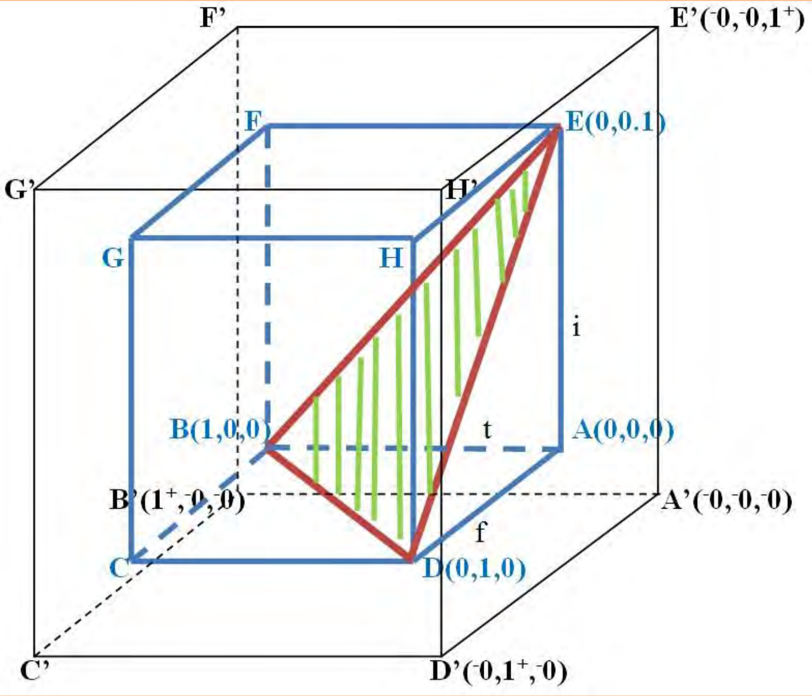
Full Issue

Vol. 17, 2017

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering



ISSN 2331-6055 (print)
ISSN 2331-608X (online)

Neutrosophic Sets and Systems

A Quarterly International Journal in Information Science and Engineering

Information for Authors and Subscribers

*"*Neutrosophic Sets and Systems*"* has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

All submissions should be designed in MS Word format using our template file:

http://fs.gallup.unm.edu/NSS/NSS-paper-template.doc.

A variety of scientific books in many languages can be downloaded freely from the Digital Library of Science:

http://fs.gallup.unm.edu/eBooks-otherformats.htm.

To submit a paper, mail the file to the Editor-in-Chief. To order printed issues, contact the Editor-in-Chief. This journal is non-commercial, academic edition. It is printed from private donations.

Information about the neutrosophics you get from the UNM website:

http://fs.gallup.unm.edu/neutrosophy.htm.

The home page of the journal is accessed on

http://fs.gallup.unm.edu/NSS.

// Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

Volume 17 **2017**

Contents

NSS

University of New Mexico

Covering-Based Rough Single Valued Neutrosophic Sets

Yingcang Ma, Wanying Zhou, Qing Wan

School of Science, Xi'an Polytechnic University, Xi'an Shaanxi 710048, China. E-mail: mayingcang@126.com, 892443522@qq.com, wqysbe@163.com

Abstract: Rough sets theory is a powerful tool to deal with uncertainty and incompleteness of knowledge in information systems. Wang et al. proposed single valued neutrosophic sets as an extension of intuitionistic fuzzy sets to deal with real-world problems. In this paper, we propose the covering-based rough single valued neutrosophic sets by combining covering-based rough sets and single valued neutrosophic sets. Firstly, three types of covering-based rough single valued neutrosophic sets models are built and the properties of lower/upper approximation operators are explored. Secondly, the lower/upper approximations in two different covering approximation spaces are studied. The sufficient and necessary condition for generating the same lower/upper approximations from two different covering approximation spaces is discussed. Moreover, the relations of the three models are discussed and the equivalence conditions for three models are given.

Keywords: covering-based rough sets, single valued neutrosophic sets, neutrosophic sets, covering-based rough single valued neutrosophic sets.

1 Introduction

Rough set theory (RST), proposed by Pawlak[1] in 1982, is one of the effective mathematical tools for processing fuzzy and uncertainty knowledge. The classical rough set theory is based on the equivalence relation on the domain. In many practical problems, the relation between objects is essentially no equivalence relation, so this equivalence relation as the basis of the classic rough set model cannot fully meet the actual needs. For this a lot of extension models of Pawlak rough set are given. One approach is to extend the equivalence realtion to similarity relations[2], tolerance relations[3], ordinary binary relations[4], reflexive and transitive relations[5] and others. The other approach is combining the other theory to get more flexible and expressive framework for modeling and processing incomplete information in information systems. Mi et al.[6] introduced the definitions for generalized fuzzy lower and upper approximation operators determined by a residual implication. Pei [7] studied generalized fuzzy rough sets. Zhang et al.[8] gave a general framework of intuitionistic fuzzy rough set theory. Yang et al. [9]proposed hesitant fuzzy rough sets and studied the models axiomatic characterizations by combining hesitant fuzzy sets and rough sets.Zhang et al.[10] further gave the construction and axiomatic characterizations of interval-valued hesitant fuzzy rough sets, and illustrated the application of the model.

Covering rough sets theory is an important rough sets theory. Covering rough set model, first proposed by Zakowski[11] in 1983, Bonikowski et al. later studied the structures of covering[12]. Chen et al. [13]discussed the covering rough sets within the framework of a completely distributive lattice. Zhu and Wang [14]proposed the reduction of covering rough sets to reduce the "redundant" members in a covering in order to find the "smallest" covering. Deng et al. [15] established fuzzy rough set models based on a covering. Li et al. [16] proposed a generalized fuzzy rough approximation operators based on fuzzy coverings. Wei et al. [17]and Xu et al. [18] established the first and second types of rough fuzzy set models based on a covering. Hu et al.[19] proposed the third type of rough fuzzy set models based on a covering. Tang et al. [20] gave the fourth type of rough fuzzy set models based on a covering.

Smarandache [21] proposed neutrosophic sets to deal with real-world problems. A neutrosophic set has three membership functions: truth membership function, indeterminacy membership function and falsity membership function, in which each membership degree is a real standard or non-standard subset of the nonstandard unit interval $]0-, 1+[$. Wang et al. [22] introduced single valued neutrosophic sets (SVNSs) that is a generalization of intuitionistic fuzzy sets, in which three membership functions are independent and their values belong to the unit interval $[0, 1]$. Further studies have done in recent years. Such as, Majumdar and Samanta [23] studied similarity and entropy of SVNSs. Ye [24] proposed correlation coefficients of SVNSs, and applied it to single valued neutrosophic decision-making problems, etc.

SVNSs and covering rough sets are two different tools of dealing with uncertainty information. In order to use the advantages of SVNSs and covering rough sets, we establish a hybrid model of SVNSs and covering rough sets. Broumi and Smarandache proposed single valued neutrosophic information systems based on rough set theory [25]. Yang et al. proposed single valued neutrosophic rough set model and single valued neutrosophic refined rough set model[26,27]. In the present paper, we shall propose covering-based rough single valued neutrosophic sets by fusing SVNSs and covering rough sets, and explore a general framework of the study of covering-based rough single valued neutrosophic sets.

The paper is organized as follows. After this introduction, In section 2, we provide the basic notions and operations of Pawlak rough sets, covering rough sets and SVNSs. Based on a SVNR, Sect. 3 proposes three types of covering-based rough single valued neutrosophic sets. Properties of lower/upper approximation operators are studied. In Sect. 4, we investigate the relations of the three types models. The last section summarizes the conclusions and gives an outlook for future research.

2 Preliminaries

In this section, we give basic notions and operations on Pawlak tough sets, covering-based rough sets and SVNSs.

Definition 2.1 *Let U be a non-empty finite university and R be an equivalence relations on U. (U, R) is called a Pawlak approximation space. $\forall X \subseteq U$, the lower and upper approximations of X, denoted by $\underline{R}(X)$ and $\overline{R}(X)$, are defined as follows, respectively:*

$\underline{R}(X = \{x \in U | [x]_R \subseteq X\}$,

$\overline{R}(X = \{x \in U | [x]_R \cap X \neq \emptyset\}$,

where $[x]_R = \{y \in U | (x, y) \in R\}$. $\underline{R}(X)$ and $\overline{R}(X)$ are called as lower and upper approximations operators, respectively. The pair $(\underline{R}(X), \overline{R}(X))$ is called a Pawlak rough set.

Definition 2.2 *Let U be a non-empty finite university, C is a family of subsets of U. If none subsets in C is empty and $\cup C = U$, then C is a covering of U.*

Definition 2.3 *Let C be a covering of U, $x \in U$. $Md_C(x) = \{K \in C \wedge (\forall S \in C \wedge x \in S \wedge S \subseteq K \Rightarrow K = S)\}$ is called the minimal description of x, When the covering is clear, we omit the lowercase C in the minimal description.*

Definition 2.4 *Let U be a space of points (objects), with a generic element in U denoted by u. A SVNS A in U is characterized by three membership functions, a truth membership function T_A, an indeterminacy membership function I_A and a falsity-membership function F_A, where $\forall u \in U, T_A(u), I_A(u), F_A(u) \in [0, 1]$. That is $T_A : U \to [0, 1], I_A : U \to [0, 1]$ and $F_A : U \to [0, 1]$. There is no restriction on the sum of $T_A(u), I_A(u)$ and $F_A(u)$, thus $0 \leq T_A(u) + I_A(u) + F_A(u) \leq 3$.*

Here A can be denoted by $A = \{\langle u, T_A(u), I_A(u), F_A(u)\rangle | u \in U\}$, $\forall u \in U, (T_A(u), I_A(u), F_A(u))$ is called a single valued neutrosophic number(SVNN).

Definition 2.5 *Let A and B be two SVNSs on U. If for any $u \in U$, $T_A(u) \leq T_B(u), I_A(u) \geq I_B(u), F_A(u) \geq F_B(u)$, then we called A is contained in B, denoted by $A \Subset B$.*

If $A \Subset B$ and $B \Subset A$, then we called A is equal to B, denoted by $A = B$.

Definition 2.6 *Let A be a SVNS on U. The complement of A is denoted by A^c, where $\forall u \in U$, $T_{A^c}(u) = F_A(u), I_{A^c}(u) = 1 - I_A(u), F_{A^c}(u) = T_A(u)$.*

Definition 2.7 *Let A and B be two SVNS on U. The union of A and B is a SVNS C, denoted by $C = A \Cup B$, where $\forall u \in U$, $T_C(u) = \max\{T_A(u), T_B(u)\}$, $I_C(u) = \min\{I_A(u), I_B(u)\}$, $F_C(u) = \min\{F_A(u), F_B(u)\}$.*

The intersection of A and B is a SVNS D, denoted by $D = A \Cap B$, where $\forall u \in U$, $T_D(u) = \min\{T_A(u), T_B(u)\}$, $I_C(u) = \max\{I_A(u), I_B(u)\}$, $F_C(u) = \max\{F_A(u), F_B(u)\}$.

Proposition 2.8 *[26] Let A and B be two SVNS on U. The following results hold:*

(1) $A \Subset A \Cup B$ and $B \Subset A \Cup B$;

(2) $A \Cap B \Subset A$ and $A \Cap B \Subset B$;

(3) $(A^c)^c = A$;

(4) $(A \Cup B)^c = A^c \Cap B^c$;

(5) $(A \Cap B)^c = A^c \Cup B^c$.

3 Covering-based rough neutrosophic sets

Definition 3.1 *Let U be a non-empty finite university, C is a covering of U, (U, C) be a covering approximation space. A is a SVNS of U. The first type of lower and upper approximations of A with respect to (U, C), denoted by $FL(A)$ and $FU(A)$, are two SVNSs whose membership functions are defined as $\forall u \in U$,*

$T_{FL(A)}(u) = \inf\{T_A(v) | v \in \cup Md(u)\}$,

$I_{FL(A)}(u) = \sup\{I_A(v) | v \in \cup Md(u)\}$,

$F_{FL(A)}(u) = \sup\{F_A(v) | v \in \cup Md(u)\}$,

$T_{FU(A)}(u) = \sup\{T_A(v) | v \in \cup Md(u)\}$,

$I_{FU(A)}(u) = \inf\{I_A(v) | v \in \cup Md(u)\}$,

$F_{FU(A)}(u) = \inf\{F_A(v) | v \in \cup Md(u)\}$.

The pair $(FL(A), FU(A))$ is called the first type of rough single valued neutrosophic set based on covering C. $FL(A)$ and $FU(A)$ are called as the first lower and upper approximations operators, respectively.

Definition 3.2 *Let U be a non-empty finite university, C is a covering of U, (U, C) be a covering approximation space. A is a SVNS of U. The second type of lower and upper approximations of A with respect to (U, C), denoted by $SL(A)$ and $SU(A)$, are two SVNSs whose membership functions are defined as $\forall u \in U$,*

$T_{SL(A)}(u) = \inf\{T_A(v) | v \in \cap Md(u)\}$,

$I_{SL(A)}(u) = \sup\{I_A(v) | v \in \cap Md(u)\}$,

$F_{SL(A)}(u) = \sup\{F_A(v) | v \in \cap Md(u)\}$,

$T_{SU(A)}(u) = \sup\{T_A(v) | v \in \cap Md(u)\}$,

$I_{SU(A)}(u) = \inf\{I_A(v) | v \in \cap Md(u)\}$,

$F_{SU(A)}(u) = \inf\{F_A(v) | v \in \cap Md(u)\}$.

The pair $(SL(A), SU(A))$ is called the second type of rough single valued neutrosophic set based on covering C. $SL(A)$ and $SU(A)$ are called as the second lower and upper approximations operators, respectively.

Definition 3.3 *Let U be a non-empty finite university, C is a covering of U, (U, C) be a covering approximation space. A is a SVNS of U. The third type of lower and upper approximations*

of A with respect to (U,C), denoted by $TL(A)$ and $TU(A)$, are two SVNSs whose membership functions are defined as $\forall u \in U$,
$T_{TL(A)}(u) = \sup_{K\in Md(u)}\{\inf_{v\in K}\{T_A(v)\}\}$,
$I_{TL(A)}(u) = \inf_{K\in Md(u)}\{\sup_{v\in K}\{I_A(v)\}\}$,
$F_{TL(A)}(u) = \inf_{K\in Md(u)}\{\sup_{v\in K}\{F_A(v)\}\}$.
$T_{TU(A)}(u) = \inf_{K\in Md(u)}\{\sup_{v\in K}\{T_A(v)\}\}$,
$I_{TU(A)}(u) = \sup_{K\in Md(u)}\{\inf_{v\in K}\{I_A(v)\}\}$,
$F_{TU(A)}(u) = \sup_{K\in Md(u)}\{\inf_{v\in K}\{F_A(v)\}\}$,
The pair $(TL(A), TU(A))$ is called the third type of rough single valued neutrosophic set based on covering C. $TL(A)$ and $TU(A)$ are called as the third lower and upper approximations operators, respectively.

Example 3.4 *Let $U = \{a,b,c,d\}$, $K_1 = \{a,b\}, K_2 = \{b,c\}, K_3 = \{c,d\}$, $C = \{K_1,K_2,K_3\}$. A single valued neutrosophic set $A = \{\langle a,(0.2,0.8,0.1)\rangle, \langle b,(1,0.3,1)\rangle, \langle c,(0.5,0.3,0)\rangle, \langle d,(0.6,0.7,0.5)\rangle\}$, then $Md(a) = \{\{a,b\}\}$, $Md(b) = \{\{a,b\},\{b,c\}\}$, $Md(c) = \{\{b,c\},\{c,d\}\}$, $Md(d) = \{\{c,d\}\}$. Thus,*
$T_{FL(A)}(a) = \inf\{T_A(v)|v \in \cup Md(a)\} = \inf\{T_A(a), T_A(b)\} = \inf\{0.2,1\} = 0.2$.
$T_{FL(A)}(b) = \inf\{T_A(v)|v \in \cup Md(b)\} = \inf\{T_A(a), T_A(b), T_A(c)\} = \inf\{0.2,1,0.5\} = 0.2$.
$T_{FL(A)}(c) = \inf\{T_A(v)|v \in \cup Md(c)\} = \inf\{T_A(b), T_A(c), T_A(d)\} = \inf\{1,0.5,0.6\} = 0.5$.
$T_{FL(A)}(d) = \inf\{T_A(v)|v \in \cup Md(d)\} = \inf\{T_A(c), T_A(d))\} = \inf\{0.5,0.6\} = 0.5$.
$T_{FU(A)}(a) = \sup\{T_A(v)|v \in \cup Md(a)\} = \sup\{T_A(a), T_A(b)\} = \sup\{0.2,1\} = 1$.
$T_{FU(A)}(b) = \sup\{T_A(v)|v \in \cup Md(b)\} = \sup\{T_A(a), T_A(b), T_A(c)\} = \sup\{0.2,1,0.5\} = 1$.
$T_{FU(A)}(c) = \sup\{T_A(v)|v \in \cup Md(c)\} = \sup\{T_A(b), T_A(c), T_A(d)\} = \sup\{1,0.5,0.6\} = 1$.
$T_{FU(A)}(d) = \sup\{T_A(v)|v \in \cup Md(d)\} = \sup\{T_A(c), T_A(d))\} = \sup\{0.5,0.6\} = 0.6$.
$I_{FL(A)}(a) = \sup\{I_A(v)|v \in \cup Md(a)\} = \sup\{I_A(a), I_A(b)\} = \sup\{0.8,0.3\} = 0.8$.
$I_{FL(A)}(b) = \sup\{I_A(v)|v \in \cup Md(b)\} = \sup\{I_A(a), I_A(b), T_A(c)\} = \sup\{0.8,0.3,0.3\} = 0.8$.
$I_{FL(A)}(c) = \sup\{I_A(v)|v \in \cup Md(c)\} = \sup\{I_A(b), I_A(c), I_A(d)\} = \sup\{0.3,0.3,0.7\} = 0.7$.
$I_{FL(A)}(d) = \sup\{I_A(v)|v \in \cup Md(d)\} = \sup\{I_A(c), I_A(d))\} = \sup\{0.3,0.7\} = 0.7$.
$I_{FU(A)}(a) = \inf\{I_A(v)|v \in \cup Md(a)\} = \inf\{I_A(a), I_A(b)\} = \inf\{0.8,0.3\} = 0.3$.
$I_{FU(A)}(b) = \inf\{I_A(v)|v \in \cup Md(b)\} = \inf\{I_A(a), I_A(b), I_A(c)\} = \inf\{0.8,0.3,0.3\} = 0.3$.
$I_{FU(A)}(c) = \inf\{I_A(v)|v \in \cup Md(c)\} = \inf\{I_A(b), I_A(c), I_A(d)\} = \inf\{0.3,0.3,0.7\} = 0.3$.
$I_{FU(A)}(d) = \inf\{I_A(v)|v \in \cup Md(d)\} = \inf\{I_A(c), I_A(d))\} = \inf\{0.3,0.7\} = 0.3$.
$F_{FL(A)}(a) = \sup\{F_A(v)|v \in \cup Md(a)\} = \sup\{F_A(a), F_A(b)\} = \sup\{0.1,1\} = 1$.
$F_{FL(A)}(b) = \sup\{F_A(v)|v \in \cup Md(b)\} = \sup\{F_A(a), F_A(b), T_A(c)\} = \sup\{0.1,1,0\} = 1$.
$F_{FL(A)}(c) = \sup\{F_A(v)|v \in \cup Md(c)\} = \sup\{F_A(b), F_A(c), F_A(d)\} = \sup\{1,0,0.5\} = 1$.
$F_{FL(A)}(d) = \sup\{F_A(v)|v \in \cup Md(d)\} = \sup\{F_A(c), F_A(d))\} = \sup\{0,0.5\} = 0.5$.
$F_{FU(A)}(a) = \inf\{F_A(v)|v \in \cup Md(a)\} = \inf\{F_A(a), F_A(b)\} = \inf\{0.1,1\} = 0.1$.
$F_{FU(A)}(b) = \inf\{F_A(v)|v \in \cup Md(b)\} = \inf\{F_A(a), F_A(b), F_A(c)\} = \inf\{0.1,1,0\} = 0$.
$F_{FU(A)}(c) = \inf\{F_A(v)|v \in \cup Md(c)\} = \inf\{F_A(b), F_A(c), F_A(d)\} = \inf\{1,0,0.5\} = 0$.
$F_{FU(A)}(d) = \inf\{F_A(v)|v \in \cup Md(d)\} = \inf\{F_A(c), F_A(d))\} = \inf\{0,0.5\} = 0$.
Thus,
$FL(A) = \{\langle a,(0.2,0.8,1)\rangle, \langle b,(0.2,0.8,1)\rangle, \langle c,(0.5,0.7,1)\rangle, \langle d,(0.5,0.7,0.5)\rangle\}$,
$FU(A) = \{\langle a,(1,0.3,0.1)\rangle, \langle b,(1,0.3,0)\rangle, \langle c,(1,0.3,0)\rangle, \langle d,(0.6,0.3,0)\rangle\}$.
Similarly,
$SL(A) = \{\langle a,(0.2,0.8,1)\rangle, \langle b,(1,0.3,1)\rangle, \langle c,(0.5,0.3,0)\rangle, \langle d,(0.5,0.7,0.5)\rangle\}$,
$SU(A) = \{\langle a,(1,0.3,0.1)\rangle, \langle b,(1,0.3,1)\rangle, \langle c,(0.5,0.3,0)\rangle, \langle d,(0.6,0.3,0)\rangle\}$.
$TL(A) = \{\langle a,(0.2,0.8,1)\rangle, \langle b,(0.5,0.3,1)\rangle, \langle c,(0.5,0.3,0.5)\rangle, \langle d,(0.5,0.7,0.5)\rangle\}$,
$TU(A) = \{\langle a,(1,0.3,0.1)\rangle, \langle b,(1,0.3,0.1)\rangle, \langle c,(0.6,0.3,0)\rangle, \langle d,(0.6,0.3,0)\rangle\}$.

Proposition 3.5 *The first type of rough single valued neutrosophic lower and upper approximation operators defined in Definition 3.1 has the following properties: $\forall A,B \in SVNS(U)$,*
(1) $FL(U) = U, FU(U) = U$;
(2) $FL(\emptyset) = \emptyset, FU(\emptyset) = \emptyset$;
(3) $FL(A) \Subset A \Subset FU(A)$;
(4) $FL(A \Cap B) = FL(A) \Cap FL(B)$, $FU(A \Cup B) = FU(A) \Cup FL(B)$;
(5) $A \Subset B \Rightarrow FL(A) \Subset FL(B)$, $A \Subset B \Rightarrow FU(A) \Subset FU(B)$;
(6) $FU(A \Cap B) \Subset FU(A) \Cap FU(B)$, $FL(A \Cup B) \Supset FL(A) \Cup FL(B)$;
(7) $FL(A^c) = (FU(A))^c$, $FU(A^c) = (FL(A))^c$.

Proof: (1) $T_{FL(U)}(u) = \inf\{T_U(v)|v \in \cup Md(u)\} = 1$, $T_{FU(U)}(u) = \sup\{T_U(v)|v \in \cup Md(u)\} = 1$, $I_{FL(U)}(u) = \sup\{I_U(v)|v \in \cup Md(u)\} = 0$, $I_{FU(U)}(u) = \inf\{I_U(v)|v \in \cup Md(u)\} = 0$, $F_{FL(U)}(u) = \sup\{F_U(v)|v \in \cup Md(u)\} = 0$, $F_{FU(U)}(u) = \inf\{F_U(v)|v \in \cup Md(u)\} = 0$, thus $FL(U) = U, FU(U) = U$.

(2) $T_{FL(\emptyset)}(u) = \inf\{T_\emptyset(v)|v \in \cup Md(u)\} = 0$, $T_{FU(\emptyset)}(u) = \sup\{T_\emptyset(v)|v \in \cup Md(u)\} = 0$, $I_{FL(\emptyset)}(u) = \sup\{I_\emptyset(v)|v \in \cup Md(u)\} = 1$, $I_{FU(\emptyset)}(u) = \inf\{I_\emptyset(v)|v \in \cup Md(u)\} = 1$, $F_{FL(\emptyset)}(u) = \sup\{F_\emptyset(v)|v \in \cup Md(u)\} = 1$, $F_{FU(\emptyset)}(u) = \inf\{F_\emptyset(v)|v \in \cup Md(u)\} = 1$, thus $FL(\emptyset) = \emptyset, FU(\emptyset) = \emptyset$.

(3) Being $u \in \cup Md(u)$, so $T_{FL(A)}(u) = \inf\{T_A(v)|v \in \cup Md(u)\} \leq T_A(u) \leq T_{FU(A)}(u) = \sup\{T_A(v)|v \in \cup Md(u)\} =$, $I_{FL(A)}(u) = \sup\{I_A(v)|v \in \cup Md(u)\} \geq$

$I_A(u) \geq I_{FU(A)}(u) = \inf\{I_A(v)|v \in \cup Md(u)\} =$, $F_{FL(A)}(u) = \sup\{F_A(v)|v \in \cup Md(u)\} \geq F_A(u) \geq F_{FU(A)}(u) = \inf\{F_A(v)|v \in \cup Md(u)\} =$, thus, $FL(A) \Subset A \Subset FU(A)$.

(4) $T_{FL}(A \Cap B)(u) = \inf\{T_{A\Cap B}(v)|v \in \cup Md(u)\} = \inf\{\min\{T_A(v), T_B(v)\}|v \in \cup Md(u)\} = \min\{\inf\{T_A(v)|v \in \cup Md(u)\}, \inf\{T_B(v)\}|v \in \cup Md(u)\} = \min\{T_{FL(A)}(u), T_{FL(B)}(u)\}$.

$I_{FL}(A \Cap B)(u) = \sup\{I_{A\Cap B}(v)|v \in \cup Md(u)\} = \sup\{\max\{I_A(v), I_B(v)\}|v \in \cup Md(u)\} = \max\{\sup\{I_A(v)|v \in \cup Md(u)\}, \sup\{I_B(v)\}|v \in \cup Md(u)\} = \max\{I_{FL(A)}(u), I_{FL(B)}(u)\}$.

$F_{FL}(A \Cap B)(u) = \sup\{F_{A\Cap B}(v)|v \in \cup Md(u)\} = \sup\{\max\{F_A(v), F_B(v)\}|v \in \cup Md(u)\} = \max\{\sup\{F_A(v)|v \in \cup Md(u)\}, \sup\{F_B(v)\}|v \in \cup Md(u)\} = \max\{F_{FL(A)}(u), F_{FL(B)}(u)\}$. Thus, $FL(A \Cap B) = FL(A) \Cap FL(B)$.

$T_{FU}(A \Cup B)(u) = \sup\{T_{A\Cup B}(v)|v \in \cup Md(u)\} = \sup\{\max\{T_A(v), T_B(v)\}|v \in \cup Md(u)\} = \max\{\sup\{T_A(v)|v \in \cup Md(u)\}, \sup\{T_B(v)\}|v \in \cup Md(u)\} = \max\{T_{FU(A)}(u), T_{FU(B)}(u)\}$.

$I_{FU}(A \Cup B)(u) = \inf\{I_{A\Cup B}(v)|v \in \cup Md(u)\} = \inf\{\min\{I_A(v), I_B(v)\}|v \in \cup Md(u)\} = \min\{\inf\{I_A(v)|v \in \cup Md(u)\}, \inf\{I_B(v)\}|v \in \cup Md(u)\} = \min\{I_{FU(A)}(u), I_{FU(B)}(u)\}$.

$F_{FU}(A \Cup B)(u) = \inf\{F_{A\Cup B}(v)|v \in \cup Md(u)\} = \inf\{\min\{F_A(v), F_B(v)\}|v \in \cup Md(u)\} = \min\{\inf\{F_A(v)|v \in \cup Md(u)\}, \inf\{F_B(v)\}|v \in \cup Md(u)\} = \min\{F_{FL(A)}(u), F_{FL(B)}(u)\}$. Thus, $FL(A \Cup B) = FL(A) \Cup FL(B)$.

So (4) holds.

(5) If $A \Subset B$, then $T_{FL(A)}(u) = \inf\{T_A(v)|v \in \cup Md(u)\} \leq \inf\{T_B(v)|v \in \cup Md(u)\} = T_{FL(B)}(u)$, $I_{FL(A)}(u) = \sup\{I_A(v)|v \in \cup Md(u)\} \geq \sup\{I_B(v)|v \in \cup Md(u)\} = I_{FL(B)}(u)$, $F_{FL(A)}(u) = \sup\{F_A(v)|v \in \cup Md(u)\} \geq \sup\{F_B(v)|v \in \cup Md(u)\} = F_{FL(B)}(u)$. So, $FL(A) \Subset FL(B)$.

The similar method we can get $A \Subset B \Rightarrow FU(A) \Subset FU(B)$. So (5) holds.

(6) Being $A \Cap B \Subset A \Subset A \Cup B$, $A \Cap B \Subset B \Subset A \Cup B$, from (5), (6) holds.

(7) $T_{FL(A^c)}(u) = \inf\{T_{A^c}(v)|v \in \cup Md(u)\} = \inf\{F_A(v)|v \in \cup Md(u)\} = F_{FU(A)}(u) = T_{(FU(A))^c})(u)$.

$I_{FL(A^c)}(u) = \sup\{I_{A^c}(v)|v \in \cup Md(u)\} = \sup\{1 - I_A(v)|v \in \cup Md(u)\} = 1 - \inf\{I_A(v)|v \in \cup Md(u)\} = 1 - I_{FU(A)})(u) = I_{(FU(A))^c}(u)$.

$F_{FL(A^c)}(u) = \sup\{F_{A^c}(v)|v \in \cup Md(u)\} = \sup\{T_A(v)|v \in \cup Md(u)\} = T_{FU(A)}(u) = F_{(FU(A))^c})(u)$.

So, $FL(A^c) = (FU(A))^c$. The similar method we can get $FU(A^c) = (FL(A))^c$, thus (7) holds.

Remark: $FL(FL(A)) = FL(A)$ and $FU(FU(A)) = FU(A)$ do not hold generally.

Similarly, we can get the following proposition.

Proposition 3.6 *The second type of rough single valued neutrosophic lower and upper approximation operators defined in Definition 3.2 has the following properties:* $\forall A, B \in SVNS(U)$,

(1) $SL(U) = U, SU(U) = U$;

(2) $SL(\emptyset) = \emptyset, SU(\emptyset) = \emptyset$;

(3) $SL(A) \Subset A \Subset SU(A)$;

(4) $SL(A \Cap B) = SL(A) \Cap SL(B)$, $SU(A \Cup B) = SU(A) \Cup SL(B)$;

(5) $A \Subset B \Rightarrow SL(A) \Subset SL(B)$, $A \Subset B \Rightarrow SU(A) \Subset SU(B)$;

(6) $SU(A \Cap B) \Subset SU(A) \Cap SU(B)$, $SL(A \Cup B) \Supset SL(A) \Cup SL(B)$;

(7) $SL(A^c) = (SU(A))^c$, $SU(A^c) = (SL(A))^c$.

Proposition 3.7 *The third type of rough single valued neutrosophic lower and upper approximation operators defined in Definition 3.3 has the following properties:* $\forall A, B \in SVNS(U)$,

(1) $TL(U) = U, TU(U) = U$;

(2) $TL(\emptyset) = \emptyset, TU(\emptyset) = \emptyset$;

(3) $TL(A) \Subset A \Subset TU(A)$;

(4) $A \Subset B \Rightarrow TL(A) \Subset TL(B)$, $A \Subset B \Rightarrow TU(A) \Subset TU(B)$;

(5) $TU(A \Cap B) \Subset TU(A) \Cap FU(B)$, $TL(A \Cup B) \Supset TL(A) \Cup TL(B)$;

(6) $TL(A^c) = (TU(A))^c$, $TU(A^c) = (TL(A))^c$.

(7) $TL(TL(A)) = TL(A)$, $TU(TU(A)) = TU(A)$.

Proof: The proofs of (1)-(6) are similar to the Proposition 3.5, we only show (7).

Let $u \in U, Md(u) = \{K_1, K_2, \cdots, K_m\}$.

$T_{TL(A)}(u) = \sup_{K\in Md(u)}\{\inf_{v\in K}(T_{(A)}(v))\} = \sup\{\inf_{v_1\in K_1}\{T_A(v_1)\}, \inf_{v_2\in K_2}\{T_A(v_2)\}, \cdots, \inf_{v_m\in K_m}\{T_A(v_m)\},\}$. Without loss of generality, let $K_i \in Md(u)$, $T_{TL(A)}(u) = \inf_{v_i\in K_i}\{T_A(v_i)\}$, then for $j \neq i, \inf_{v_i\in K_i}\{T_A(v_i)\} \geq \inf_{v_j\in K_j}\{T_A(v_j)\}$. Let $v_i \in K_i$, from Definition 3.3, we have $T_{TL(A)}(v_i) = \sup_{K\in Md(v_i)}\{\inf_{v\in K}(T_(A)(v))\} \geq \inf_{v_i\in K_i}(T_(A)(v_i)) = T_{TL(A)}(u)$. Being $\forall_{v_i\in K_i}(T_{TL(A)}(v_i) \geq T_{TL(A)(u)})$, so $\inf_{v_i\in K_i}\{T_{TL(A)(v_i)}\} = T_{TL(A)}(u)$. Let $v_j \in K_j, j \neq i$, so $\inf_{y_j\in K_j}\{T_{TL(A)}(v_j)\} \leq T_{TL(A)})(u)$ holds. Thus, $T_{TL(TL(A))}(u) = \sup_{K\in Md(u)}\{\inf_{v\in K}\{T_{TL(A)}(v)\}\} = \sup\{\inf_{v_1\in K_1}\{T_{TL(A)(v_1)}\}, \inf_{v_2\in K_2}\{T_{TL(A)(v_2)}\}, \cdots, \inf_{v_m\in K_m}\{T_{TL(A)(v_m)}\}\} = T_{TL(A)}(u)$.

$I_{TL(A)}(u) = \inf_{K\in Md(u)}\{\sup_{v\in K}(I_{(A)}(v))\} = \inf\{\sup_{v_1\in K_1}\{I_A(v_1)\}, \sup_{v_2\in K_2}\{I_A(v_2)\}, \cdots, \sup_{v_m\in K_m}\{I_A(v_m)\},\}$. Without loss of generality, let $K_i \in Md(u)$, $I_{TL(A)}(u) = \sup_{v_i\in K_i}\{I_A(v_i)\}$, then for $j \neq i, \sup_{v_i\in K_i}\{I_A(v_i)\} \leq \sup_{v_j\in K_j}\{I_A(v_j)\}$. Let $v_i \in K_i$, from Definition 3.3, we have $I_{TL(A)}(v_i) = \inf_{K\in Md(v_i)}\{\sup_{v\in K}(I_(A)(v))\} \leq \sup_{v_i\in K_i}(I_(A)(v_i)) = I_{TL(A)}(u)$. Being $\forall_{v_i\in K_i}(I_{TL(A)}(v_i) \leq I_{TL(A)(u)})$, so $\sup_{v_i\in K_i}\{I_{TL(A)(v_i)}\} = I_{TL(A)}(u)$. Let $v_j \in K_j, j \neq i$, so $\sup_{y_j\in K_j}\{I_{TL(A)}(v_j)\} \geq I_{TL(A)})(u)$ holds. Thus, $I_{TL(TL(A))}(u) = \inf_{K\in Md(u)}\{\sup_{v\in K}\{I_{TL(A)}(v)\}\} = \inf\{\sup_{v_1\in K_1}\{I_{TL(A)(v_1)}\}, \sup_{v_2\in K_2}\{I_{TL(A)(v_2)}\}, \cdots, \sup_{v_m\in K_m}\{I_{TL(A)(v_m)}\}\} = I_{TL(A)}(u)$.

$F_{TL(A)}(u) = \inf_{K\in Md(u)}\{\sup_{v\in K}(F_{(A)}(v))\} = \inf\{\sup_{v_1\in K_1}\{F_A(v_1)\}, \sup_{v_2\in K_2}\{F_A(v_2)\}, \cdots, \sup_{v_m\in K_m}\{F_A(v_m)\},\}$. Without loss of generality, let $K_i \in Md(u)$, $F_{TL(A)}(u) = \sup_{v_i\in K_i}\{F_A(v_i)\}$, then for $j \neq i, \sup_{v_i\in K_i}\{F_A(v_i)\} \leq \sup_{v_j\in K_j}\{F_A(v_j)\}$. Let $v_i \in K_i$, from Definition 3.3, we have $F_{TL(A)}(v_i) = \inf_{K\in Md(v_i)}\{\sup_{v\in K}(F(A)(v))\} \leq \sup_{v_i\in K_i}(F(A)(v_i)) = F_{TL(A)}(u)$. Being $\forall_{v_i\in K_i}(F_{TL(A)}(v_i) \leq F_{TL(A)(u)})$, so $\sup_{v_i\in K_i}\{F_{TL(A)(v_i)}\} = F_{TL(A)}(u)$. Let $v_j \in K_j, j \neq i$, so $\sup_{y_j\in K_j}\{F_{TL(A)}(v_j)\} \geq F_{TL(A)})(u)$ holds. Thus, $F_{TL(TL(A))}(u) = \inf_{K\in Md(u)}\{\sup_{v\in K}\{F_{TL(A)}(v)\}\} = \inf\{\sup_{v_1\in K_1}\{F_{TL(A)(v_1)}\}, \sup_{v_2\in K_2}\{F_{TL(A)(v_2)}\}\cdots, \sup_{v_m\in K_m}\{F_{TL(A)(v_m)}\}\} = F_{TL(A)}(u)$.

That is, $TL(TL(A)) = TL(A)$, the similar way we can get $TU(TU(A)) = TU(A)$. So (7) holds.

Remark: $TL(A \Cap B) = TL(A) \Cap TL(B)$ and $TU(A \Cup B) = TU(A) \Cup TL(B)$ do not hold generally.

4 The relations among the three types of covering-based rough single valued neutrosophic sets models

Definition 4.1 *Let C_1, C_2 are two coverings on a non-empty finite university U, $u \in U$, $\forall K \in Md_{C_1}(u)$, there exists $K' \in Md_{C_2}(u)$, such that $K' \subseteq K$, which is called C_2 is thinner than C_1, denoted by $C_2 \preceq C_1$. If $C_2 \preceq C_1$ and $C_1 \preceq C_2$, which is called C_1 equals C_2, denoted by $C_1 = C_2$. otherwise, which is called C_1 does not equal C_2, denoted by $C_1 \neq C_2$. If $C_2 \leq C_1$ and $C_1 \neq C_2$, it is called C_2 is strict thinner than C_1, denoted by $C_2 < C_1$. If $\forall K \in U, K \in C_1 \Leftrightarrow K \in C_2$, it is called C_1 identity to C_2, denoted by $C_1 \equiv C_2$.*

Proposition 4.2 *Let C_1, C_2 are two coverings on a non-empty finite university U, $C_1 \preceq C_2$, A is a single valued neutrosophic set on U. We have:*

(1) $FL_{C_2}(A) \Subset FL_{C_1}(A) \Subset A \Subset FU_{C_1}(A) \Subset FU_{C_2}(A)$;

(2) $SL_{C_2}(A) \Subset SL_{C_1}(A) \Subset A \Subset SU_{C_1}(A) \Subset SU_{C_2}(A)$;

(3) $TL_{C_2}(A) \Subset TL_{C_1}(A) \Subset A \Subset TU_{C_1}(A) \Subset TU_{C_2}(A)$.

Proof: We only show (3).

Let $u \in U$, $T_{TL_{C_1}(A)}(u) = \sup_{K\in Md(u)}\{\inf\{T_A(v)|v \in K\}\}$, $T_{TL_{C_2}(A)}(u) = \sup_{K'\in Md(u)} \{\inf\{T_A(v)|v \in K'\}\}$, being $C_1 \preceq C_2$, then $\forall K' \in Md_{C_2}(u), \exists K \in Md_{C_1}(u)$, such that $K \subseteq K'$, so $\inf_{v\in K}\{T_A(v)\} \geq \inf_{v\in K'}\{T_A(v)\}$. So $\sup_{K\in Md_{C_1}(u)}\{\inf_{v\in K}\{T_A(v)\}\} \geq \sup_{K'\in Md_{C_2}(u)}\{\inf_{v\in K'}\{T_A(v)\}\}$, that is $T_{TL_{C_1}(A)} \geq T_{TL_{C_2}(A)}$.

$I_{TL_{C_1}(A)}(u) = \inf_{K\in Md(u)}\{\sup\{I_A(v)|v \in K\}\}$, $I_{TL_{C_2}(A)}(u) = \inf_{K'\in Md(u)}\{\sup\{I_A(v)| \ v \in K'\}\}$, being $C_1 \preceq C_2$, then $\forall K' \in Md_{C_2}(u), \exists K \in Md_{C_1}(u)$, such that $K \subseteq K'$, so $\sup_{v\in K}\{I_A(v)\} \leq \sup_{v\in K'}\{T_A(v)\}$. So $\inf_{K\in Md_{C_1}(u)}\{\sup_{v\in K}\{I_A(v)\}\} \leq \inf_{K'\in Md_{C_2}(u)} \{\sup_{v\in K'}\{I_A(v)\}\}$, that is $I_{TL_{C_1}(A)} \leq I_{TL_{C_2}(A)}$.

$F_{TL_{C_1}(A)}(u) = \inf_{K\in Md(u)}\{\sup\{F_A(v)|v \in K\}\}$, $F_{TL_{C_2}(A)}(u) = \inf_{K'\in Md(u)}\{\sup\{F_A(v)| \ v \in K'\}\}$, being $C_1 \preceq C_2$, then $\forall K' \in Md_{C_2}(u), \exists K \in Md_{C_1}(u)$, such that $K \subseteq K'$, so $\sup_{v\in K}\{F_A(v)\} \leq \sup_{v\in K'}\{T_A(v)\}$. So $\inf_{K\in Md_{C_1}(u)}\{\sup_{v\in K}\{I_A(v)\}\} \leq \inf_{K'\in Md_{C_2}(u)} \{\sup_{v\in K'}\{F_A(v)\}\}$, that is $F_{TL_{C_1}(A)} \leq F_{TL_{C_2}(A)}$.

Thus we can get $TL_{C_2}(A) \Subset TL_{C_1}(A)$, the similar way we can get $TU_{C_1}(A) \Subset TU_{C_2}(A)$. According Proposition 3.7, we can get $TL_{C_2}(A) \Subset TL_{C_1}(A) \Subset A \Subset TU_{C_1}(A) \Subset TU_{C_2}(A)$ holds.

Definition 4.3 *Let C be a covering of a domain U and $K \in C$. If K is a union of some sets in $C - K$, we say K is reducible in C, otherwise K is irreducible. Let C be a covering of U. If every element in C is irreducible, we say C is irreducible; otherwise C is reducible. $\forall K \in C$, if K is reducible in C, then we can omit K from C, until C is irreducible, which is called a reduction of C, denoted by $reduct(C)$.*

Let (U, C) be a covering approximation space, $reduct(C)$ is the reduction of C, being $\forall u \in U$, $Md(u)$ is same in C and $reduct(C)$, so $C = reduct(C)$, so we can get the following result.

Proposition 4.4 *Let (U, C) be a covering approximation space, $reduct(C)$ is the reduction of C, then $\forall A \in SVNS(U)$, C and $reduct(C)$ generate the same covering-based lower/upper approximations for each type of covering-base rough single valued neutrosophic set.*

Proposition 4.5 *Let C_1, C_2 are two coverings on a non-empty finite university U, then $\forall A$, the lower/upper approximations for each type of covering-base rough single valued neutrosophic set are same in (U, C_1) and (U, C_2) iff $reduct(C_1) = reduct(C_2)$.*

Proof: $\Leftarrow$ Being $reduct(C_1) = reduct(C_2), \forall A$, A is a single valued neutrosophic set on U, from Proposition 4.2 we can get the results hold.

$\Rightarrow$ We just prove the third types of rough single valued neutrosophic set model, the others are similarly.

Proof by contradiction. Assume $reduct(C_1) \neq reduct(C_2)$, let $K \in reduct(C_1), K \notin reduct(C_2)$. We have $FL_{reduct(C_1)}(K) = K$ (here K be a single valued neutrosophic set, $T_K(u) = 1$, if $u \in K$, otherwise $T_K(u) = 0$. $I_K(u) = 0$, if $u \in K$, otherwise $I_K(u) = 1$. $F_K(u) = 0$, if $u \in K$, otherwise $F_K(u) = 1$). From Proposition 4.4, if K has the same covering-based rough single valued neutrosophic set in (U, C_1) and (U, C_2), then K has the same covering-based rough single valued neutrosophic set in $(U, reduct(C_1))$ and $(U, reduct(C_2))$, so $FL_{reduct(C_2)}(K) = K$. Being $K \notin reduct(C_2)$, then there exist $k_1, k_2, \cdots, k_n \in reduct(C_2)$, such that $K = \cup_{1\leq i\leq n} k_i$. For each $k_i \in reduct(C_2)$, there exist $k_{i1}, k_{i2}, \cdots, k_{im_i} \in reduct(C_1)$, such that $k_i = \cup_{1\leq j\leq m_i} k_{ij}$, so $K = \cup_{1\leq i\leq n} \cup_{1\leq j\leq m_i} k_{ij}$, that is K is reducible in

$reduct(C_1)$, which is contradiction that $reduct(C)$ is a reduction of C. So the result holds.

$\forall u \in U, \forall K \in Md(u)$, it is obviously that $\cap Md(u) \subseteq K \subseteq \cup Md(u)$, so we can get the following proposition.

Proposition 4.6 *Let (U, C) be a covering approximation space, A is a single valued neutrosophic set, then $FL(A) \Subset TL(A) \Subset SL(A) \Subset A \Subset SU(A) \Subset TU(A) \Subset FU(A)$.*

Proposition 4.7 *Let (U, C) be a covering approximation space, A is a single valued neutrosophic set, then the three types covering-based rough single valued neutrosophic sets are equivalence iff $\forall u \in U$, $\inf\{A(v)|v \in \cup Md(u)\} = \inf\{A(v)|v \in \cap Md(u)\}$ and $\forall u \in U$, $\sup\{A(v)|v \in \cup Md(u)\} = \sup\{A(v)|v \in \cap Md(u)\}$*

Proof: $\Leftarrow$ From Proposition 4.6 we can get $TL_{C_2}(A) \Subset TL_{C_1}(A) \Subset A \Subset TU_{C_1}(A) \Subset TU_{C_2}(A)$, being $\forall u \in U$, $\inf\{A(v)|v \in \cup Md(u)\} = \inf\{A(v)|v \in \cap Md(u)\}$, from Definition 3.1, 3.2, 3.3, we can get $FL(A) = SL(A) = TL(A)$ and $FU(A) = SU(A) = TU(A)$.

$\Rightarrow$ If the three types covering-based rough single valued neutrosophic sets are same, from Definition 3.1, 3.2, 3.3, we can easily get $\forall u \in U$, $\inf\{A(v)|v \in \cup Md(u)\} = \inf\{A(v)|v \in \cap Md(u)\}$ and $\sup\{A(v)|v \in \cup Md(u)\} = \sup\{A(v)|v \in \cap Md(u)\}$.

5 Conclusion

In this paper, we proposed the hybrid models of single valued neutrosophic refined sets, covering-based rough sets and covering-based rough single valued neutrosophic sets. Specifically, we explored the hybrid models through three different definitions and give the basic properties. Moreover, we discussed the relations of the three models. For the future prospects, we plan to explore the application of the proposed model to data mining and attribute reduction.

Acknowledgement

This work is partially supported by National Natural Science Foundation of China (Grant No. 11501435), Scholarship Fund for Studying Abroad by China Scholarship Council, Scientific Research Program Funded by Shaanxi Provincial Education Department (Program No.12JK0878) and Science and Technology Plan Project Funded by Science and Technology Bureau of Xi'an City (Program No.CXY1441(2)).

References

[1] Z. Pawlak. (1982). Rough sets. International Journal of Computer and Information Science, 11(5), 341-356.

[2] R. Slowinski, and D. Vanderpooten. (2000). A Generalized Definition of Rough Approximations Based on Similarity. IEEE Transactions on Knowledge and Data Engineering, 12(2), 331-336.

[3] A. Skowron, and J. Stepaniuk. (1996). Tolerance approximation spaces. Fundamenta Informaticae, 27(2-3), 245-253.

[4] Y.Y. Yao. (1998). Relational interpretations of neighborhood operators and rough set approximation operators. Information Sciences, 111(1-4), 239-259.

[5] K. Qin, J. Yang, and Z. Pei. (2008). Generalized rough sets based on reflexive and transitive relations. Information Sciences, 178(21), 4138-4141.

[6] J.S. Mi, and W.X. Zhang. (2004). An axiomatic characterization of a fuzzy generalization of rough sets. Information Sciences, 160, 235-249.

[7] D.W. Pei. (2005). A generalized model of fuzzy rough sets. International Journal of General Systems, 34, 603-613.

[8] X.H. Zhang, B. Zhou, and P. Li. (2012). A general frame for intuitionistic fuzzy rough sets. Information Sciences, 216, 34-49.

[9] X.B. Yang, X.N. Song, Y.S. Qi, and J.Y. Yang. (2014). Constructive and axiomatic approaches to hesitant fuzzy rough set. Soft Computing, 18, 1067C1077.

[10] H.D. Zhang, L. Shu, and S.L. Liao. (2016). On interval-valued hesitant fuzzy rough approximation operators. Soft Computing, 20(1), 189C209.

[11] W. Zakowski. (1983). Approximations in the space (U, Π). Demonstration Mathematica, 16, 761C769.

[12] Z. Bonikowski, E. Bryniarski, and U. Wybraniec. (1998). Extensions and intentions in the rough set theory. Information Sciences, 107, 149C167.

[13] D. Chen, W.X. Zhang, D. Yeung, and E.C.C. Tsang. (2006). Rough approximations on a complete completely distributive lattice with applications to generalized rough sets. Information Sciences, 176(13), 1829-1848.

[14] W. Zhu, and F.Y. Wang. Reduction and axiomization of covering generalized rough sets. Information Sciences, 152(1), 217-230.

[15] T. Deng, Y. Chen, W. Xu, and Q. Dai. (2007). A novel approach to fuzzy rough sets based on a fuzzy covering. Information Sciences, 177, 2308C2326.

[16] T.J. Li, Y. Leung, and W.X. Zhang. (2008). Generalized fuzzy rough approximation operators based on fuzzy coverings. International Journal of Approximate Reasoning, 48, 836C856.

[17] L. Wei, D.Q. Miao, F.F. Xu, and F. Xia. (2006). Research on a covering rough fuzzy set model. Journal of Computer Research and Development, 43(10), 1719-1723.

[18] Z.Y. Xu, and J.Q. Liao. (2006). On the covering fuzzy rough sets model. Fuzzy Systems and Mathematics, 20(3), 141-144.

[19] J. Hu, G.Y. Wang, and Q.H. Zhang. (2010). Covering based generalized rough fuzzy set model. Journal of Software, 21(5), 968- 977.

[20] J.G. Tang, K. She, and W. Zhu. (1012). A new type of covering-based rough fuzzy set model. Control and Decision, 21(11), 1653-1662.

[21] F. Smarandache. (1998). A Unifying Field in Logics. Neutrosophy: Neutrosophic probability, set and logic. Rehoboth: American Research Press.

[22] H. Wang, F. Smarandache, Y.Q. Zhang, and R. Sunderraman. (2010). Single valued neutrosophic sets. Multispace and Multistruct, 4, 410-413.

[23] P. Majumdar, and S.K. Samant. (2014). On similarity and entropy of neutrosophic sets. Journal of Intelligent & Fuzzy Systems, 26(3), 1245-1252.

[24] J. Ye. (2014). Improved correlation coefficients of single valued neutrosophic sets and interval neutrosophic sets for multiple attribute decision making. Journal of Intelligent & Fuzzy Systems, 27, 2453-2462.

[25] A.A. Salama, and S. Broumi. (2014). Roughness of Neutrosophic Sets. Elixir Applied Mathematics, 74, 26833-26837.

[26] H.L. Yang, C. L. Zhang, Z.L. Guo, and Y.L. Liu.(2016). A hybrid model of single valued neutrosophic sets and rough sets: single valued neutrosophic rough set model. Soft Computing, doi:10.1007/s00500- 016-2356-y.

[27] Y.L. Bao, and H. L. Yang. (2017). On single valued neutrosophic refined rough set model and its application. Journal of Intelligent & Fuzzy Systems, 33(2), 1235-1248.

[28] Abdel-Basset, M., Mohamed, M., & Sangaiah, A. K. (2017). Neutrosophic AHP-Delphi Group decision making model based on trapezoidal neutrosophic numbers. Journal of Ambient Intelligence and Humanized Computing, 1-17.https://doi.org/10.1007/s12652-017-0548-7

Received: June 26, 2017. Accepted: July 10, 2017.

NSS

University of New Mexico

Neutrosophic Regular Filters and Fuzzy Regular Filters in Pseudo-BCI Algebras

Xiaohong Zhang[1, 2], Yingcang Ma[3], and Florentin Smarandache[4]

[1] Department of Mathematics, Shaanxi University of Science & Technology, Xi'an, 710021, P.R.China. E-mail: zxhonghz@263.net

[2] Department of Mathematics, Shanghai Maritime University, Shanghai, 201306, P.R.China. E-mail: zhangxh@shmtu.edu.cn

[3] School of Science, Xi'an Polytechnic University, Xi'an 710048, P.R.China. E-mail: mayingcang@126.com

[4] Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA. E-mail: smarand@unm.edu

Abstract. Neutrosophic set is a new mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. Pseudo-BCI algebra is a kind of non-classical logic algebra in close connection with various non-commutative fuzzy logics. Recently, we applied neutrosophic set theory to pseudo-BCI algebras. In this paper, we study neutrosophic filters in pseudo-BCI algebras. The concepts of neutrosophic regular filter, neutrosophic closed filter and fuzzy regular filter in pseudo-BCI algebras are introduced, and some basic properties are discussed. Moreover, the relationships among neutrosophic regular filter, fuzzy filters and anti-grouped neutrosophic filters are presented, and the results are proved: a neutrosophic filter (fuzzy filter) is a neutrosophic regular filter (fuzzy regular filter), if and only if it is both a neutrosophic closed filter (fuzzy closed filter) and an anti-grouped neutrosophic filter (fuzzy anti-grouped filter).

Keywords: Neutrosophic set, Pseudo-BCI algebra, Neutrosophic Filter, Neutrosophic Regular Filter, Fuzzy Regular Filter.

1 Introduction

In 1998, Florentin Smarandache introduced the concept of a neutrosophic set from a philosophical point of view (see [16, 17, 18]). The neutrosophic set is a powerful general formal framework that generalizes the concept of fuzzy set and intuitionistic fuzzy set. In this paper we work with special neutrosophic sets, they are called single valued neutrosophic set (see [21]). The neutrosophic set theory is applied to many scientific fields (see [18, 19, 20]), and also applied to algebraic structures (see [1, 2, 15, 19]), it is similar to the applications of fuzzy set (soft set, rough set) theory in algebraic structures (see [11, 14, and 23]).

In 2008, W. A. Dudek and Y. B. Jun [3] introduced the notion of pseudo-BCI algebra as a generalization of BCI algebra, it is also as a generalization of pseudo-BCK algebra (which is close connection with various non-commutative fuzzy logic formal systems, see [4, 24, 26, 27, 28, and 32]). For non-classical logic algebra systems, the theory of filters (ideals) plays an important role (see [9, 12, 13, 25, and 30]). In [7], the notion of pseudo-BCI filter (ideal) of pseudo-BCI algebras is introduced. In 2009, some special pseudo-BCI filters (ideals) are discussed in [10]. Since then, some articles related filters of pseudo-BCI algebras are published (see [29, 31, 33, and 34]).

Recently, we applied neutrosophic set theory to pseudo -BCI algebras in [35]**.** This paper we further study on the applications of neutrosophic sets to pseudo-BCI algebras. We introduce the new concepts of neutrosophic regular filter, neutrosophic closed filter and fuzzy regular filter in pseudo-BCI algebras, and investigate their basic properties and present relationships among neutrosophic regular filters, anti-grouped neutrosophic filter and fuzzy filters.

Note that, the notion of pseudo-BCI algebra in this paper is a dual of the original definition in [3], so the notion of filter is a dual of (pseudo-BCI) ideal in [7, 10].

2 Some basic concepts and properties

2.1 On neutrosophic sets

Definition 2.1[17, 18, 19] Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. The functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ are real standard or non-standard subsets of $]^-0, 1^+[$. That is, $T_A(x)$: $X\to\,]^-0, 1^+[$, $I_A(x)$: $X\to\,]^-0, 1^+[$, and $F_A(x)$: $X\to\,]^-0, 1^+[$. Thus, there is no restriction on the sum of $T_A(x)$, $I_A(x)$, and $F_A(x)$, so $^-0 \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+$.

Definition 2.2[21] Let X be a space of points (objects) with generic elements in X denoted by x. A simple valued neutrosophic set A in X is characterized by truth-membership function $T_A(x)$, indeterminacy-membership function I_A(x), and falsity-membership function $F_A(x)$. Then, a simple valued neutrosophic set A can be denoted by

$$A=\{\langle x , T_A(x), I_A(x), F_A(x) \rangle \mid x\in X\},$$

where $T_A(x)$, $I_A(x)$, $F_A(x)\in[0, 1]$ for each point x in X. Therefore, the sum of $T_A(x)$, $I_A(x)$, and $F_A(x)$ satisfies the condition $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2.3[21] The complement of a simple valued neutrosophic set A is denoted by A^c and is defined as ($\forall x\in X$)

$$T_{A^c}(x) = F_A(x), I_{A^c}(x) = 1 - I_A(x), F_{A^c}(x) = T_A(x).$$

Then

$$A^c=\{\langle x , F_A(x), 1-I_A(x), T_A(x)\rangle \mid x\in X\}.$$

Definition 2.4[21] A simple valued neutrosophic set A is contained in the other simple valued neutrosophic set B, denote $A\subseteq B$, if and only if $T_A(x)\leq T_B(x)$, $I_A(x) \leq I_B(x)$, $F_A(x)\geq F_B(x)$ for any x in X.

Definition 2.5[21] Two simple valued neutrosophic sets A and B are equal, written as $A = B$, if and only if $A\subseteq B$ and $B\subseteq A$.

For convenience, "simple valued neutrosophic set" is abbreviated to "neutrosophic set" later.

Definition 2.6[21] The union of two neutrosophic sets A and B is a neutrosophic set C, written as $C=A\cup B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$T_C(x)=\max(T_A(x), T_B(x)), I_C(x)=\max(I_A(x), I_B(x)),$$
$$F_C(x)=\min(F_A(x), F_B(x)), \forall x\in X.$$

Definition 2.7[21] The intersection of two neutrosophic sets A and B is a neutrosophic set C, written as $C=A\cap B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$T_C(x)= \min(T_A(x), T_B(x)), I_C(x)=\min(I_A(x), I_B(x)),$$
$$F_C(x)=\max(F_A(x), F_B(x)), \forall x\in X.$$

Definition 2.8[20] Let A be a neutrosophic set in X and $\alpha, \beta, \gamma\in[0, 1]$ with $0\leq\alpha+\beta+\gamma\leq 3$ and (α, β, γ)-level set of A denoted by $A^{(\alpha, \beta, \gamma)}$ is defined as:

$$A^{(\alpha, \beta, \gamma)}=\{ x\in X \mid T_A(x)\geq\alpha, I_A(x)\geq\beta, F_A(x)\leq\gamma\}.$$

2.2 On pseudo-BCI algebras

Definition 2.9[3] A pseudo-BCI algebra is a structure $(X; \leq, \rightarrow, \rightsquigarrow, 1)$, where "$\leq$" is a binary relation on X, "$\rightarrow$" and "$\rightsquigarrow$" are binary operations on X and "1" is an element of X, verifying the axioms: for all $x, y, z\in X$,

(1) $y\rightarrow z\leq(z\rightarrow x)\rightsquigarrow(y\rightarrow x)$, $y\rightsquigarrow z\leq(z\rightsquigarrow x)\rightarrow(y\rightsquigarrow x)$;
(2) $x\leq(x\rightarrow y)\rightsquigarrow y$, $x\leq(x\rightsquigarrow y)\rightarrow y$;
(3) $x\leq x$;
(4) $x\leq y, y\leq x \Rightarrow x=y$;
(5) $x\leq y \Leftrightarrow x\rightarrow y =1 \Leftrightarrow x\rightsquigarrow y =1$.

If $(X; \leq, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BCI algebra satisfying $x\rightarrow y = x\rightsquigarrow y$ for all $x, y\in X$, then $(X; \rightarrow, 1)$ is a BCI-algebra.

Proposition 2.1[3, 7, 10] Let $(X; \leq, \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI algebra, then X satisfy the following properties ($\forall x, y, z\in X$):

(1) $1\leq x \Rightarrow x=1$;
(2) $x\leq y \Rightarrow y\rightarrow z\leq x\rightarrow z$, $y\rightsquigarrow z\leq x\rightsquigarrow z$;
(3) $x\leq y, y\leq z \Rightarrow x\leq z$;
(4) $x\rightsquigarrow(y\rightarrow z)=y\rightarrow(x\rightsquigarrow z)$;
(5) $x\leq y\rightarrow z \Leftrightarrow y\leq x\rightsquigarrow z$;
(6) $x\rightarrow y\leq(z\rightarrow x)\rightarrow(z\rightarrow y)$, $x\rightsquigarrow y\leq(z\rightsquigarrow x)\rightsquigarrow(z\rightsquigarrow y)$;
(7) $x\leq y \Rightarrow z\rightarrow x\leq z\rightarrow y$, $z\rightsquigarrow x\leq z\rightsquigarrow y$;
(8) $1\rightarrow x=x$, $1\rightsquigarrow x=x$;
(9) $((y\rightarrow x)\rightsquigarrow x)\rightarrow x=y\rightarrow x$, $((y\rightsquigarrow x)\rightarrow x)\rightsquigarrow x=y\rightsquigarrow x$;
(10) $x\rightarrow y\leq(y\rightarrow x)\rightsquigarrow 1$, $x\rightsquigarrow y \leq(y\rightsquigarrow x)\rightarrow 1$;
(11) $(x\rightarrow y)\rightarrow 1=(x\rightarrow 1)\rightsquigarrow(y\rightsquigarrow 1)$,
$(x\rightsquigarrow y)\rightsquigarrow 1=(x\rightsquigarrow 1)\rightarrow(y\rightarrow 1)$;
(12) $x\rightarrow 1=x\rightsquigarrow 1$.

Definition 2.10[7] A nonempty subset F of pseudo-BCI algebra X is called a pseudo-BCI filter (briefly, filter) of X if it satisfies:

(F1) $1\in F$;
(F2) $x\in F, x\rightarrow y\in F \Rightarrow y\in F$;
(F3) $x\in F, x\rightsquigarrow y\in F \Rightarrow y\in F$.

Definition 2.11[29] A pseudo-BCI algebra X is said to be anti-grouped pseudo-BCI algebra if it satisfies the following identity:

(G1) $\forall x, y, z\in X$, $(x\rightarrow y)\rightarrow(x\rightarrow z)= y\rightarrow z$,
(G2) $\forall x, y, z\in X$, $(x\rightsquigarrow y)\rightsquigarrow(x\rightsquigarrow z)= y\rightsquigarrow z$.

Proposition 2.2 [29] A pseudo-BCI algebra X is an anti-grouped pseudo-BCI algebra if and only if it satisfies:

$\forall x\in X$, $(x\rightarrow 1)\rightarrow 1=x$ or $(x\rightsquigarrow 1)\rightsquigarrow 1=x$.

Definition 2.12[29] A filter F of a pseudo-BCI algebra X is called an anti-grouped filter of X if it satisfies

(GF) $\forall x\in X$, $(x\rightarrow 1)\rightarrow 1\in F$ or $(x\rightsquigarrow 1)\rightsquigarrow 1\in F\Rightarrow x\in F$.

Definition 2.13[29] A filter F of a pseudo-BCI algebra X is called a closed filter of X if it satisfies

(CF) $\forall x\in X$, $x\rightarrow 1\in F$.

Definition 2.14[34] A filter F of pseudo-BCI algebra X is said to be regular if it satisfies:

(RF1) $\forall x, y\in X$, $y\in F$ and $x\rightarrow y\in F \Rightarrow x\in F$.
(RF2) $\forall x, y\in X$, $y\in F$ and $x\rightsquigarrow y\in F \Rightarrow x\in F$.

Proposition 2.3 [34] Let X be a pseudo-BCI algebra, F a filter of X. Then F is regular if and only if F is anti-grouped and closed.

Definition 2.15[31, 33] A fuzzy set A in pseudo-BCI algebra X is called fuzzy filter of X if it satisfies:

(FF1) $\forall x\in X$, $\mu_A(x)\leq\mu_A(1)$;

(FF2) $\forall x, y\in X$, $\min\{\mu_A(x), \mu_A(x\rightarrow y)\}\leq\mu_A(y)$;

(FF3) $\forall x, y\in X$, $\min\{\mu_A(x), \mu_A(x\rightsquigarrow y)\}\leq\mu_A(y)$.

Definition 2.16[31] A fuzzy set A: $X\rightarrow[0, 1]$ is called a fuzzy closed filter of pseudo-BCI algebra X if it is a fuzzy filter of X such that:

(FCF) $\mu_A(x\rightarrow 1)\geq\mu_A(x)$, $\forall x\in X$.

Definition 2.17[31] A fuzzy set A in pseudo-BCI algebra X is called fuzzy anti-grouped filter of X if it satisfies:

(1) $\forall x\in X$, $\mu_A(x)\leq\mu_A(1)$;

(2) $\forall x, y, z\in X$, $\min\{\mu_A(y), \mu_A((x\rightarrow y)\rightarrow(x\rightarrow z))\}\leq\mu_A(z)$;

(3) $\forall x, y, z\in X$, $\min\{\mu_A(y), \mu_A((x\rightsquigarrow y)\rightsquigarrow(x\rightsquigarrow z))\}\leq\mu_A(z)$.

Proposition 2.4[31] Let A be a fuzzy filter of pseudo-BCI algebra X. Then A is a fuzzy anti-grouped filter of X if and only if it satisfies:

$$\forall x\in X,\ \mu_A(x)\geq\mu_A((x\rightarrow 1)\rightarrow 1),\ \mu_A(x)\geq\mu_A((x\rightsquigarrow 1)\rightsquigarrow 1).$$

Definition 2.18[35] A neutrosophic set A in pseudo-BCI algebra X is called a neutrosophic filter in X if it satisfies: $\forall x, y\in X$,

(NSF1) $T_A(x)\leq T_A(1)$, $I_A(x)\leq I_A(1)$ and $F_A(x)\geq F_A(1)$;

(NSF2) $\min\{T_A(x), T_A(x\rightarrow y)\}\leq T_A(y)$, $\min\{I_A(x), I_A(x\rightarrow y)\}\leq I_A(y)$ and $\max\{F_A(x), F_A(x\rightarrow y)\}\geq F_A(y)$;

(NSF3) $\min\{T_A(x), T_A(x\rightsquigarrow y)\}\leq T_A(y)$, $\min\{I_A(x), I_A(x\rightsquigarrow y)\}\leq I_A(y)$ and $\max\{F_A(x), F_A(x\rightsquigarrow y)\}\geq F_A(y)$.

Proposition 2.5[35] Let A be a neutrosophic filter in pseudo-BCI algebra X, then $\forall x, y\in X$,

(NSF4) $x\leq y\Rightarrow T_A(x)\leq T_A(y)$, $I_A(x)\leq I_A(y)$ and $F_A(x)\geq F_A(y)$.

Definition 2.19[35] A neutrosophic set A in pseudo-BCI algebra X is called anti-grouped neutrosophic filter in X if it satisfies: $\forall x, y, z\in X$,

(1) $T_A(x)\leq T_A(1)$, $I_A(x)\leq I_A(1)$ and $F_A(x)\geq F_A(1)$;

(2) $\min\{T_A(y), T_A((x\rightarrow y)\rightarrow(x\rightarrow z))\}\leq T_A(z)$, $\min\{I_A(y), I_A((x\rightarrow y)\rightarrow(x\rightarrow z))\}\leq I_A(z)$ and $\max\{F_A(x), F_A((x\rightarrow y)\rightarrow(x\rightarrow z))\}\geq F_A(z)$;

(3) $\min\{T_A(y), T_A((x\rightsquigarrow y)\rightsquigarrow(x\rightsquigarrow z))\}\leq T_A(z)$, $\min\{I_A(y), I_A((x\rightsquigarrow y)\rightsquigarrow(x\rightsquigarrow z))\}\leq I_A(z)$ and $\max\{F_A(x), F_A((x\rightsquigarrow y)\rightsquigarrow(x\rightsquigarrow z))\}\geq F_A(z)$.

Proposition 2.6[35] Let A be a neutrosophic set in pseudo-BCI algebra X. Then A is a neutrosophic filter in X if and only if A satisfies:

(i) T_A is a fuzzy filter of X;

(ii) I_A is a fuzzy filter of X;

(iii) $1-F_A$ is a fuzzy filter of X, where $(1-F_A)(x) = 1-F_A(x)$, $\forall x\in X$.

Proposition 2.7[35] Let A be a neutrosophic set in pseudo-BCI algebra X. Then A is an anti-grouped neutrosophic filter in X if and only if A satisfies:

(i) T_A is a fuzzy anti-grouped filter of X;

(ii) I_A is a fuzzy anti-grouped filter of X;

(iii) $1-F_A$ is a fuzzy anti-grouped filter of X, where $(1-F_A)(x)=1-F_A(x)$, $\forall x\in X$.

3 Neutrosophic regular filters and neutrosophic closed filters

Definition 3.1 A neutrosophic set A in pseudo-BCI algebra X is called a neutrosophic regular filter in X if it is a neutrosophic filter in X such that: $\forall x, y\in X$,

(NSRF1) $\min\{T_A(y), T_A(x\rightarrow y)\}\leq T_A(x)$, $\min\{I_A(y), I_A(x\rightarrow y)\}\leq I_A(x)$ and $\max\{F_A(y), F_A(x\rightarrow y)\}\geq F_A(x)$;

(NSRF2) $\min\{T_A(y), T_A(x\rightsquigarrow y)\}\leq T_A(x)$, $\min\{I_A(y), I_A(x\rightsquigarrow y)\}\leq I_A(x)$ and $\max\{F_A(y), F_A(x\rightsquigarrow y)\}\geq F_A(x)$.

Definition 3.2 A neutrosophic set A in pseudo-BCI algebra X is called a neutrosophic closed filter in X if it is a neutrosophic filter in X such that: $\forall x\in X$,

(NSCF) $T_A(x\rightarrow 1)\geq T_A(x)$, $I_A(x\rightarrow 1)\geq I_A(x)$, $F_A(x\rightarrow 1)\leq F_A(x)$.

Proposition 3.1 Let A be a neutrosophic regular filter in pseudo-BCI algebra X. Then A is closed.

Proof: Suppose $x\in X$. By Definition 2.9 (2) and Proposition 2.1 (12) we have

$$x\leq(x\rightarrow 1)\rightsquigarrow 1=(x\rightarrow 1)\rightarrow 1.$$

From this and Proposition 2.5 we get

$$T_A(x)\leq T_A((x\rightarrow 1)\rightarrow 1),\ I_A(x)\leq I_A((x\rightarrow 1)\rightarrow 1),$$
$$F_A(x)\geq F_A((x\rightarrow 1)\rightarrow 1).$$

Moreover, by Definition 2.18 (NSF1) and Definition 3.1 (NSRF1)

$$T_A((x\rightarrow 1)\rightarrow 1)=\min\{T_A(1), T_A((x\rightarrow 1)\rightarrow 1)\}\leq T_A(x\rightarrow 1),$$
$$I_A((x\rightarrow 1)\rightarrow 1)=\min\{I_A(1), I_A((x\rightarrow 1)\rightarrow 1)\}\leq I_A(x\rightarrow 1),$$
$$F_A((x\rightarrow 1)\rightarrow 1)=\max\{F_A(1), F_A((x\rightarrow 1)\rightarrow 1)\}\geq F_A(x\rightarrow 1).$$

Thus,

$$T_A(x)\leq T_A((x\rightarrow 1)\rightarrow 1)\leq T_A(x\rightarrow 1),$$
$$I_A(x)\leq I_A((x\rightarrow 1)\rightarrow 1)\leq I_A(x\rightarrow 1),$$
$$F_A(x)\geq T_A((x\rightarrow 1)\rightarrow 1)\geq T_A(x\rightarrow 1).$$

By Definition 3.2 we know that A is closed.

By Proposition 2.4 and Proposition 2.7 we can get the following proposition.

Proposition 3.2 Let A be a neutrosophic filter of pseudo-BCI algebra X. Then A is an anti-grouped neutrosophic filter of X if and only if it satisfies: $\forall x\in X$,

$$T_A(x)\geq T_A((x\rightarrow 1)\rightarrow 1),\ T_A(x)\geq T_A((x\rightsquigarrow 1)\rightsquigarrow 1);$$
$$I_A(x)\geq I_A((x\rightarrow 1)\rightarrow 1),\ I_A(x)\geq I_A((x\rightsquigarrow 1)\rightsquigarrow 1);$$
$$F_A(x)\leq F_A((x\rightarrow 1)\rightarrow 1),\ F_A(x)\leq F_A((x\rightsquigarrow 1)\rightsquigarrow 1).$$

Proposition 3.3 Let A be a neutrosophic regular filter in pseudo-BCI algebra X. Then A is anti-grouped.

Proof: Suppose $x\in X$. By Definition 2.9 and Proposition 2.1 we have

$$x\rightarrow((x\rightarrow 1)\rightarrow 1)=x\rightarrow((x\rightarrow 1)\rightsquigarrow 1)=1.$$

From this we get

$$T_A(x\rightarrow((x\rightarrow 1)\rightarrow 1))=T_A(1),\ I_A(x\rightarrow((x\rightarrow 1)\rightarrow 1))=I_A(1),$$

$$F_A(x\to((x\to1)\to1))=F_A(1).$$

Thus, applying Definition 3.1 (NSRF1) we get

$$T_A(x)\geq\min\{T_A((x\to1)\to1),\ T_A(x\to((x\to1)\to1))\}$$
$$=\min\{T_A((x\to1)\to1),\ T_A(1)\}=T_A((x\to1)\to1),$$
$$I_A(x)\geq\min\{I_A((x\to1)\to1),\ I_A(x\to((x\to1)\to1))\}$$
$$=\min\{I_A((x\to1)\to1),\ I_A(1)\}=I_A((x\to1)\to1),$$
$$F_A(x)\leq\max\{F_A((x\to1)\to1),\ F_A(x\to((x\to1)\to1))\}$$
$$=\max\{F_A((x\to1)\to1),\ F_A(1)\}=F_A((x\to1)\to1).$$

Similarly, we can prove that

$$T_A(x)\geq T_A((x\rightsquigarrow1)\rightsquigarrow1), I_A(x)\geq I_A((x\rightsquigarrow1)\rightsquigarrow1),$$
$$F_A(x)\leq F_A((x\rightsquigarrow1)\rightsquigarrow1).$$

By Proposition 3.2 we know that A is anti-grouped.

Proposition 3.2 Assume that A is both an anti-grouped neutrosophic filter and a neutrosophic closed filter in pseudo-BCI algebra X. Then A satisfies: $\forall x\in X$,

$$T_A(x)=T_A(x\to1),\ I_A(x)=I_A(x\to1),\ F_A(x)=F_A(x\to1).$$

Proof: For any $x\in X$, by Definition 3.2 we have

$$T_A(x\to1)\geq T_A(x),\ I_A(x\to1)\geq I_A(x),\ F_A(x\to1)\leq F_A(x).$$

Moreover, $\forall x\in X$, by Definition 2.19 and Definition 3.2,

$$T_A(x)\geq\min\{T_A((x\to1)\to(x\to x)),\ T_A(1)\}$$
$$=\min\{T_A((x\to1)\to1),\ T_A(1)\}$$
$$=T_A((x\to1)\to1)\geq T_A(x\to1),$$
$$I_A(x)\geq\min\{I_A((x\to1)\to(x\to x)),\ I_A(1)\}$$
$$=\min\{I_A((x\to1)\to1),\ I_A(1)\}$$
$$=I_A((x\to1)\to1)\geq I_A(x\to1),$$
$$F_A(x)\leq\max\{F_A((x\to1)\to(x\to x)),\ F_A(1)\}$$
$$=\max\{F_A((x\to1)\to1),\ F_A(1)\}$$
$$=F_A((x\to1)\to1)\leq F_A(x\to1).$$

That is,

$$T_A(x)\geq T_A(x\to1),\ I_A(x)\geq I_A(x\to1),\ F_A(x)\leq F_A(x\to1).$$

Therefore,

$$\forall x\in X,\ T_A(x)=T_A(x\to1),\ I_A(x)=I_A(x\to1),\ F_A(x)=F_A(x\to1).$$

Theorem 3.1 Let A be a neutrosophic filter in pseudo-BCI algebra X. Then the following conditions are equivalent:

(i) A is both an anti-grouped neutrosophic filter and a neutrosophic closed filter in X;

(ii) A satisfies: $\forall x\in X$,

$$T_A(x)=T_A(x\to1),\ I_A(x)=I_A(x\to1),\ F_A(x)=F_A(x\to1).$$

(iii) A is a neutrosophic regular filter in X.

Proof: (i) $\Rightarrow$ (ii) See Proposition 3.2.

(iii) $\Rightarrow$ (i) See Proposition 3.1 and Proposition 3.3.

(ii) $\Rightarrow$ (iii) Suppose that A satisfies: $\forall x\in X$,

$$T_A(x)=T_A(x\to1),\ I_A(x)=I_A(x\to1),\ F_A(x)=F_A(x\to1).$$

For any $x, y\in X$, using Proposition 2.1 (6) we have

$$y\to1\leq(x\to y)\to(x\to1).$$

From this, applying Propostion 2.5,

$$T_A(y\to1)\leq T_A((x\to y)\to(x\to1)),$$
$$I_A(y\to1)\leq I_A((x\to y)\to(x\to1)),$$
$$F_A(y\to1)\geq F_A((x\to y)\to(x\to1)).$$

From these, by Definition 2.18 we get

$$\min\{T_A(y\to1),\ T_A(x\to y)\}$$
$$\leq\min\{T_A((x\to y)\to(x\to1)),\ T_A(x\to y)\}=T_A(x\to1),$$
$$\min\{I_A(y\to1),\ I_A(x\to y)\}$$
$$\leq\min\{I_A((x\to y)\to(x\to1)),\ I_A(x\to y)\}=I_A(x\to1),$$
$$\max\{F_A(y\to1),\ F_A(x\to y)\}$$
$$\geq\max\{F_A((x\to y)\to(x\to1)),\ F_A(x\to y)\}=F_A(x\to1).$$

Moreover, by condition (ii),

$$T_A(y\to1)=T_A(y),\ T_A(x\to1)=T_A(x);$$
$$I_A(y\to1)=I_A(y),\ I_A(x\to1)=I_A(x);$$
$$F_A(y\to1)=F_A(y),\ F_A(x\to1)=F_A(x).$$

Therefore,

$$\min\{T_A(y),\ T_A(x\to y)\}\leq T_A(x),$$
$$\min\{I_A(y),\ I_A(x\to y)\}\leq I_A(x),$$
$$\max\{F_A(y),\ F_A(x\to y)\}\geq F_A(x).$$

Similarly, we can get

$$\min\{T_A(y),\ T_A(x\rightsquigarrow y)\}\leq T_A(x),$$
$$\min\{I_A(y),\ I_A(x\rightsquigarrow y)\}\leq I_A(x),$$
$$\max\{F_A(y),\ F_A(x\rightsquigarrow y)\}\geq F_A(x).$$

By Definition 3.1 we know that A is a neutrosophic regular filter in X.

4 Fuzzy regular filters and neutrosophic filters

Definition 4.1 A fuzzy filter A in pseudo-BCI algebra X is called to be regular if it satisfies:

(FRF1) $\forall x, y\in X$, $\min\{\mu_A(y), \mu_A(x\to y)\}\leq\mu_A(x)$;

(FRF2) $\forall x, y\in X$, $\min\{\mu_A(y), \mu_A(x\rightsquigarrow y)\}\leq\mu_A(x)$.

Lemma 4.1[9, 33] Let X be a pseudo-BCI algebra. Then a fuzzy set μ: $X\to[0, 1]$ is a fuzzy filter of X if and only if the level set $\mu_t=\{x\in X\mid \mu(x)\geq t\}$ is filter of X for all $t\in Im(\mu)$.

Theorem 4.1 Let X be a pseudo-BCI algebra. Then a fuzzy set μ: $X\to[0, 1]$ is a fuzzy regular filter of X if and only if the level set $\mu_t=\{x\in X\mid \mu(x)\geq t\}$ is regular filter of X for all $t\in Im(\mu)$.

Proof: Assume that μ is fuzzy regular filter of X. By Lemma 4.1, for any $t\in Im(\mu)$, we have

$$\mu_t=\{x\in X\mid \mu(x)\geq t\}\text{ is filter of }X.$$

If $y\in\mu_t$ and $x\to y\in\mu_t$, then

$$\mu(y)\geq t,\ \mu(x\to y)\geq t.$$

From this and Definition 4.1 (FRF1) we get

$$\mu_A(x)\geq\min\{\mu_A(y), \mu_A(x\to y)\}\geq t.$$

This means that $x\in\mu_t$. Similarly, we can prove that

$$y\in\mu_t\text{ and }x\rightsquigarrow y\in\mu_t\Rightarrow x\in\mu_t.$$

By Definition 2.14 we know that μ_t is regular filter of X

Conversely, assume that the level set $\mu_t=\{x\in X\mid \mu(x)\geq t\}$ is regular filter of X for all $t\in Im(\mu)$. By Lemma 4.1 we know that μ: $X\to[0, 1]$ is a fuzzy filter of X. Let $x, y\in X$, denote $t_0=\min\{\mu_A(y), \mu_A(x\to y)\}$, then $t_0\in Im(\mu)$ and

$$\mu(y)\geq t_0,\ \mu(x\to y)\geq t_0.$$

This means that $y\in\mu_{t_0}$ and $x\to y\in\mu_{t_0}$. Since μ_{t_0} is regular filter of X, by Definition 2.14 we have $x\in\mu_{t_0}$, that is

$$\mu(x)\geq t_0=\min\{\mu_A(y), \mu_A(x\to y)\}.$$

It follows that Definition 4.1 (FRF1) holds. Similarly, we can prove that $\forall x, y\in X$, $\min\{\mu_A(y), \mu_A(x\rightsquigarrow y)\}\leq\mu_A(x)$. Therefore, μ: $X\to[0, 1]$ is a fuzzy regular filter of X.

Similar to Theorem 4.1 we can get the following proposition (the proofs are omitted).

Proposition 4.1 Let X be a pseudo-BCI algebra. Then a fuzzy set μ: $X\to[0, 1]$ is a fuzzy closed filter of X if and only if the level set $\mu_t=\{x\in X \mid \mu(x)\geq t\}$ is closed filter of X for all $t\in Im(\mu)$.

By Theorem 6 in [31] we have

Theorem 4.2 Let μ be a fuzzy filter of pseudo-BCI algebra X. Then the following conditions are equivalent:

(i) μ is fuzzy closed anti-grouped filter of X;

(ii) $\forall x\in X$, $\mu_A(x\to 1)=\mu_A(x)$.

(iii) μ is a fuzzy regular filter of X.

Theorem 4.3 Let A be a neutrosophic set in pseudo-BCI algebra X. Then A is a neutrosophic closed filter in X if and only if A satisfies:

(i) T_A is a fuzzy closed filter of X;

(ii) I_A is a fuzzy closed filter of X;

(iii) $1-F_A$ is a fuzzy closed filter of X, where $(1-F_A)(x)=1-F_A(x)$, $\forall x\in X$.

Proof: Assume that A is a neutrosophic closed filter in X. By Definition 3.2 we have ($\forall x\in X$)

$$T_A(x\to 1)\geq T_A(x),\ I_A(x\to 1)\geq I_A(x),\ F_A(x\to 1)\leq F_A(x).$$

Thus,

$$(1-F_A)(x\to 1)=1-F_A(x\to 1)\geq 1-F_A(x)=(1-F_A)(x).$$

Therefore, using Definition 2.16, we get that T_A, I_A and $1-F_A$ are fuzzy closed filters of X.

Conversely, assume that T_A, I_A and $1-F_A$ are fuzzy closed filters of X. Then, by Definition 2.16,

$$T_A(x\to 1)\geq T_A(x),\ I_A(x\to 1)\geq I_A(x),$$
$$(1-F_A)(x\to 1)\geq(1-F_A)(x).$$

Thus,

$$F_A(x\to 1)=1-(1-F_A)(x\to 1)\leq 1-(1-F_A)(x)=F_A(x).$$

Hence, applying Definition 3.2 we get that A is a neutrosophic closed filter A in X.

By Theorem 4.2, Theorem 4.3, Theorem 3.1 and Proposition 2.7 we can get the following results.

Theorem 4.4 Let A be a neutrosophic set in pseudo-BCI algebra X. Then A is a neutrosophic regular filter in X if and only if A satisfies:

(i) T_A is a fuzzy regular filter of X;

(ii) I_A is a fuzzy regular filter of X;

(iii) $1-F_A$ is a fuzzy regular filter of X, where $(1-F_A)(x)=1-F_A(x)$, $\forall x\in X$.

Theorem 4.5 Let X be a pseudo-BCI algebra, A be a neutrosophic set in X such that $T_A(x)\geq\alpha_0$, $I_A(x)\geq\beta_0$ and $F_A(x)\leq\gamma_0$, $\forall x\in X$, where $\alpha_0\in Im(T_A)$, $\beta_0\in Im(I_A)$ and $\gamma_0\in Im(F_A)$. Then A is a neutrosophic closed filter in X if and only if (α, β, γ)-level set $A^{(\alpha, \beta, \gamma)}$ is closed filter of X for all $\alpha\in Im(T_A)$, $\beta\in Im(I_A)$ and $\gamma\in Im(F_A)$.

Proof: Assume that A is neutrosophic closed filter in X. By Theorem 4.3 and Proposition 4.1, for any $\alpha\in Im(T_A)$, $\beta\in Im(I_A)$ and $\gamma\in Im(F_A)$, we have

$$(T_A)_\alpha=\{x\in X \mid T_A(x)\geq\alpha\},\ (I_A)_\beta=\{x\in X \mid I_A(x)\geq\beta\} \text{ and}$$
$$(1-F_A)_{1-\gamma}=\{x\in X \mid (1-F_A)(x)\geq 1-\gamma\}=\{x\in X \mid F_A(x)\leq\gamma\} \text{ are closed filters of } X.$$

Thus $(T_A)_\alpha\cap(I_A)_\beta\cap(1-F_A)_{1-\gamma}$ is a closed filters of X. Moreover, by Definition 2.8, it is easy to verify that (α, β, γ)-level set $A^{(\alpha, \beta, \gamma)}=(T_A)_\alpha\cap(I_A)_\beta\cap(1-F_A)_{1-\gamma}$. Therefore, $A^{(\alpha, \beta, \gamma)}$ is closed filter of X for all $\alpha\in Im(T_A)$, $\beta\in Im(I_A)$ and $\gamma\in Im(F_A)$.

Conversely, assume that $A^{(\alpha, \beta, \gamma)}$ is closed filter of X for all $\alpha\in Im(T_A)$, $\beta\in Im(I_A)$ and $\gamma\in Im(F_A)$. Since $T_A(x)\geq\alpha_0$, $I_A(x)\geq\beta_0$ and $F_A(x)\leq\gamma_0$, $\forall x\in X$, then

$$(T_A)_\alpha=\{x\in X \mid T_A(x)\geq\alpha\}=(T_A)_\alpha\cap X\cap X$$
$$=(T_A)_\alpha\cap(I_A)_{\beta_0}\cap(1-F_A)_{1-\gamma_0}=A^{(\alpha,\beta_0,\gamma_0)};$$
$$(I_A)_\beta=\{x\in X \mid I_A(x)\geq\beta\}=X\cap(I_A)_\beta\cap X$$
$$=(T_A)_{\alpha_0}\cap(I_A)_\beta\cap(1-F_A)_{1-\gamma_0}=A^{(\alpha_0,\beta,\gamma_0)};$$
$$(1-F_A)_{1-\gamma}=\{x\in X \mid (1-F_A)(x)\geq 1-\gamma\}$$
$$=X\cap X\cap\{x\in X \mid F_A(x)\leq\gamma\}$$
$$=(T_A)_{\alpha_0}\cap(I_A)_{\beta_0}\cap\{x\in X \mid F_A(x)\leq\gamma\}=A^{(\alpha_0,\beta_0,\gamma)}.$$

Thus,

$$(T_A)_\alpha=\{x\in X \mid T_A(x)\geq\alpha\},\ (I_A)_\beta=\{x\in X \mid I_A(x)\geq\beta\} \text{ and}$$
$$(1-F_A)_{1-\gamma}=\{x\in X \mid (1-F_A)(x)\geq 1-\gamma\}=\{x\in X \mid F_A(x)\leq\gamma\} \text{ are closed filters of } X.$$

From this, applying Proposition 4.1, we know that T_A, I_A and $1-F_A$ are fuzzy closed filters of X. By Theorem 4.3 we get that A is neutrosophic closed filter in X.

Similarly, we can get

Lemma 4.2 Let X be a pseudo-BCI algebra, A be a neutrosophic set in X such that $T_A(x)\geq\alpha_0$, $I_A(x)\geq\beta_0$ and $F_A(x)\leq\gamma_0$, $\forall x\in X$, where $\alpha_0\in Im(T_A)$, $\beta_0\in Im(I_A)$ and $\gamma_0\in Im(F_A)$. Then A is a (anti-grouped) neutrosophic filter in X if and only if (α, β, γ)-level set $A^{(\alpha, \beta, \gamma)}$ is (anti-grouped) filter of X for all $\alpha\in Im(T_A)$, $\beta\in Im(I_A)$ and $\gamma\in Im(F_A)$.

Combining Theorem 4.5, Lemma 4.2 and Theorem 3.1 we can get the following theorem.

Theorem 4.6 Let X be a pseudo-BCI algebra, A be a neutrosophic set in X such that $T_A(x)\geq\alpha_0$, $I_A(x)\geq\beta_0$ and $F_A(x)\leq\gamma_0$, $\forall x\in X$, where $\alpha_0\in Im(T_A)$, $\beta_0\in Im(I_A)$ and $\gamma_0\in Im(F_A)$. Then A is a neutrosophic regular filter in X if and only if (α, β, γ)-level set $A^{(\alpha, \beta, \gamma)}$ is regular filter of X for all $\alpha\in Im(T_A)$, $\beta\in Im(I_A)$ and $\gamma\in Im(F_A)$.

Conclusion

The neutrosophic set theory is applied to many scientific fields, and also applied to algebraic structures. This paper applied neutrosophic set theory to pseudo-BCI algebras, and some new notions of neutrosophic regular filter, neutrosophic closed filter and fuzzy regular filter in pseudo-BCI algebras are introduced. In addition to studying the basic properties of these new concepts, this paper also considered the relationships between them, and obtained some necessary and sufficient conditions.

Acknowledgment

This work was supported by National Natural Science Foundation of China (Grant No. 61573240, 61473239).

References

[1] A. A. A. Agboola, B. Davvaz, and F. Smarandache, Neutrosophic quadruple algebraic hyperstructures, Annals of Fuzzy Mathematics and Informatics, 14 (1) (2017), 29–42.
[2] R. A. Borzooei, H. Farahani, and M. Moniri, Neutrosophic deductive filters on BL-algebras, Journal of Intelligent & Fuzzy Systems, 26 (2014), 2993–3004.
[3] W. A. Dudek, and Y. B. Jun, Pseudo-BCI algebras, East Asian Mathematical Journal, 24 (2) (2008), 187–190.
[4] G. Georgescu and A. Iorgulescu, Pseudo-BCK algebras: an extension of BCK algebras, in: Combinatorics, Computability and Logic. Springer Ser. Discrete Math. Theor. Comput. Sci., 2001, 97–114.
[5] P. F. He, X. L. Xin and Y. W. Yang, On state residuated lattices, Soft Computing, 19 (8) (2015), 2083–2094.
[6] P. F. He, B. Zhao and X. L. Xin, States and internal states on semihoops, Soft Computing, 21 (11) (2017), 2941–2957.
[7] Y. B. Jun, H. S. Kim and J. Neggers, On pseudo-BCI ideals of pseudo-BCI algebras, Matematicki Vesnik, 58 (1-2) (2006), 39–46.
[8] H. S. Kim, Y. H. Kim, On BE-algebras, Sci. Math. Japon., 66(1) (2007), 113–116.
[9] M. Kondo and W.A. Dudek, On the transfer principle in fuzzy theory, Mathware & Soft Computing, 12 (2005), 41–55.
[10]K. J. Lee and C. H. Park, Some ideals of pseudo BCI-algebras, Journal of Applied Mathematics and Informatics, 27 (1-2) (2009), 217–231.
[11]L. Z. Liu, Generalized intuitionistic fuzzy filters on residuated lattices, Journal of Intelligent & Fuzzy Systems, 28 (2015), 1545–1552
[12]Z. M. Ma, B. Q. Hu, Characterizations and new subclasses of I-filters in residuated lattices, Fuzzy Sets and Systems, 247 (2014), 92–107.
[13]Z. M. Ma, W. Yang, Z. Q. Liu, Several types of filters related to the Stonean axiom in residuated lattices, Journal of Intelligent & Fuzzy Systems, 32 (1) (2017), 681–690.
[14]B. L. Meng, On filters in BE-algebras, Sci. Math. Japon., 71 (2010), 201–207.
[15]A. Rezaei, A. B. Saeid, and F. Smarandache, Neutrosophic filters in BE-algebras, Ratio Mathematica, 29 (2015), 65–79.
[16]F. Smarandache, Neutrosophy, Neutrosophic Probability, Set, and Logic, Amer. Res. Press, Rehoboth, USA, 1998.
[17]F. Smarandache, Neutrosophy and Neutrosophic Logic, Information Sciences First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability and Statistics University of New Mexico, Gallup, NM 87301, USA, 2002.
[18]F. Smarandache, Neutrosophic set–a generialization of the intuituionistics fuzzy sets, International Journal of Pure and Applied Mathematics, 24 (3) (2005), 287–297.
[19]F. Smarandache, Neutrosophic Perspectives: Triplets, Duplets, Multisets, Hybrid Operators, Modal Logic, Hedge Algebras. And Applications, Pons Publishing House, Brussels, 2017
[20]C. A. C. Sweety, I. Arockiarani, Rough sets in neutrosophic approximation space, Annals of Fuzzy Mathematics and Informatics, 13 (4) (2017), 449–463.
[21]H. Wang, F. Smarandache, Y. Q. Zhang, et al, Single valued neutrosophic sets, Multispace & Multistructure. Neutrosophic Transdisciplinarity, 4 (2010), 410–413.
[22]J. Ye, Single valued neutrosophic cross-entropy for multicriteria decision making problems, Applied Mathematical Modelling, 38 (2014), 1170–1175.
[23]J. M. Zhan, Q. Liu and Hee Sik Kim, Rough fuzzy (fuzzy rough) strong h-ideals of hemirings, Italian Journal of Pure and Applied Mathematics, 34(2015), 483–496.
[24]X. H. Zhang, Y. Q. Wang, and W. A. Dudek, T-ideals in BZ-algebras and T-type BZ-algebras, Indian Journal Pure and Applied Mathematics, 34(2003), 1559-1570.
[25]X. H. Zhang and W. H. Li, On pseudo-BL algebras and BCC-algebra, Soft Computing, 10 (2006), 941–952.
[26]X. H. Zhang, Fuzzy Logics and Algebraic Analysis, Science Press, Beijing, 2008.
[27]X. H. Zhang and W. A. Dudek, BIK+-logic and non-commutative fuzzy logics, Fuzzy Systems and Mathematics, 23 (4) (2009), 8–20.
[28]X. H. Zhang, BCC-algebras and residuated partially-ordered groupoid, Mathematica Slovaca, 63 (3) (2013), 397–410.
[29]X. H. Zhang and Y. B. Jun, Anti-grouped pseudo-BCI algebras and anti-grouped pseudo-BCI filters, Fuzzy Systems and Mathematics, 28 (2) (2014), 21–33.
[30]X. H. Zhang, H. J. Zhou and X. Y. Mao, IMTL(MV)-filters and fuzzy IMTL(MV)-filters of residuated lattices, Journal of Intelligent & Fuzzy Systems, 26 (2) (2014), 589–596.
[31]X. H. Zhang, Fuzzy commutative filters and fuzzy closed filters in pseudo-BCI algebras, Journal of Computational Information Systems, 10 (9) (2014), 3577–3584.
[32]X. H. Zhang, Fuzzy 1-type and 2-type positive implicative filters of pseudo-BCK algebras, Journal of Intelligent & Fuzzy Systems, 28 (5) (2015), 2309–2317.
[33]X. H. Zhang, Fuzzy anti-grouped filters and fuzzy normal filters in pseudo-BCI algebras, Journal of Intelligent and Fuzzy Systems, 33 (2017), 1767–1774.
[34]X. H. Zhang and Choonkil Park, On regular filters and well filters of pseudo-BCI algebras, Proceedings of the 13th International Conference on Natural Computation, Fuzzy Systems and Knowledge Discovery (ICNC-FSKD 2017), IEEE, 2017.
[35]X. H. Zhang, Y. T. Wu, and X. H. Zhai, Neutrosophic filters in pseudo-BCI algebras, submitted, 2017.
[36]Topal, S. and Smaradache, F. A Lattice-Theoretic Look: A Negated Approach to Adjectival (Intersective, Neutrosophic and Private) Phrases. The 2017 IEEE International Conference on INnovations in Intelligent SysTems and Applications (INISTA 2017); (accepted for publication).

Received: July 3, 2017. Accepted: July 19, 2017.

NSS

University of New Mexico

Competencies evaluation based on single valued neutrosophic numbers and decision analysis schema

Evelyn Jazmín Henríquez Antepara[1], Oscar Omar Apolinario Arzube[2], Jorge Arturo Chicala Arroyave[3], Eduardo Antonio Alvarado Unamuno[4], Maikel Leyva Vazquez[5]

[1]Universidad de Guayaquil, Facultad de Ciencias Matemáticas y Físicas, Guayaquil Ecuador. Email: evelyn.henriqueza@ug.edu.ec

[2]Universidad de Guayaquil, Facultad de Ciencias Matemáticas y Físicas, Guayaquil Ecuador. Email: apolinariooscar@gmail.com

[3]Universidad de Guayaquil, Facultad de Ciencias Matemáticas y Físicas, Guayaquil Ecuador. Email: jchicala@hotmail.com

[4]Universidad de Guayaquil, Facultad de Ciencias Matemáticas y Físicas, Guayaquil Ecuador. Email: eduardo.alvaradou@ug.edu.ec

[5]Universidad de Guayaquil, Facultad de Ciencias Matemáticas y Físicas, Guayaquil Ecuador. Email: mleyvaz@ug.edu.ec

Abstract. Recently, neutrosophic sets and its application to decision making have become a topic of significant im-portance for researchers and practitioners. The present work addresses one of the most complex aspects of the formative process based on competencies: evaluation. In this paper, a new method for competencies evaluation is developed in a multicriteria framework. The proposed framework is composed of four activities, framework, gathering information, ideal solution distance calculation and ranking alternatives. Student are evaluated using SVN, for the treatment of neutralities, and Euclidean distance. The paper ends with conclusion and future work proposal for the application of neutrosophy to new areas of education.

Keywords: competency, evaluation, neutrosophy, SVN numbers

1 Introduction

In this paper, one of the most complex aspects of the formative process based on competencies is addressed: evaluation. A new method for competencies evaluation is developed in a multicriteria framework based on decision analysis.

Decision analysis is a discipline, belonging to decision theory, with the goal of computing an overall assessment that summarizes the information gathered and providing useful information about each evaluated element [1]. Uncertainty is present in real world decision making problems in such cases the use of linguistic information to model and manage such an uncertainty has given good results [2]. Experts feel more comfortable providing their knowledge by using terms close to human cognitive model [3, 4].

The conventional crisp techniques have been not much effective for solving decision problems because of imprecise or fuzziness nature of the linguistic assessments. It is more reasonable to consider the values of alternatives according as single valued neutrosophic sets (SVNS) [5]. SVNS can handle indeterminate and inconsistent information, while fuzzy sets and intuitionistic fuzzy sets cannot describe them [6]. In this paper a new model competencies evaluation is developed base on single valued neutrosophic number (SVN-number) allowing the use of linguistic variables [7] and giving methodological support based on decision analysis schema.

This paper is structured as follows: Section 2 reviews some important concepts about decision analysis framework and SVN numbers. In Section 3, is presented a decision analysis framework based on SVN numbers for competencies evaluation. Section 4 shows a case study. The paper ends with conclusions and further work recommendations.

2 Decision schemes

Decision analysis is a discipline with main purpose of helping decision maker to reach a consistent decision [8]. A common decision resolution scheme consists of following phases [2, 9].

- Identify decision and objectives.
- Identify alternatives.
- Framework:
- Gathering information.
- Rating alternatives.
- Choosing the alternative/s:
- Sensitive analysis
- Decide

In the framework phase, he structures and elements of the decision problem are defined: experts, criteria, etc. The information provided by experts is collected, according to the defined framework in the gathering information phase. In line with our aims in this paper, a SVN numbers [10] approach is developed due to the fact that provide adequate computational models to deal with linguistic information [11] in decision problems allowing to include handling of indeterminate and inconsistent .

A way to compute a rating of alternatives is to use an ideal alternative. A comparison between an ideal alternative and available options in order to find the optimal choice could be used [12]. Normally, the closer alternative to the ideal, corresponds to the best alternative.

3 SVN-numbers

Neutrosophy [13] is mathematical theory developed for dealing with indeterminacy . The truth value in neutrosophic set is as follows [14]:

Let N be a set defined as: $N = \{(T, I, F) : T, I, F \subseteq [0, 1]\}$, a neutrosophic valuation n is a mapping from the set of propositional formulas to N , that is for each sentence p we have $v\,(\mathrm{p}) = (T, I, F)$.

To facilitate the real world applications of neutrosophic set and set-theoretic operators single valued neutrosophic set (SVNS) [5] was developed

A single valued neutrosophic set (SVNS) has been defined as follows [5]:

Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form:

$$A = \{\langle x, uA(x), rA(x), vA(x)\rangle : x \in X\} \qquad (1)$$

where $u_A(x): X \rightarrow [0,1]$, $r_A(x), : X \rightarrow [0,1]$ and $v_A(x): X \rightarrow [0,1]$ with $0 \leq u_A(x) + r_A(x) + v_A(x) :\leq 3$ for all $x \in X$. The intervals $u_A(x)$, $r_A(x)$ y $v_A(x)$ denote the truth- membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively.

Single valued neutrosophic numbers (SVN number) is denoted by $A = (a, b, c)$, where $a, b, c \in [0,1]$ and $a + b + c \leq 3$.

Alternatives could be rated according Euclidean distance in SVN [15, 16].

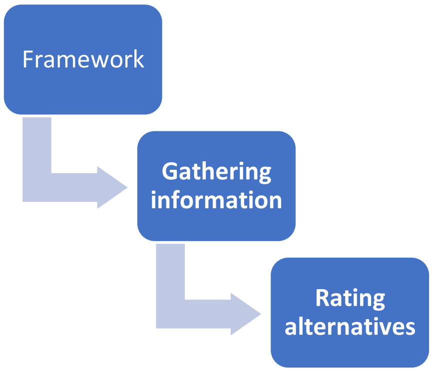
Let $A^* = (A_1^*, A_2^*, \dots, A_n^*)$ be a vector of n SVN numbers such that $A_j^* = (a_j^*, b_j^*, c_j^*)$ j=(1,2, … , n) and $B_i = (B_{i1}, B_{i2}, \dots, B_{im})$ ($i = 1,2, \dots, m$) be m vectors of n SVN numbers such that $B_{ij} = (a_{ij}, b_{ij}, c_{ij})$ ($i = 1,2, \dots, m$), ($j = 1,2, \dots, n$). Then the separation measure between $B_i's$ y A^* is defined as follows:

$$s_i = \left(\frac{1}{3}\sum_{j=1}^{n}\left\{\left(\left|a_{ij}-a_j^*\right|\right)^2+\left(\left|b_{ij}-b_j^*\right|\right)^2+\left(\left|c_{ij}-c_j^*\right|\right)^2\right\}\right)^{\frac{1}{2}} \quad (i = 1,2, \dots, m) \qquad (2)$$

In this paper linguistic variables[11] are represented using single valued neutrosophic numbers [16] for developing a framework to decision support.

2.2 Proposed framework

Our aim is to develop a framework for competencies evaluation based on for decision analysis based and SVN numbers. The model consists of the following phases (graphically, Fig. 3).



The proposed framework is composed of three activities, framework, gathering information and rating alternatives.

Framework

In this phase, the evaluation framework, the decision problem structure is defined. The framework is established as follows:

C= $\{c_1, c_2, \dots, c_l$ with $l \geq 2$, a set competencies.

E= $\{e_1, e_2, \dots, e_k\}$ with $k \geq 2$ A set of students.

Gathering information

In this phase, the assessments is provided by means of assessment vectors:

$$U = (v_{ij}, i = 1, \dots, l,\ j = 1, \dots, k) \qquad (3)$$

The assessment v_{ij}, for each criterion c_i of each student e_j, is expressed by means of SVN numbers.

Rating alternatives

For rating alternatives an ideal option is constructed [16, 17] .the evaluation criteria can be categorized into two categories, benefit and cost. Let C^+ be a collection of benefit criteria and C^- be a collection of cost criteria. The ideal alternative is defined as:

$$I = \left\{\left(max_{i=1}^{k} T_{U_j} \middle| j \in C^+, min_{i=1}^{k} T_{U_j} \middle| j \in C^-\right), \left(min_{i=1}^{k} I_{U_j} \middle| j \in C^+, max_{i=1}^{k} I_{U_j} \middle| j \in C^-\right), \left(min_{i=1}^{k} F_{U_j} \middle| j \in C^+, max_{i=1}^{k} F_{U_j} \middle| j \in C^-\right)\right\} = [v_1, v_2, \dots, v_n] \qquad (4)$$

Alternatives are rating according Euclidean distance to I (2). Ranking is based in the global distance to the ideal. If alternative x_i is closer to I the distance measure (s_i closer) better is the alternative [18].

3 Case study

A demonstrative example is given below. In the stage of establishing the evaluation framework, the domain in which the information will be verbalized is selected.
The following linguistic terms are used (Table 1).

Linguistic terms	SVNSs
Extremely good (EG)	(1,0,0)
Very very good (VVG)	(0.9, 0.1, 0.1)
Very good (VG)	(0.8,0,15,0.20)
Good (G)	(0.70,0.25,0.30)
Medium good (MG)	(0.60,0.35,0.40)
Medium (M)	(0.50,0.50,0.50)
Medium bad (MB)	(0.40,0.65,0.60)
Bad (B)	(0.30,0.75,0.70)
Very bad (VB)	(0.20,0.85,0.80)
Very very bad (VVB)	(0.10,0.90,0.90)
Extremely bad (EB)	(0,1,1)

Table 1. Linguistic terms used to provide the assessments [16].

Three core competencies are evaluated in three students.
c_1: Analyze, identify and define the requirements that must be met by a computer system to solve problems or achieve objectives of organizations and individuals.
c_2: Manage Databases through a Database Management System (DBMS).
c_3: Plan and manage software development projects.
Once the prioritization framework is established, the information is obtained.

	e_1	e_2	e_3
c_1	MDB	M	MMB
c_2	B	MMB	B
c_3	B	MDM	MB

Table 2: Preferences given by experts

From this information, the ideal alternative is calculated.
The ideal alternative results:
E^+ =(MMB, MMB, MB)
The results of the calculation of the distances allow us to order the students according to the achievement of the competences. In this case the priority order is the following: $e_3 \succ e_1 \succ e_2$

Student	**Distance**
e1	0.35355339
e2	0.59160798
e3	0.18484228

Table 3: Distance calculation

Among the advantages found by the specialists are the relative ease of the technique. The results also show the applicability of SVN-based decision support models to competency assessment.

Conclusions

In this paper, a competency assessment model was presented. The students were evaluated by means of the SVN numbers and the Euclidean distance for the treatment of neutrality.
Further works will concentrate extending the model for dealing with heterogeneous information [19] and a multi-expert setting . Another area of future work is the developing of new aggregation operators based on SVN numbers specially compensatory operators [20].

References

1. Espinilla, M., et al., *A comparative study of heterogeneous decision analysis approaches applied to sustainable energy evaluation.* International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 2012. **20**(supp01): p. 159-174.
2. Estrella, F.J., et al., *FLINTSTONES: A fuzzy linguistic decision tools enhancement suite based on the 2-tuple linguistic model and extensions.* Information Sciences, 2014. **280**: p. 152-170.
3. Rodríguez, R.M. and L. Martínez, *An analysis of symbolic linguistic computing models in decision making.* International Journal of General Systems, 2013. **42**(1): p. 121-136.
4. Leyva-Vázquez, M., et al., *The Extended Hierarchical Linguistic Model in Fuzzy Cognitive Maps*, in *Technologies and Innovation: Second International Conference, CITI 2016, Guayaquil, Ecuador, November 23-25, 2016, Proceedings*, R. Valencia-García, et al., Editors. 2016, Springer International Publishing: Cham. p. 39-50.
5. Wang, H., Smarandache, F. et al., *Single valued neutrosophic sets.* Review of the Air Force Academy, 2010(1): p. 10.
6. Akram, M. and A. Luqman, *Intuitionistic single-valued neutrosophic hypergraphs.* OPSEARCH: p. 1-17.
7. Biswas, P., S. Pramanik, and B.C. Giri, *TOPSIS method for multi-attribute group decision-making under single-valued neutrosophic environment.* Neural computing and Applications, 2016. **27**(3): p. 727-737.
8. Espinilla, M., et al. *A heterogeneous evaluation model for assessing sustainable energy: A Belgian case study.*

in *Fuzzy Systems (FUZZ), 2010 IEEE International Conference on*. 2010. IEEE.

9. Clemen, R.T., *Making Hard Decisions: An Introduction to Decision Analysis*. 1996: Duxbury Press.
10. Deli, I. and Y. Şubaş, *A ranking method of single valued neutrosophic numbers and its applications to multi-attribute decision making problems.* International Journal of Machine Learning and Cybernetics, 2016: p. 1-14.
11. Leyva-Vázquez, M., et al. *The Extended Hierarchical Linguistic Model in Fuzzy Cognitive Maps*. in *Technologies and Innovation: Second International Conference, CITI 2016, Guayaquil, Ecuador, November 23-25, 2016, Proceedings 2*. 2016. Springer.
12. Zeng, S., T. Baležentis, and C. Zhang, *A method based on OWA operator and distance measures for multiple attribute decision making with 2-tuple linguistic information.* Informatica, 2012. **23**(4): p. 665-681.
13. Smarandache, F., *A Unifying Field in Logics: Neutrosophic Logic.* Philosophy, 1999: p. 1-141.
14. Rivieccio, U., *Neutrosophic logics: Prospects and problems.* Fuzzy sets and systems, 2008. **159**(14): p. 1860-1868.
15. Ye, J., *Single-valued neutrosophic minimum spanning tree and its clustering method.* Journal of intelligent Systems, 2014. **23**(3): p. 311-324.
16. Şahin, R. and M. Yiğider, *A Multi-criteria neutrosophic group decision making metod based TOPSIS for supplier selection.* arXiv preprint arXiv:1412.5077, 2014.
17. Leyva-Vázquez, M., K. Pérez-Teruel, and R.I. John. *A model for enterprise architecture scenario analysis based on fuzzy cognitive maps and OWA operators*. in *Electronics, Communications and Computers (CONIELECOMP), 2014 International Conference on*. 2014. IEEE.
18. Leyva-Vázquez, M., et al., *Técnicas para la representación del conocimiento causal: un estudio de caso en Informática Médica.* Revista Cubana de información en ciencias de la salud, 2013. **24**(1): p. 73-83.
19. Pérez-Teruel, K., M. Leyva-Vázquez, and M. Espinilla-Estevez. *A linguistic software requirement prioritization model with heterogeneous information*. in *4th International Workshop on Knowledge Discovery, Knowledge Management and Decision Support (EUREKA 2013), Mazatlán (Mexico)*. 2013.
20. Espin-Andrade, R.A., et al., *Archimedean-Compensatory Fuzzy Logic Systems.* International Journal of Computational Intelligence Systems, 2015. **8**(sup2): p. 54-62.
21. F. Smarandache, Neutrosophic Perspectives: Triplets, Duplets, Multisets, Hybrid Operators, Modal Logic, Hedge Algebras. And Applications. Pons Editions, Bruxelles, 325 p., 2017.

Received: July 7, 2017. Accepted: July 25, 2017.

NSS

University of New Mexico

An extended TOPSIS for multi-attribute decision making problems with neutrosophic cubic information

Surapati Pramanik[1], Partha Pratim Dey[2], Bibhas C. Giri[3], and Florentin Smarandache[4]

[1] Department of Mathematics, Nandalal Ghosh B.T. College, Panpur, P.O.-Narayanpur, District - North 24 Parganas, Pin Code-743126, West Bengal, India. E-mail: sura_pati@yahoo.co.in

[2] Department of Mathematics, Jadavpur University, Kolkata-700032, West Bengal, India. E-mail: parsur.fuzz@gmail.com

[3] Department of Mathematics, Jadavpur University, Kolkata-700032, West Bengal, India. E-mail: bcgiri.ju.math@gmail.com

[4]University of New Mexico. Mathematics & Science Department, 705 Gurley Ave., Gallup, NM 87301, USA. Email: fsmarandache@gmail.com

Abstract. The paper proposes a new technique for dealing with multi-attribute decision making problems through an extended TOPSIS method under neutrosophic cubic environment. Neutrosophic cubic set is the generalized form of cubic set and is the hybridization of a neutrosophic set with an interval neutrosophic set. In this study, we have defined some operation rules for neutrosophic cubic sets and proposed the Euclidean distance between neutrosophic cubic sets. In the decision making situation, the rating of alternatives with respect to some predefined attributes are presented in terms of neutrosophic cubic information where weights of the attributes are completely unknown. In the selection process, neutrosophic cubic positive and negative ideal solutions have been defined. An extended TOPSIS method is then proposed for ranking the alternatives and finally choosing the best one. Lastly, an illustrative example is solved to demonstrate the decision making procedure and effectiveness of the developed approach.

Keywords: TOPSIS; neutrosophic sets; interval neutrosophic set; neutrosophic cubic sets; multi-attribute decision making.

1 Introduction

Smarandache [1] proposed neutrosophic set (NS) that assumes values from real standard or non-standard subsets of] $^{-}0$, 1^{+}[. Wang et al. [2] defined single valued neutrosophic set (SVNS) that assumes values from the unit interval [0, 1]. Wang et al. [3] also extended the concept of NS to interval neutrosophic set (INS) and presented the set-theoretic operators and different properties of INSs. Multi-attribute decision making (MADM) problems with neutrosophic information caught much attention in recent years due to the fact that the incomplete, indeterminate and inconsistent information about alternatives with regard to predefined attributes are easily described under neutrosophic setting. In interval neutrosophic environment, Chi and Liu [4] at first established an extended technique for order preference by similarity to ideal solution (TOPSIS) method [5] for solving MADM problems with interval neutrosophic information to get the most preferable alternative. Şahin, and Yiğider [6] discussed TOPSIS method for multi-criteria decision making (MCDM) problems with single neutrosophic values for supplier selection problem. Zhang and Wu [7] developed an extended TOPSIS for single valued neutrosophic MCDM problems where the incomplete weights are obtained by maximizing deviation method. Ye [8] proposed an extended TOPSIS method for solving MADM problems under interval neutrosophic uncertain linguistic variables. Biswas et al. [9] studied TOPSIS method for solving multi-attribute group decision making problems with single-valued neutrosophic information where weighted averaging operator is employed to aggregate the individual decision maker's opinion into group opinion.

In 2016, Ali et al. [10] proposed the notion of neutrosophic cubic set (NCS) by extending the concept of cubic set to neutrosophic cubic set. Ali et al. [10] also defined internal neutrosophic cubic set (INCS) and external neutrosophic cubic set (ENCS), $\frac{1}{3}$ -INCS ($\frac{2}{3}$ -ENCS), $\frac{2}{3}$ -INCS ($\frac{1}{3}$ -ENCS) and also proposed some relevant properties. In the same study, Ali et al. [10] proposed Hamming distance between two NCSs and developed a decision making technique via similarity measures of two NCSs in pattern recognition problems. Jun et al. [11] studied the notions of truth-internal (indeterminacy-internal, falsity-internal) neutrosophic cubic sets and truth-external (indeterminacy-external, falsity-external) neutrosophic cubic and investigated related properties. Pramanik et al. [12] defined similarity measure for neutrosophic cubic sets and proved its basic properties. In the same study, Pramanik et al. [12] developed multi criteria group decision making method with linguistic variables in neutrosophic cubic set environment.

In this paper, we develop a new approach for MADM problems with neutrosophic cubic assessments by using TOPSIS method where weights of the attributes are unknown to the decision maker (DM). We define a few operations on NCSs and propose the Euclidean distance between two NCSs. We define accumulated arithmetic operator (AAO) to convert neutrosophic cubic values (NCVs) to single neutrosophic values (SNVs). We also define neutrosophic cubic positive ideal solution (NCPIS) and neutrosophic cubic negative ideal solution (NCNIS) in this study. The rest of the paper is organized in the following way. Section 2 recalls some basic definitions which are useful for the construction of the paper. Subsection 2.1 provides several operational rules of NCSs. Section 3 is devoted to present an extended TOPSIS method for MADM problems in neutrosophic cubic set environment. In Section 4, we solve an illustrative example to demonstrate the applicability and effectiveness of the proposed approach. Finally, the last Section presents concluding remarks and future scope research.

2 The basic definitions

Definition: 1
Fuzzy sets [13]: Consider U be a universe. A fuzzy set Φ over U is defined as follows:

$$\Phi = \{\langle x,\ \mu_\Phi(x)\rangle \mid x \in U\}$$

where $\mu_\Phi(x) : U \to [0, 1]$ is termed as the membership function of Φ and $\mu_\Phi(x)$ represents the degree of membership to which $x \in \Phi$.

Definition: 2
Interval valued fuzzy sets [14]: An interval-valued fuzzy set (IVFS) Θ over U is represented as follows:

$$\Theta = \{\langle x,\ \Theta^-(x), \Theta^+(x)\rangle \mid x \in U\}$$

where $\Theta^-(x), \Theta^+(x)$ denote the lower and upper degrees of membership of the element $x \in U$ to the set Θ with $0 \le \Theta^-(x) + \Theta^+(x) \le 1$.

Definition: 3
Cubic sets [15]: A cubic set Ψ in a non-empty set U is a structure defined as follows:

$$\Psi = \{\langle x, \Theta(x), \Phi(x)\rangle \mid x \in U\}$$

where Θ and Φ respectively represent an interval valued fuzzy set and a fuzzy set. A cubic set Ψ is denoted by $\Psi = <\Theta, \Phi>$.

Definition: 4
Internal cubic sets [15]: A cubic set $\Psi = <\Theta, \Phi>$ in U is said to be internal cubic set (ICS) if $\Theta^-(x) \le \mu(x) \le \Theta^+(x)$ for all $x \in U$.

Definition: 5
External cubic sets [15]: A cubic set $\Psi = <\Theta, \Phi>$ in U is called external cubic set (ECS) if $\mu(x) \notin (\Theta^-(x), \Theta^+(x))$ for all $x \in U$.

Definition: 6
Consider $\Psi_1 = <\Theta_1, \Phi_1>$ and $\Psi_2 = <\Theta_2, \Phi_2>$ be two cubic sets in U, then we have the following relations [15].

1. (Equality) $\Psi_1 = \Psi_2$ if and only if $\Theta_1 = \Theta_2$ and $\mu_1 = \mu_2$.
2. (P-order) $\Psi_1 \subseteq_P \Psi_2$ if and only if $\Theta_1 \subseteq \Theta_2$ and $\mu_1 \le \mu_2$.
3. (R-order) $\Psi_1 \subseteq_R \Psi_2$ if and only if $\Theta_1 \subseteq \Theta_2$ and $\mu_1 \ge \mu_2$.

Definition: 7
Neutrosophic set [1]: Consider U be a space of objects, then a neutrosophic set (NS) χ on U is defined as follows:

$$\chi = \{x, \langle\alpha(x), \beta(x), \gamma(x)\rangle \mid x \in U\}$$

where $\alpha(x), \beta(x), \gamma(x) : U \to\ ]^-0, 1^+[$ define respectively the degrees of truth-membership, indeterminacy-membership, and falsity-membership of an element $x \in U$ to the set χ with $^-0 \le \sup\ \alpha(x) + \sup\ \beta(x) + \sup\ \gamma(x) \le 3^+$.

Definition: 8
Interval neutrosophic sets [9]: An INS Γ in the universal space U is defined as follows:

$$\Gamma = \{x, \langle[\Gamma_\alpha^-(x), \Gamma_\alpha^+(x)], [\Gamma_\beta^-(x), \Gamma_\beta^+(x)], [\Gamma_\gamma^-(x), \Gamma_\gamma^+(x)]\rangle \mid x \in U\}$$

where, $\Gamma_\alpha(x)$, $\Gamma_\beta(x)$, $\Gamma_\gamma(x)$ are the truth-membership function, indeterminacy-membership function, and falsity-membership function, respectively with $\Gamma_\alpha(x)$, $\Gamma_\beta(x)$, $\Gamma_\gamma(x) \subseteq [0, 1]$ for each point $x \in U$ and $0 \le \sup \Gamma_\alpha(x) + \sup \Gamma_\beta(x) + \sup \Gamma_\gamma(x) \le 3$.

Definition: 9

Neutrosophic cubic sets [15]

A neutrosophic cubic set (NCS) Ξ in a universe U is presented in the following form:

$$\Xi = \{ \langle x, \Gamma(x), \chi(x) \rangle \mid x \in U \}$$

where Γ and χ are respectively an interval neutrosophic set and a neutrosophic set in U.

However, NSs take the values from] $^-0$, 1^+[and single-valued neutrosophic set defined by Wang *et al*. [2] is appropriate for dealing with real world problems. Therefore, the definition of NCS should be modified in order to solve practical decision making purposes. Hence, a neutrosophic cubic set (NCS) Ξ in U is defined as follows:

$$\Xi = \{ \langle x, \Gamma(x), \chi(x) \rangle \mid x \in U \}$$

Here, Γ and χ are respectively an INS and a SVNS in U where $0 \le \alpha(x) + \beta(x) + \gamma(x) \le 3$ for each point $x \in U$. Generally, a NCS is denoted by $\Xi = \langle \Gamma, \chi \rangle$ and sets of all NCS over U will be represented by NCS^U.

Example 1. Assume that $U = \{u_1, u_2, u_3, u_4\}$ be a universal set. An INS A in U is defined as

$\Gamma = \{< [0.15, 0.3], [0.25, 0.4], [0.6, 0.75] >/ u_1 + < [0.25, 0.35], [0.35, 0.45], [0.4, 0.65] >/ u_2 + < [0.35, 0.5], [0.25, 0.35], [0.55, 0.85] >/ u_3 + < [0.7, 0.8], [0.15, 0.45], [0.2, 0.3] >/ u_4\}$

and a SVNS χ in U defined by

$\chi = \{< 0.35, 0.3, 0.15 >/ u_1, < 0.5, 0.1, 0.4 >/ u_2, < 0.25, 0.03, 0.35 >/ u_3, < 0.85, 0.1, 0.15 >/ u_4\}$

Then $\Xi = < A, \chi >$ is represented as a NCS in U.

Definition: 10

Internal neutrosophic cubic set [10]: Consider $\Xi = < \Gamma, \chi > \in \text{NCS}^U$, if $\Gamma^-_\alpha(x) \le \alpha(x) \le \Gamma^+_\alpha(x)$; $\Gamma^-_\beta(x) \le \beta(x) \le \Gamma^+_\beta(x)$; and $\Gamma^-_\gamma(x) \le \gamma(x) \le \Gamma^+_\gamma(x)$ for all $x \in U$, then Ξ is said to be an internal neutrosophic cubic set (INCS).

Example 2. Consider $\Xi = < \Gamma, \chi > \in \text{NCS}^U$, if $\Gamma(x) = <$ [0.65, 0.8], [0.1, 0.25], [0.2, 0.4] $>$ and $\chi(x) = <$ 0.7, 0.2, 0.3 $>$ for all $x \in U$, then $\Xi = < \Gamma, \chi >$ is an INCS.

Definition: 11

External neutrosophic cubic set [10]: Consider $\Xi = < \Gamma, \chi > \in \text{NCS}^U$, if $\alpha(x) \notin (\Gamma^-_\alpha(x), \Gamma^+_\alpha(x))$; $\beta(x) \notin (\Gamma^-_\beta(x), \Gamma^+_\beta(x))$; and $\gamma(x) \notin (\Gamma^-_\gamma(x), \Gamma^+_\gamma(x))$ for all $x \in U$, then $\Xi = < \Gamma, \chi >$ is said to be an external neutrosophic cubic set (ENCS).

Example 3. Consider $\Xi = < \Gamma, \chi > \in \text{NCS}^U$, if $\Gamma(x) = <$ [0.65, 0.8], [0.1, 0.25], [0.2, 0.4] $>$ and $\chi(x) = <$ 0.85, 0.3, 0.1 $>$ for all $x \in U$, then $\Xi = < \Gamma, \chi >$ is an ENCS.

Theorem 1. [10]

Consider $\Xi = < \Gamma, \chi > \in \text{NCS}^U$, which is not an ENCS, then there exists $x \in U$ such that $\Gamma^-_\alpha(x) \le \alpha(x) \le \Gamma^+_\alpha(x)$; $\Gamma^-_\beta(x) \le \beta(x) \le \Gamma^+_\beta(x)$; or $\Gamma^-_\gamma(x) \le \gamma(x) \le \Gamma^+_\gamma(x)$.

Definition: 12

$\frac{2}{3}$**-INCS(**$\frac{1}{3}$**-ENCS) [10]:** Consider $\Xi = < \Gamma, \chi > \in \text{NCS}^U$, if $\Gamma^-_\alpha(x) \le \alpha(x) \le \Gamma^+_\alpha(x)$; $\Gamma^-_\beta(x) \le \beta(x) \le \Gamma^+_\beta(x)$; and $\gamma(x) \notin (\Gamma^-_\gamma(x), \Gamma^+_\gamma(x))$ or $\Gamma^-_\alpha(x) \le \alpha(x) \le \Gamma^+_\alpha(x)$; $\Gamma^-_\gamma(x) \le \gamma(x) \le \Gamma^+_\gamma(x)$ and $\beta(x) \notin (\Gamma^-_\beta(x), \Gamma^+_\beta(x))$ or $\Gamma^-_\beta(x) \le \beta(x) \le \Gamma^+_\beta(x)$; and $\Gamma^-_\gamma(x) \le \gamma(x) \le \Gamma^+_\gamma(x)$ and $\alpha(x) \notin (\Gamma^-_\alpha(x), \Gamma^+_\alpha(x))$ for all $x \in U$, then $\Xi = < \Gamma, \chi >$ is said to be an $\frac{2}{3}$-INCS or $\frac{1}{3}$-ENCS.

Example 4. Consider $\Xi = < \Gamma, \chi > \in \text{NCS}^U$, if $\Gamma(x) = <$ [0. 5, 0.7], [0.1, 0.2], [0.2, 0.45] $>$ and $\chi(x) = <$ 0.65, 0.3, 0.4 $>$ for all $x \in U$, then $\Xi = < \Gamma, \chi >$ is an $\frac{2}{3}$-INCS or $\frac{1}{3}$-ENCS.

Definition: 13

$\frac{1}{3}$**-INCS (**$\frac{2}{3}$**-ENCS) [10]:** Consider $\Xi = < \Gamma, \chi > \in \text{NCS}^U$, if $\Gamma^-_\alpha(x) \le \alpha(x) \le \Gamma^+_\alpha(x)$; $\beta(x) \notin (\Gamma^-_\beta(x), \Gamma^+_\beta(x))$; and $\gamma(x) \notin (\Gamma^-_\gamma(x), \Gamma^+_\gamma(x))$ or $\Gamma^-_\gamma(x) \le \gamma(x) \le \Gamma^+_\gamma(x)$; $\alpha(x) \notin (\Gamma^-_\alpha(x), \Gamma^+_\alpha(x))$ and $\beta(x) \notin (\Gamma^-_\beta(x), \Gamma^+_\beta(x))$ or $\Gamma^-_\beta(x) \le \beta(x) \le \Gamma^+_\beta(x)$; $\alpha(x) \notin (\Gamma^-_\alpha(x), \Gamma^+_\alpha(x))$ and $\gamma(x) \notin (\Gamma^-_\gamma(x), \Gamma^+_\gamma(x))$ for all $x \in U$, then $\Xi = < \Gamma, \chi >$ is said to be an $\frac{1}{3}$-INCS or $\frac{2}{3}$-ENCS.

Example 5. Consider $\Xi = < \Gamma, \chi > \in \text{NCS}^U$, if $\Gamma(x) = <$ [0. 5, 0.8], [0.15, 0.25], [0.2, 0.35] $>$ and $\chi(x) = <$ 0.55, 0.4, 0.5 $>$ for all $x \in U$, then $\Xi = < \Gamma, \chi >$ is an $\frac{1}{3}$-INCS or $\frac{2}{3}$-ENCS.

Definition: 14 [10]

Consider $\Xi_1 = < \Gamma_1, \chi_1 >$ and $\Xi_2 = < \Gamma_2, \chi_2 >$ be two NCSs in U, then

1. (Equality) $\Xi_1 = \Xi_2$ if and only if $\Gamma_1 = \Gamma_2$ and $\chi_1 = \chi_2$.
2. (P-order) $\Xi_1 \subseteq_P \Xi_2$ if and only if $\Gamma_1 \tilde{\subseteq} \Gamma_2$ and $\chi_1 \breve{\subseteq} \chi_2$.
3. (R-order) $\Xi_1 \subseteq_R \Xi_2$ if and only if $\Gamma_1 \tilde{\subseteq} \Gamma_2$ and $\chi_1 \breve{\supseteq} \chi_2$.

For convenience, p = < ([Γ^-_{α}, $\Gamma^+_{\alpha_1}$], [$\Gamma^-_{\beta_1}$, $\Gamma^+_{\beta_1}$], [Γ^-_{γ}, Γ^+_{γ}]), (α, β, γ) > is said to represent neutrosophic cubic value (NCV)

Definition: 15
Complement [10]: Consider $\Xi = <\Gamma, \chi>$ be an NCS, then the complement of $\Xi = <\Gamma, \chi>$ is given by

$$\Xi^C = \{ \left\langle x, \Gamma^{\tilde{C}}(x), \chi^{\breve{C}}(x) \right\rangle \mid x \in U\}.$$

2.1 Several operational rules of NCVs

Consider p_1 = < ([$\Gamma^-_{\alpha_1}$, $\Gamma^+_{\alpha_1}$], [$\Gamma^-_{\beta_1}$, $\Gamma^+_{\beta_1}$], [$\Gamma^-_{\gamma_1}$, $\Gamma^+_{\gamma_1}$]), $(\alpha_1, \beta_1, \gamma_1)$ > and p_2 = < ([$\Gamma^-_{\alpha_2}$, $\Gamma^+_{\alpha_2}$], [$\Gamma^-_{\beta_2}$, $\Gamma^+_{\beta_2}$], [$\Gamma^-_{\gamma_2}$, $\Gamma^+_{\gamma_2}$]), $(\alpha_2, \beta_2, \gamma_2)$ > be two NCVs in U, then the operational rules are presented as follows:

1. The complement [10] of p_1 is p_1^C = < ([$\Gamma^-_{\gamma_1}$, $\Gamma^+_{\gamma_1}$], [1- $\Gamma^+_{\beta_1}$,1- $\Gamma^-_{\beta_1}$], [$\Gamma^-_{\alpha_1}$, $\Gamma^+_{\alpha_1}$]), $(\gamma_1, 1-\beta_1, \alpha_1)$ >.
2. The summation between p_1 and p_2 is defined as follows:

   $p_1 \oplus p_2$ = < ([$\Gamma^-_{\alpha_1} + \Gamma^-_{\alpha_2} - \Gamma^-_{\alpha_1}\Gamma^-_{\alpha_2}$, $\Gamma^+_{\alpha_1} + \Gamma^+_{\alpha_2} - \Gamma^+_{\alpha_1}\Gamma^+_{\alpha_2}$], [$\Gamma^-_{\beta_1}\Gamma^-_{\beta_2}$, $\Gamma^+_{\beta_1}\Gamma^+_{\beta_2}$], [$\Gamma^-_{\gamma_1}\Gamma^-_{\gamma_2}$, $\Gamma^+_{\gamma_1}\Gamma^+_{\gamma_2}$]), $(\alpha_1 + \alpha_2 - \alpha_1\alpha_2, \beta_1\beta_2, \gamma_1\gamma_2)$ >.
3. The multiplication between p_1 and p_2 is defined as follows:

   $p_1 \otimes p_2$ = < ([$\Gamma^-_{\alpha_1}\Gamma^-_{\alpha_2}$, $\Gamma^+_{\alpha_1}\Gamma^+_{\alpha_2}$], [$\Gamma^-_{\beta_1} + \Gamma^-_{\beta_2} - \Gamma^-_{\beta_1}\Gamma^-_{\beta_2}$, $\Gamma^+_{\beta_1} + \Gamma^+_{\beta_2} - \Gamma^+_{\beta_1}\Gamma^+_{\beta_2}$], [$\Gamma^-_{\gamma_1} + \Gamma^-_{\gamma_2} - \Gamma^-_{\gamma_1}\Gamma^-_{\gamma_2}$, $\Gamma^+_{\gamma_1} + \Gamma^+_{\gamma_2} - \Gamma^+_{\gamma_1}\Gamma^+_{\gamma_2}$]), $(\alpha_1\alpha_2, \beta_1 + \beta_2 - \beta_1\beta_2, \gamma_1 + \gamma_2 - \gamma_1\gamma_2)$ >.
4. Consider p_1 = < ([$\Gamma^-_{\alpha_1}$, $\Gamma^+_{\alpha_1}$], [$\Gamma^-_{\beta_1}$, $\Gamma^+_{\beta_1}$], [$\Gamma^-_{\gamma_1}$, $\Gamma^+_{\gamma_1}$]), $(\alpha_1, \beta_1, \gamma_1)$ > be a NCV and κ be an arbitrary positive real number, then $\kappa\, p_1$ and p_1^{κ} are defined as follows:

   (i) $\kappa\, p_1$ = < ([1- (1- $\Gamma^-_{\alpha_1})^{\kappa}$,1- (1- $\Gamma^+_{\alpha_1})^{\kappa}$], [($\Gamma^-_{\beta_1})^{\kappa}$, $(\Gamma^+_{\beta_1})^{\kappa}$],[($\Gamma^-_{\gamma_1})^{\kappa}$, $(\Gamma^+_{\gamma_1})^{\kappa}$]), (1- (1- $\alpha_1)^{\kappa}$, $(\beta_1)^{\kappa}$, $(\gamma_1)^{\kappa}$) >;

   (ii) p_1^{κ} = < ([($\Gamma^-_{\alpha_1})^{\kappa}$, ($\Gamma^+_{\alpha_1})^{\kappa}$], [1- (1- $\Gamma^-_{\beta_1})^{\kappa}$,1- (1- $\Gamma^+_{\beta_1})^{\kappa}$], [1- (1- $\Gamma^-_{\gamma_1})^{\kappa}$,1- (1- $\Gamma^+_{\gamma_1})^{\kappa}$]), ($(\alpha_1)^{\kappa}$,1-(1- $\beta_1)^{\kappa}$,1- (1- $\gamma_1)^{\kappa}$) >.

Definition: 16 [10]
Consider $\Xi_1 = <\Gamma_1, \chi_1>$ and $\Xi_2 = <\Gamma_2, \chi_2>$ be two NCSs in U, then the Hamming distance between Ξ_1 and Ξ_2 is defined as follows:

$$D_H(\Xi_1, \Xi_2) = \frac{1}{9n}\sum_{i=1}^{n} (|\Gamma^-_{1\alpha}(x_i) - \Gamma^-_{2\alpha}(x_i)| + |\Gamma^-_{1\beta}(x_i) - \Gamma^-_{2\beta}(x_i)| + |\Gamma^-_{1\gamma}(x_i) - \Gamma^-_{2\gamma}(x_i)| + |\Gamma^+_{1\alpha}(x_i) - \Gamma^+_{2\alpha}(x_i)| + |\Gamma^+_{1\beta}(x_i) - \Gamma^+_{2\beta}(x_i)| + |\Gamma^+_{1\gamma}(x_i) - \Gamma^+_{2\gamma}(x_i)| + |\alpha_1(x_i) - \alpha_2(x_i)| + |\beta_1(x_i) - \beta_2(x_i)| + |\gamma_1(x_i) - \gamma_2(x_i)|).$$

Example 7: Suppose that $\Xi_1 = <\Gamma_1, \chi_1>$ = < ([0.6, 0.75], [0.15, 0.25], [0.25, 0.45]), (0.8, 0.35, 0.15) > and $\Xi_2 = <\Gamma_2, \chi_2>$ = < ([0.45, 0.7], [0.1, 0.2], [0.05, 0.2]), (0.3, 0.15, 0.45) > be two NCSs in U, then $D_H(\Xi_1, \Xi_2)$ = 0.1944.

Definition: 17
Consider $\Xi_1 = <\Gamma_1, \chi_1>$ and $\Xi_2 = <\Gamma_2, \chi_2>$ be two NCSs in U, then the Euclidean distance between Ξ_1 and Ξ_2 is defined as given below.

$$D_E(\Xi_1, \Xi_2) = \sqrt{\frac{1}{9n}\sum_{i=1}^{n}\begin{pmatrix}(\Gamma^-_{1\alpha}(x_i) - \Gamma^-_{2\alpha}(x_i))^2 + (\Gamma^+_{1\alpha}(x_i) - \Gamma^+_{2\alpha}(x_i))^2 + (\Gamma^-_{1\beta}(x_i) - \Gamma^-_{2\beta}(x_i))^2 + \\ (\Gamma^+_{1\beta}(x_i) - \Gamma^+_{2\beta}(x_i))^2 + (\Gamma^-_{1\gamma}(x_i) - \Gamma^-_{2\gamma}(x_i))^2 + (\Gamma^+_{1\gamma}(x_i) - \Gamma^+_{2\gamma}(x_i))^2 + \\ (\alpha_1(x_i) - \alpha_2(x_i))^2 + (\beta_1(x_i) - \beta_2(x_i))^2 + (\gamma_1(x_i) - \gamma_2(x_i))^2\end{pmatrix}}$$

with the condition $0 \le D_E(\Xi_1, \Xi_2) \le 1$.

Example 8: Suppose that $\Xi_1 = <\Gamma_1, \chi_1>$ = < ([0.4, 0.5], [0.1, 0.2], [0.25, 0.5]), (0.4, 0.3, 0.25) > and $\Xi_2 = <\Gamma_2, \chi_2>$ = < ([0.5, 0.9], [0.15, 0.3], [0.05, 0.1]), (0.7, 0.1, 0.15) > be two NCSs in U, then D_E (Ξ_1, Ξ_2) = 0.2409.

3 An extended TOPSIS method for MADM problems under neutrosophic cubic set environment

In this Section, we introduce a new extended TOPSIS method to handle MADM problems involving neutrosophic cubic information. Consider $B = \{B_1, B_2, \ldots, B_m\}$, $(m \geq 2)$ be a discrete set of m feasible alternatives and $C = \{C_1, C_2, \ldots, C_n\}$, $(n \geq 2)$ be a set of attributes. Also, let $w = (w_1, w_2, \ldots, w_n)^T$ be the unknown weight vector of the attributes with $0 \leq w_j \leq 1$ such that $\sum_{j=1}^{n} w_j = 1$. Suppose that the rating of alternative B_i (i = 1, 2, …, m) with respect to the attribute C_j (j = 1, 2, …, n) is described by a_{ij} where a_{ij} = < ([$\Gamma^-_{\alpha_{ij}}, \Gamma^+_{\alpha_{ij}}$], [$\Gamma^-_{\beta_{ij}}, \Gamma^+_{\beta_{ij}}$], [$\Gamma^-_{\gamma_{ij}}, \Gamma^+_{\gamma_{ij}}$]), ($\alpha_{ij}, \beta_{ij}, \gamma_{ij}$) >. The proposed approach for ranking the alternatives under neutrosophic cubic environment is shown using the following steps:

Step 1. Construction and standardization of decision matrix with neutrosophic cubic information

Consider the rating of alternative B_i (i = 1, 2, …, m) with respect to the predefined attribute C_j, (j = 1, 2, …, n) be presented by the decision maker in the neutrosophic cubic decision matrix (See eqn. 1).

$$\left\langle a_{ij} \right\rangle_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ . & . & \ldots & . \\ . & . & \ldots & . \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \quad (1)$$

In general, there are two types of attributes appear in the decision making circumstances namely (i) benefit type attributes $\in J_1$, where the more attribute value denotes better alternative (ii) cost type attributes $\in J_2$, where the less attribute value denotes better alternative. We standardize the above decision matrix $\left\langle a_{ij} \right\rangle_{m\times n}$ in order to remove the influence of diverse physical dimensions to decision results.

Consider $\left\langle s_{ij} \right\rangle_{m\times n}$ be the standardize decision matrix, where

$$s_{ij} = < ([\dot{\Gamma}^-_{\alpha_{ij}}, \dot{\Gamma}^+_{\alpha_{ij}}], [\dot{\Gamma}^-_{\beta_{ij}}, \dot{\Gamma}^+_{\beta_{ij}}], [\dot{\Gamma}^-_{\gamma_{ij}}, \dot{\Gamma}^+_{\gamma_{ij}}]), (\dot{\alpha}_{ij}, \dot{\beta}_{ij}, \dot{\gamma}_{ij})>,$$

where

$s_{ij} = a_{ij}$, if the attribute j is benefit type;

$s_{ij} = a^c_{ij}$, if the attribute j is cost type.

Here a^c_{ij} denotes the complement of a_{ij}.

Step 2. Identify the weights of the attributes

To determine the unknown weight of attribute in the decision making situation is a difficult task for DM. Generally, weights of the attributes are dissimilar and they play a decisive role in finding the ranking order of the alternatives. Pramanik and Mondal [16] defined arithmetic averaging operator (AAO) in order to transform interval neutrosophic numbers to SVNNs. Based on the concept of Pramanik and Mondal [16], we define AAO to transform NCVs to SNVs as follows:

$$NC_{ij} < \dot{\Gamma}_{\alpha_{ij}}, \dot{\Gamma}_{\beta_{ij}}, \dot{\Gamma}_{\gamma_{ij}} > = NC_{ij}\left\langle \frac{\dot{\Gamma}^-_{\alpha_{ij}} + \dot{\Gamma}^+_{\alpha_{ij}} + \dot{\alpha}_{ij}}{3}, \frac{\dot{\Gamma}^-_{\beta_{ij}} + \dot{\Gamma}^+_{\beta_{ij}} + \dot{\beta}_{ij}}{3}, \frac{\dot{\Gamma}^-_{\gamma_{ij}} + \dot{\Gamma}^+_{\gamma_{ij}} + \dot{\gamma}_{ij}}{3} \right\rangle$$

In this paper, we utilize information entropy method to find the weights of the attributes w_j where weihgts of the attributes are unequal and fully unknown to the DM. Majumdar and Samanta [17] investigated some similarity measures and entropy measures for SVNSs where entropy is used to measure uncertain information. Here, we take the following notations:

$$T_{\Omega_P}(x_i) = \left[\frac{\dot{\Gamma}^-_{\alpha_{ij}} + \dot{\Gamma}^+_{\alpha_{ij}} + \dot{\alpha}_{ij}}{3} \right], \; I_{\Omega_P}(x_i) = \left[\frac{\dot{\Gamma}^-_{\beta_{ij}} + \dot{\Gamma}^+_{\beta_{ij}} + \dot{\beta}_{ij}}{3} \right],$$

$$F_{\Omega_P}(x_i) = \left[\frac{\dot{\Gamma}^-_{\gamma_{ij}} + \dot{\Gamma}^+_{\gamma_{ij}} + \dot{\gamma}_{ij}}{3} \right]$$

Then we can write $\Omega_P = \left\langle T_{\Omega_P}(x_i), I_{\Omega_P}(x_i), F_{\Omega_P}(x_i) \right\rangle$.

The entropy value is given as follows:

$$E_i(\Omega_P) = 1 - \frac{1}{n} \sum_{i=1}^{m} (T_{\Omega_P}(x_i) + F_{\Omega_P}(x_i)) \left| I_{\Omega_P}(x_i) - I^c_{\Omega_P}(x_i) \right|$$

which has the following properties:

(i). $E_i(\Omega_P) = 0$ if Ω_P is a crisp set and $I_{\Omega_P}(x_i) = 0$, $F_{\Omega_P}(x_i) = 0 \; \forall \; x \in E$.

(ii). $E_i(\Omega_P) = 0$ if $\left\langle T_{\Omega_P}(x_i), I_{\Omega_P}(x_i), F_{\Omega_P}(x_i) \right\rangle = < T_{\Omega_P}(x_i)$, 0.5, $F_{\Omega_P}(x_i)$ >, $\forall \; x \in E$.

(iii). $E_i(\Omega_P) \geq E_i(\Omega_Q)$ if Ω_P is more uncertain than Ω_Q i.e.

$$T_{\Omega_P}(x_i) + F_{\Omega_P}(x_i) \leq T_{\Omega_Q}(x_i) + F_{\Omega_Q}(x_i)$$

and $\left| I_{\Omega_P}(x_i) - I^c_{\Omega_P}(x_i) \right| \leq \left| I_{\Omega_Q}(x_i) - I^c_{\Omega_Q}(x_i) \right|$.

(iv). $E_i(\Omega_P) = E_i(\Omega^c_P)$, $\forall \; x \in E$.

Consequently, the entropy value Ev_j of the j-th attribute can be calculated as as follows:.

$Ev_j = 1 - \frac{1}{n}\sum_{i=1}^{m}(T_{ij}(x_i) + F_{ij}(x_i))\left|I_{ij}(x_i) - I_{ij}^C(x_i)\right|$, i = 1, 2, …, m; j = 1, 2,…, n.

We observe that $0 \le Ev_j \le 1$. Based on Hwang and Yoon [18] and Wang and Zhang [19] the entropy weight of the j-th attribute is defined as follows:

$w_j = \frac{1-\mathrm{Ev}_j}{\sum_{j=1}^{n}(1-\mathrm{Ev}_j)}$ with $0 \le w_j \le 1$ and $\sum_{j-1}^{n} w_j = 1$.

Step 3. Formulation of weighted decision matrix

The weighted decision matrix is obtained by multiplying weights of the attributes (w_j) and the standardized decision matrix $\langle s_{ij}\rangle_{m\times n}$. Therefore, the weighted neutrosophic cubic decision matrix $\langle z_{ij}\rangle_{m\times n}$ is obtained as:

$$\langle z_{ij}\rangle_{m\times n} = w_j \otimes \langle a_{ij}\rangle_{m\times n} = \begin{bmatrix} w_1s_{11} & w_2s_{12} & \cdots & w_ns_{1n} \\ w_1s_{21} & w_2s_{22} & \cdots & w_ns_{2n} \\ . & . & \cdots & . \\ w_1s_{m1} & w_2s_{m2} & \cdots & w_ns_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{bmatrix}$$

where

$z_{ij} = <([\ddot{\Gamma}^-_{\alpha_{ij}}, \ddot{\Gamma}^+_{\alpha_{ij}}], [\ddot{\Gamma}^-_{\beta_{ij}}, \ddot{\Gamma}^+_{\beta_{ij}}], [\ddot{\Gamma}^-_{\gamma_{ij}}, \ddot{\Gamma}^+_{\gamma_{ij}}]), (\ddot{\alpha}_{ij}, \ddot{\beta}_{ij}, \ddot{\gamma}_{ij}) >$

$= <([1-(1-\dot{\Gamma}^-_{\alpha_{ij}})^{w_j}, 1-(1-\dot{\Gamma}^+_{\alpha_{ij}})^{w_j}],$

$[(\dot{\Gamma}^-_{\beta_{ij}})^{w_j}, (\dot{\Gamma}^+_{\beta_{ij}})^{w_j}], [(\dot{\Gamma}^-_{\gamma_{ij}})^{w_j}, (\dot{\Gamma}^+_{\gamma_{ij}})^{w_j}]), (1-(1-\dot{\alpha}_{ij})^{w_j},$

$(\dot{\beta}_{ij})^{w_j}, (\dot{\gamma}_{ij})^{w_j}) >$

Step 4. Selection of neutrosophic cubic positive ideal solution (NCPIS) and neutrosophic cubic negative ideal solution (NCNIS)

Consider $z^U = (z_1^U, z_2^U, \ldots, z_n^U)$ and $z^L = (z_1^L, z_2^L, \ldots, z_n^L)$ be the NCPIS and NCNIS respectively, then z_j^U is defined as follows:

$z_j^U = <([(\ddot{\Gamma}^-_{\alpha_j})^U, (\ddot{\Gamma}^+_{\alpha_j})^U], [(\ddot{\Gamma}^-_{\beta_j})^U, (\ddot{\Gamma}^+_{\beta_j})^U], [(\ddot{\Gamma}^-_{\gamma_j})^U, (\ddot{\Gamma}^+_{\gamma_j})^U]), ((\ddot{\alpha}_j)^U, (\ddot{\beta}_j)^U, (\ddot{\gamma}_j)^U) >$

where

$(\ddot{\Gamma}^-_{\alpha_j})^U = \{(\max_i\{\ddot{\Gamma}^-_{\alpha_{ij}}\} | j\in J_1), (\min_i\{\ddot{\Gamma}^-_{\alpha_{ij}}\} | j\in J_2)\}$,

$(\ddot{\Gamma}^+_{\alpha_j})^U = \{(\max_i\{\ddot{\Gamma}^+_{\alpha_{ij}}\} | j\in J_1), (\min_i\{\ddot{\Gamma}^+_{\alpha_{ij}}\} | j\in J_2)\}$,

$(\ddot{\Gamma}^-_{\beta_j})^U = \{(\min_i\{\ddot{\Gamma}^-_{\beta_{ij}}\} | j\in J_1), (\max_i\{\ddot{\Gamma}^-_{\beta_{ij}}\} | j\in J_2)\}$,

$(\ddot{\Gamma}^+_{\beta_j})^U = \{(\min_i\{\ddot{\Gamma}^+_{\beta_{ij}}\} | j\in J_1), (\max_i\{\ddot{\Gamma}^+_{\beta_{ij}}\} | j\in J_2)\}$,

$(\ddot{\Gamma}^-_{\gamma_j})^U = \{(\min_i\{\ddot{\Gamma}^-_{\gamma_{ij}}\} | j\in J_1), (\max_i\{\ddot{\Gamma}^-_{\gamma_{ij}}\} | j\in J_2)\}$,

$(\ddot{\Gamma}^+_{\gamma_j})^U = \{(\min_i\{\ddot{\Gamma}^+_{\gamma_{ij}}\} | j\in J_1), (\max_i\{\ddot{\Gamma}^+_{\gamma_{ij}}\} | j\in J_2)\}$,

$(\ddot{\alpha}_j)^U = \{(\max_i\{\ddot{\alpha}_{ij}\} | j\in J_1), (\min_i\{\ddot{\alpha}_{ij}\} | j\in J_2)\}$,

$(\ddot{\beta}_j)^U = \{(\min_i\{\ddot{\beta}_{ij}\} | j\in J_1), (\max_i\{\ddot{\beta}_{ij}\} | j\in J_2)\}$,

$(\ddot{\gamma}_j)^U = \{(\min_i\{\ddot{\gamma}_{ij}\} | j\in J_1), (\max_i\{\ddot{\gamma}_{ij}\} | j\in J_2)\}$;

and z_j^L is defined as given below

$z_j^L = <[(\ddot{\Gamma}^-_{\alpha_{ij}})^L, (\ddot{\Gamma}^+_{\alpha_{ij}})^L], [(\ddot{\Gamma}^-_{\beta_{ij}})^L, (\ddot{\Gamma}^+_{\beta_{ij}})^L], [(\ddot{\Gamma}^-_{\gamma_{ij}})^L, (\ddot{\Gamma}^+_{\gamma_{ij}})^L], ((\ddot{\alpha}_{ij})^L, (\ddot{\beta}_{ij})^L, (\ddot{\gamma}_{ij})^L)>$

where $(\ddot{\Gamma}^-_{\alpha_j})^L = \{(\min_i\{\ddot{\Gamma}^-_{\alpha_{ij}}\} | j\in J_1), (\max_i\{\ddot{\Gamma}^-_{\alpha_{ij}}\} | j\in J_2)\}$, $(\ddot{\Gamma}^+_{\alpha_j})^L = \{(\min_i\{\ddot{\Gamma}^+_{\alpha_{ij}}\} | j\in J_1), (\max_i\{\ddot{\Gamma}^+_{\alpha_{ij}}\} | j\in J_2)\}$, $(\ddot{\Gamma}^-_{\beta_j})^L = \{(\max_i\{\ddot{\Gamma}^-_{\beta_{ij}}\} | j\in J_1), (\min_i\{\ddot{\Gamma}^-_{\beta_{ij}}\} | j\in J_2)\}$, $(\ddot{\Gamma}^+_{\beta_j})^L = \{(\max_i\{\ddot{\Gamma}^+_{\beta_{ij}}\} | j\in J_1), (\min_i\{\ddot{\Gamma}^+_{\beta_{ij}}\} | j\in J_2)\}$, $(\ddot{\Gamma}^-_{\gamma_j})^L = \{(\max_i\{\ddot{\Gamma}^-_{\gamma_{ij}}\} | j\in J_1), (\min_i\{\ddot{\Gamma}^-_{\gamma_{ij}}\} | j\in J_2)\}$, $(\ddot{\Gamma}^+_{\gamma_j})^L = \{(\max_i\{\ddot{\Gamma}^+_{\gamma_{ij}}\} | j\in J_1), (\min_i\{\ddot{\Gamma}^+_{\gamma_{ij}}\} | j\in J_2)\}$, $(\ddot{\alpha}_j)^L = \{(\min_i\{\ddot{\alpha}_{ij}\} | j\in J_1), (\max_i\{\ddot{\alpha}_{ij}\} | j\in J_2)\}$, $(\ddot{\beta}_j)^L = \{(\max_i\{\ddot{\beta}_{ij}\} | j\in J_1), (\min_i\{\ddot{\beta}_{ij}\} | j\in J_2)\}$, $(\ddot{\gamma}_j)^L = \{(\max_i\{\ddot{\gamma}_{ij}\} | j\in J_1), (\min_i\{\ddot{\gamma}_{ij}\} | j\in J_2)\}$.

Step 5. Calculate the distance measure of alternatives from NCPIS and NCNIS

The Euclidean distance measure of each alternative B_i, i = 1, 2, …, m from NCPIS can be defined as follows:

$$D^+_{E_i} = \sqrt{\frac{1}{9n}\sum_{j=1}^{n}\begin{pmatrix} (\ddot{\Gamma}^-_{\alpha_{ij}} - (\ddot{\Gamma}^-_{\alpha_j})^U)^2 + (\ddot{\Gamma}^+_{\alpha_{ij}} - (\ddot{\Gamma}^+_{\alpha_j})^U)^2 + (\ddot{\Gamma}^-_{\beta_{ij}} - (\ddot{\Gamma}^-_{\beta_j})^U)^2 + \\ (\ddot{\Gamma}^+_{\beta_{ij}} - (\ddot{\Gamma}^+_{\beta_j})^U)^2 + (\ddot{\Gamma}^-_{\gamma_{ij}} - (\ddot{\Gamma}^-_{\gamma_j})^U)^2 + (\ddot{\Gamma}^+_{\gamma_{ij}} - (\ddot{\Gamma}^+_{\gamma_j})^U)^2 + \\ (\ddot{\alpha}_{ij} - (\ddot{\alpha}_j)^U)^2 + (\ddot{\beta}_{ij} - (\ddot{\beta}_j)^U)^2 + (\ddot{\gamma}_{ij} - (\ddot{\gamma}_j)^U)^2 \end{pmatrix}}$$

Similarly, the Euclidean distance measure of each alternative B_i, i = 1, 2, …, m from NCNIS can be written as follows:

$$\mathrm{D}^{-}_{\mathrm{E}_i} = \sqrt{\frac{1}{9n}\sum_{j=1}^{n}\begin{pmatrix}(\ddot{\Gamma}^{-}_{\alpha_{ij}}-(\ddot{\Gamma}^{-}_{\alpha_j})^{L})^2+(\ddot{\Gamma}^{+}_{\alpha_{ij}}-(\ddot{\Gamma}^{+}_{\alpha_j})^{L})^2+(\ddot{\Gamma}^{-}_{\beta_{ij}}-(\ddot{\Gamma}^{-}_{\beta_j})^{L})^2+\\(\ddot{\Gamma}^{+}_{\beta_{ij}}-(\ddot{\Gamma}^{+}_{\beta_j})^{L})^2+(\ddot{\Gamma}^{-}_{\gamma_{ij}}-(\ddot{\Gamma}^{-}_{\gamma_j})^{L})^2+(\ddot{\Gamma}^{+}_{\gamma_{ij}}-(\ddot{\Gamma}^{+}_{\gamma_j})^{L})^2+\\(\ddot{\alpha}_{ij}-(\ddot{\alpha}_j)^{L})^2+(\ddot{\beta}_{ij}-(\ddot{\beta}_j)^{L})^2+(\ddot{\gamma}_{ij}-(\ddot{\gamma}_j)^{L})^2\end{pmatrix}}.$$

Step 6. Evaluate the relative closeness co-efficient to the neutrosophic cubic ideal solution

The relative closeness co-efficient RCC_i^* of each B_i, i = 1, 2, …, m with respect to NCPIS z_j^U, j = 1, 2, …, n is defined as follows:

$$RCC_i^* = \frac{D^{-}_{E_i}}{D^{+}_{E_i}+D^{-}_{E_i}}, \text{ i = 1, 2, …, } m.$$

Step 7. Rank the alternatives

We obtain the ranking order of the alternatives based on the RCC_i^*. The bigger value of RCC_i^* reflects the better alternative.

4. Numerical example

In this section, we consider an example of neutrosophic cubic MADM, adapted from Mondal and Pramanik [20] to demonstrate the applicability and the effectiveness of the proposed extended TOPSIS method.

Consider a legal guardian desires to select an appropriate school for his/ her child for basic education [20]. Suppose there are three possible alternatives for his/ her child:

(1) B_1, a Christian missionary school
(2) B_2, a Basic English medium school
(3) B_3, a Bengali medium kindergarten.

He/ She must take a decision based on the following four attributes:

(1) C_1 is the distance and transport,
(2) C_2 is the cost,
(3) C_3 is the staff and curriculum, and
(4) C_4 is the administrative and other facilities

Here C_1 and C_2 are cost type attributes; while C_3 and C_4 are benefit type attributes. Suppose the weights of the four attributes are unknown. Using the the following steps, we solve the problem.

Step 1. The rating of the alternative B_i, i = 1, 2, 3 with respect to the alternative C_j, j = 1, 2, 3, 4 is represented by neutrosophic cubic assessments. The decision matrix $\langle a_{ij}\rangle_{3\times 4}$ is shown in Table 1.

Table 1. Neutrosophic cubic decision matrix

	C_1	C_2	C_3	C_4
B_1	$<([0.3,0.4],[0.1,0.2],[0.2,0.35]), (0.3,0.4,0.25)>$	$<([0.6,0.7],[0.05,0.1],[0.2,0.3]), (0.5,0.1,0.25)>$	$<([0.5,0.6],[0.2,0.4],[0.1,0.3]), (0.5,0.3,0.4)>$	$<([0.4,0.6],[0.1,0.25],[0.1,0.3]), (0.5,0.2,0.4)>$
B_2	$<([0.8,0.9],[0.1,0.2],[0.15,0.3]), (0.7,0.15,0.3)>$	$<([0.3,0.5],[0.1,0.4],[0.3,0.5]), (0.4,0.3,0.2)>$	$<([0.4,0.5],[0.2,0.35],[0.05,0.2]), (0.35,0.1,0.1)>$	$<([0.2,0.3],[0.2,0.35],[0.1,0.25]), (0.4,0.1,0.1)>$
B_3	$<([0.6,0.7],[0.2,0.4],[0.25,0.4]), (0.5,0.3,0.3)>$	$<([0.2,0.35],[0.1,0.25],[0.2,0.3]), (0.3,0.3,0.4)>$	$<([0.4,0.7],[0.1,0.3],[0.15,0.25]), (0.5,0.2,0.2)>$	$<([0.5,0.7],[0.1,0.2],[0.2,0.25]), (0.3,0.1,0.2)>$

Step 2. Standardize the decision matrix.

The standardized decision matrix $\langle s_{ij}\rangle_{3\times 4}$ is constructed as follows (see Table 2):

Table 2. The standardized neutrosophic cubic decision matrix

	C_1	C_2	C_3	C_4
B_1	$<([0.2,0.35],[0.8,0.9],[0.3,0.4]), (0.25,0.6,0.3)>$	$<([0.2,0.3],[0.9,0.95],[0.6,0.7]), (0.25,0.9,0.5)>$	$<([0.5,0.6],[0.2,0.4],[0.1,0.3]), (0.5,0.3,0.4)>$	$<([0.4,0.6],[0.1,0.25],[0.1,0.3]), (0.5,0.2,0.4)>$
B_2	$<([0.15,0.3],[0.8,0.9],[0.8,0.9]), (0.3,0.85,0.7)>$	$<([0.3,0.5],[0.6,0.9],[0.3,0.5]), (0.2,0.7,0.4)>$	$<([0.4,0.5],[0.2,0.35],[0.05,0.2]), (0.35,0.1,0.1)>$	$<([0.2,0.3],[0.2,0.35],[0.1,0.25]), (0.4,0.1,0.1)>$
B_3	$<([0.25,0.4],[0.6,0.8],[0.6,0.7]), (0.3,0.7,0.5)>$	$<([0.2,0.3],[0.75,0.9],[0.2,0.35]), (0.4,0.7,0.3)>$	$<([0.4,0.7],[0.1,0.3],[0.15,0.25]), (0.5,0.2,0.2)>$	$<([0.5,0.7],[0.1,0.2],[0.2,0.25]), (0.3,0.1,0.2)>$

Step 3. Using AAO, we transform NCVs into SNVs. We calculate entropy value E_j of the j-th attribute as follows:

Ev_1 = 0.644, Ev_2 = 0.655, Ev_3 = 0.734, Ev_4 = 0.663.

The weight vector of the four attributes are obtained as:

w_1 = 0.2730, w_2 = 0.2646, w_3 = 0.2040, w_4 = 0.2584.

Step 4. After identifying the weight of the attribute (w_j), we multiply the standardized decision matrix with w_j, j = 1, 2, …, n to obtain the weighted decision matrix $\langle z_{ij}\rangle_{3\times 4}$ (see Table 3).

Table 3. The weighted neutrosophic cubic decision matrix

	C_1	C_2
B_1	$<([0.059, 0.110], [0.941, 0.972, [0.720, 0.779]), (0.075, 0.87, 0.72)>$	$<([0.057, 0.090, [0.972, 0.986], [0.874, 0.91]), (0.073, 0.972, 0.832)>$
B_2	$<([0.043, 0.093], [0.941, 0.972], [0.941, 0.972]), (0.093, 0.957, 0.907)>$	$<([0.09, 0.168], [0.874, 0.972], [0.727, 0.832]), (0.057, 0.910, 0.785)>$
B_3	$<([0.076, 0.13], [0.87, 0.941], [0.87, 0.907]), (0.093, 0.907, 0.828)>$	$<([0.057, 0.090], [0.928, 0.972], [0.653, 0.757]), (0.126, 0.910, 0.727)>$

	C_3	C_4
B_1	$<([0.132, 0.17], [0.720, 0.830], [0.625, 0.782]), (0.084, 0.625, 0.625)>$	$<([0.124, 0.211], [0.552, 0.699], [0.552, 0.733]), (0.164, 0.660, 0.789)>$
B_2	$<([0.100, 0.132], [0.720, 0.807], [0.543, 0.720]), (0.084, 0.625, 0.625)>$	$<([0.056, 0.088], [0.66, 0.762], [0.552, 0.699]), (0.124, 0.552, 0.552)>$
B_3	$<([0.100, 0.218], [0.625, 0.782], [0.679, 0.754]), (0.132, 0.720, 0.720)>$	$<([0.164, 0.267], [0.552, 0.660], [0.660, 0.699]), (0.088, 0.522, 0.660)>$

Step 5. From Table 3, the NCPIS z_j^U, j = 1, 2, 3, 4 is obtained as follows:

z_1^U = < ([0.043, 0.093], [0.941, 0.972], [0.941, 0.972]), (0.075, 0.957, 0.907) >,

z_2^U = < ([0.057, 0.09], [0.972, 0.986], [0.874, 0.91]), (0.057, 0.972, 0.832) >,

z_3^U = < ([0.132, 0.218], [0.625, 0.782], [0.543, 0.72], (0.132, 0.625, 0.625)>,

z_4^U = < [0.164, 0.267], [0.552, 0.66], [0.552, 0.699], (0.66, 0.552, 0.552)>;

The NCNIS z_j^L, j = 1, 2, 3, 4 is determined from the weighted decision matrix (see Table 3) as follows:

z_1^L = < [0.076, 0.13], [0.87, 0.941], [0.72, 0.779], (0.093, 0.87, 0.72)>,

z_2^L = < [0.09, 0.168], [0.874, 0.972], [0.653, 0.757], (0.126, 0.91, 0.727)>,

z_3^L = < [0.1, 0.132], [0.72, 0.83], [0.679, 0.782], (0.084, 0.782, 0.83)>,

z_4^L = < [0.056, 0.088], [0.66, 0.762], [0.66, 0.733], (0.088, 0.66, 0.789)>.

Step 6. The Euclidean distance measure of each alternative from NCPIS is obtained as follows:

$$D_{E_1}^{+} = 0.1232,\ D_{E_2}^{+} = 0.1110,\ D_{E_3}^{+} = 0.1200.$$

Similarly, the Euclidean distance measure of each alternative from NCNIS is computed as follows:

$$D_{E_1}^{-} = 0.0705,\ D_{E_2}^{-} = 0.0954,\ D_{E_3}^{-} = 0.0736.$$

Step 7. The relative closeness co-efficient RCC_i^*, i = 1, 2, 3 is obtained as follows:

$$RCC_1^* = 0.3640,\ RCC_2^* = 0.4622,\ RCC_3^* = 0.3802.$$

Step 8. The ranking order of the feasible alternative according to the relative closeness co-efficient of the neutrosophic cubic ideal solution is presented as follows:

$$B_2 > B_3 > B_1$$

Therefore, B_2 i.e. a Basic English medium school is the best option for the legal guardian.

5 Conclusions

In the paper, we have presented a new extended TOPSIS method for solving MADM problems with neutrosophic cubic information. We have proposed several operational rules on neutrosophic cubic sets. We have defined Euclidean distance between two neutrosophic cubic sets. We have defined arithmetic average operator for neutrosophic cubic numbers. We have employed information entropy scheme to calculate unknown weights of the attributes. We have also defined neutrosophic cubic positive ideal solution and neutrosophic cubic negative ideal solution in the decision making process. Then, the most desirable alternative is selected based on the proposed extended TOPSIS method under neutrosophic cubic environment. Finally, we have solved a numerical example of MADM problem regarding school selection for a legal guardian to illustarte the proposed TOPSIS method. We hope that the proposed TOPSIS method will be effective in dealing with different MADM problems such as medical diagnosis, pattern recognition, weaver selection, supplier selection, etc in neutrosophic cubic set environment.

References

[1] F. Smarandache. A unifying field of logics. Neutrosophy: Neutrosophic probability, set and logic, American Research Press, Rehoboth, 1998.

[2] H. Wang, F. Smarandache, Y. Zhang, and R. Sunderraman. Single valued neutrosophic sets. Multi-space and Multi-structure, 4 (2010), 410-413.

[3] H. Wang, F. Smarandache, Y. Q. Zhang, R. Sunderraman. Interval neutrosophic sets and logic, Hexis, Arizona, 2005.

[4] P. Chi, P. Liu. An extended TOPSIS method for the multiple attribute decision making problems based on interval neutrosophic set. Neutrosophic Sets and Systems, 1 (2013), 63-70.

[5] C. L. Hwang, K. Yoon. Multiple attribute decision making: methods and applications, Springer, New York, 1981.

[6] R. Şahin, M. Yiğider. A multi-criteria neutrosophic group decision making method based TOPSIS for supplier selection. Applied Mathematics and Information

Sciences, submitted.
[7] Z. Zhang and C. Wu. A novel method for single-valued neutrosophic multi-criteria decision making with incomplete weight information. Neutrosophic Sets and Systems, 4 (2015), 35-49.
[8] S. Broumi, J. Ye, and F. Smarandache. An extended TOPSIS method for multiple attribute decision making based on interval neutrosophic uncertain linguistic variables. Neutrosophic Sets and Systems, 8(2015), 23-32.
[9] P. Biswas, S. Pramanik, and B.C. Giri. TOPSIS method for multi-attribute group decision making under single valued neutrosophic environment. Neural Computing and Applications, 27 (30 (2016), 727-737.
[10] M. Ali, I. Deli, and F. Smarandache. The theory of neutrosophic cubic sets and their applications in pattern recognition. Journal of Intelligent and Fuzzy Systems, 30 (4) (2016), 1957-1963.
[11] Y.B. Jun, F. Smarandache, C.S. Kim. Neutrosophic cubic sets. New Mathematics and Natural Computation, 13 (1) (2017), 41-54.
[12] S. Pramanik, S. Dalapati, S. Alam, T. K. Roy, F Smarandache. Neutrosophic cubic MCGDM method based on similarity measure. Neutrosophic Sets and Systems 16 (2017), 44-56.
[13] L.A. Zadeh, Fuzzy sets. Information and Control, 8 (1965), 338-353.
[14] I. B. Tursen, Interval valued fuzzy sets based on normal norms. Fuzzy Sets and Systems, 20 (2) (1986), 191-210.
[15] Y. B. Jun, C. S. Kim, and K. O. Yang. Cubic sets. Annals of Fuzzy Mathematics and Informatics, 4(3) (2012), 83-98.
[16] S. Pramanik, and K. Mondal. Interval neutrosophic multi-attribute decision-making based on grey relational analysis. Neutrosophic Sets and Systems, 9 (2015), 13-22.
[17] P. Majumder, and S.K. Samanta. On similarity and entropy of neutrosophic sets. Journal of Intelligent and Fuzzy Systems, 26 (2014), 1245–1252.
[18] C.L. Hwang, K. Yoon. Multiple attribute decision making: methods and applications: a state-of-the-art survey, Springer, London, 1981.
[19] J.Q. Wang, and Z.H Zhang. Multi-criteria decision making method with incomplete certain information based on intuitionistic fuzzy number. Control and Decision, 24 (2009), 226-230.
[20] K. Mondal, and S. Pramanik. Neutrosophic decision making model of school choice. Neutrosophic Sets and Systems, 7(2015), 62-68.

Received: July 13, 2017. August 1, 2017.

NSS

University of New Mexico

Multi criteria decision making using correlation coefficient under rough neutrosophic environment

Surapati Pramanik[1], Rumi Roy[2], Tapan Kumar Roy[3] and Florentin Smarandache[4]

[1]Department of Mathematics, Nandalal Ghosh B.T. College, Panpur, PO-Narayanpur, and District: North 24 Parganas, Pin Code: 743126, West Bengal, India. Email: sura_pati@yahoo.co.in,

[2]Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah-711103, West Bengal, India. Email: roy.rumi.r@gmail.com

[3]Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah-711103, West Bengal, India. Email: roy_t_k@gmail.com

[4]University of New Mexico. Mathematics & Science Department, 705 Gurley Ave., Gallup, NM 87301, USA. Email: fsmarandache@gmail.com

Abstract In this paper, we define correlation coefficient measure between any two rough neutrosophic sets. We also prove some of its basic properties.. We develop a new multiple attribute group decision making method based on the proposed correlation coefficient measure. An illustrative example of medical diagnosis is solved to demonstrate the applicability and effecriveness of the proposed method.

Keywords: Multi-attribute group decision making; Neutrosophic set; Rough set; Rough neutrosophic set; Correlation coefficient.

1 Introduction

Smarandache established the concept of neutrosophic set and neutrosophic logic [1] to deal uncertainty, inconsistency, incompleteness and indeterminacy in 1998. Smarandache [1] and Wang et. al. [2] studied single valued neutrosophic set (SVNS), a subclass of neutrosophic set to deal realistic problems in 2010. SVNSs have been widely studied and applied in different fields such as medical diagnosis [3], multi criteria decision making [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17], image processing [18, 19, 20], etc.

Pawlak [21] defined rough set to study intelligence systems characterized by inexact, uncertain or insufficient information. Broumi et al. [22, 23] defined rough neutrosophic set by combining the rough set and single valued neutrosophic set to deal with problems involving uncertain, imprecise, incomplete and inconsistent information existing in real world problems.

Decision making in rough neutrosophic environment is a new subfield of operational resesarch. In rough neutrosophic environment, Mondal and Pramanik [24] defined accumulated geometric operator to transform rough neutrosophic number (neutrosophic pair) to single valued neutrosophic number and developed a new multi-attribute decision-making (MADM) method based on grey relational analysis. Mondal and Pramanik [25] defined accuracy score function and proved its basic properties. In the same study, Mondal and Pramanik [25] presented a new MADM method in rough neutrosophic environment. Pramanik and Mondal [26] defined cotangent similarity measure of rough neutrosophic sets and proved its basic properties. In the same study, Pramanik and Mondal [26] presented its application to medical diagnosis. Pramanik and Mondal [27] proposed cosine similarity measure of rough neutrosophic sets and its application in medical diagnosis. Pramanik and Mondal [28] also proposed Dice and Jaccard similarity measures in rough neutrosophic environment and applied them for MADM. Mondal and Pramanik [29] studied cosine, Dice and Jaccard similarity measures for interval rough neutrosophic sets and presented MADM methods based on proposed rough cosine, Dice and Jaccard similarity measures in interval rough neutrosophic environment Mondal et al. [30] presented rough trigonometric Hamming similarity measures such as cosine, sine and cotangent rough similarity measures and proved their basic properties. In the same study, Mondal et al. [30] presented new MADM methods based on cosine, sine and cotangent rough similarity measures with illustrative example. Mondal et al. [31] proposed variational coefficient similarity measures under rough neutrosophic environment and proved some of their basic properties. In the same study, Mondal et al. [31] developed a new MADM method based on the proposed variational coefficient similarity measures and presented a comparison with four existing rough similarity measures namely, rough cosine similarity measure, rough dice similarity measure, rough cotangent similarity measure and rough Jaccard similarity measure for different values of the parameter λ . Mondal et al. [32] proposed rough neutrosophic aggregate operator and weighted rough neutrosophic aggregate operator to develop TOPSIS based MADM method in rough neutrosophic environment. Pramanik et al. [33] defined projection and bidirectional projection measures between rough neutrosophic sets. In the same study, Pramanik et al. [33] proposed two new multi criteria decision making (MCDM) methods based on neutrosophic projection and bidirectional projection measures respectively.

Mondal and Pramanik [34] proposed rough tri-complex similarity measure based MADM method in rough neutrosophic environment and proved some of its basic properties. In the same study, Mondal and Pramanik [34] presented comparison of obtained results for an illustrative MADM problem with other existing rough neutrosophic similarity measures.

Mondal et al. [35] defined rough neutrosophic hyper-complex set and rough neutrosophic hyper-complex cosine function and proved some of their basic properties. In the same study, Mondal et al. [35] also proposed rough neutrosophic hyper-complex similarity measure based MADM method.

Pramanik and Mondal [36] defined bipolar rough neutrosophic sets and proved it basic properties.

The correlation coefficient is an important tool to judge the relation between two objects. The correlation coefficients [37, 38, 39, 40, 41, 42] have been widely employed to data analysis and classification, decision making, pattern recognition, and so on. Many researchers pay attention to correlation coefficients under fuzzy environments. Chiang and Lin [43] introduced the correlation of fuzzy sets. Hong [44] proposed fuzzy measures for a correlation coefficient of fuzzy numbers under Tw (the weakest t-norm)-based fuzzy arithmetic operations. As an extension of fuzzy correlations, Wang and Li [45] introduced the correlation and information energy of interval-valued fuzzy numbers. Gerstenkorn and Manko [46] developed the correlation coefficients of intuitionistic fuzzy sets IFSs). Hung and Wu [47] also proposed a method to calculate the correlation coefficients of IFSs by centroid method. Xu [48] developed another correlation measure of interval-valued intuitionistic fuzzy environment, and applied it to medical diagnosis. Ye [49] studied the fuzzy decision-making method based on the weighted correlation coefficient under intuitionistic fuzzy environment. Bustince and Burillo [50] and Hong [51] further developed the correlation coefficients for interval-valued intuitionistic fuzzy sets (IVIFSs). Hanafy et al. [52] introduced the correlation of neutrosophic data. Ye [53] presented the correlation coefficient of SVNSs based on the extension of the correlation coefficient of IFSs and proved that the cosine similarity measure of SVNSs is a special case of the correlation coefficient of SVNSs. Hanafy et al. [54] presented the centroid-based correlation coefficient of neutrosophic sets and investigated its properties. Broumi and Smarandache [55] defined correlation coefficient of interval neutrosophic set and investigated its properties.

In the literature no studies have been reported on MADM using correlation coefficient under rough neutrosophic environment. To fill the research gap, we propose correlation coefficient under rough neutrosophic environment and proved some of its basic properties. We also present a new MADM method based on proposed measure. We also present an illustrative numerical example to show the effectiveness and applicability of the proposed method.

Rest of the paper is organized as follows: Section 2 describes preliminaries of neutrosophic sets, SVNSs and rough neutrosophic set (RNS). Section 3 describes the correlation coefficient between SVNSs. Section 4 presents definition and properties of proposed correlation coefficient between RNSs. Section 5 presents a rough neutrosophic decision making method based on correlation coefficient. Section 6 presents an illustrative hypothetical medical diagnostic problem based on the proposed MADM method. Finally, section 7 presents concluding remarks and future scope of research.

2 Preliminaries

2.1 Neutrosophic sets In 1998, Smarandache offered the following definition of neutrosophic set(NS)[1].

Definition 2.1.1 [1]

Let X be a space of points(objects) with generic element in X denoted by x. A NS A in X is characterized by a truth-membership function T_A, an indeterminacy membership function I_A and a falsity membership function F_A. The functions T_A, I_A and F_A are real standard or non-standard subsets of $]0^-,1^+[$ that is $T_A : X \to]0^-,1^+[, I_A : X \to]0^-,1^+[$ and $F_A : X \to]0^-,1^+[$. It should be noted that there is no restriction on the sum of T_A, I_A and F_A i.e $0^- \le T_A + I_A + F_A \le 3^+$.

Definition 2.1.2 [1]

(Complement) The complement of a neutrosophic set A is denoted by C(A) and is defined by $T_{c(A)}(x)=\{1^+\}-T_A(x)$, $I_{c(A)}(x)=\{1^+\}-I_A(x)$, $F_{c(A)}(x)=\{1^+\}-F_A(x)$.

Definition 2.1.3 [1]

A neutrosophic set A is contained in another neutrosophic set B, denoted by $A \subseteq B$ iff inf $T_A(x) \le$ inf $T_B(x)$, sup $T_A(x) \le$ sup $T_B(x)$, inf $I_A(x) \ge$ inf $I_B(x)$, sup $I_A(x) \ge$ inf $I_B(x)$, inf $F_A(x) \ge$ inf $F_B(x)$ and sup $F_A(x) \ge$ sup $F_B(x)$ for all x in X.

Definition 2.1.4 [2]

Let X be a universal space of points (objects) with a generic element of X denoted by x. A single valued neutrosophic set A is characterized by a truth membership function $T_A(x)$, a falsity membership function $F_A(x)$ and indeterminacy function $I_A(x)$ with $T_A(x)$, $I_A(x)$ and $F_A(x) \in [0,1]$ for all x in X.

When X is continuous, a SNVS A can be written as follows: A= $\int_x < T_A(x), I_A(x), F_A(x) > / x$ for all $x \in X$ and when X is discrete, a SVNS A can be written as follows : A= $\Sigma < T_A(x), I_A(x), F_A(x) > / x$ for all $x \in X$.

For a SVNS S, $0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3$.

Definition 2.1.5 [2]

The complement of a single valued neutrosophic set A is denoted by c(A) and is defined by $T_{c(A)}(x) = F_A(x)$, $I_{c(A)}(x) = 1 - I_A(x)$, $F_{c(A)}(x) = T_A(x)$.

Definition 2.1.6 [2]

A SVNS A is contained in the other SVNS B, denoted as A $\subseteq$ B iff, $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$, $F_A(x) \geq F_B(x)$ for all x in X.

2.2 Rough Neutrosophic sets

Rough neutrosophic sets [22, 23] are the generalization of rough fuzzy sets [56, 57, 58] and rough intuitionistic fuzzy sets [59].

Definition 2.2.1 [22]

Let Y be a non-null set and R be an equivalence relation on Y. Let P be a neutrosophic set in Y with the membership function T_P, indeterminacy function I_P and non-membership function F_P. The lower and the upper approximations of P in the approximation space (Y, R) are respectively defined as:

$$\underline{N(P)} = << x, T_{\underline{N(P)}}(x), I_{\underline{N(P)}}(x), F_{\underline{N(P)}}(x) > / y \in [x]_R, x \in Y >$$

and

$$\overline{N(P)} = << x, T_{\overline{N(P)}}(x), I_{\overline{N(P)}}(x), F_{\overline{N(P)}}(x) > / y \in [x]_R, x \in Y >$$

where,

$$T_{\underline{N(P)}}(x) = \wedge z \in [x]_R T_P(Y), I_{\underline{N(P)}}(x) = \wedge z \in [x]_R I_P(Y), F_{\underline{N(P)}}(x) = \wedge z \in [x]_R F_P(Y)$$

and

$$T_{\overline{N(P)}}(x) = \vee z \in [x]_R T_P(Y), I_{\overline{N(P)}}(x) = \vee z \in [x]_R I_P(Y), F_{\overline{N(P)}}(x) = \vee z \in [x]_R F_P(Y).$$

So,

$0 \leq T_{\underline{N(P)}}(x) + I_{\underline{N(P)}}(x) + F_{\underline{N(P)}}(x) \leq 3$ and $0 \leq T_{\overline{N(P)}}(x) + I_{\overline{N(P)}}(x) + F_{\overline{N(P)}}(x) \leq 3$.

Here $\vee$ and $\wedge$ denote "max" and "min" operators respectively, $T_P(y)$, $I_P(y)$, and $F_P(y)$ are the degrees of membership, indeterminacy and non-membership of Y with respect to P.

Thus NS mapping, $\underline{N}, \overline{N} : N(Y) \to N(Y)$ are, respectively, referred to as the lower and upper rough NS approximation operators, and the pair $(\underline{N(P)}, \overline{N(P)})$ is called the rough neutrosophic set in (Y, R).

Definition 2.2.2 [22]

If $N(P) = (\underline{N}(P), \overline{N}(P))$ is a rough neutrosophic set in (Y, R), the rough complement of N(P) is the rough neutrosophic set denoted by $\sim (N(P)) = ((\underline{N}(P))^c, (\overline{N}(P))^c)$, where $(\underline{N}(P))^c$ and $(\overline{N}(P))^c$ are the complements of neutrosophic sets $\underline{N}(P)$ and $\overline{N}(P)$ respectively.

3 Correlation coefficient of SVNSs

Based on the correlation of intuitionistic fuzzy sets, Ye [53] defined the informational energy of a SVNS A, the correlation of two SVNSs A and B, and the correlation coefficient of two SVNSs A and B.

Definition 3.1 [53]

For a SVNS A in the universe of discourse X = {$x_1, x_2, \ldots, x_n$}, the informational energy of the SVNS A is defined by

$$I(A) = \sum_{i=1}^{n} [T_A^2(x_i) + I_A^2(x_i) + F_A^2(x_i)]$$

Definition 3.2 [53]

For two SVNSs A and B in the universe of discourse X = {$x_1, x_2, \ldots, x_n$}, correlation of the SVNSs A and B is defined as

$$C(A,B) = \sum_{i=1}^{n} [T_A(x_i) T_B(x_i) + I_A(x_i) I_B(x_i) + F_A(x_i) F_B(x_i)]$$

Definition 3. 3 [53]

The correlation coefficient of the SVNSs A and B is defined by the following formula: (1)

$$K(A, B) = \frac{C(A, B)}{[C(A, A).C(B, B)]^{1/2}} = \frac{\sum_{i=1}^{n} [T_A(x_i) T_B(x_i) + I_A(x_i) I_B(x_i) + F_A(x_i) F_B(x_i)]}{[\sum_{i=1}^{n} [(T_A(x_i))^2 + (I_A(x_i))^2 + (F_A(x_i))^2]]^{\frac{1}{2}} [\sum_{i=1}^{n} [(T_B(x_i))^2 + (I_B(x_i))^2 + (F_B(x_i))^2]]^{\frac{1}{2}}}$$

The correlation coefficient K(A, B) satisfies the following properties :

(1) K(A, B) = K(B, A);

(2) $0 \leq K(A, B) \leq 1$;

(3) K(A, B) = 1, if A = B.

4 Correlation coefficient of rough neutrosophic sets

Correlation coefficient between rough neutrosophic sets (RNSs) is yet to define in the literature. Therefore in this paper, we define correlation coefficient between RNSs.

Definition4.1. Assume that there are any two RNSs $A = <(\underline{T_A(x_i)}, \underline{I_A(x_i)}, \underline{F_A(x_i)}), (\overline{T_A(x_i)}, \overline{I_A(x_i)}, \overline{F_A(x_i)}), >$ and $B = <(\underline{T_B(x_i)}, \underline{I_B(x_i)}, \underline{F_B(x_i)}), (\overline{T_B(x_i)}, \overline{I_B(x_i)}, \overline{F_B(x_i)})>$. Then the correlation between the RNSs A and B is defined as

$$C(A,B) = \sum_{i=1}^{n} [\delta T_A(x_i).\delta T_B(x_i) + \delta I_A(x_i).\delta I_B(x_i) + \delta F_A(x_i).\delta F_B(x_i)]$$

where

$$\delta T_A(x_i) = \frac{\underline{T}_A(x_i) + \overline{T}_A(x_i)}{2},$$

$$\delta I_A(x_i) = \frac{\underline{I}_A(x_i) + \overline{I}_A(x_i)}{2},$$

$$\delta F_A(x_i) = \frac{\underline{F}_A(x_i) + \overline{F}_A(x_i)}{2},$$

$$\delta T_B(x_i) = \frac{\underline{T}_B(x_i) + \overline{T}_B(x_i)}{2},$$

$$\delta I_B(x_i) = \frac{\underline{I}_B(x_i) + \overline{I}_B(x_i)}{2} \quad \text{and}$$

$$\delta F_B(x_i) = \frac{\underline{F}_B(x_i) + \overline{F}_B(x_i)}{2}.$$

Definition 4.2.The correlation coefficient of the RNSs A and B is defined as

$$K(A,B) = \frac{C(A,B)}{[C(A,A).C(B,B)]^{1/2}}$$

$$= \frac{\sum_{i=1}^{n}[\delta T_A(x_i).\delta T_B(x_i) + \delta I_A(x_i).\delta I_B(x_i) + \delta F_A(x_i).\delta F_B(x_i)]}{(\sum_{i=1}^{n}[(\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2]^{\frac{1}{2}})(\sum_{i=1}^{n}[(\delta T_B(x_i))^2 + (\delta I_B(x_i))^2 + (\delta F_B(x_i))^2]^{\frac{1}{2}})} \quad ...(2)$$

The correlation coefficient K(A, B) satisfies the following properties :

(1) K(A, B) = K(B, A);

(2) $0 \le K(A,B) \le 1$;

(3) K(A, B) = 1, if A = B.

Proof

(i)

$$K(A,B) = \frac{C(A,B)}{[C(A,A).C(B,B)]^{1/2}}$$

$$= \frac{C(B,A)}{[C(B,B).C(A,A)]^{1/2}} = K(B,A)$$

(ii) As C(A, B) ≥ 0, C(A, A) ≥ 0, C(B, B) ≥ 0 so K(A, B) ≥ 0.

According to the Cauchy–Schwarz inequality:

$$(a_1b_1 + ...+ a_nb_n)^2 \le (a_1^2 + ...+ a_n^2)(b_1^2 + ...+ b_n^2)$$

where a_i , $b_i \in R$ for i=1,......,n,

So $\dfrac{(a_1b_1 + ...+ a_nb_n)}{(a_1^2 + ...+ a_n^2)^{\frac{1}{2}}(b_1^2 + ...+ b_n^2)^{\frac{1}{2}}} \le 1$

Replacing a_i by $\delta T_A(x_i)$ and b_i by $\delta T_B(x_i)$ we obtain

K(A, B) ≤1.

Therefore, 0≤ K(A, B) ≤ 1.

(iii) If A = B

then K(A,B) = K(A,A) = $\dfrac{C(A,A)}{[C(A,A).C(A,A)]^{1/2}}$

$= \dfrac{C(A,A)}{C(A,A)} = 1$

Hence proved.

Considering n = 1, we get the following: (3)

K(A,B)=

$$\frac{\delta T_A(x_i).\delta T_B(x_i) + \delta I_A(x_i).\delta I_B(x_i) + \delta F_A(x_i).\delta F_B(x_i)}{((\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2)^{\frac{1}{2}}((\delta T_B(x_i))^2 + (\delta I_B(x_i))^2 + (\delta F_B(x_i))^2)^{\frac{1}{2}}}$$

Which is the cosine similarity measure between two RNSs A and B [27].

Weighted correlation coefficient:

Let w = {w_1, w_2, …, w_n} be the weight vector of the elements x_i (i = 1, 2, …, n).

Then the weighted correlation coefficient between A and B is defined by the following formula:

$$K_W(A,B) = \frac{\sum_{i=1}^{n} w_i[\delta T_A(x_i).\delta T_B(x_i) + \delta I_A(x_i).\delta I_B(x_i) + \delta F_A(x_i).\delta F_B(x_i)]}{[(\sum_{i=1}^{n}\{w_i[(\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2]\}^{\frac{1}{2}})(\sum_{i=1}^{n}\{w_i[(\delta T_B(x_i))^2 + (\delta I_B(x_i))^2 + (\delta F_B(x_i))^2]\}^{\frac{1}{2}})]} \quad (4)$$

If w = {1/n, 1/n, …, 1/n}, then equation (4) reduces to equation (2).

Weighted correlation coefficient K_w(A, B) also satisfies the following properties:

(1) K_w(A, B) = K_w(B, A);

(2) $0 \le K_w(A,B) \le 1$;

(3) K_w(A, B) = 1, if A = B.

Proof

(i)

$$K_W(A,B) = \frac{\sum_{i=1}^{n} w_i[\delta T_A(x_i).\delta T_B(x_i) + \delta I_A(x_i).\delta I_B(x_i) + \delta F_A(x_i).\delta F_B(x_i)]}{[(\sum_{i=1}^{n}\{w_i[(\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2]\}^{\frac{1}{2}})(\sum_{i=1}^{n}\{w_i[(\delta T_B(x_i))^2 + (\delta I_B(x_i))^2 + (\delta F_B(x_i))^2]\}^{\frac{1}{2}})]}$$

$$= \frac{\sum_{i=1}^{n} w_i[\delta T_B(x_i).\delta T_A(x_i) + \delta I_B(x_i).\delta I_A(x_i) + \delta F_B(x_i).\delta F_A(x_i)]}{[(\sum_{i=1}^{n}\{w_i[(\delta T_B(x_i))^2 + (\delta I_B(x_i))^2 + (\delta F_B(x_i))^2]\}^{\frac{1}{2}})(\sum_{i=1}^{n}\{w_i[(\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2]\}^{\frac{1}{2}})]}$$

$$= K_w(B,A)$$

(ii) As $\sum_{i=1}^{n} w_i[\delta T_A(x_i).\delta T_B(x_i) + \delta I_A(x_i).\delta I_B(x_i) + \delta F_A(x_i).\delta F_B(x_i)] \ge 0,$

$\sum_{i=1}^{n}\{w_i[(\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2]\}^{\frac{1}{2}} \ge 0$

and $\sum_{i=1}^{n}\{w_i[(\delta T_B(x_i))^2 + (\delta I_B(x_i))^2 + (\delta F_B(x_i))^2]\}^{\frac{1}{2}} \ge 0$

so K_W(A,B) ≥ 0.

Using the weighted Cauchy–Schwarz inequality [60], we have

$$(w_1a_1b_1 + ...+ w_na_nb_n)^2 \le (w_1a_1^2 + ...+ w_1a_n^2)(w_1b_1^2 + ...+ w_nb_n^2)$$

where w_i , a_i , $b_i \in R$ for i =1, ... , n.

So $\dfrac{(w_1a_1b_1 + ...+ w_na_nb_n)}{(w_1a_1^2 + ...+ w_na_n^2)^{\frac{1}{2}}(w_1b_1^2 + ...+ w_nb_n^2)^{\frac{1}{2}}} \le 1$

Replacing a_i by $w_i\delta T_A(x_i)$ and b_i by $w_i\delta T_B(x_i)$ we obtain

K_W(A, B) ≤ 1.

Therefore, 0 ≤ K(A, B) ≤ 1.

(iii) If A = B, then

K(A,B) = K(A,A)

$$= \frac{\sum_{i=1}^{n} w_i[\delta T_A(x_i).\delta T_A(x_i) + \delta I_A(x_i).\delta I_A(x_i) + \delta F_A(x_i).\delta F_A(x_i)]}{[(\sum_{i=1}^{n}\{w_i[(\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2]\}^{\frac{1}{2}})(\sum_{i=1}^{n}\{w_i[(\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2]\}^{\frac{1}{2}})]}$$

$$= \frac{\sum_{i=1}^{n}\{w_i[(\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2]\}}{\sum_{i=1}^{n}\{w_i[(\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2]\}} = 1$$

Hence proved.

5 Rough neutrosophic decision making based on correlation coefficient

Let $A_1, A_2, \ldots, A_m$ be a set of elements (/objects / persons), $C_1, C_2, \ldots, C_n$ be a set of criteria for each element and $E_1, E_2, \ldots, E_k$ are the alternatives for each element.

Step 1. The relation between elements A_i (i = 1, 2, ... , m) and the criteria C_j (j = 1, 2, ... ,n) is presented in Table 1 in terms of RNSs.

Table1 : Relation between elements and criteria

$$\begin{array}{c} \\ A_1 \\ A_2 \\ \ldots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \ldots & C_n \end{array} \\ \begin{bmatrix} X_{11} & X_{12} & \ldots & X_{1n} \\ X_{21} & X_{22} & \ldots & X_{2n} \\ \ldots & \ldots & \ldots. & \ldots \\ X_{m1} & X_{m2} & \ldots & X_{mn} \end{bmatrix} \end{array}$$

where

$$X_{ij} = <(\underline{T}_{ij}, \underline{I}_{ij}, \underline{F}_{ij}), (\overline{T}_{ij}, \overline{I}_{ij}, \overline{F}_{ij})>$$

with $0 \le \underline{T}_{ij} + \underline{I}_{ij} + \underline{F}_{ij} \le 3$ and $0 \le \overline{T}_{ij} + \overline{I}_{ij} + \overline{F}_{ij} \le 3$.

The relation between criterion C_i (i = 1, 2, ... , n) and the alternative E_j (j = 1, 2, ... ,k) is presented in Table 2 in terms of RNSs.

Table 2 : Relation between criteria and alternatives

$$\begin{array}{c} \\ C_1 \\ C_2 \\ \ldots \\ C_m \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \ldots & E_k \end{array} \\ \begin{bmatrix} Y_{11} & Y_{12} & \ldots & Y_{1k} \\ Y_{21} & Y_{22} & \ldots & Y_{2k} \\ \ldots & \ldots & \ldots. & \ldots \\ Y_{n1} & Y_{n2} & \ldots & Y_{nk} \end{bmatrix} \end{array}$$

where

$$Y_{ij} = <(\underline{T}_{ij}, \underline{I}_{ij}, \underline{F}_{ij}), (\overline{T}_{ij}, \overline{I}_{ij}, \overline{F}_{ij})>$$

with

$0 \le \underline{T}_{ij} + \underline{I}_{ij} + \underline{F}_{ij} \le 3$ and $0 \le \overline{T}_{ij} + \overline{I}_{ij} + \overline{F}_{ij} \le 3$.

Step 2. Determine the correlation measure between Table 1 and Table 2 using equation 2. The obtained values are presented in Table 3.

Table 3 : Correlation coefficient between table1 and table2

$$\begin{array}{c} \\ A_1 \\ A_2 \\ \ldots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \ldots & E_k \end{array} \\ \begin{bmatrix} p_{11} & p_{12} & \ldots & p_{1k} \\ p_{21} & p_{22} & \ldots & p_{2k} \\ \ldots & \ldots & \ldots. & \ldots \\ p_{m1} & p_{m2} & \ldots & p_{mk} \end{bmatrix} \end{array}$$

Step 3. From Table 3, for each element A_i (i = 1, 2, ..., m), find the maximum correlation value of the i-th row (i = 1, 2, ..., m). If the maximum value occurs at j-th column (j = 1, 2, ..., k) (see Table 3), then E_j will be the best alternative for the element A_i (i =1, 2, ..., m).

Step 4. End.

6 Medical Diagnosis Problem

We consider a medical diagnosis problem for illustration of the proposed method. Medical diagnosis comprises of inconsistent, indeterminate and incomplete information though increased volume of information available to doctors from new medical technologies. The proposed correlation coefficients among the patients versus symptoms and symptoms versus diseases will provide medical diagnosis. Let P = {P_1, P_2, P_3} be a set of patients, D = {Viral fever, Malaria, Stomach problem, Chest problem} be a set of diseases and S = {Temperature, Headache, Stomach pain, Cough, Chest pain} be a set of symptoms. Using proposed method the doctor is to examine the patient and to determine the disease of the patient in rough neutrosophic environment.

Based on the proposed approach the considered problem is solved using the following steps:

Step 1. Construction of the rough neutrosophic decision matrix

Table 4: (Relation-1) The relation between Patients and Symptoms

	Temperature	Headache	Stomach pain	cough	Chest pain
P_1	<(.6,.4,.3), (.8,.2,.1)>	<(.4,.4,.4), (.6,.2,.2)>	<(.5,.3,.2), (.7,.1,.2)>	<(.6,.2,.4), (.8,.0,.2)>	<(.4,.4,.4), (.6,.2,.2)>
P_2	<(.5,.3,.4), (.7,.3,.2)>	<(.5,.3,.3), (.7,.3,.3)>	<(.5,.3,.4), (.7,.1,.4)>	<(.5,.3,.3), (.9,.1,.3)>	<(.5,.3,.3), (.7,.1,.3)>
P_3	<(.6,.4,.4), (.8,.2,.2)>	<(.5,.2,.3), (.7,.0,.1)>	<(.4,.3,.4), (.8,.1,.2)>	<(.6,.1,.4), (.8,.1,.2)>	<(.5,.3,.3), (.7,.1,.1)>

Table 5: (Relation-2) The relation among Symptoms and Diseases

	Viral Fever	Malaria	Stomach problem	Chest problem
Temperature	<(.6,.5,.4), (.8,.3,.2)>	<(.1,.4,.4), (.5,.2,.2)>	<(.3,.4,.4), (.5,.2,.2)>	<(.2,.4,.6), (.4,.4,.4)>
Headache	<(.5,.3,.4), (.7,.3,.2)>	<(.2,.3,.4), (.6,.3,.2)>	<(.2,.3,.3), (.4,.1,.1)>	<(.1,.5,.5), (.5,.3,.3)>
Stomach pain	<(.2,.3,.4), (.4,.3,.2)>	<(.1,.4,.4), (.3,.2,.2)>	<(.4,.3,.4), (.6,.1,.2)>	<(.1,.4,.6), (.3,.2,.4)>
cough	<(.4,.3,.3), (.6,.1,.1)>	<(.3,.3,.3), (.5,.1,.3)>	<(.1,.6,.6), (.3,.4,.4)>	<(.5,.3,.4), (.7,.1,.2)>

Chest pain	<(.2,.4,.4), (.4,.2,.2)>	<(.1,.3,.3), (.3,.1,.1)>	<(.1,.4,.4), (.3,.2,.2)>	<(.4,.4,.4), (.6,.2,.3)>

Step 2. Determination of correlation coefficient between table 1 and table 2

Table 6: The correlation measure between Relation-1 and Relation-2

	Viral Fever	Malaria	Stomach problem	Chest problem
P_1	**0.95135**	0.91141	0.84518	0.87465
P_2	**0.95033**	0.94374	0.86228	0.91731
P_3	**0.93473**	0.89549	0.82559	0.85937

Step 3. Ranking the alternatives

According to the values of correlation coefficient of each alternative shown in Table 3, the highest correlation measure occurs in column1(i.e. for the diseases viral fever. Therefore, all three patients P_1, P_2, P_3 suffer from viral fever.

7 Conclusion

In this paper, we have proposed correlation coefficient and weighted correlation coefficient between rough neutrosophic sets and proved some of their basic properties. We have developed a new multi criteria decision making method based on the correlation coefficient measure. We presented an illustrative example in medical diagnosis. We hope that the proposed method can be applied in solving realistic multi criteria group decision making problems in rough neutrosophic environment.

References

[1] F. Smarandache. A unifying field in logics: neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability, and neutrosophic statistics, Rehoboth: American Research Press. (1998).

[2] H. Wang, F. Smarandache, F. Q. Zhang, and R. Sunderraman. Single valued neutrosophic sets. Multispace and Multistructure, 4 (2010), 410–413.

[3] J. Ye. Improved cosine similarity measures of simplified neutrosophic sets for medical diagnoses. Artificial Intelligence in Medicine, 63 (2015), 171–179.

[4] J. Ye. Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment. International Journal of General Systems, 42 (2013), 386–394.

[5] J. Ye. Trapezoidal neutrosophic set and its application to multiple attribute decision-making. Neural Computing and Applications, 26 (2015), 1157–1166.

[6] P. Biswas, S. Pramanik, and B. C. Giri. Entropy based grey relational analysis method for multi-attribute decision making under single valued neutrosophic assessments. Neutrosophic Sets and Systems. 2 (2014), 102–110.

[7] P. Biswas, S. Pramanik, and B. C. Giri. Cosine similarity measure based multi-attribute decision-making with trapezoidal fuzzy neutrosophic numbers. Neutrosophic Sets and Systems, 8 (2015), 47-57.

[8] P. Biswas, S. Pramanik, and B. C. Giri. TOPSIS method for multi-attribute group decision-making under single valued neutrosophic environment. Neural Computing and Applications, 27(3) (2016), 727-737. doi: 10.1007/s00521-015-1891-2.

[9] P. Biswas, S. Pramanik, and B. C. Giri. Aggregation of triangular fuzzy neutrosophic set information and its application to multi-attribute decision making. Neutrosophic Sets and Systems. 12 (2016), 20-40.

[10] P. Biswas, S. Pramanik, and B. C. Giri. Value and ambiguity index based ranking method of single-valued trapezoidal neutrosophic numbers and its application to multi-attribute decision making. Neutrosophic Sets and Systems, 12 (2016), 127-138.

[11] P. Biswas, S. Pramanik, and B. C. Giri. Non-linear programming approach for single-valued neutrosophic TOPSIS method. New Mathematics and Natural Computation, In Press.

[12] A. Kharal. A neutrosophic multi-criteria decision making method. New Mathematics and Natural Computing, 10 (2014), 143– 162. doi: 10.1142/S1793005714500070.

[13] P. D. Liu, and H. G. Li. Multiple attribute decision-making method based on some normal neutrosophic Bonferroni mean operators. Neural Computing and Applications, 28 (2017), 179–194.

[14] R. Sahin, and P. Liu. Maximizing deviation method for neutrosophic multiple attribute decision making with incomplete weight information. Neural Computing and Applications, (2015). doi: 10.1007/s00521-015-1995-8.

[15] S. Pramanik, P. Biswas, and B. C. Giri. Hybrid vector similarity measures and their applications to multi-attribute decision making under neutrosophic environment. Neural Computing and Applications, 28(5) (2017),1163-1176.

[16] K. Mondal, and S. Pramanik. Neutrosophic decision making model of school choice. Neutrosophic Sets and Systems, 7 (2015), 62-68.

[17] K. Mondal, and S. Pramanik. Neutrosophic tangent similarity measure and its application to multiple attribute decision making. Neutrosophic sets and systems, 9 (2015), 80-87.

[18] Y. Guo, and H. D. Cheng. New neutrosophic approach to image segmentation. Pattern Recognition, 42 (2009), 587–595.

[19] Y. Guo, A. Sengur, and J. Ye. A novel image thresholding algorithm based on neutrosophic similarity score. Measurement, 58 (2014), 175–186.

[20] H. D. Cheng, and Y. Guo. A new neutrosophic approach to image thresholding. New Mathematics and Natural Computation, 4 (2008), 291–308.

[21] Z. Pawlak. Rough sets. International Journal of Information and Computer Sciences, 11(5) (1982), 341-356.

[22] S. Broumi, F. Smarandache, and M. Dhar. Rough neutrosophic sets. Italian journal of pure and applied mathematics, 32, (2014), 493−502.

[23] S. Broumi, F. Smarandache, and M. Dhar. Rough neutrosophic sets. Neutrosophic Sets and Systems, 3 (2014), 60-66.

[24] K. Mondal, and S. Pramanik. Rough neutrosophic multi-attribute decision-making based on grey relational analysis. Neutrosophic Sets and Systems, 7 (2014),8-17.

[25] K. Mondal, and S. Pramanik. Rough neutrosophic multi-attribute decision-making based on rough accuracy score function. Neutrosophic Sets and Systems, 8 (2015), 16-22.

[26] S. Pramanik, and K. Mondal. Cotangent similarity measure of rough neutrosophic sets and its application to medical diagnosis. Journal of New Theory, 4 (2015), 90-102.

[27] S. Pramanik, and K. Mondal. Cosine similarity measure of rough neutrosophic sets and its application in medical diagnosis. Global Journal of Advanced Research, 2(1) (2015), 212-220.

[28] S. Pramanik, and K. Mondal. Some rough neutrosophic similarity measure and their application to multi attribute decision making. Global Journal of Engineering Science and Research Management, 2(7) (2015), 61-74.

[29] K. Mondal, and S. Pramanik. Decision making based on some similarity measures under interval rough neutrosophic environment. Neutrosophic Sets and Systems, 10 (2015), 46-57.

[30] K. Mondal, S. Pramanik, and F. Smarandache. (2016). Several trigonometric Hamming similarity measures of rough neutrosophic sets and their applications in decision making. In F. Smarandache, & S. Pramanik (Eds.), New trends in neutrosophic theory and applications. (pp. 93-103). Brussels: Pons Editions.

[31] K. Mondal, S. Pramanik, and F. Smarandache. Multi-attribute decision making based on rough neutrosophic variational coefficient similarity measure. Neutrosophic Sets and Systems, 13(2016),3-17.

[32] K. Mondal, S. Pramanik, and F. Smarandache. Rough neutrosophic TOPSIS for multi-attribute group decision making. Neutrosophic Sets and Systems, 13(2016), 105-117.

[33] S. Pramanik, R. Roy, and T. K. Roy. (2017). Multi criteria decision making based on projection and bidirectional projection measures of rough neutrosophic sets. In F. Smarandache, & S. Pramanik (Eds.), New Trends in Neutrosophic Theory and Applications, Vol. II. Brussels: Pons Editions. In Press.

[34] K. Mondal, and S. Pramanik. Tri-complex rough neutrosophic similarity measure and its application in multi-attribute decision making. Critical Review, 11(2015g), 26-40.

[35] K. Mondal, S. Pramanik, and F. Smarandache. Rough neutrosophic hyper-complex set and its application to multi-attribute decision making. Critical Review, 13 (2016), 111-126.

[36] S. Pramanik, and K. Mondal. Rough bipolar neutrosophic set. Global Journal of Engineering Science and Research Management, 3(6) (2016),71-81.

[37] D.G. Park, Y.C. Kwun, J.H. Park, and I.Y. Park. Correlation coefficient of interval-valued intuitionistic fuzzy sets and its application to multiple attribute group decision making problems. Mathematical and Computer Modelling, 50 (2009), 1279–1293.

[38] E. Szmidt and J. Kacprzyk. Correlation of intuitionistic fuzzy sets. Lecture Notes in Computer Science, 6178 (2010), 169–177.

[39] G.W. Wei, H.J. Wang, and R. Lin. Application of correlation coefficient to interval-valued intuitionistic fuzzy multiple attribute decision-making with incomplete weight information. Knowledge and Information Systems, 26(2011),337–349.

[40] H.P. Kriegel, P. Kroger, E. Schubert, and A. Zimek. A General framework for increasing the robustness of PCA-based correlation clustering algorithms. Lecture Notes in Computer Science, 5069 (2008), 418–435.

[41] J. Ye. Multicriteria fuzzy decision-making method using entropy weights-based correlation coefficients of intervalvalued intuitionistic fuzzy sets. Applied Mathematical Modelling, 34 (2010), 3864–3870.

[42] P. Bonizzoni, G.D. Vedova, R. Dondi, and T. Jiang. Correlation clustering and consensus clustering. Lecture Notes in Computer Science, 3827 (2008), 226–235.

[43] D.A. Chiang, and N.P. Lin. Correlation of fuzzy sets. Fuzzy Sets and Systems, 102 (1999), 221–226.

[44] D.H. Hong. Fuzzy measures for a correlation coefficient of fuzzy numbers under Tw (the weakest t-norm)-based fuzzy arithmetic operations. InformationSciences, 176 (2006), 150–160.

[45] G.J. Wang, and X.P. Li. Correlation and information energy of interval-valued fuzzy numbers. Fuzzy Sets and Systems, 103 (1999), 169–175.

[46] T. Gerstenkorn, and J. Manko. Correlation of intuitionistic fuzzy sets. Fuzzy Sets and Systems, 44 (1991), 39–43.

[47] W.L. Hung and J.W. Wu. Correlation of intuitionistic fuzzy sets by centroid method. Information Sciences 144 (2002), 219–225.

[48] Z.S. Xu. On correlation measures of intuitionistic fuzzy sets. Lecture Notes in Computer Science, 4224 (2006), 16–24.

[49] J. Ye. Fuzzy decision-making method based on the weighted correlation coefficient under intuitionistic fuzzy environment. European Journal of Operational Research, 205 (2010), 202–204.

[50] H. Bustince, and P. Burillo. Correlation of interval-valued intuitionistic fuzzy sets. Fuzzy Sets and Systems, 74 (1995), 237–244.
[51] D.H. Hong. A note on correlation of interval-valued intuitionistic fuzzy sets. Fuzzy Sets and Systems, 95 (1998), 113– 117.
[52] I.M. Hanafy, A.A. Salama, and K. Mahfouz. Correlation of neutrosophic Data. International Refereed Journal of Engineering and Science. 1(2) (2012), 39-43.
[53] J. Ye. Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment. International Journal of General Systems, 42(4) (2013), 386-394.
[54] M . Hanafy, A. A. Salama, and K. M. Mahfouz. Correlation Coefficients of Neutrosophic Sets by Centroid Method. International Journal of Probability and Statistics, 2(1) (2013), 9-12.
[55] S. Broumi, F. Smarandache. Correlation coefficient of interval neutrosophic set. Applied Mechanics and Materials, 436 (2013), 511-517. doi:10.4028/www.scientific.net/AMM.436.511
[56] D. Dubios and H. Prade. Rough fuzzy sets and fuzzy rough sets. International Journal of General System, 17 (1990), 191-208.
[57] A. Nakamura. Fuzzy rough sets. Notes on Multiple Valued Logic, Japan 9 (8) (1998), 1–8.
[58] S. Nanda, and S. Majumdar. Fuzzy rough sets. Fuzzy Sets and Systems, 45 (1992), 157–160.
[59] K. V. Thomas, and L. S. Nair. Rough intuitionistic fuzzy sets in a lattice. International Mathematics Forum, 6(27) (2011), 1327–1335.
[60] S. S. Dragomir, and A. Sofo. On some inequalities of cauchy-bunyakovsky-schwarz type and applications. Tamkang Journal of Mathematics, 39(4) (2008), 291-301.

61. Abdel-Basset, M., Mohamed, M., & Sangaiah, A. K. (2017). Neutrosophic AHP-Delphi Group decision making model based on trapezoidal neutrosophic numbers. Journal of Ambient Intelligence and Humanized Computing, 1-17.https://doi.org/10.1007/s12652-017-0548-7
62. Abdel-Basset, M., Mohamed, M., Hussien, A. N., & Sangaiah, A. K. (2017). A novel group decision-making model based on triangular neutrosophic numbers. Soft Computing, 1-15. http://doi.org/10.1007/s00500-017-2758-5
63. F. Smarandache, M. Ali, *Neutrosophic Triplet as extension of Matter Plasma, Unmatter Plasma, and Antimatter Plasma***,** 69th Annual Gaseous Electronics Conference, Bochum, Germany, October 10-14, 2016, http://meetings.aps.org/Meeting/GEC16/Session/HT6.112
64. F. Smarandache, *Neutrosophic Perspectives: Triplets, Duplets, Multisets, Hybrid Operators, Modal Logic, Hedge Algebras. And Applications*. Pons Editions, Bruxelles, 325 p., 2017.
65. Topal, S. and Öner, T. Contrastings on Textual Entailmentness and Algorithms of Syllogistic Logics, Journal of Mathematical Sciences, Volume 2, Number 4, April 2015, pp 185-189.
66. Topal S. An Object- Oriented Approach to Counter-Model Constructions in A Fragment of Natural Language, BEU Journal of Science, 4(2), 103-111, 2015.
67. Topal, S. A Syllogistic Fragment of English with Ditransitive Verbs in Formal Semantics, Journal of Logic, Mathematics and Linguistics in Applied Sciences, Vol 1, No 1, 2016.
68. Topal, S. and Smaradache, F. A Lattice-Theoretic Look: A Negated Approach to Adjectival (Intersective, Neutrosophic and Private) Phrases. The 2017 IEEE International Conference on INnovations in Intelligent SysTems and Applications (INISTA 2017); (accepted for publication).
69. Taş, F. and Topal, S. Bezier Curve Modeling for Neutrosophic Data Problem. Neutrosophic Sets and Systems, Vol 16, pp. 3-5, 2017.
70. Topal, S. Equivalential Structures for Binary and Ternary Syllogistics, Journal of Logic, Language and Information, Springer, 2017, DOI: 10.1007/s10849-017-9260-4

Received: July 13, 2017. Accepted: August 1, 2017.

NSS

University of New Mexico

Decision support based on single valued neutrosophic number for information system project selection

Silvia Liliana Tejada Yepez[1]

[1]Universidad de Guayaquil, Facultad de Ciencias Matemáticas y Físicas, Guayaquil Ecuador. E-mail: silvia.tejaday@ug.edu.ec

Abstract. Neutrosophic sets and its application to decision support have become a topic of great importance for researchers and practitioners alike. In this paper, a new model for decision making in the selection of information system projects is presented based on single valued neutrosophic number (SVN-numbers) allowing the use of linguistic variables with multiples points of view from experts. The proposed framework is composed of four activities, framework, gathering information, rating alternatives and information system project selection. Project alternatives are rated based on the Euclidean distance to the ideal alternative. A case study is developed in information system, showing the applicability of the proposal. Further works will concentrate in extending the model for dealing with heterogeneous information and in developing a software tool.

Keywords: Decision Analysis, SVN Numbers, Ideal Alternative, Information Systems, project selection.
.

1 Introduction

Decision analysis is a discipline, belonging to decision theory, with the goal of computing an overall assessment that summarizes the information gathered and providing useful information about each evaluated element (Macarena Espinilla, Palomares, Martinez, & Ruan, 2012). Uncertainty is present in real world decision making problems in such cases the use of linguistic information to model and manage such an uncertainty has given good results (Estrella, Espinilla, Herrera, & Martínez, 2014). Experts feel more comfortable providing their knowledge by using terms close to human beings cognitive model (Rodríguez & Martínez, 2013) that is the rationale for using linguistic variables.

The conventional techniques have been not much effective for solving decision problems because of imprecise nature of the linguistic assessments. It is more reasonable to consider the values of alternatives according to the criteria as single valued neutrosophic sets (SVNS) (Wang, Smarandache, Zhang, & Sunderraman, 2010) for handling indeterminate and inconsistent information, while fuzzy sets and intuitionistic fuzzy sets cannot describe it properly . In this paper a new model of information system project selection is developed based on single valued neutrosophic number (SVN-number) allowing the use of linguistic variables (Biswas, Pramanik, & Giri, 2016).

This paper is structured as follows: Section 2 reviews some preliminaries concepts about decision analysis framework and SVN numbers is presented. In Section 3, a decision analysis framework based on SVN numbers for project selection. Section 4 shows a case study of the proposed model. The paper ends with conclusions and further work recommendations.

2 Preliminaries

In this section, we first provide a brief revision of a general decision scheme and the use of linguistic information using SVN numbers for information system Project selection.

2.1 Decision Scheme and Information Systems Project Selection

Decision analysis is a discipline with the main purpose of helping decision maker to reach a reliable decision (M. Espinilla, Ruan, Liu, & Martínez, 2010).

A common decision resolution scheme consists of following eight phases (Clemen, 1996; Estrella et al., 2014):

1. Identify decision and objectives.
2. Identify alternatives.
3. Framework:
4. Gathering information.
5. Rating alternatives.
6. Choosing the alternative/s:

7. Sensitive analysis
8. Make a decision

In the framework phase, the structures and elements of the decision problem are defined such as experts, criteria, options. The information provided by experts is collected, according to the defined framework.

The gathered information provided by experts is then aggregated to obtain a collective value of alternatives in the rating phase. Therefore, in rating phase, it is necessary to carry out a solving process to compute the collective assessments for the set of alternatives, using appropriate aggregation operators (Calvo, Kolesárová, Komorníková, & Mesiar, 2002).

A way to compute a rating of alternatives is by using the ideal alternative concept. A comparison between an ideal alternative and available options in order to find the optimal choice is used for the ratting of alternatives (Zeng, Baležentis, & Zhang, 2012). Normally, the closer the alternative to the ideal the better the alternative is.

Information systems project selection could be defined as a multicriteria decision problem (Lee & Kim, 2001) . This fact makes the process of selecting information systems projects suitable for decision analysis scheme model.

2.2 SVN-numbers

Neutrosophy (Smarandache, 1999) is mathematical theory developed by Florentín Smarandache for dealing with indeterminacy Neutrosophy have been the base for developing of new methods to handle indeterminate and inconsistent information like neutrosophic sets an neutrosophic logic (Smarandache, 2005; Vera, José, Menéndez Delgado, Gónzalez, & Vázquez, 2016) . It is used specially in decision making problems.

The truth value in neutrosophic set is as follows (Rivieccio, 2008):

Let N be a set defined as: $N = \{(T, I, F) : T, I, F \subseteq [0, 1]\}$, a neutrosophic valuation n is a mapping from the set of propositional formulas to N , that is for each sentence p we have $v\,(\mathrm{p}) = (T, I, F)$.

Single valued neutrosophic set (SVNS) (Wang et al., 2010) was developed with the goal of facilitate the real applications of neutrosophic set and set-theoretic operators.

A single valued neutrosophic set (SVNS) has been defined (Definition 1) (Wang et al., 2010):

Definition 1: Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form of :

$$A = \{\langle x, uA(x), rA(x), vA(x)\rangle : x \in X\} \quad (1)$$

where $u_A(x): X \rightarrow [0,1]$, $r_A(x), : X \rightarrow [0,1]$ and $v_A(x): X \rightarrow [0,1]$ with $0 \leq u_A(x) + r_A(x) + v_A(x) :\leq 3$ for all $x \in X$. The intervals $u_A(x)$, $r_A(x)$ y $v_A(x)$ denote the truth- membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A respectively.

Single valued neutrosophic numbers (SVN number) are denoted by $A = (a, b, c)$, where $a, b, c \in [0,1]$ and $a+b+c \leq 3$.

In decision analysis schema aggregation operating are important for rating options. Some aggregation operators have been proposed for SVN numbers (Ye, 2014a). Single valued neutrosophic weighted averaging (SVNWA) operator was proposed by Ye (Ye, 2014a) for SVNSs as follows(Biswas et al., 2016):

$$F_w(A_1, A_2, \dots, A_n) = \langle 1 - \prod_{j=1}^{n}\left(1 - T_{A_j}(x)\right)^{w_j}, \prod_{j=1}^{n}\left(I_{A_j}(x)\right)^{w_j}, \prod_{j=1}^{n}\left(F_{A_j}(x)\right)^{w_j} \rangle \quad (2)$$

Alternatives could be rated according Euclidean distance in SVN (Şahin & Yiğider, 2014; Ye, 2014b).

Definition 2: Let $A^* = (A_1^*, A_2^*, \dots, A_n^*)$ be a vector of n SVN numbers such that $A_j^* = (a_j^*, b_j^*, c_j^*)$ j=(1,2, … , n) and $B_i = (B_{i1}, B_{i2}, \dots, B_{im})$ $(i = 1,2, \dots, m)$ be m vectors of n SVN numbers such that $B_{ij} = (a_{ij}, b_{ij}, c_{ij})$ $(i = 1,2, \dots, m)$, $(j = 1,2, \dots, n)$. Then the separation measure between $B_i's$ y A^* is defined as follows:

$$s_i = \left(\frac{1}{3}\sum_{j=1}^{n}\left\{\left(\left|a_{ij} - a_j^*\right|\right)^2 + \left(\left|b_{ij} - b_j^*\right|\right)^2 + \left(\left|c_{ij} - c_j^*\right|\right)^2\right\}\right)^{\frac{1}{2}} \quad (3)$$

$(i = 1,2, \dots, m)$

In this paper the concept of linguistic variables (Leyva-Vázquez, Santos-Baquerizo, Peña-González, Cevallos-Torres, & Guijarro-Rodríguez, 2016) are represented using single valued neutrosophic numbers (Şahin & Yiğider, 2014)for developing a framework to decision support.

The gathering information phase is developed using SVN numbers (Deli & Şubaş, 2016) due to the fact that provides adequate computational models to deal with linguistic information (Leyva-Vázquez et al., 2016) in decision. It allow to include handling of indeterminate and inconsistent in information system project selection.

3 Proposed framework.

Our aim is to develop a framework for information system project selection based on SVN numbers. The model consists of the following phases (figure 1).

Figure 1: A framework for using SVN numbers in information system project selection

The proposed framework is composed of four activities, framework, gathering information, rating alternatives and information system project selection.

Framework

In this phase, the evaluation framework, the decision problem of information system project selection is defined. The framework is established as follows:

- C=$\{c_1, c_2, \ldots, c_n\}$ with $n \geq 2$, a set of criteria.
- E=$\{e_1, e_2, \ldots, e_k\}$ with $k \geq 2$, a set of experts.
- $X = \{x_1, x_2, \ldots, x_m\}$ with $m \geq 2$, a finite set of information systems projects alternatives. The set of experts will provide the assessments of the decision problem.

Gathering information

In this phase, each expert, e_k provides the assessments by means of assessment vectors:

$$U^K = (v^k{}_{ij}, i = 1, .., n, j = 1, .., m) \tag{5}$$

The assessment v_{ij}^k, provided by each expert e_k, for each criterion c_i of each project alternative x_j, is expressed using an SVN number.

Rating alternatives

Initial aggregation process is developed for rating alternatives. The aggregated SVN decision matrix obtained by aggregating of opinions of decision makers. In our proposal the SVNWA aggregation operator used Eq. (2).

For rating alternatives an ideal project option is constructed (Leyva-Vázquez, Pérez-Teruel, & John, 2014; Şahin & Yiğider, 2014). The evaluation criteria can be categorized into two categories, benefit and cost. Let C^+ be a collection of benefit criteria and C^- be a collection of cost criteria. The ideal alternative is defined as:

$$I = \left\{\left(max_{i=1}^{k} T_{U_j} | j \in C^+, min_{i=1}^{k} T_{U_j} | j \in C^-\right), \left(min_{i=1}^{k} I_{U_j} | j \in C^+, max_{i=1}^{k} I_{U_j} | j \in C^-\right), \left(min_{i=1}^{k} F_{U_j} | j \in C^+, max_{i=1}^{k} F_{U_j} | j \in C^-\right)\right\} = [v_1, v_2, \ldots, v_n] \tag{6}$$

Alternatives are rated according Euclidean distance I:

$$s_i = \left(\frac{1}{3}\sum_{j=1}^{n}\left\{\begin{matrix}\left(\left|T(v_{ij})\text{-}T(v_i^I)\right|\right)^2 + \left(\left|I(v_{ij})\text{-}I(v_i^J)\right|\right)^2 \\ +\left(\left|F(v_{ij})\text{-}F(v_i^I)\right|\right)^2\end{matrix}\right\}\right)^{\frac{1}{2}}$$

$$(i = 1, 2, \ldots, n) \tag{7}$$

Information System Project Selection

Ranking is based in the global distance to the ideal. If alternative project x_i is closer to I the distance measure (s_i closer) better is the project alternative (Leyva-Vázquez, Pérez-Teruel, Febles-Estrada, & Gulín-González, 2013).

4 Case study

In this section, we present an illustrative example in order to show the applicability of the proposed framework for information system project selection.

In this case study the evaluation framework is compose by 2 experts E=$\{e_1, e_2\}$ who evaluate 3 alternatives(information system projects) .

x_1: CRM

x_2: ERP

x_3: SCM

These projects are described in Table #1.

TABLE I. PROJECTS OPTIONS

id	**Name**	**Description**
1	CRM.	Custumer Relation Management Software
2	ERP	Enterprise Relationship Managemet Software
3	SCM	Supply Chain Managemet Software

3 criteria are involved, which are shown below:

c_1: Benefits

c_2: Factibility

c_3: Cost

In Table 2, we give the set of linguistic terms used for experts to provide the assessments.

TABLE II. LINGUISTIC TERMS USED TO PROVIDE THE ASSESSMENTS (ŞAHIN & YIĞIDER, 2014)

Linguistic terms	**SVNSs**
Extremely good (EG)	(1,0,0)
Very very good (VVG)	(0.9, 0.1, 0.1)
Very good (VG)	(0.8,0,15,0.20)
Good (G)	(0.70,0.25,0.30)
Medium good (MG)	(0.60,0.35,0.40)
Medium (M)	(0.50,0.50,0.50)
Medium bad (MB)	(0.40,0.65,0.60)
Bad (B)	(0.30,0.75,0.70)
Very bad (VB)	(0.20,0.85,0.80)
Very very bad (VVB)	(0.10,0.90,0.90)
Extremely bad (EB)	(0,1,1)

Once the evaluation framework has been determined the information about the projects is gathered (see Table 3).

TABLE III. RESULT OF GATHERING INFORMATION

	e_1			e_1		
	x_1	x_2	x_3	x_1	x_2	x_3
c_1	MG	EG	MB	M	VVG	M
c_2	G	MG	M	B	MB	M
c_3	MG	MG	G	MB	MG	B

For rating project alternatives, an initial aggregation process is developed. Then the aggregated SVN decision matrix obtained by aggregating of opinions of decision makers is constructed by Eq. (2). The result is given in Table 4. The importance of each expert is expressed in the weighting vector $W = [0.7,0.3]$.

TABLE IV. AGGREGATED SVN DECISION MATRIX

	x_1	x_2	x_3
c_1	(0.57,0.39,0.43)	(1,0,0)	(0.4, 0.60,0.57)
c_2	(0.65,0.31,0.35)	(0.55,0.42,0.45)	(0.50,0.50,0.50)
c_3	(0.63,0.32, 0.37)	(0.60,0.35,0.40)	(0.61,0.35,0.47)

Calculation SVN positive-ideal solution is made as Table 5.

TABLE V. SVN POSITIVE-IDEAL VALUES

	Positive-ideal
c_1	(1,0,0)
c_2	(0.65,0.31,0.35)
c_3	(0.63,0.32, 0.37)

Separation measure of each alternative from the positive-ideal solution are calculated using Eq. (4) and are given by Table 6.

TABLE VI. DISTANCE TO THE IDEAL SOLUTION

	SVN positive-ideal	Ranking
x_1	0.42	2
x_2	0.11	1
x_3	0.61 0.37)	3

According to descending order of relative closeness coefficients values, four alternatives are ranked as: $x_2 \succ x_1 \succ x_3$.

5 Conclusions.

In recent years, neutrosophic sets and its application to multiple attribute decision making have become a topic of great importance for researchers and practitioners. In this paper a new model information system project selection based on SVN-number applied allowing the use of linguistic variables for application in in complex decisions that require multiples points of view. To demonstrate the applicability of the proposal a case study. Our approach has many application information system project selection that include indeterminacy.

Further works will concentrate extending the model for dealing with heterogeneous information (Pérez-Teruel, Leyva-Vázquez, & Espinilla-Estevez, 2013). Another area of future work is the developing of new aggregation models based on SVN numbers specially compensatory operators (Espin-Andrade, González Caballero, Pedrycz, & Fernández González, 2015) and the developing of a software tool.

References

Biswas, P., Pramanik, S., & Giri, B. C. (2016). TOPSIS method for multi-attribute group decision-making under single-valued neutrosophic environment. *Neural Computing and Applications, 27*(3), 727-737.

Calvo, T., Kolesárová, A., Komorníková, M., & Mesiar, R. (2002). Aggregation operators: properties, classes and construction methods *Aggregation Operators* (pp. 3-104): Springer.

Clemen, R. T. (1996). *Making Hard Decisions: An Introduction to Decision Analysis*: Duxbury Press.

Deli, I., & Şubaş, Y. (2016). A ranking method of single valued neutrosophic numbers and its applications to multi-attribute decision making problems. *International Journal of Machine Learning and Cybernetics*, 1-14.

Espin-Andrade, R. A., González Caballero, E., Pedrycz, W., & Fernández González, E. R. (2015). Archimedean-Compensatory Fuzzy Logic Systems. *International Journal of Computational Intelligence Systems, 8*(sup2), 54-62.

Espinilla, M., Palomares, I., Martinez, L., & Ruan, D. (2012). A comparative study of heterogeneous decision analysis approaches applied to sustainable energy evaluation. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 20*(supp01), 159-174.

Espinilla, M., Ruan, D., Liu, J., & Martínez, L. (2010). *A heterogeneous evaluation model for assessing sustainable energy: A Belgian case study.* Paper presented at the Fuzzy Systems (FUZZ), 2010 IEEE International Conference on.

Estrella, F. J., Espinilla, M., Herrera, F., & Martínez, L. (2014). FLINTSTONES: A fuzzy linguistic decision tools enhancement suite based on the 2-tuple linguistic model and extensions. *Information Sciences, 280*, 152-170.

Lee, J. W., & Kim, S. H. (2001). An integrated approach for interdependent information system project selection. *International Journal of Project Management, 19*(2), 111-118.

Leyva-Vázquez, M., Pérez-Teruel, K., Febles-Estrada, A., & Gulín-González, J. (2013). Técnicas para la representación del conocimiento causal: un estudio de caso en Informática Médica. *Revista Cubana de información en ciencias de la salud, 24*(1), 73-83.

Leyva-Vázquez, M., Pérez-Teruel, K., & John, R. I. (2014). *A model for enterprise architecture scenario analysis based on fuzzy cognitive maps and OWA operators.* Paper presented at the Electronics, Communications and Computers (CONIELECOMP), 2014 International Conference on.

Leyva-Vázquez, M., Santos-Baquerizo, E., Peña-González, M., Cevallos-Torres, L., & Guijarro-Rodríguez, A. (2016). *The Extended Hierarchical Linguistic Model in Fuzzy Cognitive Maps.* Paper presented at the Technologies and Innovation: Second International Conference, CITI 2016, Guayaquil, Ecuador, November 23-25, 2016, Proceedings 2.

Pérez-Teruel, K., Leyva-Vázquez, M., & Espinilla-Estevez, M. (2013). *A linguistic software requirement prioritization model with heterogeneous information.* Paper presented at the 4th International Workshop on Knowledge Discovery, Knowledge Management and Decision Support (EUREKA 2013), Mazatlán (Mexico).

Rivieccio, U. (2008). Neutrosophic logics: Prospects and problems. *Fuzzy sets and systems, 159*(14), 1860-1868.

Rodríguez, R. M., & Martínez, L. (2013). An analysis of symbolic linguistic computing models in decision making. *International Journal of General Systems, 42*(1), 121-136.

Şahin, R., & Yiğider, M. (2014). A Multi-criteria neutrosophic group decision making metod based TOPSIS for supplier selection. *arXiv preprint arXiv:1412.5077*.

Smarandache, F. (1999). A Unifying Field in Logics: Neutrosophic Logic. *Philosophy*, 1-141.

Zeng, S., Baležentis, T., & Zhang, C. (2012). A method based on OWA operator and distance measures for multiple attribute decision making with 2-tuple linguistic information. *Informatica, 23*(4), 665-681.

Smarandache, F. (2005). *A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability: Neutrsophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability*: Infinite Study.

Vera, M., José, P., Menéndez Delgado, C. F., Gónzalez, M. P., & Vázquez, M. L. (2016). Las habilidades del marketing como determinantes que sustentaran la competitividad de la Industria del arroz en el cantón Yaguachi. Aplicación de los números SVN a la priorización de estrategias. *Neutrosophic Sets & Systems, 13*.

Wang, H., Smarandache, F., Zhang, Y., & Sunderraman, R. (2010). Single valued neutrosophic sets. *Review of the Air Force Academy*(1), 10.

Ye, J. (2014a). A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. *Journal of Intelligent & Fuzzy Systems, 26*(5), 2459-2466.

Ye, J. (2014b). Single-valued neutrosophic minimum spanning tree and its clustering method. *Journal of intelligent Systems, 23*(3), 311-324.

F. Smarandache, M. Ali, *Neutrosophic Triplet as extension of Matter Plasma, Unmatter Plasma, and Antimatter Plasma*, 69th Annual Gaseous Electronics Conference, Bochum, Germany, October 10-14, 2016, http://meetings.aps.org/Meeting/GEC16/Session/HT6.112

F. Smarandache, *Neutrosophic Perspectives: Triplets, Duplets, Multisets, Hybrid Operators, Modal Logic, Hedge Algebras. And Applications*. Pons Editions, Bruxelles, 325 p., 2017.

Received: July 18, 2017. Accepted: August 3, 2017.

NSS

University of New Mexico

Uniform Single Valued Neutrosophic Graphs

S. Broumi[1], A. Dey[2], A. Bakali[3], M. Talea[1], F. Smarandache[4]
L. H. Son[5], D. Koley[6]

[1] Laboratory of Information Processing, Faculty of Science Ben M'Sik, University Hassan II, B.P 7955, Sidi Othman, Casablanca, Morocco. E-mail: broumisaid78@gmail.com
[2]Saroj Mohan Institute of Technology, West Bengal, India, E-mail:arindam84nit@gmail.com
[3] Ecole Royale Navale, Casablanca, Morocco, e-mail: assiabakali@yahoo.fr
[4] Department of Mathematics, University of New Mexico,705 Gurley Avenue, Gallup, NM 87301, USA, E-mail: fsmarandache@gmail.com
[5]Vietnam National University, Hanoi, Vietnam, E-mail: sonlh@vnu.edu.vn
[6]Techno India Hooghly Campus, West Bengal, India, E-mail:dipakkoley973@gmail.com

Abstract. In this paper, we propose a new concept named the uniform single valued neutrosophic graph. An illustrative example and some properties are examined. Next, we develop an algorithmic approach for computing the complement of the single valued neutrosophic graph. A numerical example is demonstrated for computing the complement of single valued neutrosophic graphs and uniform single valued neutrosophic graph.

Keywords: Single valued neutrosophic sets; Uniform single valued neutrosophic graph; Complement operators

1 Introduction

In 1965, Zadeh [7] originally introduced the concept of fuzzy set(FSs) which is characterized by a membership degree in [0, 1] for each element in the dataset. It may not always be true that the degree of non-membership of an element in a fuzzy set is equal to 1 minus the truth- membership degree because there is some kind of hesitation degree. On the basis of fuzzy sets, Atanassov [4] added a non-membership in the definition of intuitionistic fuzzy sets (IFSs) and later Smarandache [2] introduced the neutrosophic sets (NSs) with the appearance of the truth-membership degree (T), the falsehood-membership degree (F), and the indeterminacy degree (I). Wang et al. [3] proposed various set theoretical operators and linked to single valued neutrosophic sets The concept of neutrosophic sets have been successfully applied to many fields [16].

Fuzzy graph has been studied extensively in the past years [5,8,9]. Later on, Smarandache [1] proposed neutrosophic graphs in some special types such as neutrosophic offgraph, neutrosophic bipolar/tripolar/ multipolar graph. Presently, works on neutrosophic vertex-edge graphs and neutrosophic edge graphs are progressing rapidly. Broumi et al.[13] introduced certain types of single valued neutrosophic graphs (in short SVNG) such as strong single valued neutrosophic graph, constant single valued neutrosophic graph, complete single valued neutrosophic graph with their properties and examples. Neighborhood degree of a vertex and closed neighborhood degree of vertex in single valued neutrosophic graph were introduced in [15]. The necessary and sufficient condition for a single valued neutrosophic graph to be an isolated single valued neutrosophic graph has been presented in [10]. Other extensions of the neutrosophic graph have been described in [11,12, 14].

Up to now, to the best of our knowledge, there has been no study on the uniform single valued neutrosophic graph. Thus, we propose in this paper a new concept named the uniform single valued neutrosophic graph. An illustrative example and some properties are examined. Next, we develop an algorithmic approach for computing the complement of the single valued neutrosophic graph.

The remainder of this paper is organized as follows. In Section 2, we present the basic definitions. In section 3, we introduce the concept of uniform single valued neutrosophic graph and investigate its properties. Section 4 introduces an algorithm for computing the complement of single valued neutrosophic graphs. A numerical example is presented in Section 5. Finally, Section 6 outlines the conclusion of this paper and suggests several directions for future research.

2 Preliminaries

In this section, we have present the basic definitions of fuzzy sets, neutrosophic sets, single valued neutrosophic sets, fuzzy graphs, uniform fuzzy graphs, complement of single valued neutrosophic graph which will be useful to our main work in the next sections.

Definition 1[1]. Let X be the universe of discourse and its elements denoted by x. In fuzzy theory, a fuzzy set

A of universe X is defined by the function $T_A(x)$, called the membership function of set A.

$$T_A: X \rightarrow [0, 1] \quad (1)$$

For any element x of universe X, $T_A(x)$ equals the degree, between 0 and 1, to which x is an element of set A, This degree represents the membership value or degree of membership of element x in set A.

Definition 2[1]**.** Let X be a space of points and let x ∈ X. A **neutrosophic set** A in X is characterized by a truth membership function T, an indeterminacy membership function I, and a falsehood membership function F which are real standard or nonstandard subsets of]−0,1+[, and T, I, F: X→]−0,1+[. The neutrosophic set can be represented as,

$$A=\{(x, T_A(x), I_A(x), F_A(x)): x \in X\} \quad (4)$$

There is no restriction on the sum of T, I, F, So

$$^{-}0 \leq T_A(x)+ I_A(x)+F_A(x) \leq 3^{+}. \quad (5)$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^{-}0,1^{+}[$. Thus it is necessary to take the interval [0,1] instead of $]^{-}0,1^{+}[$. For practical applications, it is difficult to apply $]^{-}0,1^{+}[$ in the real life applications such as engineering and scientific problems.

Definition 3[3]**.** Let X be a space of objects with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS) is characterized by truth-membership function $T_A(x)$, an indeterminate-membership function $I_A(x)$, and a falsehood-membership function $F_A(x)$. For each point x in X, $T_A(x), I_A(x), F_A(x) \in [0, 1]$. A SVNS A can be written as,

$$A=\{(x, T_A(x), I_A(x), F_A(x)): x \in X\} \quad (6)$$

Definition 4 [5]**.** ***A fuzzy graph*** is a pair of functions G = (σ, μ) where σ is a fuzzy subset of a non empty set V and μ is a symmetric fuzzy relation on σ. i.e σ : V → [0,1] and μ: VxV→[0,1] such that μ(uv) ≤ σ(u) ⋀ σ(v) for all u, v ∈ V where uv denotes the edge between u and v and σ(u) ⋀ σ(v) denotes the minimum of σ(u) and σ(v). σ is called the fuzzy vertex set of V and μ is called the fuzzy edge set of E.

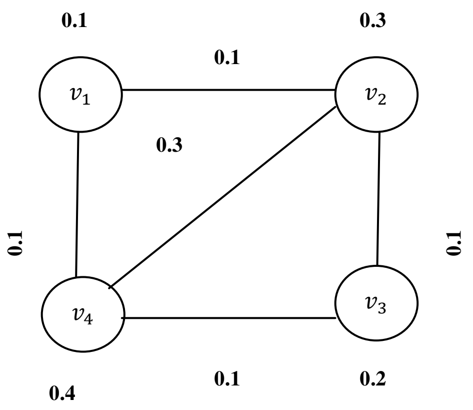


Fig.1. Fuzzy graph

Remark: The crisp graph G^* = (V, E) is a special case of the fuzzy graph G with each vertex and edge of (V, E) having degree of membership 1 (Fig. 1).

Definition5[6,8]**.** The complement of a fuzzy graph G = (σ, μ) is a fuzzy graph $\bar{G}$ = ($\bar{\sigma}$, $\bar{\mu}$) where $\bar{\sigma}$ =σ and $\bar{\mu}$(u,v) = σ(u)∧σ(v)-μ(u,v), ∀ u,v ∈ V.

Definition 6[6,8]. Let G = (σ, μ) be a fuzzy graph on a crisp graph G^* = (V, E). Let σ*={x∈ V | σ(x)> 0}.Then G is called a uniform fuzzy graph of level k if μ (x,y) = k, ∀ (x,y) ∈ (σ* × σ*) and σ(x) = k where k isa positive real such that $0 < k_1 \leq 1$**.**

Definition 7[15].Let G = (V, E) be a single valued neutrosophic graph, then the **degree of a vertex** x_i is defined by $d_G(x_i)=d_G(x)=(d_T(x), d_I(x), d_F(x))$, $d_G(x_i) = (\sum_{x\neq y} T_B(x,y), \sum_{x\neq y} I_B(x,y), \sum_{x\neq y} I_B(x,y))$.

Definition 8[15].Let G = (V, E) be a single valued neutrosophic graph, then the **total degree of a vertex** x_i is defined by $td_G(x_i) = d_G(x) = (td_T(x), td_I(x), td_F(x))$, $td_G(x_i) = (\sum_{x\neq y} T_B(x,y) + T_A(x), \sum_{x\neq y} I_B(x,y) + I_A(x), \sum_{x\neq y} I_B(x,y) + F_A(x))$.

Definition 9[13]. Let G = (V, E) be a single valued neutrosophic graph, then the complement of single valued neutrosophic graph is defined as

1. $\bar{V}$ =V
2. $\bar{T_A}$ (x)=$T_A(x)$, $\bar{I_A}(x) = I_A(x)$, $\overline{F_A}(x) = F_A(x)$ for all x∈V.
3. $\overline{T_B}(x,y)$= min $[T_A(x), T(y)] - T_B(x,y)$
$\overline{I_B}(x,y)$= max $[I_A(x), I_A(y)] - I_B(x,y)$ and
$\overline{F_B}(x,y)$= max $[F_A(x), F_A(y)] - F_B(x,y)$, for all (x, y) ∈ E

Definition 10[13]. Let G = (V, E) be a single valued neutrosophic graph. If $d_G(x_i)= (k_1, k_2, k_3)$ for all $x_i \in$ V, then the single valued neutrosophic graph is called regular SVNG of degree (k_1, k_2, k_3)

Definition 11[13]. Let G = (V, E) be a single valued neutrosophic graph. If $td_G(x_i)$= (k_1, k_2, k_3) for all $x_i \in$ V, then the single valued neutrosophic graph is called Totally regular SVNG of degree (k_1, k_2, k_3)

III. Uniform Single Valued Neutrosophic Graph

In this section, we define the concept of uniform single valued neutrosophic graphs(in short USVNGs).

Definition 8. Let G = (A, B) be a single valued neutrosophic graph where A =(T_A, I_A, F_A) is a single valued neutrosophic vertex of G and B is a single valued neutrosophic edge set of G. Let A={x∈ V | $T_A(x)$> 0,$I_A(x)$> 0 and $F_A(x) > 0$}.Then G is called Uniform single valued neutrosophic graph of level (k_1, k_2, k_3) if T_B (x,y) = k_1,$I_A(x)$=$k_2$and F_B(x,y) = k_3∀ (x,y) ∈ (V× V) and $T_A(x)$ = k_1,$I_A(x)$ = $k_2$and $F_A(x)$ = k_3 where k_1, $k_2$and k_3 are some positive real such that 0 <k_1, k_2, k_3 ≤1.

Example 1. Consider an USVNG G= (A,B) on V={v_1,v_1,v_3,v_4} as shown in Fig.2.

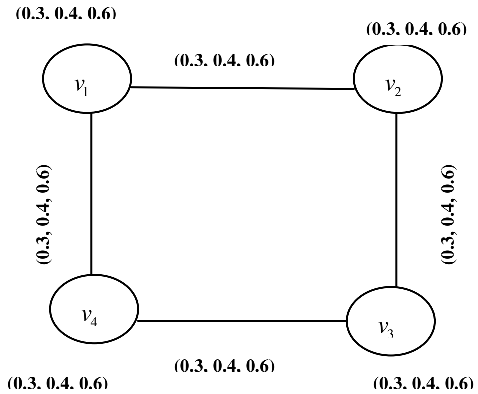


Fig. 2. USVNG.

Remark: The complement of an uniform single valued neutrosophic graph is always an empty graph.

Theorem1. If G = (A, B) is an uniform single valued neutrosophic graph of level (k_1, k_2, k_3) then G is a regular-USVNG.

Proof. Let A={x ∈ V | $T_A(x) >$ 0, $I_A(x) >$ 0 and $F_A(x) > 0$}. Suppose that G is a uniform single valued neutrosophic graph. Then T_B (x, y) = k_1,I_B (x, y) = k_2 and F_B (x,y) = k_3∀ (x,y) ∈ Eand $T_A(z)$ = k_1 , $I_A(z)$ = k_2 and $F_A(z) = k_3$∀ z ∈ V for some real k_1, $k_2$and k_3 where 0 <k_1, k_2, k_3 ≤1.

Let x ∈ V. Now $d_G(x)$=($d_T(x)$,$d_I(x)$, $d_F(x)$)

$$d_G(x) = (\sum_{x\neq y} T_B(\text{x, y}), \sum_{x\neq y} I_B(\text{x, y}), \sum_{x\neq y} F_B(\text{x, y}))$$

=($\sum_{x\neq y} k_1$, $\sum_{x\neq y} k_2$, $\sum_{x\neq y} k_3$)

=((n-1) k_1, (n-1) k_2, (n-1) k_3)

$d_G(x)$=((n-1) k_1, (n-1) k_2, (n-1) k_3) ∀ x ∈ V

Therefore, G is regular uniform single valued neutrosophic graph.

Theorem 2. If G = (A, B) is a uniform single valued neutrosophic graph of level (k_1, k_2, k_3) then G is a totally regular- USVNG.

Proof. Let A={x ∈ V | $T_A(x) >$ 0, $I_A(x) >$ 0 and $F_A(x) > 0$}. Suppose that G is a uniform single valued neutrosophic graph. Then T_B (x, y) = k_1 ,I_B (x, y) = k_2 and F_B (x,y) = k_3∀ (x,y) ∈ Eand $T_A(z)$ = k_1 , $I_A(z)$ = k_2 and $F_A(z) = k_3$∀ z ∈ V for some real k_1, k_2 and k_3 where 0 <k_1, k_2, k_3 ≤1.Let x ∈ V. Now,

$td_G(x)$=($d_T(x) + T_A(x)$,$d_I(x) + I_A(x)$,$d_F(x) + F_A(x)$)

$$td_G(x) = (\sum_{x\neq y} T_B(\text{x, y}) + T_A(x), \sum_{x\neq y} I_B(\text{x, y}) + I_A(x), \sum_{x\neq y} F_B(\text{x, y}) + F_A(x))$$

=(($\sum_{x\neq y} k_1$) + k_1, ($\sum_{x\neq y} k_2$) + k_2, ($\sum_{x\neq y} k_3$) + k_3)

=((n-1) k_1+k_1, (n-1) k_2+k_2, (n-1) k_3+k_3)

$td_G(x)$=(nk_1, n k_2, n k_3) ∀ x ∈ V.

Therefore, G is totally-regular uniform single valued neutrosophic graph.

Theorem 3. If G = (A, B) is a uniform single valued neutrosophic graph of level (k_1, k_2, k_3) on G^*=(V, E), then the order of G is O(G)= (nk_1, nk_2,nk_3).

Proof: Let A={x ∈ V | $T_A(x) >$ 0, $I_A(x) >$ 0 and $F_A(x) > 0$}. Suppose that G is a uniform single valued neutrosophic graph. Then T_B (x, y) = k_1 ,I_B (x, y) = k_2 and F_B (x,y) = k_3∀ (x,y) ∈ Eand $T_A(z)$ = k_1 , $I_A(z)$ = k_2 and $F_A(z) = k_3$∀ z ∈ V for some real k_1, k_2 and k_3 where 0 < k_1, k_2, k_3 ≤1.Let x ∈ V. Now

$O(G)$=($O_T(G)$,$O_I(G)$,$O_F(G)$)

$$O(G) = (\sum_{x\in V} T_A(\text{x}), \sum_{x\in V} I_A(\text{x}), \sum_{x\in V} f_A(\text{x}))$$

=($\sum_{x\in V} k_1$, $\sum_{x\in V} k_2$, $\sum_{x\in V} k_3$)

Then,$O(G)$=(nk_1, n k_2, n k_3).

=($\sum_{x\in V} k_1$, $\sum_{x\in V} k_2$, $\sum_{x\in V} k_3$)

Then,$O(G)$=(nk_1, n k_2, n k_3).

Theorem 4. The uniform single valued neutrosophic graph is a generalization of uniform fuzzy graph.

Proof: Straightforward.

IV. Computing Complement of Single Valued Neutrosophic Graph

In this section, we present in the last paper, a peudo-code of an algorithm computing the complement of single valued neutrosophic graph. This algorithm has the ability of computing the complement of fuzzy graphs, strong intuitionistic fuzzy graphs, uniform fuzzy graphs and also uniform single valued neutrosophic graphs.

The following flowchart demonstrates the algorithm to compute the complement operator is presented in Fig.3V.Numerical Example

In this section, we present an example to compute the complements of the uniform single valued neutrosophic graph. Consider a graph in Fig.4.

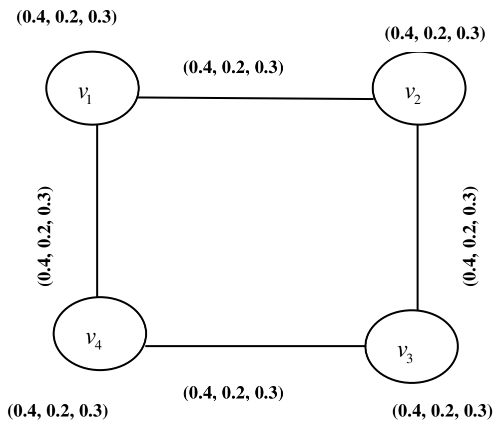
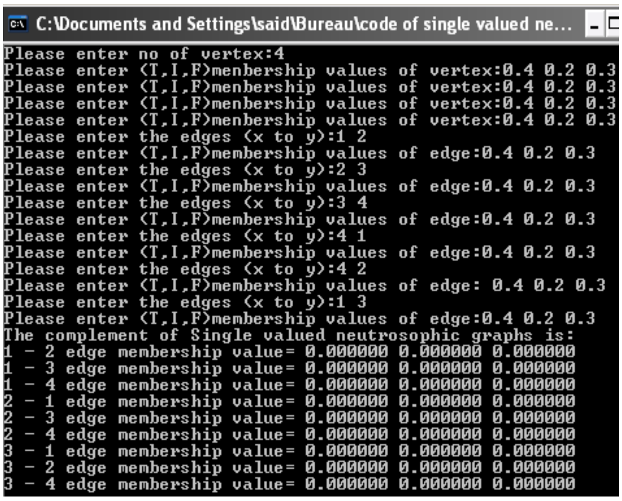


Fig. 4.A uniform single valued neutrosophic graph

Using the above pseudo code, the output result for the complement of a uniform single valued neutrosophic graph is in Fig. 5.



```
C:\Documents and Settings\said\Bureau\code of single valued ne...
Please enter no of vertex:4
Please enter (T,I,F)menbership values of vertex:0.4 0.2 0.3
Please enter (T,I,F)menbership values of vertex:0.4 0.2 0.3
Please enter (T,I,F)menbership values of vertex:0.4 0.2 0.3
Please enter (T,I,F)menbership values of vertex:0.4 0.2 0.3
Please enter the edges (x to y):1 2
Please enter (T,I,F)membership values of edge:0.4 0.2 0.3
Please enter the edges (x to y):2 3
Please enter (T,I,F)membership values of edge:0.4 0.2 0.3
Please enter the edges (x to y):3 4
Please enter (T,I,F)membership values of edge:0.4 0.2 0.3
Please enter the edges (x to y):4 1
Please enter (T,I,F)membership values of edge:0.4 0.2 0.3
Please enter the edges (x to y):4 2
Please enter (T,I,F)membership values of edge: 0.4 0.2 0.3
Please enter the edges (x to y):1 3
Please enter (T,I,F)membership values of edge:0.4 0.2 0.3
The complement of Single valued neutrosophic graphs is:
1 - 2 edge membership value= 0.000000 0.000000 0.000000
1 - 3 edge membership value= 0.000000 0.000000 0.000000
1 - 4 edge membership value= 0.000000 0.000000 0.000000
2 - 1 edge membership value= 0.000000 0.000000 0.000000
2 - 3 edge membership value= 0.000000 0.000000 0.000000
2 - 4 edge membership value= 0.000000 0.000000 0.000000
3 - 1 edge membership value= 0.000000 0.000000 0.000000
3 - 2 edge membership value= 0.000000 0.000000 0.000000
3 - 4 edge membership value= 0.000000 0.000000 0.000000
```

Fig. 5. The outputs

Example 2 Consider a fuzzy graph as shown in Fig.6

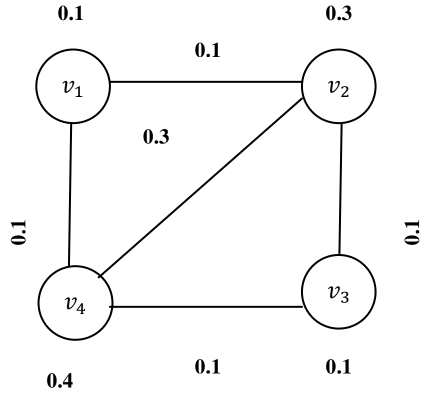
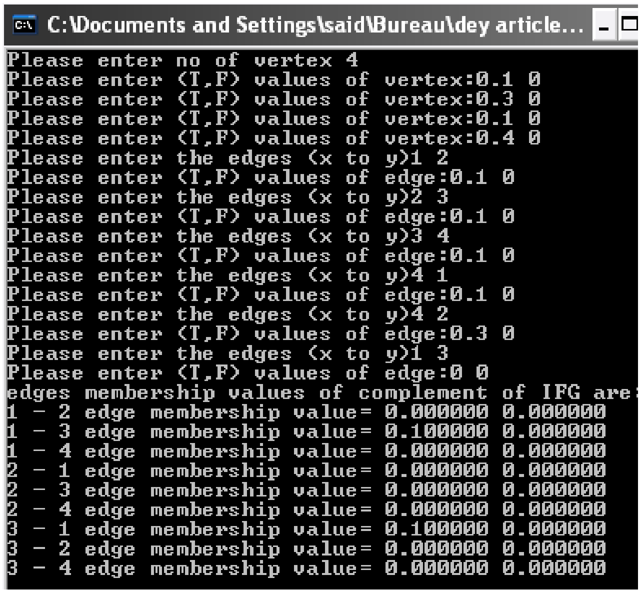


Fig. 6.Fuzzy graph

Using the above pseudo code, the output result for the complement of fuzzy graph is as follows:



```
C:\Documents and Settings\said\Bureau\dey article...
Please enter no of vertex 4
Please enter (T,F) values of vertex:0.1 0
Please enter (T,F) values of vertex:0.3 0
Please enter (T,F) values of vertex:0.1 0
Please enter (T,F) values of vertex:0.4 0
Please enter the edges (x to y)1 2
Please enter (T,F) values of edge:0.1 0
Please enter the edges (x to y)2 3
Please enter (T,F) values of edge:0.1 0
Please enter the edges (x to y)3 4
Please enter (T,F) values of edge:0.1 0
Please enter the edges (x to y)4 1
Please enter (T,F) values of edge:0.1 0
Please enter the edges (x to y)4 2
Please enter (T,F) values of edge:0.3 0
Please enter the edges (x to y)1 3
Please enter (T,F) values of edge:0 0
edges membership values of complement of IFG are:
1 - 2 edge membership value= 0.000000 0.000000
1 - 3 edge membership value= 0.100000 0.000000
1 - 4 edge membership value= 0.000000 0.000000
2 - 1 edge membership value= 0.000000 0.000000
2 - 3 edge membership value= 0.000000 0.000000
2 - 4 edge membership value= 0.000000 0.000000
3 - 1 edge membership value= 0.100000 0.000000
3 - 2 edge membership value= 0.000000 0.000000
3 - 4 edge membership value= 0.000000 0.000000
```

Example 3 Consider an uniform intuitionistic fuzzy graph as shown in Fig.7

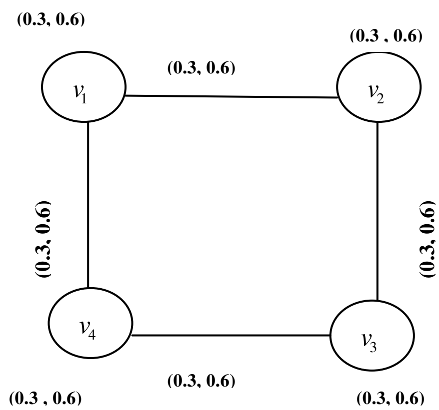
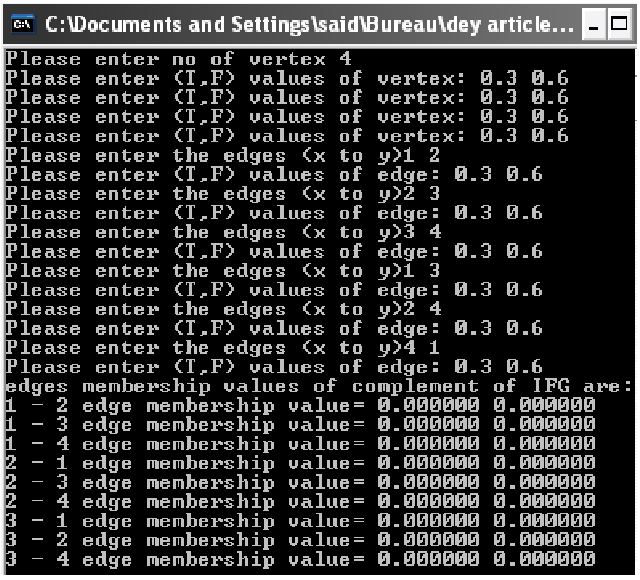


Fig.7. Uniform Intuitionistic fuzzy graph

Using the above pseudo code, the output result for the complement of uniform intuitionistic fuzzy graph is as follows



VI. Conclusion

In this paper, we propose a new uniform single valued neutrosophic graph and an algorithm for computing its complement. Some theorems of the uniform single valued neutrosophic graph have been examined. The algorithm in this research also enables us to compute the complement of uniform single valued neutrosophic graph. In the future, we plan to extended this algorithm for computing the complement of others variants of single valued neutrosophic graphs.

ACKNOWLEDGMENT

The authors are very grateful to the chief editor and reviewers for their comments and suggestions, which is helpful in improving the paper

References

[1] F. Smarandache, Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset: Similarly for Neutrosophic Over-/Under-/Off- Logic, Probability, and Statistics, Pons Editions Brussels, 2016,170p

[2] F. Smarandache, Neutrosophy. Neutrosophic Probability, Set, and Logic, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998; http://fs.gallup.unm.edu/eBook-neutrosophics6.pdf (last edition online) .

[3] H. Wang, F. Smarandache, Y. Zhang, and R. Sunderraman, Single valued Neutrosophic Sets, Multisspace and Multistructure 4, 2010, pp. 410-413.

[4] K. Atanassov, "Intuitionistic fuzzy sets," Fuzzy Sets and Systems, vol. 20, 1986, pp. 87-96.

[5] K. R. S. Narayan and M. S. Sunitha, Some Remarks on Complement of Fuzzy Graphs, IOSR Journal of Mathematics (IOSR-JM), Volume 7, Issue 4 ,2013, pp. 75-77.

[6] K. Bhanu Chander, T. V . Pradeep Kumar, A Note on Uniform Fuzzy Graphs and the Relationship between Regularity and Uniformity of a Fuzzy Graph International Journal of Innovative Research in Science, Engineering and Technology,2017,pp.2425-2431.

[7] L. Zadeh, Fuzzy sets. Inform and Control, 8, 1965, pp.338-353.

[8] M, Vijaya, Complement properties on normal product of strong fuzzy graphs, International Journal of Science and Research, Vol. 4,Issue7, 2015,pp.749-751.

[9] M.Akram, B.Davvaz, Strong intuitionstic fuzzy graph, FILOMAT, vol, 26,2012,pp.177-196.

[10] S. Broumi, A. Bakali, M, Talea, and F, Smarandache, Isolated Single Valued Neutrosophic Graphs. Neutrosophic Sets and Systems, Vol. 11, 2016, pp.74-78.

[11] S. Broumi, F. Smarandache, M. Talea and A. Bakali, An Introduction to Bipolar Single Valued Neutrosophic Graph Theory. Applied Mechanics and Materials, vol.841, 2016, pp.184-191.

[12] S. Broumi, F. Smarandache, M. Talea and A. Bakali, Decision-Making Method Based On the Interval Valued Neutrosophic Graph, Future Technologie, 2016, IEEE, pp 44-50.

[13] S. Broumi, M. Talea, A. Bakali, F. Smarandache, "Single Valued Neutrosophic Graphs," Journal of New Theory, N 10, 2016, pp. 86-101.

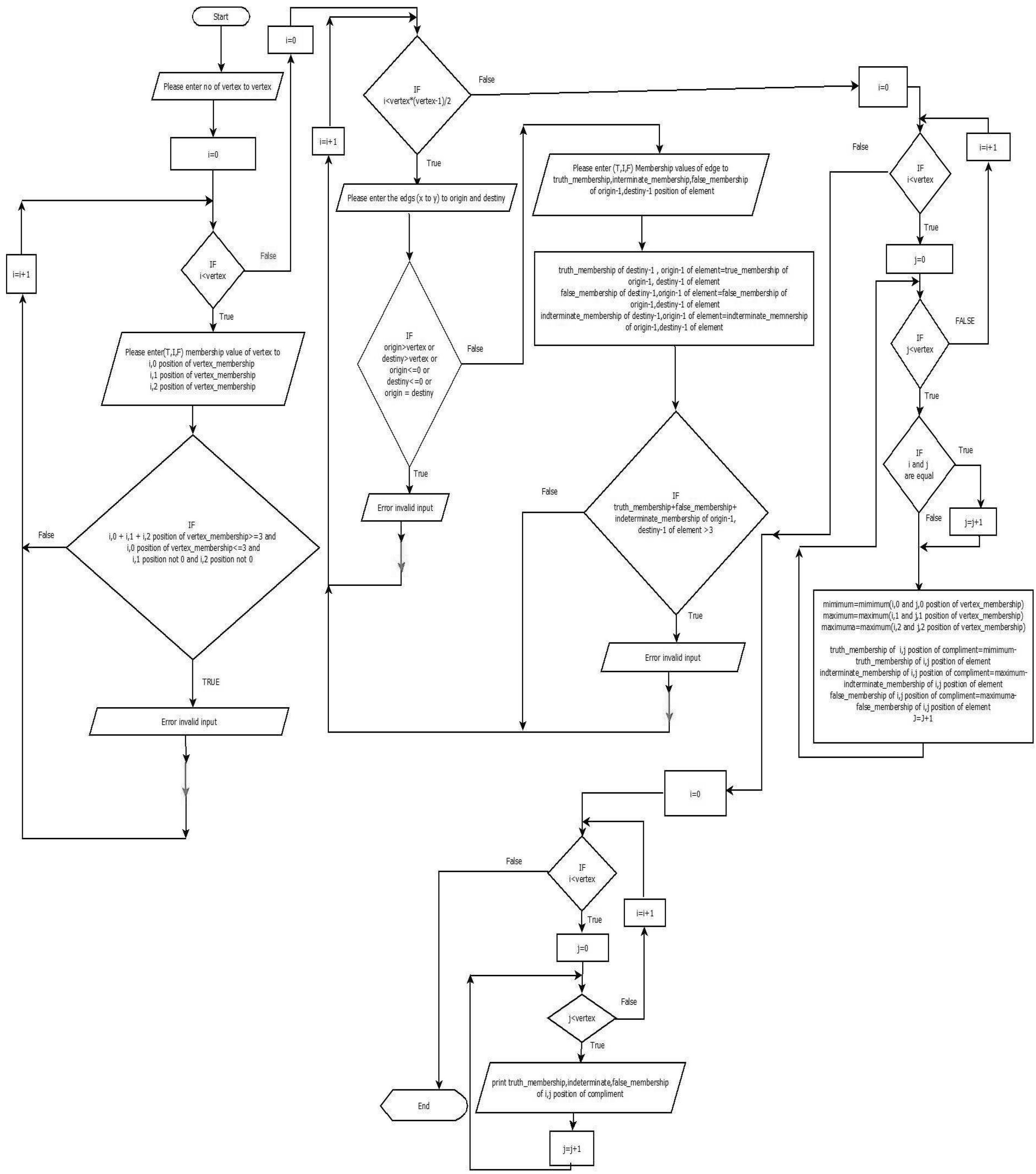
[14] S. Broumi, M. Talea, A. Bakali, F. Smarandache, Interval Valued Neutrosophic Graphs, Critical Review, XII, 2016. pp.5-33.

[15] S. Broumi, M. Talea, F. Smarandache and A. Bakali, Single Valued Neutrosophic Graphs: Degree, Order and Size. IEEE International Conference on Fuzzy Systems (FUZZ), 2016, pp. 2444-2451

[16] More information on http://fs.gallup.unm.edu/NSS/.

Appendix

```
#include<stdio.h>
#include<conio.h>
#define max 100
typedef struct {
float
Truth_membership,Indterminate_membership,False_membership;
}fuzzy;
fuzzy
element[max][max],compliment[max][max];//element store the membership value of vertex.Compliment store the value of complimented graph.
int vertex;//store total number of vertex.
```


Start
Please enter no of vertex to vertex
i=0
IF
i<vertex
False
True
i=i+1
Please enter(T,I,F) membership value of vertex to
i,0 position of vertex_membership
i,1 position of vertex_membership
i,2 position of vertex_membership
IF
i,0 + i,1 + i,2 position of vertex_membership>=3 and
i,0 position of vertex_membership<=3 and
i,1 position not 0 and i,2 position not 0
False
TRUE
Error invalid input
i=0
i=i+1
IF
i<vertex*(vertex-1)/2
False
True
Please enter the edgs (x to y) to origin and destiny
IF
origin>vertex or
destiny>vertex or
origin<=0 or
destiny<=0 or
origin = destiny
False
True
Error invalid input
Please enter (T,I,F) Membership values of edge to
truth_membership,interminate_membership,false_membership
of origin-1,destiny-1 position of element
truth_membership of destiny-1 , origin-1 of element=true_membership of
origin-1, destiny-1 of element
false_membership of destiny-1,origin-1 of element=false_membership of
origin-1,destiny-1 of element
indterminate_membership of destiny-1,origin-1 of element=indterminate_memnership
of origin-1,destiny-1 of element
IF
truth_membership+false_membership+
indeterminate_membership of origin-1,
destiny-1 of element >3
False
True
Error invalid input
i=0
False
IF
i<vertex
True
i=i+1
j=0
IF
j<vertex
FALSE
True
IF
i and j
are equal
True
False
j=j+1
mimimum=mimimum(i,0 and j,0 position of vertex_membership)
maximum=maximum(i,1 and j,1 position of vertex_membership)
maximuma=maximum(i,2 and j,2 position of vertex_membership)
truth_membership of i,j position of compliment=mimimum-
truth_membership of i,j position of element
indterminate_membership of i,j position of compliment=maximum-
indterminate_membership of i,j position of element
false_membership of i,j position of compliment=maximuma-
false_membership of i,j position of element
J=J+1
i=0
IF
i<vertex
False
True
i=i+1
j=0
j<vertex
False
True
print truth_membership,indeterminate,false_membership
of i,j position of compliment
End
j=j+1

```
float vertex_membership[max][6];//store membership value of vertex.
void input()
{
int i,origin,destiny;//origin & destiny store the no. of vertex.And i for iteration.
	printf("Please enter no of vertex:");
	scanf("%d",&vertex);
	for(i=0;i<vertex;i++)
	{
	printf("Please enter (T,I,F)menbership values of vertex:");
	scanf("%f%f%f",&vertex_membership[i][0],&vertex_membership[i][1],&vertex_membership[i][2]);//store the membership value of vertex
	if(vertex_membership[i][0]+vertex_membership[i][1]+vertex_membership[i][2]>=3&&(vertex_membership[i][0]<=3&&vertex_membership[i][1]&&vertex_membership[i][2]))
	{
	printf("Error Invalid input\n");
	i--;
	}
	}
	for(i=0;i<vertex*(vertex-1)/2;i++)
	{
	printf("Please enter the edges (x to y):");
	scanf("%d%d",&origin,&destiny);

	if(origin>vertex||destiny>vertex||origin<=0||destiny<=0||destiny==origin)
	{
	  printf("Error! Invalid input\n");
		i--;
	}
		else
	{
	printf("Please enter (T,I,F)membership values of edge:");
scanf("%f%f%f",&element[origin-1][destiny-1].Truth_membership,&element[origin-1][destiny-1].Indterminate_membership,&element[origin-1][destiny-1].False_membership);//store th membership value of edge.

element[destiny-1][origin
1].Truth_membership=element[origin-1][destiny-1].Truth_membership;//store the truth-membership value of edge.
element[destiny-1][origin-1].Indterminate_membership=element[origin-1][destiny-1].Indterminate_membership;//store the indterminate-membership value of edge.
element[destiny-1][origin-1].False_membership=element[origin-1][destiny-1].False_membership;//store the False-membership value of edge.
if(element[origin-1][destiny-1].Truth_membership+element[origin-1][destiny-1].Indterminate_membership+element[origin-1][destiny-1].False_membership>3)//store the membership value of edge.
{
printf("Error! Invalid input\n");
i--;
}
}
}


}
void output()
{
	int i,j;
	float maximum,minimum,maximuma;
	printf("The complement of Single valued neutrosophic graphs is:\n");
	for(i=0;i<vertex;i++)
{
	for(j=0;j<vertex;j++)
{
	if(i==j)
	j++;
if(vertex_membership[i][0]>vertex_membership[j][0])
minimum=vertex_membership[j][0];//find minimum value between two vertex.
else
minimum=vertex_membership[i][0];//find minimum value between two vertex.
if(vertex_membership[i][1]>vertex_membership[j][1])
maximum=vertex_membership[i][1];//find maximum value between two vertex.
else
maximum=vertex_membership[j][1];//find maximum value between two vertex.
```

```
if(vertex_membership[i][2]>vertex_membership[j][2])
maximuma=vertex_membership[i][2];//find maximum value between two vertex.
else
maximuma=vertex_membership[j][2];//find maximum value between two vertex.
compliment[i][j].Truth_membership=minimum-element[i][j].Truth_membership;//calculating compliment.
compliment[i][j].Indterminate_membership=maximum-element[i][j].Indterminate_membership;//calculating compliment.
compliment[i][j].False_membership=maximuma-element[i][j].False_membership;//calculating compliment.
}
}
        for(i=0;i<vertex-1;i++)
{
        for(j=0;j<vertex;j++)
{
        if(i==j)
        j++;
printf("%d - %d edge membership value= %f %f %f \n",i+1,j+1,compliment[i][j].Truth_membership,compliment[i][j].Indterminate_membership,compliment[i][j].False_membership);//printing complimented graph.
}
}
}
void main()
{
        input();
        output();
        getch();
        }
```

Received: July 20, 2017. Accepted: August 8, 2017.

NSS

University of New Mexico

On New Measures of Uncertainty for Neutrosophic Sets

Pinaki Majumdar

Department of Mathematics, M.U.C Women's College, Burdwan, India-713104, E-Mail: pmajumdar2@rediffmail.com

Abstract: The notion of entropy of single valued neutrosophic sets (SVNS) was first introduced by Majumdar and Samanta in [10]. In this paper some problems with the earlier definition of entropy has been pointed out and a new modified definition of entropy for SVNS has been proposed. Next four new types of entropy functions were defined with examples. Superiority of this new definition over the earlier definition of entropy has been discussed with proper examples.

Keywords: Single valued neutrosophic sets, Neutrosophic element, Neutrosophic cube, Entropy, Entropy function, Intuitionistic fuzzy sets, Measure of uncertainty.

2010 AMS Classification: 03E72, 03E75, 62C86

1. Introduction.

The first successful attempt towards incorporating non-probabilistic uncertainty, i.e. uncertainty which is not caused by randomness of an event, into mathematical modelling was made in 1965 by L. A. Zadeh [20] through his remarkable theory on fuzzy sets (FST). A fuzzy set is a set where each element of the universe belongs to it but with some 'grade' or 'degree of belongingness' which lies between 0 and 1 and such grades are called membership value of an element in that set. This gradation concept is very well suited for applications involving imprecise data such as natural language processing or in artificial intelligence, handwriting and speech recognition etc. Although Fuzzy set theory is very successful in handling uncertainties arising from vagueness or partial belongingness of an element in a set, it cannot model all sorts of uncertainties prevailing in different real physical situations specially problems involving incomplete information. Further generalization of this fuzzy set was made by K. Atanassov [1] in 1986, which is known as Intuitionistic fuzzy set (IFS). In IFS, instead of one 'membership grade', there is also a 'non-membership grade' attached with each element. Furthermore there is a restriction that the sum of these two grades is less or equal to unity. In IFS the 'degree of non-belongingness' is not independent but it is dependent on the 'degree of belongingness'. A fuzzy set can be considered as a special case of IFS where the 'degree of non-belongingness' of an element is exactly equal to 'one minus the degree of belongingness'. Intuitionistic fuzzy sets definitely have the ability to handle imprecise data of both complete and incomplete in nature. In applications like expert systems, belief systems, information fusion etc., where 'degree of non-belongingness' is equally important as 'degree of belongingness', intuitionistic fuzzy sets are quite useful. There are of course several other generalizations of Fuzzy as well as Intuitionistic fuzzy sets like L-fuzzy sets and intuitionistic L-fuzzy sets, interval valued fuzzy and intuitionistic fuzzy sets etc that have been developed and applied in solving many practical physical problems [2, 5, 6, 16]. In 1999, a new theory has been introduced by Florentin Smarandache [14] which is known as 'Neutrosophic logic'. It is a logic in which each proposition is estimated to have a degree of truth (T), a degree of indeterminacy (I) and a degree of falsity (F). A Neutrosophic set is a set where each element of the universe has a degree of truth, indeterminacy and falsity respectively and which lies between $[0^-, 1^+]$, the non-standard unit interval. Unlike in intuitionistic fuzzy sets, where the incorporated uncertainty is dependent on the degree of belongingness and degree of non belongingness, here the uncertainty present, i.e. the indeterminacy factor, is independent of truth and falsity values. Neutrosophic sets are indeed more general in nature than IFS as there are no constraints between the 'degree of truth', 'degree of indeterminacy' and 'degree of falsity'. All these degrees can individually vary within $[0^-, 1^+]$.

Smarandache [14] and Wang et. al. [17] introduced an instance of neutrosophic set known as single valued neutrosophic sets which were motivated from the practical point of view and that can be used in real scientific and engineering applications. Here the degree of truth, indeterminacy and falsity of any element of a neutrosophic set respectively lies within standard unit interval [0, 1]. The single valued neutrosophic set is a generalization of classical set, fuzzy set, intuitionistic fuzzy set and paraconsistant sets etc.

The organization of the rest of this paper is as follows: Some basic definitions and operations on SVNS are given in section 2. Section 3 discusses the notion of entropy of SVNS as defined in [10]. In section 4, some problems with the earlier definition of entropy have been pointed out using counterexample. A new definition of entropy of SVNS has been given in section 5. Section 6 concludes the paper.

2. Single Valued Neutrosophic sets.

A single valued neutrosophic set has been defined in [17] as follows:

Definition 2.1 Let X be a universal set. A Neutrosophic set A in X is characterized by a truth-membership function t_A, a indeterminacy-membership function i_A and a falsity-membership function f_A, $<t_A(x), i_A(x), f_A(x)>$: $X \to [0,1]$, are functions and $< t_A(x), i_A(x), f_A(x)$ is a single valued neutrosophic element or simply a neutrosophic element of A.

A single valued neutrosophic set A (SVNS in short) over a finite universe $X = \{x_1, x_2, x_3, \ldots, x_n\}$ is represented as

$$A = \sum_{i=1}^{n} \frac{x_i}{< t_A(x_i), i_A(x_i), f_A(x_i)}.$$

Example 2.2 Assume that $X = \{x_1, x_2, x_3\}$, where x_1 is capacity, x_2 is trustworthiness and, x_3 is price of a machine, be the universal set. The values of x_1, x_2, x_3 are in [0, 1]. They are obtained from the questionnaire of some domain experts, their option could be a degree of "good service", a degree of indeterminacy and a degree of "poor service". A is a single valued Neutrosophic set of X defined by

$$A = {< 0.3, 0.4, 0.5 >}/{x_1} + {< 0.5, 0.2, 0.3 >}/{x_2} + {< 0.7, 0.2, 0.2 >}/{x_3}.$$

The following is a graphical representation of a single valued neutrosophic set. The elements of a single valued neutrosophic set, denoted henceforth by a neutrosophic element x(t,i,f), always remain inside and on a closed unit cube which henceforth will be called a ***neutrosophic cube***. Figure 1 describes a neutrosophic cube.

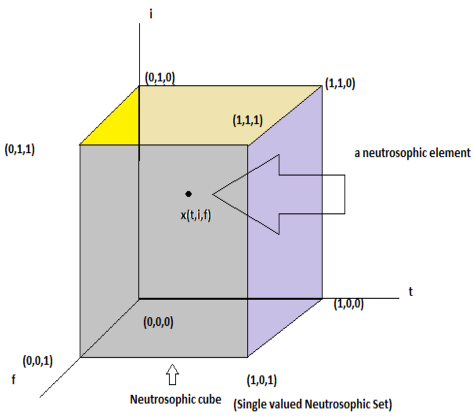


Figure 1

Next we give the definitions of complement and containment as follows:

Definition 2.3 The complement of a SVNS A is denoted by A^c and is defined by

$$t_{A^c}(x) = f_A(x);\ i_{A^c}(x) = 1 - i_A(x)$$
$$\&\ f_{A^c}(x) = t_A(x)\ \forall x \in X.$$

Definition 2.4 A SVNS A is contained in the other SVNS B, denoted as $A \subset B$, if and only if

$$t_A(x) \le t_B(x); i_A(x) \le i_B(x)$$
$$\&\ f_A(x) \ge f_B(x)\ \forall x \in X.$$

Two sets will be equal, i.e. $A = B$, iff $A \subset B \,\&\, B \subset A$.

Let us denote the collection of all SVNS over the universe X as $N(X)$.

Several operations like union and intersection has been defined on SVN sets and they satisfy most of the common algebraic properties of ordinary sets.

Definition 2.5 The union of two SVNS $A\&B$ is a SVNS C, *written as* $C = A \cup B$, which is defined as follows:

$$t_C(x) = \max(t_A(x), t_B(x)); \; i_C(x) = \max(i_A(x), i_B(x))$$
$$\& \; f_C(x) = \min(f_A(x), f_B(x)) \; \forall x \in X.$$

Definition 2.6 The intersection of two SVNS $A\&B$ is a SVNS C, *written as* $C = A \cap B$, which is defined as follows:

$$t_C(x) = \min(t_A(x), t_B(x)); \; i_C(x) = \min(i_A(x), i_B(x))$$
$$\& \; f_C(x) = \max(f_A(x), f_B(x)) \; \forall x \in X.$$

For practical purpose, throughout the rest of this chapter, we have considered only SVNS over a finite universe.

Next two operators, namely 'truth favourite' and 'falsity favourite' are defined to remove indeterminacy in the SVNS and transform it into an IFS or a paraconsistant set.

Definition 2.7 The truth favourite of a SVNS A is again a SVNS B written as $B = \Delta A$, which is defined as follows:

$$T_B(x) = \min(T_A(x) + I_A(x), 1)$$
$$I_B(x) = 0$$
$$F_B(x) = F_A(x), \; \forall x \in X.$$

Definition 2.8 The falsity favourite of a SVNS A is again a SVNS B written as $B = \nabla A$, which is defined as follows:

$$T_B(x) = T_A(x)$$
$$I_B(x) = 0$$
$$F_B(x) = \min(F_A(x) + I_A(x), 1), \; \forall x \in X.$$

The next two examples of truth & falsity favourite respectively of two given SVNS:

Example 2.9 Here the SVNS is A and the truth and falsity favourite sets are defined as follows:

$$A = {<0.3, 0.1, 0.5>}/{x_1} + {<0.5, 0.2, 0.3>}/{x_2} + {<0.4, 0.2, 0.2>}/{x_3}, \; then$$

$$B = \Delta A = {<0.4, 0.0, 0.5>}/{x_1} + {<0.7, 0.0, 0.3>}/{x_2} + {<0.6, 0.0, 0.2>}/{x_3} \; and$$

$$C = \nabla A = {<0.3, 0.0, 0.6>}/{x_1} + {<0.5, 0.0, 0.5>}/{x_2} + {<0.4, 0.0, 0.4>}/{x_3}.$$

Here both B and C are IFS.

Example 2.10 Again consider the neutrosophic set A given in example 2.2, then

$$B = \Delta A = {<0.7, 0.0, 0.5>}/{x_1} + {<0.7, 0.0, 0.3>}/{x_2} + {<0.9, 0.0, 0.2>}/{x_3} \; and$$

$$C = \nabla A = {<0.3, 0.0, 0.9>}/{x_1} + {<0.5, 0.0, 0.5>}/{x_2} + {<0.7, 0.0, 0.4>}/{x_3}.$$

Here both B and C are paraconsistant sets.

3. Entropy of Single Valued Neutrosophic sets.

Entropy can be considered as a measure of uncertainty about the information contained by a set. Generally crisp sets do not possess any entropy because there is no uncertainty about its members. But other non-crisp sets like fuzzy, intuitionistic fuzzy or vague etc, every set contain uncertain information of different types and hence there exits entropy for them. Here the SVNS are also capable of handling uncertain data, therefore as a natural consequence we are also interested in finding the entropy of a single valued neutrosophic set. Shannon [13] first introduced the notion of Probabilistic entropy. Shannon entropy has many applications in theory of communications. Entropy as a measure of fuzziness was first mentioned by Zadeh [21] in 1968. Later De Luca-Termini [4] axiomatized the non-probabilistic entropy.

$(DT1)\, E(A)=0\, iff\ A \in 2^X$
$(DT2) E(A)=1\, iff\ \mu_A(x)=0.5, \forall x \in X$
$(DT3) E(A) \le E(B)\, iff\ A\, is\, less\ fuzzy\, than\, B,$
$i.e.\ if\ \mu_A(x) \le \mu_B(x) \le 0.5\ \forall x \in X.$ (3.1)
$or\ if\ \mu_A(x) \ge \mu_B(x) \ge 0.5, \forall x \in X.$
$(DT4) E(A^c)=E(A).$

Several other authors have investigated the notion of entropy. Kaufmann [7] proposed a distance based measure of fuzzy entropy; Yager [18, 19] gave another view of entropy or the degree of fuzziness of any fuzzy set in terms of lack of distinction between the fuzzy set and its complement. Kosko [8] investigated the fuzzy entropy in relation to a measure of subset hood. Szmidt & Kacprzyk [15] studied the entropy of intuitionistic fuzzy sets etc. Several applications of fuzzy entropy in solving many practical problems like image processing, inventory, economics can be found in literatures [3, 11, 12]. In [9, 10] the notion of entropy of single valued neutrosophic sets was first introduced. The following definition of entropy of a SVNS is due to [10]:

According to them the entropy E of a fuzzy set A should satisfy the following axioms:

Definition 3.3 Here in case of SVNS also we introduce the entropy as a function $E_N : N(X) \to [0,1]$ which satisfies the following axioms:

$(i) E_N(A)=0$ if A is a crisp set

$(ii) E_N(A)=1\, if$
$(t_A(x), i_A(x), f_A(x))=(0.5,0.5,0.5) \forall x \in X$

$(iii) E_N(A) \ge E_N(B)\, if\ A\, is\, more$
$\text{uncertain than } B$

$i.e.\quad t_A(x)+f_A(x) \le t_B(x)+f_B(x)$
$and\ |i_A(x)-i_{A^c}(x)| \le |i_B(x)-i_{B^c}(x)|$ (3.2)

Now notice that in a SVNS the presence of uncertainty is due to two factors, firstly due to the partial belongingness and partial non-belongingness and secondly due to the indeterminacy factor. Considering these two factors, an entropy function E_1 for a single valued neutrosophic sets A was proposed and it is defined as follows:

$$E_1(A)=1\ -$$
$$\frac{1}{n}\sum_{x_i \in X}(t_A(x_i)+f_A(x_i)) \cdot \left|i_A(x_i)-i_{A^c}(x_i)\right| . (3.3)$$

Proposition 3.4 E_1 satisfies all the axioms given in definition 3.3.

Example 3.5 Let $X=\{a,b,c,d\}$ be the universe and A be a single valued neutrosophic set in X defined as follows:

$$A=\{\frac{a}{<0.5,0.2,0.9>},\frac{b}{<0.8,0.4,0.2>},$$
$$\frac{c}{<0.3,0.8,0.7>},\frac{d}{<0.6,0.3,0.5>}\}.$$

Then the entropy of A will be
$E_1(A)=1-0.52=0.48.$

4. Problems with the earlier definition.

In this section we point out some problems with the earlier definition of entropy given in [10].

Problem 4.1: The entropy function E_1 defined in equation 3.3 is not a correct entropy function. Especially it may not lie in [0, 1].

The following example satisfies the claim:

Example 4.2 A counter example:

In the following example we will show that E_1 is not always an entropy function for all SVN sets.

Let $X=\{a,b\}$ be the universe and let $A=\{\frac{a}{(1.0,0.01,1.0)},\frac{b}{(1.0,0.02,1.0)}\}$ be a SVNS, then

$$E_1=1-\frac{1}{2}.(2\times 0.98+2\times 0.96)$$
$$=1-1.94=-0.94<0, \quad \text{which is undesirable.}$$

This definition holds only if $t_A(x)+f_A(x) \le 1$ holds.

The figure 2 shows that actually the half cubic portion ABODEGBA, left of the yellow plane of the 'neutrosophic cubic' where formula E_1 given in equation 3.3 holds. But for the other half cube it may not hold true as described above.

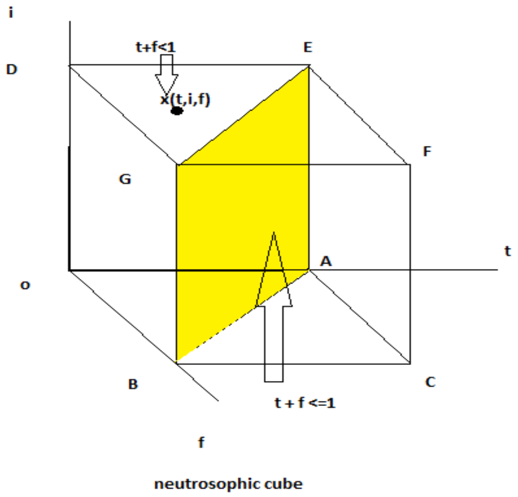


Figure 2

Problem 4.3: In definition 3.1, the most uncertain case is assumed to be (0.5, 0.5, 0.5) which is not necessarily true. Rather (0.5, 1, 0.5) is more uncertain case as here the indeterminacy factor 'i' has the maximum value. Also (1, 1, 1) is far more uncertain than (0.5, 0.5, 0.5).

More generally speaking the area indicated by pink colour in the neutrosophic cube is the place where lies the most uncertain cases. We further assume that the points D,G,F,E,J are the most uncertain neutrosophic elements because there indeterminacy is 1 and truth and falsities are also extreme. No other point in pink region can have higher uncertainty value.

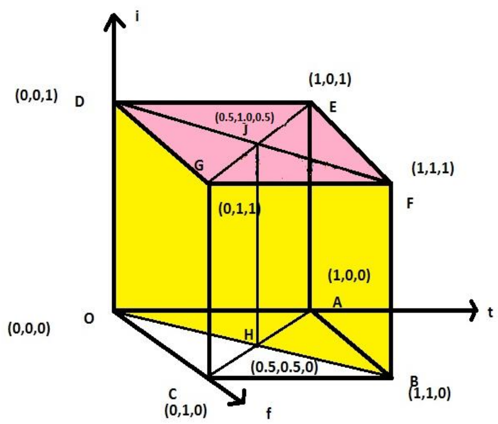


Figure 3

Problem 4.4 In case of neutrosophic sets or single valued neutrosophic sets, the degree of indeterminacy (i) of any neutrosophic element x in its complement set is defined as 1-i. This does not seem to be very reasonable. The degree of indeterminacy in the original set and its complement should be same because both bears the same amount of uncertainty. Also neutrosophic sets are generalizations of intuitionistic fuzzy sets. There the amount of uncertainty for any IF set A is measured as $\pi_A=1-t_A-f_A$. Then $\pi_A{}^c=1- f_A{}^c -t_A{}^c$ and hence $\pi_A=\pi_A{}^c$. Neutrosophic sets are generalizations of IF sets so accordingly here also $i = i^c$ should hold. Therefore we represent a new definition of complement of a SVNS as follows:

Definition 4.5 The complement of a SVNS A is denoted by A^c and is defined by

$$t_{A^c}(x)=f_A(x);\ i_{A^c}(x)=i_A(x)$$
$$\&\ f_{A^c}(x)=t_A(x)\ \forall x\in X.$$

Then A will satisfy involutive law : $(A^c)^c = A$.
Although we have to sacrifice De'Morgans Law in this case.

Considering the above problems, we propose a new modified definition of entropy of single valued neutrosophic sets in the next section.

5. New modified definition of Entropy of Neutrosophic Sets

In this section we present a modified definition of entropy for neutrosophic sets. But before that we have to introduce two new definitions, namely 'intuitionistic uncertainty' and 'more uncertain' SVNS.

Definition 5.1 For any neutrosophic set $A=\sum_{i=1}^{n}\frac{x_i}{<t_A(x_i),i_A(x_i),f_A(x_i)>}$ the Intuitionistic uncertainty of any neutrosophic element $<t_A(x),i_A(x),f_A(x>$ is defined as:

$$\pi^N_A(x)=\frac{1}{2}\times(2-t_A(x)-f\ \ (x)),\ \forall x\in X.$$

Then for the whole SVNS A the intuitionistic uncertainty will be defined as $\pi_A^N = \frac{1}{|X|}\sum_{x_i \in X} \pi_A^N(x_i)$.

Note that intuitionistic uncertainty satisfies the following properties: $(i)\, 0 \le \pi_A^N(x)$, $\pi_A^N \le 1$ *and* $(ii)\, \pi_A^N = \pi_{A^c}^N$.

But for neutrosophic elements of type $<0.5, i_A(x), 0.5>$, $\pi_A^N(x) \ne 1$ which is natural as in SVNS the uncertainty depends on both $\pi_A^N(x)$ *and* $i_A(x)$.

Consider the SVNS given in example 2.2, here $\pi_A^N(x_1) = 0.6,\ \pi_A^N(x_2) = 0.6,\ \pi_A^N(x_3) = 0.55$ *and thus* $\pi_A^N \approx 0.583$.

Example 5.2 Consider the examples 2.9 and 2.10. In the first example $\pi_A^N \approx 0.633$ *but*

$$\pi_{\Delta A}^N = \pi_B^N = \frac{1}{|X|}\sum_{i=1}^{3}\frac{(2-t_i-f_i)}{2} = \frac{1}{3}\{\frac{1.1+1+1.2}{2}\}$$
$$= 0.55 = \pi_{\nabla A}^N = \pi_C^N.$$

In the later example $\pi_A^N \approx 0.583$ *but*

$$\pi_{\Delta A}^N = \pi_B^N = \frac{1}{|X|}\sum_{i=1}^{3}\frac{(2-t_i-f_i)}{2} = \frac{1}{3}\{\frac{.8+1+.9}{2}\}$$
$$= 0.45 = \pi_{\nabla A}^N = \pi_C^N.$$

So we can also see that the Intuitionistic uncertainty of truth favourite and falsity favourite sets of a SVNS are same because the value of $(2-t_i-f_i)$ is same for every element in each set ΔA *and* ∇A, but it's different (uncertainty decreased) with the original SVNS A.

Definition 5.3 A SVNS A is said to be more uncertain than another SVNS B, denoted as $A < B$, if and only if

i.e. if $\pi_A^N(x) \ge \pi_B^N(x)$, or $t_A(x) + f_A(x) \le t_B(x) + f_B(x)$ and $i_A(x) \ge i_B(x),\ \forall x \in X$.

Example 5.4 Consider the following two SVNS's A and B defined as follows:

$$A = {<0.3, 0.8, 0.5>}/{x_1} + {<0.2, 0.7, 0.4>}/{x_2} + {<0.7, 0.6, 0.3>}/{x_3},$$

$$B = {<0.7, 0.6, 0.5>}/{x_1} + {<0.2, 0.5, 0.7>}/{x_2} + {<0.9, 0.3, 0.2>}/{x_3}.$$

Then

A	$t_A + f_A$	i_A
x_1	0.8	0.8
x_2	0.6	0.7
x_3	1.0	0.6

and

B	$t_B + f_B$	i_B
x_1	1.2	0.6
x_2	0.9	0.5
x_3	1.1	0.3

Therefore here A is more uncertain than B,

ie. A < B.

Now we introduce a new definition of Entropy for SVNS:

Definition 5.5 For any SVNS A we define entropy as a function $E_N : N(X) \to [0,1]$ which satisfies the following axioms:

$(1) E_N(A) = 0$ if A is a crisp set

$(2) E_N(A) = 1$ *for* $\forall x, x \in$ *neutrosophic elements* D, E, F, G, J

i.e. if $(t_A(x), i_A(x), f_A(x)) = J = (0.5, 1.0, 0.5) \forall x \in X$, *or* $F = (1,1,1) \forall x \in X$ *or* $D = (0,1,0) \forall x \in X$ *or* $G = (0,1,1) \forall x \in X$ *or* $E = (1,1,0) \forall x \in X$

$(3) E_N(A) \ge E_N(B)$ if A more uncertain than B, *i.e. if* $A < B$.

$(4)\ E_N(A) = E_N(A^c) \forall A \in N(X)$......(3.4)

We can also classify entropy of SVNS's into 4 classes namely type I - IV according to the point for which we get maximum entropy value.

Example 5.6 Considering these two factors we propose an entropy measure E_i of a single valued neutrosophic sets A as follows:

$$(i)\ E(x) = Min\{1, (2 - t_x - f_x).i_x\}\ , x \in A$$

$$and\ E_1(A) = \frac{1}{|X|}\sum_{x\in A} E(x) .... (3.18)$$

$$(ii)\ E(x) = \frac{1}{2}.\{|1 - t_x - f_x| + i_x\}, x \in A$$

$$and\ E_2(A) = \frac{1}{|X|}\sum_{x\in A} E(x) .... (3.19)$$

$$(iii) E(x) = \frac{1}{2}.(2 - t_x - f_x).i_x.)\ x \in A$$

$$and\ E_3(A\ = \frac{1}{|X|}\sum_{x\in A} E(x) .... (3.20)$$

$$(iv)\ E(x) = (2 - t_x - f_x).i_x, x \in A$$

$$and\ E_4(A) = \frac{1}{|X|}\sum_{x\in A} E(x) .... (3.21)$$

Here $E_1(A)$ is an entropy function for any SVNS A of Type I, $E_2(A)$is of Type II, $E_3(A)$is of Type III and $E_4(A)$ is an entropy function for any SVNS A of Type IV respectively.

Example 5.7 Consider the example 2.2. In this case we have

$$E_1(A) = E_4(A) \simeq 0.31, E_2(A) \simeq 0.22, E_3(A) \simeq 0.16.$$

So average entropy is $\frac{1}{4}\sum \ _i(A) = 0.25.$

6. Conclusion:

In this paper we have introduced a new modified definition of entropy of SVNS which is significantly different from earlier definition of entropy for SVNS.

This definition is more logical than the earlier and radically different in nature due to the introduction of new concepts like 'intuitionistic uncertainty' of a SVNS, 'more uncertain SVNs', 'most uncertain SVNs' etc.

Here we have also introduced four different types of entropy functions which are more general in nature and free from the anomalies present in the earlier entropy function.

One can further study the applications of these entropy functions in solving several decision making problems.

Acknowledgement: The author is thankful to the reviewers and editors for their constructive remarks.

References:

1. K. Atanassov, Intuitionistic Fuzzy Sets, Fuzzy Sets and Systems, 20 (1986), 87-96

2. K. Atanassov, S. Stoeva, Intuitionistic L fuzzy sets, Cybernetics and System research, 2(1984), 539-540.

3. C. C. Cheng, K.H. Liao, Parameter optimization based on entropy weight and triangular fuzzy number, International Journal of Engineering and Industries, 2(2) (2011), 62-75

4. A. De Luca, S. Termini (1972), A definition of a non-probabilistic entropy in the setting of fuzzy sets theory, Information and Control, 20 (2011), 301-312.

5. J. A. Goguen, L-fuzzy sets, Journal of Mathematical Analysis and Applications, 18 (1963), 145-174.

6. Y. Jiang, Y. Tang, Q. Chen, H. Liu, Interval-valued intuitionistic fuzzy soft sets and their properties, Computers and Mathematics with Applications, 60(3) (2010), 906-918.

7. A. Kaufmann, Introduction to the theory of Fuzzy Subsets-Vol 1: Fundamental Theoretical Elements, Academic Press, New York (1975).

8. B. Kosoko, Fuzzy entropy and conditioning, Information Sciences, 40 (2) (1986), 165-174

9. P. Majumdar, A Study of Several types of sets expressing uncertainty and some applications on them, Ph.D. Thesis, Visva-Bharati Univ., India (2013).

10. P. Majumdar, S.K. Samanta, On similarity and entropy of Neutrosophic sets, Journal of Intelligent and Fuzzy Systems, 26(2014), 1245-1252

11. N.R. Pal, J.C. Bezdek, measuring fuzzy uncertainty, IEEE Trans. Fuzzy Syst., 2, 2(1994), 107-118.

12. E. Pasha, Fuzzy entropy as cost function in image processing, Proc. of the 2nd IMT- GT regional conf. on math., Stat. and appl., Universiti Sains Malaysia, Penang (2006).

13. C.E. Shannon, A Mathematical theory of communication, Bell Syst. Tech. Jour., 27 (1948), 379-423

14. F. Smarandache, F., A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability,Set and Logic. Rehoboth: American Research Press (1999).

15. E. Szmidt, J. Kacprzyk, Entropy for intuitionistic fuzzy sets, Fuzzy Sets and systems, 118(2001), 467-477.

16. I. Tuskan, Interval valued fuzzy sets based on normal forms, Fuzzy sets and systems, 20(1986), 191-210.

17. H. Wang, F. Smarandache, Zhang, Y., Sunderraman, R., Single valued neutrosophic sets, Proc. Of 10th Int. conf. on Fuzzy Theory & Technology, Salt Lake City,Utah (2005).

18. R.R. Yagar, On the measure of fuzziness and negation, Part I: Membership in the unit interval, International Journal of General Systems, 5(1979), 189-200.

19. R.R. Yagar, On the measure of fuzziness and negation, Part II: Lattices, information and control, 44(1980), 236-260.

20. L.A. Zadeh, Fuzzy Sets, Information and Control, 8(1965), 338-353.

21. L.A. Zadeh, Probability measures of fuzzy events, Jour. Math. Anal. and Appl., 23(1968), 421-427.

22. Abdel-Basset, M., Mohamed, M., & Sangaiah, A. K. (2017). Neutrosophic AHP-Delphi Group decision making model based on trapezoidal neutrosophic numbers. Journal of Ambient Intelligence and Humanized Computing, 1-17.https://doi.org/10.1007/s12652-017-0548-7

23. Abdel-Basset, M., Mohamed, M., Hussien, A. N., & Sangaiah, A. K. (2017). A novel group decision-making model based on triangular neutrosophic numbers. Soft Computing, 1-15. http://doi.org/10.1007/s00500-017-2758-5

24. F. Smarandache, M. Ali, *Neutrosophic Triplet as extension of Matter Plasma, Unmatter Plasma, and Antimatter Plasma*, 69th Annual Gaseous Electronics Conference, Bochum, Germany, October 10-14, 2016, http://meetings.aps.org/Meeting/GEC16/Session/HT6.112

25. F. Smarandache, *Neutrosophic Perspectives: Triplets, Duplets, Multisets, Hybrid Operators, Modal Logic, Hedge Algebras. And Applications*. Pons Editions, Bruxelles, 325 p., 2017.

65. Topal, S. and Öner, T. Contrastings on Textual Entailmentness and Algorithms of Syllogistic Logics, Journal of Mathematical Sciences, Volume 2, Number 4, April 2015, pp 185-189.

Received: July 24, 2017. Accepted: August 11, 2017.

NSS

University of New Mexico

A framework for PEST analysis based on neutrosophic cognitive map: case study in a vertical farming initiative

Wilmer Ortiz Choez [1], Marcos Quinteros Cortázar [2], Shirley Huerta Cruz [3], Katiuska Rubira Carvache[4]

[1]Universidad de Guayaquil, Facultad de Comunicaciòn Social, Guayaquil Ecuador. E-mail: wilmer.ortizc@ug.edu.ec
[2]Universidad Católica de Cuenca, Extensión San Pablo de La Troncal,Cuenca Ecuador. E-mail: mpquinterosc@ucacue.edu.ec
[3]Universidad Metropolitana, Escuela Politécnica del Litoral, Guayaquil Ecuador. E-mail: shuerta@umet.edu.ec
[4]Universidad de Guayaquil, Facultad de Ingeniería Química, Guayaquil Ecuador. E-mail: katiuska.rubirac@ug.edu.ec

Abstract.

Recently, neutrosophic cognitive maps and its application in decision making have become a topic of great importance for researchers and practitioners alike. In this paper, a new model PEST analysis is presented based on neutrosophic cognitive maps static analysis. The proposed framework is composed of five activities, identifying PEST factors and sub-factors, modelling interrelation among PEST factors, calculate centrality measures, factor classification, and factors ranking. A case study developed in environment analysis for a vertical farming project was presented, ranking factor based in interrelation and incorporating indeterminacy in the analysis. Further works will concentrate extending the model for incorporating scenario analysis.

Keywords: PEST, Neutrosophy, Neutrosophic Cognitive Maps, Static Analysis, Vertical Farming.

1. Introduction

PEST (Political, Economic, Social and Technological), is used to assess these four external factors in relation to business situation [1]. When environment and legal factors are included it is name PESTEL (Political, Economic, Socio-cultural, Technological, Environment and Legal) analysis [2]. PEST analysis lacks a quantitative approach to the measurement of interrelation among it factor is generally ignored. Neutrosophic sets and logic is a generalization of fuzzy set and logic based on neutrosophy [3].

Neutrosophy can handle indeterminate and inconsistent information, while fuzzy sets and intuitionistic fuzzy sets cannot describe them appropriately [4].

In this paper a new model PEST analysis based on neutrosophic cognitive maps (NCM) [5] is presented giving methodological support and the possibility of dealing with interdependence, feedback and indeterminacy. This paper is structured as follows: Section 2 reviews some important concepts about PEST analysis framework and NCM. In Section 3, a framework for PEST analysis based on NCM static analysis is presented. Section 4 shows a case study of the proposed model applied to vertical farming project environment analysis. The paper ends with conclusions and further work recommendations.

2. Preliminaries

In this section, we first provide a brief revision PEST analysis and the interdependency of its factors. We then provide a review of the foundations of NCM.

2.1 PESTEL Analysis

PEST (Political, Economic, Social and Technological), analysis is a precondition analysis with the mains function of the identification of the environment within which the company or project operates and providing data and information that will enable the organization predictions of new situations and circumstances [6, 7].

PEST analysis in the original formulation lack a quantitative approach to measurement and the analyzed factors are generally measured and evaluated independently [2].

PEST have a hierarchical structure of objective, factor and sub-factor (Figure 1).

In [2] a proposal from analysis PEST in a multicriteria environment is presented, but only interdependency among factor is analysis.

Additionally, factors and sub-factor have ambiguity, vagueness and indeterminacy in their structure.

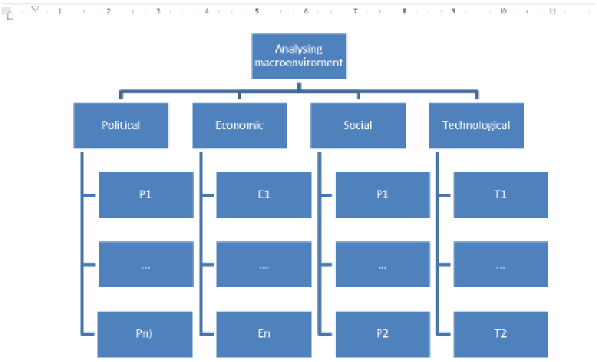


Fig. 1. The hierarchical model of PEST

This study presents a model to address problems encountered in the measurement and evaluation process of PEST analysis taking into account interdependencies among sub-factors. The integrated structure of PESTEL sub-factors were modeled by NCM and quantitative analysis is developed based on static analysis.

2.2 Neutrosophic cognitive maps

Neutrosophic Logic (NL) was introduced in 1995 as a generalization of the fuzzy logic, especially of the intuitionistic fuzzy logic [8]. A logical proposition P is characterized by three neutrosophic components:

$$NL(P) = (T, I, F) \qquad (1)$$

where T is the degree of truth, F the degree of falsehood, and I the degree of indeterminacy.

A neutrosophic matrix is a matrix where the elements $a = (a_{ij})$ have been replaced by elements in $\langle R \cup I \rangle$, where $\langle R \cup I \rangle$ is the neutrosophic integer ring [9].A neutrosophic graph is a graph in which at least one edge is a neutrosophic edge [10]. If indeterminacy is introduced in cognitive mapping it is called Neutrosophic Cognitive Map (NCM) [11].

NCM are based on neutrosophic logic to represent uncertainty and indeterminacy in cognitive maps [3]. A NCM is a directed graph in which at least one edge is an indeterminacy denoted by dotted lines [12] (Figure 2.).

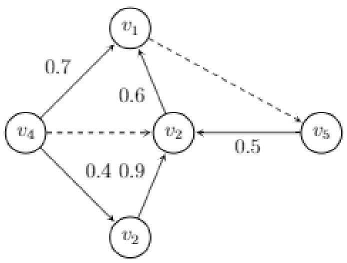


Fig. 2. Fuzzy Neutrosophic Cognitive Maps example.

In [13] a static analysis of mental model in the form of NCM is presented. The result of the static analysis result is in the form of neutrosophic numbers (a+bI, where I = indeterminacy) [14]. Finally a deneutrosophication process as proposes by Salmeron and Smarandache [15] is applied to given the final ranking value. In this paper this model is extended and detailed to deal with factor classification and prioritization.

3. Proposed Framework

Our aim is to develop a framework PEST analysis based on NCM. The model consists of the following phases (graphically, Figure 3).

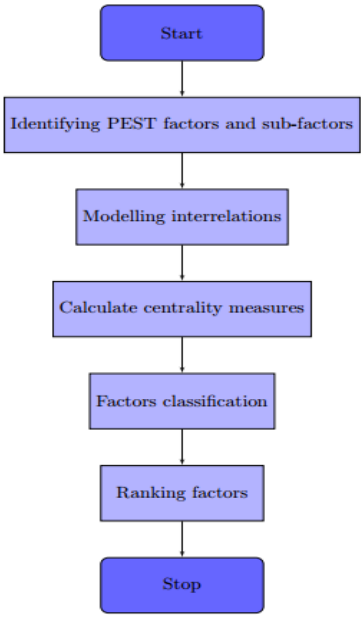


Fig. 3. Proposed framework for PEST analysis.

3.1 Identifying PEST factors and sub-factors

In this step, the relevant PESTEL factors and sub-factors are identified. PESTEL factors are derived from the themes: political, economic, socio-cultural, technological factors. Identifying PEST factors and sub-factors to form a hierarchical structure of PESTEL model (Figure 1.)

3.2 Modelling interdependencies

Causal interdependencies among PEST sub-factors are modelled. This step consists of the formation of NCM of sub-factors, according to the views of the expert team.

3.3 Calculate centrality measures

The following measures are calculated[16] with absolute values of the NCM adjacency matrix [17]:

1. Outdegree $od(v_i)$ is the row sum of absolute values of a variable in the neutrosophic adjacency matrix. It shows the cumulative strengths of connections (c_{ij}) exiting the variable.

$$od(v_i) = \sum_{i=1}^{N} c_{ij} \quad (2)$$

2. Indegree $id(v_i)$ is the column sum of absolute values of a variable. It shows the cumulative strength of variables entering the variable.

$$id(v_i) = \sum_{i=1}^{N} c_{ji} \quad (3)$$

3. The centrality (total degree $td(v_i)$), of a variable is the summation of its indegree (in-arrows) and outdegree (out-arrows)

$$td(v_i) = od(v_i) + id(v_i) \quad (4)$$

3.4 Factors classification

Factors are classified according to the following rules:

a) Transmitter variables have a positive or indeterminacy outdegree, $od(v_i)$ and zero indegree, $id(v_i)$.

b) Receiver variables have a positive indegree or indeterminacy, $id(v_i)$., and zero outdegree, $od(v_i)$.

c) Ordinary variables have both a non-zero indegree and. Ordinary variables can be more or less a receiver or transmitter variables, based on the ratio of their indegrees and outdegrees.

3.5 Ranking Factors

A de-neutrosophication process gives an interval number for centrality. This one is based on max-min values of I . A neutrosophic value is transformed in an interval with two values, the maximum and the minimum value ∈ [0,1] .

The contribution of a variable in a cognitive map can be understood by calculating its degree centrality, which shows how connected the variable is to other variables and what the cumulative strength of these connections are. The median of the extreme values [18] is used to give a centrality value :

$$\lambda([a_1, a_2]) = \frac{a_1 + a_2}{2} \quad (5)$$

Then

$$A > B \Leftrightarrow \frac{a_1 + a_2}{2} > \frac{b_1 + b_2}{2} \quad (6)$$

Finally, a ranking of variables is given. The numerical value it used for factor prioritization and/or reduction [19].

4. Case Study

Environmental concerns, including issues of ecological justice, attention to sustainability, and focus on issues of food security have gathered increased momentum in vertical farming [20]. This case study is based in a vertical farming project proposal at the University of Guayaquil.

In recent years, Guayaquil has become a city of cement with scarcity on green areas [21]. The main goal of the project is the optimization and use of spaces not suitable for cultivation, such as walls and terraces; with systems of supports helping in the beautification of the environment and allow the planting of plants of distinct types obtaining a commercial harmony sustained in the environment.

Initially factors and sub-factors were identified. Figure 3 shows the hierarchical structure.

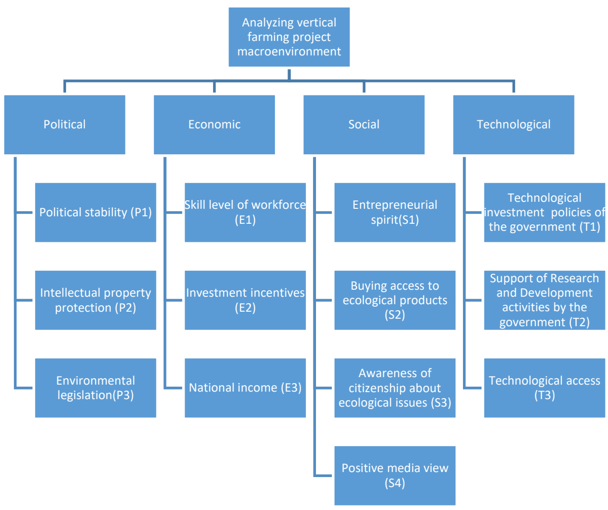


Fig. 4. The hierarchical model of PEST in the vertical farming project.

Interdependencies are identified and modelled using a NCM. NCM with weighs is represented in Table 1.

Table 1. Neutrosophic Adjacency Matrix

	P1	P2	P3	E1	E2	E3	S1	S2	S3	S4	T1	T2	T3
P1	0	0	0	0	0	0	0	0	0	0	0.7	0.6	0
P2	0	0	0	0	0.4	0	0	0	0	0	0	0	I
P3	0	0	0	0	0	0	0	0.3	0	0	0	0	0
E1	0	0	0	0	0	0	0.6	0	0	0	0	0	0
E2	0	0	0	0	0	0.3	0.4	0	0	0	0	0	0
E3	0	0	0	0	0	0	0	0.8	0	0	0	0	0
S1	0	0	0	0	0	0	0	0	0	0	0	0	0
S2	0	0	0	0	0	0	0	0	0	0	0	0	0
S3	0	0	0	0	0	0	0	0	0	0	0	0	0
S4	0	0	0	0	0	0	0	0	0	0	0	0	0
T1	0	0	0	0.2	0	0	0	0	0	0	0	0	0
T2	0	0	0	0	0	I	I	0	0.4	0.5	0	0	0
T3	0	0	0	0	0	0	0.4	0	0	0	0	0	0

The centralities measures are calculates. Outdegree and indegree measures are presentes in Table 2.

Table 2. Centrality measures, outdegree, indegree.

	id	od
P1	1.3	0
P2	0.4+I	0
P3	0.3	0
E1	0.6	0.2
E2	0.7	0.4
E3	0.8	0.3+I
S1	0	1.4+I
S2	0	1.1
S3	0	0.4
S4	0	0.5
T1	0.2	0.7
T2	0.9+2I	0.6
T3	0.4	I

Later nodes are classified. In this case, political nodes are transmitter and social nodes are received. The rest of the nodes are ordinary.

Table 3. Nodes classification

	Transmitter	Receiver	Ordinary
P1	X		
P2	X		
P3	X		
E1			X
E2			X
E3			X
S1		X	
S2		X	
S3			
S4			
T1			X
T2			X
T3			X

Total degree (Eq. 4) was calculated. Results are show in Table 5.

Table 4. Total degree

	td
P1	1.3
P2	0.4+I
P3	0.3
E1	0.8
E2	1.1
E3	1.1+I
S1	1.4+I
S2	1.1
S3	0.4
S4	0.5
T1	0.9
T2	1.5+2I
T3	0.4+I

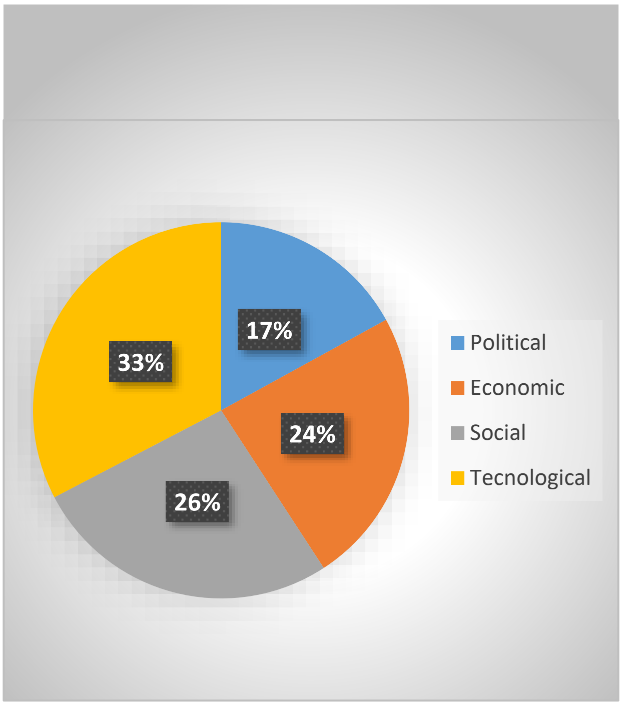
The next step is the de-neutrosophication process as proposes by Salmeron and Smarandache [15]. I ∈[0,1] is replaced by both maximum and minimum values. In Table 6 are presented as interval values.

Table 3. De-neutrosophication, total degree values

	td
P1	1.3
P2	[0.4, 1.4]
P3	0.3
E1	0.8
E2	1.1
E3	[1.1, 2.1]
S1	[1.4, 2.4]
S2	1.1
S3	0.4
S4	0.5
T1	0.9
T2	[1.5, 3.5]
T3	[0.4, 1.4]

Finally we work with the median of the extreme values (Eq 5) [18].

Table 4. Total degree using median of the extreme values



	td
P1	1.3
P2	0,9
P3	0.3
E1	0.8
E2	1.1
E3	1.6
S1	1,9
S2	1.1
S3	0.4
S4	0.5
T1	0.9
T2	2.5
T3	1.4

Graphically the result is shown in Figure 4.

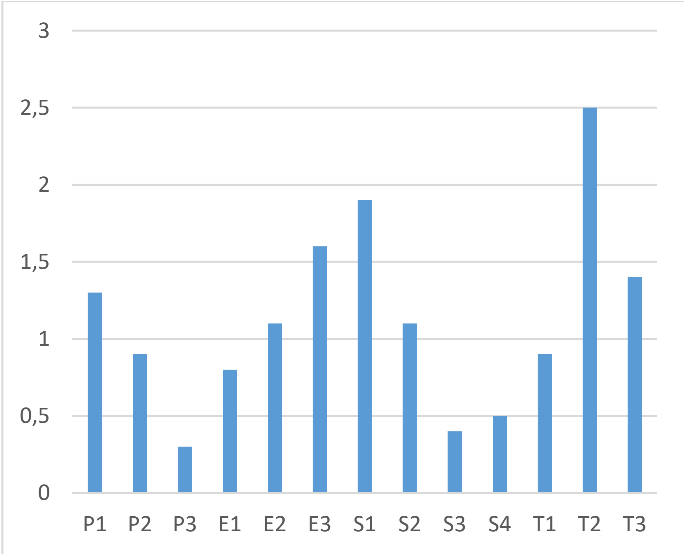


Fig. 5. Total degree measures

The ranking obtained is as follows:

$$\mathbf{T_2 \succ S_1 \succ E_3 \succ T_3 \succ P_1 \succ E_2 \sim S_2 \succ P_2 \sim T_1 \succ E_1 \succ S_4 \succ S_3 \succ P_2}$$

Support of research and development activities by the government was selected as the top environment factor at this vertical farming initiative. Centrality measures of sub factor were grouped according to its factors (Figure 6).

Fig. 6. Aggregated total centrality values by factors

After application in this case study the model is found to be practical to use. The NCM gives a high flexibility and take into account interdependencies PEST analysis

5. Conclusions

This study presents a model to address problems encountered in the measurement and evaluation process of PEST analysis taking into account interdependencies among sub-factors modeling uncertainty and indeterminacy. The integrated structure of PESTEL sub-factors were modeled by NCM and quantitative analysis is developed based on static analysis.

To demonstrate the applicability of the proposal a case study to a vertical farming project proposed at the University of Guayaquil. Most notably, this is the first study to our knowledge to integrate NCM to the PEST analysis Schema. Our approach has many applications in complex decision problem that include interdependencies among criteria, and such as complex agriculture decision support.

Further works will concentrate extending the model for dealing scenario analysis and the use of compensatory operator in static analysis [22] . Another area of future work is the developing a consensus model for NCM and the development of a software tool.

References

1. Healey, N.M., *The transition economic of central and eastern Europe: A political, economic, social and technological analysis.* The Columbia Journal of World Business, 1994. **29**(1): p. 62-70.
2. Yüksel, I., *Developing a multi-criteria decision making model for PESTEL analysis.* International Journal of Business and Management, 2012. **7**(24): p. 52-66.
3. Smarandache, F., *A unifying field in logics: neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability and statistics.* 2005: American Research Press.
4. Akram, M. and A. Luqman, *Intuitionistic single-valued neutrosophic hypergraphs.* OPSEARCH: p. 1-17.
5. Betancourt-Vázquez, A., M. Leyva-Vázquez, and K. Perez-Teruel, *Neutrosophic cognitive maps for modeling project portfolio interdependencies.* Critical Review, 2015. **10**: p. 40-44.
6. Frynas, J.G. and K. Mellahi, *Global strategic management*. 2015: Oxford University Press, USA.
7. Thompson, J.L. and F. Martin, *Strategic management: awareness & change*. 2010: Cengage Learning EMEA.
8. Smarandache, F., *Neutrosophic masses & indeterminate models.* Advances and Applications of DSmT for Information Fusion, 2015: p. 133.
9. Kandasamy, W.V. and F. Smarandache, *Fuzzy Neutrosophic Models for Social Scientists*. 2013: Education Publisher Inc.
10. Kandasamy, W.B.V. and F. Smarandache, *Fuzzy cognitive maps and neutrosophic cognitive maps*. 2003: American Research Press.
11. Kandasamy, W.V. and F. Smarandache, *Analysis of social aspects of migrant labourers living with HIV/AIDS using Fuzzy Theory and Neutrosophic Cognitive Maps*. 2004: American Research Press.
12. Salmeron, J.L. and F. Smarandache, *Processing Uncertainty and Indeterminacy in Information Systems projects success mapping*, in *Computational Modeling in Applied Problems: collected papers on econometrics, operations research, game theory and simulation*. 2006, Hexis. p. 94.
13. Pérez-Teruel, K. and M. Leyva-Vázquez, *Neutrosophic logic for mental model elicitation and analysis.* Neutrosophic Sets and Systems, 2012: p. 31-33.
14. Smarandache, F., *Refined literal indeterminacy and the multiplication law of sub-indeterminacies.* Neutrosophic Sets and Systems, 2015. **9**: p. 58-63.
15. Salmerona, J.L. and F. Smarandacheb, *Redesigning Decision Matrix Method with an indeterminacy-based inference process.* Multispace and Multistructure. Neutrosophic Transdisciplinarity (100 Collected Papers of Sciences), 2010. **4**: p. 151.
16. Lara, R.B., S.G. Espinosa, and M.Y.L. Vázquez, *Análisis estático en mapas cognitivos difusos basado en una medida de centralidad compuesta.* Ciencias de la Información, 2014. **45**(3): p. 31-36.
17. Stach, W., L. Kurgan, and W. Pedrycz, *Expert-based and computational methods for developing fuzzy cognitive maps*, in *Fuzzy Cognitive Maps*. 2010, Springer. p. 23-41.
18. Merigó, J., *New extensions to the OWA operators and its application in decision making*, in *Department of Business Administration, University of Barcelona*. 2008.
19. Altay, A. and G. Kayakutlu, *Fuzzy cognitive mapping in factor elimination: A case study for innovative power and risks.* Procedia Computer Science, 2011. **3**: p. 1111-1119.
20. Besthorn, F.H., *Vertical farming: Social work and sustainable urban agriculture in an age of global food crises.* Australian Social Work, 2013. **66**(2): p. 187-203.
21. Delgado, A., *Guayaquil.* Cities, 2013. **31**: p. 515-532.
22. Leyva-Vázquez, M.Y., R. Bello-Lara, and R.A. Espín-Andrade, *Compensatory fuzzy logic for intelligent social network analysis.* Revista Cubana de Ciencias Informáticas, 2014. **8**(4): p. 74-85.

Received: July 26, 2017. Accepted: August 15, 2017.

NSS

University of New Mexico

An Evidence Fusion Method with Importance Discounting Factors based on Neutrosophic Probability Analysis in DSmT Framework

Qiang Guo [1]*, Haipeng Wang [1], You He [1], Yong Deng [2] and Florentin Smarandache [3]

[1] Institute of Information Fusion Technology Department Naval Aeronautical and Astronautical University Yantai, China
[2] School of Computer and Information Science Sourthwest University, Chongqing, China
[3] Math. & Sciences Dept., University of New Mexico, Gallup, U.S.A.

* Correspondence. E-mail address: gq19860209@163.com; Tel.: ++8615098689289

Abstract:
To obtain effective fusion results of multi source evidences with different importance, an evidence fusion method with importance discounting factors based on neutrosopic probability analysis in DSmT framework is proposed. First, the reasonable evidence sources are selected out based on the statistical analysis of the pignistic probability functions of single focal elements. Secondly, the neutrosophic probability analysis is conducted based on the similarities of the pignistic probability functions from the prior evidence knowledge of the reasonable evidence sources. Thirdly, the importance discounting factors of the reasonable evidence sources are obtained based on the neutrosophic probability analysis and the reliability discounting factors of the real-time evidences are calculated based on probabilistic-based distances. Fourthly, the real-time evidences are discounted by the importance discounting factors and then the evidences with the mass assignments of neutrosophic empty sets are discounted by the reliability discounting factors. Finally, DSmT+PCR5 of importance discounted evidences is applied. Experimental examples show that the decision results based on the proposed fusion method are different from the results based on the existed fusion methods. Simulation experiments of recognition fusion are performed and the superiority of proposed method is testified well by the simulation results.

Keywords: Information fusion; Belief function; Dezert-Smarandache Theory; Neutrosophic probability; Importance discounting factors.

1. Introduction

As a high-level and commonly applicable key technology, information fusion can integrate partial information from multisource, and decrease potential redundant and incompatible information between different sources, thus reducing uncertainties and improving the quick and correct decision ability of high intelligence systems. It has drawn wide attention attention by scholars and has found many successful applications in the military and economy fields in recent years [1-9]. With the increment of information environmental complexity, effective highly conflict evidence reasoning has huge demands on information fusion. Belief function also called evidence theory which includes Dempster- Shafer theory (DST) and Dezert-Smarandache theory (DSmT) has made great efforts and contributions to solve this problem. Dempster-Shafer theory (DST) [10,11] has been commonly applied in information fusion field since it can represent uncertainty and full ignorance effectively and includes Bayesian theory as a special case. Although very attractive, DST has some limitations, especially in dealing with highly conflict evidences fusion [9]. DSmT, jointly proposed by Dezert and Smarandache, can be considered as an extension of DST. DSmT can solve the complex fusion problems beyond the exclusive limit of the DST discernment framework and it can get more reasonable fusion results when multisource evidences are highly conflicting and the refinement of the discernment framework is unavailable. Recently, DSmT has many successful applications in many areas, such as, Map Reconstruction of Robot [12,13], Clustering [14,15], Target Type Tracking [16,17], Image Processing [18], Data Classification [19-21], Decision Making Support [22], Sonar Imagery [23], and so on. Recently the research on the discounting factors based on DST or DSmT have been done by many scholars [24,25]. Smarandache and et al [24] put forward that discounting factors in the procedure of evidence fusion should conclude importance discounting factors and reliability discounting factors, and they also proved that effective fusion could not be carried out by Dempster combination rules when the importance discounting factors were considered. However, the method for calculating the importance discounting factors was not mentioned. A method for calculating importance or reliability discounting factors was proposed in article [25]. However, the importance and reliability discounting factors could not be distinguished and the focal element of empty set or full ignorance was processed based on DST. As the exhaustive limit of DST, it could not process empty set effectively. So, the fusion results based on importance and reliability discounting factors are the same in [25], which is not consist with real situation. In this paper, an evidence fusion method with importance discounting factors based on neutrosophic probability analysis in DSmT framework is proposed. In Section 2, basic theories including DST, DSmT and the dissimilarity measure of evidences are introduced briefly. In Section 3, the contents and procedure of the proposed fusion method are given. In Section 4, simulation experiments in the application background of recognition fusion are also performed for testifying the superiority of proposed method. In Section 5, the conclusions are given.

2. Basic Theories

2.1. DST

Let $\Theta = \{\theta_1, \theta_2, L, \theta_n\}$ be the discernment frame having *n* exhaustive and exclusive hypotheses $\theta_i, i = 1, 2, L, n$. The exhaustive and exclusive limits of DST assume that the refinement of the fusion problem is accessible and the hypotheses are precisely defined. The set of all subsets of Θ, denoted by 2^Θ, is defined as the power set of Θ. 2^Θ is under closed-world assumption. If the discernment frame Θ is defined as above, the power set can be obtained as follows [10,11]:

$$2^\Theta = \{\emptyset, \{\theta_1\}, \{\theta_2\}, L, \{\theta_n\}, \{\theta_1, \theta_2\}, L, \{\theta_1, \theta_2, L, \theta_n\}\}. \quad (1)$$

In Shafer's model, a basic belief assignment (bba) $m(.): 2^\Theta \to [0,1]$ which consists evidences is defined by

$$m_k(\emptyset) = 0 \text{ and } \sum_{A \in 2^\Theta} m(a) = 1. \quad (2)$$

The DST rule of combination (also called the Dempster combination rule) can be considered as a conjunctive normalized rule on the power set 2^Θ. The fusion results based on the Dempster combination rule are obtained by the bba's products of the focal elements from different evidences which intersect to get the focal elements of the results. DST also assumes that the evidences are independent. The i^{th} evidence source's bba is denoted m_i. The Dempster combination rule is given by [10,11]:

$$(m_1 \oplus m_2)(C) = \frac{1}{1-K} \sum_{AI\ B=C} m_1(A) m_2(B), \forall C \subseteq \Theta \quad (3)$$

$$K = \sum_{\substack{A,B \subseteq \Theta \\ AI\ B=\emptyset}} m_1(A) m_2(B) \quad (4)$$

In some applications of multisource evidences fusion, some evidences influenced by the noise or some other conditions are highly conflicting with the other evidences. The reliability of an evidence can represent its accuracy degree of describing the given problem. The reliability discounting factor α in [0, 1] is considered as the quantization of the reliability of an evidence. The reliability discounting method of DST (also called the Shafer's discounting method) is widely accepted and applied. The method consists of two steps. First, the mass assignments of focal elements are multiplied by the reliability discounting factor α. Second, all discounted mass assignments of the evidence are transferred to the focal element of full ignorance Θ. The Shafer's discounting method can be mathematically defined as follows [10,11]

$$\begin{cases} m_\alpha(X) = \alpha \cdot m(X), \text{for } X \neq \Theta \\ m_\alpha(X) = \alpha \cdot m(\Theta) + (1-\alpha) \end{cases} \quad (5)$$

where the reliability discounting factor is denoted by α and $0 \le \alpha \le 1, X$ denotes the focal element which is not the empty set, $m(.)$ denotes the original bba of evidence, $m_\alpha(.)$ denotes the bba after importance discounting.

2.2. DSmT

For many complex fusion problems, the elements can not be separated precisely and the refinement of discernment frame is inaccessible. For dealing with this situation, DSmT [9] which overcomes the exclusive limit of DST, is jointly proposed by Dezert and Smarandache. The hyper-power set in DSmT framework denoted by D^Θ consists of the unions and intersections elements in

Θ. Assume that $\Theta = \{\theta_1, \theta_2\}$, the hyper-power set of Θ can be defined as $D^{\Theta} = \{\emptyset, \theta_1, \theta_2, \theta_3, \theta_1 \cup \theta_2, \theta_1 \cap \theta_2\}$. The bba which consists the body of the evidence in DSmT framework is defined on the hyper-power set as $m(.): D^{\Theta} \rightarrow [0,1]$.

Dezert Smarandache Hybrid (DSmH) combination rule transfers partial conflicting beliefs to the union of the corresponding elements in conflicts which can be considered as partial ignorance or uncertainty. However, the way of transferring the conflicts in DSmH increases the uncertainty of fusion results and it is not convenient for decision-making based on the fusion results. The Proportional Conflict Redistribution (PCR) 1-6 rules overcome the weakness of DSmH and gives a better way of transferring the conflicts in multisource evidence fusion. PCR 1-6 rules proportionally transfer conflicting mass beliefs to the involved elements in the conflicts [9,26,27]. Each PCR rule has its own and different way of proportional redistribution of conflicts and PCR5 rule is considered as the most accurate rule among these PCR rules [9,26,27]. The combination of two independent evidences by PCR5 rule is given as follows [9,26,27]:

$$m_{1\oplus 2}(X_i) = \sum_{\substack{Y,Z\in G^{\Theta} \text{ and } Y,Z\neq\emptyset \\ Y \cap Z = X_i}} m_1(Y)\cdot m_2(Z) \tag{6}$$

$$m_{PCR5}(X_i) = \begin{cases} m_{1\oplus 2} + \sum\limits_{\substack{X_j\in G^{\Theta} \text{ and } i\neq j \\ X_i \cap X_j=\emptyset}} \left[\dfrac{m_1(X_i)^2\cdot m_2(X_j)}{m_1(X_i)+m_2(X_j)} + \dfrac{m_2(X_i)^2\cdot m_1(X_j)}{m_2(X_i)+m_1(X_j)}\right] & X_i\in G^{\Theta} \text{ and } X_i\neq\emptyset \\ 0 & X_i=\emptyset \end{cases}$$

where all denominators are more than zero, otherwise the fraction is discarded, and where G^{Θ} can be regarded as a general power set which is equivalent to the power set 2^{Θ}, the hyper-power set D^{Θ} and the super-power set S^{Θ}, if discernment of the fusion problem satisfies the Shafer's model, the hybrid DSm model, and the minimal refinement Θ^{ref} of Θ respectively [9,26,27].

Although PCR5 rule can get more reasonable fusion results than the combination rule of DST, it still has two disadvantages, first, it is not associative which means that the fusion sequence of multiple (more than 2) sources of evidences can influence the fusion results, second, with the increment of the focal element number in discernment frame, the computational complexity increases exponentially.

It is pointed out in [24] that importances and reliabilities of multisources in evidence fusion are different. The reliability of a source in DSmT framework represents the ability of describing the given problem by its real-time evidence which is the same as the notion in DST framework. The importances of sources in DSmT framework represent the weight that the fusion system designer assigns to the sources. Since the notions of importances and reliabilities of sources make no difference in DST framework, Shafer's discounting method can not be applied to evidence fusion of multisources with unequal importances.

The importance of a source in DSmT framework [24] can be characterized by an importance discounting factor, denoted β in [0,1]. The importance discounting factor β is not related with the reliability discounting factor α which is defined the same as DST framework. β can be any value in [0,1] chosen by the fusion system designer for his or her experience. The main difference of importance discounting method and reliability discounting method lies in the importance discounted mass beliefs of evidences are transferred to the empty set rather than the total ignorance Θ. The importance discounting method in DSmT framework can be mathematically defined as

$$\begin{cases} m_{\beta}(X) = \beta\cdot m(X), \text{for } X\neq\emptyset \\ m_{\beta}(\emptyset) = \beta(\emptyset) + (1-\beta) \end{cases} \tag{7}$$

where the importance discounting factor is denoted by β and $0 \leq \beta \leq 1, X$ denotes the focal element which is not the empty set, $m(.)$ denotes the original bba of evidence, $m_{\beta}(.)$ denotes the bba after importance discounting. The empty set $\emptyset$ of Equation (7) is particular in DSmT discounted framework which is not the representation of unknown elements under the open-world assumption (Smets model), but only the meaning of the discounted importance of a source. Obviously, the importance discounted mass beliefs are transferred to the empty set in DSmT discounted framework which leads to the Dempster combination rule is not suitable to solve this type of fusion problems. The fusion rule with importance discounting factors in DSmT framework for 2 sources is considered as the extension of PCR5 rule, defined as follows [24]:

$$m_{PCR5_{\emptyset}}(A) = \sum_{\substack{X_1,X_2\in G^{\Theta} \\ X_1 \cap X_2=A}} m_1(X_1)m_2(X_2) + \sum_{\substack{X\in G^{\Theta} \\ X \cap A=\emptyset}} \left[\frac{m_1(A)^2\cdot m_2(X)}{m_1(A)+m_2(X)} + \frac{m_2(A)^2\cdot m_1(X)}{m_2(A)+m_1(X)}\right] \tag{8}$$

The fusion rules with importance discounting factors considered as the extension of PCR6 and the fusion rule for multisources $(s > 2)$ as the extension of PCR5 can be seen referred in [24].

3. An Evidence Fusion Method with Importance Discounting Factors Based on Neutrosopic Probability Analysis in DSMT Framework

An evidence fusion method with importance discounting factors based on neutrosophic probability analysis in DSmT framework is proposed in this section. First, the reasonable evidence sources are selected out based on the statistical analysis of the pignistic probability functions of single focal elements. Secondly, the neutrosophic probability analysis is conducted based on the similarities of the pignistic probability functions from the prior evidence knowledge of the reasonable evidence sources. Thirdly, the importance discounting factors of the reasonable evidence sources are obtained based on the neutrosophic probability analysis and the reliability discounting factors of the real-time evidences are calculated based on probabilistic-based distances. Fourthly, the real-time evidences are discounted by the importance discounting factors and then the evidences with the mass assignments of neutrosophic empty sets are discounted by the reliability discounting factors. Finally, DSmT+PCR5 of importance discounted evidences is applied.

3.1. The reasonable evidence sources are selected out

Definition 1: Extraction function for extracting focal elements from the the pignistic probability functions of single focal elements.

$$\chi\big(P(a_i)\big) = a_i, a_i \in \{a_1, a_2, L, a_2\} \quad (11)$$

Definition 2: Reasonable sources.

The evidence sources are defined as reasonable sources if and only if the focal element which has the maximum mean value of the pignistic probability functions of all single focal elements is the element a_j which is known in prior knowledge, denoted by

$$\chi\big(P(\theta)\big) = \max\overline{\big(P(a)\big)} = a_j, 1 \le i \le z \quad (12)$$

where θ represents that the focal element which has the maximum mean value of the pignistic probability functions of all single focal elements.

Based on *Definition 2* and the prior evidence knowledge, reasonable sources are selected out. The unreasonable sources are not suggested to be considered in the following procedure for they are imprecise and unbelievable.

3.2. The neutrosophic probability analysis of the sources and the importance discounting factors in DSmT framework

The neutrosophic probability theory is proposed by Smarandache [30]. In this section, the neutrosophic probability analysis is conducted based on the similarities of the pignistic probability functions from the prior evidence knowledge of the reasonable evidence sources.

Definition 3: Similarity measure of the pignistic probability functions (SMPPF).

Assume that the distribution characteristics of pignistic probability functions of the focal elements $a_i, 1 \le i \le z$ and $a_k, k \ne i, 1 \le k \le z$ are denoted by:

$\boldsymbol{P}(a_i)$: $\{\overline{P(a_i)}, \sigma(a_i)\}$, $\boldsymbol{P}(a_k)$: $\{\overline{P(a_k)}, \sigma(a_k)\}$.

The similarity measure of the pignistic probability functions(SMPPF) is the function satisfying the following conditions:

(1) Symmetry:

$\forall a_i, a_k \in \Theta, Sim\big(\boldsymbol{P}(a_i), \boldsymbol{P}(a_k)\big) = Sim\big(\boldsymbol{P}(a_k), \boldsymbol{P}(a_i)\big)$;

(2) Consistency:

$\forall a_i \in \Theta, Sim\big(\boldsymbol{P}(a_i), \boldsymbol{P}(a_i)\big) = Sim\big(\boldsymbol{P}(a_i), \boldsymbol{P}(a_i)\big) = 1$;

(3) Nonnegativity:

$\forall a_i, a_k \in \Theta, Sim\big(\boldsymbol{P}(a_i), \boldsymbol{P}(a_k)\big) > 0$.

We will say that $\boldsymbol{P}(a_i)$ is more similar to $\boldsymbol{P}(a_k)$ than $\boldsymbol{P}(a_g)$ if and only if:

$Sim\big(\boldsymbol{P}(a_i), \boldsymbol{P}(a_k)\big) > Sim\Big(\boldsymbol{P}(a_i), \boldsymbol{P}(a_g)\Big)$.

The similarity measure of the pignistic probability functions based on the distribution characteristics of the pignistic probability functions is defined as follows:

$$similarity(a_i, a_k) = \exp\left\{-\frac{\left|\overline{P(a_i)}-\overline{P(a_k)}\right|}{2[\sigma(a_i)+\sigma(a_k)]}\right\} \quad (13)$$

Assume that a_j is known in prior knowledge, the diagram for the similarity of the pignistic probability functions of focal elements a_j and a_k which has the largest SMPPF to a_j is shown in *Fig. 1*. $\boldsymbol{P}(a_j)$ is mapped to a circle in which $\overline{P(a_j)}$ is the center and $\sigma(a_j)$ is the radius. Similarly, $\boldsymbol{P}(a_k)$ is mapped to a circle in which $\overline{P(a_k)}$ is the center and $\sigma(a_k)$ is the radius. All the evidences in the prior knowledge from the reasonable source are mapped to the drops in any circle which means that the mapping from drops in the circle of $\boldsymbol{P}(a_j)$ to the prior evidences is one-to-one mapping and similarly the mapping from drops in the circle of $\boldsymbol{P}(a_k)$ to the prior evidences is also one-to-one mapping. If $\boldsymbol{P}(a_j)$ is very similar to $\boldsymbol{P}(a_k)$, the shadow accounts for a large proportion of $\boldsymbol{P}(a_j)$ or $\boldsymbol{P}(a_k)$. If $\boldsymbol{P}(a_j)$ or $\boldsymbol{P}(a_k)$ has the random values in the shadow of the diagram, the evidences of the reasonable source can not totally and correctly support decision-making for there are two possibilities which are $P(a_j) > P(a_k)$ and $P(a_j) \le P(a_k)$. If $P(a_j) \le P(a_k)$ in the evidences, the decisions are wrong. However, if $\boldsymbol{P}(a_j)$ or $\boldsymbol{P}(a_k)$ has the random values in the blank of the diagram, there is only one possibility which is $P(a_j) > P(a_k)$ for the sources are reasonable and the decisions by these evidences are totally correct. So, we define the neutrosophic probability and the absolutely right probability of the reasonable evidence source as probability of $\boldsymbol{P}(a_j)$ in the shadow and blank of the diagram.

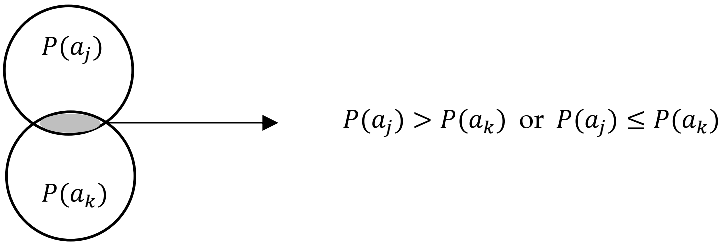


Figure 1. The diagram for the similarity.

Based on the above analysis, the neutrosophic probability and the absolutely right probability of the reasonable evidence source can be obtained by the similarity from the prior evidences for the mapping of the SMPPF of $\boldsymbol{P}(a_j)$ and $\boldsymbol{P}(a_k)$ to the probability of $\boldsymbol{P}(a_j)$ in the shadow is one-to-one mapping.

As $\forall a_i, a_k \in \Theta, 0 < similarity(\boldsymbol{P}(a_i)\ \mathrm{P}(a_k)) \le \boldsymbol{1}$, iff $a_i = a_k, similarity\big(\boldsymbol{P}(a_i)\big)$, we define that the probability of $\boldsymbol{P}\big(a_j\big)$ in the shadow is the same as $similarity(\boldsymbol{P}(a_i)\ \mathrm{P}(a_k))$.

Assume there are reasonable evidence sources for evidence fusion, denoted by $S_k, k = 1,2,\mathrm{L},h$. So, the neutrosophic probability of the the reasonable evidence source in the prior condition that a_j is known can be calculated as follows:

$$P(S_k \text{ is neutral } |a_i) = \max_{1<j<n, j\neq i}[similarity(\boldsymbol{P}(a_i)\ \mathrm{P}(a_k))] \quad (14)$$

Then, the absolutely right probability of the reasonable evidence source in the prior condition that a_j is known can be calculated as follows:

$$(S_k \text{ is absolutely right}|a_i) = 1 - P(S_k \text{ is neutral } |a_i) = 1 - \max_{1<j<n, j\neq i}[similarity(\boldsymbol{P}(a_i)\ \mathrm{P}(a_k))] \quad (15)$$

So, if the prior probability of each focal element can be obtained accurately, the absolutely right probability of the reasonable evidence source can be calculated by the equation

$$P(S_k \text{ is absolutely right}) = \sum\nolimits_{a_i\in\Theta, i=1,2,\mathrm{L},n} P\ (S_k \text{ is absolutely right}|a_i) gP(a_i). \quad (16)$$

If the prior probability of each focal element can not be obtained accurately and any focal element has no advantage in the prior knowledge, denoted by $P(a_1) = P(a_2) = \mathrm{L} = P(a_n)$, the absolutely right probability of the reasonable evidence source can be calculated as follows:

$$P(S_k \text{ is absolutely right}) = \frac{\sum_{a_i\in\Theta, i=1,2,\mathrm{L},n}(S_k \text{ is absolutely right } |a_i)}{n} \quad (17)$$

We define the discounting factors of importances in DSmT framework $\alpha_{SIG}(S_k)$ as the normalization of the absolutely right probabilities of the the reasonable evidence sources P(S_k is right), $k = 1,2,\mathrm{L},h$, denoted by

$$\alpha_{SIG}(S_k) = \frac{P(S_k \text{ is absolutely right})}{\max_{k=1,2,\mathrm{L},h}[P(S_k \text{ is absolutely right})]} \quad (18)$$

3.3. The reliablility discounting factors based on probabilistic-based distances

The Classical Pignistic Transformation(CPT) [9,10,11] is introduced briefly as follows:

$$P(A) = \sum_{X \in 2^{\Theta}} \frac{|X I A|}{|X|} m(X) \tag{19}$$

Based on CPT, if the mass assignments of the single focal elements which consist of the union set of single focal elements are equal divisions of the mass assignment of the union set of single focal elements in two evidences, the pignistic probability of two evidences are equal and the decisions of the two evidences based on CPT are also the same. From the view of decision, it is a good way to measure the similarity of the real-time evidences based on pignistic probability of evidences. Probabilistic distance based on Minkowski's distance [25] is applied in this paper to measure the similarity of real-time evidences. The method for calculating the reliability discounting factors based on Minkowski's distance [25] ($t = 1$) is given as follows.

Assume that there are h evidence sources, denoted by $S_k, k = 1,2, \mathrm{L}, h$, the real-time 2 evidences from S_i and S_j, $i \neq j$ are denoted by $\boldsymbol{m}_i$, $\boldsymbol{m}_j$ the discernment framework of the sources is $\{\theta_1, \theta_2, L, \theta_n\}$, the pignistic probabilities of single focal elements from S_i are denoted by $P_{S_i}(\theta_w), 1 < w < n$ and the pignistic probabilities of single focal elements from S_j are denoted by $P_{S_j}(\theta_w), 1 < w < n$.

1) Minkowski's distance ($t = 1$) between two real-time evidences is calculated as follows:

$$DistP(\boldsymbol{m}_i, \boldsymbol{m}_j) = \frac{1}{2} \sum_{\substack{\theta_w \in \Theta \\ |\theta_w| = 1}} \left| P_{S_i}(\theta_w) - P_{S_j}(\theta_w) \right|. \tag{20}$$

2) The similarity of the real-time evidences is obtained by

$$similary(\boldsymbol{m}_i, \boldsymbol{m}_j) = 1 - DistP(\boldsymbol{m}_i, \boldsymbol{m}_j). \tag{21}$$

3) The similarity matrix of the real-time evidences from $S_k, k = 1,2, \mathrm{L}, h$ is given

$$S = \begin{bmatrix} 1 & similarly(\boldsymbol{m}_1, \boldsymbol{m}_2) & \mathrm{L} & similarly(\boldsymbol{m}_1, \boldsymbol{m}_h) \\ similarly(\boldsymbol{m}_2, \boldsymbol{m}_1) & 1 & \mathrm{L} & similarly(\boldsymbol{m}_2, \boldsymbol{m}_h) \\ \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} \\ similarly(\boldsymbol{m}_h, \boldsymbol{m}_1) & similarly(\boldsymbol{m}_h, \boldsymbol{m}_2) & \mathrm{L} & 1 \end{bmatrix} \tag{22}$$

The average similarity of the real-time evidences from $S_k, k = 1,2, \mathrm{L}, h$ is given

$$\overline{similarly(S_k)} = \frac{\sum_{i=1,2,\mathrm{L},h, i \neq k} similarly(\boldsymbol{m}_i, \boldsymbol{m}_k)}{h-1} \tag{23}$$

4) The reliability discounting factors of the real-time evidences from $S_k, k = 1,2, \mathrm{L}, h$ is given

$$\alpha_{REL}(S_k) = \frac{\overline{similarly(S_k)}}{\max_{k=1,2,\mathrm{L},h} \left[\overline{similarly(S_k)}\right]} \tag{24}$$

3.4. The discounting method with both importance and reliability discounting factors in DSmT framework

1) Discounting evidences based on the discounting factors of importance.

Assume that the real-time evidence from the reasonable evidence source s_k is denoted by:

$\boldsymbol{m}_k = \{m(A), A \subseteq D^{\Theta}\}$, $G^{\Theta} = \{a_1 \mathrm{L}, a_2, a_1 \mathrm{I} \; \mathrm{L} \; \mathrm{I} \; a_2, a_1 \; \mathrm{UL} \; \mathrm{U} a_2\}$.

Based on the discounting factors of importances in DSmT framework $\alpha_{SIG}(s_k)$, the new evidence $\boldsymbol{m}_k^{SIG}$ after importance-discounting by $\alpha_{SIG}(s_k)$ can be calculated by:

$$\boldsymbol{m}_K^{SIG} = \begin{cases} m^{\alpha SIG}(A) = \alpha_{SIG}(S_K) g(m(A)), A \subseteq G^{\Theta} \\ m^{\alpha SIG}(\emptyset) = 1 - \alpha_{SIG}(S_K) \end{cases} \tag{25}$$

where $m^{\alpha SIG}(A)$ are the mass assignments to all focal elements of the original evidence and $m^{\alpha SIG}(\emptyset)$ is the neutrosophic probability of the source, which represents the mass assignment of paradox.

2) Discounting the real-time evidences based on reliability discounting factors after importance discounting.

As the property of the neutrosophic probability of the source, the pignistic probabilities of single focal elements are not changed after importance-discounting the real-time evidences in DSmT framework and the mass assignments of neutrosophic empty focal element Ø which represent the importances degree of sources is added to the new evidences. If some real-time evidence has larger conflict with the other real-time evidences, the evidence should be not reliable and the mass assignments of the focal elements of the evidence should be discounted based on the discounting factors of reliabilities. As one focal element of the new evidence, the mass assignment of neutrosophic empty focal element Ø of the unreliable evidence should also be discounted. So the new discounting method based on the discounting factors of reliabilities after discounting by the discounting factors of importances is given as follows

$$\boldsymbol{m}_K^{SIG} = \begin{cases} m^{\alpha SIG}(A) = \alpha_{REL}(S_k) g \alpha_{SIG}(S_k) g\big(m(A)\big), A \subseteq G^{\Theta} \\ m^{\alpha SIG}(\emptyset) = \alpha_{REL}(S_k) g[1-\alpha_{SIG}(S_k)] \\ m^{\alpha SIG}(\Theta) = 1-\alpha_{REL}(S_k) \end{cases} \quad (26)$$

3.5. The fusion method of PCR5Ø in DSmT framework is applied

After applying the new discounting method to the real-time evidences, the new evidences with the mass assignments of both the neutrosophic empty focal element Ø and the total ignorance focal elements Θ are obtained. The classic Dempster fusion rules can not be sufficient to process these evidences in DSmT framework and PCR5Ø for 2 sources in DSmT framework is applied as our fusion method as follows:

$$m_{PCR5_{\emptyset}}(A) = \sum_{\substack{X_1, X_2 \in G^{\Theta} \\ X_1 \mathrm{I} X_2 = A}} m_1(X_1) m_2(X_2) + \sum_{\substack{X \in G^{\Theta} \\ X \mathrm{I} A = \emptyset}} \left[\frac{m_1(A)^2 \cdot m_2(X)}{m_1(A) + m_2(X)} + \frac{m_2(A)^2 \cdot m_1(X)}{m_2(A) + m_1(X)} \right], A \in G^{\Theta} \text{ or } \emptyset \quad (27)$$

The mass assignment of the neutrosophic empty focal element Ø is included in the fusion results, which is not meaningful to decision. According to the principle of proportion, $m_{PCR5_{\emptyset}}(\emptyset)$ in the fusion result is redistributed to the other focal elements of the fusion result as follows:

$$\begin{aligned} m'_{PCR5_{\emptyset}}(A) &= m_{PCR5_{\emptyset}}(A) + \frac{m_{PCR5_{\emptyset}}(A)}{\sum_{A \in G^{\Theta}} m_{PCR5_{\emptyset}}(A)} \cdot m_{PCR5_{\emptyset}}(\emptyset), A \in G^{\Theta} \\ m'_{PCR5_{\emptyset}}(\emptyset) &= 0 \end{aligned} \quad (28)$$

where $m'_{PCR5_{\emptyset}}(A), A \in G^{\Theta}$ is the final fusion results of our method.

4. Simulation Experiments

The Monto Carlo simulation experiments of recognition fusion are carried out. Through the simulation experiment results comparison of the proposed method and the existed methods, included PCR5 fusion method, the method in [25] and PCR5 fusion method with the reliability discounting factors, the superiority of the proposed method is testified. (In this paper, all the simulation experiments are implemented by Matlab simulation in the hardware condition of Pentimu(R) Dual-Core CPU E5300 2.6GHz 2.59GHz, memory 1.99GB. Abscissas of the figures represent that the ratio of the the standard deviation of Gauss White noise to the maximum standard deviation of the pignistic probabilities of focal elements in prior knowledge of the evidence sources, denoted by 'the ratio of the standard deviation of GWN to the pignistic probabilities of focal elements'.)

Assume that the prior knowledge of the evidence sources is counted as the random distributions of the pignistic probability when different focal element occurs. The prior knowledge is shown in *Tabel 3* and the characteristics of random distributions are denoted by *P*(.): (mean value, variance).

Table 3. Prior knowledge of evidence sources.

Evidence sources	Prior knowledge when *a* occurs	Prior knowledge when *b* occurs
s_1	$P_1(a)$ ~ (0.6,0.3)	$P_1(a)$ ~ (0.46,0.3)
	$P_1(b)$ ~ (0.4,0.3)	$P_1(b)$ ~ (0.54,0.3)
s_2	$P_2(a)$ ~ (0.6,0.3)	$P_2(a)$ ~ (0.4,0.3)
	$P_2(b)$ ~ (0.4,0.3)	$P_2(b)$ ~ (0.6,0.3)
s_3	$P_3(a)$ ~ (0.8,0.05)	$P_3(a)$ ~ (0.2,0.05)
	$P_3(b)$ ~ (0.2,0.05)	$P_3(b)$ ~ (0.8,0.05)

5.1.1 Simulation experiments in the condition that importance discounting factors of most evidence sources are low

Assume that there are 3 evidence sources, denoted by s_1, s_2, s_3, and the discernment framework of the sources is 2 types of targets, denoted by $\{a,b\}$. The prior knowledge is shown in *Table 3*. Assume that the pignistic probabilities of the focal elements are normally distributed. The real-time evidences of 3 sources are random selected out 1000 times based on the prior knowledge in *Table 3* in the condition that a occurs and b occurs respectively. The Moto-carlo simulation experiments of recognition fusion based on the proposed method and the existed methods are carried out. With the increment of the standard deviation of Gauss White noise in the mass assignments of evidences, the fusion results comparisons in different conditions are shown in *Fig. 3* and *Fig. 4*, and the mean value of the correct recognition rates and computation time are show in *Table 11* and *Table 12*.

The fusion results comparisons in the condition that importance discounting factors of most evidence sources are low show that:

1) The method proposed in this paper has the highest correct recognition rates among the existed methods. PCR5 fusion method has the secondly highest correct recognition rates, PCR5 fusion method with reliability discounting factors has the thirdly highest correct recognition rates, the method in [25] has the lowest correct recognition rates.

2) The method proposed in this paper has the largest computation time among the existed methods. the method in [25] has the secondly largest computation time, PCR5 fusion method with reliability discounting factors has the thirdly largest computation time, PCR5 fusion method has the lowest computation time.

Table 11. The mean value of correct recognition rates.

Prior conditions	The proposed method	PCR5 fusion method	The method in [25]	PCR5 fusion method with realibility-discounting factors
a	98.9%	88.6%	80.5%	84.3%
b	98.9%	87.6%	79.0%	82.9%

Table 12. The mean value of computation time.

Prior conditions	The proposed method	PCR5 fusion method	The method in [25]	PCR5 fusion method with realibility-discounting factors
a	1.47×10^{-4}	0.48×10^{-4}	0.88×10^{-4}	0.67×10^{-4}
b	1.46×10^{-4}	0.47×10^{-4}	0.89×10^{-4}	0.66×10^{-4}

Table 13. Prior knowledge of evidence sources.

Evidence sources	Prior knowledge when a occurs	Prior knowledge when b occurs
s_1	$P_1(a)$ ~ (0.6,0.3)	$P_1(a)$ ~ (0.46,0.3)
	$P_1(b)$ ~ (0.4,0.3)	$P_1(b)$ ~ (0.54,0.3)
s_2	$P_2(a)$ ~ (0.8,0.05)	$P_2(a)$ ~ (0.2,0.05)
	$P_2(b)$ ~ (0.2,0.05)	$P_2(b)$ ~ (0.8,0.05)
s_3	$P_3(a)$ ~ (0.8,0.05)	$P_3(a)$ ~ (0.2,0.05)
	$P_3(b)$ ~ (0.2,0.05)	$P_3(b)$ ~ (0.8,0.05)

5.1.2 Simulation experiments in the condition that importance discounting factors of most evidence sources are high

Assume that there are 3 evidence sources, denoted by s_1, s_2, s_3, and the discernment framework of the sources is 2 types of targets, denoted by $\{a,b\}$. The prior knowledge is shown in *Table 13*. Assume that the pignistic probabilities of the focal elements are normally distributed. The Moto-carlo simulation experiments are carried out similarly to the *Section 4.3.1*. With the increment of the standard deviation of Gauss White noise in the mass assignments of evidences, the fusion results comparisons in different conditions are shown in *Fig. 5* and *Fig. 6*, and the mean value of the correct recognition rates and computation time are show in *Table 14* and *Table 15*. The importance factors of the evidences are calculated by Equation (18). The importance factor of s_1 is 0.19, the importance factor of s_2 and s_3 is 1.

Table 14. The mean value of correct recognition rates.

Prior conditions	The proposed method	PCR5 fusion method	The method in [25]	PCR5 fusion method with realibility-discounting factors
a	99.0%	98.8%	99.0%	99.0%
b	99.0%	98.8%	99.0%	99.0%

Table 15. The mean value of computation time.

Prior conditions	The proposed method	PCR5 fusion method	The method in [25]	PCR5 fusion method with realibility-discounting factors
a	1.45×10^{-4}	0.47×10^{-4}	0.86×10^{-4}	0.67×10^{-4}
b	1.46×10^{-4}	0.47×10^{-4}	0.87×10^{-4}	0.65×10^{-4}

The fusion results comparisons in the condition that importance discounting factors of most evidence sources are high show that:

1) The correct recognition rates of four methods are similarly closed, PCR5 fusion method has the lowest correct recognition rates among four methods.

2) The method proposed in this paper has the largest computation time among the existed methods. the method in [25] has the secondly largest computation time, PCR5 fusion method with reliability discounting factors has the thirdly largest computation time, PCR5 fusion method has the lowest computation time.

5. Conclusions

Based on the experiments results, we suggest that the fusion methods should be chosen based on the following conditions:

1) Judge whether the evidences are simple.

2) The importance discounting factors of most evidences are low or not high, the method in this paper is chosen.

The importance discounting factors of most evidences are high, PCR5 fusion method with reliability discounting factors is chosen

.

References

1. Alan J. Terry, Munir Zaman, John Illingworth. Sensor fusion by a novel algorithm for time delay estimation. *Digital Signal Processing*, **2012**, 22(3), 439-452.
2. Yilmaz Kalkan, Buyurman Baykal. Multiple target localization & data association for frequency-only widely separated MIMO radar, *Digital Signal Processing*, **2014**, 25, 51-61.
3. Donka Angelova, Lyudmila Mihaylova. Joint target tracking and classification with particle filtering and mixture Kalman filtering using kinematic radar information, *Digital Signal Processing*, **2006**, 16(2), 180-204.
4. Lauro Snidaro, Ingrid Visentini, Karna Bryan. Fusing uncertain knowledge and evidence for maritime situational awareness via Markov Logic Networks, *Information Fusion*. 21 (**2015**) 159-172.
5. Yi Yang, Deqiang Han, Chongzhao Han. Discounted combination of unreliable evidence using degree of disagreement, *International Journal of Approximate Reasoning*. 54(8) (**2013**) 1197-1216.
6. Ammar Cherchar, Messaoud Thameri, Adel Belouchrani. Performance improvement of direction finding algorithms in non-homogeneous environment through data fusion, *Digital Signal Processing*. 41 (**2015**) 41-47.

7. X. Li, J. Dezert, F. Smarandache, X. Dai. Combination of qualitative information based on 2-Tuple modelings in DSmT, *Journal of Computer Science and Technology*. 24(4) (**2009**) 786-798.
8. X. Li, J. Dezert, F. Smarandache. Fusion of imprecise qualitative information, *Applied Intelligence*. 33(3) (**2010**) 340- 351.
9. F. Smarandache, J. Dezert. *Advances and Applications of DSmT for Information Fusion*, American Research Press, Rehoboth, USA, Vol. 1, Vol. 2, Vol.3, 2004/2006/2009.
10. G. Shafer. *A Mathematical Theory of Evidence*, Princeton Univ. Press, Princeton, NJ, 1976.
11. G. Shafer, Perspectives on the Theory and Practice of Belief Functions, *International Journal of Approximate Reasoning*. 4 (**1990**) 323-362.
12. R. Singh, M. Vatsa, A. Noore. Integrated multilevel image fusion and match score fusion of visible and infrared face images for robust face recognition, *Pattern Recognition*. 41(3) (**2008**) 880-893.
13. X. Li, X. Huang, X. Dai, Z. Meng. Study on environment perception of mobile robots using DSmT-based fusion machine, *Journal of Huazhong University of Science an Technology*. 12(37) (**2009**) 64-67 [Chinese].
14. Z. Liu, Q. Pan, J. Dezert, G. Mercier. Credal c-means clustering method based on belief functions, *Knowledge Based Syst*. 74 (**2015**) 119-132.
15. T. Denoeux. Maximum likelihood estimation from Uncertain Data in the Belief Function Framework, *IEEE Transactions on Knowledge and Data Engineering*, 25(1) (**2013**) 119-130.
16. T. Denoeux, N. El Zoghby, V. Cherfaoui, A. Jouglet. Optimal object association in the Dempster-Shafer framework, *IEEE Transactions on Cybernetics*. 44(22) (**2014**) 2521-2531.
17. F. Faux, F. Luthon, Theory of evidence for face detection and tracking, *International Journal of Approximate Reasoning*. 53(5) (**2012**) 728-746.
18. Z. Liu, J. Dezert, G. Mercier, Q. Pan. Dynamic Evidential Reasoning for Change Detection in Remote Sensing Images, *IEEE T. Geoscience and Remote Sensing*. 50(5) (**2012**) 1955-1967.
19. Z. Liu, Q.Pan, J. Dezert, G. Mercier. Credal classification rule for uncertain data based on belief functions, *Pattern Recognition*. 47(7) (**2014**) 2532-2541.
20. Z. Liu, Q. Pan, G. Mercier, J. Dezert. A New Incomplete Pattern Classification Method Based on Evidential Reasoning, *IEEE T. Cybernetics*. 45(4) (**2015**) 635-646.
21. C. Lian, S. Ruan, T. Denoeux. An evidential classifier based on feature selection and two-step classification strategy, *Pattern Recognition*. 48 (**2015**) 2318-2327.
22. Z. Liu, J. Dezert, Q. Pan, Grégoire Mercier. Combination of sources of evidence with different discounting factors based on a new dissimilarity measure, *Decision Support Systems*. 52(1) (**2011**) 133-141.
23. X. Li, X. Huang, J. Dezert, et al. A successful application of DSmT in sonar grid map building and comparison with DST-based approach, *International Journal of Innovative Computing, Information and Control*. 3(3) (**2007**) 539-551.
24. Smarandache F, Dezert J, Tacnet J.-M. Fusion of sources of evidence with different importances and reliabilities[C], in Proceedings of Fusion 2010 Conference, Edinburgh, UK, July 2010.
25. Liu Zhun-ga, Dezert J, Pan Quan. Mercier Gregoire. Combination of sources of evidence with different discounting factors based on a new dissimilarity measure, *Decision Support System* 52 (**2011**), 133-141.
26. J. Dezert, F. Smarandache. DSmT: A new paradigm shift for information fusion, Cogis06 International Conference, Paris, France, 2006, pp. 15-17.
27. F. Smarandache, J. Dezert. Information Fusion Based on New Proportional Conflict Redistribution Rules, in: Proceedings of Fusion 2005 Conference, Philadelphia, July 26-29, 2005, pp. 907- 914.
28. M.C. Florea, E. Bosse, A.L. Jousselme. Metrics, distances and dissimilarity measures within Dempster–Shafer theory to characterize sources' reliability, Proceeding of Cognitive Systems with Interactive Sensors Conference (COGIS '09), 2009.
29. Anne-Laure Jousselme et al. A new distance between two bodies of evidence, *Information Fusion*. 2(2) (**2001**) 91-101.
30. Florentin Smarandache. *A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set, and Logic*, American Research Press, Rehoboth, 1999.
31. F. Smarandache, *Neutrosophic Perspectives: Triplets, Duplets, Multisets, Hybrid Operators, Modal Logic, Hedge Algebras. And Applications*. Pons Editions, Bruxelles, 325 p., 2017.
32. Topal, S. and Öner, T. Contrastings on Textual Entailmentness and Algorithms of Syllogistic Logics, Journal of Mathematical Sciences, Volume 2, Number 4, April 2015, pp 185-189.
33. Topal S. An Object- Oriented Approach to Counter-Model Constructions in A Fragment of Natural Language, BEU Journal of Science, 4(2), 103-111, 2015.

Received: July 31, 2017. Accepted: August 18, 2017.

NSS
University of New Mexico

More On P-Union and P-Intersection of Neutrosophic Soft Cubic Set

R. Anitha Cruz and F. Nirmala Irudayam

[1] R. Anitha Cruz, Department of Mathematics Nirmala College for Women, Coimbatore, 641018, India, anithacruz@gmail.com
[2] F.Nirmala Irudayam, Assistant Proffesor, Departmentof Mathematics, Nirmala College for Women, Coimbatore, 641018, India, nirmalairudayam@ymail.com

Abstract: The P-union ,P-intersection, P-OR and P-AND of neutrosophic soft cubic sets are introduced and their related properties are investigated. We show that the P-union and the P-intersection of two internal neutrosophic soft cubic sets are also internal neutrosophic soft cubic sets. The conditions for the P-union (P-intersection) of two T-external (resp. I- external, F- external) neutrosophic soft cubic sets to be T-external (resp. I- external, F- external) neutrosophic soft cubic sets is also dealt with. We provide conditions for the P-union (P-intersection) of two T-external (resp. I- external, F- external) neutrosophic soft cubic sets to be T-internal (resp. I- internal,F- internal) neutrosophic soft cubic sets. Further the conditions for the P-union (resp. P-intersection) of two neutrosophic soft cubic sets to be both T-external (resp. I- external, F- external) neutrosophic soft cubic sets and T-external (resp. I- external, F- external) neutrosophic soft cubic sets are also framed.

Keywords: Cubic set, Neutrosophic cubic set, Neutrosophic soft cubic set, T-internal (resp. I- internal,F- internal) neutrosophic soft cubic sets , T-external (resp. I- external, F- external) neutrosophic soft cubic set.

1 Introduction

Florentine Smarandache[10,11] coined neutrosophic sets and neutrosophic logic which extends the concept of the classical sets, fuzzy sets and its extensions. In neutrosophic set, indeterminacy is quantified explicity and truth-membership, indeterminacy-membership and falsity – membership are independent. This assumption is very important in many applications such as information fusion in which we try to combine the data from different sensors. Pabita Kumar Majii[18] had combined the Neutrosophic set with soft sets and introduced a new mathematical model ‘ Nuetrosophic soft set’. Y. B. Jun et al[2]., introduced a new notion, called a cubic set by using a fuzzy set and an interval-valued fuzzy set, and investigated several properties. Jun et al. [19] extended the concept of cubic sets to the neutrosophic cubic sets. [1] introduced neutrosophic soft cubic set and the notion of truth-internal (indeterminacy-internal, falsity-internal) neutrosophic soft cubic sets and truth-external (indeterminacy-internal, falsity-internal) neutrosophic soft cubic sets

As a continuation of the paper [1]We show that the P-union and the P-intersection of T-internal (resp. I-internal,F-internal) neutrosophic soft cubic sets are also T-internal (resp. I-internal,F-internal) neutrosophic soft cubic sets. We also provide conditions for the P-union (P-intersection) of two T-external (resp. I- external,F-external) neutrosophic soft cubic sets to be T-external (resp. I- external,F- external) neutrosophic soft cubic sets.

We provide conditions for the P-union (P-intersection) of two T-external (resp. I- external,F- external) neutrosophic soft cubic sets to be T-internal (resp. I-internal,F- internal) neutrosophic soft cubic sets.

We provide conditions for the P-union (resp. P-intersection) of two NSCS to be both T-external (resp. I-external,F- external) neutrosophic soft cubic sets and T-external (resp. I- external,F- external) neutrosophic soft cubic sets.

2 Preliminaries

2.1 Definition: [5] Let E be a universe. Then a fuzzy set μ over E is defined by X = { μ_x (x) / x: x ϵ E }where μ_x is called membership function of X and defined by $\mu_x : E \to [0,1]$. For each x E, the value $\mu_x(x)$ represents the degree of x belonging to the fuzzy set X.

2.2 Definition: [2] Let X be a non-empty set. By a cubic set, we mean a structure $\Xi = \{\langle x, A(x), \mu(x)\rangle \mid x \in X\}$

in which A is an interval valued fuzzy set (IVF) and μ is a fuzzy set. It is denoted by$\langle A, \mu \rangle$.

2.3 Definition: [9]Let U be an initial universe set and E be a set of parameters. Consider A ⊂ E. Let P(U) denotes the set of all neutrosophic sets of U. The collection (F, A) is termed to be the soft neutrosophic set over U, where F is a mapping given by F : A → P(U).

2.4 Definition : [4] Let X be an universe. Then a neutrosophic (NS) set λ is an object having the form

$\lambda = \{< x : T(x),I(x),F(x) >: x \in X\}$

where the functions T, I, F : X → $]^-0, 1^+[$ defines respectively the degree of Truth, the degree of indeterminacy, and the degree of Falsehood of the element x ∈ X to the set λ with the condition.

$$^{-}0 \leq T(x) + I(x) + F(x) \leq 3^{+}$$

2.5 Definition : [7] Let X be a non-empty set. An interval neutrosophic set (INS) A in X is characterized by the truth-membership function A_T, the indeterminacy-membership function A_I and the falsity-membership function A_F. For each point x ∈ X, $A_T(x), A_I(x), A_F(x) \subseteq [0,1]$.

For two INS

$A = \{<x, [A_T^-(x), A_T^+(x)], [A_I^-(x), A_I^+(x)], [A_F^-(x), A_F^+(x)]>: x \in X\}$

and

$B = \{<x, [B_T^-(x), B_T^+(x)], [B_I^-(x), B_I^+(x)], [B_F^-(x), B_F^+(x)]>: x \in X\}$

Then,

1. $A \tilde{\subseteq} B$ if and only if
   $A_T^-(x) \leq B_T^-(x), A_T^+(x) \leq B_T^+(x)$
   $A_I^-(x) \geq B_I^-(x), A_I^+(x) \geq B_I^+(x)$
   $A_F^-(x) \geq B_F^-(x), A_F^+(x) \geq B_F^+(x)$ for all x∈ X.
2. $A = B$ if and only if
   $A_T^-(x) = B_T^-(x), A_T^+(x) = B_T^+(x)$
   $A_I^-(x) = B_I^-(x), A_I^+(x) = B_I^+(x)$
   $A_F^-(x) = B_F^-(x), A_F^+(x) = B_F^+(x)$ for all x ∈ X.
3. $A^{\tilde{C}} = \{< x, [A_F^-(x), A_F^+(x)], [A_I^-(x), A_I^+(x)], [A_T^-(x), A_T^+(x)] >: x \in X\}$
4. $A \tilde{\cap} B = \{< x, [\min\{A_T^-(x), B_T^-(x)\}, \min\{A_T^+(x), B_T^+(x)\}], [\max\{A_I^-(x), B_I^-(x)\}, \max\{A_I^+(x), B_I^+(x)\}], [\max\{A_F^-(x), B_F^-(x)\}, \max\{A_F^+(x), B_F^+(x)\}] >: x \in X\}$
5. $A \tilde{\bigcup} B = \{< x, [\max\{A_T^-(x), B_T^-(x)\}, \max\{A_T^+(x), B_T^+(x)\}], [\min\{A_I^-(x), B_I^-(x)\}, \min\{A_I^+(x), B_I^+(x)\}], [\min\{A_F^-(x), B_F^-(x)\}, \min\{A_F^+(x), B_F^+(x)\}] >: x \in X\}$

2.6 Definition: [1]

Let U be an initial universe set. Let NC(U) denote the set of all neutrosophic cubic sets and E be the set of parameters. Let $A \subset E$ then $(P, A) = \{P(e_i) = \{<x, A_{e_i}(x), \lambda_{e_i}(x)>: x \in U\} e_i \in A \subset E\}$ where $A_{e_i}(x) = \{<x, A_{e_i}^T(x), A_{e_i}^I(x), A_{e_i}^F(x)> / x \in U\}$ is an interval neutrosophic set, $\lambda_{e_i}(x) = \{<x, (\lambda_{e_i}^T(x), \lambda_{e_i}^I(x), \lambda_{e_i}^T(x)> / x \in U\}$ is a neutrosophic set. The pair (P, A) is termed to be the neutrosophic soft cubic set over U where P is a mapping given by $P: A \to NC(U)$.

2.7 Definition: [1]

Let X be an initial universe set. A neutrosophic soft cubic set (P, A) in X is said to be

- truth-internal (briefly, T-internal) if the following inequality is valid
  $$(\forall x \in X, e_i \in E)\ (A_{e_i}^{-T}(x) \leq \lambda_{e_i}^T(x) \leq A_{e_i}^{+T}(x)), \quad (2.1)$$
- indeterminacy-internal (briefly, I-internal) if the following inequality is valid
  $$(\forall x \in X, e_i \in E)\ (A_{e_i}^{-I}(x) \leq \lambda_{e_i}^I(x) \leq A_{e_i}^{+I}(x)), \quad (2.2)$$
- falsity-internal (briefly, F-internal) if the following inequality is valid
  $$(\forall x \in X, e_i \in E)(A_{e_i}^{-F}(x) \leq \lambda_{e_i}^F(x) \leq A_{e_i}^{+F}(x)). \quad (2.3)$$

If a neutrosophic soft cubic set in X satisfies (2.1), (2.2) and (2.3) we say that (P, A) is an internal neutrosophic soft cubic set in X.

2.8 Definition: [1]

Let X be an initial universe set. A neutrosophic soft cubic set (P, A) in X is said to be

- truth-external (briefly, T-external) if the following inequality is valid
  $$(\forall x \in X, e_i \in E)\ (\lambda_{e_i}^T(x) \notin (A_{e_i}^{-T}(x), A_{e_i}^{+T}(x))), \quad (2.4)$$
- indeterminacy-external (briefly, I-external) if the following inequality is valid
  $$(\forall x \in X, e_i \in E)\ (\lambda_{e_i}^I(x) \notin (A_{e_i}^{-I}(x), A_{e_i}^{+I}(x))), \quad (2.5)$$
- falsity-external (briefly, F-external) if the following inequality is valid
  $$(\forall x \in X, e_i \in E)\ (\lambda_{e_i}^F(x) \notin (A_{e_i}^{-F}(x), A_{e_i}^{+F}(x))). \quad (2.6)$$

If a neutrosophic soft cubic set (P, A)) in *X* satisfies (2.4), (2.5) and (2.6), we say that (P, A) is an external neutrosophic soft cubic set in *X*.

2.9 Definition [1]

Let

$$(P, I) = \{P(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$$

and

$$(Q, J) = \{Q(e_i) = B_i = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J\}$$

be two neutrosophic soft cubic sets in X. Let I and J be any two subsets of E (set of parameters), then we have the following

1. $(P, I) = (Q, J)$ if and only if the following conditions are satisfied
   a) I = J and

b) $P(e_i)=Q(e_i)$ for all $e_i \in I$ if and only if $A_{e_i}(x)=B_{e_i}(x)$ and $\lambda_{e_i}(x)=\mu_{e_i}(x)$ for all $x \in X$ corresponding to each $e_i \in I$.

2. (P,I) and (Q,J) are two neutrosophic soft cubic set then we define and denote P-order as $(P,I) \subseteq_P (Q,J)$ if and only if the following conditions are satisfied
   c) $I \subseteq J$ and
   d) $P(e_i) \leq_P Q(e_i)$ for all $e_i \in I$ if and only if $A_{e_i}(x) \subseteq B_{e_i}(x)$ and $\lambda_{e_i}(x) \leq \mu_{e_i}(x)$ for all $x \in X$ corresponding to each $e_i \in I$.

3. (P,I) and (Q,J) are two neutrosophic soft cubic set then we define and denote P- order as $(P,I) \subseteq_R (Q,J)$ if and only if the following conditions are satisfied
   e) $I \subseteq J$ and
   f) $P(e_i) \leq_R Q(e_i)$ for all $e_i \in I$ if and only if $A_{e_i}(x) \subseteq B_{e_i}(x)$ and $\lambda_{e_i}(x) \geq \mu_{e_i}(x)$ for all $x \in X$ corresponding to each $e_i \in I$.

2.10 Definition: [1]

Let (F,I) and (G,J) be two neutrosophic soft cubic sets (NSCS) in X where I and J are any two subsets of the parameteric set E. Then we define P-union of neutrosophic soft cubic set as $(F,I) \cup_p (G,J) = (H,C)$ where $C = I \cup J$

$$H(e_i) = \begin{Bmatrix} F(e_i) & if\ e_i \in I-J \\ G(e_i) & if\ e_i \in J-I \\ F(e_i) \vee_P G(e_i) & if\ e_i \in I \cap J \end{Bmatrix}$$

where $F(e_i) \vee_P G(e_i)$ is defined as

$F(e_i) \vee_P G(e_i)$ =

$\{<x, \max\{A_{e_i}(x), B_{e_i}(x)\}, (\lambda_{e_i} \vee \mu_{e_i})(x)>: x \in X\}\ e_i \in I \cap J$

where $A_{e_i}(x), B_{e_i}(x)$ represent interval neutrosophic sets.

Hence

$F^T(e_i) \vee_P G^T(e_i)$ =

$\{<x, \max\{A^T_{e_i}(x), B^T_{e_i}(x)\}, (\lambda^T_{e_i} \vee \mu^T_{e_i})(x)>: x \in X\}\ e_i \in I \cap J$,

$F^I(e_i) \vee_P G^I(e_i)$ =

$\{<x, \max\{A^I_{e_i}(x), B^I_{e_i}(x)\}, (\lambda^I_{e_i} \vee \mu^I_{e_i})(x)>: x \in X\}\ e_i \in I \cap J$,

$F^F(e_i) \vee_P G^F(e_i)$ =

$\{<x, \max\{A^F_{e_i}(x), B^F_{e_i}(x)\}, (\lambda^F_{e_i} \vee \mu^F_{e_i})(x)>: x \in X\}\ e_i \in I \cap J$.

2.11 Definition: [1]

Let (F,I) and (G,J) be two neutrosophic soft cubic sets (NSCS) in X where I and J are any subsets of parameter's set E.

Then we define P-intersection of neutrosophic soft cubic set as $(F,I) \cap_p (G,J) = (H,C)$ where $C = I \cap J$,

$H(e_i) = F(e_i) \wedge_P G(e_i)$

$H(e_i) = F(e_i) \wedge_P G(e_i)$ and $e_i \in I \cap J$. Here $F(e_i) \wedge_P G(e_i)$ is defined as

$F(e_i) \wedge_P G(e_i) = H(e_i) =$

$\{<x, \min\{A_{e_i}(x), B_{e_i}(x)\}, (\lambda_{e_i} \wedge \mu_{e_i})(x)>: x \in X\}\ e_i \in I \cap J$

.

where $A_{e_i}(x), B_{e_i}(x)$ represent interval neutrosophic sets.

Hence

$F^T(e_i) \wedge_P G^T(e_i)$ =

$\{<x, \min\{A^T_{e_i}(x), B^T_{e_i}(x)\}, (\lambda^T_{e_i} \wedge \mu^T_{e_i})(x)>: x \in X\}\ e_i \in I \cap J$,

$F^I(e_i) \wedge_P G^I(e_i)$ =

$\{<x, \min\{A^I_{e_i}(x), B^I_{e_i}(x)\}, (\lambda^I_{e_i} \wedge \mu^I_{e_i})(x)>: x \in X\}\ e_i \in I \cap J$,

$F^F(e_i) \wedge_P G^F(e_i)$ =

$\{<x, \min\{A^F_{e_i}(x), B^F_{e_i}(x)\}, (\lambda^F_{e_i} \wedge \mu^F_{e_i})(x)>: x \in X\}\ e_i \in I \cap J$

3 More On P-union And P-intersection Of Neutrosophic Soft Cubic Set

Defintion: 3.1

Let

$(F,I) = \{F(e_i) = \{<x, A_{e_i}(x), \lambda_{e_i}(x)>: x \in X\}\ e_i \in I\}$ and

$(G,J) = \{G(e_i) = \{<x, B_{e_i}(x), \mu_{e_i}(x)>: x \in X\}\ e_i \in J\}$ be neutrososphic soft cubic set (NSCS) in X. Then

[1] P-OR is denoted by $(F,I) \vee_p (G,J)$ and defined as $(F,I) \vee_p (G,J) = (H, I \times J)$ where $H(\alpha_i, \beta_i) = F(\alpha_i) \cup_P G(\beta_i)$ for all $(\alpha_i, \beta_i) \in I \times J$.

[2] P-AND is denoted by $(F,I) \wedge_p (G,J)$ and defined as $(F,I) \wedge_p (G,J) = (H, I \times J)$ where $H(\alpha_i, \beta_i) = F(\alpha_i) \cap_P G(\beta_i)$ for all $(\alpha_i, \beta_i) \in I \times J$.

Example: 3.2

Let X = {x_1, x_2,x_3} be initial universe and E = {e_1, e_2} parameter's set. Let (F,I) be a neutrosophic soft cubic set over X and defined as $(F,I)=\{F(e_i)=\{<x, A_{e_i}(x), \lambda_{e_i}(x)>: x\in X\}\ e_i \in I\}$ and

X	F(e1)		F(e2)	
	<Ae1(x),	λ e1(x) >	<Ae2(x),	λ e2(x) >
x_1	[0.5,0.6][0.6,0.7][0.5,0.6]	[0.4,0.5,0.6]	[0.3,0.6][0.2,0.7][0.2,0.4]	[0.3,0.4,0.4]
x_2	[0.4,0.5][0.7,0.8][0.2,0.3]	[0.5,0.6,0.6]	[0.3,0.5][0.6,0.8][0.2,0.6]	[0.4,0.7,0.5]
x_3	[0.2,0.3][0.2,0.3][0.3,0.5]	[0.3,0.4,0.6]	[0.4,0.7][0.2,0.5][0.3,0.6]	[0.5,0.6,0.6]

$(G,J)=\{G(e_i)=\{<x, B_{e_i}(x), \mu_{e_i}(x)>: x\in X\}\ e_i \in J\}$

X	G(e1)		G(e2)	
	<Be1(x),	μ e1(x) >	< Ae2(x),	μ e2(x) >
x_1	[0.7,0.9][0.3,0.5][0.3,0.4]	[0.7,0.4,0.6]	[0.4,0.7][0.1,0.3][0.1,0.2]	[0.5,0.2,0.2]
x_2	[0.5,0.6][0.3,0.7][0.1,0.2]	[0.6,0.4,0.2]	[0.4,0.6][0.4,0.7][0.2,0.5]	[0.6,0.5,0.4]
x_3	[0.3,0.4][0.1,0.2][0.2,0.4]	[0.5,0.3,0.5]	[0.5,0.8][0.1,0.4][0.1,0.4]	[0.7,0.3,0.4]

P-OR is denoted by $(H, I\times J) = (F,I)\vee_P (G,J)$ where $I\times J=\{(e_1,e_1),(e_1,e_2),(e_2,e_1),(e_2,e_2)\}$ is defined

X	H(e1,e1)		H(e1,e2)		H(e2,e1)		H(e2,e2)	
	F(e1) ∪ G(e1)		F(e1) ∪ G(e2)		F(e2) ∪ G(e1)		F(e2) ∪ G(e1)	
x_1	[0.7,0.9][0.6,0.7][0.5,0.6]	[0.7,0.5,0.6]	[0.5,0.6][0.6,0.7][0.5,0.6]	[0.5,0.5,0.6]	[0.7,0.9][0.3,0.5][0.3,0.4]	[0.7,0.4,0.5]	[0.4,00.7][0.2,0.7][0.2,0.4]	[0.5,0.4,0.4]
x_2	[0.5,0.6][0.7,0.8][0.2,0.3]	[0..6,0.6,0.6]	[0.4,0.6][0.7,0.8][0.2,0.5]	[[0.6,0.6,0.6]	[0.5,0.6][0.6,0.8][0.2,0.6]	[0..6,0.7,0.5]	[0.4,0.6] [0.6,0.8] [0.2,0.6]	[0.6,0.7,0.5]
x_3	[0.3,0.4][0.2,0.3][0.3,0.5]	[0.5,0.4,0.6]	[0.5,0.8][0.2,0.3][0.3,0.5]	[0.7,0.4,0.6]	[0.4,0.7][0.2,05][0.3,06]	[0.5,0.6,0.6]	[0.5,0.8] [0.2,0.5] [0.3,0.6]	[0.7,0.6,0.6]

Definition:3.3

The complement of a neutrosophic soft cubic set $(F,I)=\{F(e_i)=\{<x, A_{e_i}(x), \lambda_{e_i}(x)>: x\in X\}\ e_i \in I\}$ is denoted by $(F,I)^C$ and defined as

$(F,I)^C = \{((F,I)^c = (F^C, \neg I)\}$, where $F^C : \neg I \to NC(X)$ and

$$F^c(e_i) = (F(\neg e_i))^c \quad for\,all \quad e_i \in \neg I$$
$$= (F(e_i))^c \quad (as\ \neg(\neg e_i) = e_i)$$

$(F,I)^C = \{(F(e_i))^C = \{<x, A^C_{e_i}(x), \lambda^C_{e_i}(x)>: x\in X\}\ e_i \in I\}$.

$(F,I)^C =$

$\{<x, \left([1-A^{+T}_{e_i}, 1-A^{-T}_{e_i}],[1-A^{+I}_{e_i}, 1-A^{-I}_{e_i}],[1-A^{+F}_{e_i}, 1-A^{-F}_{e_i}]\right), \left(1-\lambda^T_{e_i}, 1-\lambda^I_{e_i}, 1-\lambda^F_{e_i}\right) > x\in X\}\ e_i \in I$.

Example:3.4

Let X = {x_1, x_2} be initial universe and E = {e_1, e_2} parameter's set. Let (F,I) be a neutrosophic soft cubic set over X and defined as $(F,I)=\{F(e_i)=\{<x, A_{e_i}(x), \lambda_{e_i}(x)>: x\in X\}\ e_i \in I\}$

X	F(e1)		F(e2)	
	< λ e1(x) >	Ae1(x),	< λ e2(x) >	Ae2(x),
x_1	[0.3,0.5][0.1,0.4][0.5,0.8]	[0.6,0.5.0.7]	[0.4,0.6][0.5,0.7][0.6,0.9]	[0.5,0.4,0.4]
x_2	[0.6,0.8][0.4,0.7][0.4,0.7]	[0.7,0.5,0.3]	[0.2,0.4][0.4,0.7][0.3,0.6]	[0.3,0.7,0.8]

Then

$(F,I)^C = \{(F(e_i))^C = \{<x, A^C_{e_i}(x), \lambda^C_{e_i}(x)>: x\in X\}\ e_i \in I\}$

is defined as.

X	F^c (e1)		F^c (e2)	
	< λ^c e1(x) >	A^c e1(x),	< λ^c e1(x) >	A^c e1(x),
x_1	[0.5,0.7][0.6,0.9][0.2,0.5]	[0.4,0.5,0.3]	[0.4,0.6][0.3,0.5][0.1,0.4]	[0.5,0.6,0.6]
x_2	[0.2,0.4][0.3,0.6][[0.3,0.6]	[0.3,0.5,0.7]	[0.6,0.8][0.3,0.6][0.4.0.7]	[0.7,0.3,0.2]

Proposition :3.5

Let X be initial universe and I, J, L and S subsets of parametric set E. Then for any neutrosophic soft cubic sets $\mathcal{A} = (F,I)$, $\mathcal{B} = (G,J)$, $\mathcal{C} = (E,L)$, $\mathcal{D} = (T,S)$ the following properties hold

(1) if $\mathcal{A} \subseteq_P \mathcal{B}$ and $\mathcal{B} \subseteq_P \mathcal{C}$ then $A \subseteq_P \mathcal{C}$.
(2) if $\mathcal{A} \subseteq_P \mathcal{B}$ then $\mathcal{B}^c \subseteq_P \mathcal{A}^c$.
(3) if $\mathcal{A} \subseteq_P \mathcal{B}$ and $\mathcal{A} \subseteq_P C$ then $\mathcal{A} \subseteq_P \mathcal{B} \cap_P \mathcal{C}$.
(4) if $\mathcal{A} \subseteq_P \mathcal{B}$ and $\mathcal{C} \subseteq_P \mathcal{B}$ then $\mathcal{A} \cup_P \mathcal{C} \subseteq_P \mathcal{B}$.
(5) if $\mathcal{A} \subseteq_P \mathcal{B}$ and $\mathcal{C} \subseteq_P \mathcal{D}$ then $\mathcal{A} \cup_P \mathcal{C} \subseteq_P \mathcal{B} \cup_P \mathcal{D}$ and $\mathcal{A} \cap_P \mathcal{C} \subseteq_P \mathcal{B} \cap_P \mathcal{D}$.

Proof: Proof is straight forward

Theorem:3.6 Let (F,I) be a neutrosophic soft cubic set over X.

(1) If (F,I) is an internal neutrosophic soft cubic set, then $(F,I)^c$ is also an internal neutrosophic soft cubic set (INSCS).

(2) If (F,I) is an external neutrosophic soft cubic set, then $(F,I)^c$ is also an external Neutrosophic soft cubic set (ENSCS).

Proof.

(1) Given $(F,I)=\{F(e_i)=\{<x,A_{e_i}(x),\lambda_{e_i}(x)>: x\in X\}\ e_i\in I\}$ is an INSCS this implies

$A_{e_i}^{-T}(x)\le\lambda_{e_i}^{T}(x)\le A_{e_i}^{+T}(x)$,

$A_{e_i}^{-I}(x)\le\lambda_{e_i}^{I}(x)\le A_{e_i}^{+I}(x)$,

$A_{e_i}^{-F}(x)\le\lambda_{e_i}^{F}(x)\le A_{e_i}^{+F}(x)$,

for all $e_i\in I$ and for all $x\in X$.

thisimplies

$1-A_{e_i}^{+T}(x)\le 1-\lambda_{e_i}^{T}(x)\le 1-A_{e_i}^{-T}(x)$,

$1-A_{e_i}^{+I}(x)\le 1-\lambda_{e_i}^{I}(x)\le 1-A_{e_i}^{-I}(x)$,

$1-A_{e_i}^{+F}(x)\le 1-\lambda_{e_i}^{F}(x)\le 1-A_{e_i}^{-F}(x)$

for all $e_i\in I$ and for all $x\in X$.

Hence $(F,I)^c$ is an INSCS .

(2) Given $(F,I)=\{F(e_i)=\{<x,A_{e_i}(x),\lambda_{e_i}(x)>: x\in X\}\ e_i\in I\}$ is an ENSCS this implies

$\lambda_{e_i}^{T}(x)\notin(A_{e_i}^{-T}(x),A_{e_i}^{+T}(x))$,

$\lambda_{e_i}^{I}(x)\notin(A_{e_i}^{-I}(x),A_{e_i}^{+I}(x))$

$\lambda_{e_i}^{F}(x)\notin(A_{e_i}^{-F}(x),A_{e_i}^{+F}(x))$

for all $e_i\in I$ and for all $x\in X$.

Since $\lambda_{e_i}^{T}(x)\notin(A_{e_i}^{-T}(x),A_{e_i}^{+T}(x))$ &

$0\le A_{e_i}^{-T}(x)\le A_{e_i}^{+T}(x)\le 1$,

$\lambda_{e_i}^{I}(x)\notin(A_{e_i}^{-I}(x),A_{e_i}^{+I}(x)$ &

$0\le A_{e_i}^{-I}(x)\le A_{e_i}^{+I}(x)\le 1$,

$\lambda_{e_i}^{F}(x)\notin(A_{e_i}^{-F}(x),A_{e_i}^{+F}(x))$ &

$0\le A_{e_i}^{-F}(x)\le A_{e_i}^{+F}(x)\le 1$

So we have

$\lambda_{e_i}^{T}(x)\le A_{e_i}^{-T}(x)$ or $A_{e_i}^{+T}(x)\le\lambda_{e_i}^{T}(x)$,

$\lambda_{e_i}^{I}(x)\le A_{e_i}^{-I}(x)$ or $A_{e_i}^{+I}(x)\le\lambda_{e_i}^{I}(x)$,

$\lambda_{e_i}^{F}(x)\le A_{e_i}^{-F}(x)$ or $A_{e_i}^{+F}(x)\le\lambda_{e_i}^{F}(x)$

this implies

$1-\lambda_{e_i}^{T}(x)\ge 1-A_{e_i}^{-T}(x)$ or $1-A_{e_i}^{+T}(x)\ge 1-\lambda_{e_i}^{T}(x)$,

$1-\lambda_{e_i}^{I}(x)\ge 1-A_{e_i}^{-I}(x)$ or $1-A_{e_i}^{+I}(x)\ge 1-\lambda_{e_i}^{I}(x)$,

$1-\lambda_{e_i}^{F}(x)\ge 1-A_{e_i}^{-F}(x)$ or $1-A_{e_i}^{+F}(x)\ge 1-\lambda_{e_i}^{F}(x)$,

for all $e_i\in I$ and for all $x\in X$.

Thus $1-\lambda_{e_i}^{T}(x)\notin(1-A_{e_i}^{-T}(x),1-A_{e_i}^{+T}(x))$,

$1-\lambda_{e_i}^{I}(x)\notin(1-A_{e_i}^{-I}(x),1-A_{e_i}^{+I}(x))$,

$1-\lambda_{e_i}^{F}(x)\notin(1-A_{e_i}^{-F}(x),1-A_{e_i}^{+F}(x))$

Hence (F,I) is an ENSCS .

Theorem: 3.7

Let $(F,I)=\{F(e_i)=\{<x,A_{e_i}(x),\lambda_{e_i}(x)>: x\in X\}\ e_i\in I\}$ and $(G,J)=\{G(e_i)=\{<x,B_{e_i}(x),\mu_{e_i}(x)>: x\in X\}\ e_i\in J\}$ be internal nuetrosophic cubic soft sets. Then,

(1) $(F,I)\cup_p(G,J)$ is an INSCS

(2) $(F,I)\cap_p(G,J)$ is an INSCS

Proof:

(1) Since (F,I) and (G,J) are internal neutrosophic soft cubic sets. So for (F,I) we have

$A_{e_i}^{-T}(x)\le\lambda_{e_i}^{T}(x)\le A_{e_i}^{+T}(x)$,

$A_{e_i}^{-I}(x)\le\lambda_{e_i}^{I}(x)\le A_{e_i}^{+I}(x)$, $A_{e_i}^{-F}(x)\le\lambda_{e_i}^{F}(x)\le A_{e_i}^{+F}(x)$

for all $e_i\in I$ and for all $x\in X$.

Also for (G,J) we $B_{e_i}^{-T}(x)\le\mu_{e_i}^{T}(x)\le B_{e_i}^{+T}(x)$,

$B_{e_i}^{-I}(x)\le\mu_{e_i}^{I}(x)\le B_{e_i}^{+I}(x)$, $B_{e_i}^{-F}(x)\le\mu_{e_i}^{F}(x)\le B_{e_i}^{+F}(x)$

for all $e_i\in J$ and for all $x\in X$. Then we have

$\max\{A_{e_i}^{-T}(x),B_{e_i}^{-T}(x)\}\le(\lambda_{e_i}^{T}\vee\mu_{e_i}^{T})(x)\le\max\{A_{e_i}^{+T}(x),B_{e_i}^{+T}(x)\}$,

$\max\{A_{e_i}^{-I}(x),B_{e_i}^{-I}(x)\}\le(\lambda_{e_i}^{I}\vee\mu_{e_i}^{I})(x)\le\max\{A_{e_i}^{+I}(x),B_{e_i}^{+I}(x)\}$,

$\max\{A_{e_i}^{-F}(x),B_{e_i}^{-F}(x)\}\le(\lambda_{e_i}^{F}\vee\mu_{e_i}^{F})(x)\le\max\{A_{e_i}^{+F}(x),B_{e_i}^{+F}(x)\}$,

for all $e_i\in I\cup J$ and for all $x\in X$. .

Now by definition of P-union of (F,I) and (G,J), we have $(F,I)\cup_p(G,J)=(H,C)$ where $I\cup J=C$ and

$$H(e_i)=\begin{Bmatrix} F(e_i) & \text{if } e\in I-J \\ G(e_i) & \text{if } e\in J-I \\ F(e_i)\vee_p G(e_i) & \text{if } e\in I\cap J \end{Bmatrix}$$

if $e_i\in I\cap J$, then $F(e_i)\vee_p G(e_i)$ is defined as $F(e_i)\vee_p G(e_i)=H(e_i)=$

$\{<x,\max\{A_{e_i}(x),B_{e_i}(x)\},(\lambda_{e_i}\vee\mu_{e_i})(x),x\in X,e_i\in I\cap J\}$.

where

$F^T(e_i) \vee_p G^T(e_i) =$
$\{< x, \max\{A^T_{e_i}(x), B^T_{e_i}(x)\}, (\lambda_{e_i} \vee \mu_{e_i})^T(x), x \in X, e_i \in I \cap J\}$,

$F^I(e_i) \vee_p G^I(e_i) =$
$\{< x, \max\{A^I_{e_i}(x), B^I_{e_i}(x)\}, (\lambda_{e_i} \vee \mu_{e_i})^I(x), x \in X, e_i \in I \cap J\}$,

$F^F(e_i) \vee_p G^F(e_i) =$
$\{< x, \max\{A^F_{e_i}(x), B^F_{e_i}(x)\}, (\lambda_{e_i} \vee \mu_{e_i})^F(x), x \in X, e_i \in I \cap J\}$.

Thus $(F, I) \cup_p (G, J)$ is an INSCS if $e_i \in I \cap J$. If $e_i \in I - J$ or $e_i \in J - I$ then the result is trivial. Hence $(F, I) \cup_p (G, J)$ is an INSCS in all cases.

(2) Since $(F, I) \cap_p (G, J) = (H, C)$ where $I \cap J = C$ and $H(e_i) = F(e_i) \wedge_p G(e_i)$. If $e_i \in I \cap J$ then $F(e_i) \wedge_p G(e_i)$ is defined as
$H(e_i) = F(e_i) \wedge_p G(e_i) =$
$\{< x, \min\{A_{e_i}(x), B_{e_i}(x)\}, (\lambda_{e_i} \wedge \mu_{e_i})(x) > x \in X, e \in I \cap J\}$. Also given that (F, I) and (G, J) are INSCS.

So far we have

$A^{-T}_{e_i}(x) \le \lambda^T_{e_i}(x) \le A^{+T}_{e_i}(x)$, $A^{-I}_{e_i}(x) \le \lambda^I_{e_i}(x) \le A^{+I}_{e_i}(x)$,
$A^{-F}_{e_i}(x) \le \lambda^F_{e_i}(x) \le A^{+F}_{e_i}(x)$

for all $e_i \in I$ and for all $x \in X$.

And for (G, J) we have $B^{-T}_{e_i}(x) \le \mu^T_{e_i}(x) \le B^{+T}_{e_i}(x)$,
$B^{-I}_{e_i}(x) \le \mu^I_{e_i}(x) \le B^{+I}_{e_i}(x)$, $B^{-F}_{e_i}(x) \le \mu^F_{e_i}(x) \le B^{+F}_{e_i}(x)$

for all $e_i \in J$ and for all $x \in X$.

$\min\{A^{-T}_{e_i}(x), B^{-T}_{e_i}(x)\} \le (\lambda^T_{e_i} \wedge \mu^T_{e_i})(x) \le \min\{A^{+T}_{e_i}(x), B^{+T}_{e_i}(x)\}$,

$\min\{A^{-I}_{e_i}(x), B^{-I}_{e_i}(x)\} \le (\lambda^I_{e_i} \wedge \mu^I_{e_i})(x) \le \min\{A^{+I}_{e_i}(x), B^{+I}_{e_i}(x)\}$

$\min\{A^{-F}_{e_i}(x), B^{-F}_{e_i}(x)\} \le (\lambda^F_{e_i} \wedge \mu^F_{e_i})(x) \le \min\{A^{+F}_{e_i}(x), B^{+F}_{e_i}(x)\}$

for all $e_i \in I \cap J$ and for all $x \in X$.

Hence $(F, I) \cap_p (G, J)$ is an INSCS .

Definition: 3.8

Given two neutrosophic soft cubic sets (NSCS) $(F, I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and $(G, J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J\}$, if we interchange λ and μ,

Then the new neutrosophic soft cubic set (NSCS) are denoted and defined as

$(F, I)^* = \{F(e_i) = \{< x, A_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and
$(G, J)^* = \{G(e_i) = \{< x, B_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in J\}$ respectively.

Theorem 3.9

For two ENSCSs $(F, I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and $(G, J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J\}$ in X , if $(F, I)^*$ and $(G, J)^*$ are INSCS in X then $(F, I) \cup_P (G, J)$ is an INSCS in X.

Proof:

Since $(F, I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and $(G, J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in J\}$ are ENSCS.

Then for (F, I) we have $\lambda^T_{e_i}(x) \notin (A^{-T}_{e_i}(x), A^{+T}_{e_i}(x))$,
$\lambda^I_{e_i}(x) \notin (A^{-I}_{e_i}(x), A^{+I}_{e_i}(x))$, $\lambda^F_{e_i}(x) \notin (A^{-F}_{e_i}(x), A^{+F}_{e_i}(x))$
for all $e_i \in I$ and for all $x \in X$ and (G, J) we have
$\mu^T_{e_i}(x) \notin (B^{-T}_{e_i}(x), B^{+T}_{e_i}(x))$, $\mu^I_{e_i}(x) \notin (B^{-I}_{e_i}(x), B^{+I}_{e_i}(x))$,
$\mu^F_{e_i}(x) \notin (B^{-F}_{e_i}(x), B^{+F}_{e_i}(x))$
for all $e_i \in J$ and for all $x \in X$. Also given that $(F, I)^* = \{F(e_i) = \{< x, A_{e_i}(x), \mu_{e_i}(x) >: x \in X\}\ e_i \in I\}$ and $(G, J)^* = \{G(e_i) = \{< x, B_{e_i}(x), \lambda_{e_i}(x) >: x \in X\}\ e_i \in J\}$ are INSCS so this implies $A^{-T}_{e_i}(x) \le \mu^T_{e_i}(x) \le A^{+T}_{e_i}(x)$,
$A^{-I}_{e_i}(x) \le \mu^I_{e_i}(x) \le A^{+I}_{e_i}(x)$, $A^{-F}_{e_i}(x) \le \mu^F_{e_i}(x) \le A^{+F}_{e_i}(x)$
for all $e_i \in I$ and for all $x \in X$. And
$B^{-T}_{e_i}(x) \le \lambda^T_{e_i}(x) \le B^{+T}_{e_i}(x)$
$B^{-I}_{e_i}(x) \le \lambda^I_{e_i}(x) \le B^{+I}_{e_i}(x)$, $B^{-F}_{e_i}(x) \le \lambda^F_{e_i}(x) \le B^{+F}_{e_i}(x)$.
for all $e_i \in J$ and for all $x \in X$. Since (F, I) and (G, J) are ENSCS and $(F, I)^*$ and $(G, J)^*$ are INSCS. Thus by definition of ENSCS and INSCS all the possibilities are under

1) (a1) $\lambda^T_{e_i}(x) \le A^{-T}_{e_i}(x) \le \mu^T_{e_i}(x) \le A^{+T}_{e_i}(x)$

(a2) $\lambda^{+I}_{e_i}(x) \le A^{-I}_{e_i}(x) \le \mu^I_{e_i}(x) \le A^{+I}_{e_i}(x)$

(a3) $\lambda^F_{e_i}(x) \le A^{-F}_{e_i}(x) \le \mu^F_{e_i}(x) \le A^{+F}_{e_i}(x)$

(b1) $\mu^T_{e_i}(x) \le B^{-T}_{e_i}(x) \le \lambda^T_{e_i}(x) \le B^{+T}_{e_i}(x)$

(b2) $\mu^{I}_{e_i}(x)\le B^{-I}_{e_i}(x)\le \lambda^{I}_{e_i}(x)\le B^{+I}_{e_i}(x)$

(b3) $\mu^{F}_{e_i}(x)\le B^{-F}_{e_i}(x)\le \lambda^{F}_{e_i}(x)\le B^{+F}_{e_i}(x)$

2) (a1) $A^{-T}_{e_i}(x)\le \mu^{T}_{e_i}(x)\le A^{+T}_{e_i}(x)\le \lambda^{T}_{e_i}(x)$

(a2) $A^{-I}_{e_i}(x)\le \mu^{I}_{e_i}(x)\le A^{+I}_{e_i}(x)\le \lambda^{I}_{e_i}(x)$

(a3) $A^{-F}_{e_i}(x)\le \mu^{F}_{e_i}(x)\le A^{+F}_{e_i}(x)\le \lambda^{F}_{e_i}(x)$

(b1) $B^{-T}_{e_i}(x)\le \lambda^{T}_{e_i}(x)\le B^{+T}_{e_i}(x)\le \mu^{T}_{e_i}(x)$

(b2) $B^{-I}_{e_i}(x)\le \lambda^{I}_{e_i}(x)\le B^{+I}_{e_i}(x)\le \mu^{I}_{e_i}(x)$

(b3) $B^{-F}_{e_i}(x)\le \lambda^{F}_{e_i}(x)\le B^{+F}_{e_i}(x)\le \mu^{F}_{e_i}(x)$

3) (a1) $\lambda^{T}_{e_i}(x)\le A^{-T}_{e_i}(x)\le \mu^{T}_{e_i}(x)\le A^{+T}_{e_i}(x)$

(a2) $\lambda^{I}_{e_i}(x)\le A^{-I}_{e_i}(x)\le \mu^{I}_{e_i}(x)\le A^{+I}_{e_i}(x)$

(a3) $\lambda^{F}_{e_i}(x)\le A^{-F}_{e_i}(x)\le \mu^{F}_{e_i}(x)\le A^{+F}_{e_i}(x)$

(b1) $B^{-T}_{e_i}(x)\le \lambda^{T}_{e_i}(x)\le B^{+T}_{e_i}(x)\le \mu^{T}_{e_i}(x)$

(b2) $B^{-I}_{e_i}(x)\le \lambda^{I}_{e_i}(x)\le B^{+I}_{e_i}(x)\le \mu^{I}_{e_i}(x)$

(b3) $B^{-F}_{e_i}(x)\le \lambda^{F}_{e_i}(x)\le B^{+F}_{e_i}(x)\le \mu^{F}_{e_i}(x)$

4) (a2) $A^{-T}_{e_i}(x)\le \mu^{T}_{e_i}(x)\le A^{+T}_{e_i}(x)\le \lambda^{T}_{e_i}(x)$

(a2) $A^{-I}_{e_i}(x)\le \mu^{I}_{e_i}(x)\le A^{+I}_{e_i}(x)\le \lambda^{I}_{e_i}(x)$

(a2) $A^{-F}_{e_i}(x)\le \mu^{F}_{e_i}(x)\le A^{+F}_{e_i}(x)\le \lambda^{F}_{e_i}(x)$

(b1) $\mu^{T}_{e_i}(x)\le B^{-T}_{e_i}(x)\le \lambda^{T}_{e_i}(x)\le B^{+T}_{e_i}(x)$

(b2) $\mu^{I}_{e_i}(x)\le B^{-I}_{e_i}(x)\le \lambda^{I}_{e_i}(x)\le B^{+I}_{e_i}(x)$

(b2) $\mu^{F}_{e_i}(x)\le B^{-F}_{e_i}(x)\le \lambda^{F}_{e_i}(x)\le B^{+F}_{e_i}(x)$

Since P-union of (F,I) and (G,J) is denoted and defined as $(F,I)\cup_P (G,J) = (H,C)$ where $I\cup J = C$ and

$$H(e_i) = \begin{cases} F(e_i) & if\ e\in I-J \\ G(e_i) & if\ e\in J-I \\ F(e_i)\vee_P G(e_i) & if\ e\in I\cap J \end{cases}$$

$if\ e_i \in I\cap J$, then $F(e_i)\vee_R G(e_i)$ is defined as

$F(e_i)\vee_P G(e_i) =$ $H(e_i) =$ $\{< x, \max\{A_{e_i}(x), B_{e_i}(x)\}, (\lambda_{e_i}\vee\mu_{e_i})(x), x\in X, e_i\in I\cap J\}$

where

$F^T(e_i)\vee_P G^T(e_i) =$

$$\left\{< x, \max\{A^T_{e_i}(x), B^T_{e_i}(x)\}, (\lambda^T_{e_i}\vee\mu^T_{e_i})(x), x\in X, e_i\in I\cap J\right\}$$

$F^I(e_i)\vee_P G^I(e_i) =$

$$\left\{< x, \max\{A^I_{e_i}(x), B^I_{e_i}(x)\}, (\lambda^I_{e_i}\vee\mu^I_{e_i}(x), x\in X, e_i\in I\cap J\right\}$$

,

$F^F(e_i)\vee_P G^F(e_i) =$

$$\left\{< x, \max\{A^F_{e_i}(x), B^F_{e_i}(x)\}, (\lambda^F_{e_i}\vee\mu^F_{e_i}(x), x\in X, e_i\in I\cap J\right\}$$

for all $e_i\in I\cap J$ and for all $x\in X$.

.

Case: 1

If $H(e_i) = F(e_i)$ that is if $e_i\in I-J$ then from (1)(a1) and (2)(a1), we have $\lambda^T_{e_i}(x) = A^{-T}_{e_i}(x)$ and $\lambda^T_{e_i}(x) = A^{+T}_{e_i}(x)$

for all $e_i\in I$ and for all $x\in X$.

Thus

$A^{-T}_{e_i}(x)\le \lambda^{T}_{e_i}(x)\le A^{+T}_{e_i}(x)$,

for all $e_i\in I-J$ and for all $x\in X$.

Similarly we can prove for (1)(a2), (2)(a2) and (1)(a3), (2)(a3).

Thus $A^{-I}_{e_i}(x)\le \lambda^{I}_{e_i}(x)\le A^{+I}_{e_i}(x)$ and

$A^{-F}_{e_i}(x)\le \lambda^{F}_{e_i}(x)\le A^{+F}_{e_i}(x)$,

for all $e_i\in I-J$ and for all $x\in X$.

Case: 2

If $H(e_i) = G(e_i)$ that is if $e_i\in J-I$ then from (1)(b1) and (2)(b1), we have $\mu^T_{e_i}(x) = B^{-T}_{e_i}(x)$ *and* $\mu^T_{e_i}(x) = B^{+T}_{e_i}(x)$

for all $e_i\in I$ and for all $x\in X$. Thus

$B^{-T}_{e_i}(x)\le \mu^{T}_{e_i}(x)\le B^{+T}_{e_i}(x)$,

for all $e_i\in J-I$ and for all $x\in X$. Similarly we can prove for (1)(b2) and (2)(b2) and (1)(b3) and (2)(b3). Thus

$B^{-I}_{e_i}(x)\le \mu^{I}_{e_i}(x)\le B^{+I}_{e_i}(x)$ and

$B^{-F}_{e_i}(x)\le \mu^{F}_{e_i}(x)\le B^{+F}_{e_i}(x)$,

for all $e_i\in J-I$ and for all $x\in X$.

Case: 3

If $H(e_i) = F(e_i)\vee_P G(e_i)$ that is if $e_i\in I\cap J$, then from (1)(a1) and (1)(b1), we have

$A^{-T}_{e_i}(x)\le \lambda^{T}_{e_i}(x)\le A^{+T}_{e_i}(x)$ for all $e_i\in I$ and for all $x\in X$.

and

$B^{-T}_{e_i}(x)\le \mu^{T}_{e_i}(x)\le B^{+T}_{e_i}(x)$ for all $e_i\in J$ and for all $x\in X$.

Hence (i)
$e_i \in I \cap J$ then

$\max\{A_{e_i}^{-T}(x), B_{e_i}^{-T}(x)\} \le \left(\lambda_{e_i}^{T} \vee \mu_{e_i}^{T}\right)(x) \le \max\{A_{e_i}^{+T}(x), B_{e_i}^{+T}(x)\}$.

Similarly we can prove (1)(a2) , (1)(b2) and (1)(a3), (1)(b3) .
Thus

$\max\{A_{e_i}^{-I}(x), B_{e_i}^{-I}(x)\} \le \left(\lambda_{e_i}^{I} \vee \mu_{e_i}^{I}\right)(x) \le \max\{A_{e_i}^{+I}(x), B_{e_i}^{+I}(x)\}$

, $\max\{A_{e_i}^{-F}(x), B_{e_i}^{-F}(x)\} \le \left(\lambda_{e_i}^{F} \vee \mu_{e_i}^{F}\right)(x) \le \max\{A_{e_i}^{+F}(x), B_{e_i}^{+F}(x)\}$

.

Thus in all the three cases $(F, I) \cup_P (G, J)$ is an INSCS in X.

Theorem: 3.10
For two ENSCSs
$(F, I) = \{ F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\} \; e_i \in I \}$ and
$(G, J) = \{ G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\} \; e_i \in J \}$ in X if
$(F, I)^* = \{ F(e_i) = \{< x, A_{e_i}(x), \mu_{e_i}(x) >: x \in X\} \; e_i \in I \}$ and
$(G, J)^* = \{ G(e_i) = \{< x, B_{e_i}(x), \lambda_{e_i}(x) >: x \in X\} \; e_i \in J \}$ are INSCS in X then $(F, I) \cap_P (G, J)$ is an INSCS in X.

Proof: By similar way to Theorem 3.9 we can obtain the result.

Theorem: 3.11
Let
$(F, I) = \{ F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) >: x \in X\} \; e_i \in I \}$ and
$(G, J) = \{ G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) >: x \in X\} \; e_i \in J \}$ be ENSCSs in X such that
$(F, I)^* = \{ F(e_i) = \{< x, A_{e_i}(x), \mu_{e_i}(x) >: x \in X\} \; e_i \in I \}$ *and*
$(G, J)^* = \{ G(e_i) = \{< x, B_{e_i}(x), \lambda_{e_i}(x) >: x \in X\} \; e_i \in J \}$ be ENSCS in X. Then P-union of (F, I) and (G, J) is an ENSCS in X.

Proof:
Since (F, I), (G, J), $(F, I)^*$ and $(G, J)^*$ are ENSCS so by definition of an external soft cubic set for (F, I), (G, J), $(F, I)^*$ and $(G, J)^*$ we have

$\lambda_{e_i}^{T}(x) \notin (A_{e_i}^{-T}(x), A_{e_i}^{+T}(x))$, $\lambda_{e_i}^{I}(x) \notin (A_{e_i}^{-I}(x), A_{e_i}^{+I}(x))$,
$\lambda_{e_i}^{F}(x) \notin (A_{e_i}^{-F}(x), A_{e_i}^{+F}(x))$,
for all $e_i \in I$ and for all $x \in X$.

$\mu_{e_i}^{T}(x) \notin (B_{e_i}^{-T}(x), B_{e_i}^{+T}(x))$, $\mu_{e_i}^{I}(x) \notin (B_{e_i}^{-I}(x), B_{e_i}^{+I}(x))$,
$\mu_{e_i}^{F}(x) \notin (B_{e_i}^{-F}(x), B_{e_i}^{+F}(x))$ for all $e_i \in J$ and for all $x \in X$.

$\mu_{e_i}^{T}(x) \notin (A_{e_i}^{-T}(x), A_{e_i}^{+T}(x))$, $\mu_{e_i}^{I}(x) \notin (A_{e_i}^{-I}(x), A_{e_i}^{+I}(x))$,
$\mu_{e_i}^{F}(x) \notin (A_{e_i}^{-F}(x), A_{e_i}^{+F}(x))$
for all $e_i \in I$ and for all $x \in X$.

$\lambda_{e_i}^{T}(x) \notin (B_{e_i}^{-T}(x), B_{e_i}^{+T}(x))$, $\lambda_{e_i}^{I}(x) \notin (B_{e_i}^{-I}(x), B_{e_i}^{+I}(x))$,
$\lambda_{e_i}^{F}(x) \notin (B_{e_i}^{-F}(x), B_{e_i}^{+F}(x))$ for all $e_i \in J$ and for all $x \in X$ respectively.

Thus we have

$\left(\lambda_{e_i}^{T} \vee \mu_{e_i}^{T}\right)(x) \notin \{\max\{A_{e_i}^{-T}(x), B_{e_i}^{-T}(x)\}, \max\{A_{e_i}^{+T}(x), B_{e_i}^{+T}(x)\}\}$,

$\left(\lambda_{e_i}^{I} \vee \mu_{e_i}^{I}\right)(x) \notin \{\max\{A_{e_i}^{-I}(x), B_{e_i}^{-I}(x)\}, \max\{A_{e_i}^{+I}(x), B_{e_i}^{+I}(x)\}\}$,

$\left(\lambda_{e_i}^{F} \vee \mu_{e_i}^{F}\right)(x) \notin \{\max\{A_{e_i}^{-F}(x), B_{e_i}^{-F}(x)\}, \max\{A_{e_i}^{+F}(x), B_{e_i}^{+F}(x)\}\}$

for all $e_i \in I \cap J$ and for all $x \in X$. Thus we have

$\left(\lambda_{e_i} \vee \mu_{e_i}\right)(x) \notin \max\{A_{e_i}(x), B_{e_i}(x)\}$

for all $e_i \in I \cap J$ and for all $x \in X$. Also since $(F, I) \cup_P (G, J) = (H, C)$ where $I \cup J = C$ and

$$H(e_i) = \begin{cases} F(e_i) & \text{if } e \in I - J \\ G(e_i) & \text{if } e \in J - I \\ F(e_i) \vee_p G(e_i) & \text{if } e \in I \cap J \end{cases}$$

if $e \in I \cap J$, then $F(e_i) \vee_p G(e_i)$ is defined as

$F(e_i) \vee_p G(e_i) = H(e_i) =$
$\{< x, \max\{A_{e_i}(x), B_{e_i}(x)\}, (\lambda_{e_i} \vee \mu_{e_i})(x), x \in X, e_i \in I \cap J\}$.

where

$F^T(e_i) \vee_p G^T(e_i) =$

$$\left\{< x, \max\{A_{e_i}^{T}(x), B_{e_i}^{T}(x)\}, \left(\lambda_{e_i}^{T} \vee \mu_{e_i}^{T}\right)(x), x \in X, e_i \in I \cap J\right\}$$

$F^I(e_i) \vee_p G^I(e_i) =$

$$\left\{< x, \max\{A_{e_i}^{I}(x), B_{e_i}^{I}(x)\}, \left(\lambda_{e_i}^{I} \vee \mu_{e_i}^{I}\right)(x), x \in X, e_i \in I \cap J\right\}$$

$F^F(e_i) \vee_p G^F(e_i)$ =

$$\left\{ < x,\ \max\{A^F_{e_i}(x), B^F_{e_i}(x)\}, \left(\lambda^F_{e_i} \vee \mu^F_{e_i}\right)(x), x \in X, e_i \in I \cap J \right\}$$

By definition of an external soft cubic set $(F,I) \cup_P (G,J)$ is an ENSCS in X.

Example: 3.12

Let (P,I) and (Q,J) be neutrosophic soft cubic sets in X where

$(P,I) = P(e_1)$
$= \{< x, ([0.3,0.5],[0.2,0.5],[0.5,0.7]), (0.8,0.3,0.4) > e_1 \in I\}$

'$(Q,J) = Q(e_1)$
$= \{< x, ([0.7,0.9][0.6,0.8][0.4,0.7]), (0.4,0.7,0.3) > e_1 \in J\}$

for all $x \in X$

Then (P,I) and (Q,J) are T-external neutrosophic cubic sets in X and $(P,I) \cap_P (Q,J) = (P,I) \bigcap (Q,J) = P \cap Q(e_1)$
$= \{< x, ([0.3,0.5][0.2,0.5],[0.4,0.7]), (0.4,0.3,0.3) > e_1 \in I \cap J\}$

for all $x \in X$. $(P,I) \cap_P (Q,J)$ is not an T-external neutrosophic cubic set since $\left(\lambda^T_{e_1} \wedge \mu^T_{e_1}\right)(x) = 0.4 \in (0.3,0.5) = \left(\left(A^T_{e_1} \cap B^T_{e_1}\right)^-(x), \left(A^T_{e_1} \cap B^T_{e_1}\right)^+(x)\right)$

From the above example it is clear that P-intersection of T-external neutrosophic soft cubic sets may not be an T-external neutrosophic soft cubic set. We provide a condition for the P-intersection of T-external (resp. I-external and F-external) neutrosophic soft cubic sets to be T-external (resp. I-external and F-external) neutrosophic soft cubic set.

Theorem: 3.13

Let

$(F,I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) > : x \in X\}\ e_i \in I\}$ and

$(G,J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) > : x \in X\}\ e_i \in J\}$ be

T- ENSCSs in X such that

$$\left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) \in \begin{pmatrix} \max\left\{\{\min\{A^{+T}_{e_i}(x), B^{-T}_{e_i}(x)\}, \min\{A^{-T}_{e_i}(x), B^{+T}_{e_i}(x)\}\right\}, \\ \min\left\{\max\{A^{+T}_{e_i}(x), B^{-T}_{e_i}(x)\}, \max\{A^{-T}_{e_i}(x), B^{+T}_{e_i}(x)\}\right\} \end{pmatrix} \quad (3.7)$$

for all $e_i \in I$ and for all $e_i \in J$ and for all $x \in X$.

Then $(F,I) \cap_P (G,J)$ is also an T- ENSCS.

Proof

Consider $(F,I) \cap_P (G,J) = (H,C)$ where $I \cap J = C$

where $H(e_i) = F(e_i) \wedge_p G(e_i)$ is defined as

$F(e_i) \wedge_p G(e_i) = H(e_i)$
$= \left\{< x, \min\{A_{e_i}(x), B_{e_i}(x)\}, (\lambda_{e_i} \wedge \mu_{e_i})(x), x \in X, e_i \in I \cap J\right\}$.

For each $e \in I \cap J$,

Take

$\alpha^T_{e_i} = \min\left\{\max\{A^{+T}_{e_i}(x), B^{-T}_{e_i}(x)\}, \max\{A^{-T}_{e_i}(x), B^{+T}_{e_i}(x)\}\right\}$ and

$\beta^T_{e_i} = \max\left\{\{\min\{A^{+T}_{e_i}(x), B^{-T}_{e_i}(x)\}, \min\{A^{-T}_{e_i}(x), B^{+T}_{e_i}(x)\}\right\}$

.

Then $\alpha^T_{e_i}$ is one of $A^{-T}_{e_i}(x)$, $B^{-T}_{e_i}(x)$, $A^{+T}_{e_i}(x)$, $B^{+T}_{e_i}(x)$.

Now we consider $\alpha^T_{e_i}$ = $A^{-T}_{e_i}(x)$ or $A^{+T}_{e_i}(x)$ only, as the remaining cases are similar to this one.

If $\alpha^T_{e_i}$ = $A^{-T}_{e_i}(x)$ then $B^{-T}_{e_i}(x) \le B^{+T}_{e_i}(x) \le A^{-T}_{e_i}(x) \le A^{+T}_{e_i}(x)$ and so $\beta^T_{e_i}$ = $B^{+T}_{e_i}(x)$

thus $B^{-T}_{e_i}(x) = (A^T_{e_i} \cap B^T_{e_i})^-(x) \le (A^T_{e_i} \cap B^T_{e_i})^+(x) = B^{+T}_{e_i}(x) = \beta^T_{e_i} < \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x)$.

Hence $\left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) \notin \left((A^T_{e_i} \cap B^T_{e_i})^-(x), (A^T_{e_i} \cap B^T_{e_i})^+(x)\right)$

.

If $\alpha^T_{e_i}$ = $A^{+T}_{e_i}(x)$, then $B^{-T}_{e_i}(x) \le A^{+T}_{e_i}(x) \le B^{+T}_{e_i}(x)$ and so $\beta^T_{e_i}$ = $\max\{A^{-T}_{e_i}(x), B^{-T}_{e_i}(x)\}$.

Assume that $\beta^T_{e_i}$ = $A^{-T}_{e_i}(x)$ then $B^{-T}_{e_i}(x) \le A^{-T}_{e_i}(x) < \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) \le A^{+T}_{e_i}(x) \le B^{+T}_{e_i}(x)$.

So from this we can write $B^{-T}_{e_i}(x) \le A^{-T}_{e_i}(x) < \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) < A^{+T}_{e_i}(x) \le B^{+T}_{e_i}(x)$ or $B^{-T}_{e_i}(x) \le A^{-T}_{e_i}(x) < \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) = A^{+T}_{e_i}(x) \le B^{+T}_{e_i}(x)$.

For this case $B^{-T}_{e_i}(x) \le A^{-T}_{e_i}(x) < \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) < A^{+T}_{e_i}(x) \le B^{+T}_{e_i}(x)$ it is contradiction to the fact that (F,I) and (G,J) are T-ENSCS.

For the case $B^{-T}_{e_i}(x) \le A^{-T}_{e_i}(x) < \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) = A^{+T}_{e_i}(x) \le B^{+T}_{e_i}(x)$ we have $\left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) \notin$

$\left((A^T_{e_i}\cap B^T_{e_i})^-(x), A^T_{e_i}\cap B^T_{e_i})^+(x)\right)$ because $\left(\lambda^T_{e_i}\wedge\mu^T_{e_i}\right)(x)$ = $A^{+T}_{e_i}(x)$ = $(A^T_{e_i}\cap B^T_{e_i})^+(x)$. Again assume that $\beta^T_{e_i}$ = $B^{-T}_{e_i}(x)$ then $A^{-T}_{e_i}(x) \le B^{-T}_{e_i}(x) < \left(\lambda^T_{e_i}\wedge\mu^T_{e_i}\right)(x) \le A^{+T}_{e_i}(x) \le B^{+T}_{e_i}(x)$. From this we can write $A^{-T}_{e_i}(x) \le B^{-T}_{e_i}(x) < \left(\lambda^T_{e_i}\wedge\mu^T_{e_i}\right)(x) < A^{+T}_{e_i}(x) \le B^{+T}_{e_i}(x)$ or $A^{-T}_{e_i}(x) \le B^{-T}_{e_i}(x) < \left(\lambda^T_{e_i}\wedge\mu^T_{e_i}\right)(x) = A^{+T}_{e_i}(x) \le B^{+T}_{e_i}(x)$. For this case $A^{-T}_{e_i}(x) \le B^{-T}_{e_i}(x) < \left(\lambda^T_{e_i}\wedge\mu^T_{e_i}\right)(x) < A^{+T}_{e_i}(x) \le B^{+T}_{e_i}(x)$ it is contradiction to the fact that (F,I) and (G,J) are T-ENSCS. And if we take the case $A^{-T}_{e_i}(x) \le B^{-T}_{e_i}(x) < \left(\lambda^T_{e_i}\wedge\mu^T_{e_i}\right)(x) = A^{+T}_{e_i}(x) \le B^{+T}_{e_i}(x)$, we get have $\left(\lambda^T_{e_i}\wedge\mu^T_{e_i}\right)(x) \notin \left((A^T_{e_i}\cap B^T_{e_i})^-(x), A^T_{e_i}\cap B^T_{e_i})^+(x)\right)$ because $\left(\lambda^T_{e_i}\wedge\mu^T_{e_i}\right)(x)$ = $A^{+T}_{e_i}(x)$ = $(A^T_{e_i}\cap B^T_{e_i})^+(x)$. Hence in all the cases $(F,I)\cap_P (G,J)$ is an T-ENSCS in X.

Theorem: 3.14

Let
$(F,I)=\{F(e_i)=\{<x, A_{e_i}(x), \lambda_{e_i}(x)>: x\in X\}\ e_i\in I\}$ and
$(G,J)=\{G(e_i)=\{<x, B_{e_i}(x), \mu_{e_i}(x)>: x\in X\}\ e_i\in J\}$ be
I- ENSCSs in X such that

$$\left(\lambda^I_{e_i}\wedge\mu^I_{e_i}\right)(x)\in\left(\begin{array}{l}\min\left\{\max\{A^{+I}_{e_i}(x),B^{-I}_{e_i}(x)\},\max\{A^{-I}_{e_i}(x),B^{+I}_{e_i}(x)\}\right\},\\ \max\left\{\{\min\{A^{+I}_{e_i}(x),B^{-I}_{e_i}(x)\},\min\{A^{-I}_{e_i}(x),B^{+I}_{e_i}(x)\}\right\}\end{array}\right) \quad (3.8)$$

for all $e_i\in I$ and for all $e_i\in J$ and for all $x\in X$. Then $(F,I)\cap_P (G,J)$ is also an I – ENSCS

Proof:
By similar way to Theorem 3.13, we can obtain the result.

Theorem : 3.15

Let
$(F,I)=\{F(e_i)=\{<x, A_{e_i}(x), \lambda_{e_i}(x)>: x\in X\}\ e_i\in I\}$ *and*
$(G,J)=\{G(e_i)=\{<x, B_{e_i}(x), \mu_{e_i}(x)>: x\in X\}\ e_i\in J\}$ be
F- ENSCSs in X such that

$$\left(\lambda^F_{e_i}\wedge\mu^F_{e_i}\right)(x)\in\left(\begin{array}{l}\min\left\{\max\{A^{+F}_{e_i}(x),B^{-F}_{e_i}(x)\},\max\{A^{-F}_{e_i}(x),B^{+F}_{e_i}(x)\}\right\},\\ \max\left\{\{\min\{A^{+F}_{e_i}(x),B^{-F}_{e_i}(x)\},\min\{A^{-F}_{e_i}(x),B^{+F}_{e_i}(x)\}\right\}\end{array}\right)$$

…………..(3.9)

for all $e_i\in I$ and for all $e_i\in J$ and for all $x\in X$. Then $(F,I)\cap_P (G,J)$ is also an F- ENSCS.

Proof : By similar way to Theorem 3.13, we can obtain the result

Corallary:3.16

Let
$(F,I)=\{F(e_i)=\{<x, A_{e_i}(x), \lambda_{e_i}(x)>: x\in X\}\ e_i\in I\}$ *and*
$(G,J)=\{G(e_i)=\{<x, B_{e_i}(x), \mu_{e_i}(x)>: x\in X\}\ e_i\in J\}$ be
ENSCSs in X. Then P-intersection $(F,I)\cap_P (G,J)$ is also an ENSCS in X when the conditions (3.7), (3.8)and (3.9) are valid.

Theorem: 3.17

If neutrosophic soft cubic set
$(F,I)=\{F(e_i)=\{<x, A_{e_i}(x), \lambda_{e_i}(x)>: x\in X\}\ e_i\in I\}$ and
$(G,J)=\{G(e_i)=\{<x, B_{e_i}(x), \mu_{e_i}(x)>: x\in X\}\ e_i\in J\}$ in
X satisfy the following condition

$$\min\left\{\max\{A^{+T}_{e_i}(x),B^{-T}_{e_i}(x)\},\max\{A^{-T}_{e_i}(x),B^{+T}_{e_i}(x)\}\right\}$$
$$=\left(\lambda^T_{e_i}\wedge\mu^T_{e_i}\right)(x)$$
$$=\max\left\{\{\min\{A^{+T}_{e_i}(x),B^{-T}_{e_i}(x)\},\min\{A^{-T}_{e_i}(x),B^{+T}_{e_i}(x)\}\right\} \quad ….(11.1)$$

then the $(F,I)\cap_P (G,J)$ is both
an T-Internal Neutrosophic Soft Cubic Set and T-External Soft Neutrosophic Cubic Set
in X.

Proof: Consider $(F,I)\cap_P (G,J) = (H,C)$ where $I\cap J = C$ where $H(e_i) = F(e_i)\wedge_p G(e_i)$ is defined as
$F(e_i)\wedge_P G(e_i)$ = $H(e_i)$ =
$\{<x, \min\{A_{e_i}(x), B_{e_i}(x)\}, (\lambda_{e_i}\wedge\mu_{e_i})(x)>: x\in X\}\ e_i\in I\cap J\}$
where

$F^T(e_i) \wedge_P G^T(e_i) =$
$\{< x, \min\{A^T_{e_i}(x), B^T_{e_i}(x)\}, (\lambda^T_{e_i} \wedge \mu^T_{e_i})(x) > : x \in X\}\ e_i \in I \cap J\}$ .For each $e_i \in I \cap J$ Take $\alpha^T_{e_i} = \min\left\{\max\{A^{+T}_{e_i}(x), B^{-T}_{e_i}(x)\}, \max\{A^{-T}_{e_i}(x), B^{+T}_{e_i}(x)\}\right\}$ and $\beta^T_{e_i} = \max\left\{\{\min\{A^{+T}_{e_i}(x), B^{-T}_{e_i}(x)\}, \min\{A^{-T}_{e_i}(x), B^{+T}_{e_i}(x)\}\right\}$. Then $\alpha^T_{e_i}$ is one of $A^{-T}_{e_i}(x)$, $B^{-T}_{e_i}(x)$, $A^{+T}_{e_i}(x)$, $B^{+T}_{e_i}(x)$. Now we consider $\alpha^T_{e_i} = A^{-T}_{e_i}(x)$, or $A^{+T}_{e_i}(x)$ only, as the remaining cases are similar to this one. If $\alpha^T_{e_i} = A^{-T}_{e_i}(x)$ then $B^{-T}_{e_i}(x) \leq B^{+T}_{e_i}(x) \leq A^{-T}_{e_i}(x), \leq A^{+T}_{e_i}(x)$, and so $\beta^T_{e_i} = B^{+T}_{e_i}(x)$ this implies $A^{-T}_{e_i}(x) = \alpha^I_{e_i} = \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) = \beta^T_{e_i} = B^{+T}_{e_i}(x)$. Thus $B^{-T}_{e_i}(x) \leq B^{+T}_{e_i}(x) = \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) = A^{-T}_{e_i}(x) \leq A^{+T}_{e_i}(x)$, which implies that $\left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) = B^{+T}_{e_i}(x) = (A^T_{e_i} \cap B^T_{e_i})^+(x)$. Hence $\left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) \notin \left((A^T_{e_i} \cap B^T_{e_i})^-(x), (A^T_{e_i} \cap B^T_{e_i})^+(x)\right)$ and $(A^T_{e_i} \cap B^T_{e_i})^-(x) \leq \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) \leq (A^T_{e_i} \cap B^T_{e_i})^+(x)$. If $\alpha^T_{e_i} = A^{+T}_{e_i}(x)$ then $B^{-T}_{e_i}(x) \leq A^{+T}_{e_i}(x) \leq B^{+T}_{e_i}(x)$, and so $\left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) = A^{+T}_{e_i}(x) = (A^T_{e_i} \cap B^T_{e_i})^+(x)$. Hence $\left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) \notin \left((A^T_{e_i} \cap B^T_{e_i})^-(x), (A^T_{e_i} \cap B^T_{e_i})^+(x)\right)$ and $(A^T_{e_i} \cap B^T_{e_i})^-(x) \leq \left(\lambda^T_{e_i} \wedge \mu^T_{e_i}\right)(x) \leq (A^T_{e_i} \cap B^T_{e_i})^+(x)$.
Consequently we note that $(F, I) \cap_P (G, J)$ is both T-internal neutrosophic soft cubic set and T-external soft neutrosophic cubic set in X.
Similarly we have the following theorems

Theorem 3.18
If neutrosophic soft cubic set $(F, I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) > : x \in X\}\ e_i \in I\}$ and $(G, J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) > : x \in X\}\ e_i \in J\}$ in satisfy the following condition

$$\min\left\{\max\{A^{+I}_{e_i}(x), B^{-I}_{e_i}(x)\}, \max\{A^{-I}_{e_i}(x), B^{+I}_{e_i}(x)\}\right\}$$
$$= \left(\lambda^I_{e_i} \wedge \mu^I_{e_i}\right)(x)$$
$$= \max\left\{\{\min\{A^{+I}_{e_i}(x), B^{-I}_{e_i}(x)\}, \min\{A^{-I}_{e_i}(x), B^{+I}_{e_i}(x)\}\right\} \ldots\ldots.(11.2)$$

then the $(F, I) \cap_P (G, J)$ is both an I-internal neutrosophic soft cubic set and an I-external soft neutrosophic cubic set in X.

Theorem :3.19
If neutrosophic soft cubic set $(F, I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) > : x \in X\}\ e_i \in I\}$ and $(G, J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) > : x \in X\}\ e_i \in J\}$ in X satisfy the following condition

$$\min\left\{\max\{A^{+F}_{e_i}(x), B^{-F}_{e_i}(x)\}, \max\{A^{-F}_{e_i}(x), B^{+F}_{e_i}(x)\}\right\}$$
$$= \left(\lambda^F_{e_i} \wedge \mu^F_{e_i}\right)(x)$$
$$= \max\left\{\{\min\{A^{+F}_{e_i}(x), B^{-F}_{e_i}(x)\}, \min\{A^{-F}_{e_i}(x), B^{+F}_{e_i}(x)\}\right\}$$

………(11.3)then the $(F, I) \cap_P (G, J)$ is both an F-internal neutrosophic soft cubic set and an F-external soft neutrosophic cubic set in X.

Corollary:3.20
Let $(F, I) = \{F(e_i) = \{< x, A_{e_i}(x), \lambda_{e_i}(x) > : x \in X\}\ e_i \in I\}$ and $(G, J) = \{G(e_i) = \{< x, B_{e_i}(x), \mu_{e_i}(x) > : x \in X\}\ e_i \in I\}$ be NSCSs in X. Then P-intersection $(F, I) \cap_P (G, J)$ is also an ENSCS and an INSCS in X when the conditions (11.1**),** **(**11.2)and (11.3) are valid.

The following example shows that the P-union of T-external neutrosophic soft cubic sets may not be an T-external neutrosophic soft cubic set.

Example 3.21. Let (P, I) and (Q, J) be neutrosophic soft cubic sets in X where
$(P, I) = P(e_1) = \{< x, ([0.3, 0.5], [0.2, 0.5], [0.5, 0.7]), (0.8, 0.3, 0.4) > e_1 \in I\}$
$'(Q, J) = Q(e_1) = \{< x, ([0.7, 0.9][0.6, 0.8][0.4, 0.7]), (0.4, 0.7, 0.3) > e_1 \in J\}$

Then (P,I) and (Q,J) are T-external neutrosophic cubic sets in X and $(P,I)\cup(Q,J)=P\cup Q(e_1)$ $=\{<x,([0.7,0.9][0.6,0.8],[0.5,0.7]),(0.8,0.7,0.4)>\}$ $(P,I)\cup_p(Q,J)$ is not an T-external neutrosophic cubic set in X since $\left(\lambda^T_{e_1}\vee\mu^T_{e_2}\right)(x)=0.8\in(0.7,0.9)=\left(\left(A^T_{e_1}\cup B^T_{e_1}\right)^-(x),\left(A^T_{e_1}\cup B^T_{e_1}\right)^+(x)\right)$.

We consider a condition for the P-union of T-external (resp. I-external and F-external) neutrosophic soft cubic sets to be T-external (resp. I-external and F-external) neutrosophic soft cubic set.

Theorem 3.22

Let $(F,I)=\{F(e_i)=\{<x,A_{e_i}(x),\lambda_{e_i}(x)>:x\in X\}\ e_i\in I\}$ and $(G,J)=\{G(e_i)=\{<x,B_{e_i}(x),\mu_{e_i}(x)>:x\in X\}\ e_i\in J\}$ be T- ENSCSs in X such that

$$\left(\lambda^T_{e_i}\vee\mu^T_{e_i}\right)(x)\in\left(\begin{array}{l}\max\left\{\{\min\{A^{+T}_{e_i}(x),B^{-T}_{e_i}(x)\},\min\{A^{-T}_{e_i}(x),B^{+T}_{e_i}(x)\}\right\},\\ \min\left\{\max\{A^{+T}_{e_i}(x),B^{-T}_{e_i}(x)\},\max\{A^{-T}_{e_i}(x),B^{+T}_{e_i}(x)\}\right\}\end{array}\right)$$

…………..(12.1)

for all $e_i\in I$ and for all $e_i\in J$ and for all $x\in X$. Then $(F,I)\cup_P(G,J)$ is also an T- ENSCS.

Proof:

Consider $(F,I)\cup_P(G,J)=(H,C)$ where $I\cup J=C$ and

$$H(e_i)=\begin{cases}F(e_i) & if\ e\in I-J\\ G(e_i) & if\ e\in J-I\\ F(e_i)\vee_p G(e_i) & if\ e\in I\cap J\end{cases}$$

where $H(e_i)=F(e_i)\vee_p G(e_i)$ is defined as

$F(e_i)\vee_p G(e_i)=H(e_i)=$ $\{<x,\max\{A_{e_i}(x),B_{e_i}(x)\},(\lambda_{e_i}\vee\mu_{e_i})(x),x\in X,e_i\in I\cap J\}$, where

$F^T(e_i)\vee_p G^T(e_i)=$ $\{<x,\max\{A^T_{e_i}(x),B^T_{e_i}(x)\},(\lambda^T_{e_i}\vee\mu^T_{e_i})(x),x\in X,e_i\in I\cap J\}$,

If $e_i\in I\cap J$,

$$\alpha^T_{e_i}=\min\left\{\max\{A^{+T}_{e_i}(x),B^{-T}_{e_i}(x)\},\max\{A^{-T}_{e_i}(x),B^{+T}_{e_i}(x)\}\right\}$$

and

$$\beta^T_{e_i}=\max\left\{\{\min\{A^{+T}_{e_i}(x),B^{-T}_{e_i}(x)\},\min\{A^{-T}_{e_i}(x),B^{+T}_{e_i}(x)\}\right\}$$

Then $\alpha^T_{e_i}$ is one of $A^{-T}_{e_i}(x)$, $B^{-T}_{e_i}(x)$, $\alpha^T_{e_i}$ $A^{+T}_{e_i}(x)$, $B^{+T}_{e_i}(x)$. Now we consider $\alpha^T_{e_i}=A^{-T}_{e_i}(x)$ or $A^{+T}_{e_i}(x)$, only as the remaining cases are similar to this one.

If $\alpha^T_{e_i}=A^{-T}_{e_i}(x)$ then $B^{-T}_{e_i}(x)\le B^{+T}_{e_i}(x)\le A^{-T}_{e_i}(x)\le A^{+T}_{e_i}(x)$, and so $\beta^T_{e_i}=B^{+T}_{e_i}(x)$ .Thus $(A^T_{e_i}\cup B^T_{e_i})^-(x)=A^{-T}_{e_i}(x)=\alpha^T_{e_i}>\left(\lambda^T_{e_i}\vee\mu^T_{e_i}\right)(x)$. Hence $\left(\lambda^T_{e_i}\vee\mu^T_{e_i}\right)(x)\notin\left((A^T_{e_i}\cup B^T_{e_i})^-(x),A^T_{e_i}\cup B^T_{e_i})^+(x)\right)$. If $\alpha^T_{e_i}=A^{+T}_{e_i}(x)$, then $B^{-T}_{e_i}(x)\le A^{+T}_{e_i}(x)\le B^{+T}_{e_i}(x)$ and so $\beta^T_{e_i}=\max\{A^{-T}_{e_i}(x),B^{-T}_{e_i}(x)\}$. Assume that $\beta^T_{e_i}=A^{-T}_{e_i}(x)$ then $B^{-T}_{e_i}(x)\le A^{-T}_{e_i}(x)<\left(\lambda^T_{e_i}\vee\mu^T_{e_i}\right)(x)\le A^{+T}_{e_i}(x)\le B^{+T}_{e_i}(x)$. So from this we can write $B^{-T}_{e_i}(x)\le A^{-T}_{e_i}(x)<\left(\lambda^T_{e_i}\vee\mu^T_{e_i}\right)(x)<A^{+T}_{e_i}(x)\le B^{+T}_{e_i}(x)$ or $B^{-T}_{e_i}(x)\le A^{-T}_{e_i}(x)=\left(\lambda^T_{e_i}\vee\mu^T_{e_i}\right)(x)\le A^{+T}_{e_i}(x)\le B^{+T}_{e_i}(x)$.

For the case $B^{-T}_{e_i}(x)\le A^{-T}_{e_i}(x)<\left(\lambda^T_{e_i}\vee\mu^T_{e_i}\right)(x)<A^{+T}_{e_i}(x)\le B^{+T}_{e_i}(x)$ it is contradiction to the fact that (F,I) and (G,J) are T-ENSCS. For the case $B^{-T}_{e_i}(x)\le A^{-T}_{e_i}(x)=\left(\lambda^T_{e_i}\vee\mu^T_{e_i}\right)(x)\le A^{+T}_{e_i}(x)\le B^{+T}_{e_i}(x)$ we have $\left(\lambda^T_{e_i}\vee\mu^T_{e_i}\right)(x)\notin\left((A^T_{e_i}\cup B^T_{e_i})^-(x),A^T_{e_i}\cup B^T_{e_i})^+(x)\right)$ because $(A^T_{e_i}\cup B^T_{e_i})^-(x)=A^{-T}_{e_i}(x)=\left(\lambda^T_{e_i}\vee\mu^T_{e_i}\right)(x)$.

Again assume that $\beta^T_{e_i}=B^{-T}_{e_i}(x)$ then $A^{-T}_{e_i}(x)\le B^{-T}_{e_i}(x)\le\left(\lambda^T_{e_i}\vee\mu^T_{e_i}\right)(x)\le A^{+T}_{e_i}(x)\le B^{+T}_{e_i}(x)$, so from this we can write $A^{-T}_{e_i}(x) \leq B^{-T}_{e_i}(x) < \left(\lambda^{T}_{e_i} \vee \mu^{T}_{e_i}\right)(x) < A^{+T}_{e_i}(x) \leq B^{+T}_{e_i}(x)$ or $A^{-T}_{e_i}(x) \leq B^{-T}_{e_i}(x) = \left(\lambda^{T}_{e_i} \vee \mu^{T}_{e_i}\right)(x) < A^{+T}_{e_i}(x) \leq B^{+T}_{e_i}(x)$. For this case $A^{-T}_{e_i}(x) \leq B^{-T}_{e_i}(x) < \left(\lambda^{T}_{e_i} \vee \mu^{T}_{e_i}\right)(x) < A^{+T}_{e_i}(x) \leq B^{+T}_{e_i}(x)$ it is contradiction to the fact that $(\mathrm{F,I})$ and $(\mathrm{G,J})$ are T-ENSCS. And if we take the case $A^{-T}_{e_i}(x) \leq B^{-T}_{e_i}(x) = \left(\lambda^{T}_{e_i} \vee \mu^{T}_{e_i}\right)(x) \leq A^{+T}_{e_i}(x) \leq B^{+T}_{e_i}(x)$, we get have $\left(\lambda^{T}_{e_i} \vee \mu^{T}_{e_i}\right)(x) \notin \left((\mathrm{A}^{T}_{e_i} \cup B^{T}_{e_i})^{-}(x), \mathrm{A}^{T}_{e_i} \cup B^{T}_{e_i})^{+}(x)\right)$ because $(\mathrm{A}^{T}_{e_i} \cup B^{T}_{e_i})^{-}(x) = B^{-T}_{e_i}(x) = \left(\lambda^{T}_{e_i} \vee \mu^{T}_{e_i}\right)(x)$. If $\mathrm{e_i} \in \mathrm{I} - J$ *or* $\mathrm{e_i} \in J - I$, then we have trivial result. Hence $(\mathrm{F,I}) \cup_P (G,J)$ is an T-ENSCS in X.

Similarly we have the following theorems

Theorem:3.23

Let

$(\mathrm{F,I}) = \{\mathrm{F(e_i)} = \{<\mathrm{x}, \mathrm{A_{e_i}}(\mathrm{x}), \lambda_{\mathrm{e_i}}(\mathrm{x}) >: \mathrm{x} \in \mathrm{X}\}\ \mathrm{e_i} \in \mathrm{I}\}$ and

$(\mathrm{G,J}) = \{\mathrm{G(e_i)} = \{<\mathrm{x}, \mathrm{B_{e_i}}(\mathrm{x}), \mu_{\mathrm{e_i}}(\mathrm{x}) >: \mathrm{x} \in \mathrm{X}\}\ \mathrm{e_i} \in \mathrm{J}\}$ be T- ENSCSs in X such that

$$\left(\lambda^{T}_{e_i} \vee \mu^{T}_{e_i}\right)(x) \in \begin{pmatrix} \max\left\{\{\min\{A^{+T}_{e_i}(x), B^{-T}_{e_i}(x)\}, \min\{A^{-T}_{e_i}(x), B^{+T}_{e_i}(x)\}\right\}, \\ \min\left\{\max\{A^{+T}_{e_i}(x), B^{-T}_{e_i}(x)\}, \max\{A^{-T}_{e_i}(x), B^{+T}_{e_i}(x)\}\right\} \end{pmatrix}$$

…………..(12.2)

for all $\mathrm{e_i} \in \mathrm{I}$ *and* for all $\mathrm{e_i} \in J$ and for all $\mathrm{x} \in \mathrm{X}$. Then $(\mathrm{F,I}) \cup_P (G,J)$ is also an T- ENSCS.

Theorem :3.24

Let

$(\mathrm{F,I}) = \{\mathrm{F(e_i)} = \{<\mathrm{x}, \mathrm{A_{e_i}}(\mathrm{x}), \lambda_{\mathrm{e_i}}(\mathrm{x}) >: \mathrm{x} \in \mathrm{X}\}\ \mathrm{e_i} \in \mathrm{I}\}$ and

$(\mathrm{G,J}) = \{\mathrm{G(e_i)} = \{<\mathrm{x}, \mathrm{B_{e_i}}(\mathrm{x}), \mu_{\mathrm{e_i}}(\mathrm{x}) >: \mathrm{x} \in \mathrm{X}\}\ \mathrm{e_i} \in \mathrm{J}\}$ be T- ENSCSs in X such that

$$\left(\lambda^{T}_{e_i} \vee \mu^{T}_{e_i}\right)(x) \in \begin{pmatrix} \max\left\{\{\min\{A^{+T}_{e_i}(x), B^{-T}_{e_i}(x)\}, \min\{A^{-T}_{e_i}(x), B^{+T}_{e_i}(x)\}\right\}, \\ \min\left\{\max\{A^{+T}_{e_i}(x), B^{-T}_{e_i}(x)\}, \max\{A^{-T}_{e_i}(x), B^{+T}_{e_i}(x)\}\right\} \end{pmatrix}$$

…………..(12.3)

for all $\mathrm{e_i} \in \mathrm{I}$ and for all $\mathrm{e_i} \in \mathrm{J}$ and for all $\mathrm{x} \in \mathrm{X}$. Then $(\mathrm{F,I}) \cup_P (G,J)$ is also an T- ENSCS.

Corollary:3.25

Let

$(\mathrm{F,I}) = \{\mathrm{F(e_i)} = \{<\mathrm{x}, \mathrm{A_{e_i}}(\mathrm{x}), \lambda_{\mathrm{e_i}}(\mathrm{x}) >: \mathrm{x} \in \mathrm{X}\}\ \mathrm{e_i} \in \mathrm{I}\}$ and

$(\mathrm{G,J}) = \{\mathrm{G(e_i)} = \{<\mathrm{x}, \mathrm{B_{e_i}}(\mathrm{x}), \mu_{\mathrm{e_i}}(\mathrm{x}) >: \mathrm{x} \in \mathrm{X}\}\ \mathrm{e_i} \in \mathrm{J}\}$ be ENSCSs in X. Then $(\mathrm{F,I}) \cup_P (G,J)$ is also an ENSCS in X when the conditions (12.1**),** (12.2)and (12.3) are valid.

References

[1] R. Anitha Cruz and F. Nirmala Irudayam, Neutrosophic Soft Cubic Set, International Journal of Mathematics Trends and Technology, 46 (2) (2017), 88-94.

[2] Y.B. Jun, C.S. Kim and K.O. Yang, Cubic sets, Annals of Fuzzy Mathematics and Informatics 4(3) (2012), 83–98.

[3] Turksen,“Interval valued fuzzy sets based on normal forms”. Fuzzy Sets and Systems, 20, (1968), pp. 191–210.

[4] F. Smarandache, A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic, Rehoboth: American Research Press, 1999.

[5] L. A. Zadeh, Fuzzy Sets, Inform. Control 8 (1965), 338–353.

[6] I. B. Turksen, Interval-valued strict preference with Zadeh triples, Fuzzy Sets and Systems 78 (1996) 183–195.

[7] H. Wang, F. Smarandache, Y. Q. Zhangand, R. Sunderraman, Interval Neutrosophic Sets and logic: Theory and Applictionsin Computing, Hexis; Neutrosophic book series, No. 5, 2005.

[8] D. Molodtsov , Soft Set Theory–First Results, Computers and Mathematics with Application, 37 (1999) 19–31.

[9] Pabitra Kumar Maji, “Neutrosophic soft set” Annals of Fuzzy Mathematics and Informatics 5 (2013), 157–168.

[10] F. Smarandache, Neutrosophic set, a generalization of the intuitionistic fuzzy sets, Inter. J. Pu Bre Appl. Math. 24 (2005) 287–297.

[11] F. Smarandache, Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy, Neutrosophy Logic, Set, Probability and Statistics, University of New Mexico, Gallup, NM 87301, USA (2002).

[12] L. J. Kohout, W. Bandler, Fuzzy interval inference utilizing the checklist paradigm and BKrelational products, in: R.B. Kearfort et al. (Eds.), Applications of Interval Computations, Kluwer, Dordrecht, 1996, pp. 291–335.

[13] R. Sambuc, Functions Φ-Flous, Application a l’aide au Diagnostic en Pathologie Thyroidienne, These de Doctorat en Medecine, Marseille, 1975.

[14] I. B. Turksen, Interval-valued fuzzy sets and compensatory AND, Fuzzy Sets and Systems 51 (1992) 295–307.

[15] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I, Inform. Sci. 8 (1975) 199–249.

[16] P. K. Maji , R. Biswas ans A. R. Roy, "Fuzzy soft sets", Journal of Fuzzy Mathematics, Vol. 9, no. 3, 2001, pp. 589-602,

[17] P.K.Maji, R. Biswas ans A.R.Roy, "Intuitionistic Fuzzy soft sets", The journal of fuzzy Mathematics, Vol 9, (3)(2001), 677 – 692.

[18] Pabitra Kumar Maji, Neutrosophic soft set, Annals of Fuzzy Mathematics and Informatics, Volume 5, No.1,(2013).,157-168.

[19] Jun YB, Smarandache F, KimCS (2017) Neutrosophic cubic sets. New Mathematics and Natural Computation 13:41-54.

[20] Abdel-Basset, M., Mohamed, M., & Sangaiah, A. K. (2017). Neutrosophic AHP-Delphi Group decision making model based on trapezoidal neutrosophic numbers. Journal of Ambient Intelligence and Humanized Computing.

[21] Abdel-Basset, M., Mohamed, M., Hussien, A. N., & Sangaiah, A. K. (2017). A novel group decision-making model based on triangular neutrosophic numbers. Soft Computing, 1-15. http://doi.org/ 10.1007/s00500-017-2758-5

[22] F. Smarandache, M. Ali, Neutrosophic Triplet as extension of Matter Plasma, Unmatter Plasma, and Antimatter Plasma, 69th Annual Gaseous Electronics Conference, Bochum, Germany, Veranstaltung-szentrum & Audimax, Ruhr-Universitat, October 10-14, 2016.

[23] F. Smarandache, Neutrosophic Perspectives: Triplets, Duplets, Multisets, Hybrid Operators, Modal Logic, Hedge Algebras. And Applications. Pons Editions, Bruxelles, 325 p., 2017.

[24] Topal, S. and Öner, T. Contrastings on Textual Entailmentness and Algorithms of Syllogistic Logics, Journal of Mathematical Sciences, Volume 2, Number 4, April 2015, pp 185-189.

[25] Topal S. An Object- Oriented Approach to Counter-Model Constructions in A Fragment of Natural Language, BEU Journal of Science, 4(2), 103-111, 2015.

[26] Topal, S. A Syllogistic Fragment of English with Ditransitive Verbs in Formal Semantics, Journal of Logic, Mathematics and Linguistics in Applied Sciences, Vol 1, No 1, 2016.

[27] Topal, S. and Smaradache, F. A Lattice-Theoretic Look: A Negated Approach to Adjectival (Intersective, Neutrosophic and Private) Phrases. The 2017 IEEE International Conference on INnovations in Intelligent SysTems and Applications (INISTA 2017); (accepted for publication).

[28] Taş, F. and Topal, S. Bezier Curve Modeling for Neutrosophic Data Problem. Neutrosophic Sets and Systems, Vol 16, pp. 3-5, 2017.

[29] Topal, S. Equivalential Structures for Binary and Ternary Syllogistics, Journal of Logic, Language and Information, Springer, 2017, DOI: 10.1007/s10849-017-9260-4

Received: August 1, 2017. Accepted: August 21, 2017.

NSS

University of New Mexico

Extension of Crisp Functions on Neutrosophic Sets

Sabu Sebastian[1], Florentin Smarandache[2]

[1] Department of Mathematics, Nirmalagiri College, Kannur University, Kerala-670701, India
Email:sabukannur@gmail.com
[2] Department of Mathematics, University of New Mexico,705 Gurley Ave. Gallup, NM 87301, USA
E-mail: smarand@unm.edu

Abstract. In this paper, we generalize the definition of Neutrosophic sets and present a method for extending crisp functions on Neutrosophic sets and study some properties of such extended functions.

Keywords: Neutrosophic set, Multi-fuzzy set, Bridge function.

1 Introduction

L-fuzzy sets constitute a generalization of the notion of Zadeh's [26] fuzzy sets and were introduced by Goguen [8] in 1967, later Atanassov introduced the notion of the intuitionistic fuzzy sets [1] Gau and Buehrer [7] defined vague sets. Bustince and Burillo [2] showed that the notion of vague sets is the same as that of intuitionistic fuzzy sets. Deschrijver and Kerre [5] established the interrelationship between the theories of fuzzy sets, L-fuzzy sets, interval valued fuzzy sets, intuitionistic fuzzy sets, intuitionistic L-fuzzy sets, interval valued intuitionistic fuzzy sets, vague sets and gray sets [4].

2 Preliminaries

Definition 2.1. [26] Let X be a nonempty set. A fuzzy set A of X is a mapping $A : X \to [0, 1]$, that is,
$A = \{(x,\ \mu_A(x)) : \mu_A(x)$ is the grade of membership of x in A, $x \in X\}$. The set of all the fuzzy sets on X is denoted by $\mathcal{F}(X)$.

Definition 2.2. [8] Let X be a nonempty ordinary set, L a complete lattice. An L-fuzzy set on X is a mapping $A : X \to L$, that is the family of all the L-fuzzy sets on X is just L^X consisting of all the mappings from X to L.

Definition 2.3. [1] An Intuitionistic Fuzzy Set on X is a set

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\},$$

where $\mu_A(x) \in [0,\ 1]$ denotes the membership degree and $\nu_A(x) \in [0,\ 1]$ denotes the non-membership degree of x in A and

$$\mu_A(x) + \nu_A(x) \le 1, \forall x \in X.$$

The neutrosophic set (NS) was introduced by F. Smarandache [22] who introduced the degree of indeterminacy (i) as independent component in his manuscripts that was published in 1998.

Multi-fuzzy sets [12, 13, 16] was proposed in 2009 by Sabu Sebastian as an extension of fuzzy sets [8, 26] in terms of multi membership functions. In this paper we generalize the definition of neutrosophic sets and introduce extension of crisp functions on neutrosophic sets.

Definition 2.4. [22]A Neutrosophic Set on X is a set

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\},$$

where $T_A(x) \in [0,\ 1]$ denotes the truth membership degree, $I_A(x) \in [0,\ 1]$ denotes the indetermi-nancy membership degree and $F_A(x) \in [0,\ 1]$ denotes the falsity membership degree of x in A respectively and

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3, \forall x \in X.$$

For single valued neutrosophic logic (T, I, F), the sum of the components is: $0 \le T + I + F \le 3$ when all three components are independent; $0 \le T + I + F \le 2$ when two components are dependent, while the third one is independent from them; $0 \le T + I + F \le 1$ when all three components are dependent.

Definition 2.5. [12, 13, 16]Let X be a nonempty set, J be an indexing set and $\{L_j : j \in J\}$ a family of partially ordered sets. A **multi-fuzzy set A** in X is a set :

$$\mathbf{A} = \{\langle x, (\mu_j(x))_{j\in J}\rangle : x \in X,\ \mu_j \in L_j^X,\ j \in J\}.$$

The indexing set J may be uncountable. The function $\mu_{\mathbf{A}} = (\mu_j)_{j\in J}$ is called the membership function of the multi-fuzzy set $\mathbf{A}$ and $\prod_{j\in J} L_j$ is called the value domain.

If J = {1, 2, ..., n} or the set of all natural numbers, then the membership function $\mu_{\mathbf{A}} = \langle \mu_1, \mu_2, ...\rangle$ is a sequence.

In particular, if the sequence of the membership function having precisely n-terms and L_j = [0, 1], for J = {1, 2, ..., n}, then n is called the dimension and $\mathbf{M^nFS}(X)$ denotes the set of all multi-fuzzy sets in X.

Properties of multi-fuzzy sets, relations on multi-fuzzy sets and multi-fuzzy extensions of crisp functions are depend on the order relations defined in the membership functions. Most of the results in the initial papers [12, 13, 15, 16, 18] are based on product order in the membership functions. The paper [21] discussed other order relations like dictionary order, reverse dictionary order on their membership functions.

Let $\{L_j : j \in J\}$ be a family of partially ordered sets, and $\mathbf{A} = \{\langle x, (\mu_j(x))_{j\in J}\rangle : x \in$ X, $\mu_j \in L_j^X$, $j \in J\}$ and $\mathbf{B} = \{\langle x, (\nu_j(x))_{j\in J}\rangle : x \in X,\ \nu_j \in L_j^X,\ j \in J\}$ be multi-fuzzy sets in a nonempty set X. Note that, if the order relation in their membership functions are either product order, dictionary order or reverse dictionary order[16, 21], then;

- $\mathbf{A} = \mathbf{B}$ if and only if $\mu_j(x) = \nu_j(x), \forall x \in X$ and for all $j \in J$
- $\mathbf{A} \sqcup \mathbf{B} = \{\langle x, (\mu_j(x) \vee_j \nu_j(x))_{j\in J}\rangle : x \in X\}$ and
- $\mathbf{A} \sqcap \mathbf{B} = \{\langle x, (\mu_j(x) \wedge_j \nu_j(x))_{j\in J}\rangle : x \in X\}$,

where $\vee_j$ and $\wedge_j$ are the supremum and infimum defined in L_j with partial order relation $\leq_j$. Set inclusion defined as follows:

- In product order, $\mathbf{A} \subset \mathbf{B}$ if and only if $\mu_j(x) < \nu_j(x), \forall x \in X$ and for all $j \in J$.
- In dictionary order, $A \subset B$ if and only if $\mu_1(x) < \nu_1(x)$ or if $\mu_1(x) = \nu_1(x)$ and $\mu_2(x) < \nu_2(x), \forall x \in X$.

Definition 2.6. *Let L be a lattice. A mapping $'$: $L \to L$ is called an order reversing involution [25], if for all $a, b \in L$:*

1. $a \leq b \Rightarrow b' \leq a'$;
2. $(a')' = a$.

Definition 2.7. [23] Let $'$: $M \to M$ and $'$: $L \to L$ be order reversing involutions. A mapping $h : M \to L$ is called an order homomorphism, if it satisfies the conditions:

1. $h(0_M) = 0_L$;
2. $h(\vee a_i) = \vee h(a_i)$;
3. $h^{-1}(b') = (h^{-1}(b))'$,

where $h^{-1} : L \to M$ is defined by, for every $b \in L$,
$h^{-1}(b) = \vee\{a \in M : h(a) \leq b\}$.

Generalized Zadeh extension of crisp functions [24] have prime importance in the study of fuzzy mappings. Sabu Sebastian [16, 13]generalized this concept as multi-fuzzy extension of crisp functions and it is useful to map a multi-fuzzy set into another multi-fuzzy set. In the case of a crisp function, there exists infinitely many multi-fuzzy extensions, even though the domain and range of multi-fuzzy extensions are same.

Definition 2.8. [16] Let $f : X \to Y$ and $h : \prod M_i \to \prod L_j$ be a functions. The multi-fuzzy extension of f and the inverse of the extension are $f : \prod M_i^X \to \prod L_j^Y$ and $f^{-1} : \prod L_j^Y \to \prod M_i^X$ defined by

$$f(A)(y) = \bigvee_{y=f(x)} h(A(x)),\ A \in \prod M_i^X,\ y \in Y$$

and

$$f^{-1}(B)(x) = h^{-1}(B(f(x))),\ B \in \prod L_j^Y,\ x \in X;$$

where h^{-1} is the upper adjoint [23] of h. The function $h : \prod M_i \to \prod L_j$ is called the **bridge function** of the multi-fuzzy extension of f.

Remark 2.9. In particular, the multi-fuzzy extension of a crisp function $f : X \to Y$ based on the bridge function $h : I^k \to I^n$ can be written as $f : \mathbf{M^kFS}(X) \to \mathbf{M^nFS}(Y)$ and $f^{-1} : \mathbf{M^nFS}(Y) \to \mathbf{M^kFS}(X)$, where

$$f(A)(y) = \sup_{y=f(x)} h(A(x)),\ A \in \mathbf{M^kFS}(X),\ y \in Y$$

and

$$f^{-1}(B)(x) = h^{-1}(B(f(x))), B \in \mathbf{M^nFS}(Y), x \in X.$$

In the following section $\prod M_i = \prod L_j = I^3$.

Remark 2.10. There exists infinitely many bridge functions. Lattice homomorphism, order homomorphism, lattice valued fuzzy lattices and strong L-fuzzy lattices are examples of bridge functions.

Definition 2.11. [10] A function $t : [0, 1] \times [0, 1] \to [0, 1]$ is a t-norm if $\forall a, b, c \in [0, 1]$:**(1)** $t(a, 1) = a$;

(2) $t(a, b) = t(b, a)$;

(3) $t(a, t(b, c)) = t(t(a, b), c)$;

(4) $b \leq c$ implies $t(a, b) \leq t(a, c)$.

Similarly, a t-conorm (s-norm) is a commutative, associative and non-decreasing mapping s :[0, 1] $\times$ [0, 1] $\to$ [0, 1] that satisfies the boundary condition:

$s(a, 0) = a$, for all $a \in [0, 1]$.

Definition 2.12. [9] A function $c : [0, 1] \to [0, 1]$ is called a complement (fuzzy) operation, if it satisfies the following conditions:

(1) $c(0) = 1$ and $c(1) = 0$,

(2) for all $a, b \in [0, 1]$, if $a \leq b$, then $c(a) \geq c(b)$.

Definition 2.13. [9] A t-norm t and a t-conorm s are dual with respect to a fuzzy complement operation c if and only if

$c(t(a, b)) = s(c(a), c(b))$

and

$c(s(a, b)) = t(c(a), c(b))$,

for all $a, b \in [0, 1]$.

Definition 2.14. [9] Let n be an integer greater than or equal to 2. A function $m : [0, 1]^n \to [0, 1]$ is said to be an aggregation operation for fuzzy sets, if it satisfies the following conditions:

1. m is continuous;
2. m is monotonic increasing in all its arguments;
3. $m(0, 0, ..., 0) = 0$;
4. $m(1, 1, ..., 1) = 1$.

3 Neutrosophic Sets

In this section, we generalize the definition of neutrosophic sets on [0, 1]. Throughout the following sections $X$is the universe of discourse and $A \in \mathbf{M^3FS}(X)$ means A is a multi-fuzzy sets of dimension 3 with value domain I^3, where $I^3 = [0, 1] \times [0, 1] \times [0, 1]$. That is, $A \in (I^3)^X$.

Definition 3.1. Let X be a nonempty crisp set and $0 \leq \alpha \leq 3$. A multi-fuzzy set $A \in \mathbf{M^3FS}(X)$is called a neutrosophic set of order α, if

$\mathbf{A} = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X,$
$0 \leq T_A(x) + I_A(x) + F_A(x) \leq \alpha\}$.

Definition 3.2. Let A, B be neutrosophic sets in X of order 3 and let t, s, m, c be the t-norm, s-norm, aggregation operation and complement operation respectively. Then the union, intersection and complement are given by

1. $A \bigcup B = \{\langle x, s(T_A(x), T_B(x)), m(I_A(x), I_B(x)), t(F_A(x), F_B(x))\rangle : x \in X\}$;
2. $A \bigcap B = \{\langle x, t(T_A(x), T_B(x)), m(I_A(x), I_B(x)), s(F_A(x), F_B(x))\rangle : x \in X\}$;
3. $A^c = \{\langle x, c(T_A(x)), c(I_A(x)), c(F_A(x))\rangle : x \in X\}$.

4 Extension of crisp functions on neutrosophic set using order homomorphism as bridge function

Theorem 4.1. If an order homomorphism $h : I^3 \to I^3$ is the bridge function for the multi-fuzzy extension of a crisp function $f : X \to Y$, then for every $k \in K$ neutrosophic sets A_k in X and B_k in Y of order 3;

1. $A_1 \subseteq A_2$ implies $f(A_1) \subseteq f(A_2)$;
2. $f(\cup A_k) = \cup f(A_k)$;
3. $f(\cap A_k) \subseteq \cap f(A_k)$;
4. $B_1 \subseteq B_2$ implies $f^{-1}(B_1) \subseteq f^{-1}(B_2)$;
5. $f^{-1}(\cup B_k) = \cup f^{-1}(B_k)$;
6. $f^{-1}(\cap B_k) = \cap f^{-1}(B_k)$;
7. $(f^{-1}(B))' = f^{-1}(B')$;
8. $A \subseteq f^{-1}(f(A))$;
9. $f(f^{-1}(B)) \subseteq B$.

Proof.

1. $A_1 \subseteq A_2$ implies $A_1(x) \leq A_2(x), \forall x \in X$ and implies $h(A_1(x)) \leq h(A_2(x)), \forall x \in X$.

Hence $\vee\{h(A_1(x)) : x \in X, y = f(x)\} \leq \vee\{h(A_2(x)) : x \in X, y = f(x)\}$ and $f(A_1)(y) \leq f(A_2)(y)$, $\forall y \in Y$. That is, $f(A_1) \subseteq f(A_2)$.

2. For every $y \in Y$,

$$\begin{aligned} f(\cup A_k)(y) &= \vee\{h((\cup A_k)(x)) : x \in X, y = f(x)\} \\ &= \vee\{h(\vee A_k(x)) : x \in X,\ y = f(x)\} \\ &= \vee\{\vee_{k\in K} h(A_k(x)) : x \in X,\ y = f(x)\} \\ &= \vee_{k\in K} \vee \{h(A_k(x)) : x \in X,\ y = f(x)\} \\ &= \vee_{k\in K} f(A_k)(y), \end{aligned}$$

thus $f(\cup A_k) = \cup f(A_k)$.

3. For every $y \in Y$,

$$\begin{aligned} f(\cap A_k)(y) &= \vee\{h((\cap A_k)(x)) : x \in X, y = f(x)\} \\ &= \vee\{h(\wedge_{k\in K} A_k(x)) : x \in X,\ y = f(x)\} \\ &\leq \vee\{h(A_k(x)) : x \in X,\ y = f(x)\}, \end{aligned}$$

for each $k \in K$. Hence

$f(\cap A_k)(y) \leq \wedge_{k\in K} \vee \{h(A_k(x)) : x \in X, y = f(x)\} = \wedge_{k\in K} f(A_k)(y)$,

thus $f(\cap A_k) \subseteq \cap f(A_k)$.

4. $B_1 \subseteq B_2$ implies $B_1(y) \leq B_2(y), \forall y \in Y$. Hence

$f^{-1}(B_1)(x) = h^{-1}(B_1(f(x))) \leq h^{-1}(B_2(f(x))) = f^{-1}(B_2)(x), \forall x \in X$.

Therefore, $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

5. For every $x \in X$, we have

$$\begin{aligned} f^{-1}(\cup B_k)(x) &= h^{-1}((\cup B_k)(f(x))) = h^{-1}(\sup_{k\in K} B_k(f(x))) \\ &= \sup_{k\in K} h^{-1}(B_k(f(x))) = \sup_{k\in K} f^{-1}(B_k)(x) \\ &= (\cup f^{-1}(B_k))(x). \end{aligned}$$

Hence $f^{-1}(\cup B_k) = \cup f^{-1}(B_k)$.

6. For every $x \in X$, we have

$$\begin{aligned} f^{-1}(\cap B_k)(x) &= h^{-1}((\cap B_k)(f(x))) = h^{-1}(\inf_{k\in K} B_k(f(x))) \\ &= \inf_{k\in K} h^{-1}(B_k(f(x))) = \inf_{k\in K} f^{-1}(B_k)(x) \\ &= (\cap f^{-1}(B_k))(x). \end{aligned}$$

Hence $f^{-1}(\cap B_k) = \cap f^{-1}(B_k)$.

7. For every $x \in X$,

$f^{-1}(B')(x) = h^{-1}(B'(f(x))) = h^{-1}(B(f(x)))' = (f^{-1}(B))'(x)$, since $f^{-1}(B)(x) = h^{-1}(B(f(x)))$.

That is, $f^{-1}(B') = (f^{-1}(B))'$.

8. For every $x_0 \in X$,

$$\begin{aligned} A(x_0) &\leq \vee\{A(x) : x \in X,\ x \in f^{-1}(f(x_0)\} \\ &\leq h^{-1}(h(\vee\{A(x) : x \in X,\ x \in f^{-1}(f(x_0)\})) \\ &= h^{-1}(\vee\{h(A(x)) : x \in X,\ x \in f^{-1}(f(x_0))\}) \\ &= h^{-1}(f(A)(f(x_0))) \\ &= f^{-1}(f(A))(x_0). \end{aligned}$$

9. For every $y \in Y$

$$\begin{aligned} f(f^{-1}(B))(y) &= \sup_{y=f(x)} h(f^{-1}(B)(x)) \\ &= \sup_{y=f(x)} h(h^{-1}(B(f(x)))) \end{aligned}$$

$$= \quad h(h^{-1}(B(y)))$$
$$\leq \quad B(y).$$

Hence $f(f^{-1}(B)) \subseteq B$.

Proposition 4.2. If an order homomorphism $h : I^3 \to I^3$ is the bridge function for the extension of a crisp function $f : X \to Y$, then for any $k \in K$ neutrosophic sets A_k in X and B in Y:

1. $f(0_X) = 0_Y$;
2. $f(\cup A_k) = \cup f(A_k)$; and
3. $(f^{-1}(B))' = f^{-1}(B')$,

that is, the extension map f is an order homomorphism.

Acknowledgement

The authors are very grateful to referees for their constructive comments and suggestions.

References

[1] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87-96.
[2] H. Bustince and P. Burillo, Vague sets are intuitionistic fuzzy sets, Fuzzy Sets and Systems 79 (1996) 403-405.
[3] D. Coker, Fuzzy rough sets are intuitionistic L-fuzzy sets, Fuzzy Sets and Systems 96 (1998) 381-383.
[4] J.L. Deng, Introduction to grey system theory, J. rey Systems 1 (1989) 1-24.
[5] G. Deschrijver and E.E. Kerre, On the relationship between some extensions of fuzzy set theory, Fuzzy Sets and Systems 133 (2003) 227-235.
[6] C.A. Drossos, Foundations of fuzzy sets: a nonstandard approch, Fuzzy Sets and Systems 37 (1990) 287-307.
[7] W.L. Gau and D.J. Buehrer, Vague sets, IEEE Trans. Syst., Man, Cybern. 23 (1993) 610-614.
[8] J.A. Goguen, L-fuzzy sets, J. Math. Anal. Appl. 18 (1967) 145-174.
[9] G.J. Klir and B. Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice-Hall, New Delhi (1995).
[10] K. Menger, Statistical metrics, Proc. the National Academy of Sciences of the United States of America 28 (1942) 535-537.
[11] T. V. Ramakrishnan and S. Sebastian, A study on multi-fuzzy sets, Int. J. Appl. Math. 23 (2010) 713–721.
[12] S. Sebastian and T. V. Ramakrishnan, Multi-fuzzy sets, Int. Math. Forum 50 (2010) 2471–2476.
[13] S. Sebastian and T. V. Ramakrishnan, Multi-fuzzy sets: an extension of fuzzy sets, Fuzzy Inf. Eng. 1 (2011) 35–43.
[14] S. Sebastian and T. V. Ramakrishnan, Multi-fuzzy extensions of functions, Advance in Adaptive Data Analysis 3 (2011) 339 – 350.
[15] S. Sebastian and T. V. Ramakrishnan, Multi-fuzzy extension of crisp functions using bridge functions, Ann. Fuzzy Math. Inform. 2 (2011) 1–8.
[16] S. Sebastian and T. V. Ramakrishnan, Multi-fuzzy topology, Int. J. Appl. Math. 24 (2011) 117–129.
[17] S. Sebastian and T. V. Ramakrishnan, Multi-fuzzy subgroups, Int. J. Contemp. Math. Sci. 6 (2011) 365–372.
[18] S. Sebastian and T. V. Ramakrishnan, Atanassov intuitionistic fuzzy sets generating maps, J. Intell. Fuzzy Systems 25 (2013) 859–862.
[19] S. Sebastian, A Study on Multi-Fuzziness, Ph.D Thesis 2011.
[20] S. Sebastian, Multi-Fuzzy Sets, Lap Lambert Academic Publishing, Germany 2013.
[21] S. Sebastian and R. John, Multi-fuzzy sets and their correspondence to other sets, Ann. Fuzzy Math. Inform. 11(2) (2016) 341–348.
[22] F. Smarandache, Neutrosophy. Neutrosophic Probability, Set, and Logic, Amer. Res. Press, Rehoboth, USA, 105 p., 1998, 2000, 2002, 2005, 2006.
[23] G.J. Wang, Order-homomorphism on fuzzes, Fuzzy Sets and Systems, 12 (1984) 281-288.
[24] H. Wei, Generalized Zadeh function, Fuzzy Sets and Systems 97 (1998) 381-386.
[25] L. Ying-Ming and L. Mao-Kang, Fuzzy Topology, World Scientific, Singapore 1997.
[26] L.A. Zadeh, Fuzzy sets, Information and Control, 8 (1965) 338-353.
[27] F. Smarandache, M. Ali, Neutrosophic Triplet as eñtension of Matter Plasma, Unmatter Plasma, and Antimatter Plasma, 69th Annual Gaseous Electronics Conference, Bochum, Germany, Ruhr-Universitat, October 10-14, 2016.
[28] F. Smarandache, Neutrosophic Perspectives: Triplets, Duplets, Multisets, Hybrid Operators, Modal Logic, Hedge Algebras. And Applications. Pons Editions, 325 p., 2017.

Received: August 4, 2017. Accepted: August 24, 2017.

NSS
University of New Mexico

Neutrosophy for software requirement prioritization

Ronald Barriga Dias[1], Wilmer Ortiz Choez [2], Inelda Martillo Alcivar[3], Wilber Ortiz Aguilar[4]

[1]Universidad de Guayaquil, Facultad de Ciencias Matemáticas y Físicas, Guayaquil Ecuador. E-mail: ronald.barrigad@ug.edu.ec
[2]Universidad de Guayaquil, Facultad de Comunicaciòn Social, Guayaquil Ecuador. E-mail: wilmer.ortizc@ug.edu.ec
[3]Universidad de Guayaquil, Facultad de Ciencias Matemáticas y Físicas, Guayaquil Ecuador. E-mail: inelda.martilloa@ug.edu.ec
[4]Universidad de Guayaquil, Facultad de Ciencias Matemáticas y Físicas, Guayaquil Ecuador. E-mail: Wilber.ortiza@ug.edu

Abstract
Software engineers are involved in complex decisions that require multiples viewpoints. A specific case is the requirement prioritization process. This process is used to decide which software requirement to develop in certain release from a group of candidate requirements. Criteria involved in this process can involve indeterminacy. In this paper a software requirement prioritization model is develop based SVN numbers. Finally, an illustrative example is presented in order to show the proposed model.

Keywords: requirement engineering, software requirement prioritization, SVN numbers.

1. Introduction

Software quality is influenced by the ability to satisfy client and user needs obtained and described in software requirements [1]. Many models have been proposed for software requirement prioritization [1-7]. However, these proposal present limitation for dealing with indeterminacy

In order to overcome the drawbacks identified, in this contribution we propose a novel requirement prioritization process based on SVN numbers.

In software requirement prioritization intervene different stakeholders approaching to the decision problem from a different points of view. It is moreover a multidimensional problems dealing with multiple criteria of diverse nature [8]. Therefore, the proposed model is based on a decision analysis scheme [9] and the approach presented in [8]. In order to deal with heterogeneous information provided by several experts.

This paper is structured as follows: Section 2 outlines a scheme of software prioritization. Section 3 shows the theory of neutrosophy. Section 4 presents our framework for software requirements prioritization. Section 5 shows an illustrative example of the proposed model. The paper ends with conclusions and further work recommendations in Section 6.

2. Software requirement prioritization.

One frequent reason that causes low quality software is associated to problems related to identifying and selecting the most important requirements [10]. Software requirement prioritization can be modeled like a decision making problem, making it suitable to a decision analysis scheme[9]. Decision analysis is a discipline whose purpose is to help decision maker to reach a consistent decision [11].

Our proposal for a software requirement prioritization model dealing with indeterminacy is based on the classical decision analysis scheme. In this paper the software requirement prioritization process is modeled as a type of a Multi-Expert Multi-Criteria decision making problem due to the complexity of the problem where multiple criteria and experts are involved [10, 12].

In the software requirement prioritization process, it is very difficult to express reality in a quantitative way. Fuzzy set theory, introduced by Zadeh[13] in 1965, offers a mathematical model to deal with this kind of uncertainty. The fuzzy linguistic approach is based in the fuzzy set theory and especially in linguistic variable concept [14, 15]. This fact is important in software requirement prioritization where evaluation results are used to make decisions by software engineers in high complexity environment [16]. Current process of softeware prioritizationdon't deal with indeterminacy .

3. Neutrosophy

Neutrosophy [17] is a philosophy branch developed for dealing with indeterminacy (Figure 2). Neutrosophy have been the base for developing new methods to handle indeterminate and inconsistent information like neutrosophic sets an neutrosophic logic [18, 19] .

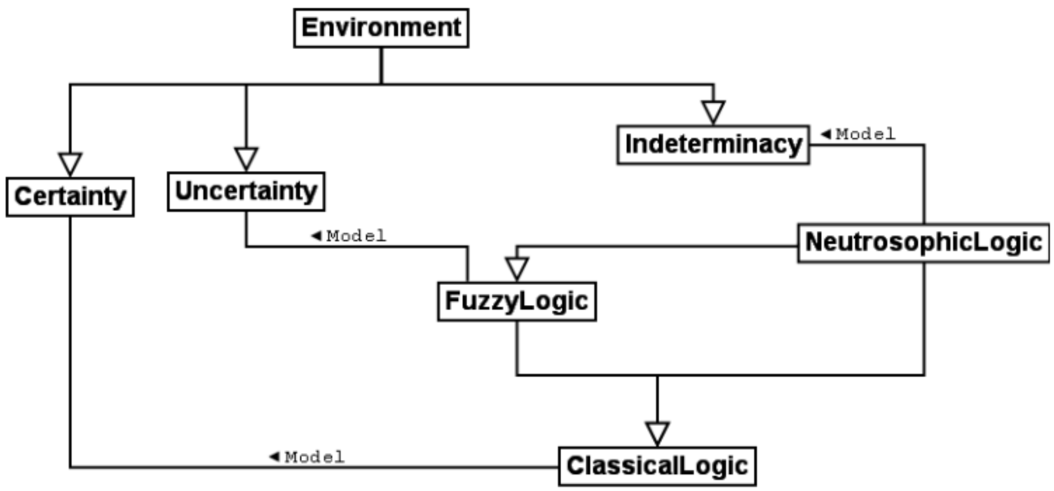


Fig. 1. Static context of Neutrosophic logic [20].

The truth value in neutrosophic set is as follows [21]:

Let N be a set defined as: $N = \{(T, I, F) : T, I, F \subseteq [0, 1]\}$, a neutrosophic valuation n is a mapping from the set of propositional formulas to N , that is for each sentence p we have v (p) $= (T, I, F)$.

Single valued neutrosophic set (SVNS) [22] was developed with the goal of facilitate the real applications of neutrosophic set and set-theoretic operators.

A single valued neutrosophic set (SVNS) has been defined as follows [22]:

Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form :

$$A = \{\langle x, uA(x), rA(x), vA(x)\rangle : x \in X\} \quad (1)$$

where $u_A(x): X \to [0,1]$, $r_A(x), : X \to [0,1]$ and $v_A(x): X \to [0,1]$ with $0 \leq u_A(x) + r_A(x) + v_A(x) :\leq 3$ for all $x \in X$. The intervals $u_A(x)$, $r_A(x)$ y $v_A(x)$ denote the truth- membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively.

Single valued neutrosophic numbers (SVN number) is denoted by A= (a,b,c), where a,b,c∈[0,1] and a+b+c≤3 .

4. A software requirement prioritization model

Our aim is develop a software requirement prioritization model based on the linguistic decision analysis schema that can deal with criteria evaluated with SVN numbers. The model consists of the following phases (graphically, Figure 2):

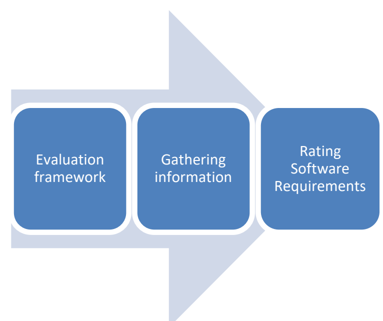


Figura 2. Scheme of the Model.

1. Evaluation framework:

In this phase, the evaluation framework is defined to fix the requirement prioritization problem structure. The framework is established as follows:

- Let E= $\{e_1, e_2, \ldots, e_n\}$ $(n \geq 2)$ be a set of experts.
- Let C=$\{c_1, c_2, \ldots, c_k\}$ $(k \geq 2)$ be a set of criteria.
- Let R=$\{r_1, r_2, \ldots, r_m\}$ $(m \geq 2)$ be a set of requirements.

Each expert can use SVN numbers to asses each criteria, attending to its nature.

2. Gathering information:

Once the framework has been defined, the knowledge of the set of experts must be obtained. Each expert provides their preferences by using utility vectors. The utility vector [23] is represented in the following way:

- $P_j^i = \{p_{j1}^i, p_{j2}^i, \ldots, p_{jh}^i\}.$,

Where $p_{jk}^i$is the preference provided to the criterion c_k of the requirement r_j by the expert e_i.

3. Rating software requirements.

The aim of this phase is to obtain a collective linguistic global assessment easily interpretable for software engineers. To do so the information is unified and aggregated.

Finally those more prioritized are identified. This phase in based the approach reviewed in the Section 3

A two-step aggregation process is developed with the aim of compute a global evaluation of each software requirement.

We obtain for each expert an assessment for each requirement.

The final aim of the rating process is to obtain a global evaluation of each requirement according to all experts. To do so, this process will aggregate all the experts' collective assessment. In decision analysis schema aggregation operating are important for rating options. Some aggregation operators have been proposed for SVN numbers [17, 24]. Single valued neutrosophic weighted averaging (SVNWA) aggregation operator was proposed by Ye [24] for SVNSs as follows[25]:

$$F_w(A_1, A_2, \dots, A_n) = \left(1 - T_{A_j}(x)\right)^{w_j}, \langle 1 - \prod_{j=1}^{n} \prod_{j=1}^{n} \left(I_{A_j}(x)\right)^{w_j}, \rangle \prod_{j=1}^{n} \left(F_{A_j}(x)\right)^{w_j} \quad (2)$$

We propose this operator to establish different weights for each expert, taking into account their knowledge and their significance in software prioritization process

Rating of the requirements

The final step in the prioritization process is to establish a ranking among software requirements, this ranking allows selecting the requirements with more value and postponing or rejecting the development of others making more effective the software development process.

For rating alternatives an ideal option is constructed [26, 27]. The evaluation criteria can be categorized into two categories, benefit and cost. Let C^+ be a collection of benefit criteria and C^- be a collection of cost criteria. The ideal alternative is defined as:

$$I = \left\{\left(max_{i=1}^{k} T_{U_j} | j \in C^+, min_{i=1}^{k} T_{U_j} | j \in C^-\right), \left(min_{i=1}^{k} I_{U_j} | j \in C^+, max_{i=1}^{k} I_{U_j} | j \in C^-\right), \left(min_{i=1}^{k} F_{U_j} | j \in C^+, max_{i=1}^{k} F_{U_j} | j \in C^-\right)\right\} = [v_1, v_2, \dots, v_n] \quad (4)$$

Alternatives are rating according Euclidean distance to I (2). Ranking is based in the global distance to the ideal. If alternative x_i is closer to I the distance measure (s_i closer) better is the alternative [28].

Alternatives could be rated according Euclidean distance in SVN [26, 29].

Let $A^* = (A_1^*, 3A_2^*, \dots, A_n^*)$ be a vector of n SVN numbers such that $A_j^* = (a_j^*, b_j^*, c_j^*)$ j=(1,2, … , n) and $B_i = (B_{i1}, B_{i2}, \dots, B_{im})$ (i = 1,2, … , m) be m vectors of n SVN numbers such that $B_{ij} = (a_{ij}, b_{ij}, c_{ij})$ (i = 1,2, … , m), (j = 1,2, … , n). Then the separation measure between $B_i's$ y A^* is defined as follows:

$$s_i = \left(\frac{1}{3}\sum_{j=1}^{n}\left\{\left(|a_{ij} - a_j^*|\right)^2 + \left(|b_{ij} - b_j^*|\right)^2 + \left(|c_{ij} - c_j^*|\right)^2\right\}\right)^{\frac{1}{2}}$$

$$(i = 1,2, \dots, m) \quad (2)$$

The best requirement is the one with the miimun distance to ideal.

5. Illustrative Example

In this section, we present an illustrative example in order to shown the applicability of the proposed model.

A. Evaluation framework

In this case study the evaluation framework is compose by: 3 experts E= {e_1, e_2, e_3, who evaluate 3 requirements R= $\{r_1, r_2, r_3\}$, where are involved 5 criteria C=$\{c_1, c_2, \dots, c_5\}$which are shown below:

- c_1: Importance for the customers
- c_2: Value
- c_3: Cost
- c_4: Technical Complexity
- c_5: Risks

The following linguistic terms are used (Table I).

Table I. Linguistic terms used to provide the assessments [26].

Linguistic terms	SVNSs
Extremely good (EG)	(1,0,0)
Very very good (VVG)	(0.9, 0.1, 0.1)
Very good (VG)	(0.8,0,15,0.20)
Good (G)	(0.70,0.25,0.30)
Medium good (MG)	(0.60,0.35,0.40)
Medium (M)	(0.50,0.50,0.50)
Medium bad (MB)	(0.40,0.65,0.60)
Bad (B)	(0.30,0.75,0.70)
Very bad (VB)	(0.20,0.85,0.80)
Very very bad (VVB)	(0.10,0.90,0.90)
Extremely bad (EB)	(0,1,1)

B. Gathering information

Once the evaluation framework has been determined the information about therequirements is gathered (see Table II).

Table II. An illustrative example of gathering information

	e_1			e_1			e_1		
	r_1	r_2	r_3	r_1	r_2	r_3	r_1	r_2	r_3
c_1	VVG	VG	EG	VVG	VG	G	M	VG	G
c_2	M	G	MB	M	VG	VG	G	M	MB
c_3	VG	M	M	VG	VVG	M	MB	G	B
c_4	G	M	VG	VG	B	VG	VG	G	G
c_5	M	G	M	G	VG	VVG	B	G	VG

C. Rating Requirements

In this example, is applied a two-step aggregation process to compute a collective evaluation for software requirements. In our problem the SVNWA is used to aggregate evaluations by requirement for each expert. In this case the weighting vectors to compute the collective evaluation is V=(0.3,0.3,0.4) .

Table III. An illustrative example of unified and aggregated information

	r_1	r_2	r_3
c_1	(0.24, 0.2, 0.12)	(0.18, 0.18, 0.14)	(0.19, 0.0, 0.0)
c_2	(0.41, 0.44, 0.35)	(0.32, 0.3, 0.25)	(0.46, 0.44, 0.35)
c_3	(0.38, 0.0, 0.17)	(0.29, 0.27, 0.19)	(0.54, 0.61, 0.5)
c_4	(0.21, 0.21, 0.17)	(0.49, 0.49, 0.41)	(0.21, 0.21, 0.17)
c_5	(0.49, 0.49, 0.41)	(0.24, 0.25, 0.2)	(0.26, 0.23, 0.16)

From this information, the ideal alternative is calculated (Table IV).

Table IV. Ideal alternative

	E^+
c_1	(0.2,0,0)
c_2	(0.4,0.3,0.25)
c_3	(0.38, 0.61,0.5)
c_4	(0.49,0.21,0.17)
c_5	(0.24,0.49,0.41)

The results of the calculation of the distances allow re-queriment.

Table V. Distance to ideal alternative

r_1	0.21
r_2	0.38
r_3	0.45

Finally, we put in order all collective evaluations and we establish a ranking among requirements with the purpose of identifying the best ones. In the case study the ranking is as follow: $r_1 \succ r_2 \succ r_3$

After application in this case study the model is found to be practical to use. The aggregation process gives a high flexibility so the model can be adapted to different situations.

6. Conclusions

In this paper, we have proposed a prioritization model based on the decision analysis scheme that can manage SVN numbers. We have applied the proposed model to an illustrative example. The model was found to be flexible and practical to use. The developing of software tool to automate the model is an area of future work.

References

1 Otero, C.E., et al. A Quality-Based Requirement Prioritization Framework Using Binary Inputs. in Fourth Asia International Conference on Mathematical/Analytical Modelling and Computer Simulation. 2010. IEEE.

2 Mead, N.R., Requirements Prioritization Case Study Using AHP. 2008, Software Engineering Institute.

3 Aurum, A. and C. Wohlin, Engineering and Managing Software Requirements. 2005, New York: Springer.

4 Wei, C.C., C.F. Chien, and M.J.J. Wang, An AHP-based approach to ERP system selection. International Journal of Production Economics, 2005. **96**(1): p. 47-62.

5 Iqbal, M.A., A.M. Zaidi, and S. Murtaza. A New Requirement Prioritization Model for Market Driven Products Using Analytical Hierarchical Process. in International Conference on Data Storage and Data Engineering. 2010. IEEE.

6 Azar, J., R.K. Smith, and D. Cordes, Value-oriented requirements prioritization in a small development organization. IEEE software, 2007: p. 32-37.

7 Logue, K. and K. McDaid. Handling uncertainty in agile requirement prioritization and scheduling using statistical simulation. 2008. IEEE.

8 Herrera-Viedma, E., et al., Applying Linguistic OWA Operators in Consensus Models under Unbalanced Linguistic Information Recent Developments in the Ordered Weighted Averaging Operators: Theory and Practice, R. Yager, J. Kacprzyk, and G. Beliakov, Editors. 2011, Springer Berlin / Heidelberg. p. 167-186.

9 Clemen, R.T., Making hard decisions: an introduction to decision analysis. 1996: Duxbury Pres.

10 Ejnioui, A., C.E. Otero, and A.A. Qureshi. Software requirement prioritization using fuzzy multi-attribute decision making. in Open Systems (ICOS), 2012 IEEE Conference on. 2012. IEEE.

11 Espinilla, M., et al. A heterogeneous evaluation model for assessing sustainable energy: A Belgian case study. in Fuzzy Systems (FUZZ), 2010 IEEE International Conference on. 2010. IEEE.

12 Ramzan, M., et al. Value based fuzzy requirement prioritization and its evaluation framework. in Innovative Computing, Information and Control (ICICIC), 2009 Fourth International Conference on. 2009. IEEE.

13 Zadeh, L.A., Fuzzy sets. Information and Control, 1965. **8**(3): p. 338-353.

14 Berthold, M.R. and D.J. Hand, Intelligent Data Analysis: An Introduction. 2010: Springer.

15 Garcia-Cascales, M.S. and M.T. Lamata, Nueva aproximación al método tópsis difuso con etiquetas lingüísticas, in ESTYLF 2010. 2010: Huelva.

16 Ruhe, G., Software engineering decision support–a new paradigm for learning software organizations, in Advances in Learning Software Organizations. 2003, Springer. p. 104-113.

17 Smarandache, F., A Unifying Field in Logics: Neutrosophic Logic. Philosophy, 1999: p. 1-141.

18 Smarandache, F., A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability: Neutrsophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability. 2005: Infinite Study.

19 Vera, M., et al., Las habilidades del marketing como determinantes que sustentaran la competitividad de la Industria del arroz en el cantón Yaguachi. Aplicación de los números SVN a la priorización de estrategias. Neutrosophic Sets & Systems, 2016. **13**.

20 Salmeron, J.L. and F. Smarandache, Redesigning Decision Matrix Method with an indeterminacy-based inference process. Multispace and Multistructure. Neutrosophic Transdisciplinarity (100 Collected Papers of Sciences), 2010. **4**: p. 151.

21 Rivieccio, U., Neutrosophic logics: Prospects and problems. Fuzzy sets and systems, 2008. **159**(14): p. 1860-1868.

22 Wang, H., Smarandache, F. et al., Single valued neutrosophic sets. Review of the Air Force Academy, 2010(1): p. 10.

23 Espinilla, M., et al., A 360-degree performance appraisal model dealing with heterogeneous information and dependent criteria. Information Sciences, 2012.

24 Ye, J., A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. Journal of Intelligent & Fuzzy Systems, 2014. **26**(5): p. 2459-2466.

25 Biswas, P., S. Pramanik, and B.C. Giri, TOPSIS method for multi-attribute group decision-making under single-valued neutrosophic environment. Neural computing and Applications, 2016. **27**(3): p. 727-737.

26 Şahin, R. and M. Yiğider, A Multi-criteria neutrosophic group decision making metod based TOPSIS for supplier selection. arXiv preprint arXiv:1412.5077, 2014.

27 Leyva-Vázquez, M., K. Pérez-Teruel, and R.I. John. A model for enterprise architecture scenario analysis

based on fuzzy cognitive maps and OWA operators. in Electronics, Communications and Computers (CONIELECOMP), 2014 International Conference on. 2014. IEEE.

28 Leyva-Vázquez, M., et al., Técnicas para la representación del conocimiento causal: un estudio de caso en Informática Médica. Revista Cubana de información en ciencias de la salud, 2013. **24**(1): p. 73-83.

29 Ye, J., Single-valued neutrosophic minimum spanning tree and its clustering method. Journal of intelligent Systems, 2014. **23**(3): p. 311-324.

30 Abdel-Basset, M., Mohamed, M., & Sangaiah, A. K. (2017). Neutrosophic AHP-Delphi Group decision making model based on trapezoidal neutrosophic numbers. Journal of Ambient Intelligence and Humanized Computing, 1-17.https://doi.org/10.1007/s12652-017-0548-7

31 Abdel-Basset, M., Mohamed, M., Hussien, A. N., & Sangaiah, A. K. (2017). A novel group decision-making model based on triangular neutrosophic numbers. Soft Computing, 1-15. http://doi.org/10.1007/s00500-017-2758-5

32 F. Smarandache, M. Ali, Neutrosophic Triplet as extension of Matter Plasma, Unmatter Plasma, and Antimatter Plasma, 69th Annual Gaseous Electronics Conference, Bochum, Germany, Veranstaltungszentrum & Audimax, Ruhr-Universitat, October 10-14, 2016, http://meetings.aps.org/Meeting/GEC16/Session/HT6.112

33 F. Smarandache, Neutrosophic Perspectives: Triplets, Duplets, Multisets, Hybrid Operators, Modal Logic, Hedge Algebras. And Applications. Pons Editions, Bruxelles, 325 p., 2017.

34 Topal, S. and Öner, T. Contrastings on Textual Entailmentness and Algorithms of Syllogistic Logics, Journal of Mathematical Sciences, Volume 2, Number 4, April 2015, pp 185-189.

35 Topal S. An Object- Oriented Approach to Counter-Model Constructions in A Fragment of Natural Language, BEU Journal of Science, 4(2), 103-111, 2015.

36 Topal, S. A Syllogistic Fragment of English with Ditransitive Verbs in Formal Semantics, Journal of Logic, Mathematics and Linguistics in Applied Sciences, Vol 1, No 1, 2016.

37 Topal, S. and Smaradache, F. A Lattice-Theoretic Look: A Negated Approach to Adjectival (Intersective, Neutrosophic and Private) Phrases. The 2017 IEEE International Conference on INnovations in Intelligent SysTems and Applications (INISTA 2017); (accepted for publication).

38 Taş, F. and Topal, S. Bezier Curve Modeling for Neutrosophic Data Problem. Neutrosophic Sets and Systems, Vol 16, pp. 3-5, 2017.

Received: August 10, 2017. Accepted: August 31, 2017.

Information about the journal:

Neutrosophic Sets and Systems has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics, and their applications in any field.

The papers should be professional, in good English, containing a brief review of a problem and obtained results.

All submissions should be designed in MS Word format using our template file:
http://fs.gallup.unm.edu/NSS/NSS-paper-template.doc

To submit a paper, mail the file to the Editor-in-Chief. To order printed issues, contact the Editor-in-Chief. This journal is non-commercial, academic edition. It is printed from private donations.

The neutrosophics website at UNM is: http://fs.gallup.unm.edu/neutrosophy.htm

The home page of the journal is accessed on http://fs.gallup.unm.edu/NSS

Editor-in-Chief:

Prof. Florentin Smarandache
Department of Mathematics and Science
University of New Mexico
705 Gurley Avenue
Gallup, NM 87301, USA

E-mails: fsmarand@unm.edu

23316055

$39.95