Neutrosophic Sets and Systems

Volume 14

Article 9

1-1-2016

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Abstract: The objective of this paper is to introduced the concept of neutrosophic cubic set to subalgebras, ideals and closed ideals of B-algebra. Links among neutrosophic cubic subalgebra with neutrosophic cubic ideals and neutrosophic closed ideals of B-algebras as well as some related properties will be investigated. This study will cover homomorphic images and inverse homomorphic images of neutrosophic cubic subalgebras, ideals and some related properties. The Cartesian product of neutrosophic cubic subalgebras will also be investigated.

Keywords: B-algebra, Neutrosophic cubic set, Neutrosophic cubic subalgebra, Neutrosophic cubic closed ideals.

1 Introduction

The concept of fuzzy sets were first introduced by Zadeh (see [31]) in 1965. After that several researchers conducted researches on generalization of fuzzy sets notion. Zadeh (see [32]) generalized the concept of fuzzy set by an interval-valued fuzzy set in 1975, as a generalization of the notion. The concept of cubic sets had been introduced by Jun et al. (see [6]) in 2012, as generalization of fuzzy set and interval-valued fuzzy set. Jun et al. (see [7]) applied the notion of cubic sets to a group, and introduced the notion of cubic subgroups in 2011. Senapati et. al. (see [25]) extended the concept of cubic set to subalgebras, ideals and closed ideals of B-algebra with lots of properties investigated. After the introduction of two classes BCK-algebra and BCI-algebra by Imai and Iseki (see [4, 5]). The concept of cubic sets to subalgebras, ideals and q-ideals in BCK/BCI-algebras was applied by Jun et al. (see [9, 10]). B-algebra was introduced by Neggers and Kim (see [12]) in 2002, which are related to extensive classes of algebras such as BCI/BCK-algebras. The relations between B-algebra and other topics were further discussed by Cho and Kim in (see [3]) 2001. Every quadratic B-algebra on field X with a BCI-algebra was obtained by Park and Kim (see [14]) in 2001. The notion of fuzzy topological B-algebra was introduced by Borumand Saeid (see [15]) in 2006. Also Saeid introduced the concept of interval-valued fuzzy subalgebra of B-algebra (see [16]) in 2006. Also some of their properties were studied by him. Walendziak (see [30]) gave some systems of axioms defining a B-algebra with the proof of the independent of axioms in 2006. Fuzzy dot subalgebras, fuzzy dot ideals, interval-valued fuzzy closed ideals of B-algebra and fuzzy subalgebras of B-algebras with respect to t-norm were introduced by Senapati et. al. (see [20, 21, 22, 23]). Also L-fuzzy G-subalgebras of G-algebras were introduced by Senapati et. al. (see [24]) in 2014 which is related to B-algebra. As a generalizations of B-algebras, lots of researches on BG-algebras (see [11]) have been done by the authors (see [26, 27, 28, 29]).

Smarandache (see [19, 18]) introduced the concept of neutrosophic cubic set is a more general research area which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Jun et. al. (see [8]) extended the concept of cubic set to neutrosophic cubic set and introduced. The notion of truth-internal (indeterminacy-internal, falsity-internal) and truth-external (indeterminacy-external, falsity-external) are introduced and related properties are investigated.

In this paper, we will introduce the concept of neutrosophic cubic set to subalgebras, ideals and closed ideals of B-algebras and introduce the notion of neutrosophic cubic set and subalgebras. Relation among neutrosophic cubic algebra with neutrosophic cubic ideals and neutrosophic closed ideals of B-algebras are studied and some related properties will be investigated. This study will cover homomorphic images and inverse homomorphic images of neutrosophic cubic subalgebras, ideals, some related properties. The Cartesian product of neutrosophic cubic subalgebras will also be investigated.

2 Preliminaries

In this section, some basic facets are included that are necessary for this paper. A B-algebra is an important class of logical algebras introduced by Neggers and Kim [12] and extendedly investigated by several researchers. This algebra is defined as follows.

A non-empty set X with constant 0 and a binary operation $*$ is called to be B-algebra [12] if it satisfies the following axioms:

B1. $x * x = 0$

B2. $x * 0 = x$

B3. $(x * y) * z = x * (z * (0 * y))$

A non-empty subset S of B-algebra X is called a subalgebra [1] of X if $x*y \in S \,\forall\, x, y \in S$. Mapping $f \mid X \to Y$ of B-algebras is called homomorphism [13] if $f(x * y) = f(x) * f(y) \,\forall\, x, y \in X$. Note that if $f \mid X \to Y$ is a $B-$homomorphism, then $f(0) = 0$. A non-empty subset I of a $B-$algebra X is called an ideal [22] if for any $x, y \in X$, (i) $0 \in I$, and (ii) $x * y \in I$ and $y \in I \Rightarrow x \in I$. An ideal I of a $B-$algebra X is called closed if $0 * y \in I \,\forall\, x \in I$.

We know review some fuzzy logic concepts as follows:

Let X be the collection of objects denoted generally by x. Then a fuzzy set [31] A in X is defined as $A = \{< x, \mu_A(x) >\mid x \in X\}$, where $\mu_A(x)$ is called the membership value of x in A and $\mu_A(x) \in [0, 1]$.

For a family $A_i = \{< x, \mu_{A_i}(x) >\mid x \in X\}$ of fuzzy sets in X, where $i \in k$ and k is index set, we define the join $(\vee)$ meet $(\wedge)$ operations as follows:

$$\bigvee_{i\in k} A_i = \Big(\bigvee_{i\in k} \mu_{A_i}\Big)(x) = sup\{\mu_{A_i} \mid i \in k\},$$

and

$$\bigwedge_{i\in k} A_i = \Big(\bigwedge_{i\in k} \mu_{A_i}\Big)(x) = inf\{\mu_{A_i} \mid i \in k\}$$

respectively, $\forall\, x \in X$.

An Interval-valued fuzzy set [32] A over X is an object having the form $A = \{< x, \tilde{\mu}_A(x) >\mid x \in X\}$, where $\tilde{\mu}_A \mid X \to D[0, 1]$, here $D[0, 1]$ is the set of all subintervals of [0,1]. The intervals $\tilde{\mu}_A x = [\mu^-_A(x), \mu^+_A(x)] \,\forall\, x \in X$ denote the degree of membership of the element x to the set A. Also $\tilde{\mu}^c_A = [1 - \mu^-_A(x), 1 - \mu^+_A(x)]$ represents the complement of $\tilde{\mu}_A$.

For a family $\{A_i \mid i \in k\}$ of interval-valued fuzzy sets in X where k is an index set, the union $G = \bigcup_{i\in k} \tilde{\mu}_{A_i}(x)$ and the intersection $F = \bigcap_{i\in k} \tilde{\mu}_{A_i}(x)$ are defined below:

$$G(x) = \Big(\bigcup_{i\in k} \tilde{\mu}_{A_i}\Big)(x) = rsup\{\tilde{\mu}_{A_i}(x) \mid i \in k\}$$

and

$$F(x) = \Big(\bigcap_{i\in k} \tilde{\mu}_{A_i}\Big)(x) = rinf\{\tilde{\mu}_{A_i}(x) \mid i \in k\},$$

respectively, $\forall\, x \in X$.

The determination of supremum and infimum between two real numbers is very simple but it is not simple for two intervals. Biswas [2] describe a method to find max/sup and min/inf between two intervals or a set of intervals.

Definition 2.1 *[2] Consider two elements $D_1, D_2 \in D[0, 1]$. If $D_1 = [a^-_1, a^+_1]$ and $D_2 = [a^-_2, a^+_2]$, then $rmax(D_1, D_2) = [max(a^-_1, a^-_2), max(a^+_1, a^+_2)]$ which is denoted by $D_1 \vee^r D_2$ and $rmin(D_1, D_2) = [min(a^-_1, a^-_2), min(a^+_a, a^+_2)]$ which is denoted by $D_1 \wedge^r D_2$. Thus, if $D_i = [a^-_i, a^+_2] \in D[0, 1]$ $for\, i = 1, 2, 3, \ldots$, then we define $rsup_i(D_i) = [sup_i(a^-_i), sup_i(a^+_i)]$, i.e., $\vee^r_i D_i = [\vee_i a^-_i, \vee_i a^+_i]$. Similarly we define $rinf_i(D_i) = [inf_i(a^-_i), inf_i(a^+_i)]$, i.e., $\wedge^r_i D_i = [\wedge_i a^-_i, \wedge_i a^+_i]$. Now we call $D_1 \geq D_2 \iff a^-_1 \geq a^-_2$ and $a^+_1 \geq a^+_2$. Similarly the relations $D_1 \leq D_2$ and $D_1 = D_2$ are defined.*

Combine the definition of subalgebra, ideal over crisp set and the idea of fuzzy set Ahn et al. [1] and senapati et al. [21] defined fuzzy subalgebra and ideal respectively, which is define bellow.

Definition 2.2 *[21, 1] A fuzzy set $A = \{< x, \mu_A(x) >\mid x \in X\}$ is called a fuzzy subalgebra of X if $\mu_A(x * y) \geq min\mu_A(x), \mu_A(y) \,\forall\, x, y \in X$,*

A fuzzy set $A = \{< x, \mu_A(x) >\mid x \in X\}$ in X is called a fuzzy ideal of X if it satisfies (i) $\mu_A(0) \geq \mu_A(x)$ and (ii) $\mu_A(x) \geq min\{\mu_A(x * y), \mu_A(y)\} \,\forall\, x, y \in X$.

Jun et al. [8] extend the concept of cubic sets to neutrosophic sets [17], and consider the notion of neutrosophic cubic sets as an extension of cubic sets, and investigated several properties.

Definition 2.3 *[8] Let X be a non-empty set. A neutrosophic cubic set in X is pair $\mathcal{C} = (\mathbf{A}, \Lambda)$ where $\mathbf{A} =: \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$ is an interval neutrosophic set in X and $\Lambda =: \{\langle x; \lambda_T(x), \lambda_I(x), \lambda_F(x)\rangle \mid x \in X\}$ is a neutrosophic set in X.*

Definition 2.4 *[8] For any $\mathcal{C}_i = (\mathbf{A}_i, \Lambda_i)$ where*
$\mathbf{A}_i =: \{\langle x; A_{iT}(x), A_{iI}(x), A_{iF}(x)\rangle \mid x \in X\}$,
$\Lambda_i =: \{\langle x; \lambda_{iT}(x), \lambda_{iI}(x), \lambda_{iF}(x)\rangle \mid x \in X\}$ for $i \in k$, P-union, P-inersection, R-union and R-intersection is defined respectively by

P-union: $\bigcup_{P\, i\in k} \mathcal{C}_i = \Big(\bigcup_{i\in k} \mathbf{A}_i, \bigvee_{i\in k} \Lambda_i\Big)$,

P-intersection: $\bigcap_{P\, i\in k} \mathcal{C}_i = \Big(\bigcap_{i\in k} \mathbf{A}_i, \bigwedge_{i\in k} \Lambda_i\Big)$

R-union: $\bigcup_{R\, i\in k} \mathcal{C}_i = \Big(\bigcup_{i\in k} \mathbf{A}_i, \bigwedge_{i\in k} \Lambda_i\Big)$,

R-intersection: $\bigcap_{R\, i\in k} \mathcal{C}_i = \Big(\bigcap_{i\in k} \mathbf{A}_i, \bigvee_{i\in k} \Lambda_i\Big)$

where

$$\bigcup_{i\in k} \mathbf{A}_i = \left\{\left\langle x; \Big(\bigcup_{i\in k} A_{iT}\Big)(x), \Big(\bigcup_{i\in k} A_{iI}\Big)(x), \Big(\bigcup_{i\in k} A_{iF}\Big)(x)\right\rangle \mid x \in X\right\},$$

$$\bigvee_{i\in k} \Lambda_i = \left\{\left\langle x; \Big(\bigvee_{i\in k} \lambda_{iT}\Big)(x), \Big(\bigvee_{i\in k} \lambda_{iI}\Big)(x), \Big(\bigvee_{i\in k} \lambda_{iF}\Big)(x)\right\rangle \mid x \in X\right\},$$

$$\bigcap_{i\in k} \mathbf{A}_i = \left\{\left\langle x; \Big(\bigcap_{i\in k} A_{iT}\Big)(x), \Big(\bigcap_{i\in k} A_{iI}\Big)(x), \Big(\bigcap_{i\in k} A_{iF}\Big)(x)\right\rangle \mid x \in X\right\},$$

$$\bigwedge_{i\in k} \Lambda_i = \left\{\left\langle x; \Big(\bigwedge_{i\in k} \lambda_{iT}\Big)(x), \Big(\bigwedge_{i\in k} \lambda_{iI}\Big)(x), \Big(\bigwedge_{i\in k} \lambda_{iF}\Big)(x)\right\rangle \mid x \in X\right\},$$

Senapati et. al. [25] defined the cubic subalgebras of B-algebra by combining the definitions of subalgebra over crisp set and the cubic sets.

Definition 2.5 *[25] Let $\mathrm{C} = \{< x, A(x), \lambda(x) >\}$ be a cubic set, where $A(x)$ is an interval-valued fuzzy set in X, $\lambda(x)$ is a fuzzy set in X and X is subalgebra. Then A is cubic subalgebra*

*under binary operation * if following condition holds:*
C1: $A(x * y) \geq rmin\{A(x), A(y)\}$,
C2: $\lambda(x * y) \leq max\{\lambda(x), \lambda(y)\} \; \forall \, x, y \in X$.

3 Neutrosophic Cubic Subalgebras Of B-algebra

Let X denote a B-algebra then the concept of cubic subalgebra can be extended to neutrosophic cubic subalgebra.

Definition 3.1 *Let $\mathcal{C} = (\mathbf{A}, \Lambda)$ be a cubic set, where X is subalgebra. Then $\mathcal{C}$ is neutrosophic cubic subalgebra under binary operation $*$ if it holds the following conditions: N1:*
$A_T(x * y) \geq rmin\{A_T(x), A_T(y)\}$
$A_I(x * y) \geq rmin\{A_I(x), A_I(y)\}$
$A_F(x * y) \geq rmin\{A_F(x), A_F(y)\}$,
N2:
$\Lambda_T(x * y) \leq max\{\Lambda_T(x), \Lambda_T(y)\}$
$\Lambda_I(x * y) \leq max\{\Lambda_I(x), \Lambda_I(y)\}$
$\Lambda_I(x * y) \leq max\{\Lambda_I(x), \Lambda_I(y)\}$
For our convenience, we will denote neutrosophic cubic set as $\mathcal{C} = (A_{T,I,F}, \lambda_{T,I,F}) = \{\langle x, A_{T,I,F}(x), \lambda_{T,I,F}(x)\rangle\}$ *and conditions N1, N2 as*
N1: $A_{T,I,F}(x * y) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$,
N2: $\lambda_{T,I,F}(x * y) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$.

Example 3.1 *Let $X = \{0, a_1, a_2, a_3, a_4, a_5\}$ be a B-algebra with the following Cayley table.*

$*$	0	a_1	a_2	a_3	a_4	a_5
0	0	a_5	a_4	a_3	a_2	a_1
a_1	a_1	0	a_5	a_4	a_3	a_2
a_2	a_2	a_1	0	a_5	a_4	a_3
a_3	a_3	a_2	a_1	0	a_5	a_4
a_4	a_4	a_3	a_2	a_1	0	a_5
a_5	a_5	a_4	a_3	a_2	a_1	0

A neutrosophic cubic set $\mathcal{C} = (A_{T,I,F}, \lambda_{T,I,F})$ of X is defined by

	0	a_1	a_2	a_3	a_4	a_5
A_T	[0.7,0.9]	[0.6,0.8]	[0.7,0.9]	[0.6,0.8]	[0.7,0.9]	[0.6,0.8]
A_I	[0.3,0.2]	[0.2,0.1]	[0.3,0.2]	[0.2,0.1]	[0.3,0.2]	[0.2,0.1]
A_F	[0.2,0.4]	[0.1,0.4]	[0.2,0.4]	[0.1,0.4]	[0.2,0.4]	[0.1,0.4]

	0	a_1	a_2	a_3	a_4	a_5
λ_T	0.1	0.3	0.1	0.3	0.1	0.3
λ_I	0.3	0.5	0.3	0.5	0.3	0.5
λ_F	0.5	0.6	0.5	0.6	0.5	0.6

.

Both the conditions of Definition 3.1 are satisfied by the set $\mathcal{C}$. Thus $\mathcal{C} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic subalgebra of X.

Proposition 3.1 *Let $\mathcal{C} = \{< x, A_{T,I,F}(x), \lambda_{T,I,F}(x) >\}$ is a neutrosophic cubic subalgebra of X, then $\forall \, x \in X$, $A_{T,I,F}(x) \geq A_{T,I,F}(0)$ and $\lambda_{T,I,F}(0) \leq \lambda_{T,I,F}(x)$. Thus, $A_{T,I,F}(0)$ and $\lambda_{T,I,F}(0)$ are the upper bounds and lower bounds of $A_{T,I,F}(x)$ and $\lambda_{T,I,F}(x)$ respectively.*

Proof: $\forall \, x \in X$, we have $A_{T,I,F}(0) = A_{T,I,F}(x * x) \geq rmin\{ A_{T,I,F}(x), A_{T,I,F}(x)\} = A_{T,I,F}(x) \Rightarrow A_{T,I,F}(0) \geq A_{T,I,F}(x)$ and $\lambda_{T,I,F}(0) = \lambda_{T,I,F}(x * x) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(x)\} = \lambda_{T,I,F}(x) \Rightarrow \lambda_{T,I,F}(0) \leq \lambda_{T,I,F}(x)$.

Theorem 3.1 *Let $\mathcal{C} = \{\langle x, A_{T,I,F}(x), \lambda_{T,I,F}(x)\rangle\}$ be a neutrosophic cubic subalgebras of X. If there exists a sequence $\{x_n\}$ of X such that $\lim\limits_{n\to\infty} A_{T,I,F}(x_n) = [1,1]$ and $\lim\limits_{n\to\infty} \lambda_{T,I,F}(x_n) = 0$. then $A_{T,I,F}(0) = [1,1]$ and $\lambda_{T,I,F}(0) = 0$.*

Proof: Using Proposition 3.1, $A_{T,I,F}(0) \geq A_{T,I,F}(x) \, \forall \, x \in X$, $\therefore A_{T,I,F}(0) \geq A_{T,I,F}(x_n)$ for $n \in \mathbf{Z}^+$. Consider, $[1,1] \geq A_{T,I,F}(0) \geq \lim\limits_{n\to\infty} A_{T,I,F}(x_n) = [1,1]$. Hence, $A_{T,I,F}(0) = [1,1]$.

Again, using Proposition 3.1, $\lambda_{T,I,F}(0) \leq \lambda_{T,I,F}(x) \; \forall \; x \in X$, $\therefore \lambda_{T,I,F}(0) \leq \lambda_{T,I,F}(x_n)$ for $n \in \mathbf{Z}^+$. Consider, $0 \geq \lambda_{T,I,F}(0) \leq \lim\limits_{n\to\infty} \lambda_{T,I,F}(x_n) = 0$. Hence, $\lambda_{T,I,F}(0) = 0$.

Theorem 3.2 *The R-intersection of any set of neutrosophic cubic subalgebras of X is also a neutrosophic cubic subalgebras of X.*

Proof: Let $\mathcal{A}_i = \{\langle x, A_{iT,I,F}, \lambda_{iT,I,F}\rangle \mid x \in X\}$ where $i \in k$, be a sets of neutrosophic cubic subalgebras of X and $x, y \in X$. Then

$$\begin{aligned}(\cap A_{iT,I,F})(x * y) &= rinf A_{iT,I,F}(x * y) \\ &\geq rinf\{rmin\{A_{iT,I,F}(x), A_{iT,I,F}(y)\}\} \\ &= rmin\{rinf A_{iT,I,F}(x), rinf A_{iT,I,F}(y)\} \\ &= rmin\{(\cap A_{iT,I,F})(x), (\cap A_{iT,I,F})(y)\} \\ \Rightarrow (\cap A_{iT,I,F})(x * y) &\geq rmin\{(\cap A_{iT,I,F})(x), (\cap A_{iT,I,F})(y)\}\end{aligned}$$

and

$$\begin{aligned}(\vee \lambda_{iT,I,F})(x * y) &= sup \lambda_{iT,I,F}(x * y) \\ &\leq sup\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(y)\}\} \\ &= max\{sup\lambda_{iT,I,F}(x), sup\lambda_{iT,I,F}(y)\} \\ &= max\{(\vee \lambda_{iT,I,F})(x), (\vee \lambda_{iT,I,F})(y)\} \\ \Rightarrow (\vee \lambda_{iT,I,F})(x * y) &\leq max\{(\vee \lambda_{iT,I,F})(x), (\vee \lambda_{iT,I,F})(y)\},\end{aligned}$$

which shows that R-intersection of $\mathcal{A}_i$ is a neutrosophic cubic subalgebra of X.

Remark 3.1 *The R-union, P-intersection and P-union of neutrosophic cubic subalgebra need not be a neutrosophic cubic subalgebra.*
Example, let $X = \{0, a_1, a_2, a_3, a_4, a_5\}$ be a B-algebra with the following Caley table. Let $\mathcal{A}_1 = (A_{1T,I,F}, \lambda_{1T,I,F})$ and $\mathcal{A}_2 = (A_{2T,I,F}, \lambda_{2T,I,F})$ be neutrosophic cubic set of X defined by

Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are neutrosophic subalgebras of X but R-union, P-union and P-intersection of $\mathcal{A}_1$ and $\mathcal{A}_2$ are not subalgebras of X because

$*$	0	a_1	a_2	a_3	a_4	a_5
0	0	a_2	a_1	a_3	a_4	a_5
a_1	a_1	0	a_2	a_5	a_3	a_4
a_2	a_2	a_1	0	a_4	a_5	a_3
a_3	a_3	a_4	a_5	0	a_1	a_2
a_4	a_4	a_5	a_3	a_2	0	a_1
a_5	a_5	a_3	a_4	a_1	a_2	0

	0	a_1	a_2	a_3	a_4	a_5
A_1T	[0.8,0.7]	[0.1,0.2]	[0.1,0.2]	[0.8,0.7]	[0.1,0.2]	[0.1,0.2]
A_1I	[0.7,0.8]	[0.2,0.3]	[0.2,0.3]	[07.,0.8]	[0.2,0.3]	[0.2,0.3]
A_1F	[0.8,0.9]	[0.3,0.4]	[0.3,0.4]	[0.8,0.9]	[0.3,0.4]	[0.3,0.4]
A_2T	[0.8,0.9]	[0.2,0.3]	[0.2,0.3]	[0.2,0.3]	[0.8,0.9]	[0.2,0.3]
A_2I	[0.7,0.6]	[0.1,0.2]	[0.1,0.2]	[0.1,0.2]	[0.7,0.6]	[0.1,0.2]
A_2F	[0.6,0.5]	[0.1,0.3]	[0.1,0.3]	[0.1,0.3]	[0.6,0.5]	[0.1,0.3]

	0	a_1	a_2	a_3	a_4	a_5
$\lambda_1 T$	0.1	0.8	0.8	0.1	0.8	0.8
$\lambda_1 I$	0.2	0.7	0.7	0.2	0.7	0.7
$\lambda_1 F$	0.4	0.6	0.6	0.4	0.6	0.6
$\lambda_2 T$	0.2	0.5	0.5	0.5	0.2	0.5
$\lambda_2 I$	0.3	0.7	0.7	0.7	0.3	0.7
$\lambda_2 F$	0.4	0.9	0.9	0.9	0.4	0.9

.

$(\cup A_{iT,I,F})(a_3 * a_4) = ([0.2, 0.3], [0.2, 0.3], [0.3, 0.4])_{T,I,F} \not\succeq ([0.8, 0.9], [0.7, 0.6], [0.6, 0.5])_{T,I,F} = rmin\{(\cup A_{iT,I,F})(a_3), (\cup A_{iT,I,F})(a_4)\}$
and
$(\wedge\lambda_{iT,I,F})(a_3 * a_4) = (0.8, 0.7, 0.9)_{T,I,F} \not\preceq (0.2, 0.3, 0, 4)_{T,I,F} = max\{(\wedge\lambda_{iT,I,F})(a_3), (\wedge\lambda_{iT,I,F})(a_4)\}$

We provide the condition that R-union, P-union and P-intersection of neutrosophic cubic subalgebras is also a neutrosophic cubic subalgebra. which are at Theorem 3.3, 3.4 and 3.5.

Theorem 3.3 *Let $\mathcal{A}_i = \{\langle x, A_{iT,I,F}, \lambda_{iT,I,F}\rangle \mid x \in X\}$ where $i \in k$, be a sets of neutrosophic cubic subalgebras of X, where $i \in k$. If $inf\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(x)\}\} = max\{inf\lambda_{iT,I,F}(x), inf\lambda_{iT,I,F}(x)\}\ \forall\ x \in X$, then the P-intersection of $\mathcal{A}_i$ is also a neutrosophic cubic subalgebras of X.*

Proof: Suppose that $\mathcal{A}_i = \{\langle x, A_{iT,I,F}, \lambda_{iT,I,F}\rangle \mid x \in X\}$ where $i \in k$, be sets of neutrosophic cubic subalgebras of X such that $inf\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(x)\}\} = max\{inf\lambda_{iT,I,F}(x), inf\lambda_{iT,I,F}(x)\}\ \forall\ x \in X$. Then for $x, y \in X$. Then

$$\begin{aligned}(\cap A_{iT,I,F})(x * y) &= rinf A_{iT,I,F}(x * y)\\ &\geq rinf\{rmin\{A_{iT,I,F}(x), A_{iT,I,F}(y)\}\}\\ &= rmin\{rinf A_{iT,I,F}(x), rinf A_{iT,I,F}(y)\}\\ &= rmin\{(\cap A_{iT,I,F})(x), (\cap A_{iT,I,F})(y)\}\\ \Rightarrow (\cap A_{iT,I,F})(x * y) &\geq rmin\{(\cap A_{iT,I,F})(x), (\cap A_{iT,I,F})(y)\}\end{aligned}$$

and

$$\begin{aligned}(\wedge\lambda_{iT,I,F})(x * y) &= inf\lambda_{iT,I,F}(x * y)\\ &\leq inf\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(y)\}\}\\ &= max\{inf\lambda_{iT,I,F}(x), inf\lambda_{iT,I,F}(y)\}\\ &= max\{(\wedge\lambda_{iT,I,F})(x), (\wedge\lambda_{iT,I,F})(y)\}\\ \Rightarrow (\wedge\lambda_{iT,I,F})(x * y) &\leq max\{(\wedge\lambda_{iT,I,F})(x), (\wedge\lambda_{iT,I,F})(y)\},\end{aligned}$$

which shows that P-intersection of $\mathcal{A}_i$ is a neutrosophic cubic subalgebra of X.

Theorem 3.4 *Let $\mathcal{A}_i = \{\langle x, A_{iT,I,F}, \lambda_{iT,I,F}\rangle \mid x \in X\}$ where $i \in k$, be a sets of neutrosophic cubic subalgebras of X. If $sup\{rmin\{A_{iT,I,F}(x), A_{iT,I,F}(x)\}\} = rmin\{supA_{iT,I,F}(x), supA_{iT,I,F}(x)\}\ \forall x \in X$, then the P-union of $\mathcal{A}_i$ is also a neutrosophic cubic subalgebra of X.*

Proof: Let $\mathcal{A}_i = \{\langle x, A_{iT,I,F}, \lambda_{iT,I,F}\rangle \mid x \in X\}$ where $i \in k$, be a sets of neutrosophic cubic subalgebras of X such that $sup\{rmin\{A_{iT,I,F}(x), A_{iT,I,F}(x)\}\} = rmin\{supA_{iT,I,F}(x), supA_{iT,I,F}(x)\}\ \forall\ x \in X$. Then for $x, y \in X$,

$$\begin{aligned}(\cup A_{iT,I,F})(x * y) &= rsupA_{iT,I,F}(x * y)\\ &\geq rsup\{rmin\{A_{iT,I,F}(x), A_{iT,I,F}(y)\}\}\\ &= rmin\{rsupA_{iT,I,F}(x), rsupA_{iT,I,F}(y)\}\\ &= rmin\{(\cup A_{iT,I,F})(x), (\cup A_{iT,I,F})(y)\}\\ (\cup A_{iT,I,F})(x * y) &\geq rmin\{(\cup A_{iT,I,F})(x), (\cup A_{iT,I,F})(y)\}\end{aligned}$$

and

$$\begin{aligned}(\vee\lambda_{iT,I,F})(x * y) &= sup\lambda_{iT,I,F}(x * y)\\ &\leq sup\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(y)\}\}\\ &= max\{sup\lambda_{iT,I,F}(x), sup\lambda_{iT,I,F}(y)\}\\ &= max\{(\vee\lambda_{iT,I,F})(x), (\vee\lambda_{iT,I,F})(y)\}\\ (\vee\lambda_{iT,I,F})(x * y) &\leq max\{(\vee\lambda_{iT,I,F})(x), (\vee\lambda_{iT,I,F})(y)\}.\end{aligned}$$

Which shows that P-union of $\mathcal{A}_i$ is a neutrosophic cubic subalgebra of X.

Theorem 3.5 *Let $\mathcal{A}_i = \{\langle x, A_{iT,I,F}, \lambda_{iT,I,F}\rangle \mid x \in X\}$ where $i \in k$, be a sets of neutrosophic cubic subalgebras of X. If $inf\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(x)\}\} = max\{inf\lambda_{iT,I,F}(x), inf\lambda_{iT,I,F}(x)\}$ and $sup\{rmin\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(x)\}\} = rmin\{sup\lambda_{iT,I,F}(x), sup\lambda_{iT,I,F}(x)\}\ \forall\ x \in X$, then the R-union of $\mathcal{A}_i$ is also a neutrosophic cubic subalgebra of X.*

Proof: Let $\mathcal{A}_i = \{\langle x, A_{iT,I,F}, \lambda_{iT,I,F}\rangle \mid x \in X\}$ where $i \in k$, be a sets of neutrosophic cubic subalgebras of X such that $inf\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(x)\}\} = max\{inf\lambda_{iT,I,F}(x), inf\lambda_{iT,I,F}(x)\}$ and $sup\{rmin\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(x)\}\} = rmin\{sup\lambda_{iT,I,F}(x), sup\lambda_{iT,I,F}(x)\}\ \forall\ x \in$

X. Then for $x, y \in X$,

$$\begin{aligned}(\cup A_{iT,I,F})(x*y) &= rsupA_{iT,I,F}(x*y)\\ &\geq rsup\{rmin\{A_{iT,I,F}(x), A_{iT,I,F}(y)\}\}\\ &= rmin\{rsupA_{iT,I,F}(x), rsupA_{iT,I,F}(y)\}\\ &= rmin\{(\cup A_{iT,I,F})(x), (\cup A_{iT,I,F})(y)\}\\ (\cup A_{iT,I,F})(x*y) &\geq rmin\{(\cup A_{iT,I,F})(x), (\cup A_{iT,I,F})(y)\}\end{aligned}$$

and

$$\begin{aligned}(\wedge\lambda_{iT,I,F})(x*y) &= inf\lambda_{iT,I,F}(x*y)\\ &\leq inf\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(y)\}\}\\ &= max\{inf\lambda_{iT,I,F}(x), inf\lambda_{iT,I,F}(y)\}\\ &= max\{(\wedge\lambda_{iT,I,F})(x), (\wedge\lambda_{iT,I,F})(y)\}\\ (\wedge\lambda_{iT,I,F})(x*y) &\leq max\{(\wedge\lambda_{iT,I,F})(x), (\wedge\lambda_{iT,I,F})(y)\}.\end{aligned}$$

Which shows that R-union of $\mathcal{A}_i$ is a neutrosophic cubic subalgebra of X.

Proposition 3.2 *If a neutrosophic cubic set* $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ *of* X *is a subalgebra, then* $\forall\ x \in X$, $A_{T,I,F}(0*x) \geq A_{T,I,F}(x)$ *and* $\lambda_{T,I,F}(0*x) \leq \lambda_{T,I,F}(x)$.

Proof: $\forall\, x \in X$, $A_{T,I,F}(0*x) \geq rmin\{A_{T,I,F}(0), A_{T,I,F}(x)\}$ $= rmin\{A_{T,I,F}(x*x), A_{T,I,F}(x)\} \geq rmin\{rmin\{A_{T,I,F}(x), A_{T,I,F}(x)\}, A_{T,I,F}(x)\} = A_{T,I,F}(x)$ and similarly $\lambda_{T,I,F}(0*x) \leq max\{\lambda_{T,I,F}(0), \lambda_{T,I,F}(x)\} = \lambda_{T,I,F}(x)$.

Lemma 3.1 *If a netrosophic cubic set* $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ *of* X *is a subalgebra, then* $\mathcal{A}(x*y) = \mathcal{A}(x*(0(0*y)))\ \forall\, x, y \in X$.

Proof: Let X be a B-algebra and $x, y \in X$. Then we know that $y = 0*(0*y)$ by ([3],lemma 3.1). Hence, $A_{T,I,F}(x*y) = A_{T,I,F}(x*(0*(0*y)))$ and $\lambda_{T,I,F}(x*y) = \lambda_{T,I,F}(x*(0*(0*y)))$. Therefore, $\mathcal{A}_{T,I,F}(x*y) = \mathcal{A}_{T,I,F}(x*(0*(0*y)))$.

Proposition 3.3 *If a nuetrosophic cubic set* $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ *of* X *is a neutrosophic cubic subalgebra, then* $\forall\, x, y \in X$, $A_{T,I,F}(x*(0*y)) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ *and* $\lambda_{T,I,F}(x*(0*y)) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$.

Proof: Let $x, y \in X$. Then we have $A_{T,I,F}(x*(0*y)) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(0*y)\} \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x*(0*y)) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(0*y)\} \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$ by Definition 3.1 and Proposition 3.2. Hence, the proof is completed.

Theorem 3.6 *If a neutrosophic cubic set* $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ *of* X *satisfies the following conditions*

1. $A_{T,I,F}(0*x) \geq A_{T,I,F}(x)$ *and* $\lambda_{T,I,F}(0*x) \leq \lambda_{T,I,F}(x)$,
2. $A_{T,I,F}(x*(0*y)) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ *and* $\lambda_{T,I,F}(x*(0*y)) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$ $\forall\, x, y \in X$.

then $\mathcal{A}$ *refers to a neutrosophic cubic subalgebra of* X.

Proof: Assume that the neutrosophic cubic set $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ of X satisfies the above conditions (1 and 2). Then by Lemma 3.1, we have $A_{T,I,F}(x*y) = A_{T,I,F}(x*(0*(0*y))) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(0*y)\} \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x*y) = \lambda_{T,I,F}(x*(0*(0*y))) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(0*y) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$ $\forall\, x, y \in X$. Hence, $\mathcal{A}$ is neutrosophic cubic subalgebra of X.

Theorem 3.7 *Nuetrosophic cubic set* $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ *of* X *is a neutrosophic cubic subalgebra of* $X \iff A^-_{T,I,F}, A^+_{T,I,F}$ *and* $\lambda_{T,I,F}$ *are fuzzy subalgebras of* X.

Proof: let $A^-_{T,I,F}, A^+_{T,I,F}$ and $\lambda_{T,I,F}$ are fuzzy subalgebra of X and $x, y \in X$. Then $A^-_{T,I,F}(x*y) \geq min\{A^-_{T,I,F}(x), A^-_{T,I,F}(y)\}$, $A^+_{T,I,F}(x*y) \geq min\{A^+_{T,I,F}(x), A^+_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x*y) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$. Now, $A_{T,I,F}(x*y) = [A^-_{T,I,F}(x*y), A^+_{T,I,F}(x*y] \geq [min\{A^-_{T,I,F}(x), A^-_{T,I,F}(y)\}, min\{A^+_{T,I,F}(x), A^+_{T,I,F}(y)\}] \geq rmin\{[A^-_{T,I,F}(x), A^+_{T,I,F}(x)], [A^-_{T,I,F}(y), A^+_{T,I,F}(y)]\} = rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$. Therefore, $\mathcal{A}$ is neutrosophic cubic subalgebra of X.

Conversely, assume that $\mathcal{A}$ is a neutrosophic cubic subalgebra of X. For any $x, y \in X$, $[A^-_{T,I,F}(x*y), A^+_{T,I,F}(x*y)] = A_{T,I,F}(x*y) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\} = rmin\{[A^-_{T,I,F}(x), A^+_{T,I,F}(x)], [A^-_{T,I,F}(y), A^+_{T,I,F}(y)]\} == [min\{A^-_{T,I,F}(x), A^-_{T,I,F}(y)\}, min\{A^+_{T,I,F}(x), A^+_{T,I,F}(y)\}]$. Thus, $A^-_{T,I,F}(x*y) \geq min\{A^-_{T,I,F}(x), A^-_{T,I,F}(y)\}$, $A^+_{T,I,F}(x*y) \geq min\{A^+_{T,I,F}(x), A^+_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x*y) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$. Hence $A^-_{T,I,F}, A^+_{T,I,F}$ and $\lambda_{T,I,F}$ are fuzzy subalgebra of X.

Theorem 3.8 *Let* $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ *be a neutrosophic cubic subalgebra of* X *and let* $n \in \mathbb{Z}^+$ *(the set of positive integer). Then*

1. $A_{T,I,F}(\prod^{n} x*x) \geq A_{T,I,F}(x)$ *for* $n \in \mathbb{O}$*(the set of odd number),*
2. $\lambda_{T,I,F}(\prod^{n} x*x) \leq A_{T,I,F}(x)$ *for* $n \in \mathbb{O}$*(the set of odd number),*
3. $A_{T,I,F}(\prod^{n} x*x) = A_{T,I,F}(x)$ *for* $n \in \mathbb{E}$*(the set of even number),*
4. $\lambda_{T,I,F}(\prod^{n} x*x) = A_{T,I,F}(x)$ *for* $n \in \mathbb{E}$*(the set of even number).*

Proof: Let $x \in X$ and assume that n is odd. Then $n = 2p-1$ for some positive integer p. We prove the theorem by induction. Now $A_{T,I,F}(x*x) = A_{T,I,F}(0) \geq A_{T,I,F}(x)$ and $\lambda_{T,I,F}(x*x) = \lambda_{T,I,F}(0) \leq \lambda_{T,I,F}(x)$. Suppose that $A_{T,I,F}(\prod^{2p-1} x*x) \geq A_{T,I,F}(x)$ and $\lambda_{T,I,F}(\prod^{2p-1} x*x) \leq \lambda_{T,I,F}(x)$. Then by assumption, $A_{T,I,F}(\prod^{2(p+1)-1} x*x) = A_{T,I,F}(\prod^{2p+1} x*x) = A_{T,I,F}(\prod^{2p-1} x*(x*(x*x))) = A_{T,I,F}(\prod^{2p-1} x*x) \geq A_{T,I,F}(x)$ and $\lambda_{T,I,F}(\prod^{2(p+1)-1} x*x) = \lambda_{T,I,F}(\prod^{2p+1} x*x) = \lambda_{T,I,F}(\prod^{2p-1} x*(x*(x*x))) = \lambda_{T,I,F}(\prod^{2p-1} x*x) \leq \lambda_{T,I,F}(x)$, which proves (1) and (2). Similarly, the proves are same to the cases (3) and (4).

The sets denoted by $I_{A_{T,I,F}}$ and $I_{\lambda_{T,I,F}}$ are also subalgebra of X. Which were defined as:
$I_{A_{T,I,F}}=\{x \in X \mid A_{T,I,F}(x) = A_{T,I,F}(0)\}$ and $I_{\lambda_{T,I,F}}=\{x \in X \mid \lambda_{T,I,F}(x) = \lambda_{T,I,F}(0)\}$.

Theorem 3.9 *Let $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ be a neutrosophic cubic subalgebra of X. Then the sets $I_{A_{T,I,F}}$ and $I_{\lambda_{T,I,F}}$ are subalgebras of X.*

Proof: Let $x, y \in I_{A_{T,I,F}}$. Then $A_{T,I,F}(x) = A_{T,I,F}(0) = A_{T,I,F}(y)$ and so, $A_{T,I,F}(x*y) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\} = A_{T,I,F}(0)$. By using Proposition 3.1, We know that $A_{T,I,F}(x*y) = A_{T,I,F}(0)$ or equivalently $x*y \in I_{A_{T,I,F}}$.

Again let $x, y \in I_{A_{T,I,F}}$. Then $\lambda_{T,I,F}(x) = \lambda_{T,I,F}(0) = \lambda_{T,I,F}(y)$ and so, $\lambda_{T,I,F}(x*y) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$ $=\lambda_{T,I,F}(0)$. Again by using Proposition 3.1, We know that $\lambda_{T,I,F}(x*y) = \lambda_{T,I,F}(0)$ or equivalently $x*y \in I_{A_{T,I,F}}$. Hence the sets $I_{A_{T,I,F}}$ and $\lambda_{A_{T,I,F}}$ are subalgebras of X.

Theorem 3.10 *Let B be a nonempty subset of X and $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ be neutrosophic cubic set of X defined by*

$$A_{T,I,F}(x) = \begin{cases} [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}], & \text{if } x \in B \\ [\beta_{T,I,F_1}, \beta_{T,I,F_2}], & \text{otherwise}, \end{cases}$$

$$\lambda_T(x) = \begin{cases} \gamma_{T,I,F}, & \text{if } x \in B \\ \delta_{T,I,F}, & \text{otherwise} \end{cases}$$

$\forall$ $[\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]$,$[\beta_{T,I,F_1}, \beta_{T,I,F_2}] \in D[0,1]$ and $\gamma_{T,I,F}$, $\delta_{T,I,F} \in [0,1]$ with $[\alpha_{T,I,F_1}, \alpha_{T,I,F_2}] \geq [\beta_{T,I,F_1}, \beta_{T,I,F_2}]$ and $\gamma_{T,I,F} \leq \delta_{T,I,F}$. Then $\mathcal{A}$ is a nuetrosophic cubic subalgebra of $X \iff B$ is a subalgebra of X. Moreover, $I_{A_{T,I,F}}= B= I_{\lambda_{T,I,F}}$.

Proof: Let $\mathcal{A}$ be a neutrosophic cubic subalgebra of X. Let $x, y \in X$ such that $x, y \in B$. Then $A_{T,I,F}(x*y) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\} = rmin\{[\alpha_{T,I,F_1}, \alpha_{T,I,F_2}], [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]\} = [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]$ and $\lambda_{T,I,F}(x*y) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\} = max\{\gamma_{T,I,F}, \gamma_{T,I,F}\} = \gamma_{T,I,F}$. Therefore $x*y \in B$. Hence, B is a subalgebra of X.

Conversely, suppose that B is a subalgebra of X. Let $x, y \in X$. We consider two cases,
Case 1: If $x, y \in B$, then $x*y \in B$, thus $A_{T,I,F}(x*y) = [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}] = rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x*y) = \gamma_{T,I,F} = max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$.
Case 2: If $x \notin B$ or $y \notin B$, then $A_{T,I,F}(x*y) \geq [\beta_{T,I,F_1}, \beta_{T,I,F_2}]$ $= rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x*y) \leq \delta_{T,I,F}$ $= max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$.

Hence $\mathcal{A}$ is a neutrosophic cubic subalgebra of X.

Now, $I_{A_{T,I,F}}=\{x \in X, A_{T,I,F}(x) = A_{T,I,F}(0)\}= \{x \in X, A_{T,I,F}(x) = [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]\} = B$ and $I_{\lambda_{T,I,F}}=\{x \in X, \lambda_{T,I,F}(x) = \lambda_{T,I,F}(0)\}=\{x \in X, \lambda_{T,I,F}(x) = \gamma_{T,I,F}\}=B$.

Definition 3.2 *Let $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ be a neutrosophic cubic set of X. For $[s_{T_1}, s_{T_2}], [s_{I_1}, s_{I_2}], [s_{F_1}, s_{F_2}] \in D[0,1]$ and $t_{T_1}, t_{I_1}, t_{F_1} \in [0,1]$, the set $U(A_{T,I,F} \mid ([s_{T_1}, s_{T_2}], [s_{I_1}, s_{I_2}], [s_{F_1}, s_{F_2}]))$ $=\{x \in X \mid A_T(x) \geq [s_{T_1}, s_{T_2}], A_I(x) \geq [s_{I_1}, s_{I_2}], A_F(x) \geq [s_{F_1}, s_{F_2}]\}$ is called upper $([s_{T_1}, s_{T_2}], [s_{I_1}, s_{I_2}], [s_{F_1}, s_{F_2}])$-level of $\mathcal{A}$ and $L(\lambda_{T,I,F} \mid (t_{T_1}, t_{I_1}, t_{F_1}))$ $=\{x \in X \mid \lambda_T(x) \leq t_{T_1}, \lambda_I(x) \leq t_{I_1}, \lambda_F(x) \leq t_{F_1}\}$ is called lower $(t_{T_1}, t_{I_1}, t_{F_1})$-level of $\mathcal{A}$.*

For our convenience we are introducing the new notation as: $U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}]$=$\{x \in X \mid A_{T,I,F}(x) \geq [s_{T,I,F_1}, s_{T,I,F_2}]\}$ is called upper $([s_{T,I,F_1}, s_{T,I,F_2}])$-level of $\mathcal{A}$ and $L(\lambda_{T,I,F} \mid t_{T,I,F_1})$=$\{x \in X \mid \lambda_{T,I,F}(x) \leq t_{T,I,F_1}\}$ is called lower t_{T,I,F_1}-level of $\mathcal{A}$.

Theorem 3.11 *If $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is neutrosophic cubic subalgebra of X, then the upper $[s_{T,I,F_1}, s_{T,I,F_2}]$-level and lower t_{T,I,F_1}-level of $\mathcal{A}$ are ones of X.*

Proof: Let $x, y \in U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$, then $A_{T,I,F}(x) \geq [s_{T,I,F_1}, s_{T,I,F_2}]$ and $A_{T,I,F}(y) \geq [s_{T,I,F_1}, s_{T,I,F_2}]$. It follows that $A_{T,I,F}(x*y) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\} \geq [s_{T,I,F_1}, s_{T,I,F_2}] \Rightarrow x*y \in U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$. Hence, $U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}]$ is a subalgebra of X.

Let $x, y \in L(\lambda_{T,I,F} \mid t_{T,I,F_1})$. Then $\lambda_{T,I,F}(x) \leq t_{T,I,F_1}$ and $\lambda_{T,I,F}(y) \leq t_{T,I,F_1}$. It follows that $\lambda_{T,I,F}(x*y) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\} \leq t_{T,I,F_1} \Rightarrow x*y \in L(\lambda_{T,I,F} \mid t_{T,I,F_1})$. Hence $L(\lambda_{T,I,F} \mid t_{T,I,F_1})$ is a subalgebra of X.

Corollary 3.1 *Let $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is neutrosophic cubic subalgebra of X. Then $A([s_{T,I,F_1}, s_{T,I,F_2}]; t_{T,I,F_1})$= $U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}]) \bigcap L(\lambda_{T,I,F} \mid t_{T,I,F_1})$=$\{x \in X \mid A_{T,I,F}(x) \geq [s_{T,I,F_1}, s_{T,I,F_2}], \lambda_{T,I,F}(x) \leq t_{T,I,F_1}\}$ is a subalgebra of X.*

Proof: Straightforward

The following example shows that the converse of Corollary 3.1 is not valid.

Example 3.2 *Let $X = \{0, a_1, a_2, a_3, a_4, a_5\}$ be a B-algebra in Remark 3.1 and $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic set defined by*

	0	a_1	a_2	a_3	a_4	a_5
A_T	[0.6,0.8]	[0.5,0.6]	[0.5,0.6]	[0.5,0.6]	[0.3,0.4]	[0.3,0.4]
A_I	[0.5,0.7]	[0.4,0.5]	[0.4,0.5]	[0.4,0.6]	[0.3,0.3]	[0.3,0.3]
A_F	[0.4,0.6]	[0.2,0.5]	[0.2,0.5]	[0.2,0.5]	[0.1,0.2]	[0.1,0.2]

	0	a_1	a_2	a_3	a_4	a_5
λ_T	0.1	0.3	0.3	0.5	0.3	0.5
λ_I	0.2	0.4	0.4	0.6	0.4	0.6
λ_F	0.3	0.5	0.5	0.7	0.5	0.7

Suppose that $[s_{T,I,F_1}, s_{T,I,F_2}]$=$([0.42, 0.49], [0.31, 0.37], [0.14, 0.18])_{T,I,F}$ *and* $t_{T,I,F_1} = (0.4, 0.5, 0.6)_{T,I,F}$, *then* $A([s_{T,I,F_1}, s_{T,I,F_2}]; t_{T,I,F_1})$=$U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}]) \bigcap L(\lambda_{T,I,F} \mid t_{T,I,F_1})$=$\{x \in X \mid A_{T,I,F}(x) \geq [s_{T,I,F_1}, s_{T,I,F_2}], \lambda_{T,I,F}(x) \leq t_{T,I,F_1}\} = \{0, a_1, a_2, a_3\} \bigcap \{0, a_1, a_2, a_4\} = \{0, a_1, a_2\}$ *is a subalgebra of* X, *but* $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ *is not a neutrosophic cubic subalgebra since* $A_T(a_1 * a_3) = [0.3, 0.4] \not\geq [0.5, 0.6] = rmin\{A_T(a_1), A_T(a_3)\}$ *and* $\lambda_T(a_2 * a_4) = 0.5 \not\leq 0.4 = max\{\lambda_T(a_2), \lambda_T(a_4)\}$.

Theorem 3.12 *Let* $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ *be a neutrosophic cubic set of* X, *such that the sets* $U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ *and* $L(\lambda_{T,I,F} \mid t_{T,I,F_1})$ *are subalgebras of* X *for every* $[s_{T,I,F_1}, s_{T,I,F_2}] \in D[0, 1]$ *and* $t_{T,I,F_1} \in [0, 1]$. *Then* $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ *is neutrosophic cubic subalgebra of* X.

Proof: Let $U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and $L(\lambda_{T,I,F} \mid t_{T,I,F_1})$ are subalgebras of X for every $[s_{T,I,F_1}, s_{T,I,F_2}] \in D[0, 1]$ and $t_{T,I,F_1} \in [0, 1]$. On the contrary, let $x_0, y_0 \in X$ be such that $A_{T,I,F}(x_0 * y_0) < rmin\{A_{T,I,F}(x_0), A_{T,I,F}(y_0)\}$. Let $A_{T,I,F}(x_0) = [\theta_1, \theta_2]$, $A_{T,I,F}(y_0) = [\theta_3, \theta_4]$ and $A_{T,I,F}(x_0 * y_0) = [s_{T,I,F_1}, s_{T,I,F_2}]$. Then $[s_{T,I,F_1}, s_{T,I,F_2}] < rmin\{[\theta_1, \theta_2], [\theta_3, \theta_4]\} = [min\{\theta_1, \theta_3\}, min\{\theta_2, \theta_4\}]$. So, $s_{T,I,F_1} < rmin\{\theta_1, \theta_3\}$ and $s_{T,I,F_2} < min\{\theta_2, \theta_4\}$. Let us consider, $[\rho_1, \rho_2] = \frac{1}{2}[A_{T,I,F}(x_0 * y_0) + rmin\{A_{T,I,F}(x_0), A_{T,I,F}(y_0)\}] = \frac{1}{2}[[s_{T,I,F_1}, s_{T,I,F_2}] + [min\{\theta_1, \theta_3\}, min\{\theta_2, \theta_4\}]] = [\frac{1}{2}(s_{T,I,F_1} + min\{\theta_1, \theta_3\}), \frac{1}{2}(s_{T,I,F_2} + min\{\theta_2, \theta_3\})]$. Therefore, $min\{\theta_1, \theta_3\} > \rho_1 = \frac{1}{2}(s_{T,I,F_1} + min\{\theta_1, \theta_3\}) > s_{T,I,F_1}$ and $min\{\theta_2, \theta_4\} > \rho_2 = \frac{1}{2}(s_{T,I,F_2} + min\{\theta_2, \theta_4\}) > s_{T,I,F_2}$. Hence, $[min\{\theta_1, \theta_3\}, min\{\theta_2, \theta_4\}] > [\rho_1, \rho_2] > [s_{T,I,F_1}, s_{T,I,F_2}]$, so that $x_0 * y_0 \notin U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ which is a contradiction since $A_{T,I,F}(x_0) = [\theta_1, \theta_2] \geq [min\{\theta_1, \theta_3\}, min\{\theta_2, \theta_4\}] > [\rho_1, \rho_2]$ and $A_{T,I,F}(y_0) = [\theta_3, \theta_4] \geq [min\{\theta_1, \theta_3\}, min\{\theta_2, \theta_4\}] > [\rho_1, \rho_2]$. This implies $x_0 * y_0 \in U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$. Thus $A_{T,I,F}(x * y) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}\ \forall\ x, y \in X$.

Again, let $x_0, y_0 \in X$ be such that $\lambda_{T,I,F}(x_0 * y_0) > max\{\lambda_{T,I,F}(x_0), \lambda_{T,I,F}(0)\}$. Let $\lambda_{T,I,F}(x_0) = \eta_{T,I,F_1}$, $\lambda_{T,I,F}(y_0) = \eta_{T,I,F_2}$ and $\lambda_{T,I,F}(x_0 * y_0) = t_{T,I,F_1}$. Then $t_{T,I,F_1} > max\{\eta_{T,I,F_1}.\eta_{T,I,F_2}\}$. Let us consider $t_{T,I,F_2} = \frac{1}{2}[\lambda_{T,I,F}(x_0 * y_0) + max\{\lambda_{T,I,F}(x_0), \lambda_{T,I,F}(0)\}]$. We get that $t_{T,I,F_2} = \frac{1}{2}(t_{T,I,F_1} + max\{\eta_{T,I,F_1}, \eta_{T,I,F_2}\})$. Therefore, $\eta_{T,I,F_1} < t_{T,I,F_2} = \frac{1}{2}(t_{T,I,F_1} + max\{\eta_{T,I,F_1}, \eta_{T,I,F_2}\}) < t_{T,I,F_1}$ and $\eta_{T,I,F_2} < t_{T,I,F_2} = \frac{1}{2}(t_{T,I,F_1} + max\{\eta_{T,I,F_1}, \eta_{T,I,F_2}\}) < t_{T,I,F_1}$. Hence, $max\{\eta_{T,I,F_1}, \eta_{T,I,F_2}\} < t_{T,I,F_2} < t_{T,I,F_1} = \lambda_{T,I,F}(x_0, y_0)$, so that $x_0 * y_0 \notin L(\lambda_{T,I,F} \mid t_{T,I,F_1})$ which is a contradiction since $\lambda_{T,I,F}(x_0) = \eta_{T,I,F_1} \leq max\{\eta_{T,I,F_1}, \eta_{T,I,F_2}\} < t_{T,I,F_2}$ and $\lambda_{T,I,F}(y_0) = \eta_{T,I,F_2} \leq max\{\eta_{T,I,F_1}, \eta_{T,I,F_2}\} < t_{T,I,F_2}$. This implies $x_0, y_0 \in L(\lambda_{T,I,F} \mid t_{T,I,F_1})$. Thus $\lambda_{T,I,F}(x * y) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}\ \forall\ x, y \in X$. Therefore, $U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and $L(\lambda_{T,I,F} \mid t_{T,I,F_1})$ are subalgebras of X. Hence, $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is neutrosophic cubic subalgebra of X.

Theorem 3.13 *Any subalgebra of* X *can be realized as both the upper* $[s_{T,I,F_1}, s_{T,I,F_2}]$*-level and lower* t_{T,I,F_1}*-level of some neutrosophic cubic subalgebra of* X.

Proof: Let $\mathcal{B}$ be a neutrosophic cubic subalgebra of X, and $\mathcal{A}$ be a neutrosophic cubic set on X defined by

$$A_{T,I,F} = \begin{cases} [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}], & \text{if } x \in \mathcal{B} \\ [0, 0] & \text{otherwise.} \end{cases},$$

$$\lambda_{T,I,F} = \begin{cases} \beta_{T,I,F_1}, & \text{if } x \in \mathcal{B} \\ 0, & \text{otherwise.} \end{cases}$$

$\forall\ [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}] \in D[0, 1]$ and $\beta_{T,I,F_1} \in [0, 1]$. We consider the following cases.

Case 1: If $\forall\ x, y \in \mathcal{B}$ then $A_{T,I,F}(x) = [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]$, $\lambda_{T,I,F}(x) = \beta_{T,I,F_1}$ and $A_{T,I,F}(y) = [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]$, $\lambda_{T,I,F}(y) = \beta_{T,I,F_1}$. Thus $A_{T,I,F}(x * y) = [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}] = rmin\{[\alpha_{T,I,F_1}, \alpha_{T,I,F_2}], [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]\} = rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x * y) = \beta_{T,I,F_1}$=$max\{\beta_{T,I,F_1}, \beta_{T,I,F_1}\} = max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$.

Case 2: If $x \in \mathcal{B}$ and $y \notin \mathcal{B}$, then $A_{T,I,F}(x) = [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]$, $\lambda_{T,I,F}(x) = \beta_{T,I,F_1}$ and $A_{T,I,F}(y) = [0, 0]$, $\lambda_{T,I,F}(y) = 1$. Thus $A_{T,I,F}(x * y) \geq [0, 0] = rmin\{[\alpha_{T,I,F_1}, \alpha_{T,I,F_2}], [0, 0]\} = rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x * y) \leq 1 = max\{\beta_{T,I,F_1}, 1\} = max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$.

Case 3: If $x \notin \mathcal{B}$ and $y \in \mathcal{B}$, then $A_{T,I,F}(x) = [0, 0]$,$\lambda_{T,I,F}(x) = 1$ and $A_{T,I,F}(y) = [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]$, $\lambda_{T,I,F}(y) = \beta_{T,I,F_1}$. Thus $A_{T,I,F}(x * y) \geq [0, 0] = rmin\{[0, 0], [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]\} = rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x * y) \leq 1 = max\{1, \beta_{T,I,F_1}\} = max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$.

Case 4: If $x \notin \mathcal{B}$ and $y \notin \mathcal{B}$, then $A_{T,I,F}(x) = [0, 0]$, $\lambda_{T,I,F}(x) = 1$ and $A_{T,I,F}(y) = [0, 0]$, $\lambda_{T,I,F}(y) = 1$. Thus $A_{T,I,F}(x * y) \geq [0, 0] = rmin\{[0, 0], [0, 0]\} = rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x * y) \leq 1 = max\{1, 1\} = max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$.

Therefore, $\mathcal{A}$ is a neutrosophic cubic subalgebra of X.

Theorem 3.14 *Let* $\mathcal{B}$ *be a subset of* X *and* $\mathcal{A}$ *be a neutrosophic cubic set on* X *which is given in the proof of Theorem 3.13. If* $\mathcal{A}$ *is realized as lower level subalgebra and upper level subalgebra of some neutrosophic cubic subalgebra of* X, *then* $\mathcal{P}$ *is a neutrosophic cubic one of* X.

Proof: Let $\mathcal{A}$ be a neutrosophic cubic subalgebra of X, and $x, y \in \mathcal{B}$. Then $A_{T,I,F}(x) = A_{T,I,F}(y) = [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]$ and $\lambda_{T,I,F}(x) = \lambda_{T,I,F}(y) = \beta_{T,I,F_1}$. Thus $A_{T,I,F}(x * y) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$=$rmin\{[\alpha_{T,I,F_1}, \alpha_{T,I,F_2}], [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]\}$ = $[\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]$ and $\lambda_{T,I,F}(x * y) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$=$max\{\beta_{T,I,F_1}, \beta_{T,I,F_1}\}$=$\beta_{T,I,F_1}$, $\Rightarrow x * y \in \mathcal{B}$. Hence, the proof is completed.

4 Images and Pre-images of Neutrosophic Cubic Subalgebras

In this section, homomorphism of neutrosophic cubic subalgebras are defined and some results are studied.

Let f be a mapping from a set X into a set Y and $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ be a neutrosophic cubic set in Y. So, the inverse-image of $\mathcal{A}$ is defined as $f^{-1}(\mathcal{A})=\{\langle x, f^{-1}(A_{T,I,F}), f^{-1}(\lambda_{T,I,F})\rangle \mid x \in X\}$ and $f^{-1}(A_{T,I,F})(x) = A_{T,I,F}(f(x))$ and $f^{-1}(\lambda_{T,I,F})(x) = \lambda_{T,I,F}(f(x))$. It can be shown that $f^{-1}(\mathcal{A})$ is a neutrosophic cubic set.

Theorem 4.1 *Suppose that $f \mid X \to Y$ be a homomorphism of B-algebras. If $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic subalgebra of Y, then the pre-image $f^{-1}(\mathcal{A})=\{\langle x, f^{-1}(A_{T,I,F}), f^{-1}(\lambda_{T,I,F})\rangle \mid x \in X\}$ of $\mathcal{A}$ under f is a neutrosophic cubic subalgebra of X.*

Proof: Assume that $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic subalgebra of Y and let $x, y \in X$. then $f^{-1}(A_{T,I,F})(x*y) = A_{T,I,F}(f(x*y)) = A_{T,I,F}(f(x)*f(y)) \geq rmin\{A_{T,I,F}(f(x)), A_{T,I,F}(f(y))\} = rmin\{f^{-1}(A_{T,I,F})(x), f^{-1}(A_{T,I,F})(y)\}$ and $f^{-1}(\lambda_{T,I,F})(x * y) = \lambda_{T,I,F}(f(x * y)) = \lambda_{T,I,F}(f(x)*f(y)) \leq max\{\lambda_{T,I,F}(f(x)), \lambda_{T,I,F}(f(y))\} = max\{f^{-1}(\lambda_{T,I,F})(x), f^{-1}(\lambda_{T,I,F})(y)\}$. $\therefore f^{-1}(\mathcal{A}) = \{\langle x, f^{-1}(A_{T,I,F}), f^{-1}(\lambda_{T,I,F})\rangle \mid x \in X\}$ is neutrosophic cubic subalgebra of X.

Theorem 4.2 *Consider $f \mid X \to Y$ be a homomorphism of B-algebras and $\mathcal{A}_j = (A_{jT,I,F}, \lambda_{jT,I,F})$ be neutrosophic cubic subalgebras of Y where $j \in k$. If $inf\{max\{\lambda_{jT,I,F}(y), \lambda_{jT,I,F}(y)\}\} = max\{inf\lambda_{jT,I,F}(y), inf\lambda_{jT,I,F}(y)\}\ \forall\ y \in Y$, then $f^{-1}(\bigcap_{R_{j\in k}}\mathcal{A}_j)$ is also a neutrosophic cubic subalgebra of X.*

Proof: Let $\mathcal{A}_j = (A_{jT,I,F}, \lambda_{jT,I,F})$ be neutrosophic cubic subalgebras of Y where $j \in k$ satisfying $inf\{max\{\lambda_{jT,I,F}(y), \lambda_{jT,I,F}(y)\}\}=max\{inf\lambda_{jT,I,F}(y), inf\lambda_{jT,I,F}(y)\}\ \forall\ y \in Y$. Then by Theorem 3.3, $\bigcap_{R_{j\in k}}\mathcal{A}_j$ is a neutrosophic cubic subalgebra of Y. Hence $f^{-1}(\bigcap_{R_{j\in k}}\mathcal{A}_j)$ is also a neutrosophic cubic subalgebra of X.

Theorem 4.3 *Let $f \mid X \to Y$ be a homomorphism of B-algebras. Assume that $\mathcal{A}_j = (A_{jT,I,F}, \lambda_{jT,I,F})$ be neutrosophic cubic subalgebras of Y where $j \in k$. If $rsup\{rmin\{A_{jT,I,F}(y_1), A_{jT,I,F}(y_1)\}\}=rmin\{rsupA_{jT,I,F}(y_1), rsupA_{jT,I,F}(y_1)\}\ \forall\ y_1, y_2 \in Y$, then $f^{-1}(\bigcup_{R_{j\in k}}\mathcal{A}_j)$ is also a neutrosophic cubic subalgebra of X.*

Proof: Let $\mathcal{A}_j = (A_{jT,I,F}, \lambda_{jT,I,F})$ be neutrosophic cubic subalgebras of Y, where $j \in k$ satisfying $rsup\{rmin\{A_{jT,I,F}(y_1), A_{jT,I,F}(y_2)\}\}=rmin\{rsupA_{jT,I,F}(y_1), rsupA_{jT,I,F}(y_2)\}\ \forall\ y_1, y_2 \in Y$. Then by Theorem 3.4, $\bigcup_{R_{j\in k}}\mathcal{A}_j$ is a neutrosophic cubic subalgebra of Y. Hence, $f^{-1}(\bigcup_{R_{j\in k}}\mathcal{A}_j)$ is also a neutrosophic cubic subalgebra of X.

Definition 4.1 *A neutrosophic cubic set $\mathcal{A}=(A_{T,I,F}, \lambda_{T,I,F})$ in the B-algebra X is said to have rsup-property and inf-property if for any subset S of X, there exist $s_0 \in T$ such that $A_{T,I,F}(s_0)= rsup_{s_0\in S}A_{T,I,F}(t_0)$ and $\lambda_{T,I,F}(t_0)=\inf_{t_0\in T}\lambda_{T,I,F}(t_0)$ respectively.*

Definition 4.2 *Let f be mapping from the set X to the set Y. If $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is neutrosphic cubic set of X, then the image of $\mathcal{A}$ under f denoted by $f(\mathcal{A})$ and is defined as $f(\mathcal{A})=\{\langle x, f_{rsup}(A_{T,I,F}), f_{inf}(A_{T,I,F})\rangle \mid x \in X\}$, where*

$$f_{rsup}(A_{T,I,F})(y) = \begin{cases} rsup_{x\in f^{-1}(y)}(A_{T,I,F})(X), & \text{if } f^{-1}(y) \neq \phi \\ [0,0], & \text{otherwise}, \end{cases}$$

and

$$f_{inf}(\lambda_{T,I,F})(y) = \begin{cases} \lambda_{T,I,F}\ (x), & \text{if } f^{-1}(y) \neq \phi \\ {}_{x\in f^{-1}(y)} & \\ 1, & \text{otherwise}. \end{cases}$$

Theorem 4.4 *suppose $f \mid X \to Y$ be a homomorphism from a B-algebra X onto a B-algebra Y. If $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic subalgebra of X, then the image $f(\mathcal{A})=\{\langle x, f_{rsup}(A_{T,I,F}), f_{inf}(A_{T,I,F})\rangle \mid x \in X\}$ of $\mathcal{A}$ under f is a neutrosophic cubic subalgebra of Y.*

Proof: Let $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ be a neutrosophic cubic subalgebra of X and let $y_1,\ y_2 \in Y$. We know that $\{x_1 * x_2 \mid x_1 \in f^{-1}(y_1)$ *and* $x_2 \in f^{-1}(y_2)\} \subseteq \{x \in X \mid x \in f^{-1}(y_1 * y_2)\}$. Now $f_{rsup}(A_{T,I,F})(y_1 * y_2)=rsup\{A_{T,I,F}(x) \mid x \in f^{-1}(y_1 * y_2)\} \geq rsup\{A_{T,I,F}(x_1 * x_2) \mid x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)\} \geq rsup\{rmin\{A_{T,I,F}(x_1), A_{T,I,F}(x_2)\} \mid x_1 \in f^{-1}(y_1)$ *and* $x_2 \in f^{-1}(y_2)\} = rmin\{rsup\{A_{T,I,F}(x_1) \mid x_1 \in f^{-1}(y_1)\}, rsup\{A_{T,I,F}(x_2) \mid x_2 \in f^{-1}(y_2)\}\} = rmin\{f_{rsup}(A_{T,I,F})(y_1), f_{rsup}(A_{T,I,F})(y_2)\}$ and $f_{inf}(\lambda_{T,I,F})(y_1 * y_2) = inf\{\lambda_{T,I,F}(x) \mid x \in f^{-1}(y_1 * y_2)\} \leq inf\{\lambda_{T,I,F}(x_1 * x_2) \mid x_1 \in f^{-1}(y_1)$ *and* $x_2 \in f^{-1}(y_2)\} \leq inf\{max\{\lambda_{T,I,F}(x_1), \lambda_{T,I,F}(x_2)\} \mid x_1 \in f^{-1}(y_1)$ *and* $x_2 \in f^{-1}(y_2)\} = max\{inf\{\lambda_{T,I,F}(x_1) \mid x_1 \in f^{-1}(y_1)\}, inf\{\lambda_{T,I,F}(x_2) \mid x_2 \in f^{-1}(y_2)\}\} = max\{f_{inf}(\lambda_{T,I,F})(y_1), f_{inf}(\lambda_{T,I,F})(y_2)\}$. Hence $f(\mathcal{A})=\{\langle x, f_{rsup}(A_{T,I,F}), f_{inf}(A_{T,I,F})\rangle \mid x \in X\}$ is a neutrosophic cubic subalgebra of Y.

Theorem 4.5 *Assume that $f \mid X \to Y$ is a homomorphism of B-algebra and $\mathcal{A}_i = (A_{iT,I,F}, \lambda_{iT,I,F})$ is a neutrosophic cubic subalgebra of X, where $i \in k$. If $inf\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(x)\}\} = max\{inf\lambda_{iT,I,F}(x), inf\lambda_{iT,I,F}(x)\}\ \forall\ x \in X$, then $f(\bigcap_{P_{i\in k}}\mathcal{A}_i)$ is a neutrosophic cubic subalgebra of Y.*

Proof: Let $\mathcal{A}_i = (A_{iT,I,F}, \lambda_{iT,I,F})$ be neutrosophic cubic subalgebra of X where $i \in k$ satisfying $inf\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(x)\}\}=max\{inf\lambda_{iT,I,F}(x), inf\lambda_{iT,I,F}(x)\}\ \forall x \in X$. Then by Theorem 3.3, $\bigcap_{P_{i\in k}}\mathcal{A}_i$ is a neutrosophic cubic algebra of X. Hence $f(\bigcap_{P_{i\in k}}\mathcal{A}_j)$ is also a neutrosophic cubic subalgebra of Y.

Theorem 4.6 *Suppose $f \mid X \rightarrow Y$ be a homomorphism of B-algebra. Let $\mathcal{A}_i$=$(A_{iT,I,F}, \lambda_{iT,I,F})$ be neutrosophic cubic subalgebras of X where $i \in k$. If $rsup\{rmin\{A_{iT,I,F}(x_1), A_{iT,I,F}(x_2)\}\}$=$rmin\{rsupA_{iT,I,F}(x_1), rsupA_{iT,I,F}(x_2)\}$ $\forall$ $x_1, x_2 \in Y$, then $f(\bigcup_{i\in k} {}_P\mathcal{A}_i)$ is also a neutrosophic cubic subalgebra of Y.*

Proof: Let $\mathcal{A}_i = (A_{iT,I,F}, \lambda_{iT,I,F})$ be neutrosophic cubic subalgebras of X where $i \in k$ satisfying $rsup\{rmin\{A_{iT,I,F}(x_1), A_{iT,I,F}(x_2)\}\}$=$rmin\{rsupA_{iT,I,F}(x_1), rsupA_{iT,I,F}(x_2)\}$ $\forall$ $x_1, x_2 \in X$. Then by Theorem 3.4, $\bigcup_{i\in k} {}_P\mathcal{A}_i$ is a neutrosophic cubic subalgebra of X. Hence $f(\bigcup_{i\in k} {}_P\mathcal{A}_i)$ is also a neutrosophic cubic subalgebra of Y.

Corollary 4.1 *For a homomorphism $f \mid X \rightarrow Y$ of B-algebras, the following results hold:*

1. *If $\forall$ $i \in k$, $\mathcal{A}_i$ are neutrosophic cubic subalgebra of X, then $f(\bigcap_{i\in k} {}_R\mathcal{A}_i)$ is neutrosophic cubic subalgebra of Y*
2. *If $\forall$ $i \in k$, $\mathcal{B}_i$ are neutrosophic cubic subalgebra of Y, then $f^{-1}(\bigcap_{i\in k} {}_R\mathcal{B}_i)$ is neutrosophic cubic subalgebra of X.*

Proof: Straightforward.

Theorem 4.7 *Let f be an isomorphism from a B-algebra X onto a B-algebra Y. If $\mathcal{A}$ is a neutrosophic cubic subalgebra of X, then $f^{-1}(f(\mathcal{A})) = \mathcal{A}$*

Proof: For any $x \in X$, let $f(x) = y$. Since f is an isomorphism, $f^{-1}(y) = \{x\}$. Thus $f(\mathcal{A})(f(x)) = f(\mathcal{A})(y) = \bigcup_{x\in f^{-1}(y)} \mathcal{A}(x)$ $= \mathcal{A}(x)$.

For any $y \in Y$, since f is an isomorphism, $f^{-1}(y) = \{x\}$ so that $f(x) = y$. Thus $f^{-1}(\mathcal{A})(x) = \mathcal{A}(f(x)) = \mathcal{A}(y)$.
Hence, $f^{-1}(f(\mathcal{A})) = f^{-1}(\mathcal{A}) = \mathcal{A}$.

Corollary 4.2 *Consider f is an Isomorphism from a B-algebra X onto a B-algebra Y. If $\mathcal{C}$ is a neutrosophic cubic subalgebra of Y, then $f(f^{-1}(\mathcal{C})) = \mathcal{C}$.*

Proof: Straightforward.

Corollary 4.3 *Let $f \mid X \rightarrow X$ be an automorphism. If $\mathcal{A}$ refers to a neutrosophic cubic subalgebra of X, then $f(\mathcal{A}) = \mathcal{A} \iff f^{-1}(\mathcal{A}) = \mathcal{A}$*

5 Neutrosophic Cubic Closed Ideals of B-algebras

In this section, neutrosophic cubic ideals and Neutrosophic cubic closed ideals of B-algebra are defined and related results are proved.

Definition 5.1 *A neutrosophic cubic set $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ of X is called a neutrosophic cubic ideal of X if it satisfies following axioms:*

N3. $A_{T,I,F}(0) \geq A_{T,I,F}(x)$ and $\lambda_{T,I,F}(0) \leq \lambda_{T,I,F}(x)$,
N4. $A_{T,I,F}(x) \geq rmin\{A_{T,I,F}(x * y), A_{T,I,F}(y)\}$,
N5. $\lambda_{T,I,F}(x) \leq max\{\lambda_{T,I,F}(x * y), \lambda_{T,I,F}(y)\}\forall$ $x, y \in X$

Example 5.1 *Consider a B-algebra $X = \{0, a_1, a_2, a_3\}$ and binary operation * is defined on X as*

$*$	0	a_1	a_2	a_3
0	0	a_1	a_2	a_3
a_1	a_1	0	a_3	a_2
a_2	a_2	a_3	0	a_1
a_3	a_3	a_2	a_1	0

Let $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ be a neutrosophic cubic set X defined as,

	0	a_1	a_2	a_3
A_T	[1,1]	[0.9,0.8]	[1,1]	[0.5,0.7]
A_I	[0.9,0.9]	[0.6,0.8]	[0.9,0.9]	[0.7,0.5]
A_F	[0.8,0.9]	[0.5,0.6]	[0.8,0.9]	[0.9,0.5]

,

	0	a_1	a_2	a_3
λ_T	0	0.9	0	0.8
λ_I	0.1	0.6	0.1	0.7
λ_F	0.3	0.4	0.3	0.5

Then it can be easy verify that $\mathcal{A}$ satisfying the conditions N3, N4 and N5. Hence $\mathcal{A}$ is a neutrosophic cubic ideal of X.

Definition 5.2 *Let $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ be a neutrosophic cubic set X then it is called neutrosophic cubic closed ideal of X if it satisfies N4, N5 and*

N6. $A_{T,I,F}(0 * x) \geq A_{T,I,F}(x)$ and $\lambda_{T,I,F}(0 * x) \leq \lambda_{T,I,F}(x)$, $\forall$ $x \in X$.

Example 5.2 *Let $X = \{0, a_1, a_2, a_3, a_4, a_5\}$ be a B-algebra in Example 3.2 and $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ be a neutrosophic cubic set X defined as*

	0	a_1	a_2	a_3	a_4	a_5
A_T	[0.3,0.6]	[0.2,0.5]	[0.2,0.5]	[0.1,0.3]	[0.1,0.3]	[0.1,0.3]
A_I	[0.4,0.7]	[0.3,0.6]	[0.3,0.6]	[0.2,0.5]	[0.2,0.5]	[0.2,0.5]
A_F	[0.5,0.8]	[0.4,0.7]	[0.4,0.7]	[0.2,0.3]	[0.2,0.3]	[0.2,0.3]

	0	a_1	a_2	a_3	a_4	a_5
λ_T	0.2	0.5	0.5	0.7	0.7	0.7
λ_I	0.3	0.4	0.4	0.6	0.6	0.6
λ_F	0.4	0.5	0.5	0.8	0.8	0.8

.

By calculations verify that $\mathcal{A}$ is a neutrosophic cubic closed ideal of X.

Proposition 5.1 *Every neutrosophic cubic closed ideal is a neutrosophic cubic ideal.*

The converse of Proposition 5.1 is not true in general as shown in the following example.

Example 5.3 *Let $X = \{0, a_1, a_2, a_3, a_4, a_5\}$ be a B-algebra in Example 3.1 and $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ be a neutrosophic cubic set in X defined as,*

	0	a_1	a_2	a_3	a_4	a_5
A_T	[0.4,0.6]	[0.3,0.5]	[0.3,0.5]	[0.2,0.3]	[0.2,0.3]	[0.2,0.3]
A_I	[0.5,0.7]	[0.4,0.6]	[0.4,0.6]	[0.3,0.5]	[0.3,0.5]	[0.3,0.5]
A_F	[0.6,0.8]	[0.5,0.7]	[0.5,0.7]	[0.4,0.3]	[0.4,0.3]	[0.4,0.3]

	0	a_1	a_2	a_3	a_4	a_5
λ_T	0.1	0.4	0.4	0.5	0.5	0.5
λ_I	0.2	0.3	0.3	0.6	0.6	0.6
λ_F	0.3	0.5	0.5	0.8	0.8	0.8

.

By calculations verify that $\mathcal{A}$ is a neutrosophic cubic ideal of X. But it is not a neutrosophic cubic closed ideal of X since $A_{T,I,F}(0*x) \geq A_{T,I,F}(x)$ and $\lambda_{T,I,F}(0*x) \leq \lambda_{T,I,F}(x)$, $\forall\ x \in X$.

Corollary 5.1 *Every neutrosophic cubic subalgebra satisfies N4 and N5 refer to a neutrosophic cubic closed ideal.*

Theorem 5.1 *Every neutrosophic cubic closed ideal of a B-algebra X works as a neutrosophic cubic subalgebra of X.*

Proof: Suppose $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ be a neutrosophic cubic closed ideal of X, then for any $x \in X$ we have $A_{T,I,F}(0*x) \geq A_{T,I,F}(x)$ and $\lambda_{T,I,F}(0*x) \leq \lambda_{T,I,F}(x)$. Now by N4, N6, ([3], Proposition 3.2), we know that $A_{T,I,F}(x*y) \geq rmin\{A_{T,I,F}((x*y)*(0*y)), A_{T,I,F}(0*y)\} = rmin\{A_{T,I,F}(x), A_{T,I,F}(0*y)\} \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x*y) \leq max\{\lambda_{T,I,F}((x*y)*(0*y)), \lambda_{T,I,F}(0*y)\} = max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(0*y)\} \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$. Hence, $\mathcal{A}$ is a neutrosophic cubic subalgeba of X.

Theorem 5.2 *The R-intersection of any set of neutrosophic cubic ideals of X is also a neutrosophic cubic ideal of X.*

Proof: Let $\mathcal{A}_i = \{A_{iT,I,F}, \lambda_{iT,I,F}\}$, where $i \in k$, be a neutrosophic cubic ideals of X and $x, y \in X$. Then

$$\begin{aligned}(\cap A_{iT,I,F})(0) &= rinf A_{iT,I,F}(0)\\ &\geq rinf A_{iT,I,F}(x)\\ &= (\cap A_{iT,I,F})(x),\end{aligned}$$

$$\begin{aligned}(\vee\lambda_{iT,I,F})(0) &= sup\lambda_{iT,I,F}(0)\\ &\leq \lambda_{iT,I,F}(x)\\ &= (\vee\lambda_{iT,I,F})(x),\end{aligned}$$

$$\begin{aligned}(\cap A_{iT,I,F})(x) &= rinf A_{iT,I,F}(x)\\ &\geq rinf\{rmin\{A_{iT,I,F}(x*y), A_{iT,I,F}(y)\}\}\\ &= rmin\{rinf A_{iT,I,F}(x*y), rinf A_{iT,I,F}(y)\}\\ &= rmin\{(\cap A_{iT,I,F})(x*y), (\cap A_{iT,I,F})(y)\}\end{aligned}$$

and

$$\begin{aligned}(\vee\lambda_{iT,I,F})(x) &= sup\lambda_{iT,I,F}(x)\\ &\leq sup\{max\{\lambda_{iT,I,F}(x*y), \lambda_{iT,I,F}(y)\}\}\\ &= max\{sup\lambda_{iT,I,F}(x*y), sup\lambda_{iT,I,F}(y)\}\\ &= max\{(\vee\lambda_{iT,I,F})(x*y), (\vee\lambda_{iT,I,F})(y)\}\end{aligned}$$

which shows that R-intersection is a neutrosophic cubic ideal of X.

Theorem 5.3 *The R-intersection of any set of neutrosophic cubic closed ideals of X is also a neutrosophic cubic closed ideal of X.*

Proof: It is similar to the proof of Theorem 5.2.

Theorem 5.4 *Neutrosophic cubic set $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ of X is a neutrosophic cubic ideal of $X \iff A^-_{T,I,F}, A^+_{T,I,F}$ and $\lambda_{T,I,F}$ are $fuzzy$ ideals of X.*

Proof: Assume that $x, y \in X$. Since $A^-_{T,I,F}(0) \geq A^-_{T,I,F}(x)$ and $A^+_{T,I,F}(0) \geq A^+_{T,I,F}(x)$, therefore, $A_{T,I,F}(0) \geq A_{T,I,F}(x)$. Also, $\lambda_{T,I,F}(0) \leq \lambda_{T,I,F}(x)$. Let $A^-_{T,I,F}$, $A^+_{T,I,F}$ and $\lambda_{T,I,F}$ are fuzzy ideals of X. Then $A_{T,I,F}(x) = [A^-_{T,I,F}(x), A^+_{T,I,F}(x)] \geq [min\{A^-_{T,I,F}(x*y), A^-_{T,I,F}(y)\}, min\{A^+_{T,I,F}(x*y), A^+_{T,I,F}(y)\}$ = $rmin\{[A^-_{T,I,F}(x*y), A^+_{T,I,F}(x*y)], [A^-_{T,I,F}(y), A^+_{T,I,F}(y)]\}$ = $rmin\{A_{T,I,F}(x*y), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x) \leq max\{\lambda_{T,I,F}(x*y), \lambda_{T,I,F}(y)\}$. Therefore $\mathcal{A}$ is a neutrosophic cubic ideal of X.

Conversely, let $\mathcal{A}$ be a neutrosophic cubic ideal of X. For any $x, y \in X$, we have $[A^-_{T,I,F}(x), A^+_{T,I,F}(x)]$ = $A_{T,I,F}(x) \geq rmin\{A_{T,I,F}(x*y), A_{T,I,F}(y)\}$ = $rmin\{[A^-_{T,I,F}(x*y), A^+_{T,I,F}(x*y)], [A^-_{T,I,F}(y), A^+_{T,I,F}(y)]\}$ = $[min\{A^-_{T,I,F}(x*y), A^-_{T,I,F}(y)\}, min\{A^+_{T,I,F}(x*y), A^+_{T,I,F}(y)\}$. Thus, $A^-_{T,I,F}(x) \geq min\{A^-_{T,I,F}(x*y), A^-_{T,I,F}(y)\}$, $A^+_{T,I,F}(x) \geq min\{A^+_{T,I,F}(x*y), A^+_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x) \leq max\{\lambda_{T,I,F}(x*y), \lambda_{T,I,F}(y)\}$. Hence, $A^-_{T,I,F}$, $A^+_{T,I,F}$ and $\lambda_{T,I,F}$ are $fuzzy\ ideals$ of X.

Theorem 5.5 *For a neutrosophic cubic ideal $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ of X, the following are valid:*

1. *if $x*y \leq z$, then $A_{T,I,F}(x) \geq rmin\{A_{T,I,F}(y), A_{T,I,F}(z)\}$ and $\lambda_{T,I,F}(x) \leq max\{\lambda_{T,I,F}(y), \lambda_{T,I,F}(z)\}$,*
2. *if $x \leq y$, then $A_{T,I,F}(x) \geq A_{T,I,F}(y)$ and $\lambda_{T,I,F}(x) \leq \lambda_{T,I,F}(y)\ \forall\ x, y, z \in X$.*

Proof: (1) Assume that $x, y, z \in X$ such that $x*y \leq z$. Then $(x*y)*z = 0$ and thus $A_{T,I,F}(x) \geq rmin\{A_{T,I,F}(x*y), A_{T,I,F}(y)\} \geq rmin\{rmin\{A_{T,I,F}((x*y)*z), A_{T,I,F}(z)\}, A_{T,I,F}(y)\}$ =$rmin\{rmin\{A_{T,I,F}(0), A_{T,I,F}(z)\}, A_{T,I,F}(y)\}$=$rmin\{A_{T,I,F}(y), A_{T,I,F}(z)\}$ and $\lambda_{T,I,F}(x) \leq max\{\lambda_{T,I,F}(x*y), \lambda_{T,I,F}(y)\} \leq max\{max\{\lambda_{T,I,F}((x*y)*z), \lambda_{T,I,F}(z)\}, \lambda_{T,I,F}(y)\}$=$max\{max\{\lambda_{T,I,F}(0), \lambda_{T,I,F}(z)\}, \lambda_{T,I,F}(y)\}$ = $max\{\lambda_{T,I,F}(y), \lambda_{T,I,F}(z)\}$.

(2) Again, take $x, y \in X$ such that $x \leq y$. Then $x * y = 0$ and thus $A_{T,I,F}(x) \geq rmin\{A_{T,I,F}(x*y), A_{T,I,F}(y)\}$ = $rmin\{A_{T,I,F}(0), A_{T,I,F}(y)\}$ = $A_{T,I,F}(y)$ and $\lambda_{T,I,F}(x) \leq rmin\{\lambda_{T,I,F}(x*y), \lambda_{T,I,F}(y)\}$ = $rmin\{\lambda_{T,I,F}(0), \lambda_{T,I,F}(y)\}$ = $\lambda_{T,I,F}(y)$.

Theorem 5.6 *Let $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ is a neutrosophic cubic ideal of X. If $x * y \leq x \; \forall \; x, y \in X$, then $\mathcal{A}$ is a neutrosophic cubic subalgebra of X.*

Proof: Assume that $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ is a neutrosophic cubic ideal of X. Suppose that $x * y \leq x \; \forall \; x, y \in X$. Then

$$\begin{aligned} A_{T,I,F}(x*y) &\geq A_{T,I,F}(x) \\ &\quad (\because By \;\; Theorem \;\; 5.5) \\ &\geq rmin\{A_{T,I,F}(x*y), A_{T,I,F}(y)\} \\ &\quad (\because By \;\; N4) \\ &\geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\} \\ &\quad (\because By \;\; Theorem \;\; 5.5) \\ \Rightarrow A_{T,I,F}(x*y) &\geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\} \end{aligned}$$

and

$$\begin{aligned} \lambda_{T,I,F}(x*y) &\leq \lambda_{T,I,F}(x) \\ &\quad (\because By \;\; Theorem \;\; 5.5) \\ &\leq max\{\lambda_{T,I,F}(x*y), \lambda_{T,I,F}(y)\} \\ &\quad (\because By \;\; N5) \\ &\leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\} \\ &\quad (\because By \;\; Theorem \;\; 5.5) \\ \Rightarrow \lambda_{T,I,F}(x*y) &\leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}. \end{aligned}$$

Hence, $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ is a neutrosophic cubic subalgebra of X.

Theorem 5.7 *If $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ is a neutrosophic cubic ideal of X, then $(...((x * a_1) * a_2) * ...) * a_n = 0$ for any $x, a_1, a_2, ..., a_n \in X$, $\Rightarrow A_{T,I,F}(x) \geq rmin\{A_{T,I,F}(a_1), A_{T,I,F}(a_2), ..., A_{T,I,F}(a_n)\}$ and $\lambda_{T,I,F}(x) \leq max\{\lambda_{T,I,F}(a_1), \lambda_{T,I,F}(a_2), ..., \lambda_{T,I,F}(a_n)\}$.*

Proof: We can prove this theorem by using induction on n and Theorem 5.5).

Theorem 5.8 *A neutrosophic cubic set $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic closed ideal of $X \iff U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and $L(\lambda_{T,I,F} \mid t_{T,I,F_1})$ are closed ideals of X for every $[s_{T,I,F_1}, s_{T,I,F_2}] \in D[0,1]$ and $t_{T,I,F_1} \in [0,1]$.*

Proof: Assume that $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic closed ideal of X. For $[s_{T,I,F_1}, s_{T,I,F_2}] \in D[0,1]$, clearly, $0 * x \in U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$, where $x \in X$. Let $x, y \in X$ be such that $x * y \in U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and $y \in U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$. Then $A_{T,I,F}(x) \geq rmin\{A_{T,I,F}(x * y), A_{T,I,F}(y)\} \geq [s_{T,I,F_1}, s_{T,I,F_2}]$, $\Rightarrow x \in U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$. Hence, $U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ is a closed ideal of X.

For $t_{T,I,F_1} \in [0,1]$. Clearly, $0 * x \in L(\lambda_{T,I,F} \mid t_{T,I,F_1})$. Let $x, y \in X$ be such that $x * y \in L(\lambda_{T,I,F} \mid t_{T,I,F_1})$ and $y \in L(\lambda_{T,I,F} \mid t_{T,I,F_1})$. Then $\lambda_{T,I,F}(x) \leq max\{\lambda_{T,I,F}(x * y), \lambda_{T,I,F}(y)\} \leq t_{T,I,F_1}$, $\Rightarrow x \in L(\lambda_{T,I,F} \mid t_{T,I,F_1})$. Hence, $L(\lambda_{T,I,F} \mid t_{T,I,F_1})$. is a neutrosophic cubic closed ideal of X.

Conversely, suppose that each non-empty level subset $U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and $L(\lambda_{T,I,F} \mid t_{T,I,F_1})$ are closed ideals of X. For any $x \in X$, let $A_{T,I,F}(x) = [s_{T,I,F_1}, s_{T,I,F_2}]$ and $\lambda_{T,I,F}(x) = t_{T,I,F_1}$. Then $x \in U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and $x \in L(\lambda_{T,I,F} \mid t_{T,I,F_1})$. Since $0 * x \in U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}]) \bigcap L(\lambda_{T,I,F} \mid t_{T,I,F_1})$, it follows that $A_{T,I,F}(0 * x) \geq [s_{T,I,F_1}, s_{T,I,F_2}] = A_{T,I,F}(x)$ and $\lambda_{T,I,F}(0 * x) \leq t_{T,I,F_1} = \lambda_{T,I,F}(x) \; \forall \; x \in X$.

If there exist $\alpha_{T,I,F_1}, \beta_{T,I,F_1} \in X$ such that $A_{T,I,F}(\alpha_{T,I,F_1}) < rmin\{A_{T,I,F}(\alpha_{T,I,F_1} * \beta_{T,I,F_1}), \beta_{T,I,F_1}\}$, then by taking $[s'_{T,I,F_1}, s'_{T,I,F_2}] = \frac{1}{2}[A_{T,I,F}(\alpha_{T,I,F_1} * \beta_{T,I,F_1}) + rmin\{A_{T,I,F}(\alpha_{T,I,F_1}), A_{T,I,F}(\beta_{T,I,F_1})\}]$, it follows that $\alpha_{T,I,F_1} * \beta_{T,I,F_1} \in U(A_{T,I,F} \mid [s'_{T,I,F_1}, s'_{T,I,F_2}])$ and $\beta_{T,I,F_1} \in U(A_{T,I,F} \mid [s'_{T,I,F_1}, s'_{T,I,F_2}])$, but $\alpha_{T,I,F_1} \notin U(A_{T,I,F} \mid [s'_{T,I,F_1}, s'_{T,I,F_2}])$, which is contradiction. Hence, $U(A_{T,I,F} \mid [s'_{T,I,F_1}, s'_{T,I,F_2}])$ is not closed ideal of X.

Again, if there exist $\gamma_{T,I,F_1}, \;\; \delta_{T,I,F_1} \in X$ such that $\lambda_{T,I,F}(\gamma_{T,I,F_1}) > max\{\lambda_{T,I,F}(\gamma_{T,I,F_1} * \delta_{T,I,F_1}), \lambda_{T,I,F}(\delta_{T,I,F_1})\}$, then by taking $t'_{T,I,F_1} = \frac{1}{2}[\lambda_{T,I,F}(\gamma_{T,I,F_1} * \delta_{T,I,F_1}) + max\{\lambda_{T,I,F}(\gamma_{T,I,F_1}), \lambda_{T,I,F}(\delta_{T,I,F_1})\}]$, it follows that $\gamma_{T,I,F_1} * \delta_{T,I,F_1} \in L(\lambda_{T,I,F} \mid t'_{T,I,F_1})$ and $\delta_{T,I,F_1} \in L(\lambda_{T,I,F} \mid t'_{T,I,F_1})$, but $\gamma_{T,I,F_1} \notin L(\lambda_{T,I,F} \mid t'_{T,I,F_1})$, which is contradiction. Hence, $L(\lambda_{T,I,F} \mid t'_{T,I,F_1})$ is not closed ideal of X. Hence, $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic closed ideal of X because it satisfies N3 and N4.

6 Investigation of Neutrosophic Cubic Ideals under Homomorphism

In this section, neutrosophic cubic ideals are investigated under homomorphism and some results are studied.

Theorem 6.1 *Suppose that $f \mid X \to Y$ is a homomorphism of B-algebra. If $\mathcal{A}$=$(A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic ideal of Y, then pre-image $f^{-1}(\mathcal{A})$=$\Big(f^{-1}(A_{T,I,F}), f^{-1}(\lambda_{T,I,F})\Big)$ of $\mathcal{A}$ under f of X is a neutrosophic cubic ideal of X.*

Proof: $\forall \; x \in X$, $f^{-1}(A_{T,I,F})(x) = A_{T,I,F}(f(x)) \leq A_{T,I,F}(0)$=$A_{T,I,F}(f(0))$=$f^{-1}(A_{T,I,F})(0)$ and $f^{-1}(\lambda_{T,I,F})(x)$ = $\lambda_{T,I,F}(f(x)) \geq \lambda_{T,I,F}(0) = \lambda_{T,I,F}(f(0)) = f^{-1}(\lambda_{T,I,F})(0)$.

Let $x, y \in X$ then $f^{-1}(A_{T,I,F})(x)$ = $A_{T,I,F}(f(x)) \geq rmin\{A_{T,I,F}(f(x) * f(y)), A_{T,I,F}(f(y))\} = rmin\{A_{T,I,F}(f(x*y)), A_{T,I,F}(f(y))\} = rmin\{f^{-1}(A_{T,I,F})(x*y), f^{-1}(A_{T,I,F})(y)\}$ and $f^{-1}(\lambda_{T,I,F})(x) = \lambda_{T,I,F}(f(x)) \leq max\{\lambda_{T,I,F}(f(x) * f(y)), \lambda_{T,I,F}(f(y))\} = max\{\lambda_{T,I,F}(f(x*y)), \lambda_{T,I,F}(f(y))\} = max\{f^{-1}(\lambda_{T,I,F})(x*y), f^{-1}(\lambda_{T,I,F})(y)\}$.

Hence, $f^{-1}(\mathcal{A}) = \Big(f^{-1}(A_{T,I,F}), f^{-1}(\lambda_{T,I,F})\Big)$ is a neutrosophic cubic ideal of X.

Corollary 6.1 *A homomorphic pre-image of a neutrosophic cubic closed ideal is a neutrosophic cubic ideal.*

Proof: Using Proposition 5.1 and Theorem 6.1, straightforward.

Corollary 6.2 *A homomorphic pre-image of a neutrosophic cubic closed ideal is also a neutrosophic cubic subalgebra.*

Proof: Straightforward, using Theorem 5.1 and Theorem 6.1.

Corollary 6.3 *Let $f \mid X \to Y$ be homomorphism of B-algebra. If $\mathcal{A}_i = (A_{iT,I,F}, \lambda_{iT,I,F})$ is a neutrosophic cubic ideals of Y where $i \in k$ then the pre-image $f^{-1}\Big(\bigcap_{i\in k_R} A_{iT,I,F}\Big)$ $= \Big(f^{-1}(\bigcap_{i\in k_R} A_{iT,I,F}), f^{-1}(\bigcap_{i\in k_R} \lambda_{iT,I,F})\Big)$ is a neutrosophic cubic ideal of X.*

Proof: Straightforward, using Theorem 5.2 and Theorem 6.1.

Corollary 6.4 *Let $f \mid X \to Y$ be homomorphism of B-algebra. If $\mathcal{A}_i = (A_{iT,I,F}, \lambda_{iT,I,F})$ is a neutrosophic cubic closed ideals of Y where $i \in k$ then the pre-image $f^{-1}\Big(\bigcap_{i\in k_R} A_{iT,I,F}\Big)$ $= \Big(f^{-1}(\bigcap_{i\in k_R} A_{iT,I,F}), f^{-1}(\bigcap_{i\in k_R} \lambda_{iT,I,F})\Big)$ is a neutrosophic cubic closed ideal of X.*

Proof: Straightforward, using theorem 5.3 and Theorem 6.1.

Theorem 6.2 *Suppose that $f \mid X \to Y$ is an epimorphism of B-algebra. Then $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic ideal of Y, if $f^{-1}(\mathcal{A}) = \Big(f^{-1}(A_{T,I,F}), f^{-1}(\lambda_{T,I,F})\Big)$ of $\mathcal{A}$ under f of X is a neutrosophic cubic ideal of X.*

Proof: For any $y \in Y$, $\exists\, x \in X$ such that $y = f(x)$. So, $A_{T,I,F}(y) = A_{T,I,F}(f(x)) = f^{-1}(A_{T,I,F})(x) \leq f^{-1}(A_{T,I,F})(0) = A_{T,I,F}(f(0)) = A_{T,I,F}(0)$ and $\lambda_{T,I,F}(y) = \lambda_{T,I,F}(f(x)) = f^{-1}(\lambda_{T,I,F})(x) \geq f^{-1}(\lambda_{T,I,F})(0) = \lambda_{T,I,F}(f(0)) = \lambda_{T,I,F}(0)$.

Suppose $y_1, y_2 \in y$. Then $f(x_1) = y_1$ and $f(x_2) = y_2$ for some $x_1,\ x_2 \in X$. Thus, $A_{T,I,F}(y_1) = A_{T,I,F}(f(x_1)) = f^{-1}(A_{T,I,F})(x_1) \geq rmin\{f^{-1}(A_{T,I,F})(x_1*x_2), f^{-1}(A_{T,I,F})(x_2)\} = rmin\{A_{T,I,F}(f(x_1*x_2)), A_{T,I,F}(f(x_2))\} = rmin\{A_{T,I,F}(f(x_1)*f(x_2)), A_{T,I,F}(f(x_2))\}$=$rmin\{A_{T,I,F}(y_1*y_2), A_{T,I,F}(y_2)\}$ and $\lambda_{T,I,F}(y_1)$=$\lambda_{T,I,F}(f(x_1))$=$f^{-1}(\lambda_{T,I,F})(x_1) \leq max\{f^{-1}(\lambda_{T,I,F})(x_1*x_2), f^{-1}(\lambda_{T,I,F})(x_2)\} = max\{\lambda_{T,I,F}(f(x_1 x_2)), \lambda_{T,I,F}(f(x_2))\} = max\{\lambda_{T,I,F}(f(x_1)*f(x_2)), \lambda_{T,I,F}(f(x_2))\} = max\{\lambda_{T,I,F}(y_1 * y_2), \lambda_{T,I,F}(y_2)\}$. Hence, $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic ideal of Y.

6.1 Product of Neutrosophic Cubic B-algebra

In this section, product of neutrosophic cubic B-algebras are defined and some corresponding results are investigated.

Definition 6.1 *Let $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ and $\mathcal{B} = (B_{T,I,F}, \upsilon_{T,I,F})$ be two neutrosophic cubic sets of X and Y respectively. The Cartesian product $\mathcal{A}\times\mathcal{B} = (X\times Y, A_{T,I,F}\times B_{T,I,F}, \lambda_{T,I,F}\times\upsilon_{T,I,F})$ is defined by $(A_{T,I,F}\times B_{T,I,F})(x,y)$ $= rmin\{A_{T,I,F}(x), B_{T,I,F}(y)\}$ and $(\lambda_{T,I,F}\times\upsilon_{T,I,F})(x,y) = max\{\lambda_{T,I,F}(x), \upsilon_{T,I,F}(y)\}$, where $A_{T,I,F}\times B_{T,I,F} \mid X\times Y \to D[0,1]$ and $\lambda_{T,I,F}\times\upsilon_{T,I,F} \mid X\times Y \to [0,1]\ \forall\,(x,y)\in X\times Y$.*

Remark 6.1 *Let X and Y be B-algebras. we define $*$ on $X\times Y$ by $(x_1,y_1)*(x_2,y_2) = (x_1*x_2, y_1*y_2)$ for every (x_1,y_1) and $(x_2,y_2)\in X\times Y$. Then clearly, $X\times Y$ is a B-algebra.*

Definition 6.2 *A neutrosophic cubic subset $\mathcal{A}\times\mathcal{B} = (X\times Y, A_{T,I,F}\times B_{T,I,F}, \lambda_{T,I,F}\times\upsilon_{T,I,F})$ is called a neutrosophic cubic subalgebra if*
N7: $(A_{T,I,F}\times B_{T,I,F})((x_1,y_1)*(x_2,y_2)) \geq rmin\{(A_{T,I,F}\times B_{T,I,F})(x_1,y_1), (A_{T,I,F}\times B_{T,I,F})(x_2,y_2)\}$
N8: $(\lambda_{T,I,F}\times\upsilon_{T,I,F})((x_1,y_1)*(x_2,y_2)) \leq max\{(\lambda_{T,I,F}\times\upsilon_{T,I,F})(x_1,y_1), (\lambda_{T,I,F}\times\upsilon_{T,I,F})(x_2,y_2)\}$
$\forall\,(x_1,y_1),(x_2,y_2)\in X\times Y$

Theorem 6.3 *Let $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ and $\mathcal{B} = (B_{T,I,F}, \upsilon_{T,I,F})$ be neutrosophic cubic subalgebra of X and Y respectively. Then $\mathcal{A}\times\mathcal{B}$ is a neutrosophic cubic subalgebra of $X\times Y$.*

Proof: Let (x_1,y_1) and $(x_2,y_2)\in X\times Y$. Then $(A_{T,I,F}\times B_{T,I,F})((x_1,y_1)*(x_2,y_2)) = (A_{T,I,F}\times B_{T,I,F})(x_1*x_2, y_1*y_2)$ = $rmin\{A_{T,I,F}(x_1*x_2), B_{T,I,F}(y_1*y_2)\} \geq rmin\{rmin\{A_{T,I,F}(x_1), A_{T,I,F}(x_2)\}, rmin\{B_{T,I,F}(y_1), B_{T,I,F}(y_2)\}\}$ = $rmin\ \{rmin\ \{A_{T,I,F}(x_1), B_{T,I,F}(y_1)\}, rmin\ \{A_{T,I,F}(x_2), B_{T,I,F}(y_2)\}\} = rmin\{(A_{T,I,F}\times B_{T,I,F})(x_1,y_1), (A_{T,I,F}\times B_{T,I,F})(x_2,y_2)\}$ and $(\lambda_{T,I,F}\times\upsilon_{T,I,F})((x_1,y_1)*(x_2,y_2)) = (\lambda_{T,I,F}\times\upsilon_{T,I,F})(x_1*x_2, y_1*y_2) = max\{\lambda_{T,I,F}(x_1*x_2), \upsilon_{T,I,F}(y_1*y_2)\} \leq max\{max\{\lambda_{T,I,F}(x_1), \lambda_{T,I,F}(x_2)\}, max\{\upsilon_{T,I,F}(y_1), \upsilon_{T,I,F}(y_2)\}\}$=$max\{max\{\lambda_{T,I,F}(x_1), \upsilon_{T,I,F}(y_1)\}, max\{\lambda_{T,I,F}(x_2), \upsilon_{T,I,F}(y_2)\}\} = max\{(\lambda_{T,I,F}\times\upsilon_{T,I,F})(x_1,y_1), (\lambda_{T,I,F}\times\upsilon_{T,I,F})(x_2,y_2)\}$. Hence $\mathcal{A}\times\mathcal{B}$ is a neutrosophic cubic subalgebra of $X\times Y$.

Definition 6.3 *A neutrosophic cubic subset $\mathcal{A}\times\mathcal{B} = (X\times Y, A_{T,I,F}\times B_{T,I,F}, \lambda_{T,I,F}\times\upsilon_{T,I,F})$ is called a neutrosophic cubic ideal if*
N9: $(A_{T,I,F}\times B_{T,I,F})(0,0) \geq (A_{T,I,F}\times B_{T,I,F})(x,y)$ *and* $(\lambda_{T,I,F}\times\upsilon_{T,I,F})(0,0) \leq (\lambda_{T,I,F}\times\upsilon_{T,I,F})(x,y)$ $\forall (x,y)\in X\times Y$,
N10: $(A_{T,I,F}\times B_{T,I,F})(x_1,y_1) \geq rmin\{(A_{T,I,F}\times B_{T,I,F})((x_1,y_1)*(x_2,y_2)), (A_{T,I,F}\times B_{T,I,F})(x_2,y_2)$
and
N11: $(\lambda_{T,I,F}\times\upsilon_{T,I,F})(x_1,y_1) \leq max\{(\lambda_{T,I,F}\times\upsilon_{T,I,F})((x_1,y_1)*(x_2,y_2)), (\lambda_{T,I,F}\times\upsilon_{T,I,F})(x_2,y_2)\}$
*and $\mathcal{A}\times\mathcal{B}$ is closed ideal if it satisfies **N9, N10, N11**, and*

N12: $(A_{T,I,F} \times B_{T,I,F})((0,0) * (x,y)) \geq (A_{T,I,F} \times B_{T,I,F})(x,y) \,\forall\, (x_1,y_1),(x_2,y_2) \in X \times Y$.

Theorem 6.4 *Let $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ and $\mathcal{B} = (B_{T,I,F}, \upsilon_{T,I,F})$ be neutrosophic cubic ideals of X and Y respectively. Then $\mathcal{A} \times \mathcal{B}$ is a neutrosophic cubic ideal of $X \times Y$.*

Proof: For any $(x,y) \in X \times Y$, we have $(A_{T,I,F} \times B_{T,I,F})(0,0) = rmin\{A_{T,I,F}(0), B_{T,I,F}(0)\} \geq rmin\{A_{T,I,F}(x), B_{T,I,F}(y)\} = (A_{T,I,F} \times B_{T,I,F})(x,y)$ and $(\lambda_{T,I,F} \times \upsilon_{T,I,F})(0,0) = max\{\lambda_{T,I,F}(0), \upsilon_{T,I,F}(0)\} \leq max\{\lambda_{T,I,F}(x), \upsilon_{T,I,F}(y)\} = (\lambda_{T,I,F} \times \upsilon_{T,I,F})(x,y)$.

Let (x_1,y_1) and $(x_2,y_2) \in X \times Y$. Then $(A_{T,I,F} \times B_{T,I,F})(x_1,y_1) = rmin\{A_{T,I,F}(x_1), B_{T,I,F}(y_1)\} \geq rmin\{rmin\{A_{T,I,F}(x_1 * x_2), A_{T,I,F}(x_2)\}, rmin\{B_{T,I,F}(y_1 * y_2), B_{T,I,F}(y_2)\}\} = rmin\{rmin\{A_{T,I,F}(x_1 * x_2), B_{T,I,F}(y_1 * y_2)\}, rmin\{A_{T,I,F}(x_2), B_{T,I,F}(y_2)\}\} = rmin\{(A_{T,I,F} \times B_{T,I,F})(x_1 * x_2, y_1 * y_2), (A_{T,I,F} \times B_{T,I,F})(x_2,y_2)\} = rmin\{(A_{T,I,F} \times B_{T,I,F})((x_1,y_1) * (x_2,y_2)), (A_{T,I,F} \times B_{T,I,F})(x_2,y_2)\}$ and $(\lambda_{T,I,F} \times \upsilon_{T,I,F})(x_1,y_1) = max\{\lambda_{T,I,F}(x_1), \upsilon_{T,I,F}(y_1)\} \leq max\{max\{\lambda_{T,I,F}(x_1 * x_2), \lambda_{T,I,F}(x_2)\}, max\{\upsilon_{T,I,F}(y_1 * y_2), \upsilon_{T,I,F}(y_2)\}\} = max\{max\{\lambda_{T,I,F}(x_1 * x_2), \upsilon_{T,I,F}(y_1 * y_2)\}, max\{\lambda_{T,I,F}(x_2), \upsilon_{T,I,F}(y_2)\}\} = max\{\lambda_{T,I,F} \times \upsilon_{T,I,F})(x_1 * x_2, y_1 * y_2), (\lambda_{T,I,F} \times \upsilon_{T,I,F})(x_2,y_2)\} = max\{(\lambda_{T,I,F} \times \upsilon_{T,I,F})((x_1,y_1) * (x_2 * y_2)), (\lambda_{T,I,F} \times \upsilon_{T,I,F})(x_2,y_2)\}$. Hence, $\mathcal{A} \times \mathcal{B}$ is a neutrosophic cubic ideal of $X \times Y$.

Theorem 6.5 *Let $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ and $\mathcal{B} = (B_{T,I,F}, \upsilon_{T,I,F})$ be neutrosophic cubic closed ideals of X and Y respectively. Then $\mathcal{A} \times \mathcal{B}$ is a neutrosophic cubic closed ideal of $X \times Y$.*

Proof: By Proposition 5.1 and Theorem 6.4, $\mathcal{A} \times \mathcal{B}$ is neutrosophic cubic ideal. Now, $(A_{T,I,F} \times B_{T,I,F})((0,0) * (x,y)) = (A_{T,I,F} \times B_{T,I,F})(0 * x, 0 * y) = rmin\{A_{T,I,F}(0 * x), B_{T,I,F}(0 * y)\} \geq rmin\{A_{T,I,F}(x), B_{T,I,F}(y)\} = (A_{T,I,F} \times B_{T,I,F})(x,y)$ and $(\lambda_{T,I,F} \times \upsilon_{T,I,F})((0,0) * (x,y)) = (\lambda_{T,I,F} \times \upsilon_{T,I,F})(0 * x, 0 * y) = max\{\lambda_{T,I,F}(0 * x), \upsilon_{T,I,F}(0 * y)\} \leq max\{\lambda_{T,I,F}(x), \upsilon_{T,I,F}(y)\} = (\lambda_{T,I,F} \times \upsilon_{T,I,F})(x,y)$. Hence, $\mathcal{A} \times \mathcal{B}$ is a neutrosophic cubic closed ideal of $X \times Y$. Hence, $\mathcal{A} \times \mathcal{B}$ is a neutrosophic cubic closed ideal of $X \times Y$.

Definition 6.4 *Let $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ and $\mathcal{B} = (B_{T,I,F}, \upsilon_{T,I,F})$ be neutrosophic cubic subalgebra of X and Y respectively. For $[s_{T,I,F_1}, s_{T,I,F_2}] \in D[0,1]$ and $t_{T,I,F_1} \in [0,1]$, the set $U(A_{T,I,F} \times B_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}]) = \{(x,y) \in X \times Y \mid (A_{T,I,F} \times B_{T,I,F})(x,y) \geq [s_{T,I,F_1}, s_{T,I,F_2}]\}$ is called upper $[s_{T,I,F_1}, s_{T,I,F_2}]$-level of $\mathcal{A} \times \mathcal{B}$ and $L(\lambda_{T,I,F} \times \upsilon_{T,I,F} \mid t_{T,I,F_1}) = \{(x,y) \in X \times Y \mid (\lambda_{T,I,F} \times \upsilon_{T,I,F})(x,y) \leq t_{T,I,F_1}$ is called lower t_{T,I,F_1}-level of $\mathcal{A} \times \mathcal{B}$.*

Theorem 6.6 *For any two neutrosophic cubic sets $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ and $\mathcal{B} = (B_{T,I,F}, \upsilon_{T,I,F})$, $\mathcal{A} \times \mathcal{B}$ is a neutrosophic cubic closed ideals of $X \times Y \iff$ the non-empty upper $[s_{T,I,F_1}, s_{T,I,F_2}]$-level cut $U(A_{T,I,F} \times B_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and the non-empty lower t_{T,I,F_1}-level $L(\lambda_{T,I,F} \times \upsilon_{T,I,F} \mid t_{T,I,F_1})$ are closed ideals of $X \times Y$ for any $[s_{T,I,F_1}, s_{T,I,F_2}] \in D[0,1]$ and $t_{T,I,F_1} \in [0,1]$.*

Proof: Suppose $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ and $\mathcal{B} = (B_{T,I,F}, \upsilon_{T,I,F})$ be neutrosophic cubic closed ideals of X. Therefore, for any $(x,y) \in X \times Y$, $(A_{T,I,F} \times B_{T,I,F})((0,0) * (x,y)) \geq (A_{T,I,F} \times B_{T,I,F})(x,y)$ and $(\lambda_{T,I,F} \times \upsilon_{T,I,F})((0,0) * (x,y)) \leq (\lambda_{T,I,F} \times \upsilon_{T,I,F})(x,y)$. For $[s_{T,I,F_1}, s_{T,I,F_2}] \in D[0,1]$, if $(A_{T,I,F} \times B_{T,I,F})(x,y) \geq [s_{T,I,F_1}, s_{T,I,F_2}]$, then $(A_{T,I,F} \times B_{T,I,F})((0,0) * (x,y)) \geq [s_{T,I,F_1}, s_{T,I,F_2}]$. $\Rightarrow (0,0) * (x,y) \in U(A_{T,I,F} \times B_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$. Let $(x,y), (x', y') \in X \times Y$ be such that $(x,y) * (x', y') \in U(A_{T,I,F} \times B_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and $(x', y') \in U(A_{T,I,F} \times B_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$. Now, $(A_{T,I,F} \times B_{T,I,F})(x,y) \geq rmin\{(A_{T,I,F} \times B_{T,I,F})((x,y) * (x', y')), (A_{T,I,F} \times B_{T,I,F})(x', y')\} \geq rmin\{[s_{T,I,F_1}, s_{T,I,F_2}], [s_{T,I,F_1}, s_{T,I,F_2}]\} = [s_{T,I,F_1}, s_{T,I,F_2}]$. $\Rightarrow (x,y) \in U(A_{T,I,F} \times B_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$. Thus $U(A_{T,I,F} \times B_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ is closed ideal of $X \times Y$. Similarly, $L(\lambda_{T,I,F} \times \upsilon_{T,I,F} \mid t_{T,I,F_1})$ is closed ideal of $X \times Y$.

Conversely, let $(x,y) \in X \times Y$ be such that $(A_{T,I,F} \times B_{T,I,F})(x,y) = [s_{T,I,F_1}, s_{T,I,F_2}]$ and $(\lambda_{T,I,F} \times \upsilon_{T,I,F})(x,y) = t_{T,I,F_1}$. This implies, $(x,y) \in U(A_{T,I,F} \times B_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and $(x,y) \in L(\lambda_{T,I,F} \times \upsilon_{T,I,F} \mid t_{T,I,F_1})$. Since $(0,0) * (x,y) \in U(A_{T,I,F} \times B_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and $(0,0) * (x,y) \in L(\lambda_{T,I,F} \times \upsilon_{T,I,F} \mid t_{T,I,F_1})$ (by N6), therefore, $(A_{T,I,F} \times B_{T,I,F})((0,0) * (x,y)) \geq [s_{T,I,F_1}, s_{T,I,F_2}]$ and $(\lambda_{T,I,F} \times \upsilon_{T,I,F})((0,0) * (x,y)) \leq t_{T,I,F_1}$. $\Rightarrow (A_{T,I,F} \times B_{T,I,F})((0,0) * (x,y)) \geq (A_{T,I,F} \times B_{T,I,F})(x,y)$ and $(\lambda_{T,I,F} \times \upsilon_{T,I,F})((0,0) * (x,y)) \leq (\lambda_{T,I,F} \times \upsilon_{T,I,F})(x,y))$. Hence $\mathcal{A} \times \mathcal{B}$ is a neutrosophic cubic closed ideals of $X \times Y$.

7 Conclusion

In this paper, the concept of neutrosophic cubic subalgebra, neutrosophic cubic ideals, neutrosophic cubic closed ideals and the product of neutrosophic cubic subalgebra of B-algebra were presented and their several useful results were canvassed. The relations among neutrosophic cubic subalgebra, neutrosophic cubic ideals and neutrosophic cubic closed ideals of B-algebra were investigated. For future work this study will be further discussed to some another algebraic system.

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Received: November 25, 2016. Accepted: November 30, 2016