\end{aligned}
$$

So $G = (X, Y, Z)$ is a bipolar neutrosophic cycle.

Definition 2.10. The BNIG $G = (X, Y, Z)$ is a *bipolar neutrosophic incidence cycle* if and only if it is a bipolar neutrosophic cycle and there exist at least two $(x, xy) \in$ supp(Z) such that

$$
\begin{aligned}
T_Z^+(x,xy) &= \min\{T_Z^+(u,vw)\ |(u,vw) \in \text{supp}(Z)\},\\
I_Z^+(x,xy) &= \min\{I_Z^+(u,vw)\ \ |(u,vw) \in \text{supp}(Z)\},\\
F_Z^+(x,xy) &= \max\{F_Z^+(u,vw)|(u,vw) \in \text{supp}(Z)\},\\
T_Z^-(x,xy) &= \max\{T_Z^-(u,vw)|(u,vw) \in \text{supp}(Z)\},\\
I_Z^-(x,xy) &= \max\{I_Z^-(u,vw)\ |(u,vw) \in \text{supp}(Z)\},\\
F_Z^-(x,xy) &= \min\{F_Z^-(u,vw)\ |(u,vw) \in \text{supp}(Z)\}.
\end{aligned}
$$

Example 2.11. In a BNIG $G = (X, Y, Z)$ as shown in Fig.2, we have

$$\begin{aligned}
T_Z^+(w,wx) &= \quad 0.1 = \min\{T_Z^+(u,vw)\,|(u,vw)\in \text{supp}(Z)\},\\
I_Z^+(w,wx) &= \quad 0.1 = \min\{I_Z^+(u,vw)\ \,|(u,vw)\in \text{supp}(Z)\},\\
F_Z^+(w,wx) &= \quad 0.8 = \max\{F_Z^+(u,vw)|(u,vw)\in \text{supp}(Z)\},\\
T_Z^-(w,wx) &= -0.2 = \max\{T_Z^-(u,vw)|(u,vw)\in \text{supp}(Z)\},\\
I_Z^-(w,wx) &= -0.2 = \max\{I_Z^-(u,vw)\,|(u,vw)\in \text{supp}(Z)\},\\
F_Z^-(w,wx) &= -0.9 = \min\{F_Z^-(u,vw)\,|(u,vw)\in \text{supp}(Z)\}.
\end{aligned}$$

and

$$\begin{aligned}
T_Z^+(y,xy) &= \quad 0.1 = \min\{T_Z^+(u,vw)\,|(u,vw)\in \text{supp}(Z)\},\\
I_Z^+(y,xy) &= \quad 0.1 = \min\{I_Z^+(u,vw)\ \,|(u,vw)\in \text{supp}(Z)\},\\
F_Z^+(y,xy) &= \quad 0.8 = \max\{F_Z^+(u,vw)|(u,vw)\in \text{supp}(Z)\},\\
T_Z^-(y,xy) &= -0.2 = \max\{T_Z^-(u,vw)|(u,vw)\in \text{supp}(Z)\},\\
I_Z^-(y,xy) &= -0.2 = \max\{I_Z^-(u,vw)\,|(u,vw)\in \text{supp}(Z)\},\\
F_Z^-(y,xy) &= -0.9 = \min\{F_Z^-(u,vw)\,|(u,vw)\in \text{supp}(Z)\}.
\end{aligned}$$

So $G=(X,Y,Z)$ is a bipolar neutrosophic incidence cycle.

Definition 2.12. If $G=(X,Y,Z)$ is a BNIG, then $H=(X^*,Y^*,Z^*)$ is a *bipolar neutrosophic incidence subgraph* of G if

$$X^*\subseteq X,\ Y^*\subseteq Y,\ Z^*\subseteq Z.$$

$H=(X^*,Y^*,Z^*)$ is a *spanning subgraph* if $X=X^*$.

Definition 2.13. *Strength of the strongest path* from x to y in BNIG $G=(X,Y,Z)$ is defined as

$$\begin{aligned}
T^+_{\rho^\infty}(x,y) &= \bigvee_{i=1}^{k} T^+_{\rho_i}(x,y), \quad I^+_{\rho^\infty}(x,y) = \bigvee_{i=1}^{k} I^+_{\rho_i}(x,y), \quad F^+_{\rho^\infty}(x,y) = \bigwedge_{i=1}^{k} F^+_{\rho_i}(x,y),\\
T^-_{\rho^\infty}(x,y) &= \bigwedge_{i=1}^{k} T^-_{\rho_i}(x,y), \quad I^-_{\rho^\infty}(x,y) = \bigwedge_{i=1}^{k} I^-_{\rho_i}(x,y), \quad F^-_{\rho^\infty}(x,y) = \bigvee_{i=1}^{k} F^-_{\rho_i}(x,y).
\end{aligned}$$

where $\rho(x,y)$ is the strength of path from x to y such that

$$\begin{aligned}
T_\rho^+(x,y) &= \wedge\,\{T_Y^+(xy)|xy\in \text{supp}(Y)\},\\
I_\rho^+(x,y) &= \wedge\,\{I_Y^+(xy)\,|xy\in \text{supp}(Y)\},\\
F_\rho^+(x,y) &= \vee\,\{F_Y^+(xy)|xy\in \text{supp}(Y)\},\\
T_\rho^-(x,y) &= \vee\,\{T_Y^-(xy)|xy\in \text{supp}(Y)\},\\
I_\rho^-(x,y) &= \vee\,\{I_Y^-(xy)\,|xy\in \text{supp}(Y)\},\\
F_\rho^-(x,y) &= \wedge\,\{F_Y^-(xy)|xy\in \text{supp}(Y)\}.
\end{aligned}$$

Definition 2.14. *Incidence strength of the strongest path* from x to wy in BNIG $G = (X, Y, Z)$ is defined as

$$T^{+}_{\psi^\infty}(x, wy) = \bigvee_{i=1}^{k} T^{+}_{\psi_i}(x, wy), \quad T^{-}_{\psi^\infty}(x, wy) = \bigwedge_{i=1}^{k} T^{-}_{\psi_i}(x, wy),$$
$$I^{+}_{\psi^\infty}(x, wy) = \bigvee_{i=1}^{k} I^{+}_{\psi_i}(x, wy), \quad I^{-}_{\psi^\infty}(x, wy) = \bigwedge_{i=1}^{k} I^{-}_{\psi_i}(x, wy),$$
$$F^{+}_{\psi^\infty}(x, wy) = \bigwedge_{i=1}^{k} F^{+}_{\psi_i}(x, wy), \quad F^{-}_{\psi^\infty}(x, wy) = \bigvee_{i=1}^{k} F^{-}_{\psi_i}(x, wy).$$

where $\psi(x, wy)$ is the incidence strength of path from x to wy such that

$$T^{+}_{\psi}(x, wy) = \wedge \{T^{+}_{Z}(x, wy) | (x, wy) \in \text{supp}(Z)\},$$
$$I^{+}_{\psi}(x, wy) = \wedge \{I^{+}_{Z}(x, wy) \ | (x, wy) \in \text{supp}(Z)\},$$
$$F^{+}_{\psi}(x, wy) = \vee \{F^{+}_{Z}(x, wy) | (x, wy) \in \text{supp}(Z)\},$$
$$T^{-}_{\psi}(x, wy) = \vee \{T^{-}_{Z}(x, wy) | (x, wy) \in \text{supp}(Z)\},$$
$$I^{-}_{\psi}(x, wy) = \vee \{I^{-}_{Z}(x, wy) \ | (x, wy) \in \text{supp}(Z)\},$$
$$F^{-}_{\psi}(x, wy) = \wedge \{F^{-}_{Z}(x, wy) | (x, wy) \in \text{supp}\}(Z).$$

Example 2.15. In a BNIG $G = (X, Y, Z)$ as shown in Fig.3
the strength of path $w, (w, wy), wy, (y, wy), y, (y, yz), yz, (z, yz), z$ is

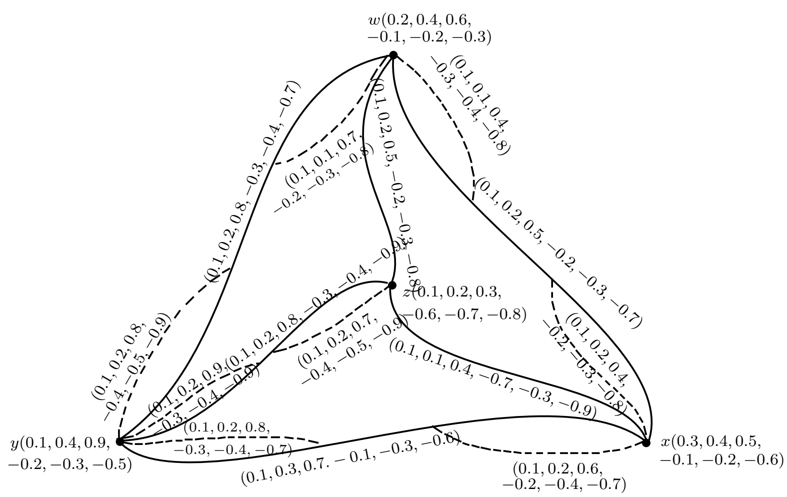

Figure 3: BNIG $G = (X, Y, Z)$

$$(0.1, 0.1, 0.8, -0.3, -0.4, -0.9),$$

the strength of path $w, (w, wx), wx, (x, wx), x, (x, xy), xy, (y, xy), y, (y, yz), yz, (z, yz), z$ is

$$(0.1, 0.2, 0.8, -0.1, -0.3, -0.9),$$

the strength of the strongest path from w to z is

$$(0.1, 0.2, 0.8, -0.3, -0.4, -0.9).$$

In a BNIG $G = (X, Y, Z)$ as shown in Fig.3
the incidence strength of the path $w, (w, wy), wy, (y, wy), y, (y, yz), yz$ is

$$(0.1, 0.1, 0.9, -0.2, -0.3, -0.9),$$

the incidence strength of the path $w, (w, wx), wx, (x, wx), x, (x, xy), xy, (y, xy), y, (y, yz), yz$ is

$$(0.1, 0.1, 0.8, -0.2, -0.3, -0.9),$$

the incidence strength of strongest path from w to yz is

$$(0.1, 0.1, 0.8, -0.2, -0.3, -0.9).$$

Definition 2.16. BNIG $G = (X, Y, Z)$ is called a *tree* if and only if supp(G)=(supp(X),supp(Y),supp(Z)) is a tree.

Definition 2.17. $G = (X, Y, Z)$ is a *bipolar single-valued neutrosophic tree* if and only if bipolar neutrosophic incidence spanning subgraph $H = (X, Y^*, Z^*)$ of $G = (X, Y, Z)$ is a tree such that

$$T_Y^+(xy) < T_{\phi^\infty}^+(x,y), \quad I_Y^+(xy) < I_{\phi^\infty}^+(x,y), \quad F_Y^+(xy) > F_{\phi^\infty}^+(x,y),$$
$$T_Y^-(xy) > T_{\phi^\infty}^-(x,y), \quad I_Y^-(xy) > I_{\phi^\infty}^-(x,y), \quad F_Y^-(xy) < F_{\phi^\infty}^-(x,y), \quad \forall xy \in \text{supp}(Y) \backslash \text{supp}(Y^*).$$

where $\phi^\infty(x, y)$ is the strength of strongest path from x to y in $H = (X, Y^*, Z^*)$.

Definition 2.18. $G = (X, Y, Z)$ is a *bipolar neutrosophic incidence tree* if and only if bipolar neutrosophic incidence spanning subgraph $H = (X, Y^*, Z^*)$ of $G = (X, Y, Z)$ is a tree such that

$$T_Z^+(x,xy) < T_{\tau^\infty}^+(x,xy), \quad I_Z^+(x,xy) < I_{\tau^\infty}^+(x,xy), \quad F_Z^+(x,xy) > F_{\tau^\infty}^+(x,xy),$$
$$T_Z^-(x,xy) > T_{\tau^\infty}^-(x,xy), \quad I_Z^-(x,xy) > I_{\tau^\infty}^-(x,xy), \quad F_Z^-(x,xy) < F_{\tau^\infty}^-(x,xy), \quad \forall (x,xy) \in \text{supp}(Z) \backslash \text{supp}(Z^*).$$

where $\tau^\infty(x, xy)$ is the strength of strongest path from x to xy in $H = (X, Y^*, Z^*)$.

Example 2.19. $G = (X, Y, Z)$ is a bipolar neutrosophic tree as shown in Fig.4 because a bipolar neutrosophic incidence spanning subgraph $H = (X, Y^*, Z^*)$ of $G = (X, Y, Z)$ as shown in Fig.5 is a tree and

$$\begin{aligned}
T_Y^+(wx) &= \quad 0.1 < \ \ 0.2 = T_{\phi^\infty}^+(w,x),\\
I_Y^+(wx) &= \quad 0.1 < \ \ 0.2 = I_{\phi^\infty}^+(w,x),\\
F_Y^+(wx) &= \quad 0.9 > \ \ 0.7 = F_{\phi^\infty}^+(w,x),\\
T_Y^-(wx) &= -0.1 > -0.2 = T_{\phi^\infty}^-(w,x),\\
I_Y^-(wx) &= -0.2 > -0.3 = I_{\phi^\infty}^-(w,x),\\
F_Y^-(wx) &= -0.9 < -0.8 = F_{\phi^\infty}^-(w,x).
\end{aligned}$$

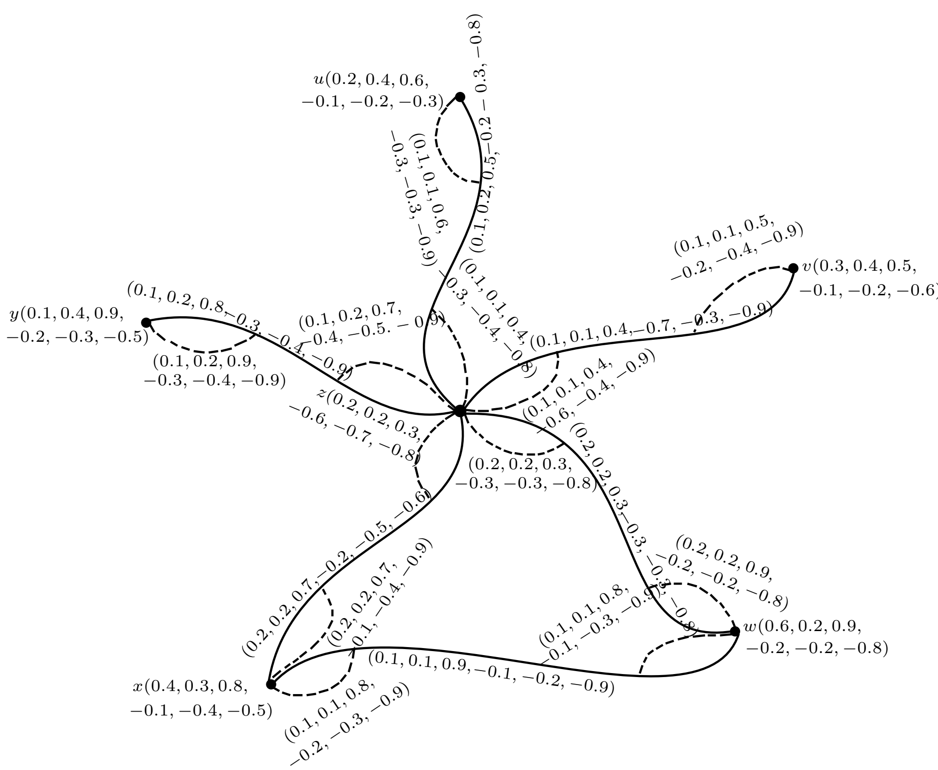

Figure 4: BNIG $G = (X, Y, Z)$

Theorem 2.20. *Let $G = (X, Y, Z)$ be a cycle. Then $G = (X, Y, Z)$ is a bipolar neutrosophic cycle if and only if $G = (X, Y, Z)$ is not a bipolar neutrosophic tree.*

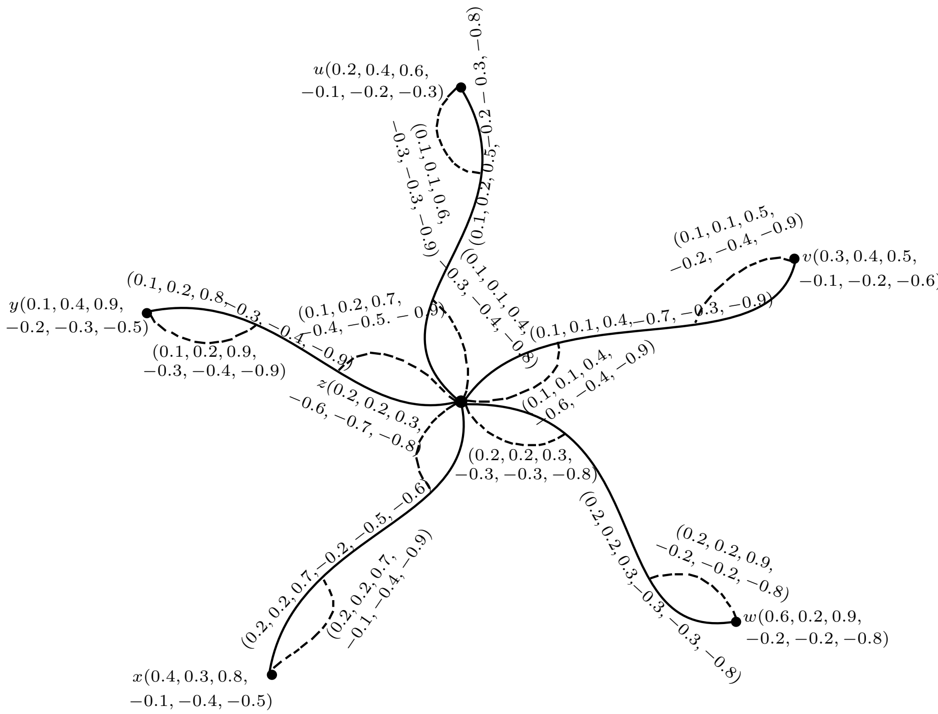

Figure 5: $H = (X, Y^{*}, Z^{*})$

Proof. Let $G = (X, Y, Z)$ be a bipolar neutrosophic cycle. So there exists $uv, xy \in \text{supp}(Y)$ such that

$$\begin{aligned}
T_{Y}^{+}(uv) &= T_{Y}^{+}(xy) = \wedge \{T_{Y}^{+}(yz) | yz \in \text{supp}(Y)\},\\
I_{Y}^{+}(uv) &= I_{Y}^{+}(xy) = \wedge \{I_{Y}^{+}(yz) | yz \in \text{supp}(Y)\},\\
F_{Y}^{+}(uv) &= F_{Y}^{+}(xy) = \vee \{F_{Y}^{+}(yz) | yz \in \text{supp}(Y)\},\\
T_{Y}^{-}(uv) &= T_{Y}^{-}(xy) = \vee \{T_{Y}^{-}(yz) | yz \in \text{supp}(Y)\},\\
I_{Y}^{-}(uv) &= I_{Y}^{-}(xy) = \vee \{I_{Y}^{-}(yz) | yz \in \text{supp}(Y)\},\\
F_{Y}^{-}(uv) &= F_{Y}^{-}(xy) = \wedge \{F_{Y}^{-}(yz) | yz \in \text{supp}(Y)\}.
\end{aligned}$$

If $H = (X, Y^{*}, Z^{*})$ is a spanning bipolar neutrosophic incidence tree of $G = (X, Y, Z)$, then $\text{supp}(Y) \backslash \text{supp}(Y^{*}) = \{yz\}$ for some $y, z \in V$ because $G = (X, Y, Z)$ is a cycle.
Hence there exists no path between y and z in $H = (X, Y^{*}, Z^{*})$ such that

$$\begin{aligned}
&T_{Y}^{+}(yz) < T_{\phi^{\infty}}^{+}(y, z), \quad I_{Y}^{+}(yz) < I_{\phi^{\infty}}^{+}(y, z), \quad F_{Y}^{+}(yz) > F_{\phi^{\infty}}^{+}(y, z),\\
&T_{Y}^{-}(yz) > T_{\phi^{\infty}}^{-}(y, z), \quad I_{Y}^{-}(yz) > I_{\phi^{\infty}}^{-}(y, z), \quad F_{Y}^{-}(yz) < F_{\phi^{\infty}}^{-}(y, z).
\end{aligned}$$

Thus, $G = (X, Y, Z)$ is not a bipolar neutrosophic tree.
Conversely, let $G = (X, Y, Z)$ be not a bipolar neutrosophic tree. Because $G = (X, Y, Z)$ is a cycle, so for all

$yz \in$ supp(Y), $H = (X, Y^*, Z^*)$ is spanning bipolar neutrosophic incidence tree in $G = (X, Y, Z)$ such that

$$T_Y^+(yz) \geq T_{\phi^\infty}^+(y,z),\ \ I_Y^+(yz) \geq I_{\phi^\infty}^+(y,z),\ \ F_Y^+(yz) \leq F_{\phi^\infty}^+(y,z),$$
$$T_Y^-(yz) \leq T_{\phi^\infty}^-(y,z),\ \ I_Y^-(yz) \leq I_{\phi^\infty}^-(y,z),\ \ F_Y^-(yz) \geq F_{\phi^\infty}^-(y,z).$$

where

$$T_{Y^*}^+(yz) = 0,\ I_{Y^*}^+(yz) = 0,\ F_{Y^*}^+(yz) = 0,$$
$$T_{Y^*}^-(yz) = 0,\ I_{Y^*}^-(yz) = 0,\ F_{Y^*}^-(yz) = 0.$$

and

$$T_{Y^*}^+(uv) = T_Y^+(uv),\ I_{Y^*}^+(uv) = I_Y^+(uv),\ F_{Y^*}^+(uv) = F_Y^+(uv),$$
$$T_{Y^*}^-(uv) = T_Y^-(uv),\ I_{Y^*}^-(uv) = I_Y^-(uv),\ F_{Y^*}^-(uv) = F_Y^-(uv), \forall\ uv \in \text{supp}(Y) \backslash \{yz\}.$$

Hence, there exists more than one edge such that

$$T_Y^+(yz) = \wedge \{T_Y^+(xy) | xy \in \text{supp}(Y)\},$$
$$I_Y^+(yz) = \wedge \{ I_Y^+(xy) | xy \in \text{supp}(Y)\},$$
$$F_Y^+(yz) = \vee \{F_Y^+(xy) | xy \in \text{supp}(Y)\},$$
$$T_Y^-(yz) = \vee \{T_Y^-(xy) | xy \in \text{supp}(Y)\},$$
$$I_Y^-(yz) = \vee \{ I_Y^-(xy) | xy \in \text{supp}(Y)\},$$
$$F_Y^-(yz) = \wedge \{F_Y^-(xy) | xy \in \text{supp}(Y)\}.$$

Thus, $G = (X, Y, Z)$ is a bipolar neutrosophic cycle. □

Theorem 2.21. *If $G = (X, Y, Z)$ is a bipolar neutrosophic tree and supp$(G) = ($supp(X), supp(Y), supp$(Z))$ is not a tree, then there exists at least one edge $xy \in$ supp(Y) such that*

$$T_Y^+(xy) < T_{\mu^\infty}^+(x,y),\ \ I_Y^+(xy) < I_{\mu^\infty}^+(x,y),\ \ F_Y^+(xy) > F_{\mu^\infty}^+(x,y),$$
$$T_Y^-(xy) > T_{\mu^\infty}^-(x,y),\ \ I_Y^-(xy) > I_{\mu^\infty}^-(x,y),\ \ F_Y^-(xy) < F_{\mu^\infty}^-(x,y).$$

where $\mu^\infty(x, y)$ is the strength of strongest path between u and v in $G = (X, Y, Z)$.

Proof. Let $G = (X, Y, Z)$ be a bipolar neutrosophic tree, then there exists a bipolar neutrosophic spanning subgraph $H = (X, Y^*, Z^*)$ that is tree and

$$T_Y^+(xy) < T_{\rho^\infty}^+(x,y),\ \ I_Y^+(xy) < I_{\rho^\infty}^+(x,y),\ \ F_Y^+(xy) > F_{\rho^\infty}^+(x,y),$$
$$T_Y^-(xy) > T_{\rho^\infty}^-(x,y),\ \ I_Y^-(xy) > I_{\rho^\infty}^-(x,y),\ \ F_Y^-(xy) < F_{\rho^\infty}^-(x,y), \forall uv \in \text{supp}(Y) \backslash \text{supp}(Y^*).$$

Also

$$T_{\rho^\infty}^+(x,y) \leq T_{\mu^\infty}^+(x,y),\ I_{\rho^\infty}^+(x,y) \leq I_{\mu^\infty}^+(x,y),\ F_{\rho^\infty}^+(x,y) \geq F_{\mu^\infty}^+(x,y),$$
$$T_{\rho^\infty}^-(x,y) \geq T_{\mu^\infty}^-(x,y),\ I_{\rho^\infty}^-(x,y) \geq I_{\mu^\infty}^-(x,y),\ F_{\rho^\infty}^-(x,y) \leq F_{\mu^\infty}^-(x,y).$$

Thus,

$$T^+_Y(xy){<}T^+_{\mu^\infty}(x,y),\ \ I^+_Y(xy)\ {<}I^+_{\mu^\infty}(x,y),\ \ F^+_Y(xy){>}F^+_{\mu^\infty}(x,y),$$
$$T^-_Y(xy){>}T^-_{\mu^\infty}(x,y),\ \ I^-_Y(xy)\ {>}I^-_{\mu^\infty}(x,y),\ \ F^-_Y(xy){<}F^-_{\mu^\infty}(x,y), \forall uv \in \text{supp}(Y)\backslash\text{supp}(Y^*)$$

and by hypothesis there exists at least one edge $xy \in$ supp(Y). □

Theorem 2.22. *Let $G = (X, Y, Z)$ be a cycle. Then $G = (X, Y, Z)$ is a bipolar neutrosophic incidence cycle if and only if $G = (X, Y, Z)$ is not a bipolar neutrosophic incidence tree.*

Proof. Let $G = (X, Y, Z)$ be a bipolar neutrosophic incidence cycle. Then there exist at least two $(x, wy) \in$ supp(Z) such that

$$\begin{aligned}
T^+_Z(x,yz) &= \min\{T^+_Z(u,vw)\ |(u,vw) \in \text{supp}(Z)\},\\
I^+_Z(x,yz) &= \min\{I^+_Z(u,vw)\ \ |(u,vw) \in \text{supp}(Z)\},\\
F^+_Z(x,yz) &= \max\{F^+_Z(u,vw)|(u,vw) \in \text{supp}(Z)\},\\
T^-_Z(x,yz) &= \max\{T^-_Z(u,vw)|(u,vw) \in \text{supp}(Z)\},\\
I^-_Z(x,yz) &= \max\{I^-_Z(u,vw)\ |(u,vw) \in \text{supp}(Z)\},\\
F^-_Z(x,yz) &= \min\{F^-_Z(u,vw)\ |(u,vw) \in \text{supp}(Z)\}.
\end{aligned}$$

If $H = (X, Y^*, Z^*)$ is a spanning bipolar neutrosophic incidence tree of $G = (X, Y, Z)$, then supp$(Z)\backslash$supp$(Z^*) = \{(x, yz)\}$ for some $x \in V$ $yz \in$ supp(Y).
Hence there exists no path between x and yz in $H = (X, Y^*, Z^*)$ such that

$$T^+_Z(x,yz){<}T^+_{\tau^\infty}(x,yz),\ \ I^+_Z(x,yz)\ {<}I^+_{\tau^\infty}(x,yz),\ \ F^+_Z(x,yz){>}F^+_{\tau^\infty}(x,yz),$$
$$T^-_Z(x,yz){>}T^-_{\tau^\infty}(x,yz),\ \ I^-_Z(x,yz)\ {>}I^-_{\tau^\infty}(x,yz),\ \ F^-_Z(x,yz){<}F^-_{\tau^\infty}(x,yz).$$

Thus, $G = (X, Y, Z)$ is not a bipolar neutrosophic incidence tree.
Conversely, let $G = (X, Y, Z)$ be not a bipolar neutrosophic incidence tree. Then for all $(x, yz) \in$ supp(Z), $H = (X, Y^*, Z^*)$ is spanning bipolar neutrosophic incidence tree in $G = (X, Y, Z)$ such that

$$T^+_Z(x,yz){\geq}T^+_{\tau^\infty}(x,yz),\ \ I^+_Z(x,yz)\ {\geq}I^+_{\tau^\infty}(x,yz),\ \ F^+_Z(x,yz){\leq}F^+_{\tau^\infty}(x,yz),$$
$$T^-_Z(x,yz){\leq}T^-_{\tau^\infty}(x,yz),\ \ I^-_Z(x,yz)\ {\leq}I^-_{\tau^\infty}(x,yz),\ \ F^-_Z(x,yz){\geq}F^-_{\tau^\infty}(x,yz).$$

where

$$T^+_{Z^*}(x,yz) = 0,\ I^+_{Z^*}(x,yz){=}\ 0,\ F^+_{Z^*}(x,yz) = 0,$$
$$T^-_{Z^*}(x,yz) = 0,\ I^-_{Z^*}(x,yz){=}\ 0,\ F^-_{Z^*}(x,yz) = 0.$$

and

$$T^+_{Z^*}(u,vw){=}T^+_Z(u,vw),\ \ I^+_{z^*}(u,vw)\ {=}I^+_Z(u,vw),\ \ F^+_{Z^*}(u,vw){=}F^+_Z(u,vw),$$
$$T^-_{Z^*}(u,vw){=}T^-_Z(u,vw),\ \ I^-_{Z^*}(u,vw)\ {=}I^-_Z(u,vw),\ \ F^-_{Z^*}(u,vw){=}F^-_Z(u,vw), \forall\ (u,vw) \in \text{supp}(Z)\backslash\{(x,yz)\}.$$

Hence, there exists more than one pair such that

$$\begin{aligned}
T_Z^+(u, vw) &= \wedge \{T_Z^+(x, yz) | (x, yz) \in \text{supp}(Z)\},\\
I_Z^+(u, vw) &= \wedge \{ I_Z^+(x, yz) | (x, yz) \in \text{supp}(Z)\},\\
F_Z^+(u, vw) &= \vee \{F_Z^+(x, yz) | (x, yz) \in \text{supp}(Z)\},\\
T_Z^-(u, vw) &= \vee \{T_Z^-(x, yz) | (x, yz) \in \text{supp}(Z)\},\\
I_Z^-(u, vw) &= \vee \{ I_Z^-(x, yz) | (x, yz) \in \text{supp}(Z)\},\\
F_Z^-(u, vw) &= \wedge \{F_Z^-(x, yz) | (x, yz) \in \text{supp}(Z)\}.
\end{aligned}$$

Thus, $G = (X, Y, Z)$ is a bipolar neutrosophic incidence cycle. □

Definition 2.23. Let $G = (X, Y, Z)$ be a BNIG. An edge xy is called a *strong edge* if

$$\begin{aligned}
T_Y^+(xy) &\geq T_{\epsilon^\infty}^+(x, y), \; T_Y^-(xy) \leq T_{\epsilon^\infty}^-(x, y),\\
I_Y^+(xy) &\geq I_{\epsilon^\infty}^+(x, y), \; I_Y^-(xy) \leq I_{\epsilon^\infty}^-(x, y),\\
F_Y^+(xy) &\leq F_{\epsilon^\infty}^+(x, y), \; F_Y^-(xy) \geq F_{\epsilon^\infty}^-(x, y).
\end{aligned}$$

An edge xy is called α-*strong* if

$$\begin{aligned}
T_Y^+(xy) &> T_{\epsilon^\infty}^+(x, y), \; T_Y^-(xy) < T_{\epsilon^\infty}^-(x, y),\\
I_Y^+(xy) &> I_{\epsilon^\infty}^+(x, y), \; I_Y^-(xy) < I_{\epsilon^\infty}^-(x, y),\\
F_Y^+(xy) &< F_{\epsilon^\infty}^+(x, y), \; F_Y^-(xy) > F_{\epsilon^\infty}^-(x, y).
\end{aligned}$$

An edge xy is called β-*strong* if

$$\begin{aligned}
T_Y^+(xy) &= T_{\epsilon^\infty}^+(x, y), \; T_Y^-(xy) = T_{\epsilon^\infty}^-(x, y),\\
I_Y^+(xy) &= I_{\epsilon^\infty}^+(x, y), \; I_Y^-(xy) = I_{\epsilon^\infty}^-(x, y),\\
F_Y^+(xy) &= F_{\epsilon^\infty}^+(x, y), \; F_Y^-(xy) = F_{\epsilon^\infty}^-(x, y).
\end{aligned}$$

where $\epsilon^\infty(x, y)$ is the strength of strongest path between x and y.

Definition 2.24. Let $G = (X, Y, Z)$ be a BNIG. An edge xy is called a δ-*edge* if

$$\begin{aligned}
T_Y^+(xy) &< T_{\epsilon^\infty}^+(x, y), \; T_Y^-(x, y) > T_{\epsilon^\infty}^-(x, y),\\
I_Y^+(xy) &< I_{\epsilon^\infty}^+(x, y), \; I_Y^-(x, y) > I_{\epsilon^\infty}^-(x, y),\\
F_Y^+(xy) &> F_{\epsilon^\infty}^+(x, y), \; F_Y^-(x, y) < F_{\epsilon^\infty}^-(x, y).
\end{aligned}$$

Definition 2.25. Let $G = (X, Y, Z)$ be a BNIG. A pair (w, xy) is called a *strong pair* if

$$\begin{aligned}
T_Z^+(w, xy) &\geq T_{\eta^\infty}^+(w, xy), \; T_Z^-(w, xy) \leq T_{\eta^\infty}^-(w, xy),\\
I_Z^+(w, xy) &\geq I_{\eta^\infty}^+(w, xy), \; I_Z^-(w, xy) \leq I_{\eta^\infty}^-(w, xy),\\
F_Z^+(w, xy) &\leq F_{\eta^\infty}^+(w, xy), \; F_Z^-(w, xy) \geq F_{\eta^\infty}^-(w, xy).
\end{aligned}$$

A pair (w, xy) is called α-*strong* if

$$\begin{aligned}
&T_Z^+(w,xy) > T_{\eta^\infty}^+(w,xy),\ \ T_Z^-(w,xy) < T_{\eta^\infty}^-(w,xy),\\
&I_Z^+(w,xy) > I_{\eta^\infty}^+(w,xy),\ \ \ I_Z^-(w,xy) < I_{\eta^\infty}^-(w,xy),\\
&F_Z^+(w,xy) < F_{\eta^\infty}^+(w,xy),\ \ F_Z^-(w,xy) > F_{\eta^\infty}^-(w,xy).
\end{aligned}$$

A pair (w, xy) is called β-*strong* if

$$\begin{aligned}
&T_Z^+(w,xy) = T_{\eta^\infty}^+(w,xy),\ \ T_Z^-(w,xy) = T_{\eta^\infty}^-(w,xy),\\
&I_Z^+(w,xy) = I_{\eta^\infty}^+(w,xy),\ \ \ I_Z^-(w,xy) = I_{\eta^\infty}^-(w,xy),\\
&F_Z^+(w,xy) = F_{\eta^\infty}^+(w,xy),\ \ F_Z^-(w,xy) = F_{\eta^\infty}^-(w,xy).
\end{aligned}$$

where $\eta^\infty(w, xy)$ is incidence strength of strongest path between w and xy.

Definition 2.26. Let $G = (X, Y, Z)$ be a BNIG. A pair (w, xy) is called a δ-*pair* if

$$\begin{aligned}
&T_Z^+(w,xy) < T_{\eta^\infty}^+(w,xy),\ \ T_Z^-(w,xy) > T_{\eta^\infty}^-(w,xy),\\
&I_Z^+(w,xy) < I_{\eta^\infty}^+(w,xy),\ \ \ I_Z^-(w,xy) > I_{\eta^\infty}^-(w,xy),\\
&F_Z^+(w,xy) > F_{\eta^\infty}^+(w,xy),\ \ F_Z^-(w,xy) < F_{\eta^\infty}^-(w,xy).
\end{aligned}$$

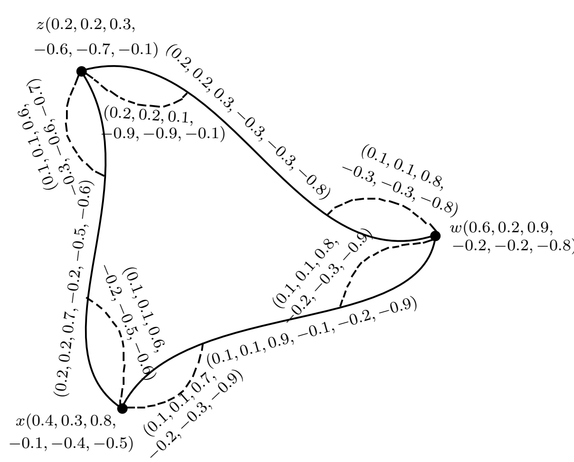

Figure 6: BNIG $G = (X, Y, Z)$

Example 2.27. In Fig.6 all edges except xw are strong. Indeed, wz and xz are $\alpha-$strong edges. whereas, a pair (z, wz) is an α-strong pair and (w, xw) is a β-strong pair.

Definition 2.28. A path P in $G = (X, Y, Z)$ is called a *strong path* if all edges and pairs of P are strong. If strong path is closed, then it is called a *strong cycle*.

Example 2.29. In Fig.7 a path $x, (x, xu), xu, (u, xu), u, (u, uw), uw, (w, uw), w$ is strong path.

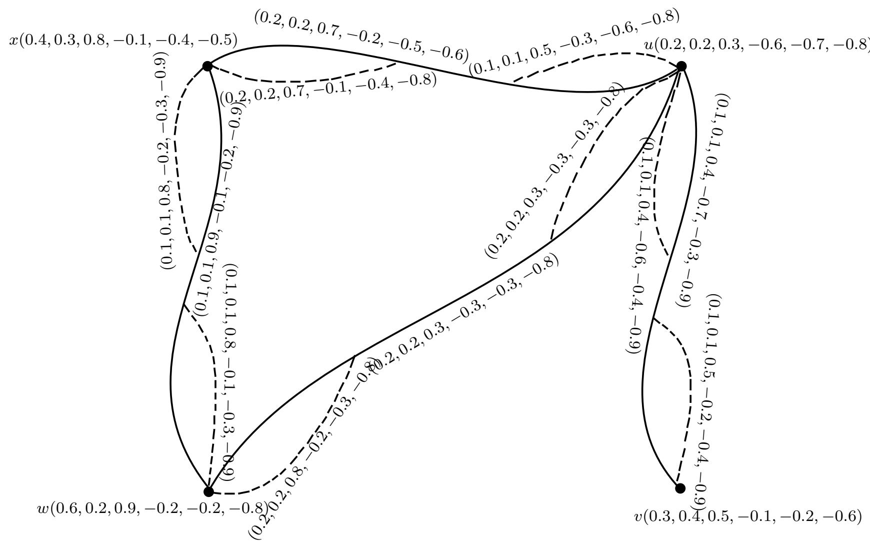

Figure 7: BNIG $G = (X, Y, Z)$

Theorem 2.30. *Let $G = (X, Y, Z)$ be a BNIG. A pair (w, xy) is strong if*

$$\begin{aligned}
T_Z^+(w, xy) &= \vee\{T_Z^+(u, vw)|(u, vw) \in \textit{supp}(Z)\},\\
I_Z^+(w, xy) &= \vee\{I_Z^+(u, vw)\ |(u, vw) \in \textit{supp}(Z)\},\\
F_Z^+(w, xy) &= \wedge\{F_Z^+(u, vw)|(u, vw) \in \textit{supp}(Z)\},\\
T_Z^-(w, xy) &= \wedge\{T_Z^-(u, vw)|(u, vw) \in \textit{supp}(Z)\},\\
I_Z^-(w, xy) &= \wedge\{I_Z^-(u, vw)\ |(u, vw) \in \textit{supp}(Z)\},\\
F_Z^-(w, xy) &= \vee\{F_Z^-(u, vw)|(u, vw) \in \textit{supp}(Z)\}.
\end{aligned}$$

Proof. Let $\psi^\infty(w, xy)$ be an incidence strength of strongest path between w and xy in $G = (X, Y, Z)$, then

$$\begin{aligned}
&T_{\psi^\infty}^+(w, xy) \le T_Z^+(w, xy),\ \ T_{\psi^\infty}^-(w, xy) \ge T_Z^-(w, xy),\\
&I_{\psi^\infty}^+(w, xy) \le I_Z^+(w, xy),\quad I_{\psi^\infty}^-(w, xy) \ge I_Z^-(w, xy),\\
&F_{\psi^\infty}^+(w, xy) \ge F_Z^+(w, xy),\ \ F_{\psi^\infty}^-(w, xy) \le F_Z^-(w, xy).
\end{aligned}$$

If (w, xy) is only one pair such that

$$\begin{aligned}
T_Z^+(w, xy) &= \vee\{T_Z^+(u, vw)|(u, vw) \in \text{supp}(Z)\},\\
I_Z^+(w, xy) &= \vee\{I_Z^+(u, vw)\ |(u, vw) \in \text{supp}(Z)\},\\
F_Z^+(w, xy) &= \wedge\{F_Z^+(u, vw)|(u, vw) \in \text{supp}(Z)\},\\
T_Z^-(w, xy) &= \wedge\{T_Z^-(u, vw)|(u, vw) \in \text{supp}(Z)\},\\
I_Z^-(w, xy) &= \wedge\{I_Z^-(u, vw)\ |(u, vw) \in \text{supp}(Z)\},\\
F_Z^-(w, xy) &= \vee\{F_Z^-(u, vw)|(u, vw) \in \text{supp}(Z)\}.
\end{aligned}$$

then for every path between u and vw, we have

$$\begin{aligned}
&T^+_{\psi\infty}(u,vw)<T^+_Z(w,xy),\ \ T^-_{\psi\infty}(u,vw) >T^-_Z(w,xy),\\
&I^+_{\psi\infty}(u,vw) <I^+_Z(w,xy),\ \ \ I^-_{\psi\infty}(u,vw) > I^-_Z(w,xy),\\
&F^+_{\psi\infty}(u,vw)>F^+_Z(w,xy),\ \ F^-_{\psi\infty}(u,vw) <F^-_Z(w,xy).
\end{aligned}$$

hence

$$\begin{aligned}
&T^+_{\psi\infty}(w,xy)<T^+_Z(w,xy),\ \ T^-_{\psi\infty}(w,xy) >T^+_Z(w,xy),\\
&I^+_{\psi\infty}(w,xy) <I^+_Z(w,xy),\ \ \ I^-_{\psi\infty}(w,xy) > I^+_Z(w,xy),\\
&F^+_{\psi\infty}(w,xy)>F^+_Z(w,xy),\ \ F^-_{\psi\infty}(w,xy) <F^+_Z(w,xy).
\end{aligned}$$

Thus, (w,xy) is an α-strong pair. If (w,xy) is not unique, then

$$\begin{aligned}
&T^+_Z(w,xy)=T^+_{\psi\infty}(w,xy),\ \ T^-_Z(w,xy) =T^-_{\psi\infty}(w,xy),\\
&I^+_Z(w,xy) =I^+_{\psi\infty}(w,xy),\ \ \ I^-_Z(w,xy) = I^-_{\psi\infty}(w,xy),\\
&F^+_Z(w,xy)=F^+_{\psi\infty}(w,xy),\ \ F^-_Z(w,xy) =F^-_{\psi\infty}(w,xy).
\end{aligned}$$

Hence (w,xy) is β-strong pair. □

Theorem 2.31. *If $G=(X,Y,Z)$ is a bipolar neutrosophic incidence tree and P is a strong path between any two vertices x and y. Then P have maximum strength between x and y.*

Proof. Let P be only one strong path between x and y. Because P is strong, all edges and pairs of P are in the spanning bipolar neutrosophic incidence tree H of G. We prove that P is a path between x and y having maximum strength.
Suppose, on contrary that P is not a path having maximum strength from x to y and P' is such a path. Then P and P' are not equal, hence P and and reversal of P' form a cycle. Since H^* is tree, so there exist no cycle in H, . Hence any edge $x'y'$ of P' must not exist in H.
By definition of G, we have

$$\begin{aligned}
&T^+_Y(x'y') < T^+_{\phi\infty}(x',y'),\ \ I^+_Y(x'y')< I^+_{\phi\infty}(x',y'),\ \ F^+_Y(x'y') > F^+_{\phi\infty}(x',y'),\\
&T^-_Y(x'y') > T^-_{\phi\infty}(x',y'),\ \ I^-_Y(x'y')> I^-_{\phi\infty}(x',y'),\ \ F^-_Y(x'y') < F^-_{\phi\infty}(x',y').
\end{aligned}$$

It means there exist a path between x' and y' in H and we can replace all edges $x'y'$ of P' which not exist in H by a path P^* from x to y in H. Hence P^* is at least as strong as P'. Hence P^* and P cannot be equal. So, P and reversal of P^* form a cycle in H, which is a contradiction to the fact that H^* is tree.
Hence our assumption P is not a path having maximum strength from x to y is wrong. □

3 Application to Illegal Migration

Suppose Mr.Kamran wants to travel from Bangladesh to India illegally. For this he use all borders line between Bangladesh and India. He have three ways, first one is a direct way, i.e. Bangladesh to India, second one is Bangladesh to Pakistan and Pakistan to India and the third one is Bangladesh to Bhutan, Bhutan to

Pakistan, Pakistan to Nepal and Nepal to India. Let V={Bangladesh(BGD), Bhutan(BTN), Pakistan(PAK), Nepal(NPL), India(IND)} be the set of countries and $E = \{(BGD, BTN), (BTN, PAK), (PAK, NPL), (NPL, IND), (BGD, PAK), (PAK, IND), (BGD, IND)\}$ a subset of $V \times V$.
Let X be the bipolar neutrosophic set on V, which is given as

$$\begin{aligned} X = \{&(BGD, 0.3, 0.2, 0.6, -0.1, -0.2, -0.5), (BTN, 0.3, 0.6, 0.9, -0.2, -0.4, -0.6),\\ &(PAK, 0.4, 0.5, 0.6, -0.1, -0.3, -0.4), (NPL, 0.9, 0.7, 0.8, -0.4, -0.3, -0.4),\\ &(IND, 0.6, 0.9, 0.1, -0.1, -0.2, -0.3)\}. \end{aligned}$$

Let Y be the bipolar neutrosophic relation on V, which is given as

$$\begin{aligned} Y = \{&((BGD, BTN), 0.1, 0.2, 0.8, -0.2, -0.3, -0.7),((BTN, PAK), 0.2, 0.5, 0.9, -0.3, -0.3, -0.7),\\ &((PAK, NPL), 0.3, 0.4, 0.7, -0.2, -0.4, -0.5),((NPL, IND), 0.5, 0.6, 0.7, -0.2, -0.3, -0.5),\\ &((BGD, PAK), 0.3, 0.1, 0.6, -0.2, -0.2, -0.6),((PAK, IND), 0.4, 0.4, 0.5, -0.1, -0.3, -0.5),\\ &((BGD, IND), 0.2, 0.1, 0.5, -0.1, -0.3, -0.6)\}. \end{aligned}$$

Let Z be the bipolar neutrosophic set on $V \times E$, which is given as

$$\begin{aligned} Z = \{&((BGD, (BGD, BTN)), 0.1, 0.1, 0.7, -0.1, -0.3, -0.8),\\ &((BTN, (BGD, BTN)), 0.1, 0.2, 0.8, -0.3, -0.3, -0.8),\\ &((BTN, (BTN, PAK)), 0.2, 0.4, 0.8, -0.2, -0.3, -0.8),\\ &((PAK, (BTN, PAK)), 0.2, 0.4, 0.8, -0.2, -0.4, -0.7),\\ &((PAK, (PAK, NPL)), 0.3, 0.3, 0.5, -0.1, -0.4, -0.5),\\ &((NPL, (PAK, NPL)), 0.2, 0.3, 0.8, -0.2, -0.3, -0.6),\\ &((NPL, (NPL, IND)), 0.4, 0.5, 0.7, -0.3, -0.3, -0.6),\\ &((IND, (NPL, IND)), 0.4, 0.5, 0.5, -0.1, -0.2, -0.7),\\ &((BGD, (BGD, PAK)), 0.1, 0.1, 0.5, -0.2, -0.3, -0.7),\\ &((PAK, (BGD, PAK)), 0.1, 0.1, 0.5, -0.2, -0.2, -0.6),\\ &((PAK, (PAK, IND)), 0.3, 0.3, 0.5, -0.1, -0.3, -0.6),\\ &((IND, (PAK, IND)), 0.4, 0.3, 0.4, -0.1, -0.3, -0.6),\\ &((BGD, (BGD, IND)), 0.1, 0.1, 0.4, -0.2, -0.2, -0.7),\\ &((IND, (BGD, IND)), 0.1, 0.1, 0.5, -0.1, -0.3, -0.8)\}. \end{aligned}$$

Thus, $G = (X, Y, Z)$ is a BNIG as shown in Fig.8.
Let $T^+_\rho(u, v)$ represent the degree of protection for an illegal immigrant to use u as origin and come to a destination v. There are three paths from BGD to IND

$$P_1 : BGD,(BGD, (BGD, IND)),(BGD, IND), (IND, (BGD, IND)),IND.$$

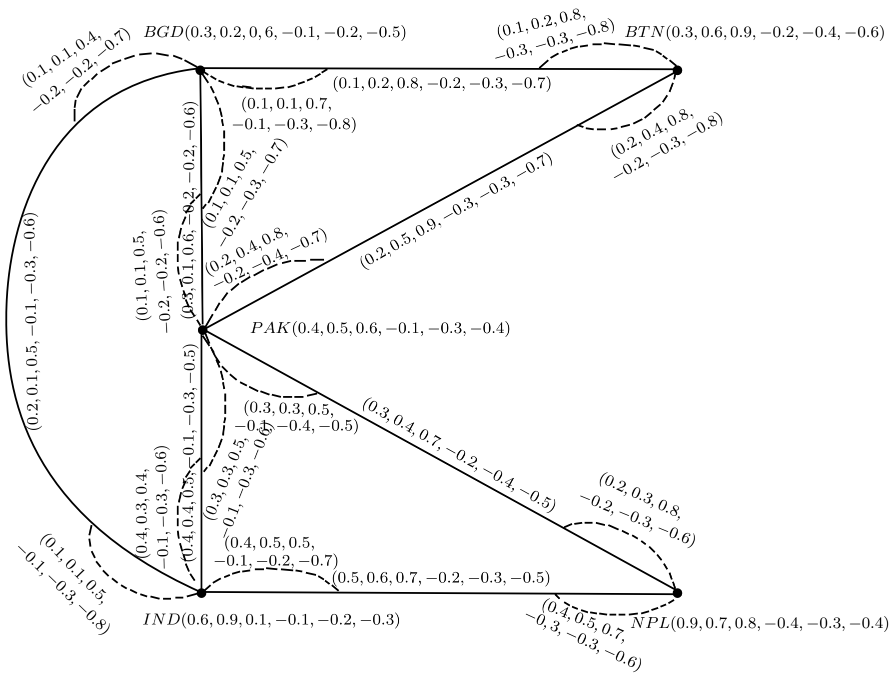

Figure 8: BNIG $G = (X, Y, Z)$

$$P_2 : BGD, (BGD, (BGD, PAK)), (BGD, PAK), (PAK, (BGD, PAK)), PAK, (PAK, (PAK, IND)), (PAK, IND), (IND, (PAK, IND)), IND.$$

$$P_3 : BGD, (BGD, (BGD, BTN)), (BGD, BTN), (BTN, (BGD, BTN)), BTN, (BTN, (BTN, PAK)), (BTN, PAK), (PAK, (BTN, PAK)), PAK, (PAK, (PAK, NPL)), (PAK, NPL), (NPL, (PAK, NPL)), NPL, (NPL, (NPL, IND)), (NPL, IND), (IND, (NPL, IND)), IND.$$

$\rho^\infty(BGD, IND)$ is the strength of strongest path between BGD and IND. This is the safest path between BGD and IND. To calculate the value of $\rho^\infty(BGD, IND)$, we need the strength of paths P_1, P_2 and P_3, which is denoted by $\rho_{P_1}(BGD, IND)$, $\rho_{P_2}(BGD, IND)$ and $\rho_{P_3}(BGD, IND)$, respectively.
By calculation, we have

$$\rho_{P_1}(BGD, IND) = (0.2, 0.1, 0.5, -0.1, -0.3, -0.6),$$

$$\rho_{P_2}(BGD, IND) = (0.3, 0.1, 0.6, -0.1, -0.2, -0.6),$$

$$\rho_{P_3}(BGD, IND) = (0.1, 0.2, 0.9, -0.2, -0.3, -0.7).$$

$\rho^\infty(BGD, IND) =(0.3, 0.2, 0.5, -0.2, -0.3, -0.6)$.
We see that

$$T^+_{\rho^\infty}(BGD, IND) = T^+_{\rho_{P_2}}(BGD, IND).$$

Hence P_2 is safest path for an illegal immigrant.
We present proposed method in the following Algorithm 3.1.

3.1 Algorithm

1. Input the vertex set V^*.

2. Input the edge set $E^* \subseteq V^* \times V^*$.

3. Input the bipolar neutrosophic set X on V^*.

4. Input the bipolar neutrosophic relation Y on V^*.

5. Input the bipolar neutrosophic set Z on $V^* \times E^*$.

6. Calculate the strength of path $\rho(x, y)$ from x to y such that

$$\begin{aligned}
T^+_\rho(x,y) &= \wedge \{T^+_Y(xy) | xy \in \text{supp}(Y)\},\\
I^+_\rho(x,y) &= \wedge \{I^+_Y(xy) \,| xy \in \text{supp}(Y)\},\\
F^+_\rho(x,y) &= \vee \{F^+_Y(xy) | xy \in \text{supp}(Y)\},\\
T^-_\rho(x,y) &= \vee \{T^-_Y(xy) | xy \in \text{supp}(Y)\},\\
I^-_\rho(x,y) &= \vee \{I^-_Y(xy) \,| xy \in \text{supp}(Y)\},\\
F^-_\rho(x,y) &= \wedge \{F^-_Y(xy) | xy \in \text{supp}(Y)\}.
\end{aligned}$$

7. Calculate the incidence strength $\rho^\infty(x, y)$ of strongest path from x to y such that

$$T^+_{\rho^\infty}(x,y) = \bigvee_{i=1}^{k} T^+_{\rho_i}(x,y),\quad I^+_{\rho^\infty}(x,y) = \bigvee_{i=1}^{k} I^+_{\rho_i}(x,y),\quad F^+_{\rho^\infty}(x,y) = \bigwedge_{i=1}^{k} F^+_{\rho_i}(x,y),$$
$$T^-_{\rho^\infty}(x,y) = \bigwedge_{i=1}^{k} T^-_{\rho_i}(x,y),\quad I^-_{\rho^\infty}(x,y) = \bigwedge_{i=1}^{k} I^-_{\rho_i}(x,y),\quad F^-_{\rho^\infty}(x,y) = \bigvee_{i=1}^{k} F^-_{\rho_i}(x,y).$$

8. The safest path is $S(v_k) = \bigvee_{i=1}^{k} T^+_{\rho_i}(x,y)$.

9. If v_k has more than one value then any path can be chosen.

4 Conclusion

Graph theory has become a branch of applied mathematics. Graph theory is considered as a mathematical tool for modeling and analyzing different mathematical structure, but it does not give the relationship between element and its relation pair. We have introduced BNIG which not only give the limitation of the relation between elements contained in a set, but also give the influence or impact of an element to its relation pair. We have defined the bipolar neutrosophic incidence cycle and tree. An application to illegal migration is presented using strength of strongest path in BNIG.

Conflict of interest: The authors declare that they have no conflict of interest.

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