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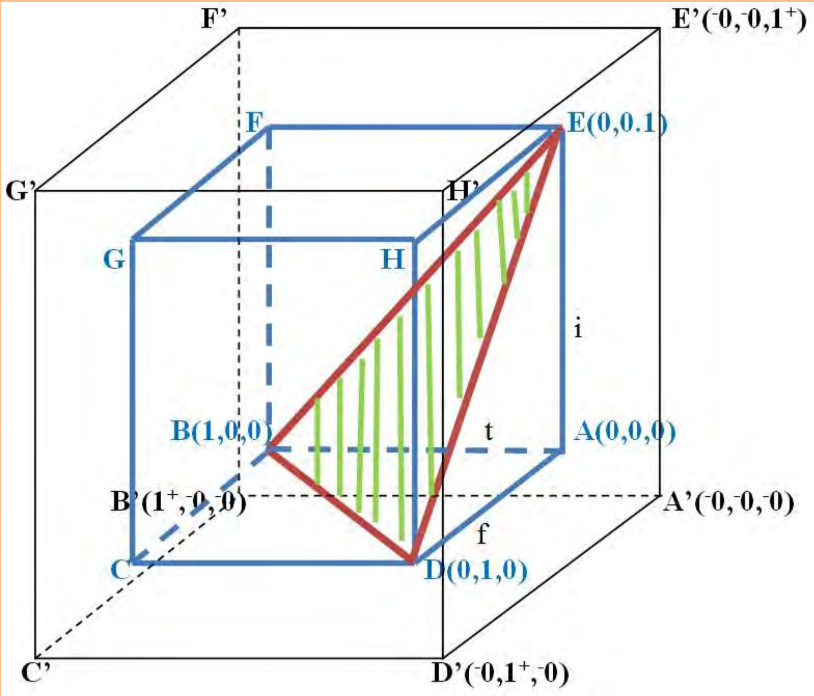

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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Multiple Criteria Evaluation Model Based on the Single Valued Neutrosophic Set

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Abstract. Gathering the attitudes of the examined respondents would be very significant in some evaluation models. Therefore, a multiple criteria approach based on the use of the neutrosophic set is considered in this paper. An example of the evaluation of restaurants is considered at the end of this paper with the aim to present in detail the proposed approach.

Keywords: neutrosophic set, single valued neutrosophic set, multiple criteria evaluation.

1. Introduction

In order to deal with indeterminate and inconsistent information, Smarandache [1] proposed a neutrosophic set (NS), thus simultaneously providing a general framework generalizing the concepts of the classical, fuzzy [2], interval-valued [3, 4], intuitionistic [5] and interval-valued intuitionistic [6] fuzzy sets.

The NS has been applied in different fields, such as: the database [7], image processing [8, 9, 10], the medical diagnosis [11, 12], decision making [13, 14], with a particular emphasis on multiple criteria decision making [15, 16, 17, 18, 19, 20].

In addition to the membership function, or the so-called truth-membership $T_A(x)$, proposed in fuzzy sets, Atanassov [5] introduced the non-membership function, or the so-called falsity-membership $F_A(x)$, which expresses non-membership to a set, thus creating the basis for the solving of a much larger number of decision-making problems.

In intuitionistic fuzzy sets, the indeterminacy $I_A(x)$ is $1-T_A(x)-F_A(x)$ by default.

In the NS, Smarandache [21] introduced independent indeterminacy-membership $I_A(x)$, thus making the NS more flexible and the most suitable for solving some complex decision-making problems, especially decision-making problems related to the use of incomplete and imprecise information, uncertainties and predictions and so on.

Smarandache [1] and Wang *et al*. [22] further proposed the single valued neutrosophic set (SVNS) suitable for solving many real-world decision-making problems.

In multiple criteria evaluation models, where evaluation is based on the ratings generated from respondents, the NS and the SVNS can provide some advantages in relation to the usage of crisp and other forms of fuzzy numbers.

Therefore, the rest of this paper is organized as follows: in Section 2, some basic definitions related to the SVNS are given. In Section 3, an approach to the determining of criteria weights is presented, while Section 4 proposes a multiple criteria evaluation model based on the use of the SVNS. In Section 5, an example is considered with the aim to explain in detail the proposed methodology. The conclusions are presented at the end of the manuscript.

2. The Single Valued Neutrosophic Set

Definition 1. [21] Let X be the universe of discourse, with a generic element in X denoted by x. Then, the Neutrosophic Set (NS) A in X is as follows:

$$A = \{x < T_A(x), I_A(x), F_A(x) > | \, x \in X\}, \tag{1}$$

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership function, the indeterminacy-membership function and the falsity-membership function, respectively, $T_A, I_A, F_A : X \to]^-0, 1^+[$ and $^-0 \le T_A(x) + I_A(x) + U_A(x) \le 3^+$

Definition 2. [1, 22] Let X be the universe of discourse. The Single Valued Neutrosophic Set (SVNS) A over X is an object having the form:

$$A = \{x < T_A(x), I_A(x), F_A(x) > | \, x \in X\}, \tag{2}$$

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership function, the intermediacy-membership function and the falsity-membership function, respectively, $T_A, I_A, F_A : X \to [0,1]$ and $0 \le T_A(x) + I_A(x) + U_A(x) \le 3$.

Definition 3. [21] For an SVNS A in X, the triple $< t_A, i_A, f_A >$ is called the single valued neutrosophic number (SVNN).

Definition 4. *SVNNs*. Let $x_1 = < t_1, i_1, f_1 >$ and $x_2 = < t_2, i_2, f_2 >$ be two SVNNs and $\lambda > 0$; then, the basic operations are defined as follows:

$$x_1 + x_2 = < t_1 + t_2 - t_1 t_2, i_1 i_2, f_1 f_2 > . \tag{3}$$

$$x_1 \cdot x_2 = < t_1 t_2, i_1 + i_2 - i_1 i_2, f_1 + f_2 - f_1 f_2 > . \tag{4}$$

$$\lambda x_1 = < 1 - (1 - t_1)^\lambda, i_1^\lambda, f_1^\lambda > . \tag{5}$$

$$x_1^\lambda = < t_1^\lambda, i_1^\lambda, 1 - (1 - f_1)^\lambda > . \tag{6}$$

Definition 5. [23] Let $x = < t_x, i_x, f_x >$ be a SVNN; then the cosine similarity measure $S_{(x)}$ between SVNN x and the ideal alternative (point) <1,0,0> can be defined as follows:

$$S_{(x)} = \frac{t}{\sqrt{t^2 + i^2 + f^2}} . \tag{7}$$

Definition 6. [23] Let $A_j = < t_j, i_j, f_j >$ be a collection of SVNSs and $W = (w_1, w_2, \ldots, w_n)^T$ be an associated weighting vector. Then the Single Valued Neutrosophic Weighted Average (SVNWA) operator of A_j is as follows:

$$\begin{aligned} SVNWA(A_1, A_2, \ldots, A_n) &= \sum_{j=1}^{n} w_j A_j \\ &= \left(1 - \prod_{j=1}^{n} (1 - t_j)^{w_j}, \prod_{j=1}^{n} (i_j)^{w_j}, \prod_{j=1}^{n} (f_j)^{w_j} \right), \end{aligned} \tag{8}$$

where: w_j is the element j of the weighting vector, $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

3. The SWARA Method

The Step-wise Weight Assessment Ratio Analysis (SWARA) technique was proposed by Kersuliene *et al.* [25]. The computational procedure of the adapted SWARA method can be shown through the following steps:

Step 1. Determine the set of the relevant evaluation criteria and sort them in descending order, based on their expected significances.

Step 2. Starting from the second criterion, determine the relative importance s_j of the criterion j in relation to the previous (j-1) criterion, and do so for each particular criterion as follows:

$$s_j = \begin{cases} >1 & \text{when significance of } C_j \succ C_{j-1} \\ 1 & \text{when significanse of } C_j = C_{j-1} \\ <1 & \text{when significance of } C_j \prec C_{j-1} \end{cases} . \tag{9}$$

By using Eq. (9), respondents are capable of expressing their opinions more realistically compared to the ordinary SWARA method, proposed by Kersuliene *et al*. [25].

Step 3. The third step in the adapted SWARA method should be performed as follows:

$$k_j = \begin{cases} 1 & j = 1 \\ 2 - s_j & j > 1 \end{cases} . \tag{10}$$

where k_j is a coefficient.

Step 4. Determine the recalculated weight q_j as follows:

$$q_j = \begin{cases} 1 & j = 1 \\ q_{j-1} / k_j & j > 1 \end{cases} . \tag{11}$$

Step 5. Determine the relative weights of the evaluation criteria as follows:

$$w_j = q_j \Big/ \sum_{k=1}^{n} q_k , \tag{12}$$

where w_j denotes the relative weight of the criterion j.

4. A Multiple Criteria Evaluation Model Based on the Use of the SVNS

For a multiple criteria evaluation problem involving the m alternatives that should be evaluated by the K respondents based on the n criteria, whereby the performances of alternatives are expressed by using the SVNS, the calculation procedure can be expressed as follows:

The determination of the criteria weights. The determination of the criteria weights can be done by applying various methods, for example by using the AHP method. However, in this approach, it is recommended that the SWARA method should be used due to its simplicity and a smaller number of pairwise comparisons compared with the well-known AHP method.

The determination of the criteria weight is done by using an interactive questionnaire made in a spreadsheet file. By using such an approach, the interviewee can see the calculated weights of the criteria, which enables him/her modify his or her answers if he or she is not satisfied with the calculated weights.

Gathering the ratings of the alternatives in relation to the selected set of the evaluation criteria. Gathering the ratings of the alternatives in relation to the chosen set of criteria is also done by using an interactive questionnaire. In this questionnaire, a declarative sentence is formed for each one of the criteria, thus giving an opportunity to the respondents to fill in their attitudes about the degree of truth, indeterminacy and falsehood of the statement.

The formation of the separated ranking order based on the weights and ratings obtained from each respondent. At this steep, the ranking order is formed for each one of the respondents, based on the respondent's respective weights and ratings, in the following manner:

– the determination of the overall ratings expressed in the form of the SVNN by using Eq. (8), for each respondent;
– the determination of the cosine similarity measure, for each respondent; and
– the determination of the ranking order, for each respondent.

The determination of the most appropriate alternative. Contrary to the commonly used approach in group decision making, no group weights and ratings are used in this approach. As a result of that, there are the *K* ranking orders of the alternatives and the most appropriate alternative is the one determined on the basis of the theory of dominance [26].

5. A Numerical Illustration

In this numerical illustration, some results adopted from a case study are used. In the said study, four traditional restaurants were evaluated based on the following criteria:

– the interior of the building and the friendly atmosphere,
– the helpfulness and friendliness of the staff,
– the variety of traditional food and drinks,
– the quality and the taste of the food and drinks, including the manner of serving them, and
– the appropriate price for the quality of the services provided.

The survey was conducted via e-mail, using an interactive questionnaire, created in a spreadsheet file. By using such an approach, the interviewee could see the calculated weights of the criteria and was also able to modify his/her answers if he or she was not satisfied with the calculated weights.

In order to explain the proposed approach, three completed surveys have been selected. The attitudes related to the weights of the criteria obtained in the first survey are shown in Table 1. Table 1 also accounts for the weights of the criteria.

Criteria	s_j	k_j	q_j	w_j
C_1		1	1	0.15
C_2	1.00	1.00	1.00	0.15
C_3	1.15	0.85	1.18	0.18
C_4	1.30	0.70	1.68	0.26
C_5	1.00	1.00	1.68	0.26

Table 1. The attitudes and the weights of the criteria obtained on the basis of the first of the three surveys

The attitudes obtained from the three surveys, as well as the appropriate weights, are accounted for in Table 2.

	E_1		E_1		E_1	
	s_j	w_j	s_j	w_j	s_j	w_j
C_1		0.15		0.16		0.19
C_2	1.00	0.15	1.00	0.16	1.00	0.19
C_3	1.15	0.18	1.20	0.20	1.05	0.20
C_4	1.30	0.26	1.10	0.22	1.10	0.22
C_5	1.00	0.26	1.10	0.25	0.95	0.21

Table 2. The attitudes and the weights obtained from the three surveys

The ratings of the alternatives expressed in terms of the SVNS obtained on the basis of the three surveys are given in Tables 3 to 5.

	C_1	C_2	C_3	C_4	C_5
w_j	0.15	0.15	0.18	0.26	0.26
A_1	<0.8,0.1,0.3>	<0.7,0.2,0.2>	<0.8,0.1,0.1>	<1,0.01,0.01>	<0.8,0.1,0.1>
A_2	<0.7,0.1,0.2>	<1.0,0.1,0.1>	<1.0,0.2,0.1>	<1,0.01,0.01>	<0.8,0.1,0.1>
A_3	<0.7,0.1,0.1>	<1.0,0.1,0.1>	<0.7,0.1,0.1>	<0.9,0.2,0.01>	<0.9,0.1,0.1>
A_4	<0.7,0.3,0.3>	<0.7,0.1,0.1>	<0.8,0.1,0.2>	<0.9,0.1,0.1>	<0.9,0.1,0.1>

Table 3. The ratings obtained based on the first survey

	C_1	C_2	C_3	C_4	C_5
w_j	0.16	0.16	0.20	0.22	0.25
A_1	<0.8,0.1,0.4>	<0.9,0.15,0.3>	<0.9,0.2,0.2>	<0.85,0.1,0.25>	<1.0,0.1,0.2>
A_2	<0.9,0.15,0.3>	<0.9,0.15,0.2>	<1.0,0.3,0.2>	<0.7,0.2,0.1>	<0.8,0.2,0.3>
A_3	<0.6,0.15,0.3>	<0.55,0.2,0.3>	<0.55,0.3,0.3>	<0.6,0.3,0.2>	<0.7,0.2,0.3>
A_4	<0.6,0.4,0.5>	<0.6,0.3,0.1>	<0.6,0.1,0.2>	<0.7,0.1,0.3>	<0.5,0.2,0.4>

Table 4. The ratings obtained based on the second survey

	C_1	C_2	C_3	C_4	C_5
w_j	0.19	0.19	0.20	0.22	0.21
A_1	<1.0,0.1,0.1>	<0.9,0.15,0.2>	<1.0,0.2,0.1>	<0.8,0.1,0.1>	<0.9,0.1,0.2>
A_2	<0.8,0.15,0.3>	<0.9,0.15,0.2>	<1,0.2,0.2>	<0.7,0.2,0.1>	<0.8,0.2,0.3>
A_3	<0.6,0.15,0.3>	<0.55,0.2,0.3>	<0.55,0.3,0.3>	<0.6,0.3,0.2>	<0.7,0.2,0.3>
A_4	<0.8,0.4,0.5>	<0.6,0.3,0.1>	<0.6,0.4,0.1>	<0.7,0.1,0.3>	<0.5,0.2,0.4>

Table 5. The ratings obtained from the third of the third survey

The calculated overall ratings obtained on the basis of the first of the three surveys expressed in the form of SVNSs are presented in Table 6. The cosine similarity measures, calculated by using Eq. (7), as well as the ranking order of the alternatives, are accounted for in Table 6.

	Overall ratings	S_i	Rank
A_1	<1.0,0.06,0.07>	0.995	2
A_2	<1.0,0.06,0.06>	0.996	1
A_3	<1.0,0.12,0.06>	0.991	3
A_4	<1.0,0.12,0.13>	0.978	4

Table 6. The ranking orders obtained on the basis of the ratings of the first survey

The ranking orders obtained based on all the three surveys are accounted for in Table 7.

	E_1 S_i	E_2 S_i	E_3 S_i	E_1 Rank	E_2 Rank	E_3 Rank
A1	0.995	0.963	0.985	2	1	1
A2	0.996	0.962	0.966	1	2	2
A3	0.991	0.864	0.867	3	4	4
A4	0.978	0.882	0.894	4	3	3

Table 7. The ranking orders obtained from the three examinees

According to Table 7, the most appropriate alternative based on the theory of dominance is the alternative denoted as A_1.

6. Conclusion

A new multiple criteria evaluation model based on using the single valued neutrosophic set is proposed in this paper. For the purpose of determining criteria weights, the SWARA method is applied due to its simplicity, whereas for the determination of the overall ratings for each respondent, the SVNN is applied. In order to intentionally avoid the group determination of weights and ratings, the final selection of the most appropriate alternative is determined by applying the theory of dominance. In order to form a simple questionnaire and obtain the respondents' real attitudes, a smaller number of the criteria were initially selected. The proposed model has proven to be far more flexible than the other MCDM-based models and is based on the conducted numerical example suitable for the solving of problems related to the selection of restaurants. The usability and efficiency of the proposed model have been demonstrated on the conducted numerical example.

References

[1] F. Smarandache. Neutrosophy: Neutrosophic probability, set, and logic, American Research Press, Rehoboth, USA, 1998.

[2] L. A. Zadeh. Fuzzy Sets. Information and Control, 8, (1965), 338-353.

[3] I. B. Turksen. Interval valued fuzzy sets based on normal forms. Fuzzy sets and systems, 20(2), (1986), 191-210.

[4] I. B. Turksen. Non-specificity and interval-valued fuzzy sets. Fuzzy sets and systems, 80(1), (1996), 87-100.

[5] K. Atanassov. Intuitionistic fuzzy sets. Fuzzy sets and Systems, 20(1), (1986), 87-96.

[6] K. Atanassov and G. Gargov. Interval valued intuitionistic fuzzy sets. Fuzzy sets and systems, 31(3), (1989), 343-349.

[7] M. Arora, R. Biswas, and U. S. Pandy. Neutrosophic relational database decompositio. International Journal of Advanced Computer Science & Applications, 1(2), (2011), 121-125.

[8] A. Salama, F. Smarandache, and M. Eisa. Introduction to Image Processing via Neutrosophic Techniques. Neutrosophic Sets and Systems, 5, (2014), 59-64.

[9] J. Mohan, A. T. S. Chandra, V. Krishnaveni, and Y. Guo. Evaluation of neutrosophic set approach filtering technique for image denoising. The International Journal of Multimedia & Its Applications, 4(4), (2012), 73-81.

[10] Y. Guo, and H. D. Cheng. New neutrosophic approach to image segmentation. Pattern Recognition, 42, (2009), 587–595.

[11] S. Broumi, and I. Deli. Correlation measure for neutrosophic refined sets and its application in medical diagnosis. Palestine Journal of Mathematics, 3(1), (2014), 11-19.

[12] Ansari, A. Q., Biswas, R., and Aggarwal, S. (2011). Proposal for applicability of neutrosophic set theory in medical AI. International Journal of Computer Applications, 27(5), 5-11.

[13] A. Kharal. A Neutrosophic Multicriteria Decision Making Method, New Mathematics and Natural Computation, Creighton University, USA, 2013.

[14] J. Ye. Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment, International Journal of General Systems, 42(4) (2013) 386--394.

[15] J. Chen, and J. Ye. A Projection Model of Neutrosophic Numbers for Multiple Attribute Decision Making of Clay-Brick Selection. NSS, 12, (2016), 139-142.

[16] J. Ye, and F. Smarandache Similarity Measure of Refined Single-Valued Neutrosophic Sets and Its Multicriteria Decision Making Method. NSS, 12, (2016), 41-44

[17] J. J. Peng, and J. Q. Wang. Multi-valued Neutrosophic Sets and its Application in Multi-criteria Decision-making Problems. Neutrosophic Sets and Systems, 10, (2015), 3-17.

[18] K. Mandal, and K. Basu. Hypercomplex Neutrosophic Similarity Measure & Its Application in Multicriteria Decision Making Problem. NSS, 9, (2015), 6-12.

[19] S. Pramanik, P. P. Dey, and B. C. Giri. TOPSIS for Single Valued Neutrosophic Soft Expert Set Based Multi-attribute Decision Making Problems. NSS, 10 (2015), 88-95

[20] Y. Jun. Another Form of Correlation Coefficient between Single Valued Neutrosophic Sets and Its Multiple Attribute Decision Making Method. NSS, 1, (2013), 8-12.

[21] F. Smarandache. A unifying field in logics. neutrosophy: Neutrosophic probability, set and logic. Rehoboth: American Research Press, 1999.

[22] H. Wang, F. Smarandache, Y. Q. Zhang and R. Sunderraman. Single valued neutrosophic sets, Multispace and Multistructure, 4, (2010), 410-413.

[23] R. Sahin. Multi-criteria neutrosophic decision making method based on score and accuracy functions under neutrosophic environment. arXiv preprint arXiv:1412.5202, 2014.

[24] J. Ye. A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. Journal of Intelligent and Fuzzy Systems, 26, (2014), 2459-2466.

[25] V. Kersuliene, E. K. Zavadskas and Z. Turskis. Selection of rational dispute resolution method by applying new step- wise weight assessment ratio analysis (SWARA). Journal of Business Economics and Management, 11(2), (2010), 243-258.

[26] W. K. M. Brauers and E. K. Zavadskas. MULTIMOORA optimization used to decide on a bank loan to buy property. Technological and Economic Development of Economy, 17(1), (2011), 174-188.

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A Neutrosophic Binomial Factorial Theorem with their Refrains

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Abstract. The Neutrosophic Precalculus and the Neutrosophic Calculus can be developed in many ways, depending on the types of indeterminacy one has and on the method used to deal with such indeterminacy. This article is innovative since the form of neutrosophic binomial factorial theorem was constructed in addition to its refrains. Two other important theorems were proven with their corollaries, and numerical examples as well. As a conjecture, we use ten (indeterminate) forms in neutrosophic calculus taking an important role in limits. To serve article's aim, some important questions had been answered.

Keyword: Neutrosophic Calculus, Binomial Factorial Theorem, Neutrosophic Limits, Indeterminate forms in Neutrosophic Logic, Indeterminate forms in Classical Logic.

1 Introduction (Important questions)

Q 1 *What are the types of indeterminacy?*

There exist two types of indeterminacy

a. Literal indeterminacy (I).

As example:

$$2+3I \tag{1}$$

b. Numerical indeterminacy.

As example:

$$x(0.6,0.3,0.4) \in A, \tag{2}$$

meaning that the indeterminacy membership = 0.3.

Other examples for the indeterminacy component can be seen in functions: $f(0)=7 \; or \; 9$ or $f(0 \;\; or \;\; 1)=5$ or $f(x)=[0.2,\;\; 0.3]\,x^2 \ldots$ etc.

Q 2 *What is the values of I to the rational power?*

1. Let

$$\sqrt{I}=x+y\,I$$
$$0+I=x^2+(2xy+y^2)I$$
$$x=0, y=\pm 1. \tag{3}$$

In general,

$$\sqrt[2k]{I}=\pm I \tag{4}$$

where $k \in z^+ = \{1,2,3,\ldots\}$.

2. Let

$$\sqrt[3]{I}=x+y\,I$$
$$0+I=x^3+3x^2y\,I+3xy^2\,I^2+y^3I^3$$
$$0+I=x^3+(3x^2y\;+3xy^2\;+y^3)I$$
$$x=0, y=1 \rightarrow \sqrt[3]{I}=I. \tag{5}$$

In general,

$$\sqrt[2k+1]{I}=I, \tag{6}$$

where $k \in z^+ = \{1,2,3,\ldots\}$.

Basic Notes

1. A component I to the zero power is undefined value, (i.e. I^0 is undefined), since $I^0=I^{1+(-1)}=I^1 * I^{-1}=\frac{I}{I}$ which is impossible case (avoid to divide by I).
2. The value of I to the negative power is undefined value (i.e. I^{-n} $,n>0$ is undefined).

Q 3 *What are the indeterminacy forms in neutrosophic calculus?*

In classical calculus, the indeterminate forms are [4]:

$$\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty^0, 0^0, 1^\infty, \infty-\infty. \tag{7}$$

The form 0 to the power I (i.e. 0^I) is an indeterminate form in Neutrosophic calculus; it is tempting to argue that an indeterminate form of type 0^I has zero value since "zero to any power is zero". However, this is fallacious, since 0^I is not a power of number, but rather a statement about limits.

Q 4 *What about the form* 1^I*?*

The base "one" pushes the form 1^I to one while the power I pushes the form 1^I to I, so 1^I is an indeterminate form in neutrosophic calculus. Indeed, the form a^I, $a \in R$ is always an indeterminate form.

Q 5 *What is the value of* a^I *, where* $a \in R$?

Let $y_1 = 2^x, x \in R, y_2 = 2^I$; it is obvious that $\lim_{x\to\infty} 2^x = \infty$, $\lim_{x\to-\infty} 2^x = 0$, $\lim_{x\to 0} 2^x = 1$; while we cannot determine if $2^I \to \infty$ *or* 0 *or* 1, therefore we can say that $y_2 = 2^I$ indeterminate form in Neutrosophic calculus. The same for a^I, where $a \in R$ [2].

2 Indeterminate forms in Neutrosophic Logic

It is obvious that there are seven types of indeterminate forms in classical calculus [4],

$$\frac{0}{0}, \frac{\infty}{\infty}, 0.\infty, 0^0, \infty^0, 1^\infty, \infty - \infty.$$

As a conjecture, we can say that there are ten forms of the indeterminate forms in Neutrosophic calculus

$$I^0, 0^I, \frac{I}{0}, I\cdot\infty, \frac{\infty}{I}, \infty^I, I^\infty, I^I,$$
$$a^I (a \in R), \infty \pm a\cdot I \ .$$

Note that:

$$\frac{I}{0} = I\cdot\frac{1}{0} = I\cdot\infty = \infty\cdot I.$$

3 Various Examples

Numerical examples on neutrosophic limits would be necessary to demonstrate the aims of this work.

Example (3.1) [1], [3]

The neutrosophic (numerical indeterminate) values can be seen in the following function:

Find $\lim_{x\to 0} f(x)$, where $f(x) = x^{[2.1,2.5]}$.

Solution:

Let $y = x^{[2.1,2.5]} \ \to \ln y = [2.1, 2.5] \ \ln x$

$$\therefore \lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{[2.1, 2.5]}{\frac{1}{\ln x}} = \frac{[2.1, 2.5]}{\frac{1}{\ln 0}}$$
$$= \frac{[2.1, 2.5]}{\frac{1}{-\infty}} = \frac{[2.1, 2.5]}{-0}$$
$$= \left[\frac{2.1}{-0}, \frac{2.5}{-0}\right] = (-\infty, -\infty)$$
$$= -\infty$$

Hence $y = e^{-\infty} = 0$

OR it can be solved briefly by

$y = x^{[2.1,2.5]} = [0^{2.1}, 0^{2.5}] = [0,0] = 0.$

Example (3.2)

$\lim_{x\to[9,11]} [3.5,5.9]x^{[1,2]} = [3.5,5.9]\ [9,11]^{[1,2]} =$ $[3.5,5.9]\ [9^1, 11^2] = [(3.5)(9), (5.9)(121)] =$ $[31.5,713.9]$.

Example (3.3)

$$\lim_{x\to\infty} [3.5,5.9]\ x^{[1,2]} = [3.5,5.9]\ \ \infty^{[1,2]}$$
$$= [3.5,5.9]\ [\infty^1, \infty^2]$$
$$= [3.5\cdot(\infty), 5.9\cdot(\infty)]$$
$$= (\infty, \infty) = \infty.$$

Example (3.4)

Find the following limit using more than one technique $\lim_{x\to 0} \frac{\sqrt{[4,5]\cdot x+1}-1}{x}$.

Solution:

The above limit will be solved firstly by using the L'Hôpital's rule and secondly by using the rationalizing technique.

<u>*Using L'Hôpital's rule*</u>

$$\lim_{x\to 0} \frac{1}{2} ([4,5]\cdot x + 1)^{-1/2}\ [4,5]$$
$$= \lim_{x\to 0} \frac{[4,5]}{2\sqrt{([4,5]\cdot x + 1)}}$$
$$= \frac{[4,5]}{2} = \left[\frac{4}{2}, \frac{5}{2}\right] = [2,2.5]$$

<u>*Rationalizing technique [3]*</u>

$$\lim_{x\to 0} \frac{\sqrt{[4,5]\cdot x + 1} - 1}{x} = \frac{\sqrt{[4,5]\cdot 0 + 1} - 1}{0}$$
$$= \frac{\sqrt{[4\cdot 0, 5\cdot 0] + 1} - 1}{0} = \frac{\sqrt{[0,0] + 1} - 1}{0}$$
$$= \frac{\sqrt{0+1} - 1}{0} = \frac{0}{0}$$
$$= \text{undefined.}$$

Multiply with the conjugate of the numerator:

$$\lim_{x\to 0}\frac{\sqrt{[4,5]x+1}-1}{x}\cdot\frac{\sqrt{[4,5]x+1}+1}{\sqrt{[4,5]x+1}+1}$$

$$=\lim_{x\to 0}\frac{\left(\sqrt{[4,5]x+1}\right)^2-(1)^2}{x\left(\sqrt{[4,5]x+1}+1\right)}$$

$$=\lim_{x\to 0}\frac{[4,5]\cdot x\cancel{+1-1}}{x\cdot\left(\sqrt{[4,5]x+1}+1\right)}$$

$$=\lim_{x\to 0}\frac{[4,5]\cancel{\cdot x}}{\cancel{x\cdot}\left(\sqrt{[4,5]x+1}+1\right)}$$

$$=\lim_{x\to 0}\frac{[4,5]}{\left(\sqrt{[4,5]x+1}+1\right)}$$

$$=\frac{[4,5]}{\left(\sqrt{[4,5]\cdot 0+1}+1\right)}=\frac{[4,5]}{\sqrt{1}+1}$$

$$=\frac{[4,5]}{2}=\left[\frac{4}{2},\frac{5}{2}\right]=[2,2.5].$$

Identical results.

Example (3.5)

Find the value of the following neutrosophic limit $\lim_{x\to -3}\frac{x^2+3x-[1,2]x-[3,6]}{x+3}$ using more than one technique .

Analytical technique [1], [3]

$\lim_{x\to -3}\frac{x^2+3x-[1,2]x-[3,6]}{x+3}$

By substituting x= -3 ,

$$\lim_{x\to -3}\frac{(-3)^2+3\cdot(-3)-[1,2]\cdot(-3)-[3,6]}{-3+3}$$

$$=\frac{9-9-[1\cdot(-3),2\cdot(-3)]-[3,6]}{0}$$

$$=\frac{0-[-6,-3]-[3,6]}{0}=\frac{[3,6]-[3,6]}{0}$$

$$=\frac{[3-6,6-3]}{0}=\frac{[-3,3]}{0},$$

which has undefined operation $\frac{0}{0}$, since $0\in[-3,3]$. Then we factor out the numerator, and simplify:

$$\lim_{x\to -3}\frac{x^2+3x-[1,2]x-[3,6]}{x+3}=$$

$$\lim_{x\to -3}\frac{(x-[1,2])\cdot(x+3)}{(x+3)}=\lim_{x\to -3}(x-[1,2])$$

$$=-3-[1,2]=[-3,-3]-[1,2]$$

$$=-([3,3]+[1,2])=[-5,-4].$$

Again, Solving by using L'Hôpital's rule

$$\lim_{x\to -3}\frac{x^2+3x-[1,2]x-[3,6]}{x+3}$$

$$=\lim_{x\to -3}\frac{2x+3-[1,2]}{1}$$

$$=\lim_{x\to -3}\frac{2(-3)+3-[1,2]}{1}$$

$$=-6+3-[1,2]$$

$$=-3-[1,2]$$

$$=[-3-1,-3-2]$$

$$=[-5,-4]$$

The above two methods are identical in results.

4 New Theorems in Neutrosophic Limits

Theorem (4.1) (Binomial Factorial)

$\lim_{x\to\infty}(I+\frac{1}{x})^x=Ie$; I is the literal indeterminacy, e = 2.7182828

Proof

$$\left(I+\frac{1}{x}\right)^x=\binom{x}{0}I^x\left(\frac{1}{x}\right)^0+\binom{x}{1}I^{x-1}\left(\frac{1}{x}\right)^1$$

$$+\binom{x}{2}I^{x-2}\left(\frac{1}{x}\right)^2+\binom{x}{3}I^{x-3}\left(\frac{1}{x}\right)^3$$

$$+\binom{x}{4}I^{x-4}\left(\frac{1}{x}\right)^4+\cdots$$

$$=I+x.I.\frac{1}{x}+\frac{I}{2!}\left(1-\frac{1}{x}\right)$$

$$+\frac{I}{3!}\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)+\frac{I}{4!}\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)$$

$$\left(1-\frac{3}{x}\right)+\cdots$$

It is clear that $\frac{1}{x}\to 0\ as\ x\ \to\infty$

$\therefore\lim_{x\to\infty}(I-\frac{1}{x})^x=I+I+\frac{I}{2!}+\frac{I}{3!}+\frac{I}{4!}+\cdots=I+\sum_{n=1}^{\infty}\frac{I^n}{n!}$

$\therefore\lim_{x\to\infty}(I+\frac{1}{x})^x=Ie$, where $e=1+\sum_{n=1}^{\infty}\frac{1}{n!}$, I is the literal indeterminacy.

Corollary (4.1.1)

$\lim_{x\to 0}(I+x)^{\frac{1}{x}}=Ie$

Proof:-

Put $y=\frac{1}{x}$

It is obvious that $y\to\infty$, as $x\to 0$

$\therefore\lim_{x\to 0}(I+x)^{\frac{1}{x}}=\lim_{y\to\infty}(I+\frac{1}{y})^y=Ie$

(using Th. 4.1)

Corollary (4.1.2)

$\lim_{x\to\infty}(I+\frac{k}{x})^x=Ie^k$, where k >0 & $k\neq 0$, I is the literal indeterminacy.

Proof

$$\lim_{x\to\infty}(I+\frac{k}{x})^x = \lim_{x\to\infty}\left[(I+\frac{k}{x})^{\frac{x}{k}}\right]^k$$

Put $y=\frac{k}{x}\to xy=k\to x=\frac{k}{y}$

Note that $y\to 0\ as\ x\to\infty$

$$\therefore \lim_{x\to\infty}\left(I+\frac{k}{x}\right)^x = \lim_{y\to 0}\left[(I+y)^{\frac{1}{y}}\right]^k$$

(using corollary 4.1.1).

$$=\left[\lim_{y\to 0}(I+y)^{\frac{1}{y}}\right]^k = (Ie)^k = I^k e^k = Ie^k$$

Corollary (4.1.3)

$\lim_{x\to 0}(I+\frac{x}{k})^{\frac{1}{x}} = (Ie)^{\frac{1}{k}} = \sqrt[k]{Ie}$,

where $k\neq 1$ & $k>0$.

Proof

The immediate substitution of the value of x in the above limit gives indeterminate form I^∞,

i.e. $\lim_{x\to 0}(I+\frac{x}{k})^{\frac{1}{x}} = \lim_{x\to 0}(I+\frac{0}{k})^{\frac{1}{0}} = I^\infty$

So we need to treat this value as follow:-

$$\lim_{x\to 0}(I+\frac{x}{k})^{\frac{1}{x}} = \lim_{x\to 0}\left[(I+\frac{x}{k})^{\frac{k}{x}}\right]^{\frac{1}{k}} = \left[\lim_{x\to 0}(I+\frac{x}{k})^{\frac{k}{x}}\right]^{\frac{1}{k}}$$

put $y=\frac{x}{k}\to x=ky\to\frac{1}{x}=\frac{1}{ky}$

As $x\to 0$, $y\to 0$

$$\lim_{x\to 0}\left(I+\frac{x}{k}\right)^{\frac{1}{x}} = \lim_{y\to 0}\left[(I+y)^{\frac{1}{y}}\right]^{\frac{1}{k}}$$

$$=\left[\lim_{y\to 0}(I+y)^{\frac{1}{y}}\right]^{\frac{1}{k}}$$

Using corollary (4.1.1)

$$=(Ie)^{\frac{I}{k}} = \sqrt[k]{Ie}$$

Theorem (4.2)

$$\lim_{x\to 0}\frac{(lna)[Ia^x-I]}{xlna+lnI} = \frac{lna}{1+lnI}$$

Where $a>0, a\neq 1$

Note that $\lim_{x\to 0}\frac{(lna)[Ia^x-I]}{xlna+lnI} = \lim_{x\to 0}\frac{Ia^x-I}{x+\frac{lnI}{lna}}$

Proof

Let $y=Ia^x-I\ \to y+I=Ia^x\ \to\ \ln(y+I)=\ln I+\ln a^x$

$\to \ln(y+I)=\ln I+xlna\to$

$$x=\frac{\ln(y+I)-lnI}{lna}$$

$$\frac{(\ln a)(Ia^x-I)}{xlna+lnI} = \frac{(Ia^x-I)}{x+\frac{lnI}{lna}}$$

$$=\frac{y}{\frac{\ln(y+I)-lnI}{lna}+\frac{lnI}{lna}}$$

$$=lna.\frac{y}{\ln(y+I)} = lna.\frac{1}{\frac{1}{y}\ln(y+I)}$$

$$=lna.\frac{1}{\ln(y+I)^{\frac{1}{y}}}$$

$$\therefore \lim_{x\to 0}\frac{(\ln a)(Ia^x-I)}{xlna+lnI} = lna\ \frac{1}{\lim_{y\to 0}ln(y+I)^{\frac{1}{y}}}$$

$$=lna\,.\frac{1}{ln\lim_{y\to 0}(y+I)^{\frac{1}{y}}}$$

$=lna\ \frac{1}{ln(Ie)}$ using corollary (4.1.1)

$$=\frac{lna}{lnI+\ lne} = \frac{lna}{lnI+1}$$

Corollary (4.2.1)

$$\lim_{x\to 0}\frac{Ia^{kx}-I}{x+\frac{lnI}{lna^k}} = \frac{k\ lna}{1+lnI}$$

Proof

Put $y=kx\to x=\frac{y}{k}$

$y\to 0\ \ as\ x\to 0$

$$\lim_{x\to 0}\frac{Ia^{kx}-I}{x+\frac{lnI}{lna^k}} = \lim_{y\to 0}\frac{Ia^y-I}{\frac{y}{k}+\frac{lnI}{k\,lna}} = k.\lim_{y\to 0}\frac{Ia^y-I}{y+\frac{lnI}{lna}}$$

using Th. (4.2)

$$=k.(\frac{lna}{1+lnI})$$

Corollary (4.2.2)

$$\lim_{x\to 0}\frac{Ie^x-I}{x+lnI} = \frac{1}{1+lnI}$$

Proof

Let $y=\ Ie^x-I$, $y\to 0\ \ as\ x\to 0$

$y+I=\ Ie^x\to\ln(y+I)=lnI+x\ lne$

$x=\ln(y+I)-lnI$

$$\therefore\ \frac{Ie^x-I}{x+\ lnI} = \frac{y}{\ln(y+I)-lnI+lnI}$$

$$=\frac{1}{\frac{1}{y}\ln(y+I)}$$

$$=\frac{1}{\ln(y+I)^{\frac{1}{y}}}$$

$$\therefore \lim_{x\to 0}\frac{Ie^x-I}{x+lnI} = \lim_{y\to 0}\frac{1}{\ln(y+I)^{\frac{1}{y}}}$$

$$=\frac{1}{\ln\ \lim_{y\to 0}(y+I)^{\frac{1}{y}}}$$

using corollary (4.1.1)

$$\frac{1}{\ln(Ie)} = \frac{1}{lnI+lne} = \frac{1}{lnI+1}$$

Corollary (4.2.3)

$$\lim_{x\to 0}\frac{Ie^{kx}-I}{x+\frac{lnI}{k}}=\frac{k}{1+lnI}$$

Proof

let $y=kx\rightarrow x=\frac{y}{k}$

$y\rightarrow 0\ \ as\ x\rightarrow 0$

$\lim_{x\to 0}\frac{Ie^{kx}-I}{x+\frac{lnI}{k}}=\lim_{y\to 0}\frac{Ie^{y}-I}{\frac{y}{k}+\frac{lnI}{k}}=k.\lim_{y\to 0}\frac{Ie^{y}-I}{y+lnI}$ using Corollary (4.2.2) to get

$$=k.\left(\frac{1}{1+lnI}\right)=\frac{k}{1+lnI}$$

Theorem (4.3)

$$\lim_{x\to 0}\frac{\ln(I+kx)}{x}=k(1+lnI)$$

Proof

$$\lim_{x\to 0}\frac{\ln(I+kx)}{x}=\lim_{x\to 0}\frac{\ln(I+kx)-lnI+lnI}{x}$$

Let $y=\ln(I+kx)-lnI\rightarrow y+lnI=\ln(I+kx)$

$$e^{y+lnI}=I+kx\rightarrow x=\frac{e^{y}e^{lnI}-I}{k}=\frac{I\,e^{y}-I}{k}$$

$y\rightarrow 0\ \ as\ x\rightarrow 0$

$$\lim_{x\to 0}\frac{\ln(I+kx)-lnI+lnI}{x}$$

$$=\lim_{y\to 0}\frac{y+lnI}{\frac{I\,e^{y}-I}{k}}$$

$$\lim_{y\to 0}\frac{k}{\frac{I\,e^{y}-I}{y+lnI}}=\frac{k}{\lim_{y\to 0}(\frac{I\,e^{y}-I}{y+lnI})}$$

using corollary (4.2.2) to get the result

$$=\frac{k}{\frac{1}{1+lnI}}=k(1+lnI)$$

Theorem (4.4)

Prove that, for any two real numbers a,b

$\lim_{x\to 0}\frac{I\mathrm{a}^{x}-I}{I\mathrm{b}^{x}-I}=1$, where $a,b>0\ \&\ a,b\neq 1$

Proof

The direct substitution of the value x in the above limit conclude that $\frac{0}{0}$,so we need to treat it as follow:

$$\lim_{x\to 0}\frac{I\mathrm{a}^{x}-I}{I\mathrm{b}^{x}-I}=\lim_{x\to 0}\frac{\frac{ln\mathrm{a}[I\mathrm{a}^{x}-I]}{xln\mathrm{a}+lnI}*\frac{xln\mathrm{a}+lnI}{ln\mathrm{a}}}{\frac{ln\mathrm{b}[I\mathrm{b}^{x}-I]}{xln\mathrm{b}+lnI}*\frac{xln\mathrm{b}+lnI}{ln\mathrm{b}}}$$

$$=\frac{\lim_{x\to x}\frac{ln\mathrm{a}[I\mathrm{a}^{x}-I]}{xln\mathrm{a}+lnI}}{\lim_{x\to x}\frac{ln\mathrm{b}[I\mathrm{b}^{x}-I]}{xln\mathrm{b}+lnI}}*\frac{\lim_{x\to 0}(xln\mathrm{a}+lnI)}{\lim_{x\to 0}(xln\mathrm{b}+lnI)}*\frac{ln\mathrm{b}}{ln\mathrm{a}}$$

(using Th.(4.2) twice (first in numerator second in denominator))

$$=\frac{\frac{ln\mathrm{a}}{1+lnI}}{\frac{ln\mathrm{b}}{1+lnI}}*\frac{lnI}{lnI}*\frac{ln\mathrm{b}}{ln\mathrm{a}}=1.$$

5 Numerical Examples

Example (5.1)

Evaluate the limit $\lim_{x\to 0}\frac{I5^{4x}-I}{x+\frac{lnI}{ln5^{4}}}$

Solution

$\lim_{x\to 0}\frac{I5^{4x}-I}{x+\frac{lnI}{ln5^{4}}}=\frac{4ln5}{1+lnI}$ (using corollary 4. 2.1)

Example (5.2)

Evaluate the limit $\lim_{x\to 0}\frac{Ie^{4x}-I}{I3^{2x}-I}$

Solution

$$\lim_{x\to 0}\frac{Ie^{4x}-I}{I3^{2x}-I}=\lim_{x\to 0}\frac{\frac{ln3[Ie^{4x}-I]}{(x+\frac{lnI}{4})}*(x+\frac{lnI}{4})}{\frac{ln3[I3^{2x}-I]}{(x+\frac{lnI}{ln3^{2}})}*(x+\frac{lnI}{ln3^{2}})}$$

$$=\frac{\lim_{x\to 0}\frac{ln3[Ie^{4x}-I]}{\left(x+\frac{lnI}{4}\right)}}{\lim_{x\to 0}\frac{ln3[I3^{2x}-I]}{\left(x+\frac{lnI}{ln3^{2}}\right)}}*\frac{\lim_{x\to 0}\left(x+\frac{lnI}{4}\right)}{\lim_{x\to 0}\left(x+\frac{lnI}{ln3^{2}}\right)}$$

(using corollary (4.2.3) on numerator & corollary (4.2.1) on denominator)

$$=\frac{\frac{4}{1+lnI}}{\frac{2ln3}{1+lnI}}*\frac{\frac{lnI}{4}}{\frac{lnI}{ln3^{2}}}=1.$$

5 Conclusion

In this article, we introduced for the first time a new version of binomial factorial theorem containing the literal indeterminacy *(I).* This theorem enhances three corollaries. As a conjecture for indeterminate forms in classical calculus, ten of new indeterminate forms in Neutrosophic calculus had been constructed. Finally, various examples had been solved.

References

[1] F. Smarandache. Neutrosophic Precalculus and Neutrosophic Calculus. EuropaNova Brussels, 2015.

[2] F. Smarandache. Introduction to Neutrosophic Statistics. Sitech and Education Publisher, Craiova, 2014.

[3] H. E. Khalid & A. K. Essa. Neutrosophic Pre-calculus and Neutrosophic Calculus. Arabic version of the book. Pons asbl 5, Quai du Batelage, Brussells, Belgium, European Union 2016.

[4] H. Anton, I. Bivens & S. Davis, Calculus, 7th Edition, John Wiley & Sons, Inc. 2002.

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The category of neutrosophic sets

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Abstract: We introduce the category $\mathbf{NSet(H)}$ consisting of neutrosophic H-sets and morphisms between them. And we study $\mathbf{NSet(H)}$ in the sense of a topological universe and prove that it is Cartesian closed over $\mathbf{Set}$, where $\mathbf{Set}$ denotes the category consisting of ordinary sets and ordinary mappings between them. Furthermore, we investigate some relationships between two categories $\mathbf{ISet(H)}$ and $\mathbf{NSet(H)}$.

Keywords: Neutrosophic crisp set, Cartesian closed category, Topological universe.

1 Introduction

In 1965, Zadeh [20] had introduced a concept of a fuzzy set as the generalization of a crisp set. In 1986, Atanassov [1] proposed the notion of intuitionistic fuzzy set as the generalization of fuzzy sets considering the degree of membership and non-membership. Moreover, in 1998, Smarandache [19] introduced the concept of a neutrosophic set considering the degree of membership, the degree of indeterminacy and the degree of non-membership.

After that time, many researchers [3, 4, 5, 6, 8, 9, 13, 15, 16, 17] have investigated fuzzy sets in the sense of category theory, for instance, $\mathbf{Set(H)}$, $\mathbf{Set_f(H)}$, $\mathbf{Set_g(H)}$, $\mathbf{Fuz(H)}$. Among them, the category $\mathbf{Set(H)}$ is the most useful one as the "standard" category, because $\mathbf{Set(H)}$ is very suitable for describing fuzzy sets and mappings between them. In particular, Carrega [3], Dubuc [4], Eytan [5], Goguen [6], Pittes [15], Ponasse [16, 17] had studied $\mathbf{Set(H)}$ in topos view-point. However Hur et al. investigated $\mathbf{Set(H)}$ in topological view-point. Moreover, Hur et al. [9] introduced the category $\mathbf{ISet(H)}$ consisting of intuitionistic H-fuzzy sets and morphisms between them, and studied $\mathbf{ISet(H)}$ in the sense of topological universe. In particular, Lim et al. [13] introduced the new category $\mathbf{VSet(H)}$ and investigated it in the sense of topological universe. Recently, Lee et al. [10] define the category composed of neutrosophic crisp sets and morphisms between neutrosophic crisp sets and study its some properties.

The concept of a topological universe was introduced by Nel [14], which implies a Cartesian closed category and a concrete quasitopos. Furthermore the concept has already been up to effective use for several areas of mathematics.

In this paper, we introduce the category $\mathbf{NSet(H)}$ consisting of neutrosophic H-sets and morphisms between them. And we study $\mathbf{NSet(H)}$ in the sense of a topological universe and prove that it is Cartesian closed over $\mathbf{Set}$, where $\mathbf{Set}$ denotes the category consisting of ordinary sets and ordinary mappings between them. Furthermore, we investigate some relationships between two categories $\mathbf{ISet(H)}$ and $\mathbf{NSet(H)}$.

2 Preliminaries

In this section, we list some basic definitions and well-known results from [7, 12, 14] which are needed in the next sections.

Definition 2.1 [12] Let $\mathbf{A}$ be a concrete category and $((Y_j, \xi_j))_J$ a family of objects in A indexed by a class J. For any set X, let $(f_j : X \to Y_j)_J$ be a source of mappings indexed by J. Then an $\mathbf{A}$-structure ξ on X is said to be initial with respect to (in short, w.r.t.) $(X, (f_j), ((Y_j, \xi_j)))_J$, if it satisfies the following conditions:

(i) for each $j \in J$, $f_j : (X, \xi) \to (Y_j, \xi_j)$ is an $\mathbf{A}$-morphism,

(ii) if (Z, ρ) is an $\mathbf{A}$-object and $g : Z \to X$ is a mapping such that for each $j \in J$, the mapping $f_j \circ g : (Z, \rho) \to (Y_j, \xi_j)$ is an $\mathbf{A}$-morphism, then $g : (Z, \rho) \to (X, \xi)$ is an $\mathbf{A}$-morphism.

In this case, $(f_j : (X, \xi) \to (Y_j, \xi_j))_J$ is called an initial source in $\mathbf{A}$.

Dual notion: cotopological category.

Result 2.2 ([12], Theorem 1.5) A concrete category $\mathbf{A}$ is topological if and only if it is cotopological.

Result 2.3 ([12], Theorem 1.6) Let $\mathbf{A}$ be a topological category over $\mathbf{Set}$, then it is complete and cocomplete.

Definition 2.4 [12] Let $\mathbf{A}$ be a concrete category.

(i) The $\mathbf{A}$-fibre of a set X is the class of all $\mathbf{A}$-structures on X.

(ii) $\mathbf{A}$ is said to be properly fibred over $\mathbf{Set}$ if it satisfies the followings:

  (a) (Fibre-smallness) for each set X, the $\mathbf{A}$-fibre of X is a set,

  (b) (Terminal separator property) for each singleton set X, the $\mathbf{A}$-fibre of X has precisely one element,

  (c) if ξ and η are $\mathbf{A}$-structures on a set X such that $id : (X, \xi) \to (X, \eta)$ and $id : (X, \eta) \to (X, \xi)$ are $\mathbf{A}$-morphisms, then $\xi = \eta$.

Definition 2.5 [7] A category $\mathbf{A}$ is said to be Cartesian closed if it satisfies the following conditions:

(i) for each $\mathbf{A}$-object A and B, there exists a product $A \times B$ in $\mathbf{A}$,

(ii) exponential objects exist in A, i.e., for each $\mathbf{A}$-object A, the functor $A \times - : \mathbf{A} \to \mathbf{A}$ has a right adjoint, i.e., for any $\mathbf{A}$-object B, there exist an $\mathbf{A}$-object B^A and an $\mathbf{A}$-morphism $e_{A,B} : A \times B^A \to B$ (called the evaluation) such that for any $\mathbf{A}$-object C and any $\mathbf{A}$-morphism $f : A \times C \to B$, there exists a unique $\mathbf{A}$-morphism $\bar{f} : C \to B^A$ such that $e_{A,B} \circ (id_A \times \bar{f}) = f$, i.e., the diagram commutes:

$$\begin{tikzcd}
A \times B^A \arrow[rr, "e_{A,B}"] & & B \\
& A \times C \arrow[ul, "\exists 1_A \times \overline{f}"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Definition 2.6 [7] A category $\mathbf{A}$ is called a topological universe over $\mathbf{Set}$ if it satisfies the following conditions:

(i) $\mathbf{A}$ is well-structured, i.e., (a) $\mathbf{A}$ is a concrete category; (b) $\mathbf{A}$ satisfies the fibre-smallness condition; (c) $\mathbf{A}$ has the terminal separator property,

(ii) $\mathbf{A}$ is cotopological over $\mathbf{Set}$,

(iii) final episinks in $\mathbf{A}$ are preserved by pullbacks, i.e., for any episink $(g_j : X_j \to Y)_J$ and any $\mathbf{A}$-morphism $f : W \to Y$, the family $(e_j : U_j \to W)_J$, obtained by taking the pullback f and g_j, for each $j \in J$, is again a final episink.

Definition 2.7 [2, 11] A lattice H is called a complete Heyting algebra if it satisfies the following conditions:

(i) it is a complete lattice,

(ii) for any $a, b \in H$, the set $\{x \in H : x \wedge a \leq b\}$ has the greatest element denoted by $a \to b$ (called the relative pseudo-complement of a and b), i.e., $x \wedge a \leq b$ if and only if $x \leq (a \to b)$.

In particular, if H is a complete Heyting algebra with the least element 0 then for each $a \in H$, $N(a) = a \to 0$ is called negation or the paudo-complement of a.

Result 2.8 ([2], Ex. 6 in p. 46) Let H be a complete Heyting algebra and $a, b \in H$.

(1) If $a \leq b$, then $N(b) \leq N(a)$, where $N : H \to H$ is an involutive order reversing operation in $(H, \leq)$.

(2) $a \leq NN(a)$.

(3) $N(a) = NNN(a)$.

(4) $N(a \vee b) = N(a) \wedge N(b)$ and $N(a \wedge b) = N(a) \vee N(b)$.

Throughout this paper, we will use H as a complete Heyting algebra with the least element 0 and the greatest element 1.

Definition 2.9 [9] Let X be a set. Then A is called an intuitionistic H-fuzzy set (in short, IHFS) in X if it satisfies the following conditions:

(i) A is of the form $A = (\mu, \nu)$, where $\mu, \nu : X \to H$ are mappings,

(ii) $\mu \leq N(\nu)$, i.e., $\mu(x) \leq N(\nu)(x)$ for each $x \in X$.

In this case, the pair (X, A) is called an intuitionistic H-fuzzy space (in short, IHFSp). We will denote the set of all IHFSs as $IHFS(X)$.

Definition 2.10 [9] The concrete category $\mathbf{ISet(H)}$ is defined as follows:

(i) each object is an IHFSp (X, A_X), where $A_X = (\mu_{A_X}, \nu_{A_X}) \in IHFS(X)$,

(ii) each morphism is a mapping $f : (X, A_X) \to (Y, A_Y)$ such that $\mu_{A_X} \leq \mu_{A_Y} \circ f$ and $\nu_{A_X} \geq \nu_{A_Y} \circ f$, i.e., $\mu_{A_X}(x) \leq \mu_{A_Y} \circ f(x)$ and $\nu_{A_X}(x) \geq \nu_{A_Y} \circ f(x)$, for each $x \in X$. In this case, the morphism $f : (X, A_X) \to (Y, A_Y)$ is called an $\mathbf{ISet(H)}$-mapping.

3 Neutrosophic sets

In [18], Salama and Smarandache introduced the concept of a neutrosophic crisp set in a set X and defined the inclusion between two neutrosophic crisp sets, the intersection [union] of two neutrosophic crisp sets, the complement of a neutrosophic crisp set, neutrosophic empty [resp., whole] set as more than two types. And they studied some properties related to neutrosophic set operations. However, by selecting only one type, we define the inclusion, the intersection [union] and the neutrosophic empty [resp., whole] set again and obtain some properties.

Definition 3.1 Let X be a non-empty set. Then A is called a neutrosophic set (in short, NS) in X, if A has the form $A = (T_A, I_A, F_A)$, where
$T_A : X \to]^-0, 1^+[$, $I_A : X \to]^-0, 1^+[$, $F_A : X \to]^-0, 1^+[$.
Since there is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, for each $x \in X$,

$$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+.$$

Moreover, for each $x \in X$, $T_A(x)$ [resp., $I_A(x)$ and $F_A(x)$] represent the degree of membership [resp., indeterminacy and non-membership] of x to A.

The neutrosophic empty [resp., whole] set, denoted by 0_N [resp., 1_N] is an NS in X defined by $0_N = (0, 0, 1)$ [resp., $1_N = (1, 1, 0)$], where $0, 1 : X \to]^-0, 1^+[$ are defined by $0(x) = 0$ and $1(x) = 1$ respectively. We will denote the set of all NSs in X as $NS(X)$.

From Example 2.1.1 in [18], we can see that every IFS (intutionistic fuzzy set) A in a non-empty set X is an NS in X having the form

$$A = (T_A, 1 - (T_A + F_A), F_A),$$

where $(1 - (T_A + F_A))(x) = 1 - (T_A(x) + F_A(x))$.

Definition 3.2 Let $A = (T_A, I_A, F_A)$, $B = (T_B, I_B, F_B) \in NS(X)$. Then

(i) A is said to be contained in B, denoted by $A \subset B$, if
$T_A(x) \leq T_B(x)$, $I_A(x) \leq I_B(x)$ and $F_A(x) \geq F_B(x)$ for each $x \in X$,

(ii) A is said to equal to B, denoted by $A = B$, if

$$A \subset B \text{ and } B \subset A,$$

(iii) the complement of A, denoted by A^c, is an NCS in X defined as:

$$A^c = (F_A, 1 - I_A, T_A),$$

(iv) the intersection of A and B, denoted by $A \cap B$, is an NCS in X defined as:

$$A \cap B = (T_A \wedge T_B, I_A \wedge I_B, F_A \vee F_B),$$

where $(T_A \wedge T_B)(x) = T_A(x) \wedge T_B(x)$, $(F_A \vee F_B) = F_A(x) \vee F_B(x)$ for each $x \in X$,

(v) the union of A and B, denoted by $A \cup B$, is an NCS in X defined as:

$$A \cup B = (T_A \vee T_B, I_A \vee I_B, F_A \wedge F_B).$$

Let $(A_j)_{j \in J} \subset NS(X)$, where $A_j = (T_{A_j}, I_{A_j}, F_{A_j})$. Then

(vi) the intersection of $(A_j)_{j \in J}$, denoted by $\bigcap_{j \in J} A_j$ (simply, $\bigcap A_j$), is an NS in X defined as:

$$\bigcap A_j = (\bigwedge T_{A_j}, \bigwedge I_{A_j}, \bigvee F_{A_j}),$$

(vii) the union of $(A_j)_{j \in J}$, denoted by $\bigcup_{j \in J} A_j$ (simply, $\bigcup A_j$), is an NCS in X defined as:

$$\bigcup A_j = (\bigvee T_{A_j}, \bigvee I_{A_j}, \bigwedge F_{A_j}).$$

The followings are the immediate results of Definition 3.2.

Proposition 3.3 *Let $A, B, C \in NS(X)$. Then*
(1) $0_N \subset A \subset 1_N$,
(2) *if* $A \subset B$ *and* $B \subset C$, *then* $A \subset C$,
(3) $A \cap B \subset A$ *and* $A \cap B \subset B$,
(4) $A \subset A \cup B$ *and* $B \subset A \cup B$,
(5) $A \subset B$ *if and only if* $A \cap B = A$,
(6) $A \subset B$ *if and only if* $A \cup B = B$.

Also the followings are the immediate results of Definition 3.2.

Proposition 3.4 *Let $A, B, C \in NS(X)$. Then*
(1) (Idempotent laws): $A \cup A = A$, $A \cap A = A$,
(2) (Commutative laws): $A \cup B = B \cup A$, $A \cap B = B \cap A$,
(3) (Associative laws): $A \cup (B \cup C) = (A \cup B) \cup C$,
$A \cap (B \cap C) = (A \cap B) \cap C$,
(4) (Distributive laws): $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
(5) (Absorption laws): $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$,
(6) (De Morgan's laws): $(A \cup B)^c = A^c \cap B^c$,
$(A \cap B)^c = A^c \cup B^c$,
(7) $(A^c)^c = A$,
(8) (8a) $A \cup 0_N = A$, $A \cap 0_N = 0_N$,
(8b) $A \cup 1_N = 1_N$, $A \cap 1_N = A$,
(8c) $1_N^c = 0_N$, $0_N^c = 1_N$,
(8d) *in general,* $A \cup A^c \neq 1_N$, $A \cap A^c \neq 0_N$.

Proposition 3.5 *Let $A \in NS(X)$ and let $(A_j)_{j \in J} \subset NS(X)$. Then*
(1) $(\bigcap A_j)^c = \bigcup A_j^c$, $(\bigcup A_j)^c = \bigcap A_j^c$,
(2) $A \cap (\bigcup A_j) = \bigcup (A \cap A_j)$, $A \cup (\bigcap A_j) = \bigcap (A \cup A_j)$.

Proof. (1) Let $A_j = (T_{A_j}, I_{A_j}, F_{A_j})$.
Then $\bigcap A_j = (\bigwedge T_{A_j}, \bigwedge I_{A_j}, \bigvee F_{A_j})$.

Thus

$$\begin{aligned}(\bigcap A_j)^c &= (\bigvee F_{A_j}, 1-\bigwedge I_{A_j}, \bigwedge T_{A_j})\\ &= (\bigvee F_{A_j}, \bigvee(1-I_{A_j}), \bigwedge T_{A_j})\\ &= \bigcup A_j^c\end{aligned}$$

Similarly, the second part is proved.
(2) Let $A=(T_A, I_A, F_A)$ and $A_j=(T_{A_j}, I_{A_j}, F_{A_j})$.
Then

$$\begin{aligned}A\cup(\bigcap A_j) &= (T_A\vee(\bigwedge T_{A_j}, I_A\vee(\bigwedge I_{A_j}), F_A\wedge(\bigvee F_{A_j}))\\ &= (\bigwedge(T_A\vee T_{A_j}), \bigwedge(I_A\vee I_{A_j}), \bigvee(F_A\wedge F_{A_j})\\ &= \bigcap(A\cup A_j).\end{aligned}$$

Similarly, the first part is proved. □

Definition 3.6 Let $f: X\to Y$ be a mapping and let $A\subset X$, $B\subset Y$. Then

(i) the image of A under f, denoted by $f(A)$, is an NS in Y defined as:

$$f(A)=(f(T_A), f(I_A), f(F_A)),$$

where for each $y\in Y$,

$$[f(T_A)](y)=\begin{cases}\bigvee_{x\in f^{-1}(y)} T_A(x) & \text{if } f^{-1}(y)\neq\phi\\ 0 & \text{if } f^{-1}(y)=\phi,\end{cases}$$

(ii) the preimage of B, denoted by $f^{-1}(B)$, is an NCS in X defined as:

$$f^{-1}(B)=(f^{-1}(T_B), f^{-1}(I_B), f^{-1}(F_B)),$$

where $f^{-1}(T_B)(x)=T_B(f(x))$ for each $x\in X$,
in fact, $f^{-1}(B)=(T_B\circ f, I_B\circ f, F_B\circ f)$.

Proposition 3.7 *Let $f: X\to Y$ be a mapping and let $A, B, C\in NCS(X)$, $(A_j)_{j\in J}\subset NCS(X)$ and $D, E, F\in NCS(Y)$, $(D_k)_{k\in K}\subset NCS(Y)$. Then the followings hold:*
(1) *if $B\subset C$, then $f(B)\subset f(C)$ and
if $E\subset F$, then $f^{-1}(E)\subset f^{-1}(F)$.*
(2) *$A\subset f^{-1}f(A))$ and
if f is injective, then $A=f^{-1}f(A))$,*
(3) *$f(f^{-1}(D))\subset D$ and
if f is surjective, then $f(f^{-1}(D))=D$,*
(4) *$f^{-1}(\bigcup D_k)=\bigcup f^{-1}(D_k)$, $f^{-1}(\bigcap D_k)=\bigcap f^{-1}(D_k)$,*
(5) *$f(\bigcup D_k)=\bigcup f(D_k)$, $f(\bigcap D_k)\subset\bigcap f(D_k)$,*
(6) *$f(A)=0_N$ if and only if $A=0_N$ and hence $f(0_N)=0_N$,
in particular if f is surjective, then $f(1_{X,N})=1_{Y,N}$,*
(7) *$f^{-1}(1_{Y,N})=1_{X,N}$, $f^{-1}(0_{Y,N})=0_{X,N}$.*

4 Properties of $\mathbf{NSet(H)}$

Definition 4.1 A is called a neutrosophic H-set (in short, NHS) in a non-empty set X if it satisfies the following conditions:

(i) A has the form $A=(T_A, I_A, F_A)$, where $T_A, I_A, F_A) : X\to H$ are mappings,

(ii) $T_A\leq N(F_A)$ and $I_A\geq N(F_A)$.

In this case, the pair (X, A) is called a neutrosophic H-space (in short, NHSp). We will denote the set of all the NHSs as $NHS(X)$.

Definition 4.2 Let (X, A_X), (Y, A_Y) be two NHSps and let $f: X\to Y$ be a mapping. Then $f:(X, A_X)\to(Y, A_Y)$ is called a morphism if $A_X\subset f^{-1}(A_Y)$, i.e.,

$T_{A_X}\leq T_{A_Y}\circ f$, $I_{A_X}\leq I_{A_Y}\circ f$ and $F_{A_X}\geq F_{A_Y}\circ f$.

In particular, $f:(X, A_X)\to(Y, A_Y)$ is called an epimorphism [resp., a monomorphism and an isomorphism], if it is surjective [resp., injective and bijective].

The following is the immediate result of Definition 4.2.

Proposition 4.3 *For each NHSp (X, A_X), the identity mapping $id:(X, A_X)\to(X, A_X)$ is a morphism.*

Proposition 4.4 *Let (X, A_X), (Y, A_Y), (Z, A_Z) be NHSps and let $f: X\to Y$, $g: Y\to Z$ be mappings. If $f:(X, A_X)\to(Y, A_Y)$ and $f:(Y, A_Y)\to(Z, A_Z)$ are morphisms, then $g\circ f:(X, A_X)\to(Z, A_Z)$ is a morphism.*

Proof. Let $A_X=(T_{A_X}, I_{A_X}, F_{A_X})$, $A_Y=(T_{A_Y}, I_{A_Y}, F_{A_Y})$ and $A_Z=(T_{A_Z}, I_{A_Z}, F_{A_Z})$. Then by the hypotheses and Definition 4.2, $A_X\subset f^{-1}(A_Y)$ and $A_Y\subset g^{-1}(A_Z)$, i.e.,

$T_{A_X}\leq T_{A_Y}\circ f$, $I_{A_X}\leq I_{A_Y}\circ f$, $F_{A_X}\geq F_{A_Y}\circ f$

and

$T_{A_Y}\leq T_{A_Z}\circ g$, $I_{A_Y}\leq I_{A_Z}\circ g$, $F_{A_Z}\geq F_{A_Z}\circ g$.

Thus $T_{A_X}\leq(T_{A_Z}\circ g)\circ f$, $I_{A_X}\leq(I_{A_Z}\circ g)\circ f$, $F_{A_X}\geq(F_{A_Z}\circ g)\circ f$.

So $T_{A_X}\leq T_{A_Z}\circ(g\circ f)$, $I_{A_X}\leq I_{A_Z}\circ(g\circ f)$, $F_{A_X}\geq F_{A_Z}\circ(g\circ f)$.

Hence $g\circ f$ is a morphism. □

From Propositions 4.3 and 4.4, we can form the concrete category $\mathbf{NSet(H)}$ consisting of NHSs and morphisms between them. Every $\mathbf{NSet(H)}$-morphism will be called an $\mathbf{NSet(H)}$-mapping.

Lemma 4.5 *The category* $\mathbf{NSet}$ *is topological over* $\mathbf{Set}$.

Proof. Let X be any set and let $((X_j, A_j))_{j\in J}$ be any family of NHSps indexed by a class J, where $A_j=(T_{A_j}, I_{A_j}, F_{A_j})$. Suppose $(f_j: X\to(X_j, A_j)_J$ is a source of ordinary mappings. We define mappings $T_{A_X}, I_{A_X}, F_{A_X}: X\to H$ as follows: for each $x\in X$,

$T_{A_X}(x)=\bigwedge(T_{A_j}\circ f_j)(x)$, $I_{A_X}(x)=\bigwedge(I_{A_j}\circ f_j)(x)$, $F_{A_X}(x)=\bigvee(F_{A_j}\circ f_j)(x)$.

Let $j \in J$ and $x \in X$.
Since $A_j = (T_{A_j}, I_{A_j}, F_{A_j}) \in NHS(X)$,
$T_{A_j} \leq N(F_{A_X})$ and $I_{A_j} \geq N(F_{A_X})$. Then

$$\begin{aligned} N(F_{A_X}(x)) &= N(\bigvee(F_{A_j} \circ f_j)(x)) \\ &= \bigwedge N(F_{A_j}(f_j(x))) \\ &\geq \bigwedge T_{A_j}(f_j(x)) \\ &= \bigwedge T_{A_j} \circ f_j(x) \\ &= T_{A_X}(x) \end{aligned}$$

and

$$\begin{aligned} N(F_{A_X}(x)) &= \bigwedge N(F_{A_j}(f_j(x))) \\ &\leq \bigwedge I_{A_j}(f_j(x)) \\ &= \bigwedge I_{A_j} \circ f_j(x) \\ &= I_{A_X}(x) \end{aligned}$$

Thus $T_{A_X} \leq N(F_{A_X})$ and $I_{A_X} \geq N(F_{A_X})$.
So $A_X = \bigcap f_j^{-1}(A_j) \in NHS(X)$ and thus (X, A_X) is an NHSp. Moreover, by the definition of A_X,

$$T_{A_X} \leq T_{A_j} \circ f_j, I_{A_X} \leq I_{A_j} \circ f_j, F_{A_X} \geq F_{A_j} \circ f_j.$$

Hence $A_X \subset f_j^{-1}(A_j)$.

Therefore each $f_j : (X, A_X) \to (X_j, A_j)$ is an $\mathbf{NSet(H)}$-mapping.

Now let (Y, A_Y) be any NHSp and suppose $g : Y \to X$ is an ordinary mapping for which $f_j \circ g : (Y, A_Y) \to (X_j, A_j)$ is an $\mathbf{NSet(H)}$-mapping for each $j \in J$. Then

$A_Y \subset (f_j \circ g)^{-1}(A_j) = g^{-1}(f_j^{-1}(A_j))$ for each $j \in J$.

Thus

$$A_Y \subset g^{-1}(\bigcap f_j^{-1}(A_j)) = g^{-1}(A_X).$$

So $g : (Y, A_Y) \to (X, A_X)$ is an $\mathbf{NSet(H)}$-mapping. Hence $(f_j : (X, A_X) \to (X_j, A_j))_J$ is an initial source in $\mathbf{NSet(H)}$. This completes the proof. □

Example 4.6 (1) Let X be a set, let (Y, A_Y) be an NHSp and let $f : X \to Y$ be an ordinary mapping. Then clearly, there exists a unique NHS $A_X \in NHS(X)$ for which $f : (X, A_X) \to (Y, A_Y)$ is an $\mathbf{NSet(H)}$-mapping. In fact, $A_X = f^{-1}(A_Y)$.

In this case, A_X is called the inverse image under f of the NHS structure A_Y.

(2) Let $((X_j, A_j))_{j\in J}$ be any family of NHSps and let $X = \Pi_{j\in J} X_j$. For each $j \in J$, let $pr_j : X \to X_j$ be the ordinary projection. Then there exists a unique NHS $A_X \in NHS(X)$ for which $pr_j : (X, A_X) \to (X_j, A_j)$ is an $\mathbf{NSet(H)}$-mapping for each $j \in J$.

In this case, A_X is called the product of $(A_j)_J$, denoted by

$$A_X = \Pi_{j\in J} A_j = (\Pi_{j\in J} T_{A_j}, \Pi_{j\in J} I_{A_j}, \Pi_{j\in J} F_{A_j})$$

and (X, A_X) is called the product NHSp of $((X_j, A_j))_J$.
In fact, $A_X = \bigcap_{j\in J} pr^{-1}(A_j)$
and

$$\Pi_{j\in J} T_{A_j} = \bigwedge T_{A_j} \circ pr_j, \quad \Pi_{j\in J} I_{A_j} = \bigwedge I_{A_j} \circ pr_j,$$
$$\Pi_{j\in J} F_{A_j} = \bigvee F_{A_j} \circ pr_j.$$

In particular, if $J = \{1, 2\}$, then

$$\Pi_{j\in J} T_{A_j} = T_{A_1} \times T_{A_2} = (T_{A_1} \circ pr_1) \wedge (T_{A_2} \circ pr_2),$$
$$\Pi_{j\in J} I_{A_j} = I_{A_1} \times I_{A_2} = (I_{A_1} \circ pr_1) \wedge (I_{A_2} \circ pr_2),$$
$$\Pi_{j\in J} F_{A_j} = F_{A_1} \times F_{A_2} = (F_{A_1} \circ pr_1) \vee (F_{A_2} \circ pr_2).$$

The following is the immediate result of Lemma 4.5 and Result 2.3.

Corollary 4.7 *The category* $\mathbf{NSet(H)}$ *is complete and cocomplete.*

The following is obvious from Result 2.2. But we show directly it.

Corollary 4.8 *The category* $\mathbf{NCSet}$ *is cotopological over* $\mathbf{Set}$.

Proof. Let X be any set and let $((X_j, A_j))_J$ be any family of NHSps indexed by a class J. Suppose $(f_j : X_j \to X)_J$ is a sink of ordinary mappings. We define mappings $T_{A_X}, I_{A_X}, F_{A_X} : X \to H$ as follows: for each $x \in X$,

$$T_{A_X}(x) = \begin{cases} \bigvee_J \bigvee_{x_j \in f_j^{-1}(x)} T_{A_j}(x_j) & \text{if } f_j^{-1}(x) \neq \phi \text{ for all } j \\ 0 & \text{if } f_j^{-1}(x) = \phi \text{ for some } j, \end{cases}$$

$$I_{A_X}(x) = \begin{cases} \bigvee_J \bigvee_{x_j \in f_j^{-1}(x)} I_{A_j}(x_j) & \text{if } f_j^{-1}(x) \neq \phi \text{ for all } j \\ 0 & \text{if } f_j^{-1} = \phi \text{ for some } j, \end{cases}$$

$$F_{A_X}(x) = \begin{cases} \bigwedge_J \bigwedge_{x_j \in f_j^{-1}(x)} F_{A_j}(x_j) & \text{if } f_j^{-1} \neq \phi \text{ for all } j \\ 1 & \text{if } f_j^{-1} = \phi \text{ for some } j. \end{cases}$$

Since $((X_j, A_j))_J$ is a family of NHSps, $T_{A_j} \leq N(F_{A_j})$ and $I_{A_j} \geq N(F_{A_j})$ for each $j \in J$. We may assume that $f_j^{-1} \neq \phi$ without loss of generality. Let $x \in X$. Then

$$\begin{aligned} N(F_{A_X}(x)) &= N(\bigwedge_J \bigwedge_{x_j \in f_j^{-1}(x)} F_{A_j}(x_j)) \\ &= \bigvee_J \bigvee_{x_j \in f_j^{-1}(x)} N(F_{A_j}(x_j)) \\ &\geq \bigvee_J \bigvee_{x_j \in f_j^{-1}(x)} T_{A_j}(x_j). \\ &= T_{A_X}(x). \end{aligned}$$

and

$$\begin{aligned} N(F_{A_X}(x)) &= \bigvee_J \bigvee_{x_j \in f_j^{-1}(x)} N(F_{A_j}(x_j)) \\ &\leq \bigvee_J \bigvee_{x_j \in f_j^{-1}(x)} I_{A_j}(x_j). \\ &= I_{A_X}(x). \end{aligned}$$

Thus $T_{A_X} \leq N(F_{A_X})$ and $I_{A_X} \geq N(F_{A_X})$.
So (X, A_X) is an NHSp. Moreover, for each $j \in J$,

$$f_j^{-1}(A_X) = f_j^{-1}(\bigcup f_j(A_j)) = \bigcup f_j^{-1}(f_j(A_j)) \supset A_j.$$

Hence each $f_j : (X_j, A_j) \to (X, A_X)$ is an $\mathbf{NSet(H)}$-mapping.

Now for each NHSp (Y, A_Y), let $g : X \to Y$ be an ordinary mapping for which each $g \circ f_j : (X_j, A_j) \to (Y, A_Y)$ is an $\mathbf{NSet(H)}$-mapping. Then clearly for each $j \in J$,

$A_j \subset (g \circ f_j)^{-1}(A_Y)$, i.e., $A_j \subset f_j^{-1}(g^{-1}(A_Y))$.

Thus $\bigcup A_j \subset \bigcup f_j^{-1}(g^{-1}(A_Y))$.
So $f_j(\bigcup A_j) \subset f_j(\bigcup f_j^{-1}(g^{-1}(A_Y)))$. By Proposition 3.7 and the definition of A_X,

$$f_j(\bigcup A_j) = \bigcup f_j(A_j) = A_X$$

and

$$f_j(\bigcup f_j^{-1}(g^{-1}(A_Y))) = \bigcup (f_j \circ f_j^{-1})(g^{-1}(A_Y)) = g^{-1}(A_Y).$$

Hence $A_X \subset g^{-1}(A_Y)$. Therefore $g : (X, A_X) \to (Y, A_Y)$ is an $\mathbf{NSet(H)}$-mapping. This completes the proof. $\square$

Example 4.9 (1) Let $(X, A_X) \in \mathbf{NSet(H)}$, let R be an ordinary equivalence relation on X and let $\varphi : X \to X/R$ be the canonical mapping. Then there exists the final NHS structure $A_{X/R}$ in X/R for which $\varphi : (X, A_X) \to (X/R, A_{X/R})$ is an $\mathbf{NSet(H)}$-mapping, where $A_{X/R} = (T_{A_{X/R}}, I_{A_{X/R}}, F_{A_{X/R}}) = (\varphi(T_{A_X}), \varphi(I_{A_X}), \varphi(F_{A_X}))$.

In this case, $A_{X/R}$ is called the neutrosophic H-quotient set structure of X by R.

(2) Let $((X_\alpha, A_\alpha))_{\alpha\in\Gamma}$ be a family of NHSs, let X be the sum of $(X_\alpha)_{\alpha\in\Gamma}$, i.e., $X = \bigcup(X_\alpha \times \{\alpha\})$ and let $j_\alpha : X_\alpha \to X$ the canonical (injective) mapping for each $\alpha \in \Gamma$. Then there exists the final NHS A_X in X. In fact, $A_X = (T_{A_X}, I_{A_X}, F_{A_X})$, where for each $(x, \alpha) \in X$,

$$T_{A_X}(x, \alpha) = \bigvee_\Gamma T_{A_\alpha}(x), \quad I_{A_X}(x, \alpha) = \bigvee_\Gamma I_{A_\alpha}(x),$$
$$F_{A_X}(x, \alpha) = \bigwedge_\Gamma F_{A_\alpha}(x).$$

In this case, A_X is called the sum of $((X_\alpha, A_\alpha))_{\alpha\in\Gamma}$.

Lemma 4.10 *Final episinks in* $\mathbf{NSet(H)}$ *are prserved by pullbacks.*

Proof. Let $(g_j : (X_j, A_j) \to (Y, A_Y))_J$ be any final episink in $\mathbf{NSet(H)}$ and let $f : (W, A_W) \to (Y, A_Y)$ be any $\mathbf{NSet(H)}$-mapping. For each $j \in J$, let

$$U_j = \{(w, x_j) \in W \times X_j : f(w) = g_j(x_j)\}.$$

For each $j \in J$, we define mappings $T_{A_{U_j}}, I_{A_{U_j}}, F_{A_{U_j}} : U_j \to H$ as follows: for each $(w, x_j) \in U_j$,

$$T_{A_{U_j}}(w, x_j) = T_{A_W}(w) \wedge T_{A_j}(x_j),$$

$$I_{A_{U_j}}(w, x_j) = I_{A_W}(w) \wedge I_{A_j}(x_j),$$

$$F_{A_{U_j}}(w, x_j) = F_{A_W}(w) \vee F_{A_j}(x_j).$$

Then clearly, $A_{U_j} = (T_{A_{U_j}}, I_{A_{U_j}}, F_{A_{U_j}}) = (A_W \times A_j)_* \in NHS(U_j)$. Thus (U_j, A_{U_j}) is an NHSp, where $(A_W \times A_j)_*$ denotes the restriction of $A_W \times A_j$ under U_j.

Let e_j and p_j be ordinary projections of U_j. Let $j \in J$. Then clearly,

$$A_{U_j} \subset e_j^{-1}(A_Y) \text{ and } A_{U_j} \subset p_j^{-1}(A_j).$$

Thus $e_j : (U_j, A_{U_j}) \to (W, A_W)$ and $p_j : (U_j, A_{U_j}) \to (X_j, A_j)$ are $\mathbf{NSet(H)}$-mappings. Moreover, $g_h \circ p_h = f \circ e_j$ for each $j \in J$, i.e., the diagram is a pullback square in $\mathbf{NCSet}$:

$$\begin{tikzcd}
(U_j, A_{U_j}) \arrow[r, "p_j"] \arrow[d, "e_j"'] & (X_j, A_j) \arrow[d, "g_j"] \\
(W, A_W) \arrow[r, "f"'] & (Y, A_Y).
\end{tikzcd}$$

Now in order to prove that $(e_j)_J$ is an episink in $\mathbf{NSet(H)}$, i.e., each e_j is surjective, let $w \in W$. Since $(g_j)_J$ is an episink, there exists $j \in J$ such that $g_j(x_j) = f(w)$ for some $x_j \in X_j$. Thus $(w, x_j) \in U_j$ and $w = e_j(w, x_j)$. So $(e_j)_J$ is an episink in $\mathbf{NSet(H)}$.

Finally, let us show that $(e_j)_J$ is final in $\mathbf{NSet(H)}$. Let A_W^* be the final structure in W w.r.t. $(e_j)_J$ and let $w \in W$. Then

$$\begin{aligned}
T_{A_W}(w) &= T_{A_W}(w) \wedge T_{A_W}(w) \\
&\leq T_{A_W}(w) \wedge f^{-1}(T_{A_Y}(w)) \\
&\qquad \text{[since } f : (W, A_W) \to (Y, A_Y))_J) \text{ is an } \mathbf{NSet(H)}\text{-mapping]} \\
&= T_{A_W}(w) \wedge T_{A_Y}(f(w)) \\
&= T_{A_W}(w) \wedge (\textstyle\bigvee_J \bigvee_{x_j \in g_j^{-1}(f(w))} T_{A_j}(x_j)) \\
&\qquad \text{[since } (g_j)_J \text{ is final in } \mathbf{NSet(H)}] \\
&= \textstyle\bigvee_J \bigvee_{x_j \in g_j^{-1}(f(w))} (T_{A_W}(w) \wedge T_{A_j}(x_j)) \\
&= \textstyle\bigvee_J \bigvee_{(w, x_j) \in e_j^{-1}(w)} (T_{U_j}(w, x_j)) \\
&= T_{A_W^*}(w).
\end{aligned}$$

Thus $T_{A_W} \leq T_{A_W^*}$. Similarly, we can see that $I_{A_W} \leq I_{A_W^*}$ and $F_{A_W} \geq F_{A_W^*}$. So $A_W \subset A_W^*$. On the other hand, since $e_j : (U_j, A_{U_j}) \to (W, A_W^*)$ is final, $id_W : (W, A_W^*) \to (W, A_W)$ is an $\mathbf{NSet(H)}$-mapping. So $A_W^* \subset A_W$. Hence $A_W = A_W^*$. This completes the proof. $\square$

For any singleton set $\{a\}$, since the NHS structure $A_{\{a\}}$ on $\{a\}$ is not unique, the category $\mathbf{NSet(H)}$ is not properly fibred over $\mathbf{Set}$. Then by Lemmas 4.5,4.9 and Definition 2.6, we obtain the following result.

Theorem 4.11 *The category* $\mathbf{NSet(H)}$ *satisfies all the conditions of a topological universe over* $\mathbf{Set}$ *except the terminal separator property.*

Theorem 4.12 *The category* $\mathbf{NSet(H)}$ *is Cartesian closed over* $\mathbf{Set}$.

Proof. From Lemma 4.5, it is clear that $\mathbf{NSet(H)}$ has products. So it is sufficient to prove that $\mathbf{NSet(H)}$ has exponential objects.

For any NHSs $\mathbf{X} = (X, A_X)$ and $\mathbf{Y} = (Y, A_Y)$, let Y^X be the set of all ordinary mappings from X to Y. We define mappings $T_{A_{Y^X}}, I_{A_{Y^X}}, F_{A_{Y^X}} : Y^X \to H$ as follows: for each $f \in Y^X$,

$$T_{A_{Y^X}}(f) = \bigvee\{h \in H : T_{A_X}(x) \wedge h \leq T_{A_Y}(f(x)),$$
$$\text{for each } x \in X\},$$

$$I_{A_{Y^X}}(f) = \bigvee\{h \in H : I_{A_X}(x) \wedge h \leq I_{A_Y}(f(x)),$$

for each $x \in X\}$,

$$F_{A_{Y^X}}(f) = \bigwedge\{h \in H : F_{A_X}(x) \vee h \geq F_{A_Y}(f(x)),$$

for each $x \in X\}$.

Then clearly, $A_{Y^X} = (T_{A_{Y^X}}, I_{A_{Y^X}}, F_{A_{Y^X}}) \in NHS(Y^X)$ and thus (Y^X, A_{Y^X}) is an NHSp. Let $\mathbf{Y^X} = (Y^X, A_{Y^X})$ and let $f \in Y^X$, $x \in X$. Then by the definition of A_{Y^X},

$$T_{A_X}(x) \wedge T_{A_{Y^X}}(f) \leq T_{A_Y}(f(x)),$$

$$I_{A_X}(x) \wedge I_{A_{Y^X}}(f) \leq I_{A_Y}(f(x)),$$

$$F_{A_X}(x) \vee F_{A_{Y^X}}(f) \geq F_{A_Y}(f(x)).$$

We define a mapping $e_{X,Y} : X \times Y^X \to Y$ as follows: for each $(x, f) \in X \times Y^X$,

$$e_{X,Y}(x, f) = f(x).$$

Then clearly, $A_X \times A_{Y^X} \in NHS(X \times Y^X)$, where $A_X = (T_{A_X}, I_{A_X}, F_{A_X})$
and for each $(x, f) \in X \times Y^X$,

$$T_{A_X \times A_{Y^X}}(x, f) = T_{A_X}(x) \wedge T_{A_{Y^X}}(f),$$
$$I_{A_X \times A_{Y^X}}(x, f) = I_{A_X}(x) \wedge I_{A_{Y^X}}(f),$$
$$F_{A_X \times A_{Y^X}}(x, f) = F_{A_X}(x) \vee F_{A_{Y^X}}(f).$$

Let us show that $A_X \times A_{Y^X} \subset e^{-1}_{X,Y}(A_Y)$. Let $(x, f) \in X \times Y^X$. Then

$$e^{-1}_{X,Y}(A_Y)(x, f) = A_Y(e_{X,Y}(x, f)) = A_Y(f(x)).$$

Thus

$$\begin{aligned} T_{e^{-1}_{X,Y}(A_Y)}(x, f) &= T_{A_Y}(f(x)) \\ &\geq T_{A_X}(x) \wedge T_{A_{Y^X}}(f) \\ &= T_{A_X \times A_{Y^X}}(x, f), \end{aligned}$$

$$\begin{aligned} I_{e^{-1}_{X,Y}(A_Y)}(x, f) &= I_{A_Y}(f(x)) \\ &\geq I_{A_X}(x) \wedge I_{A_{Y^X}}(f) \\ &= I_{A_X \times A_{Y^X}}(x, f), \end{aligned}$$

$$\begin{aligned} F_{e^{-1}_{X,Y}(A_Y)}(x, f) &= F_{A_Y}(f(x)) \\ &\leq F_{A_X}(x) \vee F_{A_{Y^X}}(f) \\ &= F_{A_X \times A_{Y^X}}(x, f). \end{aligned}$$

So $A_X \times A_{Y^X} \subset e^{-1}_{X,Y}(A_Y)$. Hence $e_{X,Y} : \mathbf{X} \times \mathbf{Y^X} \to \mathbf{Y}$ is an $\mathbf{NSet(H)}$-mapping, where
$\mathbf{X} \times \mathbf{Y^X} = (X \times Y^X, A_X \times A_{Y^X})$ and $\mathbf{Y} = (Y, A_Y)$.

For any $\mathbf{Z} = (Z, A_Z) \in \mathbf{NSet(H)}$, let $h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ be an $\mathbf{NSet(H)}$-mapping where $\mathbf{X} \times \mathbf{Z} = (X \times Z, A_X \times A_Z)$. We define a mapping $\bar{h} : Z \to Y^X$ as follows:

$$(\bar{h}(z))(x) = h(x, z),$$

for each $z \in Z$ and each $x \in X$. Let $(x, z) \in X \times Z$. Then

$$\begin{aligned} T_{A_X \times A_Z}(x, z) &= T_{A_X}(x) \wedge T_{A_Z}(z) \\ &\leq T_{A_Y}(h(x, z)) \text{ [since } h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y} \\ &\qquad \text{is an } \mathbf{NSet(H)}\text{-mapping]} \\ &= T_{A_Y}(\bar{h}(z))(x). \end{aligned}$$

Thus by the definition of A_{Y^X},

$$T_{A_Z}(z) \leq T_{A_{Y^X}}(\bar{h}(z)) = \bar{h}^{-1}(T_{A_{Y^X}})(z).$$

So $T_{A_Z} \leq \bar{h}^{-1}(T_{A_{Y^X}})$. Similarly, we can see that $I_{A_Z} \leq \bar{h}^{-1}(I_{A_{Y^X}})$ and $F_{A_Z} \geq \bar{h}^{-1}(F_{A_{Y^X}})$. Hence $\bar{h} : \mathbf{Z} \to \mathbf{Y^X}$ is an $\mathbf{NSet(H)}$-mapping, where $\mathbf{Y^X} = (Y^X, A_{Y^X})$. Furthermore, we can prove that $\bar{h}$ is a unique $\mathbf{NSet(H)}$-mapping such that $e_{X,Y} \circ (id_X \times \bar{h}) = h$. □

5 The relation between $\mathbf{NSet(H)}$ and $\mathbf{ISet(H)}$

Lemma 5.1 *Define $G_1, G_2 : \mathbf{NSet(H)} \to \mathbf{ISet(H)}$ by:*

$$G_1(X, (T, I, F)) = (X, (T, F)),$$

$$G_2(X, (T, I, F)) = (X, (T, N(T)))$$

and

$$G_1(f) = G_2(f) = f.$$

Then G_1 and G_2 are functors.

Proof. It is clear that $G_1(X, (T, I, F)) = (X, (T, F)) \in \mathbf{ISet(H)}$ for each $(X, (T, I, F) \in \mathbf{NSet(H)}$.

Let $(X, (T_X, I_X, F_X)), (Y, (T_Y, I_Y, F_Y)) \in \mathbf{NSet(H)}$ and let $f : (X, (T_X, I_X, F_X)) \to (Y, (T_Y, I_Y, F_Y))$ be an $\mathbf{NSet(H)}$-mapping. Then

$$T_X \leq T_Y \circ f \text{ and } F_X \geq F_Y \circ f.$$

Thus $G_1(f) = f$ is an $\mathbf{ISet(H)}$-mapping. So $G_1 : \mathbf{NSet(H)} \to \mathbf{ISet(H)}$ is a functor.

Now let $(X, (T, I, F)) \in \mathbf{NSet(H)}$ and consider $(X, (T, N(T)))$. Then by Result 2.8, $T \leq NN(T)$. Thus $G_2(X, (T, I, F)) = (X, (T, N(T))) \in \mathbf{NSet(H)}$.

Let $(X, (T_X, I_X, F_X)), (Y, (T_Y, I_Y, F_Y)) \in \mathbf{NSet(H)}$ and let $f : (X, (T_X, I_X, F_X)) \to (Y, (T_Y, I_Y, F_Y))$ be an $\mathbf{NSet(H)}$-mapping. Then $T_X \leq T_Y \circ f$. Thus $N(T_X) \geq N(T_Y) \circ f$.
So $G_2(f) = f : (X, (T_X, N(T_X)) \to (Y, (T_Y, N(T_Y))$ is an $\mathbf{ISet(H)}$-mapping. Hence $G_2 : \mathbf{NSet(H)} \to \mathbf{ISet(H)}$ is a functor. □

Lemma 5.2 *Define* $F_1 : \mathbf{ISet}(\mathbf{H}) \to \mathbf{NSet}(\mathbf{H})$ *by:*
$F_1(X, (\mu, \nu)) = (X, (\mu, N(\nu), \nu))$ *and* $F_1(f) = f$.
Then F_1 *is a functor.*

Proof. Let $(X, (\mu, \nu)) \in \mathbf{ISet}(\mathbf{H})$. Then
$\mu \leq N(\nu)$ and $N(\nu) \leq N(\nu)$.
Thus $F_1(X, (\mu, \nu)) = (X, (\mu, N(\nu), \nu)) \in \mathbf{NSet}(\mathbf{H})$.
Let $(X, (\mu_X, \nu_X)), (Y, (\mu_Y, \nu_Y)) \in \mathbf{ISet}(\mathbf{H})$ and let
$f : (X, (\mu_X, \nu_X)) \to (Y, (\mu_Y, \nu_Y))$ be an $\mathbf{ISet}(\mathbf{H})$-mapping.

Consider the mapping

$$F_1(f) = f : F_1(X, (\mu_X, \nu_X)) \to F_1(Y, (\mu_Y, \nu_Y)),$$

where

$$F_1(X, (\mu_X, \nu_X)) = (X, (\mu_X, N(\nu_X), \nu_X))$$

and

$$F_1(Y, (\mu_Y, \nu_Y)) = (Y, (\mu_Y, N(\nu_Y), \nu_Y)).$$

Since $f : (X, (\mu_X, \nu_X)) \to (Y, (\mu_Y, \nu_Y))$ is an $\mathbf{ISet}(\mathbf{H})$-mapping, $\mu_X \leq \mu_Y \circ f$ and $\nu_X \geq \nu_Y \circ f$. Thus $N(\nu_X) \leq N(\nu_Y) \circ f$. So $F_1(f) = f : (X, (\mu_X, N(\nu_X), \nu_X)) \to (Y, (\mu_Y, N(\nu_Y), \nu_Y))$ is an $\mathbf{NSet}(\mathbf{H})$-mapping. Hence F_1 is a functor. □

Lemma 5.3 *Define* $F_2 : \mathbf{ISet}(\mathbf{H}) \to \mathbf{NSet}(\mathbf{H})$ *by:*

$$F_2(X, (\mu, \nu)) = (X, (\mu, N(\nu), N(\mu)) \text{ and } F_2(f) = f.$$

Then F_2 *is a functor.*

Proof. Let $(X, (\mu, \nu)) \in \mathbf{ISet}(\mathbf{H})$. Then $\mu \leq N(\nu)$ and $\mu \leq NN(\mu)$, by Result 2.8. Also by Result 2.8, $NN(\mu) \leq NNN(\nu) = N(\nu)$. Thus $\mu \leq NN(\mu) \leq N(\nu)$. So $F_2(X, (\mu, \nu)) = (X, (\mu, N(\nu), N(\mu))) \in \mathbf{NSet}(H)$.

Let $(X, (\mu_X, \nu_X)), (Y, (\mu_Y, \nu_Y)) \in \mathbf{ISet}(H)$ and $f : (X, (\mu_X, \nu_X)) \to (Y, (\mu_Y, \nu_Y))$ be an $\mathbf{ISet}(H)$-mapping. Then
$\mu_X \leq \mu_Y \circ f^2$ and $\nu_X \geq \nu_Y \circ f^2$.
Thus $N(\nu_X) \leq N(\nu_Y) \circ f^2$. So $L(f) = f : (X, (\mu_X, N(\nu_X), N(\mu_X))) \to (Y, (\mu_Y, N(\nu_Y), N(\mu_Y)))$ is an $\mathbf{NSet}(H)$-mapping. Hence F_2 is a functor. □

Theorem 5.4 *The functor* $F_1 : \mathbf{ISet}(\mathbf{H}) \to \mathbf{NSet}(\mathbf{H})$ *is a left adjoint of the functor* $G_1 : \mathbf{NSet}(\mathbf{H}) \to \mathbf{ISet}(\mathbf{H})$.

Proof. For each $(X, (\mu, \nu)) \in \mathbf{ISet}(\mathbf{H})$, $1_X : (X, (\mu, \nu)) \to G_1F_1(X, (\mu, \nu)) = (X, (\mu, \nu))$ is an $\mathbf{ISet}(\mathbf{H})$-mapping. Let $(Y, (T_Y, I_Y, F_Y)) \in \mathbf{NSet}(\mathbf{H})$ and let $f : (X, (\mu, \nu)) \to G_1(Y, (T_Y, I_Y, F_Y)) = (Y, (T_Y, F_Y))$ be an $\mathbf{ISet}(\mathbf{H})$-mapping.

We will show that $f : F_1(X, (\mu, \nu)) = (X, (\mu, N(\nu), \nu)) \to (Y, (T_Y, I_Y, F_Y))$ is an $\mathbf{NSet}(\mathbf{H})$-mapping. Since $f : (X, (\mu, \nu)) \to (Y, (T_Y, F_Y))$ is an $\mathbf{ISet}(\mathbf{H})$-mapping,
$\mu \leq T_Y \circ f$ and $\nu \geq F_Y \circ f$.
Then $N(\nu) \leq N(F_Y) \circ f$. Since $(Y, (T_Y, I_Y, F_Y)) \in \mathbf{NSet}(\mathbf{H})$, $I_Y \geq N(F_Y))$. Thus $N(\nu) \leq I_Y \circ f$. So $f : F_1(X, (\mu, \nu)) = (X, (\mu, N(\nu), \nu)) \to (Y, (T_Y, I_Y, F_Y))$ is an $\mathbf{NSet}(\mathbf{H})$-mapping. Hence 1_X is a G_1-universal mapping for $(X, (\mu, \nu)) \in \mathbf{ISet}(\mathbf{H})$. This completes the proof. □

For each $(X, (\mu, \nu)) \in \mathbf{ISet}(\mathbf{H})$, $F_1(X, (\mu, \nu)) = (X, (\mu, N(\nu), \nu))$ is called a neutrosophic H-space induced by $(X, (\mu, \nu))$. Let us denote the category of all induced neutrosophic H-spaces and $\mathbf{NSet}(\mathbf{H})$-mappings as $\mathbf{NSet}^*(\mathbf{H})$. Then $\mathbf{NSet}^*(\mathbf{H})$ is a full subcategory of $\mathbf{NSet}(\mathbf{H})$.

Theorem 5.5 *Two categories* $\mathbf{ISet}(\mathbf{H})$ *and* $\mathbf{NSet}^*(\mathbf{H})$ *are isomorphic.*

Proof. From Lemma 5.2, it is clear that $F_1 : \mathbf{ISet}(\mathbf{H}) \to \mathbf{NSet}^*(\mathbf{H})$ is a functor. Consider the restriction $G_1 : \mathbf{NSet}^*(\mathbf{H}) \to \mathbf{ISet}(\mathbf{H})$ of the functor G_1 in Lemma 5.1. Let $(X, (\mu, \nu)) \in \mathbf{ISet}(\mathbf{H})$. Then by Lemma 5.2, $F_1(X, (\mu, \nu)) = (X, (\mu, N(\nu), \nu))$. Thus $G_1F_1(X, (\mu, \nu)) = G_1(X, (\mu, N(\nu), \nu)) = (X, (\mu, \nu))$. So $G_1 \circ F_1 = 1_{\mathbf{ISet}(\mathbf{H})}$.

Now let $(X, (T_X, I_X, F_X)) \in \mathbf{NSet}^*(\mathbf{H})$. Then by definition of $\mathbf{NSet}^*(\mathbf{H})$, there exists $(X, (\mu, N(\nu), \nu))$ such that

$$F_1(X, (\mu, \nu)) = (X, (\mu, N(\nu), \nu)) = (X, (T_X, I_X, F_X)).$$

Thus by Lemma 5.1,

$$\begin{aligned} G_1(X, (T_X, I_X, F_X)) &= G_1(X, (\mu, N(\nu), \nu)) \\ &= (X, (\mu, \nu)). \end{aligned}$$

So

$$\begin{aligned} F_1G_1(X, (T_X, I_X, F_X)) &= F_1(X, (\mu, \nu)) \\ &= (X, (T_X, I_X, F_X)). \end{aligned}$$

Hence $F_1 \circ G_1 = 1_{\mathbf{NSet}^*(\mathbf{H})}$. Therefore $F_1 : \mathbf{ISet}(\mathbf{H}) \to \mathbf{NSet}^*(\mathbf{H})$ is an isomorphism. This completes the proof. □

6 Conclusions

In the future, we will form a category $\mathbf{NCRel}$ composed of neutrosophic crisp relations and morphisms between them [resp., $\mathbf{NRel}(\mathbf{H})$ composed of neutrosophic relations and morphisms between them, $\mathbf{NCTop}$ composed of neutrosophic crisp topological spaces and morphisms between them and $\mathbf{NTop}$ composed of neutrosophic topological spaces and morphisms between them] and investigate each category in view points of topological universe. Moreover, we will form some subcategories of each category and study their properties.

References

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.

[2] G. Birkhoff, Lattice Theory, A. M. S. Colloquim Publication, Vol. 25 1975.

[3] J. C. Carrega, The category $Set(H)$ and $Fzz(H)$, Fuzzy sets and systems 9 (1983) 327–332.

[4] E. J. Dubuc, Concrete quasitopoi Applications of Sheaves, Proc. Dunham 1977, Lect. Notes in Math. 753 (1979) 239–254.

[5] M. Eytan, Fuzzy sets:a topological point of view, Fuzzy sets and systems 5 (1981) 47–67.

[6] J. A. Goguen, Categories of V-sets, Bull. Amer. Math. Soc. 75 (1969) 622–624.

[7] H. Herrlich, Catesian closed topological categories, Math. Coll. Univ. Cape Town 9 (1974) 1–16.

[8] K. Hur, A Note on the category $Set(H)$, Honam Math. J. 10 (1988) 89–94.

[9] K. Hur, H. W. Kang and J. H. Ryou, Intutionistic H-fuzzy sets, J. Korea Soc. Math. Edu. Ser. B:Pure Appl. Math. 12 (1) (2005) 33–45.

[10] K. Hur, P. K. Lim, J. G. Lee, J. Kim, The category of neutrosophic crisp sets, To be submitted.

[11] P. T. Jhonstone, Stone Spaces, Cambridge University Press 1982.

[12] C. Y. Kim, S. S. Hong, Y. H. Hong and P. H. Park, Algebras in Cartesian closed topological categories, Lecture Note Series Vol. 1 1985.

[13] P. K. Lim, S. R. Kim and K. Hur, The category $VSet(H)$, International Journal of Fuzzy Logic and Intelligent Systems 10 (1) (2010) 73–81.

[14] L. D. Nel, Topological universes and smooth Gelfand Naimark duality, mathematical applications of category theory, Prc. A. M. S. Spec. Sessopn Denver,1983, Contemporary Mathematics 30 (1984) 224–276.

[15] A. M. Pittes, Fuzzy sets do not form a topos, Fuzzy sets and Systems 8 (1982) 338–358.

[16] D. Ponasse, Some remarks on the category $Fuz(H)$ of M. Eytan, Fuzzy sets and Systems 9 (1983) 199–204.

[17] D. Ponasse, Categorical studies of fuzzy sets, Fuzzy sets and Systems 28 (1988) 235–244.

[18] A. A. Salama and Florentin Smarandache, Neutrosophic Crisp Set Theory, The Educational Publisher Columbus, Ohio 2015.

[19] F. Smarandache, Neutrosophy, Neutrisophic Probability, Set, and Logic, Amer Res Press, Rehoboth, USA 1998.

[20] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

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On Single-Valued Neutrosophic Entropy of order α

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Abstract: Entropy is one of the measures which is used for measuring the fuzziness of the set. In this article, we have presented an entropy measure of order α under the single-valued neutrosophic set environment by considering the pair of their membership functions as well as the hesitation degree between them. Based on this measure, some of its desirable properties have been proposed and validated by taking an example of structure linguistic variable. Furthermore, an approach based on the proposed measure has been presented to deal with the multi criteria decision-making problems. Finally, a practical example is provided to illustrate the decision-making process.

Keywords: Entropy measure, neutrosophic set, multi criteria decision-making, linguistic variable.

1 Introduction

In a real world, due to complexity of decision making or various constraints in today's life, it is difficult for the decision makers to give their opinions in a precise form. To handle these situations, fuzzy set (FS) theory [1], intuitionistic fuzzy set (IFS) theory [2] are successful theories for dealing the uncertainties in the data. After their pioneer works, various researchers have worked on these theories under the different domains such as on entropy measures, on correlation coefficients, on aggregation operators, and many others [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. However, both FS and IFS theories are not able to deal with the indeterminate and inconsistent information. For example, if an expert take an opinion from a certain person about the certain object, then a person may say that 0.5 is the possibility that statement is true, 0.7 say that the statement is false and 0.2 says that he or she is not sure of it. To resolve this, Smarandache [13] introduced a new component called as "indeterminacy-membership function" and added into the "truth membership function" and "falsity membership function", all are independent components lies in $]0^+, 1^+[$, and hence the corresponding set is known as Neutrosophic sets (NSs), which is the generalization of IFS and FS. However, without specification, NSs are difficult to apply in real-life problems. Thus, an extension of the NS, called a single-valued NSs (SVNSs) has been proposed by Wang et al. [14]. After their pioneer work, researchers are engaged in their extensions and their applications in the different disciplines. However, the most important task for the decision maker is to rank the objects so as to get the desired one(s). For it, researchers have incorporating the idea of SVNS theory into the measure theory and applied in many practically uncertain situations such as decision making, pattern recognition, medical diagnosis by using similarity measures [15, 16], distance measures [17, 18], cosine similarity measure [19, 20, 21, 22]. Thus, it has been concluded that the information measures such as entropy, divergence, distance, similarity etc., are of key importance in a number of theoretical and applied statistical inference and data processing problems.

But it has been observed from the above studies that all their measures do not incorporate the idea of the decision-maker preferences into the measure. Furthermore, the existing measure is in linear order, and hence it does not give the exact nature of the alternative. Therefore, keeping the criteria of flexibility and efficiency of neutrosophic sets, this paper presents a new parametric entropy measure of order α for measuring the fuzziness degree of a set. For this, a entropy measure of order α has been presented which makes the decision makers more reliable and flexible for the different values of these parameters. Based on it, some desirable properties of these measures have been studied.

The rest of the manuscript is summarized as follows. Section 2 presents some basic definition about the NS. In Section 3, a new entropy of order α is proposed and its axiomatic justification is established. Further, various desirable properties of it in terms of joint, and conditional entropies have been studied. An illustrative example to show their superiority has been described for structural linguistic variable. Section 4 presents the MCDM method based on the proposed generalized entropy measure along with an illustrative example for selecting the best alternative. Finally a conclusion has been drawn in Section 5.

2 Preliminaries

In this section, some needed basic concepts and definitions related to neutrosophic sets (NS) are introduced.

Definition 2.1. *[13] A NS 'A' in X is defined by its "truth membership function" $(T_A(x))$, a "indeterminacy-membership function" $(I_A(x))$ and a "falsity membership function" $(F_A(x))$ where all are the subset of* $]0^-, 1^+[$ *such that* $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$ *for all* $x \in X$.

Definition 2.2. *[14] A NS 'A' is defined by*

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\}$$

and is called as SVNS where $T_A(x), I_A(x), F_A(x) \in [0, 1]$. *For each point x in X,* $T_A(x), I_A(x), F_A(x) \in [0, 1]$ *and* $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$. *The pairs of these is called as single-valued neutrosophic numbers (SVNNs) denoted by*

$$\alpha = \langle \mu_A(x), \rho_A(x), \nu_A(x) \mid x \in X\rangle$$

and class of SVNSs is denoted by $\Phi(X)$.

Definition 2.3. *Let $A=\langle \mu_A(x),\rho_A(x),\nu_A(x) \mid x\in X\rangle$ and $B=\langle \mu_B(x),\rho_B(x),\nu_B(x) \mid x\in X\rangle$ be two SVNSs. Then the following expressions are defined by [14]*

(i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x), \rho_A(x) \geq \rho_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all x in X;

(ii) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

(iii) $A^c = \{\langle \nu_A(x), \rho_A(x), \mu_A(x) \mid x \in X\rangle\}$

(iv) $A \cap B = \langle \min(\mu_A(x), \mu_B(x)), \max(\rho_A(x), \rho_B(x)), \max(\nu_A(x), \nu_B(x))\rangle$

(v) $A \cup B = \langle \max(\mu_A(x), \mu_B(x)), \min(\rho_A(x), \rho_B(x)), \min(\nu_A(x), \nu_B(x))\rangle$

Majumdar and Samant [16] define the concept of entropy for neutrosophic sets which has been defined as below.

Definition 2.4. *An entropy on $SVNS(X)$ is defined as real valued function $E : SVNS(X) \to [0, 1]$ which satisfies following axioms [16]:*

(P1) $E(A) = 0$ if A is crisp set.

(P2) $E(A) = 1$ if $\mu_A(x) = \rho_A(x) = \nu_A(x)$

(P3) $E(A) = E_(A^c)$ for all $A \in SVNS(X)$

(P4) $E(A) \leq E(B)$ if $A \subseteq B$ that is , $\mu_A(x) \leq \mu_B(x), \nu_A(x) \geq \nu_B(x)$ and $\rho_A(x) \geq \rho_B(x)$ for $\mu_B(x) \leq \nu_B(x)$ and $\mu_B(x) \leq \rho_B(x)$.

3 Entropy of order-α

In this section we proposed parametric entropy for $SVNS$

Definition 3.1. *The entropy of order- α for $SVNS$ A is defined as:*

$$\begin{aligned} E_\alpha(A) &= \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} \log_3 \Big[\big(\mu_A^\alpha(x_i) + \rho_A^\alpha(x_i) + \nu_A^\alpha(x_i)\big) \\ &\times \big(\mu_A(x_i) + \rho_A(x_i) + \nu_A(x_i)\big)^{1-\alpha} \\ &+ 3^{1-\alpha}\big(1 - \mu_A(x_i) - \rho_A(x_i) - \nu_A(x_i)\big)\Big], \end{aligned} \tag{1}$$

where $\alpha > 0, \alpha \neq 1$.

Theorem 1. *$E_\alpha(A)$ as defined in Definition 3.1 is entropy for $SVNS$.*

Proof. In order to proof $E_\alpha(A)$ is a valid measure, we have to proof that it satisfies the axioms as given in Definition 2.4.

(P1) Let A be a crisp set i.e. $A = (1, 0, 0)$ or $A = (0, 0, 1)$. Then from Definition 3.1 we get $E_\alpha(A) = 0$.

(P2) Let $\mu_A(x_i) = \rho_A(x_i) = \nu_A(x_i)$ for all $x_i \in X$ which implies that $E_\alpha(A)$ becomes

$$\begin{aligned} & E_\alpha(A) \\ =\ & \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} \log_3 \big[\big(\mu_A^\alpha(x_i) + \mu_A^\alpha(x_i) + \mu_A^\alpha(x_i)\big) \times \\ & \quad +\big(\mu_A(x_i) + \mu_A(x_i) + \mu_A(x_i)\big)^{(1-\alpha)} \\ & \quad +3^{1-\alpha}\big(1 - \mu_A(x_i) - \mu_A(x_i) - \mu_A(x_i)\big)\big] \\ =\ & \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} \log_3 \big[\big(3\mu_A^\alpha(x_i)\big)\big(3\mu_A(x_i)\big)^{1-\alpha} \\ & \quad +3^{1-\alpha}\big(1 - 3\mu_A(x_i)\big)\big] \\ =\ & \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} \log_3 \big[3^{2-\alpha}\mu_A(x_i) \\ & \quad +3^{1-\alpha} - 3^{2-\alpha}\mu_A(x_i)\big] \\ =\ & 1 \end{aligned}$$

Now, let $E_\alpha(A) = 1$, that is,

$$\begin{aligned} & \sum_{i=1}^{n} \log_3 \Big[\big(\mu_A^\alpha(x_i) + \rho_A^\alpha(x_i) + \nu_A^\alpha(x_i)\big)\big(\mu_A(x_i) + \rho_A(x_i) \\ & +\nu_A(x_i)\big)^{1-\alpha} + 3^{1-\alpha}\big(1 - \mu_A(x_i) - \rho_A(x_i) - \nu_A(x_i)\big)\Big] \\ & = n(1-\alpha) \\ \Rightarrow\ & \log_3 \Big[\big(\mu_A^\alpha(x_i) + \rho_A^\alpha(x_i) + \nu_A^\alpha(x_i)\big)\big(\mu_A(x_i) + \rho_A(x_i) + \\ & \nu_A(x_i)\big)^{1-\alpha} + 3^{1-\alpha}\big(1 - \mu_A(x_i) - \rho_A(x_i) - \nu_A(x_i)\big)\Big] \\ & = (1-\alpha) \\ \Rightarrow\ & \big(\mu_A^\alpha(x_i) + \rho_A^\alpha(x_i) + \nu_A^\alpha(x_i)\big)\Big(\mu_A(x_i) + \rho_A(x_i) + \\ & \nu_A(x_i)\Big)^{1-\alpha} + 3^{1-\alpha}\Big(1 - \mu_A(x_i) - \rho_A(x_i) - \nu_A(x_i)\Big) \\ & = 3^{1-\alpha} \\ \Rightarrow\ & \Big(\mu_A(x_i) + \rho_A(x_i) + \nu_A(x_i)\Big)\Bigg[\frac{\mu_A^\alpha(x_i) + \rho^\alpha(x_i) + \nu_A^\alpha(x_i)}{3} \\ & -\Big(\frac{\mu_A(x_i) + \rho_A(x_i) + \nu_A(x_i)}{3}\Big)^\alpha\Bigg] = 0 \end{aligned} \tag{2}$$

From Eq. (2) we get, either $\mu_A(x_i)+\rho_A(x_i)+\nu_A(x_i) = 0$ implies that

$$\mu_A(x_i) = \rho_A(x_i) = \nu_A(x_i) = 0 \text{ for all } x_i \in X \tag{3}$$

or

$$\frac{\mu_A^\alpha(x_i) + \rho^\alpha(x_i) + \nu_A^\alpha(x_i)}{3} - \Big(\frac{\mu_A(x_i) + \rho_A(x_i) + \nu_A(x_i)}{3}\Big)^\alpha = 0 \tag{4}$$

Now, consider the following function

$$g(\zeta) = \zeta^\alpha \text{ where } \zeta \in [0, 1]$$

Differentiate it with respect to ζ, we get

$$g'(\zeta) = \alpha\zeta^{\alpha-1}$$

$$g''(\zeta) = \alpha(\alpha-1)\zeta^{\alpha-2}$$

because $g''(\zeta) > 0$ for $\alpha > 1$ and $g''(\zeta) < 0$ for $\alpha < 1$ therefore $g(\zeta)$ is convex or concave according to $\alpha > 1$ or $\alpha < 1$. So, for any points ζ_1, ζ_2 and ζ_3 in $[0,1]$, we have

$$\frac{g(\zeta_1)+g(\zeta_2)+g(\zeta_3)}{3} - g\Big(\frac{\zeta_1+\zeta_2+\zeta_3}{3}\Big) \geq 0 \text{ for } \alpha > 1 \quad (5)$$

$$\frac{g(\zeta_1)+g(\zeta_2)+g(\zeta_3)}{3} - g\Big(\frac{\zeta_1+\zeta_2+\zeta_3}{3}\Big) \leq 0 \text{ for } \alpha < 1 \quad (6)$$

In above, equality holds only if $\zeta_1 = \zeta_2 = \zeta_3$. Hence from Eqs. (3),(4), (5) and (6) we conclude Eqs. (2) and (4) holds only when $\mu_A(x_i) = \rho_A(x_i) = \nu_A(x_i)$ for all $x_i \in X$.

(P3) Since $A^c = \{\langle x, \nu_A(x), \rho_A(x), \mu_A(x) \mid x \in X\rangle\}$ which implies that $E_\alpha(A^c) = E_\alpha(A)$.

(P4) Rewrite the entropy function as

$$f(x,y,z) = \frac{1}{1-\alpha}\sum_{i=1}^{n}\log_3\Big[\big(x^\alpha+y^\alpha+z^\alpha\big)\big(x+y+z\big)^{1-\alpha} + 3^{1-\alpha}(1-x-y-z)\Big] \quad (7)$$

where $x,y,z \in [0,1]$. In order to proof the proposed entropy satisfies (P4), it is sufficient to prove that the function f defined in Eq. (7) is an increasing function with respect to x and decreasing with respect to y and z. For it, take a partial derivative of the function with respect to x, y and z and hence we get.

$$\frac{\partial f}{\partial x} = \frac{(1-\alpha)(x^\alpha+y^\alpha+z^\alpha)(x+y+z)^{-\alpha} + \alpha(x+y+z)^{1-\alpha}x^{\alpha-1} - 3^{1-\alpha}}{(x^\alpha+y^\alpha+z^\alpha)(x+y+z)^{1-\alpha} + 3^{1-\alpha}(1-x-y-z)]} \quad (8)$$

$$\frac{\partial f}{\partial y} = \frac{(1-\alpha)(x^\alpha+y^\alpha+z^\alpha)(x+y+z)^{-\alpha} + \alpha(x+y+z)^{1-\alpha}y^{\alpha-1} - 3^{1-\alpha}}{(1-\alpha)[(x^\alpha+y^\alpha+z^\alpha)(x+y+z)^{1-\alpha} + 3^{1-\alpha}(1-x-y-z)]} \quad (9)$$

$$\frac{\partial f}{\partial z} = \frac{(1-\alpha)(x^\alpha+y^\alpha+z^\alpha)(x+y+z)^{-\alpha} + \alpha(x+y+z)^{1-\alpha}z^{\alpha-1} - 3^{1-\alpha}}{(1-\alpha)[(x^\alpha+y^\alpha+z^\alpha)(x+y+z)^{1-\alpha} + 3^{1-\alpha}(1-x-y-z)]} \quad (10)$$

After setting $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial z} = 0$, we get $x = y = z$. Also,

$$\frac{\partial f}{\partial x} \geq 0, \text{ whenever } x \leq y, x \leq z, \alpha > 0, \alpha \neq 0 \quad (11)$$

$$\frac{\partial f}{\partial x} \leq 0, \text{ whenever } x \geq y, x \geq z, \alpha > 0, \alpha \neq 0. \quad (12)$$

Thus, $f(x,y,z)$ is increasing function with respect to x for $x \leq y, x \leq z$ and decreasing when $x \geq y, x \geq z$. Similarly, we have

$$\frac{\partial f}{\partial y} \leq 0 \text{ and } \frac{\partial f}{\partial z} \leq 0, \text{whenever } x \leq y, x \leq z. \quad (13)$$

$$\frac{\partial f}{\partial y} \geq 0 \text{ and } \frac{\partial f}{\partial z} \geq 0, \text{whenever } x \geq y, x \geq z. \quad (14)$$

Thus, $f(x,y,z)$ is decreasing function with respect to y and z for $x \leq y, x \leq z$ and increasing when $x \geq y, x \geq z$.

Therefore from monotonicity of function f, and by taking two $SVNSs$ $A \subseteq B$, i.e., $\mu_A(x) \leq \mu_B(x), \nu_A(x) \geq \nu_B(x)$ and $\rho_A(x) \geq \rho_B(x)$ for $\mu_B(x) \leq \nu_B(x)$ and $\mu_B(x) \leq \rho_B(x)$, we get $E_\alpha(A) \leq E_\alpha(B)$.

□

Example 3.1. *Let A be $SVNS$ in universe of discourse $X = \{x_1, x_2, x_3, x_4\}$ defined as $A = \{\langle x_1, 0.4, 0.3, 0.9\rangle, \langle x_2, 0.7, 0.5, 0.3\rangle, \langle x_3, 0.2, 0.9, 0.8\rangle, \langle x_4, 0.5, 0.4, 0.6\rangle\}$. Then entropies values for different values of α is $E_{0.2}(A) = 0.9710; E_{0.5}(A) = 0.9303; E_2(A) = 0.7978; E_5(A) = 0.7246; E_{10}(A) = 0.7039$. It is clearly seen from this result that with the increase of α, the values of $E_\alpha(A)$ is decreases.*

The above proposed entropy measure of order α satisfies the following additional properties.

Consider two $SVNSs$ A and B defined over $X = \{x_1, x_2, \ldots, x_n\}$. Take partition of X as $X_1 = \{x_i \in X : A \subseteq B\}$, $X_2 = \{x_i \in X : A \supseteq B\}$. Then we define the joint and conditional entropies between them as follows

(i) **Joint entropy**

$$\begin{aligned}
& E_\alpha(A \cup B) \\
= & \frac{1}{n(1-\alpha)}\sum_{i=1}^{n}\log_3\Big[\big(\mu^\alpha_{A\cup B}(x_i) + \rho^\alpha_{A\cup B}(x_i) + \nu^\alpha_{A\cup B}(x_i)\big) \times \\
& \big(\mu_{A\cup B}(x_i) + \rho_{A\cup B}(x_i) + \nu_{A\cup B}(x_i)\big)^{1-\alpha} \\
& + 3^{1-\alpha}\big(1 - \mu_{A\cup B}(x_i) - \rho_{A\cup B}(x_i) - \nu_{A\cup B}(x_i)\big)\Big] \\
= & \frac{1}{n(1-\alpha)}\Bigg\{\sum_{x_i \in X_1}\log_3\Big[\Big(\mu^\alpha_B(x_i) + \rho^\alpha_B(x_i) + \nu^\alpha_B(x_i)\Big) \times \\
& \Big(\mu_B(x_i) + \rho_B(x_i) + \nu_B(x_i)\Big)^{1-\alpha} + 3^{1-\alpha}\Big(1 - \mu_B(x_i) \\
& - \rho_B(x_i) - \nu_B(x_i)\Big)\Big] + \sum_{x_i \in X_2}\log_3\Big[\Big(\mu^\alpha_A(x_i) + \rho^\alpha_A(x_i) \\
& + \nu^\alpha_A(x_i)\Big)\Big(\mu_A(x_i) + \rho_A(x_i) + \nu_A(x_i)\Big)^{1-\alpha} \\
& + 3^{1-\alpha}\Big(1 - \mu_A(x_i) - \rho_A(x_i) - \nu_A(x_i)\Big)\Big]\Bigg\}
\end{aligned} \quad (15)$$

(ii) **Conditional entropy**

$$
\begin{aligned}
& E_\alpha(A|B) \\
= \; & \frac{1}{n(1-\alpha)} \sum_{x_i \in X_2} \Bigg\{ \log_3 \Bigg[\Big(\mu_A^\alpha(x_i) + \rho_A^\alpha(x_i) + \nu_A^\alpha(x_i) \Big) \times \\
& \Big(\mu_A(x_i) + \rho_A(x_i) + \nu_A(x_i) \Big)^{1-\alpha} + 3^{1-\alpha} \Big(1 - \mu_A(x_i) \\
& - \rho_A(x_i) - \nu_A(x_i) \Big) \Bigg] - \log_3 \Bigg[\Big(\mu_B^\alpha(x_i) + \rho_B^\alpha(x_i) + \nu_B^\alpha(x_i) \Big) \times \\
& \Big(\mu_B(x_i) + \rho_B(x_i) + \nu_B(x_i) \Big)^{1-\alpha} + 3^{1-\alpha} \Big(1 - \mu_B(x_i) \\
& - \rho_B(x_i) - \nu_B(x_i) \Big) \Bigg] \Bigg\}
\end{aligned}
$$

and

$$
\begin{aligned}
& E_\alpha(B|A) \\
= \; & \frac{1}{n(1-\alpha)} \sum_{x_i \in X_1} \Bigg\{ \log_3 \Bigg[\Big(\mu_B^\alpha(x_i) + \rho_B^\alpha(x_i) + \nu_B^\alpha(x_i) \Big) \times \\
& \Big(\mu_B(x_i) + \rho_B(x_i) + \nu_B(x_i) \Big)^{1-\alpha} + 3^{1-\alpha} \Big(1 - \mu_B(x_i) \\
& - \rho_B(x_i) - \nu_B(x_i) \Big) \Bigg] - \log_3 \Bigg[\Big(\mu_A^\alpha(x_i) + \rho_A^\alpha(x_i) + \nu_A^\alpha(x_i) \Big) \times \\
& \Big(\mu_A(x_i) + \rho_A(x_i) + \nu_A(x_i) \Big)^{1-\alpha} + 3^{1-\alpha} \Big(1 - \mu_A(x_i) \\
& - \rho_A(x_i) - \nu_A(x_i) \Big) \Bigg] \Bigg\}
\end{aligned}
$$

Here $E_\alpha(A|B)$ is "entropy of A given B".

Theorem 2. *For $SVNSs$ A and B following statements hold*

(i) $E_\alpha(A \cup B) = E_\alpha(A) + E_\alpha(B|A)$

(ii) $E_\alpha(A \cup B) = E_\alpha(B) + E_\alpha(A|B)$

(iii) $E_\alpha(A \cup B) = E_\alpha(A) + E_\alpha(B|A) = E_\alpha(B) + E_\alpha(A|B)$

(iv) $E_\alpha(A \cup B) + E_\alpha(A \cap B) = E_\alpha(A) + E_\alpha(B)$.

Proof. (i) Here, we have to proof (i) only, (ii) and (iii) can be follows from it.

$$
\begin{aligned}
& E_\alpha(A) + E_\alpha(B|A) - E_\alpha(A \cup B) \\
= \; & \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} \log_3 \Bigg[\Big(\mu_A^\alpha(x_i) + \rho_A^\alpha(x_i) + \nu_A^\alpha(x_i) \Big) \times \\
& \Big(\mu_A(x_i) + \rho_A(x_i) + \nu_A(x_i) \Big)^{1-\alpha} + 3^{1-\alpha} \Big(1 - \mu_A(x_i) \\
& - \rho_A(x_i) - \nu_A(x_i) \Big) \Bigg] \\
+ \; & \frac{1}{n(1-\alpha)} \sum_{x_i \in X_1} \Bigg\{ \log_3 \Bigg[\Big(\mu_B^\alpha(x_i) + \rho_B^\alpha(x_i) + \nu_B^\alpha(x_i) \Big) \times \\
& \Big(\mu_B(x_i) + \rho_A(x_i) + \nu_B(x_i) \Big)^{1-\alpha} + 3^{1-\alpha} \Big(1 - \mu_B(x_i) \\
& - \rho_B^{(}x_i) - \nu_B(x_i) \Big) \Bigg] - \log_3 \Bigg[\Big(\mu_A^\alpha(x_i) + \rho_A^\alpha(x_i) + \nu_A^\alpha(x_i) \Big) \times \\
& \Big(\mu_A(x_i) + \rho_A(x_i) + \nu_A(x_i) \Big)^{1-\alpha} + 3^{1-\alpha} \Big(1 - \mu_A(x_i) \\
& - \rho_A(x_i) - \nu_A(x_i) \Big) \Bigg] \Bigg\} \\
- \; & \frac{1}{n(1-\alpha)} \Bigg\{ \sum_{x_i \in X_1} \log_3 \Bigg[\Big(\mu_B^\alpha(x_i) + \rho_B^\alpha(x_i) + \nu_B^\alpha(x_i) \Big) \times \\
& \Big(\mu_B(x_i) + \rho_B(x_i) + \nu_B(x_i) \Big)^{1-\alpha} + 3^{1-\alpha} \Big(1 - \mu_B(x_i) \\
& - \rho_B(x_i) - \nu_B(x_i) \Big) \Bigg] - \sum_{x_i \in X_2} \log_3 \Bigg[\Big(\mu_A^\alpha(x_i) + \rho_A^\alpha(x_i) \nu_A^\alpha(x_i) \Big) \\
& \Big(\mu_A(x_i) + \rho_A(x_i) + \nu_A(x_i) \Big)^{1-\alpha} + 3^{1-\alpha} \Big(1 - \mu_A(x_i) \\
& - \rho_A(x_i) - \nu_A(x_i) \Big) \Bigg] \Bigg\} \\
= \; & \frac{1}{n(1-\alpha)} \Bigg\{ \sum_{x_i \in X_1} \log_3 \Bigg[\Big(\mu_A^\alpha(x_i) + \rho_A^\alpha(x_i) + \nu_A^\alpha(x_i) \Big) \times \\
& \Big(\mu_A(x_i) + \rho_A(x_i) + \nu_A(x_i) \Big)^{1-\alpha} + 3^{1-\alpha} \Big(1 - \mu_A(x_i) \\
& - \nu_A(x_i) \Big) \Bigg] + \sum_{x_i \in X_2} \log_3 \Bigg[\Big(\mu_A^\alpha(x_i) + \rho_A^\alpha(x_i) + \nu_A^\alpha(x_i) \Big) \times \\
& \Big(\mu_A(x_i) + \rho_A(x_i) + \nu_A(x_i) \Big)^{1-\alpha} + 3^{1-\alpha} \Big(1 - \mu_A(x_i) \\
& - \rho_A(x_i) - \nu_A(x_i) \Big) \Bigg] \Bigg\} \\
+ \; & \frac{1}{n(1-\alpha)} \sum_{x_i \in X_1} \Bigg\{ \log_3 \Bigg[\Big(\mu_B^\alpha(x_i) + \rho_B^\alpha(x_i) + \nu_B^\alpha(x_i) \Big) \times \\
& \Big(\mu_B(x_i) + \rho_A(x_i) + \nu_B(x_i) \Big)^{1-\alpha} + 3^{1-\alpha} \Big(1 - \mu_B(x_i) \\
& - \rho_B(x_i) - \nu_B(x_i) \Big) \Bigg] - \log_3 \Bigg[\Big(\mu_A^\alpha(x_i) + \rho_A^\alpha(x_i) + \nu_A^\alpha(x_i) \Big) \times \\
& \Big(\mu_A(x_i) + \rho_A(x_i) + \nu_A(x_i) \Big)^{1-\alpha} + 3^{1-\alpha} \Big(1 - \mu_A(x_i) \\
& - \rho_A(x_i) - \nu_A(x_i) \Big) \Bigg] \Bigg\} \\
- \; & \frac{1}{n(1-\alpha)} \Bigg\{ \sum_{x \in X_1} \log_3 \Bigg[\Big(\mu_B^\alpha(x_i) + \rho_B^\alpha(x_i) + \nu_B^\alpha(x_i) \Big) \times \\
& \Big(\mu_B(x_i) + \rho_B(x_i) + \nu_B(x_i) \Big)^{1-\alpha} + 3^{1-\alpha} \Big(1 - \mu_B(x_i) \\
& - \rho_B(x_i) - \nu_B(x_i) \Big) \Bigg] - \sum_{x_i \in X_2} \log_3 \Bigg[\Big(\mu_A^\alpha(x_i) + \nu_A^\alpha(x_i) \Big) \times \\
& \Big(\mu_A(x_i) + \rho_A(x_i) + \nu_A(x_i) \Big)^{1-\alpha} + 3^{1-\alpha} \Big(1 - \mu_A(x_i) \\
& - \rho_A(x_i) - \nu_A(x_i) \Big) \Bigg] \Bigg\} \\
= \; & 0
\end{aligned}
$$

(iv) For an $SVNSs$ A and B, we have

$$\begin{aligned}
& E_\alpha(A\cap B) \\
= & \frac{1}{n(1-\alpha)}\sum_{i=1}^{n}\log_3\Big[\Big(\mu^\alpha_{A\cap B}(x_i)+\rho^\alpha_{A\cap B}(x_i)+\nu^\alpha_{A\cap B}(x_i)\Big)\times \\
& \Big(\mu_{A\cap B}(x_i)+\rho_{A\cap B}(x_i)+\nu_{A\cap B}(x_i)\Big)^{1-\alpha}+ \\
& 3^{1-\alpha}\Big(1-\mu_{A\cap B}(x_i)-\nu_{A\cap B}(x_i)\Big)\Big] \\
= & \frac{1}{n(1-\alpha)}\Big\{\sum_{x\in X_1}\log_3\Big[\Big(\mu^\alpha_A(x_i)+\rho^\alpha_A(x_i)+\nu^\alpha_A(x_i)\Big)\times \\
& \Big(\mu_A(x_i)+\rho_A(x_i)+\nu_A(x_i)\Big)^{(1-\alpha)}+3^{1-\alpha}\Big(1-\mu_A(x_i) \\
& -\rho_A(x_i)-\nu_A(x_i)\Big)\Big]+\sum_{x\in X_2}\log_3\Big[\Big(\mu^\alpha_B(x_i)+\rho^\alpha_B(x_i)+\nu^\alpha_B(x_i)\Big) \\
& \Big(\mu_B(x_i)+\rho_B(x_i)+\nu_B(x_i)\Big)^{1-\alpha}+3^{1-\alpha}\Big(1-\mu_B(x_i) \\
& -\rho_B(x_i)-\nu_B(x_i)\Big)\Big]\Big\}
\end{aligned}$$

Hence, by the definition of joint entropy $E_\alpha(A\cup B)$ given in Eq. (15), we get

$$E_\alpha(A\cup B)+E_\alpha(A\cap B)=E_\alpha(A)+E_\alpha(B)$$

□

Theorem 3. *For $SVNSs$ A and B following statements holds*

(i) $E_\alpha(A)-E_\alpha(A\cap B)=E_\alpha(A|B)$

(ii) $E_\alpha(B)-E_\alpha(A\cap B)=E_\alpha(A|B)$

Proof. We prove (i) part only, other can be proven similarly. Consider

$$\begin{aligned}
& E_\alpha(A)-E_\alpha(A\cap B) \\
= & \frac{1}{n(1-\alpha)}\Big\{\sum_{i=1}^{n}\log_3\Big[\big(\mu^\alpha_A(x_i)+\rho^\alpha_A(x_i)+\nu^\alpha_A(x_i)\Big)\times \\
& \Big(\mu_A(x_i)+\rho_A(x_i)+\nu_A(x_i)\Big)^{1-\alpha}+3^{1-\alpha}\Big(1-\mu_A(x_i) \\
& -\rho_A(x_i)-\nu_A(x_i)\Big)\Big]-\sum_{i=1}^{n}\log_3\Big[\Big(\mu^\alpha_{A\cap B}(x_i)+\rho^\alpha_{A\cap B}(x_i) \\
& +\nu^\alpha_{A\cap B}(x_i))\Big(\mu_{A\cap B}(x_i)+\rho_{A\cap B}(x_i)+\nu_{A\cap B}(x_i)\Big)^{1-\alpha} \\
& +3^{1-\alpha}\Big(1-\mu_{A\cap B}(x_i)-\rho_{A\cap B}(x_i)-\nu_{A\cap B}(x_i)\Big)\Big]\Big\} \\
= & \frac{1}{n(1-\alpha)}\Big\{\sum_{x\in X_1}\log_3\Big[\Big(\mu^\alpha_A(x_i)+\rho^\alpha_A(x_i)+\nu^\alpha_A(x_i)\Big)\times \\
& \Big(\mu_A(x_i)+\rho_A(x_i)+\nu_A(x_i)\Big)^{1-\alpha}+3^{1-\alpha}\Big(1-\mu_A(x_i) \\
& -\nu_A(x_i)\Big)\Big]+\sum_{x\in X_2}\log_3\Big[\Big(\mu^\alpha_A(x_i)+\rho^\alpha_A(x_i)+\nu^\alpha_A(x_i)\Big)\times \\
& \Big(\mu_A(x_i)+\rho_A(x_i)+\nu_A(x_i))^{1-\alpha}+3^{1-\alpha}\Big(1-\mu_A(x_i) \\
& \rho_A(x_i)-\nu_A(x_i)\Big)\Big]-\sum_{x\in X_1}\log_3\Big[\Big(\mu^\alpha_A(x_i)+\rho^\alpha_A(x_i)+\nu^\alpha_A(x_i)\Big)\times \\
& \Big(\mu_A(x_i)+\rho_A(x_i)+\nu_A(x_i)\Big)^{1-\alpha}+3^{1-\alpha}\Big(1-\mu_A(x_i) \\
& -\rho_A(x_i)-\nu_A(x_i)\Big)\Big]-\sum_{x\in X_2}\log_3\Big[\Big(\mu^\alpha_B(x_i)+\rho^\alpha_B(x_i)+\nu^\alpha_B(x_i)\Big)\times \\
& \Big(\mu_B(x_i)+\rho_B(x_i)+\nu_B(x_i)\Big)^{1-\alpha}+3^{1-\alpha}\Big(1-\mu_B(x_i) \\
& -\rho_B(x_i)-\nu_B(x_i)\Big)\Big]\Big\} \\
+ & \frac{1}{n(1-\alpha)}\sum_{x\in X_2}\Big\{\log_3\Big[\Big(\mu^\alpha_A(x_i)+\rho^\alpha_A(x_i)+\nu^\alpha_A(x_i)\Big)\times \\
& \Big(\mu_A(x_i)+\rho_A(x_i)+\nu_A(x_i)\Big)^{(1-\alpha)}+3^{1-\alpha}\Big(1-\mu_A(x_i) \\
& -\rho_A(x_i)-\nu_A(x_i)\Big)\Big]-\log_3\Big[\Big(\mu^\alpha_B(x_i)+\rho^\alpha_B(x_i)+\nu^\alpha_B(x_i)\Big)\times \\
& \Big(\mu_B(x_i)+\rho_B(x_i)+\nu_B(x_i)\Big)^{1-\alpha}+3^{1-\alpha}\Big(1-\mu_B(x_i) \\
& -\rho_B(x_i)-\nu_B(x_i)\Big)\Big]\Big\} \\
= & E_\alpha(A|B)
\end{aligned}$$

This completes the proof. □

Let $A=\langle x,\mu_A(x),\rho_A(x),\nu_A(x)|x\in X\rangle$ be $SVNS$ in X. For n be any positive real number, Zhang et al. [23] defined A^n as follows

$$A^n=\langle x,\mu_A(x)^n,1-(1-\rho_A(x))^n,1-(1-\nu_A(x))^n\rangle \quad (16)$$

Definition 4. *Contraction of $SVNS$ A in universe of discourse X is defined by*

$$CON(A)=\langle x,\mu_{CON(A)}(x),\rho_{CON(A)}(x),\nu_{CON(A)}(x)\rangle$$

where $\mu_{CON(A)}(x)=[\mu_A(x)]^2$; $\rho_{CON(A)}(x)=1-[1-\rho_A(x)]^2$; $\nu_{CON(A)}(x)=1-[1-\nu_A(x)]^2$ *i.e.* $CON(A)=A^2$

Definition 5. *Dilation of $SVNS$ A in universe of discourse X is defined by*

$$DIL(A)=\langle x,\mu_{DIL(A)}(x),\rho_{DIL(A)}(x),\nu_{DIL(A)}(x)\rangle$$

where $\mu_{DIL(A)}(x)=[\mu_A(x)]^{1/2}$; $\rho_{DIL(A)}(x)=1-[1-\rho_A(x)]^{1/2}$; $\nu_{DIL(A)}(x)=1-[1-\nu_A(x)]^{1/2}$ *i.e.* $DIL(A)=$

$A^{1/2}$

An illustrative example has been tested on the concentration and dilation for comparing the performance of proposed entropy with the some existing entropies as given below.

(i) Entropy defined by [5];

$$E_{SK}(A) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\min(\mu_A(x_i), \nu_A(x_i)) + \pi_A(x_i)}{\max(\mu_A(x_i), \nu_A(x_i)) + \pi_A(x_i)}\right]$$

(ii) Entropy defined by [3];

$$E_{BB}(A) = \frac{1}{n}\sum_{i=1}^{n}\pi_A(x_i)$$

(iii) Entropy defined by [8];

$$E_{ZJ}(A) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\mu_A(x_i)\bigwedge \nu A(x_i)}{\mu_A(x_i)\bigvee \nu A(x_i)}\right)$$

(iv) Entropy defined by [4];

$$E_{ZL}(A) = 1 - \frac{1}{n}\sum_{i=1}^{n}|\mu_A(x_i) - \nu_A(x_i)|$$

Example 3.2.

Let $X = \{x_1, x_2, ..., x_5\}$ be universe of discourse and a $SVNS\ A$ "LARGE" on X may be defined as $A = \{\langle x_1, 0.1, 0.7, 0.8\rangle, \langle x_2, 0.3, 0.6, 0.5\rangle, \langle x_3, 0.5, 0.3, 0.4\rangle, \langle x_4, 0.9, 0.2, 0.0\rangle, \langle x_5, 1.0, 0.1, 0.0\rangle\}$. Using the operations defined in Eq. (16) on $SVNS$, we can generate following $SVNSs$

$$A, A^{1/2}, A^2, A^3$$

which can be defined as
$A^{1/2}$ may treated as "More or Less LARGE",
A^2 may treated as "Very LARGE",
A^3 may treated as "Quite Very LARGE"
and these corresponding sets are computed as

$A^{1/2} = \{\langle x_1, 0.3162, 0.4523, 0.5528\rangle, \langle x_2, 0.5477, 0.3675, 0.2929\rangle, \langle x_3, 0.7071, 0.1633, 0.2254\rangle, \langle x_4, 0.9487, 0.1056, 0\rangle, \langle x_5, 1.0000, 0.0513, 0\rangle\}$;
$A^1 = \{\langle x_1, 0.1, 0.7, 0.8\rangle, \langle x_2, 0.3, 0.6, 0.5\rangle, \langle x_3, 0.5, 0.3, 0.4\rangle, \langle x_4, 0.9, 0.2, 0.0\rangle, \langle x_5, 1.0, 0.1, 0\rangle\}$;
$A^2 = \{\langle x_1, 0.01, 0.91, 0.96\rangle, \langle x_2, 0.09, 0.84, 0.75\rangle, \langle x_3, 0.25, 0.51, 0.64\rangle, \langle x_4, 0.81, 0.36, 0\rangle, \langle x_5, 1.00, 0.19, 0\rangle\}$;
$A^3 = \{\langle x_1, 0.0010, 0.9730, 0.9920\rangle, \langle x_2, 0.0270, 0.9360, 0.8750\rangle, \langle x_3, 0.1250, 0.6570, 0.7840\rangle, \langle x_4, 0.7290, 0.4880, 0\rangle, \langle x_5, 1.000, 0.2710, 0\rangle\}$

The entropy measures values corresponding to existing measures as well as the proposed measures for different values of α are summarized in Table 1 for these different linguistic variable $SVNSs$. From this table, it has been concluded that with the increase of the parameter α, the entropy measure for the linguistic variable "More or Less LARGE", "LARGE', "VERY LARGE" are decreases. Also it has been observed that whenever the values of α are increases from 0 to 15 then the pattern for the variable "LARGE" is $E_\alpha(A) > E_\alpha(A^{1/2}) > E_\alpha(A^2) > E_\alpha(A^3)$ and the results coincides with the existing measures results. On the other hand, whenever the value of α are increases beyond the 15 then the order the patterns are slightly different. Hence the proposed entropy measure is used as an alternative measure for computing the order value of the linguistic variable as compared to existing. Moreover, the proposed measure is more generalized as the different different values of α will give the different choices of the decision-maker for assessing the results, and hence more reliable from linguistic variable point-of-view.

Table 1: Values of different entropy measure for IFS

Entropy measure	$A^{1/2}$	A	A^2	A^3	Ranking
E_{BB}[3]	0.0818	0.100	0.0980	0.0934	(2341)
E_{ZL}[4]	0.4156	0.4200	0.2380	0.1546	(2134)
E_{SK}[5]	0.3446	0.3740	0.1970	0.1309	(2134)
${E_{hc}}^2$[7]	0.3416	0.3440	0.2610	0.1993	(2134)
${E_r}^{1/2}$[7]	0.6672	0.6777	0.5813	0.4805	(2134)
E_{ZJ}[8]	0.2851	0.3050	0.1042	0.0383	(2134)
$E_\alpha(A)$ (Proposed measure)					
$\alpha = 0.3$	0.7548	0.7566	0.6704	0.5774	(2134)
$\alpha = 0.5$	0.7070	0.7139	0.6101	0.5137	(2134)
$\alpha = 0.8$	0.6517	0.6637	0.5579	0.4731	(2134)
$\alpha \to 1$	0.6238	0.6385	0.5372	0.4611	(2134)
$\alpha = 2$	0.5442	0.5727	0.4956	0.4513	(2134)
$\alpha = 5$	0.4725	0.5317	0.4858	0.4793	(2341)
$\alpha = 10$	0.4418	0.5173	0.4916	0.4999	(2431)
$\alpha = 15$	0.4312	0.5112	0.4937	0.5064	(2431)
$\alpha = 50$	0.4166	0.4994	0.4937	0.5064	(4231)
$\alpha = 100$	0.4137	0.4965	0.4612	0.5112	(4231)

4 MCDM problem on proposed entropy measure

In this section, we discuss the method for solving the MCDM problem based on the proposed entropy measure.

4.1 MCDM method based on proposed Entropy measure

Consider the set of different alternatives $A = \{A_1, A_2, ..., A_m\}$ having the different criteria $C = \{C_1, C_2, ..., C_n\}$ in neutrosophic environment and the steps for computing the best alternative is summarized as follows

Step 1: Construction of decision making matrix :
Arrange the each alternatives A_i under the criteria C_j according to preferences of the decision maker in the form of neutrosophic matrix $D_{m\times n} = \langle\mu_{ij}, \nu_{ij}, \rho_{ij}\rangle$ where μ_{ij} represents the degree that alternative A_i satisfies the criteria C_j, ρ_{ij} represents the degree that alternative A_i indeterminant about the criteria C_j and ν_{ij} represents the degree that alternative A_i doesn't satisfies the criteria C_j, where $0 \leq \mu_{ij}, \rho_{ij}, \nu_{ij} \leq 1$ and $\mu_{ij} + \rho_{ij} + \nu_{ij} \leq 3$;

$i = 1, 2, ..., m$; $j = 1, 2, ...n$. The decision matrix given below

$$D_{m\times n}(x_{ij}) = \begin{bmatrix} \langle\mu_{11},\rho_{11},\nu_{11}\rangle & \langle\mu_{12},\rho_{12},\nu_{12}\rangle & \cdots & \langle\mu_{1n},\rho_{1n},\nu_{1n}\rangle \\ \langle\mu_{21},\rho_{21},\nu_{21}\rangle & \langle\mu_{22},\rho_{22},\nu_{22}\rangle & \cdots & \langle\mu_{2n},\rho_{2n},\nu_{2n}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\mu_{m1},\rho_{m1},\nu_{m1}\rangle & \langle\mu_{m2},\rho_{m2},\nu_{m2}\rangle & \cdots & \langle\mu_{mn},\rho_{mn},\nu_{mn}\rangle \end{bmatrix}$$

Step 2: Normalized the decision making : Criterion of alternatives may be of same type or of different types . If the all criterion are of same kind then there is no need of normalization. On the other hand , we should convert the benefit type criterion values to the cost types in C by using the following method-

$$r_{ij} = \begin{cases} \beta^c_{ij}; & j \in B \\ \beta_{ij}; & j \in C \end{cases} \tag{17}$$

where $\beta^c_{ij} = \langle\nu_{ij}, \rho_{ij}, \mu_{ij}\rangle$ is complement of $\beta_{ij} = \langle\mu_{ij}, \rho_{ij}, \nu_{ij}\rangle$. Hence, we obtain the normalized NS decision making $R = [r_{ij}]_{m\times n}$.

Step 3: Compute the aggregated value of the alternatives: By using the proposed entropy measure aggregated the rating values corresponding to each alternatives $A_i(i = 1, 2, ..., m)$ and get the overall value r_i.

Step 4: Rank the Alternatives: Rank all the alternatives $A_i(i = 1, 2, ..., m)$ according to the values of proposed entropy obtained from Step 3 and get the most desirable alternative.

4.2 Illustrative Example

Let us consider multi-criteria decision making problem. There is investment company,which wants to invest a sum of money in best option. There is a panel with four possible alternatives to invest the money, namely

(i) A_1 is food company;

(ii) A_2 is transport company;

(iii) A_3 is an electronic company;

(iv) A_4 is an tyre company.

Decision maker take decision according to three criteria given below:

a) C_1 is growth analysis;

b) C_2 is risk analysis;

c) C_3 is enviroment impact analysis.

Then the following procedure has been followed for computing the best alternative as an investment.

Step 1: The value of an alternative $A_i(i = 1, 2, 3, 4)$ with respect to criteria $C_j(j = 1, 2, 3)$ obtained from questionnaire of domain expert. Thus, when the four possible alternatives with respect to the above three criteria are evaluated by the expert, we obtain the following single valued neutrosophic decision matrix:

$$D = \begin{bmatrix} \langle 0.5, 0.2, 0.3\rangle & \langle 0.5, 0.1, 0.4\rangle & \langle 0.7, 0.1, 0.2\rangle \\ \langle 0.4, 0.2, 0.3\rangle & \langle 0.3, 0.2, 0.4\rangle & \langle 0.8, 0.3, 0.2\rangle \\ \langle 0.4, 0.3, 0.1\rangle & \langle 0.5, 0.1, 0.3\rangle & \langle 0.5, 0.1, 0.4\rangle \\ \langle 0.6, 0.1, 0.2\rangle & \langle 0.2, 0.2, 0.5\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$$

Step 2: Since the criteria C_1 is the benefit criteria and C_2,C_3 are cost criteria, so we above decision matrix transformed into following normalized matrix $R = \langle T_{ij}, I_{ij}, F_{ij}\rangle$ as follows

$$R = \begin{bmatrix} \langle 0.3, 0.2, 0.5\rangle & \langle 0.5, 0.1, 0.4\rangle & \langle 0.7, 0.1, 0.2\rangle \\ \langle 0.3, 0.2, 0.4\rangle & \langle 0.3, 0.2, 0.4\rangle & \langle 0.8, 0.3, 0.2\rangle \\ \langle 0.1, 0.3, 0.4\rangle & \langle 0.5, 0.1, 0.3\rangle & \langle 0.5, 0.1, 0.4\rangle \\ \langle 0.2, 0.1, 0.6\rangle & \langle 0.2, 0.2, 0.5\rangle & \langle 0.4, 0.3, 0.2\rangle \end{bmatrix}$$

Step 3: Utilizing the proposed entropy measure corresponding to $\alpha = 2$ to get the aggregated values r_{ij} of all the alternatives, which are as following $E_\alpha(A_1) = 0.7437$; $E_\alpha(A_2) = 0.8425$; $E_\alpha(A_3) = 0.8092$; $E_\alpha(A_4) = 0.8089$

Step 4: Based on above values, we conclude that ranking of given alternatives is

$$E_\alpha(A_2) > E_\alpha(A_3) > E_\alpha(A_4) > E_\alpha(A_1)$$

Hence, A_2 is best alternative i.e., Investment company should invest in transport company.

5 Conclusion

In this article, we have introduced the entropy measure of order α for single valued neutrosophic numbers for measuring the degree of the fuzziness of the set in which the uncertainties present in the data are characterized into the truth, the indeterminacy and the falsity membership degrees. Some desirable properties corresponding to these entropy have also been illustrated. A structure linguistic variable has been taken as an illustration. Finally, a decision-making method has been proposed based on entropy measures. To demonstrate the efficiency of the proposed coefficients, numerical example from the investment field has been taken. A comparative study as well as the effect of the parameters on the ranking of the alternative will support the theory and hence demonstrate that the proposed measures place an alternative way for solving the decision-making problems.

References

[1] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

[2] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87 – 96.

[3] P. Burillo, H. Bustince, Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets, Fuzzy sets and systems 78 (3) (1996) 305 – 316.

[4] W. Zeng, H. Li, Relationship between similarity measure and entropy of interval-valued fuzzy sets, Fuzzy Sets and Systems 157 (11) (2006) 1477 – 1484.

[5] E. Szmidt, J. Kacprzyk, Entropy for intuitionistic fuzzy sets, Fuzzy Sets and Systems 118 (3) (2001) 467 – 477.

[6] H. Garg, N. Agarwal, A. Tripathi, Generalized intuitionistic fuzzy entropy measure of order α and degree β and its applications to multi-criteria decision making problem, International Journal of Fuzzy System Applications 6 (1) (2017) 86 – 107.

[7] W. L. Hung, M. S. Yang, Fuzzy entropy on intuitionistic fuzzy sets, Intenational Journal of Intelligent Systems 21 (2006) 443 – 451.

[8] Q. S. Zhang, S. Y. Jiang, A note on information entropy measure for vague sets, Information Science 178 (2008) 4184 – 4191.

[9] H. Garg, A new generalized improved score function of interval-valued intuitionistic fuzzy sets and applications in expert systems, Applied Soft Computing 38 (2016) 988 – 999.

[10] H. Garg, A new generalized pythagorean fuzzy information aggregation using einstein operations and its application to decision making, International Journal of Intelligent Systems 31 (9) (2016) 886 – 920.

[11] H. Garg, A novel correlation coefficients between pythagorean fuzzy sets and its applications to decision-making processes, International Journal of Intelligent Systems 31 (12) (2016) 1234 – 1253.

[12] H. Garg, Generalized intuitionistic fuzzy interactive geometric interaction operators using einstein t-norm and t-conorm and their application to decision making, Computer and Industrial Engineering 101 (2016) 53 – 69.

[13] F. Smarandache, A unifying field in logics. Neutrosophy: neutrosophic probability, set and logic, American Research Press, Rehoboth, 1999.

[14] H. Wang, F. Smarandache, Y. Q. Zhang, R. Sunderraman, Single valued neutrosophic sets, Multispace Multistructure 4 (2010) 410 – 413.

[15] J. Ye, Multiple attribute group decision-making method with completely unknown weights based on similarity measures under single valued neutrosophic environment, Journal of Intelligent and Fuzzy Systems 27 (6) (2014) 2927 – 2935.

[16] P. Majumdar, S. K. Samant, On similarity and entropy of neutrosophic sets, Journal of Intelligent and Fuzzy Systems 26 (3) (2014) 1245 – 1252.

[17] J. Ye, Similarity measures between interval neutrosophic sets and their applications in multicriteria decision-making, International Journal of Intelligent Fuzzy Systems 26 (1) (2014) 165 – 172.

[18] H. L. Huang, New distance measure of single-valued neutrosophic sets and its application, International Journal of Intelligent Systems 31 (10) (2016) 1021 – 1032.

[19] S. Broumi, F. Smarandache, Correlation coefficient of interval neutrosophic set, Applied Mechanics and Material 436 (2013) 511 – 517.

[20] I. Hanafy, A. Salama, K. Mahfouz, Correlation coefficients of neutrosophic sets by centroid method, International Journal of Probability and Statistics 2 (1) (2013) 9–12.

[21] S. Broumi, F. Smarandache, Cosine similarity measure of interval valued neutrosophic sets, Neutrosophic Sets and Systems 5 (2014) 15 – 20.

[22] J. Ye, Improved cosine similarity measures of simplified neutrosophic sets for medical diagnoses, Artificial Intelligence in Medicine 63 (2015) 171 – 179.

[23] H. Y. Zhang, J. Q. Wang, X. H. Chen. Interval neutrosophic sets and their application in multicriteria decision making problems, The Scientific World Journal Volume 2014 (2014) Article ID 645953, 15 pages.

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Fuzzy Logic vs. Neutrosophic Logic: Operations Logic

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Abstract. The goal of this research is first to show how different, thorough, widespread and effective are the operations logic of the neutrosophic logic compared to the fuzzy logic's operations logical. The second aim is to observe how a fully new logic, the neutrosophic logic, is established starting by changing the previous logical perspective fuzzy logic, and by changing that, we mean changing the truth values from the truth and falsity degrees membership in fuzzy logic, to the truth, falsity and indeterminacy degrees membership in neutrosophic logic; and thirdly, to observe that there is no limit to the logical discoveries - we only change the principle, then the system changes completely.

Keywords: Fuzzy Logic, Neutrosophic Logic, Logical Connectives, Operations Logic, New Logic.

1 Introduction:

There is no doubt in the fact that the mathematical logic as an intellectual practice has not been far from contemplation and the philosophical discourse, and disconnecting it from philosophy seems to be more of a systematic disconnection than a real one, because throughout the history of philosophy, the philosophers and what they have built as intellectual landmark, closed or opened, is standing on a logical foundation even if it did not come out as a symbolic mathematical logic.

Since the day Aristotle established the first logic theory which combines the first rules of the innate conclusion mechanism of the human being, it was a far-reaching step-forward to all those who came after him up till today, and that led to the epiphany that : the universe with all its physical and metaphysical notions is in fact a logical structure that needs an incredible accuracy in abstraction to show it for the beauty of the different notions in it, and the emotional impressions it makes in the common sense keeps the brain from the real perception of its logical structure. Many scientists and philosophers paid attention to the matter which is reflected in the variety and the difference of the systems, the logical references and mathematics in the different scientific fields. Among these scientists and philosophers who have strived to find this logical structure are: Professor Lotfi A. Zadeh, founder of the fuzzy logic (FL) idea, which he established in 1965 [7], and Professor Florentin Smarandache, founder of the neutrosophic logic (NL) idea, which he established in 1995 [1]. In this research and using the logical operations only of the two theories that we have sampled from the two systems, we will manage to observe which one is wider and more comprehensive to express more precisely the hidden logical structure of the universe.

2 Definition of Fuzzy and Neutrosophic Logical Connectives (Operations Logic):

The connectives (rules of inference, or operators), in any non-bivalent logic, can be defined in various ways, giving rise to lots of distinct logics. A single change in one of any connective's truth table is enough to form a (completely) different logic [2]. For example, Fuzzy Logic and Neutrosophic Logic.

2.1 One notes the fuzzy logical values of the propositions (A) and $(B)$by:

$$FL(A) = (T_A, F_A)\text{ , and } FL(B) = (T_B, F_B)$$

A fuzzy propositions (A) and (B) are real standard subsets in universal set(U), which is characterized by a truth-membership function T_A , T_B , and a falsity-membership function F_A , F_B, of $[0,1]$. That is

$$T_A : U \rightarrow [0,1]$$
$$F_A : U \rightarrow [0,1]$$
And
$$T_B : U \rightarrow [0,1]$$
$$F_B : U \rightarrow [0,1]$$

There is no restriction on the sum of T_A, F_A or T_B, F_B, so $0 \leq supT_A + supF_A \leq 1$, and $0 \leq supT_B + supF_B \leq 1$.

2.2 Two notes the neutrosophic logical values of the propositions (A) and (B) by[2]:

$$NL(A) = (T_A, I_A, F_A)\text{ , and } NL(B) = (T_B, I_B, F_B)$$

A neutrosophic propositions (A) and (B) are real standard or non-standard subsets in universal set(U), which is characterized by a truth-membership function T_A , T_B , a indeterminacy-membership function I_A , I_B and a falsity-membership function F_A , F_B, of $]^{-}0,1^{+}[$. That is

$$T_A : U \rightarrow]^{-}0,1^{+}[$$
$$I_A : U \rightarrow]^{-}0,1^{+}[$$
$$F_A : U \rightarrow]^{-}0,1^{+}[$$
And
$$T_B : U \rightarrow]^{-}0,1^{+}[$$
$$I_B : U \rightarrow]^{-}0,1^{+}[$$
$$F_B : U \rightarrow]^{-}0,1^{+}[$$

There is no restriction on the sum of T_A, I_A, F_A or T_B, I_B, F_B, so $^{-}0 \le supT_A + supI_A + supF_A \le 3^{+}$, and $^{-}0 \le supT_B + supI_B + supF_B \le 3^{+}$.[3]

2.3 Negation:

2.3.1 In Fuzzy Logic:

Negation the fuzzy propositions (A) and (B) is the following :

$$FL(\neg A) = (\{1\} - T_A, \{1\} - F_A)$$
And
$$FL(\neg B) = (\{1\} - T_B, \{1\} - F_B)$$

The negation link of the two fuzzy propositions (A) and (B) in the following truth table [6]:

(1,0)	(1,0)	(0,1)	(0,1)
(1,0)	(0,1)	(0,1)	(1,0)
(0,1)	(1,0)	(1,0)	(0,1)
(0,1)	(0,1)	(1,0)	(1,0)

2.3.2 In Neutrosophic Logic:

Negation the neutrosophic propositions (A) and (B) is the following [4]:

$$NL(\neg A) = (\{1\} \ominus T_A, \{1\} \ominus I_A, \{1\} \ominus F_A)$$
And
$$NL(\neg B) = (\{1\} \ominus T_B, \{1\} \ominus I_B, \{1\} \ominus F_B)$$

The negation link of the two neutrosophic propositions (A) and (B) in the following truth table :

A	B	$\neg A$	$\neg B$
(1,0,0)	(1,0,0)	(0,1,1)	(0,1,1)
(1,0,0)	(0,0,1)	(0,1,1)	(1,1,0)
(0,0,1)	(0,1,0)	(1,1,0)	(1,0,1)
(0,0,1)	(1,0,0)	(1,1,0)	(0,1,1)
(0,1,0)	(0,0,1)	(1,0,1)	(1,1,0)
(0,1,0)	(0,1,0)	(1,0,1)	(1,0,1)

2.4 Conjunction :

2.4.1 In Fuzzy Logic:

Conjunction the fuzzy propositions (A) and (B) is the following :

$$FL(A \wedge B) = (T_A \cdot T_B, F_A \cdot F_B)$$

(And, in similar way, generalized for n propositions)

The conjunction link of the two fuzzy propositions (A) and (B) in the following truth table [6] :

A	B	$A \wedge B$
(1,0)	(1,0)	(1,0)
(1,0)	(0,1)	(0,0)
(0,1)	(1,0)	(0,0)
(0,1)	(0,1)	(0,1)

2.4.2 In Neutrosophic Logic:

Conjunction the neutrosophic propositions (A) and (B) is the following [5]:

$$NL(A \wedge B) = (T_A \odot T_B, I_A \odot I_B, F_A \odot F_B)$$

(And, in similar way, generalized for n propositions)

The conjunction link of the two neutrosophic propositions (A) and (B) in the following truth table :

A	B	$A \wedge B$
(1,0,0)	(1,0,0)	(1,0,0)
(1,0,0)	(0,0,1)	(0,0,0)
(0,0,1)	(0,1,0)	(0,0,0)
(0,0,1)	(1,0,0)	(0,0,0)
(0,1,0)	(0,0,1)	(0,0,0)
(0,1,0)	(0,1,0)	(0,1,0)

2.5 Weak or inclusive disjunction:

2.5.1 In Fuzzy Logic:

Inclusive disjunction the fuzzy propositions (A) and (B) is the following :

$$FL(A \vee B) = \big((T_A + T_B) - (T_A \cdot T_B), (F_A + F_B) - (F_A \cdot F_B)\big)$$

(And, in similar way, generalized for n propositions)

The inclusive disjunction link of the two fuzzy propositions (A) and (B) in the following truth table [6]:

A	B	$A \vee B$
(1,0)	(1,0)	(1,0)
(1,0)	(0,1)	(1,1)
(0,1)	(1,0)	(1,1)
(0,1)	(0,1)	(0,1)

2.5.2 In Neutrosophic Logic:

Inclusive disjunction the neutrosophic propositions (A) and (B) is the following [4]:

$$NL(A \vee B) = (T_A \oplus T_B \ominus T_A \odot T_B, I_A \oplus I_B \ominus I_A \odot I_B, F_A \oplus F_B \ominus F_A \odot F_B)$$

(And, in similar way, generalized for n propositions)

The inclusive disjunction link of the two neutrosophic propositions (A) and (B) in the following truth table :

A	B	$A \vee B$
$(1,0,0)$	$(1,0,0)$	$(1,0,0)$
$(1,0,0)$	$(0,0,1)$	$(1,0,1)$
$(0,0,1)$	$(0,1,0)$	$(0,1,1)$
$(0,0,1)$	$(1,0,0)$	$(1,0,1)$
$(0,1,0)$	$(0,0,1)$	$(0,1,1)$
$(0,1,0)$	$(0,1,0)$	$(0,1,0)$

2.6Strong or exclusive disjunction:

2.6.1 In Fuzzy Logic:

Exclusive disjunction the fuzzy propositions (A) and (B) is the following :

$$\boldsymbol{FL(A \vee\!\vee B)} = \begin{pmatrix} (\boldsymbol{T_A} \cdot (\{\boldsymbol{1}\} - \boldsymbol{T_B}) + \boldsymbol{T_B} \cdot (\{\boldsymbol{1}\} - \boldsymbol{T_A}) - \boldsymbol{T_A} \cdot \boldsymbol{T_B} \cdot (\{\boldsymbol{1}\} - \boldsymbol{T_A}) \cdot (\{\boldsymbol{1}\} - \boldsymbol{T_B}), \\ (\boldsymbol{F_A} \cdot (\{\boldsymbol{1}\} - \boldsymbol{F_B}) + \boldsymbol{F_B} \cdot (\{\boldsymbol{1}\} - \boldsymbol{F_A}) - \boldsymbol{F_A} \cdot \boldsymbol{F_B} \cdot (\{\boldsymbol{1}\} - \boldsymbol{F_A}) \cdot (\{\boldsymbol{1}\} - \boldsymbol{F_B}) \end{pmatrix}$$

(And, in similar way, generalized for n propositions)

The exclusive disjunction link of the two fuzzy propositions (A) and (B) in the following truth table [6]:

A	B	$A \vee\!\vee B$
$(1,0)$	$(1,0)$	$(0,0)$
$(1,0)$	$(0,1)$	$(1,1)$
$(0,1)$	$(1,0)$	$(1,1)$
$(0,1)$	$(0,1)$	$(0,0)$

2.6.2 In Neutrosophic Logic:

Exclusive disjunction the neutrosophic propositions (A) and (B) is the following [5]:

$$\boldsymbol{NL(A \vee\!\vee B)} = \begin{pmatrix} (\boldsymbol{T_A} \odot (\{\boldsymbol{1}\} \ominus \boldsymbol{T_B}) \oplus \boldsymbol{T_B} \odot (\{\boldsymbol{1}\} \ominus \boldsymbol{T_A}) \ominus \boldsymbol{T_A} \odot \boldsymbol{T_B} \odot (\{\boldsymbol{1}\} \ominus \boldsymbol{T_A}) \odot (\{\boldsymbol{1}\} \ominus \boldsymbol{T_B}), \\ (\boldsymbol{I_A} \odot (\{\boldsymbol{1}\} \ominus \boldsymbol{I_B}) \oplus \boldsymbol{I_B} \odot (\{\boldsymbol{1}\} \ominus \boldsymbol{I_A}) \ominus \boldsymbol{I_A} \odot \boldsymbol{I_B} \odot (\{\boldsymbol{1}\} \ominus \boldsymbol{I_A}) \odot (\{\boldsymbol{1}\} \ominus \boldsymbol{I_B}), \\ (\boldsymbol{F_A} \odot (\{\boldsymbol{1}\} \ominus \boldsymbol{F_B}) \oplus \boldsymbol{F_B} \odot (\{\boldsymbol{1}\} \ominus \boldsymbol{F_A}) \ominus \boldsymbol{F_A} \odot \boldsymbol{F_B} \odot (\{\boldsymbol{1}\} \ominus \boldsymbol{F_A}) \odot (\{\boldsymbol{1}\} \ominus \boldsymbol{F_B}) \end{pmatrix}$$

(And, in similar way, generalized for n propositions)

The exclusive disjunction link of the two neutrosophic propositions (A) and (B) in the following truth table :

A	B	$A \vee\!\vee B$
$(1,0,0)$	$(1,0,0)$	$(0,0,0)$
$(1,0,0)$	$(0,0,1)$	$(1,0,1)$
$(0,0,1)$	$(0,1,0)$	$(0,1,1)$
$(0,0,1)$	$(1,0,0)$	$(1,0,1)$
$(0,1,0)$	$(0,0,1)$	$(0,1,1)$
$(0,1,0)$	$(0,1,0)$	$(0,0,0)$

2.7 Material conditional (implication) :

2.7.1 In Fuzzy Logic:

Implication the fuzzy propositions (A) and (B) is the following :

$$FL(A \to B) = (\{1\} - T_A + T_A \cdot T_B, \{1\} - F_A + F_A \cdot F_B)$$

The implication link of the two fuzzy propositions (A) and (B) in the following truth table [6]:

A	B	$A \to B$
$(1,0)$	$(1,0)$	$(1,0)$
$(1,0)$	$(0,1)$	$(0,1)$
$(0,1)$	$(1,0)$	$(1,0)$
$(0,1)$	$(0,1)$	$(0,1)$

2.7.2 In Neutrosophic Logic:

Implication the neutrosophic propositions (A) and (B) is the following [4]:

$$\boldsymbol{NL(A \to B)} = (\{\boldsymbol{1}\} \ominus \boldsymbol{T_A} \oplus \boldsymbol{T_A} \odot \boldsymbol{T_B}, \{\boldsymbol{1}\} \ominus \boldsymbol{I_A} \oplus \boldsymbol{I_A} \odot \boldsymbol{I_B}, \{\boldsymbol{1}\} \ominus \boldsymbol{F_A} \oplus \boldsymbol{F_A} \odot \boldsymbol{F_B})$$

The implication link of the two neutrosophic propositions (A) and (B) in the following truth table :

A	B	$A \to B$
$(1,0,0)$	$(1,0,0)$	$(1,1,1)$
$(1,0,0)$	$(0,0,1)$	$(0,1,1)$
$(0,0,1)$	$(0,1,0)$	$(1,1,0)$
$(0,0,1)$	$(1,0,0)$	$(1,1,0)$
$(0,1,0)$	$(0,0,1)$	$(1,0,1)$
$(0,1,0)$	$(0,1,0)$	$(1,1,1)$

2.8 Material biconditional (equivalence) :

2.8.1 In Fuzzy Logic:

Equivalencethe fuzzy propositions (A) and (B) is the following :

$$FL(A \leftrightarrow B) = \begin{pmatrix} ((\{1\} - T_A + T_A \cdot T_B) \cdot (\{1\} - T_B + T_A \cdot T_B)), \\ ((\{1\} - F_A + F_A \cdot F_B) \cdot (\{1\} - F_B + F_A \cdot F_B)) \end{pmatrix}$$

The equivalence link of the two fuzzy propositions (A) and (B) in the following truth table :

A	B	$A \leftrightarrow B$
$(1,0)$	$(1,0)$	$(1,1)$
$(1,0)$	$(0,1)$	$(0,0)$
$(0,1)$	$(1,0)$	$(0,0)$
$(0,1)$	$(0,1)$	$(1,1)$

2.8.2 In Neutrosophic Logic:

Equivalencethe neutrosophic propositions (A) and (B) is the following [5]:

$$NL(A \leftrightarrow B) = \begin{pmatrix} ((\{1\} \ominus T_A \oplus T_A \odot T_B) \odot (\{1\} \ominus T_B \oplus T_A \odot T_B)), \\ ((\{1\} \ominus I_A \oplus I_A \odot I_B) \odot (\{1\} \ominus I_B \oplus I_A \odot I_B)), \\ ((\{1\} \ominus F_A \oplus F_A \odot F_B) \odot (\{1\} \ominus F_B \oplus F_A \odot F_B)) \end{pmatrix}$$

The equivalence link of the two neutrosophic propositions (A) and (B) in the following truth table :

A	B	$A \leftrightarrow B$
$(1,0,0)$	$(1,0,0)$	$(1,1,1)$
$(1,0,0)$	$(0,0,1)$	$(0,1,0)$
$(0,0,1)$	$(0,1,0)$	$(1,0,0)$
$(0,0,1)$	$(1,0,0)$	$(0,1,0)$
$(0,1,0)$	$(0,0,1)$	$(1,0,0)$
$(0,1,0)$	$(0,1,0)$	$(1,1,1)$

2.9 Sheffer's connector:

2.9.1 In Fuzzy Logic:

The result of the sheffer's connector between the two fuzzy propositions (A) and (B) :

$$FL(A|\,B) = FL(\neg A \vee \neg B) = (\{1\} - T_A \cdot T_B, \{1\} - F_A \cdot F_B)$$

The result of the sheffer's connector between the two fuzzy propositions (A) and (B) in the following truth table :

A	B	$\neg A$	$\neg B$	$\neg A \vee \neg B$	$A\|B$
(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(0,1)
(1,0)	(0,1)	(0,1)	(1,0)	(1,1)	(1,1)
(0,1)	(1,0)	(1,0)	(0,1)	(1,1)	(1,1)
(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)

2.9.2 In Neutrosophic Logic:

The result of the sheffer's connector between the two neutrosophic propositions (A) and (B)[4]:

$$\boldsymbol{NL(A|\,B) = NL(\neg A \vee \neg B) = (\{1\} \ominus T_A \odot T_B, \{1\} \ominus I_A \odot I_B, \{1\} \ominus F_A \odot F_B)}$$

The result of the sheffer's connector between the two neutrosophic propositions (A) and (B) in the following truth table :

A	B	$\neg A$	$\neg B$	$\neg A \vee \neg B$	$A\|B$
(1,0,0)	(1,0,0)	(0,1,1)	(0,1,1)	(0,1,1)	(0,1,1)
(1,0,0)	(0,0,1)	(0,1,1)	(1,1,0)	(1,1,1)	(1,1,1)
(0,0,1)	(0,1,0)	(1,1,0)	(1,0,1)	(1,1,1)	(1,1,1)
(0,0,1)	(1,0,0)	(1,1,0)	(0,1,1)	(1,1,1)	(1,1,1)
(0,1,0)	(0,0,1)	(1,0,1)	(1,1,0)	(1,1,1)	(1,1,1)
(0,1,0)	(0,1,0)	(1,0,1)	(1,0,1)	(1,0,1)	(1,0,1)

2.10 Peirce's connector:

2.10.1 In Fuzzy Logic:

The result of the Peirce's connectorbetween the two fuzzy propositions $(A)$and (B) :

$$\boldsymbol{FL(A \downarrow B) = FL(\neg A \wedge \neg B) = \big((\{1\} - T_A) \cdot (\{1\} - T_B), (\{1\} - F_A) \cdot (\{1\} - F_B)\big)}$$

The result of the peirce's connectorbetween the two fuzzy propositions (A) and (B) in the following truth table :

A	B	$\neg A$	$\neg B$	$\neg A \wedge \neg B$	$A \downarrow B$
(1,0)	(1,0)	(0,1)	(0,1)	(0,1)	(0,1)
(1,0)	(0,1)	(0,1)	(1,0)	(0,0)	(0,0)
(0,1)	(1,0)	(1,0)	(0,1)	(0,0)	(0,0)
(0,1)	(0,1)	(1,0)	(1,0)	(1,0)	(1,0)

2.10.2 In Neutrosophic Logic:

The result of the Peirce's connectorbetween the two neutrosophic propositions (A) and (B)[5]:

$$\boldsymbol{NL(A \downarrow B) = NL(\neg A \wedge \neg B) = \big((\{1\} \ominus T_A) \odot (\{1\} \ominus T_B), (\{1\} \ominus I_A) \odot (\{1\} \ominus I_B), (\{1\} \ominus F_A) \odot (\{1\} \ominus F_B)\big)}$$

The result of the peirce's connectorbetween the two neutrosophic propositions (A) and (B) in the following truth table :

A	B	$\neg A$	$\neg B$	$\neg A \wedge \neg B$	$A \downarrow B$
(1,0,0)	(1,0,0)	(0,1,1)	(0,1,1)	(0,1,1)	(0,1,1)
(1,0,0)	(0,0,1)	(0,1,1)	(1,1,0)	(0,1,0)	(0,1,0)
(0,0,1)	(0,1,0)	(1,1,0)	(1,0,1)	(1,0,1)	(1,0,1)
(0,0,1)	(1,0,0)	(1,1,0)	(0,1,1)	(0,1,0)	(0,1,0)
(0,1,0)	(0,0,1)	(1,0,1)	(1,1,0)	(1,0,0)	(1,0,0)
(0,1,0)	(0,1,0)	(1,0,1)	(1,0,1)	(1,0,1)	(1,0,1)

3 Conclusion :

From what has been discussed previously, we can ultimately reach three points :

3.1 We see that the logical operations of the neutrosophic logic (NL) are different from the logical operations of the fuzzy logic (FL) in terms of width, comprehensiveness and effectiveness. The reason behind that is the addition of professor Florentin Smarandache of anew field to the real values, the truth and falsity interval in (FL) and that is what he called « the indeterminacy interval » which is expressed in the function I_A or I_B in the logical operations of (NL) as we have seen, and that is what makes (NL) the closest and most precise image of the hidden logical structure of the universe.

3.2 We see that (NL) is a fully new logic, that has been established starting by changing a principle (FL), we mean by this principle changing the real values of the truth and falsity membership degrees only in (FL) to the truth and indeterminacy then falsity membership degrees in (NL).

3.3 We see that there is no limit to the logical discoveries, we only have to change the principle and that leads to completely change the system. So what if we also change the truth values from the truth and indeterminacy and falsity membership degrees in (NL), and that is by **doubling** it, as follows :

The neutrosophic propositions (A) is real standard or non-standard subsets in universal set(U), which is characterized by a truth-membership function T_A , a indeterminacy-membership function I_A , and a falsity-membership function F_A , of $]\,^{-}0, 1^{+}[$. That is

$$T_A : U \rightarrow]\,^{-}0, 1^{+}[$$
$$I_A : U \rightarrow]\,^{-}0, 1^{+}[$$
$$F_A : U \rightarrow]\,^{-}0, 1^{+}[$$

Let T_A , is real standard or non-standard subset in universal set(U), which is characterized by a truth-truth membership function T_{T_A} , a indeterminacy-truth membership function I_{T_A} , and a falsity-truth membership function F_{T_A} , of $]\,^{-}0, 1^{+}[$. That is

$$T_{T_A} : U \rightarrow]\,^{-}0, 1^{+}[$$
$$I_{T_A} : U \rightarrow]\,^{-}0, 1^{+}[$$
$$F_{T_A} : U \rightarrow]\,^{-}0, 1^{+}[$$

There is no restriction on the sum of $T_{T_A}, I_{T_A}, F_{T_A}$, so $^{-}0 \leq supT_{T_A} + supI_{T_A} + supF_{T_A} \leq 3^{+}$.

Let I_A , is real standard or non-standard subset in universal set(U), which is characterized by a truth-indeterminacy membership function T_{I_A} , a indeterminacy-indeterminacy membership function I_{I_A} , and a falsity-indeterminacy membership function F_{I_A} , of $]^{-}0, 1^{+}[$. That is

$$T_{I_A} : U \to]^{-}0, 1^{+}[$$
$$I_{I_A} : U \to]^{-}0, 1^{+}[$$
$$F_{I_A} : U \to]^{-}0, 1^{+}[$$

There is no restriction on the sum of $T_{I_A}, I_{I_A}, F_{I_A}$, so $^{-}0 \leq supT_{I_A} + supI_{I_A} + supF_{I_A} \leq 3^{+}$.

Let F_A , is real standard or non-standard subset in universal set(U), which is characterized by a truth-falsity membership function T_{F_A} , a indeterminacy-falsity membership function I_{F_A} , and a falsity-falsity membership function F_{F_A} , of $]^{-}0, 1^{+}[$. That is

$$T_{F_A} : U \to]^{-}0, 1^{+}[$$
$$I_{F_A} : U \to]^{-}0, 1^{+}[$$
$$F_{F_A} : U \to]^{-}0, 1^{+}[$$

There is no restriction on the sum of $T_{F_A}, I_{F_A}, F_{F_A}$,so $^{-}0 \leq supT_{F_A} + supI_{F_A} + supF_{F_A} \leq 3^{+}$.

Therefore :

$$T_{TA} + I_{T_A} + F_{T_A} : U \to]^{-}0, 3^{+}[$$

$$T_{I_A} + I_{I_A} + F_{I_A} : U \to]^{-}0, 3^{+}[$$

$$T_{F_A} + I_{F_A} + F_{F_A} : U \to]^{-}0, 3^{+}[$$

There is no restriction on the sum of $T_{T_A}, I_{T_A}, F_{T_A}$, and of$T_{I_A}, I_{I_A}, F_{I_A}$, and of$T_{F_A}, I_{F_A}, F_{F_A}$, so $^{-}0 \leq supT_{T_A} + supI_{T_A} + supF_{T_A} + supT_{I_A} + supI_{I_A} + supF_{I_A} + supT_{F_A} + supI_{F_A} + supF_{F_A} \leq 9^{+}$.

Therefore :

$$(T_{T_A}, I_{T_A}, F_{T_A}), (T_{I_A}, I_{I_A}, F_{I_A}), (T_{F_A}, I_{F_A}, F_{F_A})): U \to]^{-}0, 1^{+}[\text{^}9$$

This example: we suggest to be named: **Double Neutrosophic Logic** (**DNL**).

This is a particular case of Neutrosophic Logic and Set of Type-2 (and Type-n), introduced by Smarandache [8] in 2017, as follows:

> *“Definition of Type-2 (and Type-n) Neutrosophic Set (and Logic).*
>
> Type-2 Neutrosophic Set is actually a neutrosophic set of a neutrosophic set.
>
> See an example for a type-2 single-valued neutrosophic set below:
>
> Let x(0.4 <0.3, 0.2, 0.4>, 0.1 <0.0, 0.3, 0.8>, 0.7 <0.5, 0.2, 0.2>) be an element in the neutrosophic set A, which means the following: x(0.4, 0.1, 0.7) belongs to the neutrosophic set A in the following way, the truth value of x is 0.4, the indeterminacy value of x is 0.1, and the falsity value of x is 0.7 [this is type-1 neutrosophic set]; but the neutrosophic probability that the truth value of x is 0.4 with respect to the neutrosophic set A is <0.3, 0.2, 0.4>, the neutrosophic probability that the indeterminacy value of x is 0.1 with respect to the neutrosophic set A is <0.0, 0.3, 0.8>, and the neutrosophic probability that the falsity value of x is 0.7 with respect to the neutrosophic set A is <0.5, 0.2, 0.2> [now this is type-2 neutrosophic set].
>
> So, in a type-2 neutrosophic set, when an element x(t, i, f) belongs to a neutrosophic set A, we are not sure about the values of t, i, f, we only get each of them with a given neutrosophic probability.
>
> Neutrosophic Probability (NP) of an event E is defined as: NP(E) = (chance that E occurs, indeterminate chance about E occurrence, chance that E does not occur).
> Similarly, a type-2 fuzzy set is a fuzzy set of a fuzzy set.
> And a type-2 intuitionistic fuzzy set is an intuitionistic fuzzy set of an intuitionistic fuzzy set.
> Surely, one can define a type-3 neutrosophic set (which is a neutrosophic set of a neutrosophic set of a neutrosophic set), and so on (*type-n neutrosophic set*, for n ≥ 2), but they become useless and confusing.
> Neither in fuzzy set nor in intuitionistic fuzzy set the researchers went further that type-2.“

Hence : $(FL) \to (NL) \to (\boldsymbol{DNL}) \to NLn.$

Especially in quantum theory, there is an uncertainty about the energy and the momentum of particles. And, because the particles in the subatomic world don’t have exact positions, we better calculate their **double neutrosophic probabilities** (i.e. computation a truth-truth percent, indeterminacy-truth percent, falsity-truth percent, and truth-indeterminacy percent, indeterminacy-indeterminacy percent, falsity-indeterminacy percent, and truth-falsity percent, indeterminacy-falsity percent, falsity-falsity percent) of being at some particular points than their neutrosophic probabilities.

3.4 Definition of Double Neutrosophic Logical Connectives (Operations Logic) :

One notes the double neutrosophic logical values of the propositions (A) and (B) by:

$$DNL(A) = \left((T_{T_A}, I_{T_A}, F_{T_A}), (T_{I_A}, I_{I_A}, F_{I_A}), (T_{F_A}, I_{F_A}, F_{F_A})\right)$$

And

$$DNL(B) = \left((T_{T_B}, I_{T_B}, F_{T_B}), (T_{I_B}, I_{I_B}, F_{I_B}), (T_{F_B}, I_{F_B}, F_{F_B})\right)$$

3.4.1 Negation:

$$\boldsymbol{DNL(\neg A)} =$$
$$\left((\{1\} \ominus \boldsymbol{T_{T_A}}, \{1\} \ominus \boldsymbol{I_{T_A}}\{1\} \ominus \boldsymbol{F_{T_A}}), (\{1\} \ominus \boldsymbol{T_{I_A}}, \{1\} \ominus \boldsymbol{I_{I_A}}\{1\} \ominus \boldsymbol{F_{I_A}}), (\{1\} \ominus \boldsymbol{T_{F_A}}, \{1\} \ominus \boldsymbol{I_{F_A}}\{1\} \ominus \boldsymbol{F_{F_A}})\right)$$

And

$$\boldsymbol{DNL(\neg B)} =$$
$$\left((\{1\} \ominus \boldsymbol{T_{T_B}}, \{1\} \ominus \boldsymbol{I_{T_B}}\{1\} \ominus \boldsymbol{F_{T_B}}), (\{1\} \ominus \boldsymbol{T_{I_B}}, \{1\} \ominus \boldsymbol{I_{I_B}}\{1\} \ominus \boldsymbol{F_{I_B}}), (\{1\} \ominus \boldsymbol{T_{F_B}}, \{1\} \ominus \boldsymbol{I_{F_B}}\{1\} \ominus \boldsymbol{F_{F_B}})\right)$$

3.4.2 Conjunction :

$$DNL(A \wedge B) =$$

$$(T_{T_A} \odot T_{T_B}, I_{T_A} \odot I_{T_B}, F_{T_A} \odot F_{T_B}), (T_{I_A} \odot T_{I_B}, I_{I_A} \odot I_{I_B}, F_{I_A} \odot F_{I_B}), (T_{F_A} \odot T_{F_B}, I_{F_A} \odot I_{F_B}, F_{F_A} \odot F_{F_B})$$

(And, in similar way, generalized for n propositions)

3.4.3 Weak or inclusive disjunction :

$$DNL(A \vee B) =$$

$$\left.\begin{array}{c} (T_{T_A} \oplus T_{T_B} \ominus T_{T_A} \odot T_{T_B}, I_{T_A} \oplus I_{T_B} \ominus I_{T_A} \odot I_{T_B}, F_{T_A} \oplus F_{T_B} \ominus F_{T_A} \odot F_{T_B}), \\ (T_{I_A} \oplus T_{I_B} \ominus T_{I_A} \odot T_{I_B}, I_{I_A} \oplus I_{I_B} \ominus I_{I_A} \odot I_{I_B}, F_{I_A} \oplus F_{I_B} \ominus F_{I_A} \odot F_{I_B}), \\ (T_{F_A} \oplus T_{F_B} \ominus T_{F_A} \odot T_{F_B}, I_{F_A} \oplus I_{F_B} \ominus I_{F_A} \odot I_{F_B}, F_{F_A} \oplus F_{F_B} \ominus F_{F_A} \odot F_{F_B}) \end{array}\right)$$

(And, in similar way, generalized for n propositions)

3.4.4 Strong or exclusive disjunction :

$$DNL(A \veebar B) =$$

$$\left(\begin{array}{c} \left(\begin{array}{c} (T_{T_A} \odot (\{1\} \ominus T_{T_B}) \oplus T_{T_B} \odot (\{1\} \ominus T_{T_A}) \ominus T_{T_A} \odot T_{T_B} \odot (\{1\} \ominus T_{T_A}) \odot (\{1\} \ominus T_{T_B}), \\ (I_{T_A} \odot (\{1\} \ominus I_{T_B}) \oplus I_{T_B} \odot (\{1\} \ominus I_{T_A}) \ominus I_{T_A} \odot I_{T_B} \odot (\{1\} \ominus I_{T_A}) \odot (\{1\} \ominus I_{T_B}), \\ F_{T_A} \odot (\{1\} \ominus F_{T_B}) \oplus F_{T_B} \odot (\{1\} \ominus F_{T_A}) \ominus F_{T_A} \odot F_{T_B} \odot (\{1\} \ominus F_{T_A}) \odot (\{1\} \ominus F_{T_B} \end{array}\right), \\ \left(\begin{array}{c} T_{I_A} \odot (\{1\} \ominus T_{I_B}) \oplus T_{I_B} \odot (\{1\} \ominus T_{I_A}) \ominus T_{I_A} \odot T_{I_B} \odot (\{1\} \ominus T_{I_A}) \odot (\{1\} \ominus T_{I_B}), \\ I_{I_A} \odot (\{1\} \ominus I_{I_B}) \oplus I_{I_B} \odot (\{1\} \ominus I_{I_A}) \ominus I_{I_A} \odot I_{I_B} \odot (\{1\} \ominus I_{I_A}) \odot (\{1\} \ominus I_{I_B}), \\ F_{I_A} \odot (\{1\} \ominus F_{I_B}) \oplus F_{I_B} \odot (\{1\} \ominus F_{I_A}) \ominus F_{I_A} \odot F_{I_B} \odot (\{1\} \ominus F_{I_A}) \odot (\{1\} \ominus F_{I_B} \end{array}\right), \\ \left(\begin{array}{c} (T_{F_A} \odot (\{1\} \ominus T_{F_B}) \oplus T_{F_B} \odot (\{1\} \ominus T_{F_A}) \ominus T_{F_A} \odot T_{F_B} \odot (\{1\} \ominus T_{F_A}) \odot (\{1\} \ominus T_{F_B}), \\ (I_{F_A} \odot (\{1\} \ominus I_{F_B}) \oplus I_{F_B} \odot (\{1\} \ominus I_{F_A}) \ominus I_{F_A} \odot I_{F_B} \odot (\{1\} \ominus I_{F_A}) \odot (\{1\} \ominus I_{F_B}), \\ (F_{F_A} \odot (\{1\} \ominus F_{F_B}) \oplus F_{F_B} \odot (\{1\} \ominus F_{F_A}) \ominus F_{F_A} \odot F_{F_B} \odot (\{1\} \ominus F_{F_A}) \odot (\{1\} \ominus F_{F_B} \end{array}\right) \end{array}\right)$$

(And, in similar way, generalized for n propositions)

3.4.5 Material conditional (implication) :

$$DNL(A \rightarrow B) =$$

$$\left(\begin{array}{c} \{1\} \ominus T_{T_A} \oplus T_{T_A} \odot T_{T_B}, \{1\} \ominus I_{T_A} \oplus I_{T_A} \odot I_{T_B}, \{1\} \ominus F_{T_A} \oplus F_{T_A} \odot F_{T_B}), \\ \{1\} \ominus T_{I_A} \oplus T_{I_A} \odot T_{I_B}, \{1\} \ominus I_{I_A} \oplus I_{I_A} \odot I_{I_B}, \{1\} \ominus F_{I_A} \oplus F_{I_A} \odot F_{I_B}), \\ \{1\} \ominus T_{F_A} \oplus T_{F_A} \odot T_{F_B}, \{1\} \ominus I_{F_A} \oplus I_{F_A} \odot I_{F_B}, \{1\} \ominus F_{F_A} \oplus F_{F_A} \odot F_{F_B} \end{array}\right)$$

3.4.6 Material biconditional (equivalence) :

$$DNL(A \leftrightarrow B) =$$

$$\left(\begin{array}{c} \left(\begin{array}{c} ((\{1\} \ominus T_{T_A} \oplus T_{T_A} \odot T_{T_B}) \odot (\{1\} \ominus T_{T_B} \oplus T_{T_A} \odot T_{T_B})), \\ ((\{1\} \ominus I_{T_A} \oplus I_{T_A} \odot I_{T_B}) \odot (\{1\} \ominus I_{T_B} \oplus I_{T_A} \odot I_{T_B})), \\ ((\{1\} \ominus F_{T_A} \oplus F_{T_A} \odot F_{T_B}) \odot (\{1\} \ominus F_{T_B} \oplus F_{T_A} \odot F_{T_B}) \end{array}\right), \\ \left(\begin{array}{c} ((\{1\} \ominus T_{I_A} \oplus T_{I_A} \odot T_{I_B}) \odot (\{1\} \ominus T_{I_B} \oplus T_{I_A} \odot T_{I_B})), \\ ((\{1\} \ominus I_{I_A} \oplus I_{I_A} \odot I_{I_B}) \odot (\{1\} \ominus I_{I_B} \oplus I_{I_A} \odot I_{I_B})), \\ ((\{1\} \ominus F_{I_A} \oplus F_{I_A} \odot F_{I_B}) \odot (\{1\} \ominus F_{I_B} \oplus F_{I_A} \odot F_{I_B}) \end{array}\right), \\ \left(\begin{array}{c} ((\{1\} \ominus T_{F_A} \oplus T_{F_A} \odot T_{F_B}) \odot (\{1\} \ominus T_{F_B} \oplus T_{F_A} \odot T_{F_B})), \\ ((\{1\} \ominus I_{F_A} \oplus I_{F_A} \odot I_{F_B}) \odot (\{1\} \ominus I_{F_B} \oplus I_{F_A} \odot I_{F_B})), \\ ((\{1\} \ominus F_{F_A} \oplus F_{F_A} \odot F_{F_B}) \odot (\{1\} \ominus F_{F_B} \oplus F_{F_A} \odot F_{F_B}) \end{array}\right) \end{array}\right)$$

3.4.7 Sheffer's connector :

$$DNL(A| B) = DNL(\neg A \vee \neg B) =$$

$$\left(\begin{array}{c} \{1\} \ominus T_{T_A} \odot T_{T_B}, \{1\} \ominus I_{T_A} \odot I_{T_B}, \{1\} \ominus F_{T_A} \odot F_{T_B}), \\ \{1\} \ominus T_{I_A} \odot T_{I_B}, \{1\} \ominus I_{I_A} \odot I_{I_B}, \{1\} \ominus F_{I_A} \odot F_{I_B}), \\ \{1\} \ominus T_{F_A} \odot T_{F_B}, \{1\} \ominus I_{F_A} \odot I_{F_B}, \{1\} \ominus F_{F_A} \odot F_{F_B} \end{array}\right)$$

3.4.8 Peirce's connector :

$$DNL(A \downarrow B) = DNL(\neg A \wedge \neg B) =$$

$$\left(\begin{array}{c} \{1\} \ominus T_{T_A} \odot \{1\} \ominus T_{T_B}, \{1\} \ominus I_{T_A} \odot \{1\} \ominus I_{T_B}, \{1\} \ominus F_{T_A} \odot \{1\} \ominus F_{T_B}), \\ \{1\} \ominus T_{I_A} \odot \{1\} \ominus T_{I_B}, \{1\} \ominus I_{I_A} \odot \{1\} \ominus I_{I_B}, \{1\} \ominus F_{I_A} \odot \{1\} \ominus F_{I_B}), \\ \{1\} \ominus T_{F_A} \odot \{1\} \ominus T_{F_B}, \{1\} \ominus I_{F_A} \odot \{1\} \ominus I_{F_B}, \{1\} \ominus F_{F_A} \odot \{1\} \ominus F_{F_B} \end{array}\right)$$

References :

[1] Charles Ashbacher, *Introduction to NeutrosophicLogic*, AmericanResearch,Rehoboth, 2002, p. 52.

[2] Florentin Smarandache , Salah Osman, *Netrosophy in Arabic Philosophy*, United States of America, Renaissance High Press, 2007, p. 64.

[3] Haibin Wang, Florentin Smarandache, Yan-qing Zhang, RajshekharSunderaman, *Interval NeutrosophicSets and Logic: Theory and Applications in Computing*,neutrosophic book series, no.5, Hexis Arizona, United States of America, 2005, p. 4.

[4] Florentin Smarandache, *A Unifying Field in Logic : Neutrosophic Logic, Neutrosophy,, Neutrosophic Set, NeutrosophicProbability and Statistics*, American R. Press, Rehoboth, fourth edition, 2005, pp. 119-120.

[5] Florentin Smarandache , *Proceedings of the First International Conference on Neutrosophy , Neutrosophic Logic, Neutrosophic Set , Neutrosophic Probability and Statistics*, University of New Mexico – Gallup, second printed edition, 1-3 December 2001, pp. 11-12.

[6] J. Nirmala, G.Suvitha, *Fuzzy Logic Gates in Electronic Circuits*, International Journal of Scientific and Research Publications, Volume 3, Issue 1, January 2013, pp. 2-3.

[7] Lotfi A. Zadeh, *«Fuzzy Sets»,Information and Control*,8, 1965.

[8] Florentin Smarandache, *Definition of Type-2 (and Type-n) Neutrosophic Set*, in Nidus idearum. Scilogs, III: Viva la Neutrosophia!, Section 92, pp. 102-103, Brussels, 2017.

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Interval-valued Possibility Quadripartitioned Single Valued Neutrosophic Soft Sets and some uncertainty based measures on them

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Abstract: The theory of quadripartitioned single valued neutrosophic sets was proposed very recently as an extension to the existing theory of single valued neutrosophic sets. In this paper the notion of possibility fuzzy soft sets has been generalized into a new concept viz. interval-valued possibility quadripartitioned single valued neutrosophic soft sets. Some basic set-theoretic operations have been defined on them. Some distance, similarity, entropy and inclusion measures for possibility quadripartitioned single valued neutrosophic sets have been proposed. An application in a decision making problem has been shown.

Keywords: Neutrosophic set, entropy measure, inclusion measure, distance measure, similarity measure.

1 Introduction

The theory of soft sets (introduced by D. Molodstov, in 1999) ([10],[15]) provided a unique approach of dealing with uncertainty with the implementation of an adequate parameterization technique. In a very basic sense, given a crisp universe, a soft set is a parameterized representation or parameter-wise classification of the subsets of that universe of discourse with respect to a given set of parameters. It was further shown that fuzzy sets could be represented as a particular class of soft sets when the set of parameters was considered to be $[0, 1]$. Since soft sets could be implemented without the rigorous process of defining a suitable membership function, the theory of soft sets, which seemed much easier to deal with, underwent rapid developments in fields pertaining to analysis as well as applications (as can be seen from the works of [1],[6],[7],[12],[14],[16],[17] etc.)

On the otherhand, hybridized structures, often designed and obtained as a result of combining two or more existing structures, have most of the inherent properties of the combined structures and thus provide for a stronger tool in handling application oriented problems. Likewise, the potential of the theory of soft sets was enhanced to a greater extent with the introduction of hybridized structures like those of the fuzzy soft sets [8], intuitionistic fuzzy soft sets [9], generalized fuzzy soft sets [13], neutrosophic soft sets [11], possibility fuzzy soft sets [2], possibility intuitionistic fuzzy soft sets [3] etc. to name a few.

While in case of generalized fuzzy soft sets, corresponding to each parameter a degree of possibility is assigned to the corresponding fuzzy subset of the universe; possibility fuzzy sets, a further modification of the generalized fuzzy soft sets, characterize each element of the universe with a possible degree of belongingness along with a degree of membership. Based on Belnap's four-valued logic [4] and Smarandache's n-valued refined neutrosophic set [18], the theory of quadripartitioned single valued neutrosophic sets [5] was proposed as a generalization of the existing theory of single valued neutrosophic sets [19]. In this paper the concept of interval valued possibility quadripartitioned single valued neutrosophic soft sets (IPQSVNSS, in short) has been proposed. In the existing literature studies pertaining to a possibility degree has been dealt with so far. Interval valued possibility assigns a closed sub-interval of $[0, 1]$ as the degree of chance or possibility instead of a number in $[0, 1]$ and thus it is a generalization of the existing concept of a possibility degree. The proposed structure can be viewed as a generalization of the existing theories of possibility fuzzy soft sets and possibility intuitionistic fuzzy soft sets.

The organization of the rest of the paper is as follows: a couple of preliminary results have been stated in Section 2, some basic set-theoretic operations on IPQSVNSS have been defined in Section 3, some uncertainty based measures viz. entropy, inclusion measure, distance measure and similarity measure, have been defined in Section 4 and their properties, applications and inter-relations have been studied. Section 5 concludes the paper.

2 Preliminaries

In this section some preliminary results have been outlined which would be useful for the smooth reading of the work that follows.

2.1 An outline on soft sets and possibility intuitionistic fuzzy soft sets

Definition 1 [15]. Let X be an initial universe and E be a set of parameters. Let $\mathcal{P}(X)$ denotes the power set of X and $A \subset E$. A pair (F, A) is called a soft set iff F is a mapping of A into

$\mathcal{P}(X)$.
The following results are due to [3].

Definition 2 [3]. Let $U = \{x_1, x_2, ..., x_n\}$ be the universal sets of elements and let $E = \{e_1, e_1, ..., e_m\}$ be the universal set of parameters. The pair (U, E) will be called a soft universe. Let $F : E \to (I \times I)^U \times I^U$ where $(I \times I)^U$ is the collection of all intuitionistic fuzzy subsets of U and I^U is the collection of all fuzzy subsets of U. Let p be a mapping such that $p : E \to I^U$ and let $F_p : E \to (I \times I)^U \times I^U$ be a function defined as follows:
$F_p(e) = (F(e)(x), p(e)(x))$, where $F(e)(x) = (\mu_e(x), \nu_e(x)) \, \forall x \epsilon U$.
Then F_p is called a possibility intuitionistic fuzzy soft set (PIFSS in short) over the soft universe (U, E). For each parameter e_i, $F_p(e_i)$ can be represented as:
$F_p(e_i) = \left\{ \left(\frac{x_1}{F(e_i)(x_1)}, p(e_i)(x_1) \right), ..., \left(\frac{x_n}{F(e_i)(x_n)}, p(e_i)(x_n) \right) \right\}$

Definition 3 [3]. Let F_p and G_q be two PIFSS over (U, E). Then the following operations were defined over PIFSS as follows:
Containment: F_p is said to be a possibility intuitionistic fuzzy soft subset (PIFS subset) of G_q and one writes $F_p \subseteq G_q$ if
(i) $p(e)$ is a fuzzy subset of $q(e)$, for all $e \epsilon E$,
$(ii) F(e)$ is an intuitionistic fuzzy subset of $G(e)$, for all $e \epsilon E$.
Equality: F_p and G_q are said to be equal and one writes $F_p = G_q$ if F_p is a PIFS subset of G_q and G_q is a PIFS subset of F_p
Union: $F_p \tilde{\cup} G_q = H_r$, $H_r : E \to (I \times I)^U \times I^U$ is defined by $H_r(e) = (H(e)(x), r(e)(x))$, $\forall e \epsilon E$ such that $H(e) = \cup_{Atan}(F(e), G(e))$ and $r(e) = s(p(e), q(e))$, where $\cup_{Atan}$ is Atanassov union and s is a triangular conorm.
Intersection: $F_p \tilde{\cap} G_q = H_r$, $H_r : E \to (I \times I)^U \times I^U$ is defined by $H_r(e) = (H(e)(x), r(e)(x))$, $\forall e \epsilon E$ such that $H(e) = \cap_{Atan}(F(e), G(e))$ and $r(e) = t(p(e), q(e))$, where $\cap_{Atan}$ is Atanassov intersection and t is a triangular norm.

Definition 4 [3]. A PIFSS is said to be a possibility absolute intuitionistic fuzzy soft set, denoted by A_1, if $A_1 : E \to (I \times I)^U \times I^U$ is such that $A_1(e) = (F(e)(x), P(e)(x))$, $\forall e \epsilon E$ where $F(e) = (1, 0)$ and $P(e) = 1, \forall e \epsilon E$.

Definition 5 [3]. A PIFSS is said to be a possibility null intuitionistic fuzzy soft set, denoted by ϕ_0, if $\phi_0 : E \to (I \times I)^U \times I^U$ is such that $\phi_0 = (F(e)(x), p(e)(x))$, $\forall e \epsilon E$ where $F(e) = (0, 1)$ and $p(e) = 0, \forall e \epsilon E$.

2.2 An outline on quadripartitioned single valued neutrosophic sets

Definition 6 [5]. Let X be a non-empty set. A quadripartitioned neutrosophic set (QSVNS) A, over X characterizes each element x in X by a truth-membership function T_A, a contradiction-membership function C_A, an ignorance-membership function U_A and a falsity membership function F_A such that for each $x \epsilon X, T_A, C_A, U_A, F_A \epsilon [0, 1]$
When X is discrete, A is represented as,
$A = \sum_{i=1}^{n} \langle T_A(x_i), C_A(x_i), U_A(x_i), F_A(x_i) \rangle / x_i, \, x_i \epsilon X$.
However, when the universe of discourse is continuous, A is represented as,
$A = \langle T_A(x), C_A(x), U_A(x), F_A(x) \rangle / x, x \epsilon X$

Definition 7 [5]. A QSVNS is said to be an absolute QSVNS, denoted by $\mathcal{A}$, iff its membership values are respectively defined as $T_{\mathcal{A}}(x) = 1$, $C_{\mathcal{A}}(x) = 1$, $U_{\mathcal{A}}(x) = 0$ and $F_{\mathcal{A}}(x) = 0, \forall x \epsilon X$.

Definition 8 [5]. A QSVNS is said to be a null QSVNS, denoted by Θ, iff its membership values are respectively defined as $T_{\Theta}(x) = 0$, $C_{\Theta}(x) = 0$, $U_{\Theta}(x) = 1$ and $F_{\Theta}(x) = 1, \forall x \epsilon X$

Definition 9 [5]. Let A and B be two QSVNS over X. Then the following operations can be defined:
Containment: $A \subseteq B$ iff $T_A(x) \leq T_B(x)$, $C_A(x) \leq C_A(x)$, $U_A(x) \geq U_A(x)$ and $F_A(x) \geq F_A(x)$, $\forall x \epsilon X$.
Complement: $A^c = \sum_{i=1}^{n} \langle F_A(x_i), U_A(x_i), C_A(x_i), T_A(x_i) \rangle / x_{i,} x_i \, \epsilon \, X$
i.e. $T_{A^c}(x_i) = F_A(x_i)$, $C_{A^c}(x_i) = U_A(x_i)$, $U_{A^c}(x_i) = C_A(x_i)$ and $F_{A^c}(x_i) = T_A(x_i)$, $x_i \, \epsilon \, X$
Union: $A \cup B = \sum_{i=1}^{n} < (T_A(x_i) \vee T_B(x_i)), (C_A(x_i) \vee C_B(x_i)), (U_A(x_i) \wedge U_B(x_i)), (F_A(x) \wedge F_B(x)) > / x_{i,} x_i \, \epsilon \, X$
Intersection: $A \cap B = \sum_{i=1}^{n} < (T_A(x_i) \wedge T_B(x_i)), (C_A(x_i) \wedge C_B(x_i)), (U_A(x_i) \vee U_B(x_i)), (F_A(x_i) \vee F_B(x_i)) > / x_{i,} x_i \, \epsilon \, X$

Proposition 1[5]. Quadripartitioned single valued neutrosophic sets satisfy the following properties under the aforementioned set-theoretic operations:

1.(i) $A \cup B = B \cup A$
(ii) $A \cap B = B \cap A$
2.(i) $A \cup (B \cup C) = (A \cup B) \cup C$
(ii) $A \cap (B \cap C) = (A \cap B) \cap C$
3.(i) $A \cup (A \cap B) = A$
(ii) $A \cap (A \cup B) = A$
4.(i) $(A^c)^c = A$
(ii) $\mathcal{A}^c = \Theta$
(iii) $\Theta^c = \mathcal{A}$
(iv) De-Morgan's laws hold viz. $(A \cup B)^c = A^c \cap B^c$; $(A \cap B)^c = A^c \cup B$
5.(i) $A \cup \mathcal{A} = \mathcal{A}$
(ii) $A \cap \mathcal{A} = A$
(iii) $A \cup \Theta = A$
(iv) $A \cap \Theta = \Theta$

3 Interval-valued possibility quadripartitioned single valued neutrosophic soft sets and some of their properties

Definition 10. Let X be an initial crisp universe and E be a set of parameters. Let $I=[0,1]$, $QSVNS(X)$ represents the collection of all quadripartitioned single valued neutrosophic sets over X , $Int([0,1])$ denotes the set of all closed subintervals of $[0,1]$ and $(Int([0,1]))^X$ denotes the collection of interval valued fuzzy subsets over X . An interval-valued possibility quadripartitioned single valued neutrosophic soft set (IPQSVNSS, in short) is a mapping of the form $F_\rho : E \to QSVNS(X) \times (Int([0,1]))^X$ and is defined as $F_\rho(e)=(F_e,\rho_e)$, $e\epsilon E$, where, for each $x\epsilon X$, $F_e(x)$ is the quadruple which represents the truth membership, the contradiction-membership, the ignorance-membership and the falsity membership of each element x of the universe of discourse X viz. $F_e(x)=\langle t_F^e(x), c_F^e(x), u_F^e(x), f_F^e(x)\rangle$,$\forall x\epsilon X$ and $\rho_e(x)=[\rho_e^-(x),\rho_e^+(x)]\epsilon Int([0,1])$. If $X=\{x_1,x_2,...,x_n\}$ and $E=\{e_1,e_2,...,e_m\}$, an interval-valued possibility quadripartitioned single valued neutrosophic soft set over the soft universe (X,E) is represented as, $F_\rho(e_i)=\{\left(\frac{x_1}{F_{e_i}(x_1)},\rho_{e_i}(x_1)\right),\left(\frac{x_2}{F_{e_i}(x_2)},\rho_{e_i}(x_2)\right),...,$ $\left(\frac{x_n}{F_{e_i}(x_n)},\rho_{e_i}(x_n)\right)\}$ viz.

$$F_\rho(e_i)=\{\left(\frac{x_1}{\langle t_F^{e_i}(x_1),c_F^{e_i}(x_1),u_F^{e_i}(x_1),f_F^{e_i}(x_1)\rangle},[\rho_{e_i}^-(x_1),\rho_{e_i}^+(x_1)]\right), ..., \left(\frac{x_n}{\langle t_F^{e_i}(x_n),c_F^{e_i}(x_n),u_F^{e_i}(x_n),f_F^{e_i}(x_n)\rangle},[\rho_{e_i}^-(x_n),\rho_{e_i}^+(x_n)]\right)\}, e_i\epsilon E,$$
$i=1,2,...,m.$

Example 1. Let $X=\{x_1,x_2,x_3\}$ and $E=\{e_1,e_2\}$. Define an IPQSVNSS over the soft universe (X,E), $F_\rho : E\to QSVNS(X)\times(Int([0,1]))^X$ as,

$F_\rho(e_1)=\{\left(\frac{x_1}{\langle 0.3,0.1,0.4,0.5\rangle},[0.5,0.6]\right),$
$\left(\frac{x_2}{\langle 0.6,0.2,0.1,0.01\rangle},[0.25,0.3]\right),\left(\frac{x_3}{\langle 0.7,0.3,0.4,0.6\rangle},[0.6,0.7]\right)\}$
$F_\rho(e_2)=\{\left(\frac{x_1}{\langle 0.7,0.3,0.5,0.2\rangle},[0.1,0.2]\right),$
$\left(\frac{x_2}{\langle 0.1,0.2,0.6,0.7\rangle},[0.45,0.6]\right),\left(\frac{x_3}{\langle 0.5,0.5,0.3,0.2\rangle},[0.3,0.4]\right)\}$

Another IPQSVNSS G_μ can be defined over (X,E) as

$G_\mu(e_1)=\{\left(\frac{x_1}{\langle 0.8,0.6,0.3,0.4\rangle},[0.8,0.85]\right),$
$\left(\frac{x_2}{\langle 0.2,0.1,0.1,0.6\rangle},[0.4,0.5]\right),\left(\frac{x_3}{\langle 0.5,0.5,0.3,0.4\rangle},[0.4,0.6]\right)\}$
$G_\mu(e_2)=\{\left(\frac{x_1}{\langle 0.2,0.6,0.3,0.7\rangle},[0.6,0.75]\right),$
$\left(\frac{x_2}{\langle 0.4,0.2,0.2,0.7\rangle},[0.8,0.9]\right),\left(\frac{x_3}{\langle 0.9,0.7,0.1,0.6\rangle},[0.35,0.5]\right)\}$

Definition 11. The absolute IPQSVNSS over (X,E) is denoted by $\tilde{A}_{\bar{1}}$ such that for each $e\epsilon E$ and $\forall x\epsilon X$, $\tilde{A}_e(x)=\langle 1,1,0,0\rangle$ and $\bar{1}_e(x)=[1,1]$

Definition 12. The null IPQSVNSS over (X,E) is denoted by $\tilde{\theta}_{\bar{0}}$ such that for each $e\epsilon E$ and $\forall x\epsilon X$, $\tilde{\theta}_e(x)=\langle 0,0,1,1\rangle$ and $\bar{0}_e(x)=[0,0]$

3.1 Operations over IPQSVNSS

Definition 13. Let F_ρ and G_μ be two IPQSVNSS over the common soft universe (X,E). Some elementary set-theoretic operations on IPQSVNSS are defined as,

(i) Union: $F_\rho\tilde{\cup}G_\mu=H_\eta$ such that for each $e\epsilon E$ and $\forall x\epsilon X$,
$H_e(x)=\langle t_F^e(x)\vee t_G^e(x), c_F^e(x)\vee c_G^e(x), u_F^e(x)\wedge u_G^e(x), f_F^e(x)\wedge f_G^e(x)\rangle$ and
$\eta_e(x)=[sup(\rho_e^-(x),\mu_e^-(x)), sup(\rho_e^+(x),\mu_e^+(x))]$.

(ii) Intersection: $F_\rho\tilde{\cap}G_\mu=H_\eta$ such that for each $e\epsilon E$ and $\forall x\epsilon X$, $H_e(x)=\langle t_F^e(x)\wedge t_G^e(x), c_F^e(x)\wedge c_G^e(x), u_F^e(x)\vee u_G^e(x), f_F^e(x)\vee f_G^e(x)\rangle$ and
$\eta_e(x)=[inf(\rho_e^-(x),\mu_e^-(x)), inf(\rho_e^+(x),\mu_e^+(x))]$.

(iii) Complement: $(F_\rho)^c=F_\rho^c$ such that for each $e\epsilon E$ and $\forall x\epsilon X$, $F_e^c(x)=\langle f_F^e(x),u_F^e(x),c_F^e(x),t_F^e(x)\rangle$ and $\rho_e^c(x)=[1-\rho_e^+(x),1-\rho_e^-(x)]$

(iv) Containment: $F_\rho\tilde{\subseteq}G_\mu$ if for each $e\epsilon E$ and $\forall x\epsilon X$, $t_F^e(x)\le t_G^e(x), c_F^e(x)\le c_G^e(x), u_F^e(x)\ge u_G^e(x), f_F^e(x)\ge f_G^e(x)$ and $\rho_e^-(x)\le\mu_e^-(x), \rho_e^+(x)\le\mu_e^+(x)$.

Example 2. Consider the IPQSNSS F_ρ and G_μ over the same soft universe (X,E) defined in example 1. Then, F_ρ^c is obtained as,

$F_\rho^c(e_1)=\{\left(\frac{x_1}{\langle 0.5,0.4,0.1,0.3\rangle},[0.4,0.5]\right),$
$\left(\frac{x_2}{\langle 0.01,0.1,0.2,0.6\rangle},[0.7,0.75]\right),\left(\frac{x_3}{\langle 0.6,0.4,0.3,0.7\rangle},[0.3,0.4]\right)\}$
$F_\rho^c(e_2)=\{\left(\frac{x_1}{\langle 0.2,0.5,0.3,0.7\rangle},[0.8,0.9]\right),$
$\left(\frac{x_2}{\langle 0.7,0.6,0.2,0.1\rangle},[0.4,0.55]\right),\left(\frac{x_3}{\langle 0.2,0.3,0.5,0.5\rangle},[0.6,0.7]\right)\}$

$H_\eta=F_\rho\tilde{\cup}G_\mu$ is obtained as,

$H_\eta(e_1)=\{\left(\frac{x_1}{\langle 0.8,0.6,0.3,0.4\rangle},[0.8,0.85]\right),$
$\left(\frac{x_2}{\langle 0.6,0.2,0.1,0.01\rangle},[0.4,0.5]\right),\left(\frac{x_3}{\langle 0.7,0.5,0.3,0.4\rangle},[0.6,0.7]\right)\}$
$H_\eta(e_2)=\{\left(\frac{x_1}{\langle 0.7,0.6,0.3,0.2\rangle},[0.6,0.75]\right),$
$\left(\frac{x_2}{\langle 0.4,0.2,0.2,0.7\rangle},[0.8,0.9]\right),\left(\frac{x_3}{\langle 0.9,0.7,0.1,0.2\rangle},[0.35,0.5]\right)\}$

Also, the intersection $K_\delta=F_\rho\tilde{\cap}G_\mu$ is defined as,

$K_\delta(e_1)=\{\left(\frac{x_1}{\langle 0.3,0.1,0.4,0.5\rangle},[0.5,0.6]\right),$
$\left(\frac{x_2}{\langle 0.2,0.1,0.1,0.6\rangle},[0.25,0.3]\right),\left(\frac{x_3}{\langle 0.5,0.3,0.4,0.6\rangle},[0.4,0.6]\right)\}$
$K_\delta(e_2)=\{\left(\frac{x_1}{\langle 0.2,0.3,0.5,0.7\rangle},[0.1,0.2]\right),$
$\left(\frac{x_2}{\langle 0.1,0.2,0.6,0.7\rangle},[0.45,0.6]\right),\left(\frac{x_3}{\langle 0.5,0.5,0.3,0.6\rangle},[0.3,0.4]\right)\}$

Proposition 2. For any $F_\rho, G_\mu, H_\eta\epsilon IPQSVNSS(X,E)$, the following results hold:

1. (i) $F_\rho\tilde{\cup}G_\mu=G_\mu\tilde{\cup}F_\rho$
(ii) $F_\rho\tilde{\cap}G_\mu=G_\mu\tilde{\cap}F_\rho$
2. (i) $F_\rho\tilde{\cup}(G_\mu\tilde{\cup}H_\eta)=(F_\rho\tilde{\cup}G_\mu)\tilde{\cup}H_\eta$
(ii) $F_\rho\tilde{\cap}(G_\mu\tilde{\cap}H_\eta)=(F_\rho\tilde{\cap}G_\mu)\tilde{\cap}H_\eta$

3. $(i)\ F_\rho \tilde{\cup} \tilde{\theta}_{\bar{0}} = F_\rho$
$(ii)\ F_\rho \tilde{\cap} \tilde{\theta}_{\bar{0}} = \tilde{\theta}_{\bar{0}}$
$(iii)\ F_\rho \tilde{\cup} \tilde{A}_{\bar{1}} = \tilde{A}_{\bar{1}}$
$(iv)\ F_\rho \tilde{\cap} \tilde{A}_{\bar{1}} = F_\rho$
4. $(i)\ \left(F_\rho^c\right)^c = F_\rho$
$(ii)\ \tilde{A}_{\bar{1}}^c = \tilde{\theta}_{\bar{0}}$
$(iii)\ \left(\tilde{\theta}_{\bar{0}}\right)^c = \tilde{A}_{\bar{1}}$
5. $(i)\ (F_\rho \tilde{\cup} G_\mu)^c = (F_\rho)^c \tilde{\cap} (G_\mu)^c$
$(ii)\ (F_\rho \tilde{\cap} G_\mu)^c = (F_\rho)^c \tilde{\cup} (G_\mu)^c$

Proofs are straight-forward.

4 Some uncertainty-based measures on IPQSVNSS

4.1 Entropy measure

Definition 14. Let $IPQSVNSS(X,E)$ denotes the set of all IPQSVNSS over the soft universe (X,E). A mapping $\varepsilon : IPQSVNSS(X,E) \to [0,1]$ is said to be a measure of entropy if it satisfies the following properties:
$(e1)\ \varepsilon\left(F_\rho^c\right) = \varepsilon\left(F_\rho\right)$
$(e2) \varepsilon\left(F_\rho\right) \leq \varepsilon\left(G_\mu\right)$ whenever $F_\rho \tilde{\subseteq} G_\mu$ with $f_F^e(x) \geq f_G^e(x) \geq t_G^e(x) \geq t_F^e(x)$, $u_F^e(x) \geq u_G^e(x) \geq c_G^e(x) \geq c_F^e(x)$ and $\rho_e^-(x) + \rho_e^+(x) \leq 1$.
$(e3)\ \varepsilon\left(F_\rho\right) = 1$ iff $t_F^e(x) = f_F^e(x)$, $c_F^e(x) = u_F^e(x)$ and $\rho_e^-(x) + \rho_e^+(x) = 1$, $\forall x \epsilon X$ and $\forall e \epsilon E$.

Theorem 1. *The mapping* $e : IPQSVNSS(X,E) \to [0,1]$ *defined as,* $\varepsilon\left(F_\rho\right) = 1 - \frac{1}{||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}|t_F^e(x) - f_F^e(x)|.|c_F^e(x) - u_F^e(x)|.|1 - \{\rho_e^+(x) + \rho_e^-(x)\}|$ *is an entropy measure for IPQSVNSS.*

Proof:

$(i)\ \varepsilon\left(F_\rho^c\right) = 1 - \frac{1}{||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}|f_F^e(x) - t_F^e(x)|.|u_F^e(x) - c_F^e(x)|.|1 - \{(1-\rho_e^-(x)) + (1-\rho_e^+(x))\}|$
$= 1 - \frac{1}{||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}|t_F^e(x) - f_F^e(x)|.|c_F^e(x) - u_F^e(x)|.|1 - \{\rho_e^+(x) + \rho_e^-(x)\}| = \varepsilon\left(F_\rho\right)$.

(ii) Suppose that $F_\rho \tilde{\subseteq} G_\mu$ and $f_G^e(x) \geq t_G^e(x)$, $u_G^e(x) \geq c_G^e(x)$, $\rho_e^-(x) + \rho_e^+(x) \leq 1$. Automatically, $\mu_e^-(x) + \mu_e^+(x) \leq 1$. Thus, $f_F^e(x) \geq f_G^e(x), t_G^e(x) \geq t_F^e(x)$, $u_F^e(x) \geq u_G^e(x), c_G^e(x) \geq c_F^e(x), \mu_e^-(x) \geq \rho_e^-(x)$,
$\mu_e^+(x) \geq \rho_e^+(x)$, and $f_G^e(x) \geq t_G^e(x)$, $u_G^e(x) \geq c_G^e(x)$, $\rho_e^-(x) + \rho_e^+(x) \leq 1$.
$\Rightarrow f_F^e(x) \geq f_G^e(x) \geq t_G^e(x) \geq t_F^e(x)$, $u_F^e(x) \geq u_G^e(x) \geq c_G^e(x) \geq c_F^e(x)$, $\mu_e^-(x) \geq \rho_e^-(x), \mu_e^+(x) \geq \rho_e^+(x)$ and $\rho_e^-(x) + \rho_e^+(x) \leq 1, \mu_e^-(x) + \mu_e^+(x) \leq 1$.
From the above relations it follows that $t_G^e(x) - f_G^e(x) \geq t_F^e(x) - f_F^e(x)$ but $t_G^e(x) - f_G^e(x) \leq 0, t_F^e(x) - f_F^e(x) \leq 0$
$\Rightarrow |t_G^e(x) - f_G^e(x)| \leq |t_F^e(x) - f_F^e(x)|$. Similarly, $|c_G^e(x) - u_G^e(x)| \leq |c_F^e(x) - u_F^e(x)|$ and $|1 - \{\mu_e^+(x) + \mu_e^-(x)\}| \leq |1 - \{\rho_e^+(x) + \rho_e^-(x)\}|$, $\forall x \epsilon X$, $\forall e \epsilon E$. Then,
$|t_G^e(x) - f_G^e(x)|.|c_G^e(x) - u_G^e(x)|.|1 - \{\mu_e^+(x) + \mu_e^-(x)\}|$
$\leq |t_F^e(x) - f_F^e(x)|.|c_F^e(x) - u_F^e(x)|.|1 - \{\rho_e^+(x) + \rho_e^-(x)\}|$
$\Rightarrow 1 - \frac{1}{||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}|t_F^e(x) - f_F^e(x)|.|c_F^e(x) - u_F^e(x)|.|1 - \{\rho_e^+(x) + \rho_e^-(x)\}|$
$\leq 1 - \frac{1}{||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}|t_G^e(x) - f_G^e(x)|.|c_G^e(x) - u_G^e(x)|.|1 - \{\mu_e^+(x) + \mu_e^-(x)\}|$
$\Rightarrow \varepsilon\left(F_\rho\right) \leq \varepsilon\left(G_\mu\right)$

$(iii)\ \varepsilon\left(F_\rho\right) = 1$
$\Leftrightarrow 1 - \frac{1}{||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}|t_F^e(x) - f_F^e(x)|.|c_F^e(x) - u_F^e(x)|.|1 - \{\rho_e^+(x) + \rho_e^-(x)\}| = 1$
$\Leftrightarrow \frac{1}{||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}|t_F^e(x) - f_F^e(x)|.|c_F^e(x) - u_F^e(x)|.|1 - \{\rho_e^+(x) + \rho_e^-(x)\}| = 0$
$\Leftrightarrow |t_F^e(x) - f_F^e(x)| = 0$, $|c_F^e(x) - u_F^e(x)| = 0$, $|1 - \{\rho_e^+(x) + \rho_e^-(x)\}| = 0$, for each $x \epsilon X$ and each $e \epsilon E$.
$\Leftrightarrow t_F^e(x) = f_F^e(x)$, $c_G^e(x) = u_G^e(x)$, $\rho_e^+(x) + \rho_e^-(x) = 1$, for each $x \epsilon X$ and each $e \epsilon E$.

Remark 1. In particular, from *Theorem 1*, it follows that, $\varepsilon\left(\tilde{A}_{\bar{1}}\right) = 0$ and $\varepsilon\left(\tilde{\theta}_{\bar{0}}\right) = 0$.

Proof is straight-forward.

4.1.1 An application of entropy measure in decision making problem

The entropy measure not only provides an all over information about the amount of uncertainty ingrained in a particular structure, it can also be implemented as an efficient tool in decision making processes. Often while dealing with a selection process subject to a predefined set of requisitions, the procedure involves allocation of weights in order to signify the order of preference of the criteria under consideration. In what follows next, the entropy measure corresponding to an IPQSVNSS has been utilized in defining weights corresponding to each of the elements of the parameter set over which the IPQSVNSS has been defined.

The algorithm is defined as follows:

Step 1: Represent the data in hand in the form of an IPQSVNSS, say F_ρ.
Step 2: Calculate the entropy measure $\varepsilon\left(F_\rho\right)$, as defined in Theorem 1.
Step 3: For each $\alpha \epsilon E$, assign weights $\omega_F(\alpha)$, given by the formula,
$\omega_F(\alpha) = \frac{\varepsilon(F_\rho)}{\kappa_F(\alpha)}$, where $\kappa_F(\alpha) = 1 - \frac{1}{||X||.||E||}\sum_{x\epsilon X}|t_F^\alpha(x) - f_F^\alpha(x)|.|c_F^\alpha(x) - u_F^\alpha(x)|.|1 - \{\rho_\alpha^+(x) + \rho_\alpha^-(x)\}|$.
Step 4: Corresponding to each option $x \epsilon X$, calculate the net score, defined as,
$score(x_i) = \sum_e \omega_F(\alpha).[t_F^\alpha(x_i) + c_F^\alpha(x_i) + \{1 - u_F^\alpha(x_i)\} + \{1 - f_F^\alpha(x_i)\}].\{\frac{\rho_\alpha^+(x_i) + \rho_\alpha^-(x_i)}{2}\}$.

Step 5: Arrange $score(x_i)$ in the decreasing order of values.
Step 6: Select $max_i\{score(x_i)\}$. If $max_i\{score(x_i)\} = score(x_m), x_m \epsilon X$, then x_m is the selected option.

Theorem 2. *Corresponding to each parameter $\alpha\epsilon E$, $\omega_F(\alpha) = \frac{\varepsilon(F_\rho)}{\kappa_F(\alpha)}$ is such that $0 \leq \omega_F(\alpha) \leq 1$.*

Proof:

From the definition of $\kappa_F(\alpha)$ and $\varepsilon(F_\rho)$, it is clear that $\omega_F(\alpha) \geq 0$.
Consider $|t^\alpha_F(x) - f^\alpha_F(x)|.|c^\alpha_F(x) - u^\alpha_F(x)|.|1 - \{\rho^+_\alpha(x) + \rho^-_\alpha(x)\}|$. It follows that,
$\sum_{\alpha\epsilon E}\sum_{x\epsilon X}|t^\alpha_F(x) - f^\alpha_F(x)|.|c^\alpha_F(x) - u^\alpha_F(x)|.|1 - \{\rho^+_\alpha(x) + \rho^-_\alpha(x)\}| \geq \sum_{x\epsilon X}|t^\alpha_F(x) - f^\alpha_F(x)|.|c^\alpha_F(x) - u^\alpha_F(x)|.|1 - \{\rho^+_\alpha(x) + \rho^-_\alpha(x)\}|$, whenever $||X|| \geq 1$.
$\Rightarrow 1 - \frac{1}{||X||.||E||}\sum_{\alpha\epsilon E}\sum_{x\epsilon X}|t^\alpha_F(x) - f^\alpha_F(x)|.|c^\alpha_F(x) - u^\alpha_F(x)|.|1 - \{\rho^+_\alpha(x) + \rho^-_\alpha(x)\}| \leq 1 - \frac{1}{||X||.||E||}\sum_{x\epsilon X}|t^\alpha_F(x) - f^\alpha_F(x)|.|c^\alpha_F(x) - u^\alpha_F(x)|.|1 - \{\rho^+_\alpha(x) + \rho^-_\alpha(x)\}|$
$\Rightarrow \varepsilon(F_\rho) \leq \kappa_F(\alpha)$
$\Rightarrow \omega_F(\alpha) = \frac{\varepsilon(F_\rho)}{\kappa_F(\alpha)} \leq 1$, for each $\alpha\epsilon E$.

Example 3. Suppose a person wishes to buy a phone and the judging parameters he has set are a: appearance, c: cost, b: battery performance, s: storage and l: longevity. Further suppose that he has to choose between 3 available models, say x_1, x_2, x_3 of the desired product. After a survey has been conducted by the buyer both by word of mouth from the current users and the salespersons, the resultant information is represented in the form of an IPQSVNSS, say F_ρ as follows, where it is assumed that corresponding to an available option, a higher degree of belongingness signifies a higher degree of agreement with the concerned parameter:

$F_\rho(a) = \{\left(\frac{x_1}{\langle 0.4,0.3,0.1,0.5\rangle}, [0.5, 0.6]\right),$
$\left(\frac{x_2}{\langle 0.8,0.1,0.0,0.01\rangle}, [0.6, 0.7]\right), \left(\frac{x_3}{\langle 0.6,0.3,0.2,0.5\rangle}, [0.45, 0.5]\right)\}$
$F_\rho(c) = \{\left(\frac{x_1}{\langle 0.8,0.1,0.1,0.2\rangle}, [0.7, 0.75]\right),$
$\left(\frac{x_2}{\langle 0.5,0.01,0.1,0.6\rangle}, [0.4, 0.55]\right), \left(\frac{x_3}{\langle 0.7,0.2,0.1,0.1\rangle}, [0.6, 0.65]\right)\}$
$F_\rho(b) = \{\left(\frac{x_1}{\langle 0.65,0.3,0.1,0.2\rangle}, [0.6, 0.65]\right),$
$\left(\frac{x_2}{\langle 0.8,0.2,0.1,0.0\rangle}, [0.75, 0.8]\right), \left(\frac{x_3}{\langle 0.4,0.5,0.3,0.6\rangle}, [0.7, 0.8]\right)\}$
$F_\rho(s) = \{\left(\frac{x_1}{\langle 0.5,0.4,0.3,0.6\rangle}, [0.7, 0.8]\right),$
$\left(\frac{x_2}{\langle 0.85,0.1,0.0,0.01\rangle}, [0.8, 0.85]\right), \left(\frac{x_3}{\langle 0.8,0.2,0.1,0.02\rangle}, [0.85, 0.9]\right)\}$
$F_\rho(l) = \{\left(\frac{x_1}{\langle 0.6,0.3,0.2,0.5\rangle}, [0.45, 0.55]\right),$
$\left(\frac{x_2}{\langle 0.75,0.3,0.3,0.2\rangle}, [0.67, 0.75]\right), \left(\frac{x_3}{\langle 0.75,0.3,0.2,0.2\rangle}, [0.7, 0.75]\right)\}$

Following steps 2-6, we have the following results:

(2) $\varepsilon(F_\rho) = 0.982$
(3) $\omega_F(a) = 0.984, \omega_F(c) = 0.983, \omega_F(b) = 0.988, \omega_F(s) = 0.99, \omega_F(l) = 0.984$
(4) $score(x_1) = 7.193, score(x_2) = 9.097, score(x_3) = 8.554$
(5) $score(x_2) > score(x_3) > score(x_1)$
(6) x_2 is the chosen model.

4.2 Inclusion measure

Definition 15. A mapping $I : IPQSVNSS(X,E) \times IPQSVNSS(X,E) \rightarrow [0,1]$ is said to be an inclusion measure for IPQSVNSS over the soft universe (X,E) if it satisfies the following properties:
$(I1)$ $I\left(\tilde{A}_{\bar{1}}, \tilde{\theta}_{\bar{0}}\right) = 0$
$(I2)$ $I(F_\rho, G_\mu) = 1 \Leftrightarrow F_\rho \tilde{\subseteq} G_\mu$
$(I3)$ if $F_\rho \tilde{\subseteq} G_\mu \tilde{\subseteq} H_\eta$ then $I(H_\eta, F_\rho) \leq I(G_\mu, F_\rho)$ and $I(H_\eta, F_\rho) \leq I(H_\eta, G_\mu)$

Theorem 3. *The mapping $I : IPQSVNSS(X,E) \rightarrow [0,1]$ defined as,*
$I(F_\rho, G_\mu) = 1 - \frac{1}{6||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}[|t^e_F(x) - min\{t^e_F(x), t^e_G(x)\}| + |c^e_F(x) - min\{c^e_F(x), c^e_G(x)\}| + |max\{u^e_F(x), u^e_G(x)\} - u^e_F(x)| + |max\{f^e_F(x), f^e_G(x)\} - f^e_F(x)| + |\rho^-_e(x) - min\{\rho^-_e(x), \mu^-_e(x)\}| + |\rho^+_e(x) - min\{\rho^+_e(x), \mu^+_e(x)\}|]$, *is an inclusion measure for IPQSVNSS.*

Proof:

(i) Clearly, according to the definition of the proposed measure, $I\left(\tilde{A}_{\bar{1}}, \tilde{\theta}_{\bar{0}}\right) = 0$

(ii) From the definition of the proposed measure, it follows that,
$I(F_\rho, G_\mu) = 1$,
$\Leftrightarrow \sum_{e\epsilon E}\sum_{x\epsilon X}[|t^e_F(x) - min\{t^e_F(x), t^e_G(x)\}| + |c^e_F(x) - min\{c^e_F(x), c^e_G(x)\}| + |max\{u^e_F(x), u^e_G(x)\} - u^e_F(x)| + |max\{f^e_F(x), f^e_G(x)\} - f^e_F(x)| + |\rho^-_e(x) - min\{\rho^-_e(x), \mu^-_e(x)\}| + |\rho^+_e(x) - min\{\rho^+_e(x), \mu^+_e(x)\}|] = 0, \forall x\epsilon X, \forall e\epsilon E$.
$\Leftrightarrow |t^e_F(x) - min\{t^e_F(x), t^e_G(x)\}| = 0$, $|c^e_F(x) - min\{c^e_F(x), c^e_G(x)\}| = 0$, $|max\{u^e_F(x), u^e_G(x)\} - u^e_F(x)| = 0$, $|max\{f^e_F(x), f^e_G(x)\} - f^e_F(x)| = 0$, $|\rho^-_e(x) - min\{\rho^-_e(x), \mu^-_e(x)\}| = 0$ and $|\rho^+_e(x) - min\{\rho^+_e(x), \mu^+_e(x)\}| = 0, \forall x\epsilon X, \forall e\epsilon E$.
Now, $|t^e_F(x) - min\{t^e_F(x), t^e_G(x)\}| = 0 \Leftrightarrow t^e_F(x) \leq t^e_G(x)$.
Similarly, it can be shown that, $c^e_F(x) \leq c^e_G(x), u^e_F(x) \geq u^e_G(x), f^e_F(x) \geq f^e_G(x), \rho^-_e(x) \leq \mu^-_e(x)$ and $\rho^+_e(x) \leq \mu^+_e(x), \forall x\epsilon X, \forall e\epsilon E$ which proves $F_\rho \tilde{\subseteq} G_\mu$.

(iii) Suppose, $F_\rho \tilde{\subseteq} G_\mu \tilde{\subseteq} H_\eta$. Thus we have, $t^e_F(x) \leq t^e_G(x) \leq t^e_H(x)$, $c^e_F(x) \leq c^e_G(x) \leq c^e_H(x)$, $u^e_F(x) \geq u^e_G(x) \geq u^e_H(x)$, $f^e_F(x) \geq f^e_G(x) \geq f^e_H(x)$, $\rho^-_e(x) \leq \mu^-_e(x) \leq \eta^-_e(x)$ and $\rho^+_e(x) \leq \mu^+_e(x) \leq \eta^+_e(x)$ for all $x\epsilon X$ and $e\epsilon E$.
$\Rightarrow I(H_\eta, F_\rho) \leq I(G_\mu, F_\rho)$.
In an exactly analogous manner, it can be shown that, $I(H_\eta, F_\rho) \leq I(H_\eta, G_\mu)$. This completes the proof.

Example 4. Consider IPQSVNSS F_ρ, G_μ in *Example 1*, then $I(F_\rho, G_\mu) = 0.493$.

4.3 Distance measure

Definition 16. A mapping $d : IPQSVNSS(X,E) \times IPQSVNSS(X,E) \to R^+$ is said to be a distance measure between IPQSVNSS if for any $F_\rho, G_\mu, H_\eta \epsilon IPQSVNSS(X,E)$ it satisfies the following properties:
$(d1)$ $d(F_\rho, G_\mu) = d(G_\mu, F_\rho)$
$(d2)$ $d(F_\rho, G_\mu) \geq 0$ and $d(F_\rho, G_\mu) = 0 \Leftrightarrow F_\rho = G_\mu$
$(d3)$ $d(F_\rho, H_\eta) \leq d(F_\rho, G_\mu) + d(G_\mu, H_\eta)$
In addition to the above conditions, if the mapping d satisfies the condition
$(d4)$ $d(F_\rho, G_\mu) \leq 1, \forall F_\rho, G_\mu \epsilon IPQSVNSS(X,E)$
it is called a Normalized distance measure for IPQSVNSS.

Theorem 4. *The mapping* $d_h : IPQSVNSS(X,E) \times IPQSVNSS(X,E) \to R^+$ *defined as,*
$d_h(F_\rho, G_\mu) = \sum_{e\epsilon E} \sum_{x\epsilon X} (|t_F^e(x) - t_G^e(x)| + |c_F^e(x) - c_G^e(x)| + |u_F^e(x) - u_G^e(x)| + |f_F^e(x) - f_G^e(x)| + |\rho_e^-(x) - \mu_e^-(x)| + |\rho_e^+(x) - \mu_e^+(x)|)$ *is a distance measure for IPQSVNSS. It is known as the Hamming Distance.*

Proofs are straight-forward.

Definition 17. The corresponding Normalized Hamming distance for IPQSVNSS is defined as $d_h^N(F_\rho, G_\mu) = \frac{1}{6||X||.||E||} d_h(F_\rho, G_\mu)$, where $||.||$ denotes the cardinality of a set.

Theorem 5. *The mapping* $d_E : IPQSVNSS(X,E) \times IPQSVNSS(X,E) \to R^+$ *defined as,*
$d_E(F_\rho, G_\mu) = \sum_{e\epsilon E} \sum_{x\epsilon X} \{(t_F^e(x) - t_G^e(x))^2 + (c_F^e(x) - c_G^e(x))^2 + (u_F^e(x) - u_G^e(x))^2 + (f_F^e(x) - f_G^e(x))^2 + (\rho_e^-(x) - \mu_e^-(x))^2 + (\rho_e^+(x) - \mu_e^+(x))^2\}^{\frac{1}{2}}$ *is a distance measure for IPQSVNSS. It is known as the Euclidean Distance.*

Proofs are straight-forward.

Definition 18. The corresponding Normalized Hamming distance for IPQSVNSS is defined as $d_E^N(F_\rho, G_\mu) = \frac{1}{6||X||.||E||} d_E(F_\rho, G_\mu)$.

Proposition 3. $F_\rho \tilde{\subseteq} G_\mu \tilde{\subseteq} H_\eta$ iff
(i) $d_h(F_\rho, H_\eta) = d_h(F_\rho, G_\mu) + d_h(G_\mu, H_\eta)$
(ii) $d_h^N(F_\rho, H_\eta) = d_h^N(F_\rho, G_\mu) + d_h^N(G_\mu, H_\eta)$

Proofs are straight-forward.

Example 5. Consider the IPQSVNSS given in *Example 1*. The various distance measures between the sets are obtained as, $d_h(F_\rho, G_\mu) = 5.29$, $d_h^N(F_\rho, G_\mu) = 0.882$, $d_E(F_\rho, G_\mu) = 4.387$, $d_E^N(F_\rho, G_\mu) = 0.731$

4.4 Similarity measure

Definition 19. A mapping $s : IPQSVNSS(X,E) \times IPQSVNSS(X,E) \to R^+$ is said to be a quasi-similarity measure between IPQSVNSS if for any $F_\rho, G_\mu, H_\eta \epsilon IPQSVNSS(X,E)$ it satisfies the following properties:
$(s1)$ $s(F_\rho, G_\mu) = s(G_\mu, F_\rho)$
$(s2)$ $0 \leq s(F_\rho, G_\mu) \leq 1$ and $s(F_\rho, G_\mu) = 1 \Leftrightarrow F_\rho = G_\mu$
In addition, if it satisfies
$(s3)$ if $F_\rho \tilde{\subseteq} G_\mu \tilde{\subseteq} H_\eta$ then $s(F_\rho, H_\eta) \leq s(F_\rho, G_\mu) \wedge s(G_\mu, H_\eta)$
then it is known as a similarity measure between IPQSVNSS.

Various similarity measures for quadripartitioned single valued neutrosophic sets were proposed in [5]. Undertaking a similar line of approach, as in our previous work [5] we propose a similarity measure for IPQSVNSS as follows:

Definition 20. Consider $F_\rho, G_\mu \epsilon IPQSVNSS(X,E)$. Define functions $\tau_{i,e}^{F,G} : X \to [0,1]$, $i = 1,2,..,5$ such that for each $x\epsilon X, e\epsilon E$
$\tau_{1,e}^{F,G}(x) = |t_G^e(x) - t_F^e(x)|$
$\tau_{2,e}^{F,G}(x) = |f_F^e(x) - f_G^e(x)|$
$\tau_{3,e}^{F,G}(x) = |c_G^e(x) - c_F^e(x)|$
$\tau_{4,e}^{F,G}(x) = |u_F^e(x) - u_G^e(x)|$
$\tau_{5,e}^{F,G}(x) = |\rho_e^-(x) - \mu_e^-(x)|$
$\tau_{6,e}^{F,G}(x) = |\rho_e^+(x) - \mu_e^+(x)|$
Finally, define a mapping $s : IPQSVNSS(X,E) \times IPQSVNSS(X,E) \to R^+$ as, $s(F_\rho, G_\mu) = 1 - \frac{1}{6||X||.||E||} \sum_{e\epsilon E} \sum_{x\epsilon X} \sum_{i=1}^{6} \tau_{i,e}^{F,G}(x)$

Theorem 6. *The mapping* s (F_ρ, G_μ) *defined above is a similarity measure.*

Proof:

(i) It is easy to prove that $s(F_\rho, G_\mu) = s(G_\mu, F_\rho)$.

(ii) We have, $t_F^e(x), c_F^e(x), u_F^e(x), f_F^e(x) \epsilon [0,1]$ and $\rho_e(x), \mu_e(x) \epsilon Int([0,1])$ for each $x\epsilon X, e\epsilon E$. Thus, $\tau_{1,e}^{F,G}(x)$ attains its maximum value if either one of $t_F^e(x)$ or $t_G^e(x)$ is equal to 1 while the other is 0 and in that case the maximum value is 1. Similarly, it attains a minimum value 0 if $t_F^e(x) = t_G^e(x)$. So, it follows that $0 \leq \tau_{1,e}^{F,G}(x) \leq 1$, for each $x\epsilon X$. Similarly it can be shown that $\tau_{i,e}^{F,G}(x)$, $i = 2, ..., 6$ lies within $[0,1]$ for each $x\epsilon X$. So,
$0 \leq \sum_{i=1}^{6} \tau_{i,e}^{F,G}(x) \leq 6$
$\Rightarrow 0 \leq \sum_{e\epsilon E} \sum_{x\epsilon X} \sum_{i=1}^{n} \tau_{i,e}^{F,G}(x) \leq 6||X||.||E||$
which implies $0 \leq s(F_\rho, G_\mu) \leq 1$.
Now $s(F_\rho, G_\mu) = 1$ iff $\sum_{i=1}^{n} \tau_{i,e}(x) = 0$ for each $x\epsilon X, e\epsilon E$ $\Leftrightarrow$ $t_F^e(x) = t_G^e(x)$, $c_F^e(x) = c_G^e(x)$, $u_F^e(x) = u_G^e(x)$,

$f_F^e(x) = f_G^e(x)$ and $\rho_e^-(x) = \mu_e^-(x)$, $\rho_e^+(x) = \mu_e^+(x)$, for all $x\epsilon X, e\epsilon E$ i.e.. iff F_ρ, G_μ.

(iii) Suppose $F_\rho \tilde{\subseteq} G_\mu \tilde{\subseteq} H_\eta$. then, we have, $t_F^e(x) \le t_G^e(x) \le t_H^e(x)$, $c_F^e(x) \le c_G^e(x) \le c_H^e(x)$, $u_F^e(x) \ge u_G^e(x) \ge u_H^e(x)$, $f_F^e(x) \ge f_G^e(x) \ge f_H^e(x)$, $\rho_e^-(x) \le \mu_e^-(x) \le \eta_e^-(x)$ and $\rho_e^+(x) \le \mu_e^+(x) \le \eta_e^+(x)$ for all $x\epsilon X$ and $e\epsilon E$. Consider $\tau_{1,e}^{F,G}(x)$ and $\tau_{2,e}^{F,G}(x)$. Since $t_F^e(x) \le t_G^e(x)$ holds, it follows that, $|t_G^e(x) - t_F^e(x)| \le |t_H^e(x) - t_F^e(x)| \Rightarrow \tau_{1,e}^{F,G}(x) \le \tau_{1,e}^{F,H}(x)$. Similarly it can be shown that $\tau_{i,e}^{F,G}(x) \le \tau_{i,e}^{F,H}(x)$, for $i = 3, 5, 6$ and all $x\epsilon X$. Next, consider $\tau_{2,e}^{F,G}(x)$.
Since, $f_F^e(x) \ge f_G^e(x) \ge f_H^e(x)$, it follows that $f_F^e(x) - f_G^e(x) \le f_F^e(x) - f_H^e(x)$ where $f_F^e(x) - f_G^e(x) \ge 0$, $f_F^e(x) - f_H^e(x) \ge 0$. Thus, $|f_F^e(x) - f_G^e(x)| \le |f_F^e(x) - f_H^e(x)| \Rightarrow \tau_{3,e}^{F,G}(x) \le \tau_{3,e}^{F,H}(x)$.
Also, it can be shown that $\tau_{4,e}^{F,G}(x) \le \tau_{4,e}^{F,H}(x)$ respectively for each $x\epsilon X$.
Thus, we have, $\sum_{e\epsilon E}\sum_{x\epsilon X}\sum_{i=1}^{n}\tau_{i,e}^{F,G}(x) \le \sum_{e\epsilon E}\sum_{x\epsilon X}\sum_{i=1}^{n}\tau_{i,e}^{F,H}(x)$
$\Rightarrow 1 - \frac{1}{6||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}\sum_{i=1}^{n}\tau_{i,e}^{F,H}(x) \le 1 - \frac{1}{6||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}\sum_{i=1}^{n}\tau_{i,e}^{F,G}(x)$
$\Rightarrow s(F_\rho, H_\eta) \le s(F_\rho, G_\mu)$
In an analogous manner, it can be shown that $s(F_\rho, H_\eta) \le s(G_\mu, H_\eta)$. Thus, we have, $s(F_\rho, H_\eta) \le s(F_\rho, G_\mu) \wedge s(G_\mu, H_\eta)$

Remark 2. $s(\tilde{A}_{\bar{1}}, \tilde{\theta}_{\bar{0}}) = 0$.

Proof :

For each $x\epsilon X$ and $e\epsilon E$,
$\tau_1^{\tilde{A}_{\bar{1}},\tilde{\theta}_{\bar{0}}}(x) = |t_{\tilde{\theta}_{\bar{0}}}^e(x) - t_{\tilde{A}_{\bar{1}}}^e(x)| = 1$, $\tau_2^{\tilde{A}_{\bar{1}},\tilde{\theta}_{\bar{0}}}(x) = |f_{\tilde{A}_{\bar{1}}}^e(x) - f_{\tilde{\theta}_{\bar{0}}}^e(x)| = 1$
$\tau_3^{\tilde{A}_{\bar{1}},\tilde{\theta}_{\bar{0}}}(x) = |c_{\tilde{\theta}_{\bar{0}}}^e(x) - c_{\tilde{A}_{\bar{1}}}^e(x)| = 1$, $\tau_4^{\tilde{A}_{\bar{1}},\tilde{\theta}_{\bar{0}}}(x) = |u_{\tilde{A}_{\bar{1}}}^e(x) - u_{\tilde{\theta}_{\bar{0}}}^e(x)| = 1$
$\tau_5^{\tilde{A}_{\bar{1}},\tilde{\theta}_{\bar{0}}}(x) = |\rho_e^-(x) - \mu_e^-(x)| = 1$, $\tau_6^{\tilde{A}_{\bar{1}},\tilde{\theta}_{\bar{0}}}(x) = |\rho_e^+(x) - \mu_e^+(x)| = 1$
which yields $\sum_{e\epsilon E}\sum_{x\epsilon X}\sum_{i=1}^{6}\tau_i^{\tilde{A}_{\bar{1}},\tilde{\theta}_{\bar{0}}}(x) = 6||X||.||E||$
$\Rightarrow s(\tilde{A}_{\bar{1}}, \tilde{\theta}_{\bar{0}}) = 1 - \frac{1}{6||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}\sum_{i=1}^{6}\tau_i^{\tilde{A}_{\bar{1}},\tilde{\theta}_{\bar{0}}}(x) = 0$.

Definition 21. Suppose $F_\rho, G_\mu \epsilon IPQSVNSS(X, E)$. Consider functions $\tau_{i,e}^{F,G} : X \rightarrow [0, 1]$, $i = 1, 2, .., 5$ as in *Definition 1*. Define a mapping $s_\omega : IPQSVNSS(X, E) \times IPQSVNSS(X, E) \rightarrow R^+$ as, $s_\omega(F_\rho, G_\mu) = 1 - \frac{\sum_{e\epsilon E}\sum_{x\epsilon X}\sum_{i=1}^{6}\omega(e)\tau_{i,e}^{F,G}(x)}{6||X||.||E||\sum_{e\epsilon E}\omega(e)}$, where $\omega(e)$ is the weight allocated to the parameter $e\epsilon E$ and $\omega(e)\epsilon[0, 1]$, for each $e\epsilon E$.

Theorem 7. *$s_\omega(F_\rho, G_\mu)$ is a similarity measure.*

Proof is similar to that of *Theorem 6*.

Remark 3. $s_\omega(F_\rho, G_\mu)$ is the weighted similarity measure between any two IPQSVNSS F_ρ and G_μ.

4.4.1 Allocation of entropy-based weights in calculating weighted similarity

It was shown in *Section 4.1.1* how entropy measure could be implemented to allocate specific weights to the elements of the parameter set. In this section, it is shown how the entropy-based weights can be implemented in calculating weighted similarity. Consider an IPQSVNSS F_ρ defined over the soft universe (X, E). Let $\omega_F(e)\epsilon[0, 1]$ be the weight allocated to an element $e\epsilon E$, w.r.t. the IPQSVNSS F_ρ.
Define $\omega_F(\alpha)$ as before, viz.
$\omega_F(\alpha) = \frac{\varepsilon(F_\rho)}{\kappa_F(\alpha)}$, where $\kappa_F(\alpha) = 1 - \frac{1}{||X||.||E||}\sum_{x\epsilon X}|t_F^\alpha(x) - f_F^\alpha(x)|.|c_F^\alpha(x) - u_F^\alpha(x)|.|1 - \{\rho_\alpha^+(x) + \rho_\alpha^-(x)\}|$
Consider any two IPQSVNSS $F_\rho, G_\mu \epsilon IPQSVNSS(X)$. Following Definition C, the weighted similarity measure between these two sets can be defined as
$s_\omega(F_\rho, G_\mu) = 1 - \frac{\sum_{e\epsilon E}\omega(\alpha)\{\sum_{x\epsilon X}\sum_{i=1}^{6}\tau_i^{F,G}(x)\}}{6||X||.||E||\sum_{e\epsilon E}\omega(\alpha)}$, where $\omega(\alpha) = \frac{\omega_F(\alpha)+\omega_G(\alpha)}{2}$, and $\omega_G(\alpha) = \frac{\varepsilon(G_\mu)}{\kappa_G(\alpha)}$ is the weight allocated to the parameter $\alpha\epsilon E$ w.r.t. the IPQSVNSS G_μ.
From previous results clearly, $\omega_F(\alpha), \omega_G(\alpha)\epsilon[0, 1] \Rightarrow \omega(\alpha)\epsilon[0, 1]$.

Example 6. Consider $F_\rho, G_\mu \epsilon IPQSVNSS(X)$ as defined in Example 1. Then $s(F_\rho, G_\mu) = 0.738$. Also, $\omega_F(e_1) = 0.983, \omega_G(e_1) = 0.987, \omega_F(e_2) = 0.993, \omega_G(e_2) = 0.988$, which gives, $\omega(e_1) = 0.985, \omega(e_2) = 0.991$ which finally yields $s_\omega(F_\rho, G_\mu) = 0.869$.

5 Relation between the various uncertainty based measures

Theorem 8. *$s_d^1(F_\rho, G_\mu) = 1 - d_h^N(F_\rho, G_\mu)$ is a similarity measure.*

Proof:

(i) $d_h^N(F_\rho, G_\mu) = d_h^N(G_\mu, F_\rho) \Rightarrow s_d^1(F_\rho, G_\mu) = s_d^1(G_\mu, F_\rho)$
(ii) $0 \le d_h^N(F_\rho, G_\mu) \le 1 \Rightarrow 0 \le s_d^1(F_\rho, G_\mu) \le 1$
Also, $s_d^1(F_\rho, G_\mu) = 1 \Leftrightarrow d_h^N(F_\rho, G_\mu) = 0 \Leftrightarrow F_\rho = G_\mu$.
(iii) Whenever $F_\rho \tilde{\subseteq} G_\mu \tilde{\subseteq} H_\eta$, $d_h^N(F_\rho, H_\eta) = d_h^N(F_\rho, G_\mu) + d_h^N(G_\mu, H_\eta)$. Thus,
$s_d^1(F_\rho, G_\mu) - s_d^1(F_\rho, H_\eta) = 1 - d_h^N(F_\rho, G_\mu) - 1 + d_h^N(F_\rho, H_\eta) = d_h^N(F_\rho, H_\eta) - d_h^N(F_\rho, G_\mu) = d_h^N(G_\mu, H_\eta) \ge 0$, from property of distance measure.
$\Rightarrow s_d^1(F_\rho, H_\eta) \le s_d^1(F_\rho, G_\mu)$.
Similarly, it can be shown that, $s_d^1(F_\rho, H_\eta) \le s_d^1(G_\mu, H_\eta)$.

Hence, $s_d^1(F_\rho, H_\eta) \leq s_d^1(F_\rho, G_\mu) \wedge s_d^1(G_\mu, H_\eta)$.

Remark 4. For any similarity measure $s(F_\rho, G_\mu)$, $1-s(F_\rho, G_\mu)$ may not be a distance measure.

Theorem 9. $s_d^2(F_\rho, G_\mu) = \frac{1}{1+d_h(F_\rho, G_\mu)}$ *is a similarity measure.*

Proof:

(i) $d_h(F_\rho, G_\mu) = d_h(G_\mu, F_\rho) \Rightarrow s_d^2(F_\rho, G_\mu) = s_d^2(G_\mu, F_\rho)$
(ii) $d_h(F_\rho, G_\mu) \geq 0 \Rightarrow 0 \leq s_d^2(F_\rho, G_\mu) \leq 1$. Also, $s_d^2(F_\rho, G_\mu) = 1 \Leftrightarrow d_h(F_\rho, G_\mu) = 0 \Leftrightarrow F_\rho = G_\mu$.
(iii) $d_h(F_\rho, H_\eta) = d_h(F_\rho, G_\mu) + d_h(G_\mu, H_\eta)$ whenever $F_\rho \tilde{\subseteq} G_\mu \tilde{\subseteq} H_\eta$.
$\Rightarrow d_h(F_\rho, H_\eta) \geq d_h(F_\rho, G_\mu)$ and $d_h(F_\rho, H_\eta) \geq d_h(G_\mu, H_\eta)$.
$\Rightarrow \frac{1}{1+d_h(F_\rho, H_\eta)} \leq \frac{1}{1+d_h(F_\rho, G_\mu)} \Rightarrow s_d^2(G_\mu, F_\rho) \leq s_d^2(F_\rho, G_\mu)$.
Similarly, it can be shown that, $s_d^2(G_\mu, F_\rho) \leq s_d^2(G_\mu, H_\eta)$.

Corollary 1. $s_d^3(F_\rho, G_\mu) = \frac{1}{1+d_h^N(F_\rho, G_\mu)}$ *is a similarity measure.*

Proofs follow in the exactly same way as the previous theorem.

Remark 5. For any similarity measure $s(F_\rho, G_\mu)$, $\frac{1}{s(F_\rho, G_\mu)} - 1$ may not be a distance measure.

Theorem 10 *Consider the similarity measure* $s(F_\rho, G_\mu)$. $s(F_\rho, F_\rho \tilde{\cap} G_\mu)$ *is an inclusion measure.*

Proof:

(i) Choose $F_\rho = \tilde{A}_{\bar{1}}$ and $G_\mu = \tilde{\theta}_{\bar{0}}$. Then, $s(F_\rho, F_\rho \tilde{\cap} G_\mu) = s(\tilde{A}_{\bar{1}}, \tilde{\theta}_{\bar{0}}) = 0$, from previous result.
(ii) $s(F_\rho, F_\rho \tilde{\cap} G_\mu) = 1 \Leftrightarrow F_\rho = F_\rho \tilde{\cap} G_\mu \Leftrightarrow F_\rho \tilde{\subseteq} G_\mu$.
(iii) Let $F_\rho \tilde{\subseteq} G_\mu \tilde{\subseteq} H_\eta$. Then, $s(F_\rho, H_\eta) \leq s(F_\rho, G_\mu)$ and $s(F_\rho, H_\eta) \leq s(G_\mu, H_\eta)$ hold. Consider $s(F_\rho, H_\eta) \leq s(F_\rho, G_\mu)$. From commutative property of similarity measure, it follows that, $s(H_\eta, F_\rho) \leq s(G_\mu, F_\rho) \Rightarrow s(H_\eta, H_\eta \tilde{\cap} F_\rho) \leq s(G_\mu, G_\mu \tilde{\cap} F_\rho)$. Similarly, $s(H_\eta, H_\eta \tilde{\cap} F_\rho) \leq s(F_\rho, F_\rho \tilde{\cap} G_\mu)$.

Theorem 11. $1 - d_h(F_\rho, F_\rho \tilde{\cap} G_\mu)$ *is an inclusion measure.*

Proof follows from the results of *Theorem 8* and *Theorem 10*.

Theorem 12. $\frac{1}{1+d_h(F_\rho, F_\rho \tilde{\cap} G_\mu)}$ *and* $\frac{1}{1+d_h^N(F_\rho, F_\rho \tilde{\cap} G_\mu)}$ *are inclusion measures.*

Proofs follow from *Theorem 9*,*Corollary 1* and *Theorem 10*.

Theorem 13. *Let* $e : IPQSVNSS(X, E) \to [0, 1]$ *be a measure of entropy such that* $\varepsilon(F_\rho) \leq \varepsilon(G_\mu) \Rightarrow F_\rho \tilde{\subseteq} G_\mu$. *Then* $|\varepsilon(F_\rho) - \varepsilon(G_\mu)|$ *is a distance measure.*

Proof:

(i) $|\varepsilon(F_\rho) - \varepsilon(G_\mu)| = |\varepsilon(G_\mu) - \varepsilon(F_\rho)|$
(ii) $|\varepsilon(F_\rho) - \varepsilon(G_\mu)| \geq 0$ and in particular, $|\varepsilon(F_\rho) - \varepsilon(G_\mu)| = 0 \Leftrightarrow \varepsilon(F_\rho) = \varepsilon(G_\mu) \Leftrightarrow \varepsilon(F_\rho) \leq \varepsilon(G_\mu)$ and $\varepsilon(F_\rho) \geq \varepsilon(G_\mu) \Leftrightarrow F_\rho = G_\mu$
(iii) Triangle inequality follows from the fact that, $|\varepsilon(F_\rho) - \varepsilon(H_\eta)| \leq |\varepsilon(F_\rho) - \varepsilon(G_\mu)| + |\varepsilon(G_\mu) - \varepsilon(H_\eta)|$ for any $F_\rho, G_\mu, H_\eta \epsilon IPQSVNSS(X, E)$.

6 Conclusions and Discussions

In this paper, the concept of interval possibility quadripartitioned single valued neutrosophic sets has been proposed. In the present set-theoretic structure an interval valued gradation of possibility viz. the chance of occurrence of an element with respect to a certain criteria is assigned and depending on that possibility of occurrence the degree of belongingness, non-belongingness, contradiction and ignorance are assigned thereafter. Thus, this structure comes as a generalization of the existing structures involving the theory of possibility namely, possibility fuzzy soft sets and possibility intuitionistic fuzzy soft sets. In the present work, the relationship between the various uncertainty based measures have been established. Applications have been shown where the entropy measure has been utilized to assign weights to the elements of the parameter set which were later implemented in a decision making problem and also in calculating a weighted similarity measure. The proposed theory is expected to have wide applications in processes where parameter-based selection is involved.

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References

[1] H. Aktas and N. Cagman Soft sets and soft groups, Information Sciences, 177 (2007), 2726–2735.

[2] S. Alkhazaleh, A. R. Salleh and Nasruddin Hassan Possibility Fuzzy Soft Set, Advances in Decision Sciences, 2011 (2011), 1-18.

[3] M. Bashir, A. R. Salleh and S. Alkhazaleh Possibility Intuitionistic Fuzzy Soft Set, Advances in Decision Sciences, 2012 (2012), 1-24.

[4] N. D. Belnap Jr. A useful four valued logic, Modern Uses of Multiple-Valued Logic, 1904 (1977), 9-37.

[5] R. Chatterjee, P. Majumdar and S. K. Samanta On some similarity measures and entropy on quadripartitioned single valued neutrosophic sets, Journal of Intelligent and Fuzzy Systems, 30(4) (2016), 2475-2485.

[6] D. Chen, E. C. C. Tsang, D. S. Yeung and X. Wang The parametrized reduction of soft sets and its applications, Computers and Mathematics with Applications, 49 (2009), 757-763.

[7] S. Das and S. K. Samanta Soft real sets, soft real numbers and their properties, Journal of Fuzzy Mathematics, 20(3) (2012), 551-576.

[8] P. K. Maji, R. Biswas and A. R. Roy Fuzzy soft sets, Journal of fuzzy mathematics, 9(3) (2001), 589-602.

[9] P. K. Maji, R. Biswas and A. R. Roy Intuitionistic fuzzy soft sets, Journal of fuzzy mathematics, 9(3) (2001), 677-692.

[10] P. K. Maji, R. Biswas and A. R. Roy Soft set theory, Computers and Mathematics with Applications,45 (2003), 555-562.

[11] P. K. Maji Neutrosophic soft set, Annals of Fuzzy Mathematics and Informatics, 5(1) (2013), 157-168.

[12] P. Majumdar and S. K. Samanta Similarity measureof soft sets, New Mathematics and Natural Computation, 4(1) (2008), 1-12.

[13] P. Majumdar and S. K. Samanta Generalized fuzzy soft sets, Computers and Mathematics with Applications, 59 (2010), 1425-1432.

[14] P. Majumdar and S. K. Samanta Softness of a soft set: soft set entropy, Annals of Fuzzy Mathematics with Informatics, 6(1) (2013), 50-68.

[15] D. Molodstov Soft set theory-First results, Computers and Mathematics with Applications, 37 (1999), 19-31.

[16] M. M. Mushrif, S. Sengupta and A. K. Roy Texture classification using a novel soft-set theory based classification algorithm, Springer-Verlag Berlin Heidelberg (2006), 246-254.

[17] D. Pei and D. Miao From soft sets to information systems, Proceedings of Granular Computing 2 IEEE (2005), 617-621.

[18] F. Smarandache n-valued Refined Neutrosophic Logic and Its Applications to Physics, arXiv preprint arXiv:1407.1041 (2014).

[19] H. Wang, F. Smarandache, Y. Zhang and R Sunderraman Single Valued Neutrosophic Sets, Multispace and Multistructure, 4 (2010), 410-413.

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Modified Collatz conjecture or *(3a + 1) + (3b + 1)I* Conjecture for Neutrosophic Numbers $\langle Z \cup I \rangle$

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Abstract: In this paper, a modified form of Collatz conjecture for neutrosophic numbers $\langle Z \cup I \rangle$ is defined. We see for any $n \in \langle Z \cup I \rangle$ the related sequence using the formula $(3a + 1) + (3b + 1)I$ converges to any one of the 55 elements mentioned in this paper. Using the akin formula of Collatz conjecture viz. $(3a - 1) + (3b - 1)I$ the neutrosophic numbers converges to any one of the 55 elements mentioned with appropriate modifications. Thus, it is conjectured that every $n \in \langle Z \cup I \rangle$ has a finite sequence which converges to any one of the 55 elements.

Keywords: Collatz Conjecture, Modified Collatz Conjecture, Neutrosophic Numbers.

1 Introduction

The Collatz conjecture was proposed by Lothar Collatz in 1937. Till date this conjecture remains open. The 3n – 1 conjecture was proposed by authors [9]. Later in [9] the 3n ± p conjecture; a generalization of Collatz Conjecture was proposed in 2016 [9].

However, to the best of authors knowledge, no one has studied the Collatz Conjecture in the context of neutrosophic numbers $\langle Z \cup I \rangle = \{a + bI / a, b \in Z; I^2 = I\}$ where I is the neutrosophic element or indeterminancy introduced by [7]. Several properties about neutrosophic numbers have been studied. In this paper, authors for the first time study Collatz Conjecture for neutrosophic numbers. This paper is organized into three sections.

Section one is introductory. Section two defines / describes Collatz conjecture for neutrosophic numbers. Final section gives conclusions based on this study. Extensive study of Collatz conjecture by researchers can be found in [1-6]. Collatz conjecture or 3n + 1 conjecture can be described as for any positive integer n perform the following operations.

If n is even divide by 2 and get $\frac{n}{2}$ if $\frac{n}{2}$ is even divide by 2 and proceed till $\frac{n}{2^t}$ is odd.

If n is odd multiply n by 3 and add 1 to it and find 3n + 1. Repeat the process (which has been called Half of Triple Plus One or HTPO) indefinitely. The conjecture puts forth the following hypothesis; whatever positive number one starts with one will always eventually reach 1 after a finite number of steps.

Let n = 3, the related sequence is 3n + 1, 10, 5, 16, 8, 4, 2, 1.

Let n = 11, the related sequence is 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.

Let n = 15, the related sequence is 15, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1.

In simple notation of mod 2 this conjecture can be viewed as

$$f(n) = \begin{cases} n/2 & \text{if } n \equiv 0 \pmod 2 \\ 3n+1 & \text{if } n \equiv 1 \pmod 2 \end{cases}.$$

The total stopping time for very large numbers have been calculated. The 3n – 1 conjecture is a kin to Collatz conjecture.

Take any positive integer n. If n is even divide by 2 and get $\frac{n}{2}$ if $\frac{n}{2}$ is odd multiply it by 3 and subtract 1 to i.e. 3n – 1, repeat this process indefinitely, [9] calls this method as Half Or Triple Minus One (HOTMO).

The conjecture state for all positive n, the number will converge to 1 or 5 or 17.

In other words, the 3n – 1 conjecture can be described as follows.

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ 3n-1 & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

Let n = 3, 3n – 1 = 8, 4, 2, 1.
Let n = 28, 14, 7, 20, 10, 5.
n = 17, 50, 25, 74, 37, 110, 55, 164, 82, 41, 122, 61, 182, 91, 272, 136, 68, 34, 17.

Several interesting features about the 3n – 1 conjecture is derived and described explicitly in [9].

It is pertinent to keep on record in the Coltaz conjecture 3n + 1 if n is taken as a negative number than using 3n + 1 for negative values sequence terminate only at – 1 or – 5 or – 17. Further the 3n – 1 conjecture for any negative n, the sequence ends only in – 1.

Thus, for using 3n + 1 any integer positive or negative the sequence terminates at any one of the values {–17, –5, –1, 0, 1} and using 3n – 1 the sequence for any integer n positive or negative terminates at any one of the values {–1, 0, 1, 5, 17}.

2 Collatz Conjecture for the neutrosophic numbers ⟨Z ∪ I⟩

In this section, we introduce the modified form of Collatz conjecture in case of neutrosophic numbers ⟨Z ∪ I⟩ = {a + bI / a, b ∈ Z and I^2 = I} where I is the neutrosophic element or the indeterminancy introduced by [7]. For more info, please refer to [7].

Now, we will see how elements of ⟨Z ∪ I⟩ behave when we try to apply the modified form of Collatz conjecture.

The modified formula for Collatz conjecture for neutrosophic numbers n = a + bI is (3a + 1) + (3b + 1)I; if a = 0 then 3bI + I = (3b + 1)I is taken if b = 0 then 3a + 1 term is taken, however iteration is taken the same number of times for a and bI in n = a + bI.

If n ∈ ⟨Z ∪ I⟩ is of the form n = a, a ∈ Z then Collatz conjecture is the same, when n = aI, a ∈ I, I^2 = I then also the Collatz conjecture takes the value I; for we say aI is even if a is even and aI is odd is a is odd.

For 3I, 9I, 27I, 15I, 45I, 19I, 35I, 47I, 105I, 101I, 125I are all odd neutrosophic numbers.

Now 12I, 16I, 248I, 256I etc. are even neutrosophic numbers.

The working is instead of adding 1 after multiplying with 3 we add I after multiplying with 3.

For instance consider n = 12I, the sequence for n = 12I is as follows:

12I, 6I, 3I, 3 × 3I + I = 10I, 5I, 16I, 8I, 4I, 2I, I.

So the element n = 12I has a sequence which terminates at I.

Consider n = 256I, the sequence is 256I, 128I, 64I, 32I, 16I, 8I, 4I, 2I, I so converges to I.

Take n = 31I, 31I is odd so the sequence for n = 31I is

31I, 94I, 47I, 142I, 71I, 214I, 107I, 322I, 161I, 484I, 242I, 121I, 364I, 182I, 91I, 274I, 137I, 412I, 206I, 103I, 310I, 155I, 466I, 233I, 700I, 350I, 175I, 526I, 263I, 790I, 385I, 1156I, 578I, 289I, 868I, 434I, 217I, 652I, 326I, 163I, 490I, 245I, 736I, 368I, 184I, 92I, 46I, 23I, 70I, 35I, 106I, 53I, 160I, 80I, 40I, 20I, 10I, 5I, 16I, 8I, 4I, 2I, I.

Let n = 45I the sequence is 45I, 136I, 68I, 34I, 17I, 52I, 26I, 13I, 40I, 20I, 10I, 5I, 16I, 8I, 4I, 2I, I.

So if n ∈ Z then as usual by the Collatz conjecture the sequence converges to 1. If n ∈ ZI then by applying the Collatz conjecture it converges to I. Now if x ∈ ⟨Z ∪ I⟩ that is x = a + bI how does x converge.

We will illustrate this by an example.

Now if x = a + bI, a, b ∈ Z \ {0}; is it even or odd? We cannot define or put the element x to be odd or to be even. Thus to apply Collatz conjecture one is forced to define in a very different way. We apply the Collatz conjecture separately for a and for bI, but maintain the number of iterations to be the same as for that of a + bI. We will illustrate this situation by some examples.

Consider n = 3I + 14 ∈ ⟨Z ∪ I⟩. n is neither odd nor even. We use (3a + 1) + (3b + 1)I formula in the following way

3I + 14, 10I + 7, 5I + 22, 16I + 11, 8I + 34, 4I + 17, 2I + 52, I + 26, 4I + 13, 2I + 40, I + 20, 4I + 10, 2I + 5, I + 16, 4I + 8, 2I + 4, I + 2, 4I + 1, 2I + 4, I + 2, 4I +1, I + 4, I + 2.

So the sequence terminates at I + 2.

Consider n = 3I – 14 ∈ ⟨Z ∪ I⟩, n is neither even nor odd.

The sequence for this n is as follows.

3I – 14, 10I – 7, 5I – 20, 16I – 10, 8I – 5, 4I – 14, 2I – 7, I – 20, 4I – 10, 2I – 5, I – 14, 4I – 7, 2I – 20, I – 10, 4I – 5, 2I – 14, I – 7, 4I – 20, 2I – 10, I – 5, 4I – 14, 2I – 7, I – 20, 4I – 10, 2I – 5, ... , I – 5.

So for n = 3I – 14 the sequence converges to 2I – 5.

Consider n = – 5I – 34; – 5I – 34, –14I –17, –7I – 50, –20I –25, –10I –74, –5I –37, –14I, –110, –7I –55, –20I – 164, –10I – 82, –5I –41, –14I –122, –7I –61, –20I –182, –10I –91, –5I – 272, –14I –136, –7I – 68, –20I – 34, –10I –17, –5I – 50, –14I –25, –7I – 74, –20I –37, –10I –110, –5I –55, –14I –164, –7I –82, –20I –41, –10I –122, –5I –61, –14I –182, –7I –91, –20I –272, –10I –136, –5I –68, –14I –34, –7I –17, –20I –50, –10I –25, –5I –74, –14I –37, –7I –110, –20I –55, –10I –164, –5I –82, –14I –41, –7I –122, –20I –61, –10I –182, –5I –91, –14I –272, –7I –136, –20I –68, –10I –34, –5I – 17. (1)

n = – 5I – 34, converges to –5I – 17.

Let n = –10I –17, –5I – 50, –14I –25, –7I –74, –20I – 37, –10I –110, –5I – 55, –14I – 164, –7I –82, –20I – 41, –10I – 122, –5I –61, –14I –182, –7I –91, –20I –272, –10I – 136, –5I – 68, –14I – 34, –7I –17, – 20I – 50, –10I –25, –5I –74, –14I –37, –7I – 110, –20I – 55, –10I – 164, –5I – 82, –14I – 41, 7I – 122, –20I – 61, –10I – 182, –5I – 91, –14I – 272, –7I –136, –20I –68, –10I – 34, –5I – 17.

Thus, by using the modified form of Collatz conjecture for neutrosophic numbers ⟨Z ∪ I⟩ we get the following collection A of numbers as the limits of finite sequences after performing the above discussed operations using the modified formula 3(a + bI) + 1 + I or (3a + 1) + (3b + 1)I; a, b ∈ Z \ {0} if a = 0 then (3b + 1)I formula and if b = 0 then 3a + 1 formula is used.

A = {1, –1, 0, I, –I, 1 + I, –I + 1, –1 + I, –1 – I, –17, –5, –17I, –5I, 1 + 2I, 1 – 2I, –1 – 2I, –1 + 2I, 2 – I, 2 + I, –2 – I, –2 + I, –5 + I, –5 + 2I, –5 – 17I, –5 – I, –5 – 2I, –51 + 1, –5I + 2, –5I – 2, –5I – 1, –5I – 17, –17 – I, –17 + I, –17I + 1, –17I – 1, –17 – 2I, –17 + 2I, –17I + 2, –17I – 2, 1 + 4I, 4I + 1, 4 – I, 4I – 1, –34 –5I, –17I –10, – 17 – 10I, –34I – 5, –17 – 20I, –17I – 20, – 68I – 5, – 68 – 5I, – 5I + 4, – 5 + 4I, –17 + 4I, –17I +4}.

Thus, the modified 3n + 1 Collatz conjecture for neutrosophic numbers ⟨Z ∪ I⟩ is (3a + 1) + (3b + 1) I for n = a + bI ∈ ⟨Z ∪ I⟩, a, b ∈ Z \ {0}.

If a = 0 then we use the formula (3b + 1)I and if b = 0 then use the classical Collatz conjecture formula 3a + 1. It is conjectured that using (3a + 1) + (3b + 1)I where a, b ∈ Z \ {0} or 3a + 1 if b = 0 or (3b + 1)I if a = 0, formula every n ∈ ⟨Z ∪ I⟩ ends after a finite number of iterations to one and only one of the 55 elements from the set A given above. Prove or disprove.

Now the 3n – 1 conjecture for neutrosophic numbers ⟨Z ∪ I⟩ reads as (3a – 1) + (3bI – I) where n = a + bI; a, b ∈ Z \ {0}; if a = 0 then (3b – 1)I = 3bI – I is used instead of 3n – 1 or (3a – 1) + (3b – 1) I.

If b = 0 then 3a – 1 that is formula 3n – 1 is used.

Now every n ∈ ⟨Z ∪ I⟩ the sequence converges to using the modified 3n – 1 Collatz conjecture (3a – 1) + (3b – 1)I to one of the elements in the set B; where B = {1, 0, –1, I, 5I, 5, 17, 17I, –I, 1 + 2I, 1 – 2I, –1 + 2I, –1 –2I, 1 + I, I – 2, I + 2, –I –2, –I + 2, I – 1, –I – 1, 5 + I, 5 – I, 5 – 2I, 5 + 2I, –I + 1, 5 + 17I, 17 – I, 17 + I, 17 – 2I, 17 + 2I, 17 + 5I, 5I – 1, 5I – 2, 5I + 1, 5I + 2, 17I – 1, 17I – 2, 17I + 1, 17I + 2, 17 + 10I, 17I + 10, 34 + 5I, 34I + 5, 17 + 20I, 20 + 17I, 68 + 5I, 68I + 5, 5I – 4, 5 – 4I, 17 – 4I, 17I – 4, –4I + 1, –4I – 1, –4 + I, –4 – I }.

We will just illustrate how the (3a – 1) + (3b – 1)I formula functions on ⟨Z ∪ I⟩.

Consider 12 + 17I ∈ ⟨Z ∪ I⟩ the sequence attached to it is 12 + 17I, 6 + 50I, 3 + 25I, 8 + 74I, 4 + 37I, 2 + 110I, 1 + 55I, 2 + 164I, 1 + 82I, 2 + 41I, 1 + 122I, 2 + 61I, 1 + 182I, 2 + 91I, 1 + 272I, 2 + 136I, 1 + 68I, 2 + 34I, 1 + 17I, 2 + 50I, 1 + 25I, 2 + 74I, 1 + 37I, 2 + 110I, 1 + 55I, 2 + 164I, 1 + 82I, 2 + 41I, 1 + 122I, 2 + 61I, 1 + 182I, 2 + 91I, 1 + 272I, 2 + 136I, 1 + 68I, 2 + 34I, 1 + 17I.

The sequence associated with 12 + 17I terminates at 1 + 17I.

Thus, it is conjectured that every n ∈ ⟨Z ∪ I⟩ using the modified Collatz conjecture (3a – 1) + (3b – 1)I; a, b ∈ Z \ {0} or 3a – 1 if b = 0 or (3b + 1)I if a = 0, has a finite sequence which terminates at only one of the elements from the set B.

3 Conclusions

In this paper, the modified form of 3n ± 1 Collatz conjecture for neutrosophic numbers ⟨Z ∪ I⟩ is defined and described. It is defined analogously as (3a ± 1) + (3b ± 1) I where a + bI ∈ ⟨Z ∪ I⟩ with a ≠ 0 and b ≠ 0.

If a = 0 the formula reduces to (3b ± 1)I and if b = 0 the formula reduces to (3a ± 1).

It is conjectured every n ∈ ⟨Z ∪ I⟩ using the modified form of Collatz conjecture has a finite sequence which terminates at one and only element from the set A or B according as (3a + 1) + (3b + 1)I formula is used or (3a – 1) + (3b – 1)I formula is used respectively. Thus, when a neutrosophic number is used from ⟨Z ∪ I⟩ the number of values to which the sequence terminates after a finite number of steps is increased from 5 in case of 3n ±1 Collatz conjecture to 55 when using (3a ± 1) + (3b ± 1)I the modified Collatz conjecture.

References

[1] Stefan Andrei and Cristian Masalagiu. About the Collatz conjecture. *Acta Informatica*, 35(2):167-179, 1998.

[2] Lynn E Garner. On the Collatz 3n+ 1 algorithm. *Proceedings of the American Mathematical Society*, 82(1):19-22, 1981.

[3] Michael A Idowu. A novel theoretical framework formulated for information discovery from number system and Collatz conjecture data. *Procedia Computer Science*, 61:105-111, 2015.

[4] Jefrey C Lagarias. The 3x+ 1 problem and its generalizations. *The American Mathematical Monthly*, 92(1):3-23, 1985.

[5] K LaTourette et al. Explorations of the Collatz conjecture. *Moravian College Senior Honours Thesis*, 2007.

[6] Jean Paul Van Bendegem. The Collatz conjecture. a case study in mathematical problem solving. *Logic and Logical Philosophy*, 14(1):7-23, 2005.

[7] Florentin Smarandache, Neturosophic logic - Generalization of Intuitionistic Fuzzy Logic, presented at the special session on Intuitionistic Fuzzy Sets and Related Concepts, of International EUSFLAT Conference, Zittau, Germany, 10-2 September 2003.

[8] Vasantha Kandasamy and Florentin Smarandache, Basic Neutrosophic Algebraic Structures and their Application to Fuzzy and Neutrosophic Models, Hexis, US, 2004.

[9] Vasantha Kandasamy, Ilanthenral and Florentin Smarandache, The 3n ± p conjecture: A generalization of Collatz Conjecture, *Journal of Ultra Scientist of Physical Sciences*, 29(2):83 -88, 2017.

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Neutrosophic Cubic Subalgebras and Neutrosophic Cubic Closed Ideals of B-algebras

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Abstract: The objective of this paper is to introduced the concept of neutrosophic cubic set to subalgebras, ideals and closed ideals of B-algebra. Links among neutrosophic cubic subalgebra with neutrosophic cubic ideals and neutrosophic closed ideals of B-algebras as well as some related properties will be investigated. This study will cover homomorphic images and inverse homomorphic images of neutrosophic cubic subalgebras, ideals and some related properties. The Cartesian product of neutrosophic cubic subalgebras will also be investigated.

Keywords: B-algebra, Neutrosophic cubic set, Neutrosophic cubic subalgebra, Neutrosophic cubic closed ideals.

1 Introduction

The concept of fuzzy sets were first introduced by Zadeh (see [31]) in 1965. After that several researchers conducted researches on generalization of fuzzy sets notion. Zadeh (see [32]) generalized the concept of fuzzy set by an interval-valued fuzzy set in 1975, as a generalization of the notion. The concept of cubic sets had been introduced by Jun et al. (see [6]) in 2012, as generalization of fuzzy set and interval-valued fuzzy set. Jun et al. (see [7]) applied the notion of cubic sets to a group, and introduced the notion of cubic subgroups in 2011. Senapati et. al. (see [25]) extended the concept of cubic set to subalgebras, ideals and closed ideals of B-algebra with lots of properties investigated. After the introduction of two classes BCK-algebra and BCI-algebra by Imai and Iseki (see [4, 5]). The concept of cubic sets to subalgebras, ideals and q-ideals in BCK/BCI-algebras was applied by Jun et al. (see [9, 10]). B-algebra was introduced by Neggers and Kim (see [12]) in 2002, which are related to extensive classes of algebras such as BCI/BCK-algebras. The relations between B-algebra and other topics were further discussed by Cho and Kim in (see [3]) 2001. Every quadratic B-algebra on field X with a BCI-algebra was obtained by Park and Kim (see [14]) in 2001. The notion of fuzzy topological B-algebra was introduced by Borumand Saeid (see [15]) in 2006. Also Saeid introduced the concept of interval-valued fuzzy subalgebra of B-algebra (see [16]) in 2006. Also some of their properties were studied by him. Walendziak (see [30]) gave some systems of axioms defining a B-algebra with the proof of the independent of axioms in 2006. Fuzzy dot subalgebras, fuzzy dot ideals, interval-valued fuzzy closed ideals of B-algebra and fuzzy subalgebras of B-algebras with respect to t-norm were introduced by Senapati et. al. (see [20, 21, 22, 23]). Also L-fuzzy G-subalgebras of G-algebras were introduced by Senapati et. al. (see [24]) in 2014 which is related to B-algebra. As a generalizations of B-algebras, lots of researches on BG-algebras (see [11]) have been done by the authors (see [26, 27, 28, 29]).

Smarandache (see [19, 18]) introduced the concept of neutrosophic cubic set is a more general research area which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Jun et. al. (see [8]) extended the concept of cubic set to neutrosophic cubic set and introduced. The notion of truth-internal (indeterminacy-internal, falsity-internal) and truth-external (indeterminacy-external, falsity-external) are introduced and related properties are investigated.

In this paper, we will introduce the concept of neutrosophic cubic set to subalgebras, ideals and closed ideals of B-algebras and introduce the notion of neutrosophic cubic set and subalgebras. Relation among neutrosophic cubic algebra with neutrosophic cubic ideals and neutrosophic closed ideals of B-algebras are studied and some related properties will be investigated. This study will cover homomorphic images and inverse homomorphic images of neutrosophic cubic subalgebras, ideals, some related properties. The Cartesian product of neutrosophic cubic subalgebras will also be investigated.

2 Preliminaries

In this section, some basic facets are included that are necessary for this paper. A B-algebra is an important class of logical algebras introduced by Neggers and Kim [12] and extendedly investigated by several researchers. This algebra is defined as follows.

A non-empty set X with constant 0 and a binary operation $*$ is called to be B-algebra [12] if it satisfies the following axioms:

B1. $x * x = 0$

B2. $x * 0 = x$
B3. $(x * y) * z = x * (z * (0 * y))$
A non-empty subset S of B-algebra X is called a subalgebra [1] of X if $x*y \in S \,\forall\, x, y \in S$. Mapping $f \mid X \to Y$ of B-algebras is called homomorphism [13] if $f(x * y) = f(x) * f(y) \,\forall\, x, y \in X$. Note that if $f \mid X \to Y$ is a $B-$homomorphism, then $f(0) = 0$. A non-empty subset I of a $B-$algebra X is called an ideal [22] if for any $x, y \in X$, (i) $0 \in I$, and (ii) $x * y \in I$ and $y \in I \Rightarrow x \in I$. An ideal I of a $B-$algebra X is called closed if $0 * y \in I \,\forall\, x \in I$.

We know review some fuzzy logic concepts as follows:

Let X be the collection of objects denoted generally by x. Then a fuzzy set [31] A in X is defined as $A = \{< x, \mu_A(x) > \mid x \in X\}$, where $\mu_A(x)$ is called the membership value of x in A and $\mu_A(x) \in [0, 1]$.

For a family $A_i = \{< x, \mu_{A_i}(x) > \mid x \in X\}$ of fuzzy sets in X, where $i \in k$ and k is index set, we define the join ($\vee$) meet ($\wedge$) operations as follows:

$$\bigvee_{i\in k} A_i = \Big(\bigvee_{i\in k} \mu_{A_i}\Big)(x) = sup\{\mu_{A_i} \mid i \in k\},$$

and

$$\bigwedge_{i\in k} A_i = \Big(\bigwedge_{i\in k} \mu_{A_i}\Big)(x) = inf\{\mu_{A_i} \mid i \in k\}$$

respectively, $\forall\, x \in X$.

An Interval-valued fuzzy set [32] A over X is an object having the form $A = \{< x, \tilde{\mu}_A(x) > \mid x \in X\}$, where $\tilde{\mu}_A \mid X \to D[0, 1]$, here $D[0, 1]$ is the set of all subintervals of [0,1]. The intervals $\tilde{\mu}_A x = [\mu_A^-(x), \mu_A^+(x)] \,\forall\, x \in X$ denote the degree of membership of the element x to the set A. Also $\tilde{\mu}_A^c = [1 - \mu_A^-(x), 1 - \mu_A^+(x)]$ represents the complement of $\tilde{\mu}_A$.

For a family $\{A_i \mid i \in k\}$ of interval-valued fuzzy sets in X where k is an index set, the union $G = \bigcup_{i\in k} \tilde{\mu}_{A_i}(x)$ and the intersection $F = \bigcap_{i\in k} \tilde{\mu}_{A_i}(x)$ are defined below:

$$G(x) = \Big(\bigcup_{i\in k} \tilde{\mu}_{A_i}\Big)(x) = rsup\{\tilde{\mu}_{A_i}(x) \mid i \in k\}$$

and

$$F(x) = \Big(\bigcap_{i\in k} \tilde{\mu}_{A_i}\Big)(x) = rinf\{\tilde{\mu}_{A_i}(x) \mid i \in k\},$$

respectively, $\forall\, x \in X$.

The determination of supremum and infimum between two real numbers is very simple but it is not simple for two intervals. Biswas [2] describe a method to find max/sup and min/inf between two intervals or a set of intervals.

Definition 2.1 *[2] Consider two elements $D_1, D_2 \in D[0, 1]$. If $D_1 = [a_1^-, a_1^+]$ and $D_2 = [a_2^-, a_2^+]$, then $rmax(D_1, D_2) = [max(a_1^-, a_2^-), max(a_1^+, a_2^+)]$ which is denoted by $D_1 \vee^r D_2$ and $rmin(D_1, D_2) = [min(a_1^-, a_2^-), min(a_a^+, a_2^+)]$ which is denoted by $D_1 \wedge^r D_2$. Thus, if $D_i = [a_i^-, a_2^+] \in D[0, 1]\, for\, i = 1, 2, 3, \ldots,$ then we define $rsup_i(D_i) = [sup_i(a_i^-), sup_i(a_i^+)]$, i.e., $\vee_i^r D_i = [\vee_i a_i^-, \vee_i a_i^+]$. Similarly we define $rinf_i(D_i) = [inf_i(a_i^-), inf_i(a_i^+)]$, i.e., $\wedge_i^r D_i = [\wedge_i a_i^-, \wedge_i a_i^+]$. Now we call $D_1 \geq D_2 \iff a_1^- \geq a_2^-$ and $a_1^+ \geq a_2^+$. Similarly the relations $D_1 \leq D_2$ and $D_1 = D_2$ are defined.*

Combine the definition of subalgebra, ideal over crisp set and the idea of fuzzy set Ahn et al. [1] and senapati et al. [21] defined fuzzy subalgebra and ideal respectively, which is define bellow.

Definition 2.2 *[21, 1] A fuzzy set $A = \{< x, \mu_A(x) > \mid x \in X\}$ is called a fuzzy subalgebra of X if $\mu_A(x * y) \geq min\mu_A(x), \mu_A(y) \,\forall\, x, y \in X,$*

A fuzzy set $A = \{< x, \mu_A(x) > \mid x \in X\}$ in X is called a fuzzy ideal of X if it satisfies (i) $\mu_A(0) \geq \mu_A(x)$ and (ii) $\mu_A(x) \geq min\{\mu_A(x * y), \mu_A(y)\} \,\forall\, x, y \in X$.

Jun et al. [8] extend the concept of cubic sets to neutrosophic sets [17], and consider the notion of neutrosophic cubic sets as an extension of cubic sets, and investigated several properties.

Definition 2.3 *[8] Let X be a non-empty set. A neutrosophic cubic set in X is pair $\mathcal{C} = (\mathbf{A}, \Lambda)$ where $\mathbf{A} =: \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$ is an interval neutrosophic set in X and $\Lambda =: \{\langle x; \lambda_T(x), \lambda_I(x), \lambda_F(x)\rangle \mid x \in X\}$ is a neutrosophic set in X.*

Definition 2.4 *[8] For any $\mathcal{C}_i = (\mathbf{A}_i, \Lambda_i)$ where*
$\mathbf{A}_i =: \{\langle x; A_{iT}(x), A_{iI}(x), A_{iF}(x)\rangle \mid x \in X\}$,
$\Lambda_i =: \{\langle x; \lambda_{iT}(x), \lambda_{iI}(x), \lambda_{iF}(x)\rangle \mid x \in X\}$ *for $i \in k$, P-union, P-inersection, R-union and R-intersection is defined respectively by*

P-union: $\bigcup_{P_{i\in k}} \mathcal{C}_i = \Big(\bigcup_{i\in k} \mathbf{A}_i, \bigvee_{i\in k} \Lambda_i\Big)$,

P-intersection: $\bigcap_{P_{i\in k}} \mathcal{C}_i = \Big(\bigcap_{i\in k} \mathbf{A}_i, \bigwedge_{i\in k} \Lambda_i\Big)$

R-union: $\bigcup_{R_{i\in k}} \mathcal{C}_i = \Big(\bigcup_{i\in k} \mathbf{A}_i, \bigwedge_{i\in k} \Lambda_i\Big)$,

R-intersection: $\bigcap_{R_{i\in k}} \mathcal{C}_i = \Big(\bigcap_{i\in k} \mathbf{A}_i, \bigvee_{i\in k} \Lambda_i\Big)$

where

$$\bigcup_{i\in k} \mathbf{A}_i = \left\{\left\langle x; \Big(\bigcup_{i\in k} A_{iT}\Big)(x), \Big(\bigcup_{i\in k} A_{iI}\Big)(x), \Big(\bigcup_{i\in k} A_{iF}\Big)(x)\right\rangle \mid x \in X\right\},$$

$$\bigvee_{i\in k} \Lambda_i = \left\{\left\langle x; \Big(\bigvee_{i\in k} \lambda_{iT}\Big)(x), \Big(\bigvee_{i\in k} \lambda_{iI}\Big)(x), \Big(\bigvee_{i\in k} \lambda_{iF}\Big)(x)\right\rangle \mid x \in X\right\},$$

$$\bigcap_{i\in k} \mathbf{A}_i = \left\{\left\langle x; \Big(\bigcap_{i\in k} A_{iT}\Big)(x), \Big(\bigcap_{i\in k} A_{iI}\Big)(x), \Big(\bigcap_{i\in k} A_{iF}\Big)(x)\right\rangle \mid x \in X\right\},$$

$$\bigwedge_{i\in k} \Lambda_i = \left\{\left\langle x; \Big(\bigwedge_{i\in k} \lambda_{iT}\Big)(x), \Big(\bigwedge_{i\in k} \lambda_{iI}\Big)(x), \Big(\bigwedge_{i\in k} \lambda_{iF}\Big)(x)\right\rangle \mid x \in X\right\},$$

Senapati et. al. [25] defined the cubic subalgebras of B-algebra by combining the definitions of subalgebra over crisp set and the cubic sets.

Definition 2.5 *[25] Let $\mathrm{C} = \{< x, A(x), \lambda(x) >\}$ be a cubic set, where $A(x)$ is an interval-valued fuzzy set in X, $\lambda(x)$ is a fuzzy set in X and X is subalgebra. Then A is cubic subalgebra*

*under binary operation * if following condition holds:*
C1: $A(x*y) \geq rmin\{A(x), A(y)\}$,
C2: $\lambda(x*y) \leq max\{\lambda(x), \lambda(y)\}\ \forall\, x, y \in X$.

3 Neutrosophic Cubic Subalgebras Of B-algebra

Let X denote a B-algebra then the concept of cubic subalgebra can be extended to neutrosophic cubic subalgebra.

Definition 3.1 *Let $\mathcal{C} = (\mathbf{A}, \Lambda)$ be a cubic set, where X is subalgebra. Then $\mathcal{C}$ is neutrosophic cubic subalgebra under binary operation $*$ if it holds the following conditions: N1:*
$A_T(x*y) \geq rmin\{A_T(x), A_T(y)\}$
$A_I(x*y) \geq rmin\{A_I(x), A_I(y)\}$
$A_F(x*y) \geq rmin\{A_F(x), A_F(y)\}$,
N2:
$\Lambda_T(x*y) \leq max\{\Lambda_T(x), \Lambda_T(y)\}$
$\Lambda_I(x*y) \leq max\{\Lambda_I(x), \Lambda_I(y)\}$
$\Lambda_I(x*y) \leq max\{\Lambda_I(x), \Lambda_I(y)\}$
For our convenience, we will denote neutrosophic cubic set as
$\mathcal{C} = (A_{T,I,F}, \lambda_{T,I,F}) = \{\langle x, A_{T,I,F}(x), \lambda_{T,I,F}(x)\rangle\}$
and conditions N1, N2 as
N1: $A_{T,I,F}(x*y) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$,
N2: $\lambda_{T,I,F}(x*y) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$.

Example 3.1 *Let $X = \{0, a_1, a_2, a_3, a_4, a_5\}$ be a B-algebra with the following Cayley table.*

$*$	0	a_1	a_2	a_3	a_4	a_5
0	0	a_5	a_4	a_3	a_2	a_1
a_1	a_1	0	a_5	a_4	a_3	a_2
a_2	a_2	a_1	0	a_5	a_4	a_3
a_3	a_3	a_2	a_1	0	a_5	a_4
a_4	a_4	a_3	a_2	a_1	0	a_5
a_5	a_5	a_4	a_3	a_2	a_1	0

A neutrosophic cubic set $\mathcal{C} = (A_{T,I,F}, \lambda_{T,I,F})$ of X is defined by

	0	a_1	a_2	a_3	a_4	a_5
A_T	[0.7,0.9]	[0.6,0.8]	[0.7,0.9]	[0.6,0.8]	[0.7,0.9]	[0.6,0.8]
A_I	[0.3,0.2]	[0.2,0.1]	[0.3,0.2]	[0.2,0.1]	[0.3,0.2]	[0.2,0.1]
A_F	[0.2,0.4]	[0.1,0.4]	[0.2,0.4]	[0.1,0.4]	[0.2,0.4]	[0.1,0.4]

	0	a_1	a_2	a_3	a_4	a_5
λ_T	0.1	0.3	0.1	0.3	0.1	0.3
λ_I	0.3	0.5	0.3	0.5	0.3	0.5
λ_F	0.5	0.6	0.5	0.6	0.5	0.6

.

Both the conditions of Definition 3.1 are satisfied by the set $\mathcal{C}$. Thus $\mathcal{C} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic subalgebra of X.

Proposition 3.1 *Let $\mathcal{C} = \{< x, A_{T,I,F}(x), \lambda_{T,I,F}(x) >\}$ is a neutrosophic cubic subalgebra of X, then $\forall\, x \in X$, $A_{T,I,F}(x) \geq A_{T,I,F}(0)$ and $\lambda_{T,I,F}(0) \leq \lambda_{T,I,F}(x)$. Thus, $A_{T,I,F}(0)$ and $\lambda_{T,I,F}(0)$ are the upper bounds and lower bounds of $A_{T,I,F}(x)$ and $\lambda_{T,I,F}(x)$ respectively.*

Proof: $\forall\, x \in X$, we have $A_{T,I,F}(0) = A_{T,I,F}(x*x) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(x)\} = A_{T,I,F}(x) \Rightarrow A_{T,I,F}(0) \geq A_{T,I,F}(x)$ and $\lambda_{T,I,F}(0) = \lambda_{T,I,F}(x*x) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(x)\} = \lambda_{T,I,F}(x) \Rightarrow \lambda_{T,I,F}(0) \leq \lambda_{T,I,F}(x)$.

Theorem 3.1 *Let $\mathcal{C} = \{\langle x, A_{T,I,F}(x), \lambda_{T,I,F}(x)\rangle\}$ be a neutrosophic cubic subalgebras of X. If there exists a sequence $\{x_n\}$ of X such that $\lim\limits_{n\to\infty} A_{T,I,F}(x_n) = [1,1]$ and $\lim\limits_{n\to\infty} \lambda_{T,I,F}(x_n) = 0$. then $A_{T,I,F}(0) = [1,1]$ and $\lambda_{T,I,F}(0) = 0$.*

Proof: Using Proposition 3.1, $A_{T,I,F}(0) \geq A_{T,I,F}(x)\ \forall\, x \in X$, $\therefore A_{T,I,F}(0) \geq A_{T,I,F}(x_n)$ for $n \in \mathbf{Z}^+$. Consider, $[1,1] \geq A_{T,I,F}(0) \geq \lim\limits_{n\to\infty} A_{T,I,F}(x_n) = [1,1]$. Hence, $A_{T,I,F}(0) = [1,1]$.

Again, using Proposition 3.1, $\lambda_{T,I,F}(0) \leq \lambda_{T,I,F}(x)\ \forall\, x \in X$, $\therefore \lambda_{T,I,F}(0) \leq \lambda_{T,I,F}(x_n)$ for $n \in \mathbf{Z}^+$. Consider, $0 \geq \lambda_{T,I,F}(0) \leq \lim\limits_{n\to\infty} \lambda_{T,I,F}(x_n) = 0$. Hence, $\lambda_{T,I,F}(0) = 0$.

Theorem 3.2 *The R-intersection of any set of neutrosophic cubic subalgebras of X is also a neutrosophic cubic subalgebras of X.*

Proof: Let $\mathcal{A}_i = \{\langle x, A_{iT,I,F}, \lambda_{iT,I,F}\rangle \mid x \in X\}$ where $i \in k$, be a sets of neutrosophic cubic subalgebras of X and $x, y \in X$. Then

$$\begin{aligned}
(\cap A_{iT,I,F})(x*y) &= rinf A_{iT,I,F}(x*y) \\
&\geq rinf\{rmin\{A_{iT,I,F}(x), A_{iT,I,F}(y)\}\} \\
&= rmin\{rinf A_{iT,I,F}(x), rinf A_{iT,I,F}(y)\} \\
&= rmin\{(\cap A_{iT,I,F})(x), (\cap A_{iT,I,F})(y)\} \\
\Rightarrow (\cap A_{iT,I,F})(x*y) &\geq rmin\{(\cap A_{iT,I,F})(x), (\cap A_{iT,I,F})(y)\}
\end{aligned}$$

and

$$\begin{aligned}
(\vee \lambda_{iT,I,F})(x*y) &= sup\lambda_{iT,I,F}(x*y) \\
&\leq sup\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(y)\}\} \\
&= max\{sup\lambda_{iT,I,F}(x), sup\lambda_{iT,I,F}(y)\} \\
&= max\{(\vee \lambda_{iT,I,F})(x), (\vee \lambda_{iT,I,F})(y)\} \\
\Rightarrow (\vee \lambda_{iT,I,F})(x*y) &\leq max\{(\vee \lambda_{iT,I,F})(x), (\vee \lambda_{iT,I,F})(y)\},
\end{aligned}$$

which shows that R-intersection of $\mathcal{A}_i$ is a neutrosophic cubic subalgebra of X.

Remark 3.1 *The R-union, P-intersection and P-union of neutrosophic cubic subalgebra need not be a neutrosophic cubic subalgebra.*
Example, let $X = \{0, a_1, a_2, a_3, a_4, a_5\}$ be a B-algebra with the following Caley table. Let $\mathcal{A}_1 = (A_{1T,I,F}, \lambda_{1T,I,F})$ and $\mathcal{A}_2 = (A_{2T,I,F}, \lambda_{2T,I,F})$ be neutrosophic cubic set of X defined by

Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are neutrosophic subalgebras of X but R-union, P-union and P-intersection of $\mathcal{A}_1$ and $\mathcal{A}_2$ are not subalgebras of X because

$*$	0	a_1	a_2	a_3	a_4	a_5
0	0	a_2	a_1	a_3	a_4	a_5
a_1	a_1	0	a_2	a_5	a_3	a_4
a_2	a_2	a_1	0	a_4	a_5	a_3
a_3	a_3	a_4	a_5	0	a_1	a_2
a_4	a_4	a_5	a_3	a_2	0	a_1
a_5	a_5	a_3	a_4	a_1	a_2	0

	0	a_1	a_2	a_3	a_4	a_5
A_1T	[0.8,0.7]	[0.1,0.2]	[0.1,0.2]	[0.8,0.7]	[0.1,0.2]	[0.1,0.2]
A_1I	[0.7,0.8]	[0.2,0.3]	[0.2,0.3]	[07.,0.8]	[0.2,0.3]	[0.2,0.3]
A_1F	[0.8,0.9]	[0.3,0.4]	[0.3,0.4]	[0.8,0.9]	[0.3,0.4]	[0.3,0.4]
A_2T	[0.8,0.9]	[0.2,0.3]	[0.2,0.3]	[0.2,0.3]	[0.8,0.9]	[0.2,0.3]
A_2I	[0.7,0.6]	[0.1,0.2]	[0.1,0.2]	[0.1,0.2]	[0.7,0.6]	[0.1,0.2]
A_2F	[0.6,0.5]	[0.1,0.3]	[0.1,0.3]	[0.1,0.3]	[0.6,0.5]	[0.1,0.3]

	0	a_1	a_2	a_3	a_4	a_5
$\lambda_1 T$	0.1	0.8	0.8	0.1	0.8	0.8
$\lambda_1 I$	0.2	0.7	0.7	0.2	0.7	0.7
$\lambda_1 F$	0.4	0.6	0.6	0.4	0.6	0.6
$\lambda_2 T$	0.2	0.5	0.5	0.5	0.2	0.5
$\lambda_2 I$	0.3	0.7	0.7	0.7	0.3	0.7
$\lambda_2 F$	0.4	0.9	0.9	0.9	0.4	0.9

.

$(\cup A_{iT,I,F})(a_3 * a_4) = ([0.2, 0.3], [0.2, 0.3], [0.3, 0.4])_{T,I,F} \not\succeq ([0.8, 0.9], [0.7, 0.6], [0.6, 0.5])_{T,I,F} = rmin\{(\cup A_{iT,I,F})(a_3), (\cup A_{iT,I,F})(a_4)\}$
and
$(\wedge\lambda_{iT,I,F})(a_3 * a_4) = (0.8, 0.7, 0.9)_{T,I,F} \not\leq (0.2, 0.3, 0, 4)_{T,I,F}$ $= max\{(\wedge\lambda_{iT,I,F})(a_3), (\wedge\lambda_{iT,I,F})(a_4)\}$

We provide the condition that R-union, P-union and P-intersection of neutrosophic cubic subalgebras is also a neutrosophic cubic subalgebra. which are at Theorem 3.3, 3.4 and 3.5.

Theorem 3.3 *Let $\mathcal{A}_i = \{\langle x, A_{iT,I,F}, \lambda_{iT,I,F}\rangle \mid x \in X\}$ where $i \in k$, be a sets of neutrosophic cubic subalgebras of X, where $i \in k$. If $inf\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(x)\}\} = max\{inf\lambda_{iT,I,F}(x), inf\lambda_{iT,I,F}(x)\}\ \forall\ x \in X$, then the P-intersection of $\mathcal{A}_i$ is also a neutrosophic cubic subalgebras of X.*

Proof: Suppose that $\mathcal{A}_i = \{\langle x, A_{iT,I,F}, \lambda_{iT,I,F}\rangle \mid x \in X\}$ where $i \in k$, be sets of neutrosophic cubic subalgebras of X such that $inf\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(x)\}\} = max\{inf\lambda_{iT,I,F}(x), inf\lambda_{iT,I,F}(x)\}\ \forall\ x \in X$. Then for $x, y \in X$. Then

$$\begin{aligned}(\cap A_{iT,I,F})(x*y) &= rinf A_{iT,I,F}(x*y)\\ &\geq rinf\{rmin\{A_{iT,I,F}(x), A_{iT,I,F}(y)\}\}\\ &= rmin\{rinf A_{iT,I,F}(x), rinf A_{iT,I,F}(y)\}\\ &= rmin\{(\cap A_{iT,I,F})(x), (\cap A_{iT,I,F})(y)\}\\ \Rightarrow (\cap A_{iT,I,F})(x*y) &\geq rmin\{(\cap A_{iT,I,F})(x), (\cap A_{iT,I,F})(y)\}\end{aligned}$$

and

$$\begin{aligned}(\wedge\lambda_{iT,I,F})(x*y) &= inf\lambda_{iT,I,F}(x*y)\\ &\leq inf\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(y)\}\}\\ &= max\{inf\lambda_{iT,I,F}(x), inf\lambda_{iT,I,F}(y)\}\\ &= max\{(\wedge\lambda_{iT,I,F})(x), (\wedge\lambda_{iT,I,F})(y)\}\\ \Rightarrow (\wedge\lambda_{iT,I,F})(x*y) &\leq max\{(\wedge\lambda_{iT,I,F})(x), (\wedge\lambda_{iT,I,F})(y)\},\end{aligned}$$

which shows that P-intersection of $\mathcal{A}_i$ is a neutrosophic cubic subalgebra of X.

Theorem 3.4 *Let $\mathcal{A}_i = \{\langle x, A_{iT,I,F}, \lambda_{iT,I,F}\rangle \mid x \in X\}$ where $i \in k$, be a sets of neutrosophic cubic subalgebras of X. If $sup\{rmin\{A_{iT,I,F}(x), A_{iT,I,F}(x)\}\} = rmin\{supA_{iT,I,F}(x), supA_{iT,I,F}(x)\}\ \forall x \in X$, then the P-union of $\mathcal{A}_i$ is also a neutrosophic cubic subalgebra of X.*

Proof: Let $\mathcal{A}_i = \{\langle x, A_{iT,I,F}, \lambda_{iT,I,F}\rangle \mid x \in X\}$ where $i \in k$, be a sets of neutrosophic cubic subalgebras of X such that $sup\{rmin\{A_{iT,I,F}(x), A_{iT,I,F}(x)\}\} = rmin\{supA_{iT,I,F}(x), supA_{iT,I,F}(x)\}\ \forall\ x \in X$. Then for $x, y \in X$,

$$\begin{aligned}(\cup A_{iT,I,F})(x*y) &= rsupA_{iT,I,F}(x*y)\\ &\geq rsup\{rmin\{A_{iT,I,F}(x), A_{iT,I,F}(y)\}\}\\ &= rmin\{rsupA_{iT,I,F}(x), rsupA_{iT,I,F}(y)\}\\ &= rmin\{(\cup A_{iT,I,F})(x), (\cup A_{iT,I,F})(y)\}\\ (\cup A_{iT,I,F})(x*y) &\geq rmin\{(\cup A_{iT,I,F})(x), (\cup A_{iT,I,F})(y)\}\end{aligned}$$

and

$$\begin{aligned}(\vee\lambda_{iT,I,F})(x*y) &= sup\lambda_{iT,I,F}(x*y)\\ &\leq sup\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(y)\}\}\\ &= max\{sup\lambda_{iT,I,F}(x), sup\lambda_{iT,I,F}(y)\}\\ &= max\{(\vee\lambda_{iT,I,F})(x), (\vee\lambda_{iT,I,F})(y)\}\\ (\vee\lambda_{iT,I,F})(x*y) &\leq max\{(\vee\lambda_{iT,I,F})(x), (\vee\lambda_{iT,I,F})(y)\}.\end{aligned}$$

Which shows that P-union of $\mathcal{A}_i$ is a neutrosophic cubic subalgebra of X.

Theorem 3.5 *Let $\mathcal{A}_i = \{\langle x, A_{iT,I,F}, \lambda_{iT,I,F}\rangle \mid x \in X\}$ where $i \in k$, be a sets of neutrosophic cubic subalgebras of X. If $inf\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(x)\}\} = max\{inf\lambda_{iT,I,F}(x), inf\lambda_{iT,I,F}(x)\}$ and $sup\{rmin\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(x)\}\} = rmin\{sup\lambda_{iT,I,F}(x), sup\lambda_{iT,I,F}(x)\}\ \forall\ x \in X$, then the R-union of $\mathcal{A}_i$ is also a neutrosophic cubic subalgebra of X.*

Proof: Let $\mathcal{A}_i = \{\langle x, A_{iT,I,F}, \lambda_{iT,I,F}\rangle \mid x \in X\}$ where $i \in k$, be a sets of neutrosophic cubic subalgebras of X such that $inf\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(x)\}\} = max\{inf\lambda_{iT,I,F}(x), inf\lambda_{iT,I,F}(x)\}$ and $sup\{rmin\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(x)\}\} = rmin\{sup\lambda_{iT,I,F}(x), sup\lambda_{iT,I,F}(x)\}\ \forall\ x \in$

X. Then for $x, y \in X$,

$$\begin{aligned}(\cup A_{iT,I,F})(x*y) &= rsupA_{iT,I,F}(x*y)\\ &\geq rsup\{rmin\{A_{iT,I,F}(x), A_{iT,I,F}(y)\}\}\\ &= rmin\{rsupA_{iT,I,F}(x), rsupA_{iT,I,F}(y)\}\\ &= rmin\{(\cup A_{iT,I,F})(x), (\cup A_{iT,I,F})(y)\}\\ (\cup A_{iT,I,F})(x*y) &\geq rmin\{(\cup A_{iT,I,F})(x), (\cup A_{iT,I,F})(y)\}\end{aligned}$$

and

$$\begin{aligned}(\wedge\lambda_{iT,I,F})(x*y) &= inf\lambda_{iT,I,F}(x*y)\\ &\leq inf\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(y)\}\}\\ &= max\{inf\lambda_{iT,I,F}(x), inf\lambda_{iT,I,F}(y)\}\\ &= max\{(\wedge\lambda_{iT,I,F})(x), (\wedge\lambda_{iT,I,F})(y)\}\\ (\wedge\lambda_{iT,I,F})(x*y) &\leq max\{(\wedge\lambda_{iT,I,F})(x), (\wedge\lambda_{iT,I,F})(y)\}.\end{aligned}$$

Which shows that R-union of $\mathcal{A}_i$ is a neutrosophic cubic subalgebra of X.

Proposition 3.2 *If a neutrosophic cubic set $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ of X is a subalgebra, then $\forall\ x \in X$, $A_{T,I,F}(0*x) \geq A_{T,I,F}(x)$ and $\lambda_{T,I,F}(0*x) \leq \lambda_{T,I,F}(x)$.*

Proof: $\forall\, x \in X$, $A_{T,I,F}(0*x) \geq rmin\{A_{T,I,F}(0), A_{T,I,F}(x)\} = rmin\{A_{T,I,F}(x*x), A_{T,I,F}(x)\} \geq rmin\{rmin\{A_{T,I,F}(x), A_{T,I,F}(x)\}, A_{T,I,F}(x)\} = A_{T,I,F}(x)$ and similarly $\lambda_{T,I,F}(0*x) \leq max\{\lambda_{T,I,F}(0), \lambda_{T,I,F}(x)\} = \lambda_{T,I,F}(x)$.

Lemma 3.1 *If a netrosophic cubic set $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ of X is a subalgebra, then $\mathcal{A}(x*y) = \mathcal{A}(x*(0(0*y)))\ \forall\, x, y \in X$.*

Proof: Let X be a B-algebra and $x, y \in X$. Then we know that $y = 0*(0*y)$ by ([3],lemma 3.1). Hence, $A_{T,I,F}(x*y) = A_{T,I,F}(x*(0*(0*y)))$ and $\lambda_{T,I,F}(x*y) = \lambda_{T,I,F}(x*(0*(0*y)))$. Therefore, $\mathcal{A}_{T,I,F}(x*y) = \mathcal{A}_{T,I,F}(x*(0*(0*y)))$.

Proposition 3.3 *If a nuetrosophic cubic set $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ of X is a neutrosophic cubic subalgebra, then $\forall\, x, y \in X$, $A_{T,I,F}(x*(0*y)) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x*(0*y)) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$.*

Proof: Let $x, y \in X$. Then we have $A_{T,I,F}(x*(0*y)) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(0*y)\} \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x*(0*y)) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(0*y)\} \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$ by Definition 3.1 and Proposition 3.2. Hence, the proof is completed.

Theorem 3.6 *If a neutrosophic cubic set $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ of X satisfies the following conditions*

1. $A_{T,I,F}(0*x) \geq A_{T,I,F}(x)$ *and* $\lambda_{T,I,F}(0*x) \leq \lambda_{T,I,F}(x)$,
2. $A_{T,I,F}(x*(0*y)) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ *and* $\lambda_{T,I,F}(x*(0*y)) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$ $\forall\, x, y \in X$.

then $\mathcal{A}$ refers to a neutrosophic cubic subalgebra of X.

Proof: Assume that the neutrosophic cubic set $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ of X satisfies the above conditions (1 and 2). Then by Lemma 3.1, we have $A_{T,I,F}(x*y) = A_{T,I,F}(x*(0*(0*y))) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(0*y)\} \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x*y) = \lambda_{T,I,F}(x*(0*(0*y))) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(0*y) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$ $\forall\, x, y \in X$. Hence, $\mathcal{A}$ is neutrosophic cubic subalgebra of X.

Theorem 3.7 *Nuetrosophic cubic set $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ of X is a neutrosophic cubic subalgebra of $X \iff A^-_{T,I,F}, A^+_{T,I,F}$ and $\lambda_{T,I,F}$ are fuzzy subalgebras of X.*

Proof: let $A^-_{T,I,F}, A^+_{T,I,F}$ and $\lambda_{T,I,F}$ are fuzzy subalgebra of X and $x, y \in X$. Then $A^-_{T,I,F}(x*y) \geq min\{A^-_{T,I,F}(x), A^-_{T,I,F}(y)\}$, $A^+_{T,I,F}(x*y) \geq min\{A^+_{T,I,F}(x), A^+_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x*y) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$. Now, $A_{T,I,F}(x*y) = [A^-_{T,I,F}(x*y), A^+_{T,I,F}(x*y] \geq [min\{A^-_{T,I,F}(x), A^-_{T,I,F}(y)\}, min\{A^+_{T,I,F}(x), A^+_{T,I,F}(y)\}] \geq rmin\{[A^-_{T,I,F}(x), A^+_{T,I,F}(x)], [A^-_{T,I,F}(y), A^+_{T,I,F}(y)]\} = rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$. Therefore, $\mathcal{A}$ is neutrosophic cubic subalgebra of X.

Conversely, assume that $\mathcal{A}$ is a neutrosophic cubic subalgebra of X. For any $x, y \in X$, $[A^-_{T,I,F}(x*y), A^+_{T,I,F}(x*y)] = A_{T,I,F}(x*y) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\} = rmin\{[A^-_{T,I,F}(x), A^+_{T,I,F}(x)], [A^-_{T,I,F}(y), A^+_{T,I,F}(y)]\} = = [min\{A^-_{T,I,F}(x), A^-_{T,I,F}(y)\}, min\{A^+_{T,I,F}(x), A^+_{T,I,F}(y)\}]$. Thus, $A^-_{T,I,F}(x*y) \geq min\{A^-_{T,I,F}(x), A^-_{T,I,F}(y)\}$, $A^+_{T,I,F}(x*y) \geq min\{A^+_{T,I,F}(x), A^+_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x*y) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$. Hence $A^-_{T,I,F}, A^+_{T,I,F}$ and $\lambda_{T,I,F}$ are fuzzy subalgebra of X.

Theorem 3.8 *Let $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ be a neutrosophic cubic subalgebra of X and let $n \in \mathbb{Z}^+$(the set of positive integer). Then*

1. $A_{T,I,F}(\prod^{n} x*x) \geq A_{T,I,F}(x)$ *for $n \in \mathbb{O}$(the set of odd number),*
2. $\lambda_{T,I,F}(\prod^{n} x*x) \leq A_{T,I,F}(x)$ *for $n \in \mathbb{O}$(the set of odd number),*
3. $A_{T,I,F}(\prod^{n} x*x) = A_{T,I,F}(x)$ *for $n \in \mathbb{E}$(the set of even number),*
4. $\lambda_{T,I,F}(\prod^{n} x*x) = A_{T,I,F}(x)$ *for $n \in \mathbb{E}$(the set of even number).*

Proof: Let $x \in X$ and assume that n is odd. Then $n = 2p-1$ for some positive integer p. We prove the theorem by induction. Now $A_{T,I,F}(x*x) = A_{T,I,F}(0) \geq A_{T,I,F}(x)$ and $\lambda_{T,I,F}(x*x) = \lambda_{T,I,F}(0) \leq \lambda_{T,I,F}(x)$. Suppose that $A_{T,I,F}(\prod^{2p-1} x*x) \geq A_{T,I,F}(x)$ and $\lambda_{T,I,F}(\prod^{2p-1} x*x) \leq \lambda_{T,I,F}(x)$. Then by assumption, $A_{T,I,F}(\prod^{2(p+1)-1} x*x) = A_{T,I,F}(\prod^{2p+1} x*x) = A_{T,I,F}(\prod^{2p-1} x*(x*(x*x))) = A_{T,I,F}(\prod^{2p-1} x*x) \geq A_{T,I,F}(x)$ and $\lambda_{T,I,F}(\prod^{2(p+1)-1} x*x) = \lambda_{T,I,F}(\prod^{2p+1} x*x) = \lambda_{T,I,F}(\prod^{2p-1} x*(x*(x*x))) = \lambda_{T,I,F}(\prod^{2p-1} x*x) \leq \lambda_{T,I,F}(x)$, which proves (1) and (2). Similarly, the proves are same to the cases (3) and (4).

The sets denoted by $I_{A_{T,I,F}}$ and $I_{\lambda_{T,I,F}}$ are also subalgebra of X. Which were defined as:
$I_{A_{T,I,F}}$={$x \in X \mid A_{T,I,F}(x) = A_{T,I,F}(0)$} and $I_{\lambda_{T,I,F}}$={$x \in X \mid \lambda_{T,I,F}(x) = \lambda_{T,I,F}(0)$}.

Theorem 3.9 *Let $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ be a neutrosophic cubic subalgebra of X. Then the sets $I_{A_{T,I,F}}$ and $I_{\lambda_{T,I,F}}$ are subalgebras of X.*

Proof: Let $x, y \in I_{A_{T,I,F}}$. Then $A_{T,I,F}(x) = A_{T,I,F}(0) = A_{T,I,F}(y)$ and so, $A_{T,I,F}(x*y) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\} = A_{T,I,F}(0)$. By using Proposition 3.1, We know that $A_{T,I,F}(x*y) = A_{T,I,F}(0)$ or equivalently $x*y \in I_{A_{T,I,F}}$.

Again let $x, y \in I_{A_{T,I,F}}$. Then $\lambda_{T,I,F}(x) = \lambda_{T,I,F}(0) = \lambda_{T,I,F}(y)$ and so, $\lambda_{T,I,F}(x*y) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$ =$\lambda_{T,I,F}(0)$. Again by using Proposition 3.1, We know that $\lambda_{T,I,F}(x*y) = \lambda_{T,I,F}(0)$ or equivalently $x*y \in I_{A_{T,I,F}}$. Hence the sets $I_{A_{T,I,F}}$ and $\lambda_{A_{T,I,F}}$ are subalgebras of X.

Theorem 3.10 *Let B be a nonempty subset of X and $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ be neutrosophic cubic set of X defined by*

$$A_{T,I,F}(x) = \begin{cases} [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}], & \textit{if } x \in B \\ [\beta_{T,I,F_1}, \beta_{T,I,F_2}], & \textit{otherwise}, \end{cases}$$

$$\lambda_T(x) = \begin{cases} \gamma_{T,I,F}, & \textit{if } x \in B \\ \delta_{T,I,F}, & \textit{otherwise} \end{cases}$$

$\forall$ $[\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]$,$[\beta_{T,I,F_1}, \beta_{T,I,F_2}] \in D[0,1]$ and $\gamma_{T,I,F}$, $\delta_{T,I,F} \in [0,1]$ with $[\alpha_{T,I,F_1}, \alpha_{T,I,F_2}] \geq [\beta_{T,I,F_1}, \beta_{T,I,F_2}]$ and $\gamma_{T,I,F} \leq \delta_{T,I,F}$. Then $\mathcal{A}$ is a nuetrosophic cubic subalgebra of $X \iff B$ is a subalgebra of X. Moreover, $I_{A_{T,I,F}}$= B= $I_{\lambda_{T,I,F}}$.

Proof: Let $\mathcal{A}$ be a neutrosophic cubic subalgebra of X. Let $x, y \in X$ such that $x, y \in B$. Then $A_{T,I,F}(x*y) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\} = rmin\{[\alpha_{T,I,F_1}, \alpha_{T,I,F_2}], [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]\} = [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]$ and $\lambda_{T,I,F}(x*y) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\} = max\{\gamma_{T,I,F}, \gamma_{T,I,F}\} = \gamma_{T,I,F}$. Therefore $x*y \in B$. Hence, B is a subalgebra of X.

Conversely, suppose that B is a subalgebra of X. Let $x, y \in X$. We consider two cases,
Case 1: If $x, y \in B$, then $x*y \in B$, thus $A_{T,I,F}(x*y) = [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}] = rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x*y) = \gamma_{T,I,F} = max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$.
Case 2: If $x \notin B$ or $y \notin B$, then $A_{T,I,F}(x*y) \geq [\beta_{T,I,F_1}, \beta_{T,I,F_2}] = rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x*y) \leq \delta_{T,I,F} = max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$.

Hence $\mathcal{A}$ is a neutrosophic cubic subalgebra of X.

Now, $I_{A_{T,I,F}}$={$x \in X, A_{T,I,F}(x) = A_{T,I,F}(0)$}= {$x \in X, A_{T,I,F}(x) = [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]$} $= B$ and $I_{\lambda_{T,I,F}}$={$x \in X, \lambda_{T,I,F}(x) = \lambda_{T,I,F}(0)$}={$x \in X, \lambda_{T,I,F}(x) = \gamma_{T,I,F}$}=$B$.

Definition 3.2 *Let $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ be a neutrosophic cubic set of X. For $[s_{T_1}, s_{T_2}], [s_{I_1}, s_{I_2}], [s_{F_1}, s_{F_2}] \in D[0,1]$ and $t_{T_1}, t_{I_1}, t_{F_1} \in [0,1]$, the set $U(A_{T,I,F} \mid ([s_{T_1}, s_{T_2}], [s_{I_1}, s_{I_2}], [s_{F_1}, s_{F_2}]))$ ={$x \in X \mid A_T(x) \geq [s_{T_1}, s_{T_2}], A_I(x) \geq [s_{I_1}, s_{I_2}], A_F(x) \geq [s_{F_1}, s_{F_2}]$} is called upper $([s_{T_1}, s_{T_2}], [s_{I_1}, s_{I_2}], [s_{F_1}, s_{F_2}])$-level of $\mathcal{A}$ and $L(\lambda_{T,I,F} \mid (t_{T_1}, t_{I_1}, t_{F_1}))$ ={$x \in X \mid \lambda_T(x) \leq t_{T_1}, \lambda_I(x) \leq t_{I_1}, \lambda_F(x) \leq t_{F_1}$} is called lower $(t_{T_1}, t_{I_1}, t_{F_1})$-level of $\mathcal{A}$.*

For our convenience we are introducing the new notation as: $U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}]$={$x \in X \mid A_{T,I,F}(x) \geq [s_{T,I,F_1}, s_{T,I,F_2}]$} is called upper $([s_{T,I,F_1}, s_{T,I,F_2}])$-level of $\mathcal{A}$ and $L(\lambda_{T,I,F} \mid t_{T,I,F_1})$={$x \in X \mid \lambda_{T,I,F}(x) \leq t_{T,I,F_1}$} is called lower t_{T,I,F_1}-level of $\mathcal{A}$.

Theorem 3.11 *If $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is neutrosophic cubic subalgebra of X, then the upper $[s_{T,I,F_1}, s_{T,I,F_2}]$-level and lower t_{T,I,F_1}-level of $\mathcal{A}$ are ones of X.*

Proof: Let $x, y \in U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$, then $A_{T,I,F}(x) \geq [s_{T,I,F_1}, s_{T,I,F_2}]$ and $A_{T,I,F}(y) \geq [s_{T,I,F_1}, s_{T,I,F_2}]$. It follows that $A_{T,I,F}(x*y) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\} \geq [s_{T,I,F_1}, s_{T,I,F_2}] \Rightarrow x*y \in U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$. Hence, $U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}]$ is a subalgebra of X.

Let $x, y \in L(\lambda_{T,I,F} \mid t_{T,I,F_1})$. Then $\lambda_{T,I,F}(x) \leq t_{T,I,F_1}$ and $\lambda_{T,I,F}(y) \leq t_{T,I,F_1}$. It follows that $\lambda_{T,I,F}(x*y) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\} \leq t_{T,I,F_1} \Rightarrow x*y \in L(\lambda_{T,I,F} \mid t_{T,I,F_1})$. Hence $L(\lambda_{T,I,F} \mid t_{T,I,F_1})$ is a subalgebra of X.

Corollary 3.1 *Let $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is neutrosophic cubic subalgebra of X. Then $A([s_{T,I,F_1}, s_{T,I,F_2}]; t_{T,I,F_1})$= $U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}]) \bigcap L(\lambda_{T,I,F} \mid t_{T,I,F_1})$={$x \in X \mid A_{T,I,F}(x) \geq [s_{T,I,F_1}, s_{T,I,F_2}], \lambda_{T,I,F}(x) \leq t_{T,I,F_1}$} is a subalgebra of X.*

Proof: Straightforward

The following example shows that the converse of Corollary 3.1 is not valid.

Example 3.2 *Let $X = \{0, a_1, a_2, a_3, a_4, a_5\}$ be a B-algebra in Remark 3.1 and $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic set defined by*

	0	a_1	a_2	a_3	a_4	a_5
A_T	[0.6,0.8]	[0.5,0.6]	[0.5,0.6]	[0.5,0.6]	[0.3,0.4]	[0.3,0.4]
A_I	[0.5,0.7]	[0.4,0.5]	[0.4,0.5]	[0.4,0.6]	[0.3,0.3]	[0.3,0.3]
A_F	[0.4,0.6]	[0.2,0.5]	[0.2,0.5]	[0.2,0.5]	[0.1,0.2]	[0.1,0.2]

	0	a_1	a_2	a_3	a_4	a_5
λ_T	0.1	0.3	0.3	0.5	0.3	0.5
λ_I	0.2	0.4	0.4	0.6	0.4	0.6
λ_F	0.3	0.5	0.5	0.7	0.5	0.7

Suppose that $[s_{T,I,F_1}, s_{T,I,F_2}]$=$([0.42, 0.49], [0.31, 0.37], [0.14, 0.18])_{T,I,F}$ *and* $t_{T,I,F_1} = (0.4, 0.5, 0.6)_{T,I,F}$*, then* $A([s_{T,I,F_1}, s_{T,I,F_2}]; t_{T,I,F_1})$=$U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}]) \bigcap L(\lambda_{T,I,F} \mid t_{T,I,F_1})$=$\{x \in X \mid A_{T,I,F}(x) \geq [s_{T,I,F_1}, s_{T,I,F_2}], \lambda_{T,I,F}(x) \leq t_{T,I,F_1}\} = \{0, a_1, a_2, a_3\} \bigcap \{0, a_1, a_2, a_4\} = \{0, a_1, a_2\}$ *is a subalgebra of* X*, but* $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ *is not a neutrosophic cubic subalgebra since* $A_T(a_1 * a_3) = [0.3, 0.4] \ngeq [0.5, 0.6] = rmin\{A_T(a_1), A_T(a_3)\}$ *and* $\lambda_T(a_2 * a_4) = 0.5 \nleq 0.4 = max\{\lambda_T(a_2), \lambda_T(a_4)\}$.

Theorem 3.12 *Let* $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ *be a neutrosophic cubic set of* X*, such that the sets* $U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ *and* $L(\lambda_{T,I,F} \mid t_{T,I,F_1})$ *are subalgebras of* X *for every* $[s_{T,I,F_1}, s_{T,I,F_2}] \in D[0, 1]$ *and* $t_{T,I,F_1} \in [0, 1]$*. Then* $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ *is neutrosophic cubic subalgebra of* X.

Proof: Let $U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and $L(\lambda_{T,I,F} \mid t_{T,I,F_1})$ are subalgebras of X for every $[s_{T,I,F_1}, s_{T,I,F_2}] \in D[0, 1]$ and $t_{T,I,F_1} \in [0, 1]$. On the contrary, let $x_0, y_0 \in X$ be such that $A_{T,I,F}(x_0 * y_0) < rmin\{A_{T,I,F}(x_0), A_{T,I,F}(y_0)\}$. Let $A_{T,I,F}(x_0) = [\theta_1, \theta_2]$, $A_{T,I,F}(y_0) = [\theta_3, \theta_4]$ and $A_{T,I,F}(x_0 * y_0) = [s_{T,I,F_1}, s_{T,I,F_2}]$. Then $[s_{T,I,F_1}, s_{T,I,F_2}] < rmin\{[\theta_1, \theta_2], [\theta_3, \theta_4]\} = [min\{\theta_1, \theta_3\}, min\{\theta_2, \theta_4\}]$. So, $s_{T,I,F_1} < rmin\{\theta_1, \theta_3\}$ and $s_{T,I,F_2} < min\{\theta_2, \theta_4\}$. Let us consider, $[\rho_1, \rho_2] = \frac{1}{2}[A_{T,I,F}(x_0 * y_0) + rmin\{A_{T,I,F}(x_0), A_{T,I,F}(y_0)\}] = \frac{1}{2}[[s_{T,I,F_1}, s_{T,I,F_2}] + [min\{\theta_1, \theta_3\}, min\{\theta_2, \theta_4\}]] = [\frac{1}{2}(s_{T,I,F_1} + min\{\theta_1, \theta_3\}), \frac{1}{2}(s_{T,I,F_2} + min\{\theta_2, \theta_3\})]$. Therefore, $min\{\theta_1, \theta_3\} > \rho_1 = \frac{1}{2}(s_{T,I,F_1} + min\{\theta_1, \theta_3\}) > s_{T,I,F_1}$ and $min\{\theta_2, \theta_4\} > \rho_2 = \frac{1}{2}(s_{T,I,F_2} + min\{\theta_2, \theta_4\}) > s_{T,I,F_2}$. Hence, $[min\{\theta_1, \theta_3\}, min\{\theta_2, \theta_4\}] > [\rho_1, \rho_2] > [s_{T,I,F_1}, s_{T,I,F_2}]$, so that $x_0 * y_0 \notin U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ which is a contradiction since $A_{T,I,F}(x_0) = [\theta_1, \theta_2] \geq [min\{\theta_1, \theta_3\}, min\{\theta_2, \theta_4\}] > [\rho_1, \rho_2]$ and $A_{T,I,F}(y_0) = [\theta_3, \theta_4] \geq [min\{\theta_1, \theta_3\}, min\{\theta_2, \theta_4\}] > [\rho_1, \rho_2]$. This implies $x_0 * y_0 \in U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$. Thus $A_{T,I,F}(x * y) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ $\forall\ x, y \in X$.

Again, let $x_0, y_0 \in X$ be such that $\lambda_{T,I,F}(x_0 * y_0) > max\{\lambda_{T,I,F}(x_0), \lambda_{T,I,F}(0)\}$. Let $\lambda_{T,I,F}(x_0) = \eta_{T,I,F_1}$, $\lambda_{T,I,F}(y_0) = \eta_{T,I,F_2}$ and $\lambda_{T,I,F}(x_0 * y_0) = t_{T,I,F_1}$. Then $t_{T,I,F_1} > max\{\eta_{T,I,F_1}.\eta_{T,I,F_2}\}$. Let us consider $t_{T,I,F_2} = \frac{1}{2}[\lambda_{T,I,F}(x_0 * y_0) + max\{\lambda_{T,I,F}(x_0), \lambda_{T,I,F}(0)\}]$. We get that $t_{T,I,F_2} = \frac{1}{2}(t_{T,I,F_1} + max\{\eta_{T,I,F_1}, \eta_{T,I,F_2}\})$. Therefore, $\eta_{T,I,F_1} < t_{T,I,F_2} = \frac{1}{2}(t_{T,I,F_1} + max\{\eta_{T,I,F_1}, \eta_{T,I,F_2}\}) < t_{T,I,F_1}$ and $\eta_{T,I,F_2} < t_{T,I,F_2} = \frac{1}{2}(t_{T,I,F_1} + max\{\eta_{T,I,F_1}, \eta_{T,I,F_2}\}) < t_{T,I,F_1}$. Hence, $max\{\eta_{T,I,F_1}, \eta_{T,I,F_2}\} < t_{T,I,F_2} < t_{T,I,F_1} = \lambda_{T,I,F}(x_0, y_0)$, so that $x_0 * y_0 \notin L(\lambda_{T,I,F} \mid t_{T,I,F_1})$ which is a contradiction since $\lambda_{T,I,F}(x_0) = \eta_{T,I,F_1} \leq max\{\eta_{T,I,F_1}, \eta_{T,I,F_2}\} < t_{T,I,F_2}$ and $\lambda_{T,I,F}(y_0) = \eta_{T,I,F_2} \leq max\{\eta_{T,I,F_1}, \eta_{T,I,F_2}\} < t_{T,I,F_2}$. This implies $x_0, y_0 \in L(\lambda_{T,I,F} \mid t_{T,I,F_1})$. Thus $\lambda_{T,I,F}(x * y) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$ $\forall\ x, y \in X$. Therefore, $U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and $L(\lambda_{T,I,F} \mid t_{T,I,F_1})$ are subalgebras of X. Hence, $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is neutrosophic cubic subalgebra of X.

Theorem 3.13 *Any subalgebra of* X *can be realized as both the upper* $[s_{T,I,F_1}, s_{T,I,F_2}]$*-level and lower* t_{T,I,F_1}*-level of some neutrosophic cubic subalgebra of* X.

Proof: Let $\mathcal{B}$ be a neutrosophic cubic subalgebra of X, and $\mathcal{A}$ be a neutrosophic cubic set on X defined by

$$A_{T,I,F} = \begin{cases} [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}], & \text{if } x \in \mathcal{B} \\ [0, 0] & \text{otherwise.} \end{cases},$$

$$\lambda_{T,I,F} = \begin{cases} \beta_{T,I,F_1}, & \text{if } x \in \mathcal{B} \\ 0, & \text{otherwise.} \end{cases}$$

$\forall\ [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}] \in D[0, 1]$ and $\beta_{T,I,F_1} \in [0, 1]$. We consider the following cases.

Case 1: If $\forall\ x, y \in \mathcal{B}$ then $A_{T,I,F}(x) = [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]$, $\lambda_{T,I,F}(x) = \beta_{T,I,F_1}$ and $A_{T,I,F}(y) = [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]$, $\lambda_{T,I,F}(y) = \beta_{T,I,F_1}$. Thus $A_{T,I,F}(x * y) = [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}] = rmin\{[\alpha_{T,I,F_1}, \alpha_{T,I,F_2}], [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]\} = rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x * y) = \beta_{T,I,F_1}$=$max\{\beta_{T,I,F_1}, \beta_{T,I,F_1}\} = max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$.

Case 2: If $x \in \mathcal{B}$ and $y \notin \mathcal{B}$, then $A_{T,I,F}(x) = [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]$, $\lambda_{T,I,F}(x) = \beta_{T,I,F_1}$ and $A_{T,I,F}(y) = [0, 0]$, $\lambda_{T,I,F}(y) = 1$. Thus $A_{T,I,F}(x * y) \geq [0, 0] = rmin\{[\alpha_{T,I,F_1}, \alpha_{T,I,F_2}], [0, 0]\} = rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x * y) \leq 1 = max\{\beta_{T,I,F_1}, 1\} = max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$.

Case 3: If $x \notin \mathcal{B}$ and $y \in \mathcal{B}$, then $A_{T,I,F}(x) = [0, 0]$,$\lambda_{T,I,F}(x) = 1$ and $A_{T,I,F}(y) = [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]$, $\lambda_{T,I,F}(y) = \beta_{T,I,F_1}$. Thus $A_{T,I,F}(x * y) \geq [0, 0] = rmin\{[0, 0], [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]\} = rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x * y) \leq 1 = max\{1, \beta_{T,I,F_1}\} = max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$.

Case 4: If $x \notin \mathcal{B}$ and $y \notin \mathcal{B}$, then $A_{T,I,F}(x) = [0, 0]$, $\lambda_{T,I,F}(x) = 1$ and $A_{T,I,F}(y) = [0, 0]$, $\lambda_{T,I,F}(y) = 1$. Thus $A_{T,I,F}(x * y) \geq [0, 0] = rmin\{[0, 0], [0, 0]\} = rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x * y) \leq 1 = max\{1, 1\} = max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$.

Therefore, $\mathcal{A}$ is a neutrosophic cubic subalgebra of X.

Theorem 3.14 *Let* $\mathcal{B}$ *be a subset of* X *and* $\mathcal{A}$ *be a neutrosophic cubic set on* X *which is given in the proof of Theorem 3.13. If* $\mathcal{A}$ *is realized as lower level subalgebra and upper level subalgebra of some neutrosophic cubic subalgebra of* X*, then* $\mathcal{P}$ *is a neutrosophic cubic one of* X.

Proof: Let $\mathcal{A}$ be a neutrosophic cubic subalgebra of X, and $x, y \in \mathcal{B}$. Then $A_{T,I,F}(x) = A_{T,I,F}(y) = [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]$ and $\lambda_{T,I,F}(x) = \lambda_{T,I,F}(y) = \beta_{T,I,F_1}$. Thus $A_{T,I,F}(x * y) \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$=$rmin\{[\alpha_{T,I,F_1}, \alpha_{T,I,F_2}], [\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]\}$ = $[\alpha_{T,I,F_1}, \alpha_{T,I,F_2}]$ and $\lambda_{T,I,F}(x * y) \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$=$max\{\beta_{T,I,F_1}, \beta_{T,I,F_1}\}$=$\beta_{T,I,F_1}$, $\Rightarrow x * y \in \mathcal{B}$. Hence, the proof is completed.

4 Images and Pre-images of Neutrosophic Cubic Subalgebras

In this section, homomorphism of neutrosophic cubic subalgebras are defined and some results are studied.

Let f be a mapping from a set X into a set Y and $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ be a neutrosophic cubic set in Y. So, the inverse-image of $\mathcal{A}$ is defined as $f^{-1}(\mathcal{A})$=$\{\langle x, f^{-1}(A_{T,I,F}), f^{-1}(\lambda_{T,I,F})\rangle \mid x \in X\}$ and $f^{-1}(A_{T,I,F})(x) = A_{T,I,F}(f(x))$ and $f^{-1}(\lambda_{T,I,F})(x) = \lambda_{T,I,F}(f(x))$. It can be shown that $f^{-1}(\mathcal{A})$ is a neutrosophic cubic set.

Theorem 4.1 *Suppose that $f \mid X \to Y$ be a homomorphism of B-algebras. If $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic subalgebra of Y, then the pre-image $f^{-1}(\mathcal{A})$=$\{\langle x, f^{-1}(A_{T,I,F}), f^{-1}(\lambda_{T,I,F})\rangle \mid x \in X\}$ of $\mathcal{A}$ under f is a neutrosophic cubic subalgebra of X.*

Proof: Assume that $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic subalgebra of Y and let $x, y \in X$. then $f^{-1}(A_{T,I,F})(x*y) = A_{T,I,F}(f(x*y)) = A_{T,I,F}(f(x)*f(y)) \geq rmin\{A_{T,I,F}(f(x)), A_{T,I,F}(f(y))\} = rmin\{f^{-1}(A_{T,I,F})(x), f^{-1}(A_{T,I,F})(y)\}$ and $f^{-1}(\lambda_{T,I,F})(x*y) = \lambda_{T,I,F}(f(x*y)) = \lambda_{T,I,F}(f(x)*f(y)) \leq max\{\lambda_{T,I,F}(f(x)), \lambda_{T,I,F}(f(y))\} = max\{f^{-1}(\lambda_{T,I,F})(x), f^{-1}(\lambda_{T,I,F})(y)\}$. $\therefore$ $f^{-1}(\mathcal{A}) = \{\langle x, f^{-1}(A_{T,I,F}), f^{-1}(\lambda_{T,I,F})\rangle \mid x \in X\}$ is neutrosophic cubic subalgebra of X.

Theorem 4.2 *Consider $f \mid X \to Y$ be a homomorphism of B-algebras and $\mathcal{A}_j = (A_{jT,I,F}, \lambda_{jT,I,F})$ be neutrosophic cubic subalgebras of Y where $j \in k$. If $inf\{max\{\lambda_{jT,I,F}(y), \lambda_{jT,I,F}(y)\}\} = max\{inf\lambda_{jT,I,F}(y), inf\lambda_{jT,I,F}(y)\}$ $\forall$ $y \in Y$, then $f^{-1}(\bigcap_{R\, j\in k}\mathcal{A}_j)$ is also a neutrosophic cubic subalgebra of X.*

Proof: Let $\mathcal{A}_j = (A_{jT,I,F}, \lambda_{jT,I,F})$ be neutrosophic cubic subalgebras of Y where $j \in k$ satisfying $inf\{max\{\lambda_{jT,I,F}(y), \lambda_{jT,I,F}(y)\}\}$=$max\{inf\lambda_{jT,I,F}(y), inf\lambda_{jT,I,F}(y)\}$ $\forall$ $y \in Y$. Then by Theorem 3.3, $\bigcap_{R\, j\in k}\mathcal{A}_j$ is a neutrosophic cubic subalgebra of Y. Hence $f^{-1}(\bigcap_{R\, j\in k}\mathcal{A}_j)$ is also a neutrosophic cubic subalgebra of X.

Theorem 4.3 *Let $f \mid X \to Y$ be a homomorphism of B-algebras. Assume that $\mathcal{A}_j = (A_{jT,I,F}, \lambda_{jT,I,F})$ be neutrosophic cubic subalgebras of Y where $j \in k$. If $rsup\{rmin\{A_{jT,I,F}(y_1), A_{jT,I,F}(y_1)\}\}$=$rmin\{rsupA_{jT,I,F}(y_1), rsupA_{jT,I,F}(y_1)\}$ $\forall$ $y_1, y_2 \in Y$, then $f^{-1}(\bigcup_{R\, j\in k}\mathcal{A}_j)$ is also a neutrosophic cubic subalgebra of X.*

Proof: Let $\mathcal{A}_j = (A_{jT,I,F}, \lambda_{jT,I,F})$ be neutrosophic cubic subalgebras of Y, where $j \in k$ satisfying $rsup\{rmin\{A_{jT,I,F}(y_1), A_{jT,I,F}(y_2)\}\}$=$rmin\{rsupA_{jT,I,F}(y_1), rsupA_{jT,I,F}(y_2)\}$ $\forall$ $y_1, y_2 \in Y$. Then by Theorem 3.4, $\bigcup_{R\, j\in k}\mathcal{A}_j$ is a neutrosophic cubic subalgebra of Y. Hence, $f^{-1}(\bigcup_{R\, j\in k}\mathcal{A}_j)$ is also a neutrosophic cubic subalgebra of X.

Definition 4.1 *A neutrosophic cubic set $\mathcal{A}$=$(A_{T,I,F}, \lambda_{T,I,F})$ in the B-algebra X is said to have rsup-property and inf-property if for any subset S of X, there exist $s_0 \in T$ such that $A_{T,I,F}(s_0)$= $rsup_{s_0\in S}A_{T,I,F}(t_0)$ and $\lambda_{T,I,F}(t_0)$=$\inf_{t_0\in T}\lambda_{T,I,F}(t_0)$ respectively.*

Definition 4.2 *Let f be mapping from the set X to the set Y. If $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is neutrosphic cubic set of X, then the image of $\mathcal{A}$ under f denoted by $f(\mathcal{A})$ and is defined as $f(\mathcal{A})$=$\{\langle x, f_{rsup}(A_{T,I,F}), f_{inf}(A_{T,I,F})\rangle \mid x \in X\}$, where*

$$f_{rsup}(A_{T,I,F})(y) = \begin{cases} rsup_{x\in f^{-1}(y)}(A_{T,I,F})(X), & \textit{if } f^{-1}(y) \neq \phi \\ [0,0], & \textit{otherwise}, \end{cases}$$

and

$$f_{inf}(\lambda_{T,I,F})(y) = \begin{cases} \underset{x\in f^{-1}(y)}{\lambda_{T,I,F}}(x), & \textit{if } f^{-1}(y) \neq \phi \\ 1, & \textit{otherwise}. \end{cases}$$

Theorem 4.4 *suppose $f \mid X \to Y$ be a homomorphism from a B-algebra X onto a B-algebra Y. If $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic subalgebra of X, then the image $f(\mathcal{A})$=$\{\langle x, f_{rsup}(A_{T,I,F}), f_{inf}(A_{T,I,F})\rangle \mid x \in X\}$ of $\mathcal{A}$ under f is a neutrosophic cubic subalgebra of Y.*

Proof: Let $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ be a neutrosophic cubic subalgebra of X and let $y_1, \ y_2 \in Y$. We know that $\{x_1 * x_2 \mid x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)\} \subseteq \{x \in X \mid x \in f^{-1}(y_1 * y_2)\}$. Now $f_{rsup}(A_{T,I,F})(y_1 * y_2)$=$rsup\{A_{T,I,F}(x) \mid x \in f^{-1}(y_1 * y_2)\} \geq rsup\{A_{T,I,F}(x_1 * x_2) \mid x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)\} \geq rsup\{rmin\{A_{T,I,F}(x_1), A_{T,I,F}(x_2)\} \mid x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)\} = rmin\{rsup\{A_{T,I,F}(x_1) \mid x_1 \in f^{-1}(y_1)\}, rsup\{A_{T,I,F}(x_2) \mid x_2 \in f^{-1}(y_2)\}\} = rmin\{f_{rsup}(A_{T,I,F})(y_1), f_{rsup}(A_{T,I,F})(y_2)\}$ and $f_{inf}(\lambda_{T,I,F})(y_1 * y_2) = inf\{\lambda_{T,I,F}(x) \mid x \in f^{-1}(y_1*y_2)\} \leq inf\{\lambda_{T,I,F}(x_1*x_2) \mid x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)\} \leq inf\{max\{\lambda_{T,I,F}(x_1), \lambda_{T,I,F}(x_2)\} \mid x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)\} = max\{inf\{\lambda_{T,I,F}(x_1) \mid x_1 \in f^{-1}(y_1)\}, inf\{\lambda_{T,I,F}(x_2) \mid x_2 \in f^{-1}(y_2)\}\} = max\{f_{inf}(\lambda_{T,I,F})(y_1), f_{inf}(\lambda_{T,I,F})(y_2)\}$. Hence $f(\mathcal{A})$=$\{\langle x, f_{rsup}(A_{T,I,F}), f_{inf}(A_{T,I,F})\rangle \mid x \in X\}$ is a neutrosophic cubic subalgebra of Y.

Theorem 4.5 *Assume that $f \mid X \to Y$ is a homomorphism of B-algebra and $\mathcal{A}_i = (A_{iT,I,F}, \lambda_{iT,I,F})$ is a neutrosophic cubic subalgebra of X, where $i \in k$. If $inf\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(x)\}\} = max\{inf\lambda_{iT,I,F}(x), inf\lambda_{iT,I,F}(x)\}$ $\forall$ $x \in X$, then $f(\bigcap_{P\, i\in k}\mathcal{A}_i)$ is a neutrosophic cubic subalgebra of Y.*

Proof: Let $\mathcal{A}_i = (A_{iT,I,F}, \lambda_{iT,I,F})$ be neutrosophic cubic subalgebra of X where $i \in k$ satisfying $inf\{max\{\lambda_{iT,I,F}(x), \lambda_{iT,I,F}(x)\}\}$=$max\{inf\lambda_{iT,I,F}(x), inf\lambda_{iT,I,F}(x)\}$ $\forall x \in X$. Then by Theorem 3.3, $\bigcap_{P\, i\in k}\mathcal{A}_i$ is a neutrosophic cubic algebra of X. Hence $f(\bigcap_{P\, i\in k}\mathcal{A}_j)$ is also a neutrosophic cubic subalgebra of Y.

Theorem 4.6 *Suppose $f \mid X \to Y$ be a homomorphism of B-algebra. Let $\mathcal{A}_i$=$(A_{iT,I,F}, \lambda_{iT,I,F})$ be neutrosophic cubic subalgebras of X where $i \in k$. If $rsup\{rmin\{A_{iT,I,F}(x_1), A_{iT,I,F}(x_2)\}\}$=$rmin\{rsupA_{iT,I,F}(x_1), rsupA_{iT,I,F}(x_2)\}$ $\forall$ $x_1, x_2 \in Y$, then $f(\bigcup_{i\in k}{}_P\mathcal{A}_i)$ is also a neutrosophic cubic subalgebra of Y.*

Proof: Let $\mathcal{A}_i = (A_{iT,I,F}, \lambda_{iT,I,F})$ be neutrosophic cubic subalgebras of X where $i \in k$ satisfying $rsup\{rmin\{A_{iT,I,F}(x_1), A_{iT,I,F}(x_2)\}\}$=$rmin\{rsupA_{iT,I,F}(x_1), rsupA_{iT,I,F}(x_2)\}$ $\forall$ $x_1, x_2 \in X$. Then by Theorem 3.4, $\bigcup_{i\in k}{}_P\mathcal{A}_i$ is a neutrosophic cubic subalgebra of X. Hence $f(\bigcup_{i\in k}{}_P\mathcal{A}_i)$ is also a neutrosophic cubic subalgebra of Y.

Corollary 4.1 *For a homomorphism $f \mid X \to Y$ of B-algebras, the following results hold:*

1. *If $\forall$ $i \in k$, $\mathcal{A}_i$ are neutrosophic cubic subalgebra of X, then $f(\bigcap_{i\in k}{}_R\mathcal{A}_i)$ is neutrosophic cubic subalgebra of Y*
2. *If$\forall$ $i \in k$, $\mathcal{B}_i$ are neutrosophic cubic subalgebra of Y, then $f^{-1}(\bigcap_{i\in k}{}_R\mathcal{B}_i)$ is neutrosophic cubic subalgebra of X.*

Proof: Straightforward.

Theorem 4.7 *Let f be an isomorphism from a B-algebra X onto a B-algebra Y. If $\mathcal{A}$ is a neutrosophic cubic subalgebra of X, then $f^{-1}(f(\mathcal{A})) = \mathcal{A}$*

Proof: For any $x \in X$, let $f(x) = y$. Since f is an isomorphism, $f^{-1}(y) = \{x\}$. Thus $f(\mathcal{A})(f(x)) = f(\mathcal{A})(y) = \bigcup_{x\in f^{-1}(y)} \mathcal{A}(x)$ $= \mathcal{A}(x)$.

For any $y \in Y$, since f is an isomorphism, $f^{-1}(y) = \{x\}$ so that $f(x) = y$. Thus $f^{-1}(\mathcal{A})(x) = \mathcal{A}(f(x)) = \mathcal{A}(y)$.
Hence, $f^{-1}(f(\mathcal{A})) = f^{-1}(\mathcal{A}) = \mathcal{A}$.

Corollary 4.2 *Consider f is an Isomorphism from a B-algebra X onto a B-algebra Y. If $\mathcal{C}$ is a neutrosophic cubic subalgebra of Y, then $f(f^{-1}(\mathcal{C})) = \mathcal{C}$.*

Proof: Straightforward.

Corollary 4.3 *Let $f \mid X \to X$ be an automorphism. If $\mathcal{A}$ refers to a neutrosophic cubic subalgebra of X, then $f(\mathcal{A}) = \mathcal{A} \iff f^{-1}(\mathcal{A}) = \mathcal{A}$*

5 Neutrosophic Cubic Closed Ideals of B-algebras

In this section, neutrosophic cubic ideals and Neutrosophic cubic closed ideals of B-algebra are defined and related results are proved.

Definition 5.1 *A neutrosophic cubic set $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ of X is called a neutrosophic cubic ideal of X if it satisfies following axioms:*

N3. $A_{T,I,F}(0) \geq A_{T,I,F}(x)$ and $\lambda_{T,I,F}(0) \leq \lambda_{T,I,F}(x)$,
N4. $A_{T,I,F}(x) \geq rmin\{A_{T,I,F}(x * y), A_{T,I,F}(y)\}$,
N5. $\lambda_{T,I,F}(x) \leq max\{\lambda_{T,I,F}(x * y), \lambda_{T,I,F}(y)\}\forall$ $x, y \in X$

Example 5.1 *Consider a B-algebra $X = \{0, a_1, a_2, a_3\}$ and binary operation * is defined on X as*

$*$	0	a_1	a_2	a_3
0	0	a_1	a_2	a_3
a_1	a_1	0	a_3	a_2
a_2	a_2	a_3	0	a_1
a_3	a_3	a_2	a_1	0

Let $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ be a neutrosophic cubic set X defined as,

	0	a_1	a_2	a_3
A_T	[1,1]	[0.9,0.8]	[1,1]	[0.5,0.7]
A_I	[0.9,0.9]	[0.6,0.8]	[0.9,0.9]	[0.7,0.5]
A_F	[0.8,0.9]	[0.5,0.6]	[0.8,0.9]	[0.9,0.5]

,

	0	a_1	a_2	a_3
λ_T	0	0.9	0	0.8
λ_I	0.1	0.6	0.1	0.7
λ_F	0.3	0.4	0.3	0.5

Then it can be easy verify that $\mathcal{A}$ satisfying the conditions N3, N4 and N5. Hence $\mathcal{A}$ is a neutrosophic cubic ideal of X.

Definition 5.2 *Let $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ be a neutrosophic cubic set X then it is called neutrosophic cubic closed ideal of X if it satisfies N4, N5 and*

N6. $A_{T,I,F}(0 * x) \geq A_{T,I,F}(x)$ and $\lambda_{T,I,F}(0 * x) \leq \lambda_{T,I,F}(x)$, $\forall$ $x \in X$.

Example 5.2 *Let $X = \{0, a_1, a_2, a_3, a_4, a_5\}$ be a B-algebra in Example 3.2 and $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ be a neutrosophic cubic set X defined as*

	0	a_1	a_2	a_3	a_4	a_5
A_T	[0.3,0.6]	[0.2,0.5]	[0.2,0.5]	[0.1,0.3]	[0.1,0.3]	[0.1,0.3]
A_I	[0.4,0.7]	[0.3,0.6]	[0.3,0.6]	[0.2,0.5]	[0.2,0.5]	[0.2,0.5]
A_F	[0.5,0.8]	[0.4,0.7]	[0.4,0.7]	[0.2,0.3]	[0.2,0.3]	[0.2,0.3]

	0	a_1	a_2	a_3	a_4	a_5
λ_T	0.2	0.5	0.5	0.7	0.7	0.7
λ_I	0.3	0.4	0.4	0.6	0.6	0.6
λ_F	0.4	0.5	0.5	0.8	0.8	0.8

.

By calculations verify that $\mathcal{A}$ is a neutrosophic cubic closed ideal of X.

Proposition 5.1 *Every neutrosophic cubic closed ideal is a neutrosophic cubic ideal.*

The converse of Proposition 5.1 is not true in general as shown in the following example.

Example 5.3 *Let $X = \{0, a_1, a_2, a_3, a_4, a_5\}$ be a B-algebra in Example 3.1 and $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ be a neutrosophic cubic set in X defined as,*

	0	a_1	a_2	a_3	a_4	a_5
A_T	[0.4,0.6]	[0.3,0.5]	[0.3,0.5]	[0.2,0.3]	[0.2,0.3]	[0.2,0.3]
A_I	[0.5,0.7]	[0.4,0.6]	[0.4,0.6]	[0.3,0.5]	[0.3,0.5]	[0.3,0.5]
A_F	[0.6,0.8]	[0.5,0.7]	[0.5,0.7]	[0.4,0.3]	[0.4,0.3]	[0.4,0.3]

	0	a_1	a_2	a_3	a_4	a_5
λ_T	0.1	0.4	0.4	0.5	0.5	0.5
λ_I	0.2	0.3	0.3	0.6	0.6	0.6
λ_F	0.3	0.5	0.5	0.8	0.8	0.8

.

By calculations verify that $\mathcal{A}$ is a neutrosophic cubic ideal of X. But it is not a neutrosophic cubic closed ideal of X since $A_{T,I,F}(0 * x) \geq A_{T,I,F}(x)$ and $\lambda_{T,I,F}(0 * x) \leq \lambda_{T,I,F}(x)$, $\forall \ x \in X$.

Corollary 5.1 *Every neutrosophic cubic subalgebra satisfies N4 and N5 refer to a neutrosophic cubic closed ideal.*

Theorem 5.1 *Every neutrosophic cubic closed ideal of a B-algebra X works as a neutrosophic cubic subalgebra of X.*

Proof: Suppose $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ be a neutrosophic cubic closed ideal of X, then for any $x \in X$ we have $A_{T,I,F}(0 * x) \geq A_{T,I,F}(x)$ and $\lambda_{T,I,F}(0 * x) \leq \lambda_{T,I,F}(x)$. Now by N4, N6, ([3], Proposition 3.2), we know that $A_{T,I,F}(x*y) \geq rmin\{A_{T,I,F}((x y)*(0*y)), A_{T,I,F}(0*y)\} = rmin\{A_{T,I,F}(x), A_{T,I,F}(0*y)\} \geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x*y) \leq max\{\lambda_{T,I,F}((x*y)*(0*y)), \lambda_{T,I,F}(0*y)\} = max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(0* y)\} \leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}$. Hence, $\mathcal{A}$ is a neutrosophic cubic subalgeba of X.

Theorem 5.2 *The R-intersection of any set of neutrosophic cubic ideals of X is also a neutrosophic cubic ideal of X.*

Proof: Let $\mathcal{A}_i = \{A_{iT,I,F}, \lambda_{iT,I,F}\}$, where $i \in k$, be a neutrosophic cubic ideals of X and $x, y \in X$. Then

$$\begin{aligned}(\cap A_{iT,I,F})(0) &= rinf A_{iT,I,F}(0)\\ &\geq rinf A_{iT,I,F}(x)\\ &= (\cap A_{iT,I,F})(x),\end{aligned}$$

$$\begin{aligned}(\vee \lambda_{iT,I,F})(0) &= sup\lambda_{iT,I,F}(0)\\ &\leq \lambda_{iT,I,F}(x)\\ &= (\vee \lambda_{iT,I,F})(x),\end{aligned}$$

$$\begin{aligned}(\cap A_{iT,I,F})(x) &= rinf A_{iT,I,F}(x)\\ &\geq rinf\{rmin\{A_{iT,I,F}(x*y), A_{iT,I,F}(y)\}\}\\ &= rmin\{rinf A_{iT,I,F}(x*y), rinf A_{iT,I,F}(y)\}\\ &= rmin\{(\cap A_{iT,I,F})(x*y), (\cap A_{iT,I,F})(y)\}\end{aligned}$$

and

$$\begin{aligned}(\vee \lambda_{iT,I,F})(x) &= sup\lambda_{iT,I,F}(x)\\ &\leq sup\{max\{\lambda_{iT,I,F}(x*y), \lambda_{iT,I,F}(y)\}\}\\ &= max\{sup\lambda_{iT,I,F}(x*y), sup\lambda_{iT,I,F}(y)\}\\ &= max\{(\vee \lambda_{iT,I,F})(x*y), (\vee \lambda_{iT,I,F})(y)\}\end{aligned}$$

which shows that R-intersection is a neutrosophic cubic ideal of X.

Theorem 5.3 *The R-intersection of any set of neutrosophic cubic closed ideals of X is also a neutrosophic cubic closed ideal of X.*

Proof: It is similar to the proof of Theorem 5.2.

Theorem 5.4 *Neutrosophic cubic set $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ of X is a neutrosophic cubic ideal of $X \iff A^-_{T,I,F}, A^+_{T,I,F}$ and $\lambda_{T,I,F}$ are $fuzzy$ ideals of X.*

Proof: Assume that $x, y \in X$. Since $A^-_{T,I,F}(0) \geq A^-_{T,I,F}(x)$ and $A^+_{T,I,F}(0) \geq A^+_{T,I,F}(x)$, therefore, $A_{T,I,F}(0) \geq A_{T,I,F}(x)$. Also, $\lambda_{T,I,F}(0) \leq \lambda_{T,I,F}(x)$. Let $A^-_{T,I,F}$, $A^+_{T,I,F}$ and $\lambda_{T,I,F}$ are fuzzy ideals of X. Then $A_{T,I,F}(x) = [A^-_{T,I,F}(x), A^+_{T,I,F}(x)] \geq [min\{A^-_{T,I,F}(x*y), A^-_{T,I,F}(y)\}, min\{A^+_{T,I,F}(x*y), A^+_{T,I,F}(y)\}$ = $rmin\{[A^-_{T,I,F}(x*y), A^+_{T,I,F}(x*y)], [A^-_{T,I,F}(y), A^+_{T,I,F}(y)]\}$ = $rmin\{A_{T,I,F}(x*y), A_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x) \leq max\{\lambda_{T,I,F}(x*y), \lambda_{T,I,F}(y)\}$. Therefore $\mathcal{A}$ is a neutrosophic cubic ideal of X.

Conversely, let $\mathcal{A}$ be a neutrosophic cubic ideal of X. For any $x, y \in X$, we have $[A^-_{T,I,F}(x), A^+_{T,I,F}(x)]$ = $A_{T,I,F}(x) \geq rmin\{A_{T,I,F}(x*y), A_{T,I,F}(y)\}$ = $rmin\{[A^-_{T,I,F}(x*y), A^+_{T,I,F}(x*y)], [A^-_{T,I,F}(y), A^+_{T,I,F}(y)]\}$ = $[min\{A^-_{T,I,F}(x*y), A^-_{T,I,F}(y)\}, min\{A^+_{T,I,F}(x*y), A^+_{T,I,F}(y)\}$. Thus, $A^-_{T,I,F}(x) \geq min\{A^-_{T,I,F}(x*y), A^-_{T,I,F}(y)\}$, $A^+_{T,I,F}(x) \geq min\{A^+_{T,I,F}(x*y), A^+_{T,I,F}(y)\}$ and $\lambda_{T,I,F}(x) \leq max\{\lambda_{T,I,F}(x*y), \lambda_{T,I,F}(y)\}$. Hence, $A^-_{T,I,F}$, $A^+_{T,I,F}$ and $\lambda_{T,I,F}$ are $fuzzy\ ideals$ of X.

Theorem 5.5 *For a neutrosophic cubic ideal $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ of X, the following are valid:*

1. *if $x * y \leq z$, then $A_{T,I,F}(x) \geq rmin\{A_{T,I,F}(y), A_{T,I,F}(z)\}$ and $\lambda_{T,I,F}(x) \leq max\{\lambda_{T,I,F}(y), \lambda_{T,I,F}(z)\}$,*
2. *if $x \leq y$, then $A_{T,I,F}(x) \geq A_{T,I,F}(y)$ and $\lambda_{T,I,F}(x) \leq \lambda_{T,I,F}(y)$ $\forall\, x, y, z \in X$.*

Proof: (1) Assume that $x, y, z \in X$ such that $x * y \leq z$. Then $(x*y)*z = 0$ and thus $A_{T,I,F}(x) \geq rmin\{A_{T,I,F}(x*y), A_{T,I,F}(y)\} \geq rmin\{rmin\{A_{T,I,F}((x*y)*z), A_{T,I,F}(z)\}, A_{T,I,F}(y)\}$ =$rmin\{rmin\{A_{T,I,F}(0), A_{T,I,F}(z)\}, A_{T,I,F}(y)\}$=$rmin\{A_{T,I,F}(y), A_{T,I,F}(z)\}$ and $\lambda_{T,I,F}(x) \leq max\{\lambda_{T,I,F}(x*y), \lambda_{T,I,F}(y)\} \leq max\{max\{\lambda_{T,I,F}((x*y)*z), \lambda_{T,I,F}(z)\}, \lambda_{T,I,F}(y)\}$=$max\{max\{\lambda_{T,I,F}(0), \lambda_{T,I,F}(z)\}, \lambda_{T,I,F}(y)\}$ = $max\{\lambda_{T,I,F}(y), \lambda_{T,I,F}(z)\}$.

(2) Again, take $x, y \in X$ such that $x \leq y$. Then $x * y = 0$ and thus $A_{T,I,F}(x) \geq rmin\{A_{T,I,F}(x*y), A_{T,I,F}(y)\}$ = $rmin\{A_{T,I,F}(0), A_{T,I,F}(y)\}$ = $A_{T,I,F}(y)$ and $\lambda_{T,I,F}(x) \leq rmin\{\lambda_{T,I,F}(x*y), \lambda_{T,I,F}(y)\}$ = $rmin\{\lambda_{T,I,F}(0), \lambda_{T,I,F}(y)\}$ = $\lambda_{T,I,F}(y)$.

Theorem 5.6 *Let $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ is a neutrosophic cubic ideal of X. If $x * y \leq x \; \forall \; x, y \in X$, then $\mathcal{A}$ is a neutrosophic cubic subalgebra of X.*

Proof: Assume that $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ is a neutrosophic cubic ideal of X. Suppose that $x * y \leq x \; \forall \; x, y \in X$. Then

$$\begin{aligned}
A_{T,I,F}(x*y) &\geq A_{T,I,F}(x) \\
&\quad (\because By \;\; Theorem \;\; 5.5) \\
&\geq rmin\{A_{T,I,F}(x*y), A_{T,I,F}(y)\} \\
&\quad (\because By \;\; N4) \\
&\geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\} \\
&\quad (\because By \;\; Theorem \;\; 5.5) \\
\Rightarrow A_{T,I,F}(x*y) &\geq rmin\{A_{T,I,F}(x), A_{T,I,F}(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{T,I,F}(x*y) &\leq \lambda_{T,I,F}(x) \\
&\quad (\because By \;\; Theorem \;\; 5.5) \\
&\leq max\{\lambda_{T,I,F}(x*y), \lambda_{T,I,F}(y)\} \\
&\quad (\because By \;\; N5) \\
&\leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\} \\
&\quad (\because By \;\; Theorem \;\; 5.5) \\
\Rightarrow \lambda_{T,I,F}(x*y) &\leq max\{\lambda_{T,I,F}(x), \lambda_{T,I,F}(y)\}.
\end{aligned}$$

Hence, $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ is a neutrosophic cubic subalgebra of X.

Theorem 5.7 *If $\mathcal{A} = \{A_{T,I,F}, \lambda_{T,I,F}\}$ is a neutrosophic cubic ideal of X, then $(...((x * a_1) * a_2) * ...) * a_n = 0$ for any $x, a_1, a_2, ..., a_n \in X$, $\Rightarrow A_{T,I,F}(x) \geq rmin\{A_{T,I,F}(a_1), A_{T,I,F}(a_2), ..., A_{T,I,F}(a_n)\}$ and $\lambda_{T,I,F}(x) \leq max\{\lambda_{T,I,F}(a_1), \lambda_{T,I,F}(a_2), ..., \lambda_{T,I,F}(a_n)\}$.*

Proof: We can prove this theorem by using induction on n and Theorem 5.5).

Theorem 5.8 *A neutrosophic cubic set $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic closed ideal of $X \iff U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and $L(\lambda_{T,I,F} \mid t_{T,I,F_1})$ are closed ideals of X for every $[s_{T,I,F_1}, s_{T,I,F_2}] \in D[0,1]$ and $t_{T,I,F_1} \in [0,1]$.*

Proof: Assume that $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic closed ideal of X. For $[s_{T,I,F_1}, s_{T,I,F_2}] \in D[0,1]$, clearly, $0 * x \in U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$, where $x \in X$. Let $x, y \in X$ be such that $x * y \in U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and $y \in U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$. Then $A_{T,I,F}(x) \geq rmin\{A_{T,I,F}(x * y), A_{T,I,F}(y)\} \geq [s_{T,I,F_1}, s_{T,I,F_2}]$, $\Rightarrow x \in U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$. Hence, $U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ is a closed ideal of X.

For $t_{T,I,F_1} \in [0,1]$. Clearly, $0 * x \in L(\lambda_{T,I,F} \mid t_{T,I,F_1})$. Let $x, y \in X$ be such that $x * y \in L(\lambda_{T,I,F} \mid t_{T,I,F_1})$ and $y \in L(\lambda_{T,I,F} \mid t_{T,I,F_1})$. Then $\lambda_{T,I,F}(x) \leq max\{\lambda_{T,I,F}(x * y), \lambda_{T,I,F}(y)\} \leq t_{T,I,F_1}$, $\Rightarrow x \in L(\lambda_{T,I,F} \mid t_{T,I,F_1})$. Hence, $L(\lambda_{T,I,F} \mid t_{T,I,F_1})$. is a neutrosophic cubic closed ideal of X.

Conversely, suppose that each non-empty level subset $U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and $L(\lambda_{T,I,F} \mid t_{T,I,F_1})$ are closed ideals of X. For any $x \in X$, let $A_{T,I,F}(x) = [s_{T,I,F_1}, s_{T,I,F_2}]$ and $\lambda_{T,I,F}(x) = t_{T,I,F_1}$. Then $x \in U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and $x \in L(\lambda_{T,I,F} \mid t_{T,I,F_1})$. Since $0 * x \in U(A_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}]) \bigcap L(\lambda_{T,I,F} \mid t_{T,I,F_1})$, it follows that $A_{T,I,F}(0 * x) \geq [s_{T,I,F_1}, s_{T,I,F_2}] = A_{T,I,F}(x)$ and $\lambda_{T,I,F}(0 * x) \leq t_{T,I,F_1} = \lambda_{T,I,F}(x) \; \forall \; x \in X$.

If there exist $\alpha_{T,I,F_1}, \beta_{T,I,F_1} \in X$ such that $A_{T,I,F}(\alpha_{T,I,F_1}) < rmin\{A_{T,I,F}(\alpha_{T,I,F_1} * \beta_{T,I,F_1}), \beta_{T,I,F_1}\}$, then by taking $[s'_{T,I,F_1}, s'_{T,I,F_2}] = \frac{1}{2}[A_{T,I,F}(\alpha_{T,I,F_1} * \beta_{T,I,F_1}) + rmin\{A_{T,I,F}(\alpha_{T,I,F_1}), A_{T,I,F}(\beta_{T,I,F_1})\}]$, it follows that $\alpha_{T,I,F_1} * \beta_{T,I,F_1} \in U(A_{T,I,F} \mid [s'_{T,I,F_1}, s'_{T,I,F_2}])$ and $\beta_{T,I,F_1} \in U(A_{T,I,F} \mid [s'_{T,I,F_1}, s'_{T,I,F_2}])$, but $\alpha_{T,I,F_1} \notin U(A_{T,I,F} \mid [s'_{T,I,F_1}, s'_{T,I,F_2}])$, which is contradiction. Hence, $U(A_{T,I,F} \mid [s'_{T,I,F_1}, s'_{T,I,F_2}])$ is not closed ideal of X.

Again, if there exist $\gamma_{T,I,F_1}, \;\; \delta_{T,I,F_1} \in X$ such that $\lambda_{T,I,F}(\gamma_{T,I,F_1}) > max\{\lambda_{T,I,F}(\gamma_{T,I,F_1} * \delta_{T,I,F_1}), \lambda_{T,I,F}(\delta_{T,I,F_1})\}$, then by taking $t'_{T,I,F_1} = \frac{1}{2}[\lambda_{T,I,F}(\gamma_{T,I,F_1} * \delta_{T,I,F_1}) + max\{\lambda_{T,I,F}(\gamma_{T,I,F_1}), \lambda_{T,I,F}(\delta_{T,I,F_1})\}]$, it follows that $\gamma_{T,I,F_1} * \delta_{T,I,F_1} \in L(\lambda_{T,I,F} \mid t'_{T,I,F_1})$ and $\delta_{T,I,F_1} \in L(\lambda_{T,I,F} \mid t'_{T,I,F_1})$, but $\gamma_{T,I,F_1} \notin L(\lambda_{T,I,F} \mid t'_{T,I,F_1})$, which is contradiction. Hence, $L(\lambda_{T,I,F} \mid t'_{T,I,F_1})$ is not closed ideal of X. Hence, $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic closed ideal of X because it satisfies N3 and N4.

6 Investigation of Neutrosophic Cubic Ideals under Homomorphism

In this section, neutrosophic cubic ideals are investigated under homomorphism and some results are studied.

Theorem 6.1 *Suppose that $f \mid X \to Y$ is a homomorphism of B-algebra. If $\mathcal{A}$=$(A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic ideal of Y, then pre-image $f^{-1}(\mathcal{A})$=$\Big(f^{-1}(A_{T,I,F}), f^{-1}(\lambda_{T,I,F})\Big)$ of $\mathcal{A}$ under f of X is a neutrosophic cubic ideal of X.*

Proof: $\forall \; x \in X$, $f^{-1}(A_{T,I,F})(x) = A_{T,I,F}(f(x)) \leq A_{T,I,F}(0)$=$A_{T,I,F}(f(0))$=$f^{-1}(A_{T,I,F})(0)$ and $f^{-1}(\lambda_{T,I,F})(x) = \lambda_{T,I,F}(f(x)) \geq \lambda_{T,I,F}(0) = \lambda_{T,I,F}(f(0)) = f^{-1}(\lambda_{T,I,F})(0)$.

Let $x, y \in X$ then $f^{-1}(A_{T,I,F})(x) = A_{T,I,F}(f(x)) \geq rmin\{A_{T,I,F}(f(x) * f(y)), A_{T,I,F}(f(y))\} = rmin\{A_{T,I,F}(f(x*y)), A_{T,I,F}(f(y))\} = rmin\{f^{-1}(A_{T,I,F})(x*y), f^{-1}(A_{T,I,F})(y)\}$ and $f^{-1}(\lambda_{T,I,F})(x) = \lambda_{T,I,F}(f(x)) \leq max\{\lambda_{T,I,F}(f(x) * f(y)), \lambda_{T,I,F}(f(y))\} = max\{\lambda_{T,I,F}(f(x*y)), \lambda_{T,I,F}(f(y))\} = max\{f^{-1}(\lambda_{T,I,F})(x*y), f^{-1}(\lambda_{T,I,F})(y)\}$.

Hence, $f^{-1}(\mathcal{A}) = \Big(f^{-1}(A_{T,I,F}), f^{-1}(\lambda_{T,I,F})\Big)$ is a neutrosophic cubic ideal of X.

Corollary 6.1 *A homomorphic pre-image of a neutrosophic cubic closed ideal is a neutrosophic cubic ideal.*

Proof: Using Proposition 5.1 and Theorem 6.1, straightforward.

Corollary 6.2 *A homomorphic pre-image of a neutrosophic cubic closed ideal is also a neutrosophic cubic subalgebra.*

Proof: Straightforward, using Theorem 5.1 and Theorem 6.1.

Corollary 6.3 *Let $f \mid X \to Y$ be homomorphism of B-algebra. If $\mathcal{A}_i = (A_{iT,I,F}, \lambda_{iT,I,F})$ is a neutrosophic cubic ideals of Y where $i \in k$ then the pre-image $f^{-1}\Big(\bigcap_{i\in k_R} A_{iT,I,F}\Big)$ $= \Big(f^{-1}(\bigcap_{i\in k_R} A_{iT,I,F}), f^{-1}(\bigcap_{i\in k_R} \lambda_{iT,I,F})\Big)$ is a neutrosophic cubic ideal of X.*

Proof: Straightforward, using Theorem 5.2 and Theorem 6.1.

Corollary 6.4 *Let $f \mid X \to Y$ be homomorphism of B-algebra. If $\mathcal{A}_i = (A_{iT,I,F}, \lambda_{iT,I,F})$ is a neutrosophic cubic closed ideals of Y where $i \in k$ then the pre-image $f^{-1}\Big(\bigcap_{i\in k_R} A_{iT,I,F}\Big)$ $= \Big(f^{-1}(\bigcap_{i\in k_R} A_{iT,I,F}), f^{-1}(\bigcap_{i\in k_R} \lambda_{iT,I,F})\Big)$ is a neutrosophic cubic closed ideal of X.*

Proof: Straightforward, using theorem 5.3 and Theorem 6.1.

Theorem 6.2 *Suppose that $f \mid X \to Y$ is an epimorphism of B-algebra. Then $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic ideal of Y, if $f^{-1}(\mathcal{A}) = \Big(f^{-1}(A_{T,I,F}), f^{-1}(\lambda_{T,I,F})\Big)$ of $\mathcal{A}$ under f of X is a neutrosophic cubic ideal of X.*

Proof: For any $y \in Y$, $\exists\, x \in X$ such that $y = f(x)$. So, $A_{T,I,F}(y) = A_{T,I,F}(f(x)) = f^{-1}(A_{T,I,F})(x) \leq f^{-1}(A_{T,I,F})(0) = A_{T,I,F}(f(0)) = A_{T,I,F}(0)$ and $\lambda_{T,I,F}(y) = \lambda_{T,I,F}(f(x)) = f^{-1}(\lambda_{T,I,F})(x) \geq f^{-1}(\lambda_{T,I,F})(0) = \lambda_{T,I,F}(f(0)) = \lambda_{T,I,F}(0)$.

Suppose $y_1, y_2 \in y$. Then $f(x_1) = y_1$ and $f(x_2) = y_2$ for some $x_1,\ x_2 \in X$. Thus, $A_{T,I,F}(y_1) = A_{T,I,F}(f(x_1)) = f^{-1}(A_{T,I,F})(x_1) \geq rmin\{f^{-1}(A_{T,I,F})(x_1 * x_2), f^{-1}(A_{T,I,F})(x_2)\} = rmin\{A_{T,I,F}(f(x_1 * x_2)), A_{T,I,F}(f(x_2))\} = rmin\{A_{T,I,F}(f(x_1) * f(x_2)), A_{T,I,F}(f(x_2))\} = rmin\{A_{T,I,F}(y_1 * y_2), A_{T,I,F}(y_2)\}$ and $\lambda_{T,I,F}(y_1) = \lambda_{T,I,F}(f(x_1)) = f^{-1}(\lambda_{T,I,F})(x_1) \leq max\{f^{-1}(\lambda_{T,I,F})(x_1 * x_2), f^{-1}(\lambda_{T,I,F})(x_2)\} = max\{\lambda_{T,I,F}(f(x_1 x_2)), \lambda_{T,I,F}(f(x_2))\} = max\{\lambda_{T,I,F}(f(x_1) * f(x_2)), \lambda_{T,I,F}(f(x_2))\} = max\{\lambda_{T,I,F}(y_1 * y_2), \lambda_{T,I,F}(y_2)\}$. Hence, $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ is a neutrosophic cubic ideal of Y.

6.1 Product of Neutrosophic Cubic B-algebra

In this section, product of neutrosophic cubic B-algebras are defined and some corresponding results are investigated.

Definition 6.1 *Let $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ and $\mathcal{B} = (B_{T,I,F}, \upsilon_{T,I,F})$ be two neutrosophic cubic sets of X and Y respectively. The Cartesian product $\mathcal{A} \times \mathcal{B} = (X \times Y, A_{T,I,F} \times B_{T,I,F}, \lambda_{T,I,F} \times \upsilon_{T,I,F})$ is defined by $(A_{T,I,F} \times B_{T,I,F})(x,y) = rmin\{A_{T,I,F}(x), B_{T,I,F}(y)\}$ and $(\lambda_{T,I,F} \times \upsilon_{T,I,F})(x,y) = max\{\lambda_{T,I,F}(x), \upsilon_{T,I,F}(y)\}$, where $A_{T,I,F} \times B_{T,I,F} \mid X \times Y \to D[0,1]$ and $\lambda_{T,I,F} \times \upsilon_{T,I,F} \mid X \times Y \to [0,1]\ \forall\, (x,y) \in X \times Y$.*

Remark 6.1 *Let X and Y be B-algebras. we define $*$ on $X \times Y$ by $(x_1, y_1) * (x_2, y_2) = (x_1 * x_2, y_1 * y_2)$ for every (x_1, y_1) and $(x_2, y_2) \in X \times Y$. Then clearly, $X \times Y$ is a B-algebra.*

Definition 6.2 *A neutrosophic cubic subset $\mathcal{A} \times \mathcal{B} = (X \times Y, A_{T,I,F} \times B_{T,I,F}, \lambda_{T,I,F} \times \upsilon_{T,I,F})$ is called a neutrosophic cubic subalgebra if*
N7: $(A_{T,I,F} \times B_{T,I,F})((x_1, y_1) * (x_2, y_2)) \geq rmin\{(A_{T,I,F} \times B_{T,I,F})(x_1, y_1), (A_{T,I,F} \times B_{T,I,F})(x_2, y_2)\}$
N8: $(\lambda_{T,I,F} \times \upsilon_{T,I,F})((x_1, y_1) * (x_2, y_2)) \leq max\{(\lambda_{T,I,F} \times \upsilon_{T,I,F})(x_1, y_1), (\lambda_{T,I,F} \times \upsilon_{T,I,F})(x_2, y_2)\}$
$\forall\, (x_1, y_1), (x_2, y_2) \in X \times Y$

Theorem 6.3 *Let $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ and $\mathcal{B} = (B_{T,I,F}, \upsilon_{T,I,F})$ be neutrosophic cubic subalgebra of X and Y respectively. Then $\mathcal{A} \times \mathcal{B}$ is a neutrosophic cubic subalgebra of $X \times Y$.*

Proof: Let (x_1, y_1) and $(x_2, y_2) \in X \times Y$. Then $(A_{T,I,F} \times B_{T,I,F})((x_1, y_1) * (x_2, y_2)) = (A_{T,I,F} \times B_{T,I,F})(x_1 * x_2, y_1 * y_2) = rmin\{A_{T,I,F}(x_1 * x_2), B_{T,I,F}(y_1 * y_2)\} \geq rmin\{rmin\{A_{T,I,F}(x_1), A_{T,I,F}(x_2)\}, rmin\{B_{T,I,F}(y_1), B_{T,I,F}(y_2)\}\} = rmin\ \{rmin\ \{A_{T,I,F}(x_1), B_{T,I,F}(y_1)\}, rmin\ \{A_{T,I,F}(x_2), B_{T,I,F}(y_2)\}\} = rmin\{(A_{T,I,F} \times B_{T,I,F})(x_1, y_1), (A_{T,I,F} \times B_{T,I,F})(x_2, y_2)\}$ and $(\lambda_{T,I,F} \times \upsilon_{T,I,F})((x_1, y_1) * (x_2, y_2)) = (\lambda_{T,I,F} \times \upsilon_{T,I,F})(x_1 * x_2, y_1 * y_2) = max\{\lambda_{T,I,F}(x_1 * x_2), \upsilon_{T,I,F}(y_1 * y_2)\} \leq max\{max\{\lambda_{T,I,F}(x_1), \lambda_{T,I,F}(x_2)\}, max\{\upsilon_{T,I,F}(y_1), \upsilon_{T,I,F}(y_2)\}\} = max\{max\{\lambda_{T,I,F}(x_1), \upsilon_{T,I,F}(y_1)\}, max\{\lambda_{T,I,F}(x_2), \upsilon_{T,I,F}(y_2)\}\} = max\{(\lambda_{T,I,F} \times \upsilon_{T,I,F})(x_1, y_1), (\lambda_{T,I,F} \times \upsilon_{T,I,F})(x_2, y_2)\}$. Hence $\mathcal{A} \times \mathcal{B}$ is a neutrosophic cubic subalgebra of $X \times Y$.

Definition 6.3 *A neutrosophic cubic subset $\mathcal{A} \times \mathcal{B} = (X \times Y, A_{T,I,F} \times B_{T,I,F}, \lambda_{T,I,F} \times \upsilon_{T,I,F})$ is called a neutrosophic cubic ideal if*
N9: $(A_{T,I,F} \times B_{T,I,F})(0,0) \geq (A_{T,I,F} \times B_{T,I,F})(x,y)$ *and* $(\lambda_{T,I,F} \times \upsilon_{T,I,F})(0,0) \leq (\lambda_{T,I,F} \times \upsilon_{T,I,F})(x,y)$ $\forall (x,y) \in X \times Y$,
N10: $(A_{T,I,F} \times B_{T,I,F})(x_1, y_1) \geq rmin\{(A_{T,I,F} \times B_{T,I,F})((x_1, y_1) * (x_2, y_2)), (A_{T,I,F} \times B_{T,I,F})(x_2, y_2)$
and
N11: $(\lambda_{T,I,F} \times \upsilon_{T,I,F})(x_1, y_1) \leq max\{(\lambda_{T,I,F} \times \upsilon_{T,I,F})((x_1, y_1) * (x_2, y_2)), (\lambda_{T,I,F} \times \upsilon_{T,I,F})(x_2, y_2)\}$
*and $\mathcal{A} \times \mathcal{B}$ is closed ideal if it satisfies **N9**, **N10**, **N11**, and*

N12: $(A_{T,I,F} \times B_{T,I,F})((0,0) * (x,y)) \geq (A_{T,I,F} \times B_{T,I,F})(x,y) \; \forall \; (x_1,y_1),(x_2,y_2) \in X \times Y$.

Theorem 6.4 *Let $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ and $\mathcal{B} = (B_{T,I,F}, \upsilon_{T,I,F})$ be neutrosophic cubic ideals of X and Y respectively. Then $\mathcal{A} \times \mathcal{B}$ is a neutrosophic cubic ideal of $X \times Y$.*

Proof: For any $(x,y) \in X \times Y$, we have $(A_{T,I,F} \times B_{T,I,F})(0,0) = rmin\{A_{T,I,F}(0), B_{T,I,F}(0)\} \geq rmin\{A_{T,I,F}(x), B_{T,I,F}(y)\} = (A_{T,I,F} \times B_{T,I,F})(x,y)$ and $(\lambda_{T,I,F} \times \upsilon_{T,I,F})(0,0) = max\{\lambda_{T,I,F}(0), \upsilon_{T,I,F}(0)\} \leq max\{\lambda_{T,I,F}(x), \upsilon_{T,I,F}(y)\} = (\lambda_{T,I,F} \times \upsilon_{T,I,F})(x,y)$.

Let (x_1,y_1) and $(x_2,y_2) \in X \times Y$. Then $(A_{T,I,F} \times B_{T,I,F})(x_1,y_1) = rmin\{A_{T,I,F}(x_1), B_{T,I,F}(y_1)\} \geq rmin\{rmin\{A_{T,I,F}(x_1 * x_2), A_{T,I,F}(x_2)\}, rmin\{B_{T,I,F}(y_1 * y_2), B_{T,I,F}(y_2)\}\} = rmin\{rmin\{A_{T,I,F}(x_1*x_2), B_{T,I,F}(y_1*y_2)\}, rmin\{A_{T,I,F}(x_2), B_{T,I,F}(y_2)\}\} = rmin\{(A_{T,I,F} \times B_{T,I,F})(x_1*x_2, y_1*y_2), (A_{T,I,F} \times B_{T,I,F})(x_2,y_2)\} = rmin\{(A_{T,I,F} \times B_{T,I,F})((x_1,y_1) * (x_2,y_2)), (A_{T,I,F} \times B_{T,I,F})(x_2,y_2)\}$ and $(\lambda_{T,I,F} \times \upsilon_{T,I,F})(x_1,y_1) = max\{\lambda_{T,I,F}(x_1), \upsilon_{T,I,F}(y_1)\} \leq max\{max\{\lambda_{T,I,F}(x_1 * x_2), \lambda_{T,I,F}(x_2)\}, max\{\upsilon_{T,I,F}(y_1 * y_2), \upsilon_{T,I,F}(y_2)\}\} = max\{max\{\lambda_{T,I,F}(x_1 * x_2), \upsilon_{T,I,F}(y_1 * y_2)\}, max\{\lambda_{T,I,F}(x_2), \upsilon_{T,I,F}(y_2)\}\} = max\{\lambda_{T,I,F} \times \upsilon_{T,I,F})(x_1 * x_2, y_1 * y_2), (\lambda_{T,I,F} \times \upsilon_{T,I,F})(x_2,y_2)\} = max\{(\lambda_{T,I,F} \times \upsilon_{T,I,F})((x_1,y_1)*(x_2*y_2)), (\lambda_{T,I,F} \times \upsilon_{T,I,F})(x_2,y_2)\}$. Hence, $\mathcal{A} \times \mathcal{B}$ is a neutrosophic cubic ideal of $X \times Y$.

Theorem 6.5 *Let $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ and $\mathcal{B} = (B_{T,I,F}, \upsilon_{T,I,F})$ be neutrosophic cubic closed ideals of X and Y respectively. Then $\mathcal{A} \times \mathcal{B}$ is a neutrosophic cubic closed ideal of $X \times Y$.*

Proof: By Proposition 5.1 and Theorem 6.4, $\mathcal{A} \times \mathcal{B}$ is neutrosophic cubic ideal. Now, $(A_{T,I,F} \times B_{T,I,F})((0,0) * (x,y)) = (A_{T,I,F} \times B_{T,I,F})(0 * x, 0 * y) = rmin\{A_{T,I,F}(0 * x), B_{T,I,F}(0 * y)\} \geq rmin\{A_{T,I,F}(x), B_{T,I,F}(y)\} = (A_{T,I,F} \times B_{T,I,F})(x,y)$ and $(\lambda_{T,I,F} \times \upsilon_{T,I,F})((0,0) * (x,y)) = (\lambda_{T,I,F} \times \upsilon_{T,I,F})(0 * x, 0 * y) = max\{\lambda_{T,I,F}(0 * x), \upsilon_{T,I,F}(0 * y)\} \leq max\{\lambda_{T,I,F}(x), \upsilon_{T,I,F}(y)\} = (\lambda_{T,I,F} \times \upsilon_{T,I,F})(x,y)$. Hence, $\mathcal{A} \times \mathcal{B}$ is a neutrosophic cubic closed ideal of $X \times Y$. Hence, $\mathcal{A} \times \mathcal{B}$ is a neutrosophic cubic closed ideal of $X \times Y$.

Definition 6.4 *Let $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ and $\mathcal{B} = (B_{T,I,F}, \upsilon_{T,I,F})$ be neutrosophic cubic subalgebra of X and Y respectively. For $[s_{T,I,F_1}, s_{T,I,F_2}] \in D[0,1]$ and $t_{T,I,F_1} \in [0,1]$, the set $U(A_{T,I,F} \times B_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}]) = \{(x,y) \in X \times Y \mid (A_{T,I,F} \times B_{T,I,F})(x,y) \geq [s_{T,I,F_1}, s_{T,I,F_2}]\}$ is called upper $[s_{T,I,F_1}, s_{T,I,F_2}]$-level of $\mathcal{A} \times \mathcal{B}$ and $L(\lambda_{T,I,F} \times \upsilon_{T,I,F} \mid t_{T,I,F_1}) = \{(x,y) \in X \times Y \mid (\lambda_{T,I,F} \times \upsilon_{T,I,F})(x,y) \leq t_{T,I,F_1}$ is called lower t_{T,I,F_1}-level of $\mathcal{A} \times \mathcal{B}$.*

Theorem 6.6 *For any two neutrosophic cubic sets $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ and $\mathcal{B} = (B_{T,I,F}, \upsilon_{T,I,F})$, $\mathcal{A} \times \mathcal{B}$ is a neutrosophic cubic closed ideals of $X \times Y \iff$ the non-empty upper $[s_{T,I,F_1}, s_{T,I,F_2}]$-level cut $U(A_{T,I,F} \times B_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and the non-empty lower t_{T,I,F_1}-level $L(\lambda_{T,I,F} \times \upsilon_{T,I,F} \mid t_{T,I,F_1})$ are closed ideals of $X \times Y$ for any $[s_{T,I,F_1}, s_{T,I,F_2}] \in D[0,1]$ and $t_{T,I,F_1} \in [0,1]$.*

Proof: Suppose $\mathcal{A} = (A_{T,I,F}, \lambda_{T,I,F})$ and $\mathcal{B} = (B_{T,I,F}, \upsilon_{T,I,F})$ be neutrosophic cubic closed ideals of X. Therefore, for any $(x,y) \in X \times Y$, $(A_{T,I,F} \times B_{T,I,F})((0,0) * (x,y)) \geq (A_{T,I,F} \times B_{T,I,F})(x,y)$ and $(\lambda_{T,I,F} \times \upsilon_{T,I,F})((0,0) * (x,y)) \leq (\lambda_{T,I,F} \times \upsilon_{T,I,F})(x,y)$. For $[s_{T,I,F_1}, s_{T,I,F_2}] \in D[0,1]$, if $(A_{T,I,F} \times B_{T,I,F})(x,y) \geq [s_{T,I,F_1}, s_{T,I,F_2}]$, then $(A_{T,I,F} \times B_{T,I,F})((0,0) * (x,y)) \geq [s_{T,I,F_1}, s_{T,I,F_2}]$. $\Rightarrow (0,0) * (x,y) \in U(A_{T,I,F} \times B_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$. Let $(x,y), (x', y') \in X \times Y$ be such that $(x,y) * (x', y') \in U(A_{T,I,F} \times B_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and $(x', y') \in U(A_{T,I,F} \times B_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$. Now, $(A_{T,I,F} \times B_{T,I,F})(x,y) \geq rmin\{(A_{T,I,F} \times B_{T,I,F})((x,y) * (x', y')), (A_{T,I,F} \times B_{T,I,F})(x', y')\} \geq rmin\{[s_{T,I,F_1}, s_{T,I,F_2}], [s_{T,I,F_1}, s_{T,I,F_2}]\} = [s_{T,I,F_1}, s_{T,I,F_2}]$. $\Rightarrow (x,y) \in U(A_{T,I,F} \times B_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$. Thus $U(A_{T,I,F} \times B_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ is closed ideal of $X \times Y$. Similarly, $L(\lambda_{T,I,F} \times \upsilon_{T,I,F} \mid t_{T,I,F_1})$ is closed ideal of $X \times Y$.

Conversely, let $(x,y) \in X \times Y$ be such that $(A_{T,I,F} \times B_{T,I,F})(x,y) = [s_{T,I,F_1}, s_{T,I,F_2}]$ and $(\lambda_{T,I,F} \times \upsilon_{T,I,F})(x,y) = t_{T,I,F_1}$. This implies, $(x,y) \in U(A_{T,I,F} \times B_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and $(x,y) \in L(\lambda_{T,I,F} \times \upsilon_{T,I,F} \mid t_{T,I,F_1})$. Since $(0,0) * (x,y) \in U(A_{T,I,F} \times B_{T,I,F} \mid [s_{T,I,F_1}, s_{T,I,F_2}])$ and $(0,0) * (x,y) \in L(\lambda_{T,I,F} \times \upsilon_{T,I,F} \mid t_{T,I,F_1})$ (by N6), therefore, $(A_{T,I,F} \times B_{T,I,F})((0,0) * (x,y)) \geq [s_{T,I,F_1}, s_{T,I,F_2}]$ and $(\lambda_{T,I,F} \times \upsilon_{T,I,F})((0,0) * (x,y)) \leq t_{T,I,F_1}$. $\Rightarrow (A_{T,I,F} \times B_{T,I,F})((0,0) * (x,y)) \geq (A_{T,I,F} \times B_{T,I,F})(x,y)$ and $(\lambda_{T,I,F} \times \upsilon_{T,I,F})((0,0) * (x,y)) \leq (\lambda_{T,I,F} \times \upsilon_{T,I,F})(x,y))$. Hence $\mathcal{A} \times \mathcal{B}$ is a neutrosophic cubic closed ideals of $X \times Y$.

7 Conclusion

In this paper, the concept of neutrosophic cubic subalgebra, neutrosophic cubic ideals, neutrosophic cubic closed ideals and the product of neutrosophic cubic subalgebra of B-algebra were presented and their several useful results were canvassed. The relations among neutrosophic cubic subalgebra, neutrosophic cubic ideals and neutrosophic cubic closed ideals of B-algebra were investigated. For future work this study will be further discussed to some another algebraic system.

References

[1] S. S. Ahn and K. Bang, On fuzzy subalgebrasbin B-algebra, Communications of the Korean Mathematical Society, 18(2003), 429-437.

[2] R. Biswas, Rosenfeld's fuzzy subgroup with interval valued membership function, Fuzzy Sets and Systems, 63(1994), 87-90.

[3] J. R. Cho and H. S. Kim, On B-algebras and quasigroups, Quasigroups and Related System, 8(2001), 1-6.

[4] Y. Imai and K. Iseki, On Axiom systems of Propositional calculi XIV, Proc. Japan Academy, 42(1966), 19-22.

[5] K. Iseki, An algebra related with a propositional calculus, Proc. Japan Academy, 42(1966), 26-29.

[6] Y. B. Jun, C. S. Kim and K. O. Yang, Cubic sets, Annuals of Fuzzy Mathematics and Informatics, 4(2012), 83-98.

[7] Y. B. JUN, S. T. JUNG AND M. S. KIM, Cubic subgroup, Annals of Fuzzy Mathematics and Infirmatics, 2(2011), 9-15.

[8] Y. B. JUN, F. SMARANDACHE AND C. S. KIM, Neutrosophic cubic sets, New Mathematics and Natural Computation, (2015), 8-41.

[9] Y. B. JUN, C. S. KIM AND M. S. KANG, Cubic subalgebras and ideals of BCK/BCI-algebra, Far East Journal of Mathematical Sciences, 44(2010), 239-250.

[10] Y. B. JUN, C. S. KIM AND J. G. KANG, Cubic q-Ideal of BCI-algebras, Annals of Fuzzy Mathematics and Informatics, 1(2011), 25-31.

[11] C. B. KIM AND H. S. KIM, On BG-algebra, Demonstration Mathematica, 41(2008), 497-505.

[12] J. NEGGERS AND H. S. KIM, On B-algebras, Mathematichki Vensnik, 54(2002), 21-29.

[13] J. NEGGERS AND H. S. KIM, A fundamental theorem of B-homomorphism for B-algebras, International Mathematical Journal, 2(2002), 215-219.

[14] H. K. PARK AND H. S. KIM, On quadratic B-algebras, Qausigroups and Related System, 7(2001), 67-72.

[15] A. B. SAEID, Fuzzy topological B-algebra, International Journal of Fuzzy Systems, 8(2006), 160-164.

[16] A. B. SAEID, Interval-valued fuzzy B-algebras, Iranian Journal of Fuzzy System, 3(2006), 63-73.

[17] T. SENAPATI, Bipolar fuzzy structure of BG-algebras, The Journal of Fuzzy Mathematics, 23(2015), 209-220.

[18] F. SMARANDACHE, Neutrosophic set a generalization of the intuitionistic fuzzy set, International Journal of Pure and Applied Mathematics, 24(3)(2005), 287-297.

[19] F. SMARANDACHE, A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set and Neutrosophic Probability, (American Reserch Press, Rehoboth, NM, 1999).

[20] T. SENAPATI, M. BHOWMIK AND M. PAL, Fuzzy dot subalgebras and fuzzy dot ideals of B-algebra, Journal of Uncertain System, 8(2014), 22-30.

[21] T. SENAPATI, M. BHOWMIK AND M. PAL, Fuzzy closed ideals of B-algebras, International Journal of Computer Science, Engineering and Technology, 1(2011), 669-673.

[22] T. SENAPATI, M. BHOWMIK AND M. PAL, Fuzzy closed ideals of B-algebras, with interval-valued membership function, International Journal of Fuzzy Mathematical Archive, 1(2013), 79-91.

[23] T. SENAPATI, M. BHOWMIK AND M. PAL, Fuzzy B-subalgebras of B-algebra with resepect to t-norm, Journal of Fuzzy Set Valued Analysis, (2012), doi: 10.5899/2012/jfsva-00111.

[24] T. SENAPATI, C. JANA, M. BHOWMIK AND M. PAL, L-fuzzy G-subalgebra of G-algebras, Journal of the Egyptian Mathematical Society, 23(2014), 219223.

[25] T. SENAPATI, C.H. KIM, M. BHOWMIK AND M. PAL, Cubic subalgebras and cubic closed ideals of B-algebras, Fuzzy Information and Engineering, 7(2015), 129-149.

[26] T. SENAPATI, M. BHOWMIK AND M. PAL, Intuitionistic L-fuzzy ideals of BG-algebras, Afrika Matematika, 25(2014), 577-590.

[27] T. SENAPATI, M. BHOWMIK AND M. PAL, Interval-valued intuitionistic fuzzy BG-subalgebras, The Journal of Fuzzy Mathematics, 20(2012), 707-720.

[28] T. SENAPATI, M. BHOWMIK AND M. PAL, Interval-valued intuitionistic fuzzy closed ideals BG-algebras and their products, International Journal of Fuzzy Logic Systems, 2(2012), 27-44.

[29] T. SENAPATI, M. BHOWMIK AND M. PAL, Intuitionistic fuzzifications of ideals in BG-algebra, Mathematica Aeterna, 2(2012), 761-778.

[30] A. WALENDZIAK, Some axiomation of B-algebras, Mathematics Slovaca, 56(2006), 301-306.

[31] L. A. ZADEH, Fuzzy sets, Information and control, 8(1965) 338-353.

[32] L. A. ZADEH, The concept of a linguistic variable and its application to approximate reasoning-I, Information science, 8(1975), 199-249.

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Static analysis in neutrosophic cognitive maps

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Abstract.
Static analysis is developed in neutrosophic cognitive maps to define the importance of each node based on centrality measures. In this paper a framework static analysis of neutrosophic cognitive maps is presented. The analysis results are given in the form of neutrosophic numbers. Variables are classified and a de-neutrosophication process gives an interval number for centrality. Finally the nodes are ordered. An illustrative example based on critical success factor of customer relationship management (CRM) systems implementation is provided to show the applicability of the proposal. The paper ends with conclusion and future research directions.

Keywords: mental model, neutrosophic Logic, neutrosophic cognitive maps, static analysis

1 Introduction

Neutrosophic Cognitive Maps (NCM) [1] was introduced as a generalization of Fuzzy Cognitive Maps (FCM) [2]. A special feature of NCMs is their ability to handle indeterminacy in relations between two concepts, which is denoted by 'I'. NCM are capable of giving results with greater sensitivity than the FCM . It also allows a larger liberty for expert to express not just the positive, negative and absence of relations but also the indeterminacy of causal relations.

Static analysis is develop to define the importance of each node based on centrality measures [3].In this paper, we propose the use of an innovative technique for static analysis in neutrosophic cognitive maps.

The outline of this paper is as follows: Section 2 is dedicated to neutrosophic cognitve maps and static anlysis. The proposed framework is presented in Section 3. An illustrative example is discussed in Section 4. The paper closes with concluding remarks, and discussion of future work in Section 5.

2 Neutrosophic cognitive maps

Neutrosophic logic is a generalization of fuzzy logic based on neutrosophy [4]. A neutrosophic matrix is a matrix where the elements $a = (a_{ij})$ have been replaced by elements in $\langle R \cup I \rangle$, where $\langle R \cup I \rangle$ is the neutrosophic integer ring [5]. A neutrosophic graph is a graph in which at least one edge or one vertex is neutrosophic [6]. If indeterminacy is introduced in cognitive mapping it is called Neutrosophic Cognitive Map (NCM) [7].

NCM are based on neutrosophic logic to represent uncertainty and indeterminacy in cognitive maps [4]. A NCM is a directed graph in which at least one edge is an indeterminacy denoted by dotted lines [8].

In [9] a static analysis of mental model in the form of NCM is presented. The result of the static analysis result is in the form of neutrosophic numbers (a+bI, where I = indeterminacy) [10]. Finally, a de-neutrosophication process as proposed by Salmeron and Smarandache [11] is applied to give the final ranking value In this paper this model is extended and detailed to deal with nodes classification.

3 Proposed Framework

The following steps will be used to establish a framework static analysis in NCM (Fig. 1).

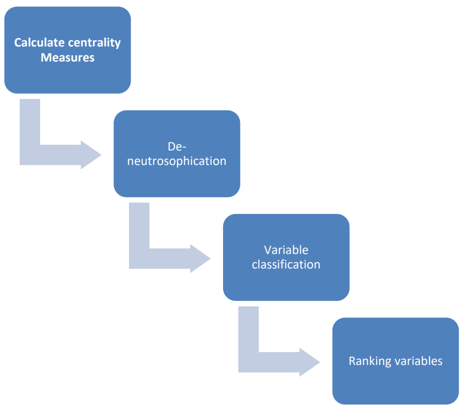

Figura 1Proposed framework

- **Calculate centrality Measures**

  The following measures are calculated [12] with absolute values of the NCM adjacency matrix [13]:

  1. Outdegree $od(v_i)$ is the row sum of absolute values of a variable in the neutrosophic adjacency matrix. It shows the cumulative strengths of connections (a_{ij}) exiting the variable.

  2. Indegree $id(v_i)$ is the column sum of absolute values of a variable. It shows the cumulative strength of variables entering the variable.

  3. The centrality (total degree $td(v_i)$), of a variable is the summation of its indegree (in-arrows) and outdegree (out-arrows)

$$td(v_i)= od(v_i)+ id(v_i) \quad (1)$$

- Variable classification

  Variables are classified according to the following rules:

  a) Transmitter variables have a positive or indeterminacy outdegree, $od(v_i)$ and zero indegree, $id(v_i)$.

  b) Receiver variables have a positive indegree or indeterminacy, $id(v_i)$, and zero outdegree, $od(v_i)$.

  c) Ordinary variables have both a non-zero indegree and outdegree. Ordinary variables can be more or less receiver or transmitter variables, based on the ratio of their indegrees and outdegrees.

- Ranking variables

  A de-neutrosophication process gives an interval number for centrality. Finally the nodes are ordered.

  The contribution of a variable in a cognitive map can be understood by calculating its degree centrality, which shows how connected the variable is to other variables and what the cumulative strength of these connections are. The median of the extreme values [14] is used :

$$\lambda([a_1, a_2]) = \frac{a_1 + a_2}{2} \quad (2)$$

Then

$$A > B \Leftrightarrow \frac{a_1 + a_2}{2} > \frac{b_1 + b_2}{2} \quad (3)$$

Finally a ranking of variables is given.

4 Illustrative example

In this section, we present an illustrative example in order to show the applicability of the proposed framework. We selected a critical sucess factor(CSF) of custumer relationship managemente (CRM)sytems implementation [15] for modeling interdependencies in the form of NCM [16]. Building a NCM allows dealing with indeterminacy, making easy the elicitation of interdependencies CSF [17].

Node	Description
A	Market orientation
B	Flexibility
C	Managers support
D	Organizational changes inclusion
F	Users' commitment and presence.
G	Time

TABLE I. NCM NODES

The NCM is developed integrating knowledge. The NCM with weighs is represented in tale II.

0	0	0.4	0	0	0
I	0	0	0	0	-0.7
0	0	0	0	I	-0.5
0	I	0	0	0	0
0	0	0	0	0	-0.7
0	0	0.6	0	0	0

TABLE II. ADJACENCY MATRIX

The centralities measures are presented.

A	0.4
B	0.7+I
C	0.5+I
D	I
E	0.7
F	0.6

TABLE III. OUTDEGREE

A	I
B	I
C	1
D	0
E	I
F	1.4

TABLE III. INDEGREE

A	0.4+I
B	0.7+2I
C	1.5+I
D	I
E	0.7+I
F	2.0

TABLE III. TOTAL DEGREE

Later nodes are clasified. In this case node D: "Organizational changes inclusion" is Transmitter, the rest of the nodes are Ordinary.
The next step is the de-neutrosophication process as proposes by Salmeron and Smarandache [11]. $I \in [0,1]$ is repalaced by both maximum and minimum values.

A	[0.4, 1.4]
B	[0.7, 2.7]
C	[1.5, 2.5]
D	[0, 1]
E	[0.7, 1.7]
F	2.0

TABLE III. DE-NEUTRIFICATION

Finally we work with the median of the extreme values [14] (3) .

A	0,9
B	1,7
C	2.0
D	0.5
E	1.2
F	2.0

TABLE III. MEDIAN OF THE EXTREME VALUES

The ranking is as follows:

$$C \sim F \succ B \succ E \succ A \succ D$$

"Managers support" and "Users commitment and presence "are the more important factors in his model.

5 Conclusions

In this paper, we propose a new framework for processing uncertainty and indeterminacy in static analysis of NCM. A case study was presented showing the applicability of the proposal. The analysis results are given in the form of neutrosophic numbers. Variables are classified and a de-neutrosophication process gives an interval number for centrality allowing the ranking of the variables.
Future research will focus on conducting further real life experiments and the development of a tool to automate the process. The calculation of other metrics is another area of future research.

References

1. Kandasamy, W.V. and F. Smarandache, *Fuzzy cognitive maps and neutrosophic cognitive maps*. 2003: Infinite Study.
2. Kosko, B., *Fuzzy cognitive maps.* International journal of man-machine studies, 1986. **24**(1): p. 65-75.
3. Samarasinghea, S. and G. Strickert, *A New Method for Identifying the Central Nodes in Fuzzy Cognitive Maps using Consensus Centrality Measure*, in *19th International Congress on Modelling and Simulation*. 2011: Perth, Australia.
4. Smarandache, F., *A unifying field in logics: neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability and statistics.* 2005: American Research Press.
5. Kandasamy, W.V. and F. Smarandache, *Fuzzy Neutrosophic Models for Social Scientists*. 2013: Education Publisher Inc.
6. Kandasamy, W.B.V. and F. Smarandache, *Fuzzy cognitive maps and neutrosophic cognitive maps*. 2003: American Research Press.
7. Kandasamy, W.V. and F. Smarandache, *Analysis of social aspects of migrant labourers living with HIV/AIDS using Fuzzy Theory and Neutrosophic Cognitive Maps*. 2004: American Research Press.

8. Salmeron, J.L. and F. Smarandache, *Processing Uncertainty and Indeterminacy in Information Systems projects success mapping*, in *Computational Modeling in Applied Problems: collected papers on econometrics, operations research, game theory and simulation*. 2006, Hexis. p. 94.
9. Pérez-Teruel, K. and M. Leyva-Vázquez, *Neutrosophic logic for mental model elicitation and analysis.* Neutrosophic Sets and Systems, 2012: p. 31-3.
10. Smarandache, F., *Refined literal indeterminacy and the multiplication law of sub-indeterminacies.* Neutrosophic Sets and Systems, 2015. **9**: p. 58-63.
11. Salmerona, J.L. and F. Smarandacheb, *Redesigning Decision Matrix Method with an indeterminacy-based inference process.* Multispace and Multistructure. Neutrosophic Transdisciplinarity (100 Collected Papers of Sciences), 2010. **4**: p. 151.
12. Lara, R.B., S.G. Espinosa, and M.Y.L. Vázquez, *Análisis estático en mapas cognitivos difusos basado en una medida de centralidad compuesta.* Ciencias de la Información, 2014. **45**(3): p. 31-36.
13. Stach, W., L. Kurgan, and W. Pedrycz, *Expert-based and computational methods for developing fuzzy cognitive maps*, in *Fuzzy Cognitive Maps*. 2010, Springer. p. 23-41.
14. Merigó, J., *New extensions to the OWA operators and its application in decision making*, in *Department of Business Administration, University of Barcelona*. 2008.
15. Wilson, H., E. Daniel, and M. McDonald, *Factors for success in customer relationship management (CRM) systems.* Journal of marketing management, 2002. **18**(1-2): p. 193-219.
16. Salmeron, J.L. and F. Smarandache, *Processing Uncertainty and Indeterminacy in Information Systems success mapping.* arXiv preprint cs/0512047, 2005.
17. Betancourt-Vázquez, A., K. Pérez-Teruel, and M. Leyva-Vázquez, *Modeling and analyzing non-functional requirements interdependencies with neutrosofic logic.* Neutrosophic Sets and Systems, 2015: p. 44.

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$(\mathcal{I},\mathcal{T})$ –Standard neutrosophic rough set and its topologies properties

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Abstract. In this paper, we defined $(\mathcal{I},\mathcal{T})-$ standard neutrosophic rough sets based on an implicator $\mathcal{I}$ and a t-norm $\mathcal{T}$ on D^*; lower and upper approximations of standard neutrosophic sets in a standard neutrosophic approximation are defined. Some properties of $(\mathcal{I},\mathcal{T})-$ standard neutrosophic rough sets are investigated. We consider the case when the neutrosophic components (truth, indeterminacy, and falsehood) are totally dependent, single-valued, and hence their sum is ≤ 1.

Keywords: standard neutrosophic, $(\mathcal{I},\mathcal{T})-$ standard neutrosophic rough sets

1. Introduction

Rough set theory was introduced by Z. Pawlak in 1980s [1]. It becomes a useful mathematical tool for data mining, especially for redundant and uncertain data. At first, the establishment of the rough set theory is based on equivalence relation. The set of equivalence classes of the universal set, obtained by an equivalence relation, is the basis for the construction of upper and lower approximation of the subset of the universal set.

Fuzzy set theory was introduced by L.Zadeh since 1965 [2]. Immediately, it became a useful method to study the problems of imprecision and uncertainty. Since, a lot of new theories treating imprecision and uncertainty have been introduced. For instance, Intuitionistic fuzzy sets were introduced in1986, by K. Atanassov [3], which is a generalization of the notion of a fuzzy set. When fuzzy set give the degree of membership of an element in a given set, Intuitionistic fuzzy set give a degree of membership and a degree of non-membership of an element in a given set. In 1998 [22], F. Smarandache gave the concept of neutrosophic set which generalized fuzzy set and intuitionistic fuzzy set. This new concept is difficult to apply in the real appliction. It is a set in which each proposition is estimated to have a degree of truth (T), adegree of indeterminacy (I) and a degree of falsity (F). Over time, the subclass of neutrosophic sets was proposed. They are also more advantageous in the practical application. Wang et al. [11] proposed interval neutrosophic sets and some operators of them. Smarandache [22] and Wang et al. [12] proposed a single valued neutrosophic set as an instance of the neutrosophic set accompanied with various set theoretic operators and properties. Ye [13] defined the concept of simplified neutrosophic sets, it is a set where each element of the universe has a degree of truth, indeterminacy, and falsity respectively and which lie between [0, 1] and some operational laws for simplified neutrosophic sets and to propose two aggregation operators, including a simplified neutrosophic weighted arithmetic average operator and a simplified neutrosophic weighted geometric average operator. In 2013, B.C. Cuong and V. Kreinovich introduced the concept of picture fuzzy set [4,5], and picture fuzzy set is regarded the standard neutrosophic set [6].

More recently, rough set have been developed into the fuzzy environment and obtained many interesting results. The approximation of rough (or fuzzy) sets in fuzzy approximation space gives us the fuzzy rough set [7,8,9]; and the approximation of fuzzy sets in crisp approximation space gives us the rough fuzzy set [8, 9]. In 2014, X.T. Nguyen introduces the rough picture fuzzy set as the result of approximation of a picture fuzzy set with respect to a crisp approximation space [18]. Radzikowska and Kerre defined $(\mathcal{I},\mathcal{T})-$ fuzzy rough sets [19], which determined by an implicator $\mathcal{I}$ and a t-norm $\mathcal{T}$ on [0,1]. In 2008, L. Zhou et al. [20] constructed $(\mathcal{I},\mathcal{T})-$ intuitionistic fuzzy rough sets determined by an implicator $\mathcal{I}$ and a t-norm $\mathcal{T}$ on L^*.

In this paper, we considered the case when the neutrosophic components are single valued numbers in [0, 1] and they are totally dependent [17], which means that their sum is ≤ 1. We defined $(\mathcal{I},\mathcal{T})-$ standard neutrosophic rough sets based on an implicator $\mathcal{I}$ and a t-norm $\mathcal{T}$ on D^*; in which, implicator $\mathcal{I}$ and a t-norm $\mathcal{T}$ on D^* is investigated in [21].

2. Standard neutrosophic logic

We consider the set D^* defined by the following definition.

Definition 1. We denote:

$$D^* = \{x = (x_1, x_2, x_3) | x_1 + x_2 + x_3 \leq 1, x_i \in [0,1], i = 1,2,3\}$$

For $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in D^*$, we define:

$x \leq_{D^*} y$ iff $((x_1 < y_1) \wedge (x_3 \geq y_3)) \vee ((x_1 = y_1) \wedge (x_3 > y_3)) \vee ((x_1 = y_1) \wedge (x_3 = y_3) \wedge (x_2 \leq y_2))$, and $x = y \Leftrightarrow (x \leq_{D^*} y) \wedge (y \leq_{D^*} x)$.

Then $(D^*, \leq_{D^*})$ is a lattice, in which $0_{D^*} = (0,0,1) \leq x \leq 1_{D^*} = (1,0,0), \forall x = (x_1, x_2, x_3) \in D^*$. The meet operator $\wedge$ and the join operator $\vee$ on $(D^*, \leq_{D^*})$ are defined as follows:

For $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in D^*$,

$$x \wedge y = (\min(x_1, y_1), \min(x_2, y_2), \max(x_3, y_3)),$$
$$x \vee y = (\max(x_1, y_1), \min(x_2, y_2), \min(x_3, y_3)).$$

On D^*, we consider logic operators as negation, t-norm, t-conorm, implication.

2.1. Standard neutrosophic negation

Definition 2. A standard neutrosophic negation is any nonincreasing $D^* \to D^*$ mapping n satisfying $n(0_{D^*}) = 1_{D^*}$ và $n(1_{D^*}) = 0_{D^*}$.

Example 1. For all $x = (x_1, x_2, x_3) \in D^*$, we have some standard neutrosophic negations on D^* as follows:

+ $n_0(x) = (x_3, 0, x_1)$
+ $n_1(x) = (x_3, x_4, x_2)$ where $x_4 = 1 - x_1 - x_2 - x_3$.

2.2. Standard neutrosophic t-norm

For $x = (x_1, x_2, x_3) \in D^*$, we denote

$$\Gamma(x) = \{y \in D^* : y = (x_1, y_2, x_3), 0 \leq y_2 \leq x_2\}$$

Obviously, we have $\Gamma(0_{D^*}) = 0_{D^*}$, $\Gamma(1_{D^*}) = 1_{D^*}$.

Definition 3. A standard neutrosophic t-norm is an $(D^*)^2 \to D^*$ mapping $\boldsymbol{\mathcal{T}}$ satisfying the following conditions

(T1) $\boldsymbol{\mathcal{T}}(x, y) = \boldsymbol{\mathcal{T}}(y, x), \forall x, y \in D^*$

(T2) $\boldsymbol{\mathcal{T}}(x, \boldsymbol{\mathcal{T}}(y, z)) = \boldsymbol{\mathcal{T}}(\boldsymbol{\mathcal{T}}(x, y), z)), \forall x, y, z \in D^*$

(T3) $\boldsymbol{\mathcal{T}}(x, y) \leq \boldsymbol{\mathcal{T}}(x, z), \forall x, y, z \in D^*$ and $y \leq_{D^*} z$

(T4) $\boldsymbol{\mathcal{T}}(1_{D^*}, x) \in \Gamma(x)$.

Example 2. Some standard neutrosophic t-norm, for all $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in D^*$

+ t-norm min: $\boldsymbol{\mathcal{T}}_M(x, y) = (x_1 \wedge y_1, x_2 \wedge y_2, x_3 \vee y_3)$
+ t-norm product: $\boldsymbol{\mathcal{T}}_P(x, y) = (x_1 y_1, x_2 y_2, x_3 + y_3 - x_3 y_3)$
+ t-norm Lukasiewicz: $\boldsymbol{\mathcal{T}}_L(x, y) = (\max(0, x_1 + y_1 - 1), \max(0, x_2 + y_2 - 1), \min(1, x_3 + y_3))$.

Remark 1.

+ $\boldsymbol{\mathcal{T}}(0_{D^*}, x) = 0_{D^*}$ for all $x \in D^*$. Indeed, for all $x \in D^*$ we have $\boldsymbol{\mathcal{T}}(0_{D^*}, x) \leq \boldsymbol{\mathcal{T}}(0_{D^*,} 1_{D^*}) = 0_{D^*}$
+ $\boldsymbol{\mathcal{T}}(1_{D^*}, 1_{D^*}) = 1_{D^*}$ (obvious)

2.3. Standard neutrosophic t-conorm

Definition 4. A standard neutrosophic t-conorm is an $(D^*)^2 \to D^*$ mapping S satisfying the following conditions

(S1) $S(x, y) = S(y, x), \forall x, y \in D^*$

(S2) $S(x, S(y, z)) = S(S(x, y), z)), \forall x, y, z \in D^*$

(S3) $S(x, y) \leq S(x, z), \forall x, y, z \in D^*$ and $y \leq_{D^*} z$

(S4) $S(0_{D^*}, x) \in \Gamma(x)$

Example 3. Some standard neutrosophic t-norm, for all $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in D^*$

+ t-conorm max: $S_M(x, y) = (x_1 \vee y_1, x_2 \wedge y_2, x_3 \wedge y_3)$
+ t-conorm product: $S_P(x, y) = (x_1 + y_1 - x_1 y_1, x_2 y_2, x_3 y_3)$
+ t-conorm Luksiewicz: $S_L(x, y) = (\min(1, x_1 + y_1), \max(0, x_2 + y_2 - 1), \max(0, x_3 + y_3 - 1))$.

Remark 2.

+ $S(1_{D^*}, x) = 1_{D^*}$ for all $x \in D^*$. Indeed, for all $x \in D^*$ we have $S(0_{D^*}, 1_{D^*}) \in \Gamma(1_{D^*}) = 1_{D^*}$ so that $\leq S(0_{D^*}, 1_{D^*}) \leq S(0_{D^*,} x) \leq 1_{D^*}$.
+ $S(0_{D^*}, 0_{D^*}) = 0_{D^*}$ (obvious).

A standard neutrosophic t-norm $\boldsymbol{\mathcal{T}}$ and a standard neutrosophic t-conorm S on D^* are said to be dual with respect to (w.r.t) a standard neutrosophic negation n if

$$\boldsymbol{\mathcal{T}}(n(x), n(y)) = nS(x, y) \quad \forall x, y \in D^*,$$
$$S(n(x), n(y)) = n\boldsymbol{\mathcal{T}}(x, y) \quad \forall x, y \in D^*.$$

Example 4. With negation $n_0(x) = (x_3, 0, x_1)$ we have some t-norm and t-conorm dual as follows:

a. $\boldsymbol{\mathcal{T}}_M$ and S_M
b. $\boldsymbol{\mathcal{T}}_P$ and S_P
c. $\boldsymbol{\mathcal{T}}_L$ and S_L

Many properties of t-norms, t-conorms, negations should be given in [21].

2.4 Standard neutrosophic implication operators

In this section, we recall two classes of standard neutrosophic implication in [21].

A standard neutrosophic implication off class 1.

Definition 5. A mapping $\boldsymbol{\mathcal{I}}: (D^*)^2 \to D^*$ is referred to as a standard neutrosophic implicator off class 1 on D^* if it satisfying following conditions:

$$\boldsymbol{\mathcal{I}}(0_{D^*}, 0_{D^*}) = 1_{D^*}; \boldsymbol{\mathcal{I}}(0_{D^*}, 1_{D^*}) = 1_{D^*}; \boldsymbol{\mathcal{I}}(1_{D^*}, 1_{D^*}) = 1_{D^*};$$
$$I(1_{D^*}, 0_{D^*}) = 0_{D^*}$$

Proposition 1. Let $\boldsymbol{\mathcal{T}}, S$ and n be standard neutrosophic t-norm $\boldsymbol{\mathcal{T}}$, a standard neutrosophic t-conorm S and a standard neutrosophic negation on D^*, respectively. Then, we have a standard neutrosophic implication on D^*, which defined as following:

$$\boldsymbol{\mathcal{I}}_{S,\mathcal{T},n}(x, y) = S(\boldsymbol{\mathcal{T}}(x, y), n(x)), \forall x, y \in D^*.$$

Proof.

We consider border conditions in definition 5.

$\boldsymbol{\mathcal{I}}(0_{D^*}, 0_{D^*}) = S(\boldsymbol{\mathcal{T}}(0_{D^*}, 0_{D^*}), n(0_{D^*})) = S(0_{D^*}, 1_{D^*}) = 1_{D^*}$,

$\boldsymbol{\mathcal{I}}(0_{D^*}, 1_{D^*}) = S(\boldsymbol{\mathcal{T}}(0_{D^*}, 1_{D^*}), n(0_{D^*})) = S(0_{D^*}, 1_{D^*}) = 1_{D^*}$,

$\boldsymbol{\mathcal{I}}(1_{D^*}, 1_{D^*}) = S(\boldsymbol{\mathcal{T}}(1_{D^*}, 1_{D^*}), n(1_{D^*})) = S(1_{D^*}, 0_{D^*}) = 1_{D^*}$, and

$\mathcal{J}(1_{D^*}, 0_{D^*}) = S(\mathcal{T}(1_{D^*}, 0_{D^*}), n(1_{D^*})) = S(0_{D^*}, 0_{D^*}) = 0_{D^*}$.

We have the proof.□

Example 5. For all $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in D$, we have some standard neutrosophic implication of class 1 on D^* based on proposition 1 as follows

a. If $\mathcal{T} = \mathcal{T}_M, S = S_M$ and $n_0(x) = (x_3, 0, x_1)$ then $\mathcal{J}_{S_M,\mathcal{T}_M,n_0}(x, y) = (\max(\min(x_1, y_1), x_3), 0, \min(\max(x_3, y_3), x_1)$.

b. If $\mathcal{T} = \mathcal{T}_P, S = S_P$ and $n_1(x) = (x_3, x_4, x_1)$ then $\mathcal{J}_{S_P,\mathcal{T}_P,n_1}(x, y) = (x_1y_1+x_3 - x_1y_1x_3, x_2y_2x_4, x_1(x_3 + y_3 - x_3y_3))$.

A standard neutrosophic implication off cals 2.

Definition 6. A mapping $\mathcal{J}: (D^*)^2 \to D^*$ is referred to as a standard neutrosophic implicator off class 2 on D^* if it is decreasing in its first component, increasing in its second component and satisfying following conditions:

$$\mathcal{J}(0_{D^*}, 0_{D^*}) = 1_{D^*};\ \mathcal{J}(1_{D^*}, 1_{D^*}) = 1_{D^*};$$
$$\mathcal{J}(1_{D^*}, 0_{D^*}) = 0_{D^*}$$

Definition 7. A standard neutrosophic implicator $\mathcal{J}$ off class 2 is called boder standard neutrosophic implication if $\mathcal{J}(1_{D^*}, x) = x$ for all $x \in D^*$.

Proposition 2. Let $\mathcal{T}, S$ and n be standard neutrosophic t-norm $\mathcal{T}$, a standard neutrosophic t-conorm S and a standard neutrosophic negation on D^*, respectively. Then, we have a standard neutrosophic implication on D^*, which defined as following:

$$\mathcal{J}_{S,n}(x, y) = S(n(x), y), \forall x, y \in D^*.$$

Example 6. For all $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in D$, we have some standard neutrosophic implication of class 1 on D^* based on proposition ? as follows

a. If $S = S_M$ and $n_0(x) = (x_3, 0, x_1)$ then $\mathcal{J}_{S_M,n_0}(x, y) = (\max(x_3, y_1), 0, \min(x_1, y_3))$

b. If $S = S_P$ and $n_1(x) = (x_3, x_4, x_1)$ then $\mathcal{J}_{S_P,n_1}(x, y) = (x_3+y_1 - x_3y_1, x_4y_2, x_1y_3)$

Note that, we can define the negation operators from implication operators, such as, the mapping $n_{\mathcal{J}}(x) = \mathcal{J}(x, 0_{D^*}), \forall x \in D^*$, is a standard negation on D^*. For example, if $\mathcal{J}_{S_P,n_1}(x, y) = (x_3+y_1 - x_3y_1, x_4y_2, x_1y_3)$ then we obtain $n_{I_{S_P,n_1}}(x) = \mathcal{J}_{S_P,n_1}(x, 0_{D^*}) = (x_3, 0, x_1) = n_0(x)$.

2.5 Standard neutrosophic set

Definition 8. Let U be a universal set. A standard neutrosophic (PF) set A on the universe U is an object of the form $A = \{(x, \mu_A(x), \eta_A(x), \gamma_A(x)) \mid x \in U\}$ where $\mu_A(x)(\in [0,1])$ is called the "degree of positive membership of x in A", $\eta_A(x)(\in [0,1])$ is called the "degree of neutral membership of x in A" and $\gamma_A(x)(\in[0,1])\gamma_A(x)(\in [0,1])$ is called the "degree of negative membership of x in A", and where μ_A, η_A $\mu_A, \gamma_A$and γ_A $\eta_A$satisfy the following condition:

$$\mu_A(x) + \eta_A(x) + \gamma_A(x) \le 1, (\forall x \in X)\mu_A(x) + \gamma_A(x) + \eta_A(x)) \le 1, (\forall x \in X).$$

The family of all standard neutrosophic set in U is denoted by PFS(U).

3. Standard neutrosophic rough set

Definition 9.

Suppose that R is a standard neutrosophic relation on the set of universe U. $\mathcal{T}$ is a t −norm on D^*, $\mathcal{J}$ an implication on D^*, for all $F \in PFS(U)$, we denote $F(v) = (\mu_F(v), \eta_F(v), \gamma_F(v))$. Then (U, R) is a standard neutrosophic approximation space. We define the upper and lower approximation set of F on (U, R) as following

$$\bar{R}^{\mathcal{T}}(F)(u) = \bigvee_{v\in U} \mathcal{T}(R(u, v), F(v)), \forall u \in U$$

and

$$\underline{R}_{\mathcal{J}}(F)(u) = \bigwedge_{v\in U} \mathcal{J}(R(u, v), F(v)), u \in U.$$

Example 7. Let $U = \{a, b, c\}$ be an universe and R is a standard neutrosophic relation on U

$$R = \begin{pmatrix} (0.7,0.2,0.1) & (0.6,0.2,0.1) & (0.5,0.3,0.2) \\ (0.5,0.4,0.1) & (0.6,0.1,0.2) & (0.5,0.1,0.2) \\ (0.3,0.5,0.1) & (0.4,0.2,0.3) & (0.7,0.1,0.1) \end{pmatrix}$$

A standard neutrosophic on U is $F = \{\langle a, 0{,}6{,}0.2{,}0.2\rangle, \langle b, 0.5{,}0.3{,}0.1\rangle, \langle c, (0.7{,}0.2{,}0.1)\rangle\}$. Let $\mathcal{T}_M(x, y) = (x_1 \wedge y_1, x_2 \wedge y_2, x_3 \vee y_3)$ be a t-norm on D^*, and $\mathcal{J}(x, y) = (x_3 \vee y_1, x_2 \wedge y_2, x_1 \wedge y_3)$ be an implication on D^*, forall $x = (x_1, x_2, x_3) \in D^*$ and $y = (y_1, y_2, y_3) \in D^*$, We compute

$$\mathcal{T}(R(a, a), F(a)) = \mathcal{T}((0.7{,}0.2, 0.1), (0.6{,}0.2{,}0.2)) = (0.6{,}0.2{,}0.2)$$
$$\mathcal{T}(R(a, b), F(b)) = \mathcal{T}((0.6{,}0.2{,}0.1), (0.5{,}0.3{,}0.1)) = (0.5{,}0.2{,}0.1)$$
$$\mathcal{T}(R(a, c), F(c)) = \mathcal{T}((0.5{,}0.3{,}0.2), (0.7{,}0.2{,}0.1)) = (0.5{,}0.2{,}0.2)$$

Hence $\bar{R}^{T}(F)(a) = \bigvee_{v\in U} \mathcal{T}(R(a, v), F(v)) = (0.6{,}0.2{,}0.1)$.

And

$$\mathcal{T}(R(b, a), F(a)) = \mathcal{T}((0.5{,}0.4, 0.1), (0.6{,}0.2{,}0.2)) = (0.5{,}0.2{,}0.2)$$
$$\mathcal{T}(R(b, b), F(b)) = \mathcal{T}((0.6{,}0.1{,}0.2), (0.5{,}0.3{,}0.1)) = (0.5{,}0.1{,}0.3)$$
$$\mathcal{T}(R(b, c), F(c)) = \mathcal{T}((0.5{,}0.1{,}0.2), (0.7{,}0.2{,}0.1)) = (0.5{,}0.1{,}0.2)$$

Hence $\bar{R}^{\mathcal{T}}(F)(b) = \bigvee_{v\in U} \mathcal{T}(R(b, v), F(v)) = (0.5{,}0.1{,}0.2)$

$$\boldsymbol{\mathcal{T}}\big(R(c,a),F(a)\big)=\boldsymbol{\mathcal{T}}\big((0.3,0.5,0.1),(0.6,0.2,0.2)\big)=(0.3,0.2,0.2)$$
$$\boldsymbol{\mathcal{T}}\big(R(c,b),F(b)\big)=\boldsymbol{\mathcal{T}}\big((0.4,0.2,0.3),(0.5,0.3,0.1)\big)=(0.4,0.2,0.3)$$
$$\boldsymbol{\mathcal{T}}\big(R(c,c),F(c)\big)=\boldsymbol{\mathcal{T}}\big((0.7,0.1,0.1),(0.7,0.2,0.1)\big)=(0.7,0.1,0.1)$$

So that $\bar{R}^{\mathcal{T}}(F)(c)=\bigvee_{v\in U}\boldsymbol{\mathcal{T}}\big(R(c,v),F(v)\big)=(0.7,0.1,0.1)$.

We obtain the upper approximation $\bar{R}^{T}(F)=\frac{(0.6,0.2,0.1)}{a}+\frac{(0.5,0.1,0.2)}{b}+\frac{(0.7,0.1,0.1)}{c}$.

Similarly, computing with the lower approximation set, we have $\boldsymbol{\mathcal{I}}\big((0.7,0.2,0.1),(0.6,0.2,0.2)\big)=(0.1,0.2,0.7)\vee(0.6,0.2,0.2)=(0.6,0.2,0.2)$

$$\boldsymbol{\mathcal{I}}\big(R(a,b),F(b)\big)=\boldsymbol{\mathcal{I}}\big((0.6,0.2,0.1),(0.5,0.3,0.1)\big)=(0.1,0.2,0.6)\vee(0.5,0.3,0.1)=(0.5,0.2,0.1)$$
$$\boldsymbol{\mathcal{I}}\big(R(a,c),F(c)\big)=\boldsymbol{\mathcal{I}}\big((0.5,0.3,0.2),(0.7,0.2,0.1)\big)=(0.2,0.3,0.5)\vee(0.7,0.2,0.1)=(0.7,0.2,0.1)$$

$\underline{R}_{\mathcal{I}}(F)(a)=\bigwedge_{v\in U}\boldsymbol{\mathcal{I}}\big(R(a,v),F(v)\big)=(0.5,0.2,0.2)$.

And

$$\boldsymbol{\mathcal{I}}\big(R(b,a),F(a)\big)=\boldsymbol{\mathcal{I}}\big((0.5,0.4,0.1),(0.6,0.2,0.2)\big)=(0.6,0.2,0.1)$$
$$\boldsymbol{\mathcal{I}}\big(R(b,b),F(b)\big)=\boldsymbol{\mathcal{I}}\big((0.6,0.1,0.2),(0.5,0.3,0.1)\big)=(0.5,0.1,0.1)$$
$$\boldsymbol{\mathcal{I}}\big(R(b,c),F(c)\big)=\boldsymbol{\mathcal{I}}\big((0.5,0.1,0.2),(0.7,0.2,0.1)\big)=(0.7,0.1,0.1)$$

$\underline{R}_{\mathcal{I}}(F)(b)=\bigwedge_{v\in U}\boldsymbol{\mathcal{I}}\big(T(b,v),F(v)\big)=(0.5,0.1,0.1)$.

$$\boldsymbol{\mathcal{I}}\big(R(c,a),F(a)\big)=\boldsymbol{\mathcal{I}}\big((0.3,0.5,0.1),(0.6,0.2,0.2)\big)=(0.6,0.2,0.1)$$
$$\boldsymbol{\mathcal{I}}\big(R(c,b),F(b)\big)=\boldsymbol{\mathcal{I}}\big((0.4,0.2,0.3),(0.5,0.3,0.1)\big)=(0.5,0.2,0.1)$$
$$\boldsymbol{\mathcal{I}}\big(R(c,c),F(c)\big)=\boldsymbol{\mathcal{I}}\big((0.7,0.1,0.1),(0.7,0.2,0.1)\big)=(0.7,0.1,0.1)$$

Hence $\underline{R}_{\mathcal{I}}(F)(c)=\bigwedge_{v\in U}\boldsymbol{\mathcal{I}}\big(R(c,v),F(v)\big)=(0.5,0.1,0.1)$.

So that

$\underline{R}_{\mathcal{I}}(F)=\frac{(0.5,0.2,0.2)}{a}+\frac{(0.5,0.1,0.1)}{b}+\frac{(0.5,0.1,0.1)}{c}$.

Now, we have the upper and lower approximations of $F=\frac{(0,6,0.2,0.2)}{a}+\frac{(0.5,0.3,0.1)}{b}+\frac{(0.7,0.2,0.1)}{c}$ are

$$\bar{R}^{\mathcal{T}}(F)=\frac{(0,6,0.2,0.1)}{a}+\frac{(0.5,0.1,0.2)}{b}+\frac{(0.7,0.1,0.1)}{c}$$

and

$$\underline{R}_{\mathcal{I}}(F)=\frac{(0.5,0.2,0.2)}{a}+\frac{(0.5,0.1,0.1)}{b}+\frac{(0.5,0.1,0.1)}{c}$$

Example 8. Let $U=\{a,b,c\}$ be an universe set. And R is a standard neutrosophic relation on U with

$$R=\begin{pmatrix}(1,0,0) & (0.6,0.3,0) & (0.6,0.3,0)\\(0.6,0.3,0) & (1,0,0) & (0.6,0.3,0)\\(0.6,0.3,0) & (0.6,0.3,0) & (1,0,0)\end{pmatrix}$$

Let $F=\frac{(0.4,0.3,0.3)}{a}+\frac{(0.5,0.2,0.3)}{b}+\frac{(0.4,0.4,0.1)}{c}$ be standard neutrosophic set on U. A $t-$ norm $\boldsymbol{\mathcal{T}}(x,y)=(x_1\wedge y_1,x_2\wedge y_2,x_3\vee y_3)$, and an implication operator $\boldsymbol{\mathcal{I}}(x,y)=(x_3\vee y_1,x_2\wedge y_2,x_1\wedge y_3)$ for all $x=(x_1,x_2,x_3)\in D^*$, $y=(y_1,y_2,y_3)\in D^*$, we put

$$\boldsymbol{\mathcal{T}}\big(R(a,a),F(a)\big)=\boldsymbol{\mathcal{T}}\big((1,0,0),(0.7,0.2,0.1)\big)=(0.7,0,0.1)$$
$$\boldsymbol{\mathcal{T}}\big(R(a,b),F(b)\big)=\boldsymbol{\mathcal{T}}\big((0.6,0.3,0),(0.5,0.2,0.3)\big)=(0.5,0.2,0.3)$$
$$\boldsymbol{\mathcal{T}}\big(R(a,c),F(c)\big)=\boldsymbol{\mathcal{T}}\big((0.6,0.3,0),(0.4,0.4,0.1)\big)=(0.4,0.3,0.1)$$

Then $\bar{R}^{\mathcal{T}}(F)(a)=\bigvee_{v\in U}\boldsymbol{\mathcal{T}}\big(R(a,v),F(v)\big)=(0.7,0,0.1)$.

$$\boldsymbol{\mathcal{T}}\big(R(b,a),F(a)\big)=\boldsymbol{\mathcal{T}}\big((0.6,0.3,0),(0.7,0.2,0.1)\big)=(0.6,0.2,0.1)$$
$$\boldsymbol{\mathcal{T}}\big(R(b,b),F(b)\big)=\boldsymbol{\mathcal{T}}\big((1,0,0),(0.5,0.2,0.3)\big)=(0.5,0,0.3)$$
$$\boldsymbol{\mathcal{T}}\big(R(b,c),F(c)\big)=\boldsymbol{\mathcal{T}}\big((0.6,0.3,0),(0.4,0.4,0.1)\big)=(0.4,0.3,0.1)$$

Hence $\bar{R}^{\mathcal{T}}(F)(b)=\bigvee_{v\in U}\boldsymbol{\mathcal{T}}\big(R(b,v),F(v)\big)=(0.6,0,0.1)$.

$$\boldsymbol{\mathcal{T}}\big(R(c,a),F(a)\big)=\boldsymbol{\mathcal{T}}\big((0.6,0.3,0),(0.7,0.2,0.1)\big)=(0.6,0.2,0.1)$$
$$\boldsymbol{\mathcal{T}}\big(R(c,b),F(b)\big)=\boldsymbol{\mathcal{T}}\big((0.6,0.3,0),(0.5,0.2,0.3)\big)=(0.5,0.2,0.3)$$
$$\boldsymbol{\mathcal{T}}\big(R(c,c),F(c)\big)=\boldsymbol{\mathcal{T}}\big((1,0,0),(0.4,0.4,0.1)\big)=(0.4,0,0.1)$$
$$\bar{R}^{\mathcal{T}}(F)(a)=\bigvee_{v\in U}\boldsymbol{\mathcal{T}}\big(R(a,v),F(v)\big)=(0.6,0,0.1).$$

We obtain the upper approximation set $\bar{R}^{\mathcal{T}}(F)=\frac{(0.7,0,0.1)}{a}+\frac{(0.6,0,0.1)}{b}+\frac{(0.6,0,0.1)}{c}$.

Similarly, computing with the lower approximation, we have

$$\boldsymbol{\mathcal{I}}\big(R(a,a),F(a)\big)=\boldsymbol{\mathcal{I}}\big((1,0,0),(0.7,0.2,0.1)\big)=(0,0,1)\vee(0.7,0.2,0.1)=(0.7,0,0.1)$$
$$\boldsymbol{\mathcal{I}}\big(R(a,b),F(b)\big)=\boldsymbol{\mathcal{I}}\big((0.6,0.3,0),(0.5,0.2,0.3)\big)=(0,0.3,0.6)\vee(0.5,0.2,0.3)=(0.5,0.2,0.3)$$
$$\boldsymbol{\mathcal{I}}\big(R(a,c),F(c)\big)=\boldsymbol{\mathcal{I}}\big((0.6,0.3,0),(0.4,0.4,0.1)\big)=(0,0.3,0.6)\vee(0.4,0.4,0.1)=(0.4,0.3,0.1)$$

$\underline{R}_{\mathcal{I}}(F)(a)=\bigwedge_{v\in U}\boldsymbol{\mathcal{I}}\big(T(a,v),F(v)\big)=(0.4,0,0.3)$.

Compute

$$\boldsymbol{\mathcal{I}}\big(R(b,a),F(a)\big)=\boldsymbol{\mathcal{I}}\big((0.6,0.3,0),(0.7,0.2,0.1)\big)=(0,0.3,0.6)\vee(0.7,0.2,0.1)=(0.7,0.2,0.1)$$
$$\boldsymbol{\mathcal{I}}\big(R(b,b),F(b)\big)=\boldsymbol{\mathcal{I}}\big((1,0,0),(0.5,0.2,0.3)\big)=(0,0,1)\vee(0.5,0.2,0.3)=(0.5,0,0.3)$$

$\mathcal{I}(R(b,c),F(c)) = \mathcal{I}((0.6,0.3,0),(0.4,0.4,0.1))$
$= (0,0.3,0.6) \vee (0.4,0.4,0.1)$
$= (0.4,0.3,0.1)$
$\underline{R}_{\mathcal{I}}(F)(b) = \bigwedge_{v\in U} \mathcal{I}(T(b,v),F(v)) = (0.4,0,0.3)$.
and
$\mathcal{I}(R(c,a),F(a)) = \mathcal{I}((0.6,0.3,0),(0.7,0.2,0.1))$
$= (0,0.3,0.6) \vee (0.7,0.2,0.1)$
$= (0.7,0.2,0.1)$
$\mathcal{I}(R(c,b),F(b)) = \mathcal{I}((0.6,0.3,0),(0.5,0.2,0.3))$
$= (0,0.3,0.6) \vee (0.5,0.2,0.3)$
$= (0.5,0.2,0.3)$
$\mathcal{I}(R(c,c),F(c)) = \mathcal{I}((1,0,0),(0.4,0.4,0.1))$
$= (0,0,1) \vee (0.4,0.4,0.1) = (0.4,0,0.1)$
$\underline{R}_{\mathcal{I}}(F)(c) = \bigwedge_{v\in U} \mathcal{I}(T(c,v),F(v)) = (0.4,0,0.3)$.
Hence
$$\underline{R}_{\mathcal{I}}(F) = \frac{(0.4,0,0.1)}{a} + \frac{(0.4,0,0.3)}{b} + \frac{(0.4,0,0.3)}{c}$$
Now, we have the upper and lower approximation sets of $F = \frac{(0.4,0.3,0.3)}{a} + \frac{(0.5,0.2,0.3)}{b} + \frac{(0.4,0.4,0.1)}{c}$ as following
$$\bar{R}^{\mathcal{T}}(F) = \frac{(0.7,0,0.1)}{a} + \frac{(0.6,0,0.1)}{b} + \frac{(0.6,0,0.1)}{c}$$
and
$$\underline{R}_{\mathcal{I}}(F) = \frac{(0.4,0,0.3)}{a} + \frac{(0.4,0,0.3)}{b} + \frac{(0.4,0,0.3)}{c}.$$

Remark 3. If R is reflexive, symmetric transitive then $\underline{R}_{\mathcal{I}}(F) \subset F \subset \bar{R}^{\mathcal{T}}(F)$.

4. Some properties of standard neutrosophic rough set

Theorem 1. Let (U,R) be the standard neutrosophic approximation space. Let $\mathcal{T}, S$ be the t-norm , and t –conorm D^*, n is a negative on D^*. If S and T are dual w.r.t n then
(i) $\sim_n \underline{R}_{\mathcal{I}}(A) = \bar{R}^{\mathcal{T}}(\sim_n A)$
(ii) $\sim_n \bar{R}^{\mathcal{T}}(A) = \underline{R}_{\mathcal{I}}(\sim A)$
where $\mathcal{I}(x,y) = S(n(x),y), \forall x,y \in D^*$.

Proof.

(i) $\sim_n \bar{R}^{\mathcal{T}}(\sim_n A) = \underline{R}_{\mathcal{I}}(A)$.

Indeed, for all $x \in U$, we have
$$\bar{R}^{\mathcal{T}}(\sim_n A)(x) = \bigvee_{y\in U} \mathcal{T}[R(x,y),\sim_n A(y)]$$
$$= \bigvee_{y\in U} nS[nR(x,y),n(\sim_n A(y))]$$
$$= \bigvee_{y\in U} nS[nR(x,y),A(y)] .$$
Moreover,
$$\underline{R}_{\mathcal{I}}(A)(x) = \bigwedge_{y\in U} \mathcal{I}(R(x,y),A(y))$$
$$= \bigwedge_{y\in U} S[nR(x,y),A(y)]$$
Hence
$$\sim_n \underline{R}_{\mathcal{I}}(A)(x)(x) = n(\bigwedge_{y\in U} S[nR(x,y),A(y)])$$
$$= = \bigvee_{y\in U} nS[nR(x,y),A(y)]$$
and $\bar{R}^{T}(\sim_n A)(x) =\sim_n \underline{R}_{\mathcal{I}}(A)(x), \forall x \in U.$

(ii) $\underline{R}_{\mathcal{I}}(\sim_n A) =\sim_n \bar{R}^{\mathcal{T}}(A)$

Indeed, for all $x \in U$ we have
$\underline{R}_{\mathcal{I}}(\sim_n A)(x) = \bigwedge_{y\in U} \mathcal{I}(R(x,y),\sim_n A(y)), x \in U = \bigwedge_{y\in U} S[nR(x,y),\sim_n A(y)]$
And $\sim_n \bar{R}^{T}(A)(x) = n(\bigvee_{y\in U} \mathcal{T}[R(x,y),A(y))]) = \bigvee_{y\in U} n\mathcal{T}[R(x,y),A(y)]$
$$= \bigwedge_{y\in U} S[nR(x,y),\sim_n A(y)]$$
It means that $\underline{R}_{\mathcal{I}}(\sim_n A)(x) =\sim_n \bar{R}^{\mathcal{T}}(A)(x), \forall x \in U.$ □

Theorem 2. a) $\bar{R}^{\mathcal{T}}((\widehat{\alpha,\beta,\theta})) \subset (\widehat{\alpha,\beta,\theta})$, where $(\widehat{\alpha,\beta,\theta})x = (\alpha,\beta,\theta)$, $\forall x \in U$

b) $\underline{R}_{\mathcal{I}}((\widehat{\alpha,\beta,\theta})) \supset (\widehat{\alpha,\beta,\theta})$, where I is a border implication in class 2.

Proof.

a) We have
$\bar{R}^{\mathcal{T}}((\widehat{\alpha,\beta,\theta}))(u) =$
$\bigvee_{v\in U} \mathcal{T}\left(R(u,v),(\widehat{\alpha,\beta,\theta})(v)\right) =$
$\mathcal{T}\left(\bigvee_{v\in U} R(u,v),(\alpha,\beta,\theta)\right) \leq_{D^*} \mathcal{T}(1_{D^*},(\alpha,\beta,\theta))$
$= (\alpha,\beta,\theta) = (\widehat{\alpha,\beta,\theta})(u), \ \forall u \in U$

b) We have
$\underline{R}_{\mathcal{I}}((\widehat{\alpha,\beta,\theta}))(u) =$
$\bigwedge_{v\in U} \mathcal{I}\begin{pmatrix} R(u,v), \\ (\widehat{\alpha,\beta,\theta})(v)\end{pmatrix} = \bigwedge_{v\in U} \mathcal{I}\begin{pmatrix} R(u,v), \\ (\alpha,\beta,\theta)\end{pmatrix} \geq_{D^*} \bigwedge_{v\in U} \mathcal{I}(1_{D^*},(\alpha,\beta,\theta)) =$
$(\alpha,\beta,\theta) = (\widehat{\alpha,\beta,\theta})(u), \ \forall u \in U$□

5. Conclusion

In this paper, we introduce the $(\mathcal{I},\mathcal{T})-$ standard neutrosophic rough sets based on an implicator $\mathcal{I}$ and a t-norm $\mathcal{T}$ on D^*, lower and upper approximations of standard neutrosophic sets in a standard neutrosophic approximation are first introduced. We also have some notes on logic operations. Some properties of $(\mathcal{I},\mathcal{T})-$ standard neutrosophic rough sets are investigated. In the feature, we will investigate more properties on $(\mathcal{I},\mathcal{T})-$ standard neutrosophic rough sets.

References

[1] Z. Pawlak, *Rough sets*, International Journal of Computer and Information Sciences, vol. 11, no.5 , pp 341 – 356, 1982.

[2] L. A. Zadeh, *Fuzzy Sets*, Information and Control, Vol. 8, No. 3 (1965), p 338-353.

[3] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy set and systems, vol.20, pp.87-96, 1986.

[4] B.C. Cuong, V. Kreinovick, *Picture fuzzy sets – a new concept for computational intelligence problems*, Published in proceedings of the third world congress on information and communication technologies WICT'2013, Hanoi, Vietnam, December 15-18, pp 1-6, 2013.

[5] B.C. Cuong, Picture Fuzzy Sets, Journal of Computer Science and Cybernetics, Vol.30, n.4, 2014, 409-420.

[6] B.C. Cuong, P.H.Phong and F. Smarandache, Standard Neutrosophic Soft Theory: Some First Results, Neutrosophic Sets and Systems, vol.12, 2016, pp.80-91.

[7] D. Dubois, H Prade, *Rough fuzzy sets and fuzzy rough sets*, International journal of general systems, Vol. 17, p 191-209, 1990.

[8] Y.Y.Yao, *Combination of rough and fuzzy sets based on level sets*, Rough sets and Data mining: analysis for imprecise data, Kluwer Academic Publisher, Boston, p 301 – 321, 1997. $\alpha -$

[9] W. Z. Wu, J. S. Mi, W. X. Zhang, *Generalized fuzzy rough sets*, Information Sciences 151, p. 263-282, 2003.

[10] F. Smarandache, *A unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic*, American Research Press, Rehoboth 1999.

[11] H. Wang, F. Smarandache, Y.Q. Zhang et al., *Interval neutrosophic sets and logic: Theory and applications in computing*, Hexis, Phoenix, AZ 2005.

[12] H. Wang, F. Smarandache, Y.Q. Zhang, et al., *Single valued neutrosophic sets*, Multispace and Multistructure 4 (2010), 410-413.

[13] J. Ye, *A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets*, Journal of Intelligent & Fuzzy Systems 26 (2014) 2459-2466.

[14] P. Majumdar, *Neutrosophic sets and its applications to decision making*, Computation intelligentce for big data analysis (2015), V.19, pp 97-115.

[15] J. Peng, J. Q. Wang, J. Wang, H. Zhang, X. Chen, *Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems*, International journal of systems science (2016), V.47, issue 10, pp 2342-2358.

[16] Florentin Smarandache, Degrees of Membership > 1 and < 0 of the Elements With Respect to a Neutrosophic OffSet, Neutrosophic Sets and Systems, vol. 12, 2016, pp. 3-8.

[17] Florentin Smarandache, Degree of Dependence and Independence of the (Sub) Components of Fuzzy Set and Neutrosophic Set, Neutrosophic Sets and Systems, vol. 11, 2016, pp. 95-97;

http://fs.gallup.unm.edu/NSS/DegreeOfDependenceAndIndependence.pdf.

[18] N.X. Thao, N.V. Dinh, *Rough picture fuzzy set and picture fuzzy topologies*, Science Computer and Cybernetics, *Vol 31, No 3 (2015), pp 245-254.*

[19] A.M. Radzikowska, E.E. Kerre, *A comparative study of fuzzy rough sets*, Fuzzy set and Systems 126 (2002), pp 137-155.

[20] L. Zhou, W. Z. Wu, W. X. Zhang, *On character of intuitionistic fuzzy rough sets based on intuitionistic fuzzy implicators*, Information Sciences 179 (2009), pp 883-898.

[21] B.C. Cuong, P.V. Hai, *Some fuzzy logic operators for picture fuzzy sets*, 2015 Seventh International Conference on Knowledge and Systems Engineering, pp 132-137.

[22] F. Smarandache, *Neutrosophy. Neutrosophic Probability, Set, and Logic*, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998; http://fs.gallup.unm.edu/eBook-neutrosophics6.pdf (last edition online).

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Real Life Decision Optimization Model

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Abstract: In real life scientific and engineering problems decision making is common practice. Decision making include single decision maker or group of decision makers. Decision maker's expressions consists imprecise, inconsistent and indeterminate information. Also, the decision maker cannot select the best solution in unidirectional (single goal) way. Therefore, proposed model adopts decision makers' opinions in Neutrosophic Values (SVNS/INV) which effectively deals imprecise, inconsistent and indeterminate information, Multi goal (criteria) decision making and creditability (due to partial knowledge of decision maker) associated decision makers' expressions. Then partially known or unknown priorities (weights) of Multi Criteria Group Decision Making (MCGDM) problem is determined by establishing Correlation Coefficient (CC) established from improved cross entropy linear programming technique. The Multi Goal Linear equation was solved using a Novel Self Adaptive Harmonic Search Algorithm. The (NSAH) alternate solutions were ranked by weighted correlation coefficients of each alternative (lower the CC higher will be the rank). The validation of proposed method was demonstrated with an illustrative examples and compare with recent advancements. Hence, the proposed method was effective, flexible and accurate.

Keywords: MCGDM, Creditability, Improved Cross Entropy, Correlational Coefficient, and NSAH.

1 Introduction

In process of decision making real life scientific and engineering problems includes conflicting, non-commensurable, multi criteria and innumerable alternatives. The input information of decision making problem may involve decision maker's qualitative information and actual quantitative information. Hence, Multi Criteria Decision Making **(MCDM)** is a strategy of evaluating practical complex problems based on various qualitative or quantitative criteria in certain or uncertain environments to recommend best choice among various alternatives. Several comparative studies **[1]** have been taken to demonstrate its vast applicability **[2, 3, 4].** Briefing MCDM methods **[5]** will give clear understanding over techniques available **[6]** and benefits **[1].** More than one decision maker comprise in decision making process stated as Multi Criteria Group Decision Making **(MCGDM).**

In evaluation process MCDM had undergone quantification of decision makers' subjective information. Fundamental stages MCDM uses crisp information to represent decision makers' opinions. Crisp values can induce imprecision and confusion to the decision makers resulting inaccurate results. Real world decision making conflicting, inconsistent, indeterminate information cannot be expressed in terms of crisp values. To reduce fuzziness and vagueness of subjective information Zadeh **[7]** proposed Fuzzy Set **(FS)** theory and the decision making methods have developed by Bellman and Zadeh **[8]** using fuzzy theory. Subsequent research had been conducted to reduce uncertainty in decision maker's opinion under fuzzy environment.

F. Smarandache **[8]** represents truth function which describes decision maker acceptance value to alternative categorized by an attribute. But the constraint lies, it doesn't represent false (rejection value) function. Therefore, Atanassov introduce Intuitionistic Fuzzy Sets **(IFS)** **[9, 10]** which can represent truth membership function T(x) as well as falsity membership function F(x), they satisfy the condition $T(x), F(x) \in [0,1]$ and $0 \leq T(x) + F(x) \leq 1$. In IFS the indeterminate function is rest of truth and false functions $1-T(x) - F(x)$, here indeterminate and inconsistence functions are not clearly defined.

Smarandache [11] generalized FS, IFS, and Interval Valued Intuitionistic Fuzzy Set **(IVIFS) [10]** so on as Neutrosophic Set **(NS)** by adding indeterminate information. In NS the truth membership, indeterminacy membership, false membership functions are completely independent. Recently, NS became interesting area for researcher in decision making which can express supporting, nondeterministic, rejection values in terms of NS Values. Wang **[13]** propose Single Valued Neutrosophic Sets (SVNS) and Ye **[14]** gives correlation coefficient and weighted correlation coefficient in SVNS similar to IVIFS. Wang **[15]** proposed Interval Neutrosophic Sets (INS) in which the truth memberships, indeterminacy membership, false membership functions were extended to interval values. Ye **[16]** given similarity measures between INSs based on hamming and Euclidean distances and demonstrate with a MCDM problem.

Ye [18] developed a simplified neutrosophic weighted arithmetic averaging (SNWAA) operator, a simplified neutrosophic weighted geometric averaging (SNWGA) operator and applied to multiple attribute decision making under simplified neutrosophic environment. **Tian et al (2015) [19]** proposed a simplified neutrosophic linguistic normalized weighted Bonferroni mean operator (SNNWB) and constructed a multi criteria decision-making model based on SNNWB. But, the current aggregation operators for SVNNs and INNs ignore the knowledge background of the decision maker and his corresponding credibility on every evaluation value of SVNNs/INNs for each attributes.

Inspired by this idea **Jun Ye (2015) [20]** put forward a concept of Credibility-Induced Interval Neutrosophic Weighted Arithmetic Averaging (CIINWAA) operator and a Credibility-Induced Interval Neutrosophic Weighted Geometric Averaging (CIINWGA) operator by taking the importance of attribute weights and the credibility of the evaluation values of attributes into account. He also applied CIINWAA and CIINWGA to MCGDM problem; ranking of alternatives are based on INNs projection measures under creditability information.

Ye [22] reviewed evolution of cross entropy and its applicability in scientific and engineering applications. He proposed Improved cross entropy measures for SVNS and INS by overcome drawbacks (fail to fulfill the symmetric property) of cross entropy measures proposed by **Ye [21]**. Also he developed MCDM model based on improved cross entropy measures for SVNS and INS by taking advantage of ability of producing accurate results and minimizing information loss.

Jun Ye [23] presents correlational coefficients and weighted correlational coefficients of SVNS. He also introduced cosine similarity measure for SVNS. **Surapati et al [24]** proposed TOPSIS for single valued neutrosophic sets to solve multi criteria decision making problem which has unknown attribute weights and group of decision makers. The unknown weights of attributes derived from maximizing deviation method and rating of alternatives based on TOPSIS with imprecise and indeterminate information. **Said Broumi et al [25]** proposed extended TOPSIS using interval neutrosophic linguistic information for multi attribute decision making problems in which attribute weights are unknown.

Pranab Biswas et al (2016) [26] defined Triangular Fuzzy Number Neutrosophic Sets (TFNNS) by combining Triangular Fuzzy Numbers (TFN) and Single Valued Neutrosophic Sets (SVNS). He also proposed its operational rules based on TFN, SVNS and aggregation operators for TFNNS by extending Single Valued Neutrosophic Weighted Arithmetic (SVNWA) and Single Valued Neutrosophic Weighted Geometric (SVNWG) operators. Then, he developed MADM model based on TFNNS aggregation operators, score and accuracy functions. He also **[27]** introduced Single Valued Trapezoidal Neutrosophic Numbers (SVTrNN) and their operational rules, cut sets. The neutrosophic trapezoidal numbers express the truth function (T), indeterminate function (I) and false function (F) independently. He presents cosine similarity measures based multi criteria decision making method using trapezoidal fuzzy nutrosophic sets (TFNS). The ranking method is proposed after defining value and ambiguity indices of truth, false, indeterminate membership functions. The validity and applicability is shown by illustrative tablet selection problem. He also **[28]** proposed cosine similarity measures between two trapezoidal neutrosophic sets and its properties.

Jun Ye [29] introduced simplified neutrosophic harmonic averaging projection measures for multi criteria decision making problems. Projection measures are very suitable tool for dealing MCDM problems because it considers not only distance between alternatives but also its direction. The projection measures have extended flexibility of handling various types of information for instance **[30, 31]** uncertain and fuzzy based projection measures applied in multi attribute decision making. Ye observed drawbacks of general projection measures and proposed bidirectional projection measures **[32]** by overcoming shortcomings of general projection measures. He extends the applications of bidirectional projection measures in complex group decision making under neutrosophic environment.

Surapati and Kalyan [33] defined Accumulated Arithmetic Operator (AAO) to transform interval neutrosophic set to single valued neutrosophic sets. He also extended single valued Gray Relation Analysis (GRA) to interval valued numbers in multi criteria decision making. Then he proposed entropy based GRA for unknown attributes in MCDM problems under INN environment. **Rıdvan Şahin [34]** proposed two transformation methods for interval neutrosophic values to fuzzy sets and single valued neutrosophic sets. He developed two methodologies based on extended cross entropy to MCDM problems using interval valued numbers. But the transformation of INN to SVNS may results inaccurate outcomes.

Kalyan and Surapati [35] present quality bricks selection based on multi criteria decision making with single valued neutrosophic grey relational analysis. The weights of attributes are determined using experts opinions. Ranking is based on gray relation coefficient that derived from hamming distance between alternative to ideal neutrosophic estimate reliable solution and ideal neutrosophic estimates unreliable solution then neutrosophic relational degree used to select the quality brick. **Jun Ye [36]** proposed exponential similarity measures between two neutrosophic numbers. The advantages of exponential measures are that indicates stronger discrimination and higher sensitivity with respect than cosine similarity measure of neutrosophic numbers. He applied exponential similarity measures to the vibration fault diagnosis of steam turbine under indeterminate information. The proposed method not only analysis fault type but also predicts fault trends based on relation indices.

Tian et al (2016) [37] extends uncertain linguistic variable and simplified neutrosophic sets to simplified neutrosophic uncertain linguistic sets which integrates qualitative as well as quantitative evaluation. It reflects decision maker's expressions having inconsistence, incompleteness, indeterminate information. After reviewing relevant literature he developed Generalized Simplified Neutrosophic Uncertain Linguistic Prioritized Weighted Aggregation (GSNULPWA) operators and applied to solving MCDM problems.

Bipolarity refers to the propensity of the human mind to reason and make decisions on the basis of positive and negative effects**. Irfan Deli et al [38]** introduced bipolar sets which is the extension of fuzzy sets, bipolar fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets. He also developed the Bipolar Neutrosophic Weighted Average (BNWA) Operators and Bipolar Neutrosophic Weighted Geometric (BNWG) operators to aggregate the bipolar neutrosophic information. Then he proposed multi criteria decision making model using bipolar neutrosophic sets and its operators of certainty, score and accuracy functions.

Roy and Dos [39] developed neutrosophic based linear goal programming and lexicographic goal programming for multi objective linear programming (MOLP) problem. He describes evolution of neutrosophic theory and its operations in linear programming models. He also proposed two models for MOLP, applied to bank there investment problem by varying the weights. **Feng Li (2011) [40]** reduced process complexity and computation time after developing the closeness coefficient based non-linear programming model for MCDM problem. The nonlinear equation based on closeness coefficient applied to searching algorithm to obtain attribute weights and the ranking of alternatives estimated based on optimal membership degrees. The proposed methodology validated with real example and demonstrates its applicability.

Tian et al (2015) [41] put forward the concept of multi criteria decision making based on cross entropy under interval neutrosophic sets. The INS values are transformed to SVNS for ease of calculations and formulated a linear equation for deriving weights of attributes. These two linear equations are constructed from decision maker's indeterminate and inconsistent information.

Then the linear programming techniques are used to determine weights of attributes here constraints established by partially known indeterminate weights. After obtaining attribute weights possibility degree method ranked the alternatives.

After rigorous investigation on literature and research gap analysis the proposed model considered performance factors such as it should adopt practical/ real world problems, flexible to operate, accurate in results and effective. Real life decision making includes group of decision makers, their limited knowledge about specific attributes (creditability) and unknown priorities of multi objectives (attributes) to choose best out of existing alternatives.

Therefore considering shortcomings of recent methods we proposed new Multi criteria Group Decision Making Model for unknown attribute weights in continuous space and finite set of alternatives in discrete space in Neutrosophic environment.

The rest of the paper is organized as follows. Section 2 briefly describes some basic concepts of neutrosophic numbers and its operational functions. Section 3 proposes new approaches to solve real world decision making problems under neutrosophic environment. In Section 5, illustrative examples are presented to demonstrate the application of the proposed method, and then the effectiveness and advantages of the proposed methods are demonstrated by the comparative analysis with existing relative methods in sections 6. Finally, Section 7 contains conclusions and applications of present work.

2 Preliminaries

2.1 Single Valued Neutrosophic Sets (SVNS)

Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form $A=\{\langle x, u_A(x), w_A(x), v_A(x)\rangle : x\in X\}$where $u_A(x)$: $X\rightarrow[0,1]$, $w_A(x)$:$X\rightarrow[0,1]$ and $v_A(x)$:$X\rightarrow[0,1]$ with $0 \le u_A(x) + w_A(x) + v_A(x) \le 3$ for all $x\in X$. The intervals (x), w_A (x) and (x) denote the truth membership degree, the indeterminacy membership degree and the falsity membership degree of x to A, respectively.

2.2 Geometric Weighted Average Operator (GWA) for SVNC

Let A_k (k=1, 2,…, n) ∈ SVNS (X). The single valued neutrosophic weighted geometric average operator is defined by $G_\omega = (A_1, A_2,\ldots, A_n) = \prod_{k=1}^{n} A_k^{w_k}$

=

$$\left(\prod_{k=1}^{n}\left(u_{A_k}(x)\right)^{w_k}, \quad 1-\prod_{k=1}^{n}\left(1-w_{A_k}(x)\right)^{w_k}, 1-\prod_{k=1}^{n}\left(1-v_{A_k}(x)\right)^{w_k}\right) \tag{2}$$

Where ω_k is the weight of A_k (k=1, 2,…,n), $\omega_k\in[0,1]$ and $\sum_{k=1}^{n} w_k = 1$. Principally, assume ω_k=1/n (k=1, 2,…, n), then G_ω is called a geometric average for SVNSs.

2.3 Compliment of SVNS

The complement of an SVNS A is denoted by Ac and is defined as uAc $(x) = v$ (x), wAc (x) = 1−(x), and $vAc(x)$ = uA (x) for all $x \in X$. That is, $Ac = \{\langle x, vA\ (x), 1-wA\ (x), uA\ (x)\ \rangle : x\in X\}$.

2.4 Improved Cross Entropy Measures of SVNS

For any two SVNSs A and B in a universe of discourse X = $\{x_1, x_2,\ldots, x_n\}$. Let weight of each element is w_i, $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$ then the weighted cross entropy between SVNSs A from B is defined as follows:

$$N_w(A,B)=\sum_{i=1}^{n} w_i \left[\begin{array}{c} \sqrt{\left[\frac{T_A^2(x_i)+T_B^2(x_i)}{2}\right]} - \left(\frac{\sqrt{T_A(x_i)}+\sqrt{T_B(x_i)}}{2}\right)^2 \\ +\sqrt{\left[\frac{I_A^2(x_i)+I_B^2(x_i)}{2}\right]} - \left(\frac{\sqrt{I_A(x_i)}+\sqrt{I_B(x_i)}}{2}\right)^2 \\ +\sqrt{\frac{[1-I_A(x_i)]^2+[1-I_B(x_i)]^2}{2}} \\ -\left(\frac{\sqrt{[1-I_A(x_i)]}+\sqrt{[1-I_B(x_i)]}}{2}\right)^2 \\ +\sqrt{\left[\frac{F_A^2(x_i)+F_B^2(x_i)}{2}\right]} \\ -\left(\frac{\sqrt{F_A(x_i)}+\sqrt{F_B(x_i)}}{2}\right)^2 \end{array}\right]$$

2.5 Interval Valued Neutrosophic Sets (INS)

The real scientific and engineering applications can be expressed as INS values.
Let X be a space of points (objects) and int [0,1] be the set of all closed subsets of [0,1]. For convenience, if let $u\tilde{A}(x)$ = $[u\tilde{A}-(x), u\tilde{A}+(x)]$, $w\tilde{A}(x)$ = $[w\tilde{A}-(x), w\tilde{A}+(x)]$ and $v\tilde{A}$ (x) = $[v\tilde{A}-(x), v\tilde{A}+(x)]$, then $\tilde{A}=\{\langle x, [u\tilde{A}-(x),u\tilde{A}+(x)], [w\tilde{A}-(x),w\tilde{A}+(x)], [v\tilde{A}-(x),v\tilde{A}+(x)]\rangle : x\in X\}$ with the condition, $0\le \sup u\tilde{A}(x)+\sup w\tilde{A}(x)+\sup v\tilde{A}\ (x)\le 3$ for all $x\in X$. Here, we only consider the sub-unitary interval of [0, 1]. Therefore, an INS is clearly neutrosophic set.

2.6 Compliment of INS

The complement of an INS $\tilde{A}$ is denoted by $\tilde{A}c$ and is defined as $u\tilde{A}c(x) = v(x)$, $(w\tilde{A}-)c(x) = 1-w\tilde{A}+(x)$, $(w\tilde{A}+)c(x) = 1-w\tilde{A}-(x)$ and $v\tilde{A}c(x) = u(x)$ for all $x \in X$. That is, $\tilde{A}c=\{\langle x, [v\tilde{A}-(x),v\tilde{A}+(x)], [1-w\tilde{A}+(x),1-w\tilde{A}-(x)], [u\tilde{A}-(x),u\tilde{A}+(x)]\rangle : x\in X\}$.

2.7 Geometric Aggregation Operator for INS

Let $\tilde{A}k$ (k=1,2,…,n) ∈ INS(X). The interval neutrosophic weighted geometric average operator is defined by
$G\omega=(\tilde{A}1,\tilde{A}2,\ldots,\tilde{A}n) = \prod_{k=1}^{n} A_k^{w_k}$

$$\left(\begin{array}{c} \left[\prod_{k=1}^{n}\left(u^{-}{}_{\tilde{A}_k}(x)\right)^{w_k}, \prod_{k=1}^{n}\left(u^{+}{}_{\tilde{A}_k}(x)\right)^{w_k}\right], \\ \left[1-\prod_{k=1}^{n}\left(1-w^{-}{}_{A_k}(x)\right)^{w_k}, 1-\prod_{k=1}^{n}\left(1-w^{+}{}_{A_k}(x)\right)^{w_k}\right], \\ \left[1-\prod_{k=1}^{n}\left(1-v^{-}{}_{A_k}(x)\right)^{w_k}, \quad \left(1-\prod_{k=1}^{n}\left(1-v^{+}{}_{A_k}(x)\right)^{w_k}\right)\right] \end{array}\right) \tag{4}$$

Where ωk is the weight of $\tilde{A}k$ (k=1,2,…,n), $\omega k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$. Principally, assume ωk=1/n (k=1,2,…,n), then $G\omega$ is called a geometric average for INSs.

2.8 Improved Cross Entropy Measures of INS

For any two SVNSs A and B in a universe of discourse X = {x1, x2,⋯, xn}. Let weight of each element is wi, ωi ∈ [0,1] and Σ_(i=1)^n w_i =1 then the weighted cross entropy between SVNSs A from B is defined as follows:

$$M(A,B)= \frac{1}{2}\left\{\sum_{i=1}^{n} w_i\left[\sqrt{\left[\frac{infT_A^2(x_i)+infT_B^2(x_i)}{2}\right]}-\left(\frac{\sqrt{infT_A(x_i)}+\sqrt{infT_B(x_i)}}{2}\right)^2+\sqrt{\left[\frac{infI_A^2(x_i)+infI_B^2(x_i)}{2}\right]}-\left(\frac{\sqrt{infI_A(x_i)}+\sqrt{infI_B(x_i)}}{2}\right)^2+\sqrt{\frac{[1-infI_A(x_i)]^2+[1-infI_B(x_i)]^2}{2}}-\left(\frac{\sqrt{[1-infI_A(x_i)]}+\sqrt{[1-infI_B(x_i)]}}{2}\right)^2+\sqrt{\left[\frac{infF_A^2(x_i)+infF_B^2(x_i)}{2}\right]}-\left(\frac{\sqrt{infF_A(x_i)}+\sqrt{infF_B(x_i)}}{2}\right)^2\right]+\sum_{i=1}^{n} w_i\left[\sqrt{\left[\frac{supT_A^2(x_i)+supT_B^2(x_i)}{2}\right]}-\left(\frac{\sqrt{supT_A(x_i)}+\sqrt{supT_B(x_i)}}{2}\right)^2+\sqrt{\left[\frac{supI_A^2(x_i)+supI_B^2(x_i)}{2}\right]}-\left(\frac{\sqrt{supI_A(x_i)}+\sqrt{supI_B(x_i)}}{2}\right)^2+\sqrt{\frac{[1-supI_A(x_i)]^2+[1-supI_B(x_i)]^2}{2}}-\left(\frac{\sqrt{[1-supI_A(x_i)]}+\sqrt{[1-supI_B(x_i)]}}{2}\right)^2+\sqrt{\left[\frac{supF_A^2(x_i)+supF_B^2(x_i)}{2}\right]}-\left(\frac{\sqrt{supF_A(x_i)}+\sqrt{supF_B(x_i)}}{2}\right)^2\right]\right\}$$

3 Proposed Methodology

In real life problems decision makers' expressions are inconsistence, indeterminate, incomplete. The Neutrosophic sets are most popular in dealing with such a vague and imprecise decision makers' opinions. The decision maker is not always aware of all the attributes in complex decision making problems. So, the results tend to unreasonable or incredible if the evaluations of the decision maker for all the attributes imply the same credibility.

Therefore, the credibility of the attribute evaluations given by the decision maker in the aggregation process of the attribute values should consider to avoiding the unreasonable or incredible judgments in decision making. In reality, decision making is multi-dimensional (Multi Goal) and prioritized goals are considered for evaluations.

The unknown priorities (weights) of goals (attributes) are determined by constructing Multi Goal Linear Programming (MGLP). While construction MGLP [46, 47] adopts maximizing deviation method and weighted distance methods. Some limitations observed as complexity in calculations, improper results due to distance measures which are not effective for discriminating any two NS and MGLP is solved using trade off/ heuristic techniques these focused on local optima implies inaccurate results. Then ranking of alternatives using score and accuracy or distance measures from PIS may loss valid information or produces indefinite outcomes.

Therefore the proposed method is developed by overcoming shortcomings of recent models and designed for real world problems focused on performance factors such as accuracy, flexibility and effectiveness. The proposed MCGDM problem solving procedure described as follows.

In a multiple attribute group decision-making problem with neutrosophic numbers, let S = {S1, S2… Sm} be a set of alternatives, Ai = {A1, A2… Am} be a set of attributes, and Dk = {D1, D2… Ds} be a set of decision makers or experts. The weight vector of attributes is Wj = (w1, w2,…, wn) with wj∈ [0, 1] and ∑_(j=1)^n 〖w_j=1〗 the creditability weight vector of Decision makers is λ = {λ1, λ2, . . . , λs}.with with λ k∈ [0, 1] and ∑_(k=1)^s 〖λ_k=1〗 .

Step: 1 Obtain decision matrices D_s from each decision maker. Decision makers' expressions of each alternative to corresponding attributes represented in SVNS/INS.

Step: 2 Establish grouped decision matrix D_ij by aggregating individual decision matrices using Equation 2 in case of SVNS or Equation 7 in case of INS values.

Step: 3 Normalize group decision matrix (r_ij) if required (contains cost & benefit attributes) using Equation 3 for SVNS or Equation 6 for INS values.

Step: 4 Construct Multi Goal Linear Programming using min ∑_(i=1)^m∑_(j=1)^n 〖(d^+ (r_ij,r^+))/(d^+ (r_ij, r^+)+d^- (r_ij,r^-)) w_j 〗 where d^+ (r_ij, r^+) ,d^- (r_ij, r^-) are symmetric discrimination measures of r_ij to r^+ and r^- respectively. Here r^+ is PIS assumed as (1,0,0) and r^- is NIS assumed as (0,1,1)

Step: 5 Determine priorities of goal by solving MGLP applying Novel Self Adaptive Harmonic Search algorithm [46].

Step: 6 Rank the alternatives based on weighted correlational coefficient derived from improved cross entropy i.e.

$$A_i=\sum_{j=1}^{n}\frac{d^+(r_{ij},r^+)}{d^+(r_{ij},r^+)+d^-(r_{ij},r^-)}w_j$$

lower the Ai value higher will be the rank.

4 Illustrative Examples

Example: 1 here, we choose the decision making problem adapted from [47]. An automotive company is desired to select the most appropriate supplier for one of the key elements in its manufacturing process. After preevaluation, four suppliers have remained as alternatives for further evaluation. In order to evaluate alternative suppliers, a committee composed of four decision makers has been formed. The committee selects four attributes to evaluate the alternatives: (1) $C1$: product quality, (2) $C2$: relationship closeness, (3) $C3$: delivery performance and (4) $C4$: price. Suppose that there are four decision makers, denoted by D1, D2, D3, D4, whose corresponding weight vector is λ = (0.25, 0.25, 0.25, 0.25).

Step: 1 Decision matrices of each decision maker

$$D1 = \begin{bmatrix} \{0.4, 0.2, 0.3\} & \{0.4, 0.2, 0.3\} & \{0.2, 0.2, 0.5\} & \{0.7, 0.2, 0.3\} \\ \{0.6, 0.1, 0.2\} & \{0.6, 0.1, 0.2\} & \{0.5, 0.2, 0.3\} & \{0.5, 0.1, 0.2\} \\ \{0.3, 0.2, 0.3\} & \{0.5, 0.2, 0.3\} & \{0.1, 0.5, 0.2\} & \{0.1, 0.4, 0.5\} \\ \{0.7, 0.2, 0.1\} & \{0.6, 0.1, 0.2\} & \{0.4, 0.3, 0.2\} & \{0.4, 0.5, 0.1\} \end{bmatrix}$$

$$D2 = \begin{bmatrix} \{0.1, 0.3, 0.5\} & \{0.5, 0.1, 0.5\} & \{0.3, 0.1, 0.6\} & \{0.4, 0.1, 0.4\} \\ \{0.2, 0.5, 0.4\} & \{0.3, 0.4, 0.3\} & \{0.2, 0.3, 0.1\} & \{0.2, 0.3, 0.5\} \\ \{0.5, 0.2, 0.6\} & \{0.2, 0.4, 0.3\} & \{0.5, 0.2, 0.5\} & \{0.1, 0.5, 0.3\} \\ \{0.2, 0.4, 0.2\} & \{0.1, 0.1, 0.3\} & \{0.1, 0.5, 0.4\} & \{0.5, 0.3, 0.1\} \end{bmatrix}$$

$$D3 = \begin{bmatrix} \{0.3, 0.2, 0.1\} & \{0.3, 0.1, 0.3\} & \{0.1, 0.4, 0.5\} & \{0.2, 0.3, 0.5\} \\ \{0.6, 0.1, 0.4\} & \{0.6, 0.4, 0.2\} & \{0.5, 0.4, 0.1\} & \{0.5, 0.2, 0.4\} \\ \{0.3, 0.3, 0.6\} & \{0.4, 0.2, 0.4\} & \{0.2, 0.3, 0.2\} & \{0.3, 0.5, 0.1\} \\ \{0.3, 0.6, 0.1\} & \{0.5, 0.3, 0.2\} & \{0.3, 0.3, 0.6\} & \{0.4, 0.3, 0.2\} \end{bmatrix}$$

$$D4 = \begin{bmatrix} \{0.2, 0.2, 0.3\} & \{0.3, 0.2, 0.3\} & \{0.2, 0.3, 0.5\} & \{0.4, 0.2, 0.5\} \\ \{0.4, 0.1, 0.2\} & \{0.6, 0.3, 0.5\} & \{0.1, 0.2, 0.2\} & \{0.5, 0.1, 0.2\} \\ \{0.3, 0.5, 0.1\} & \{0.2, 0.2, 0.3\} & \{0.5, 0.4, 0.3\} & \{0.5, 0.3, 0.2\} \\ \{0.3, 0.1, 0.1\} & \{0.2, 0.1, 0.4\} & \{0.2, 0.3, 0.2\} & \{0.3, 0.1, 0.6\} \end{bmatrix}$$

Step: 2 Group Decision Matrix after aggregation with decision maker's creditability

$$\begin{bmatrix} \{0.2213, 0.9906, 0.9953\} & \{0.3663, 0.9790, 0.9972\} & \{0.1861, 0.9923, 0.9995\} & \{0.3869, 0.9875, 0.9987\} \\ \{0.4120, 0.9867, 0.9954\} & \{0.5045, 0.9952, 0.9955\} & \{0.2659, 0.9942, 0.9835\} & \{0.3976, 0.9835, 0.9964\} \\ \{0.3409, 0.9955, 0.9984\} & \{0.2991, 0.9926, 0.9964\} & \{0.2659, 0.9972, 0.9955\} & \{0.1968, 0.9987, 0.9940\} \\ \{0.3350, 0.9964, 0.9722\} & \{0.2783, 0.9782, 0.9942\} & \{0.2213, 0.9972, 0.9973\} & \{0.3936, 0.9953, 0.9924 \end{bmatrix}$$

Step: 3 Normalized group decision matrix (criteria 4 is cost type attribute) apply Equation: 3 to step 2 to normalize so that all attributes are in benefit type.

$$\begin{bmatrix} \{0.2213, 0.9906, 0.9953\} & \{0.3663, 0.9790, 0.9972\} & \{0.1861, 0.9923, 0.9995\} & \{0.9987, 0.0125, 0.3869\} \\ \{0.4120, 0.9867, 0.9954\} & \{0.5045, 0.9952, 0.9955\} & \{0.2659, 0.9942, 0.9835\} & \{0.9964, 0.0165, 0.3976\} \\ \{0.3409, 0.9955, 0.9984\} & \{0.2991, 0.9926, 0.9964\} & \{0.2659, 0.9972, 0.9955\} & \{0.9964, 0.0013, 0.1968\} \\ \{0.3350, 0.9964, 0.9722\} & \{0.2783, 0.9782, 0.9942\} & \{0.2213, 0.9972, 0.9973\} & \{0.9924, 0.0047, 0.3936\} \end{bmatrix}$$

Step: 4 Multi Goal Linear Equation formed as

$$\min\ 3.6171\acute{\omega}_1 + 3.5687\acute{\omega}_2 + 3.7290\acute{\omega}_3 + 0.4031\acute{\omega}_4$$

Subjected to

Case: 1 completely unknown weights $\sum_{i=1}^{n} w_i = 1$ and $w_j \in [0, 1]$ here j=1, 2, 3, 4

Step: 5 Priorities of attributes obtain after solving MGLP with unknown weights using NSAH are

$$w_1 = 0.1996, w_2 = 0.2126, w_3 = 0.3278, w_4 = 0.3587$$

Step: 6 Ranking based on weighted correlation coefficients of each alternatives

$$A_i = \sum_{j=1}^{n} \frac{d^+(r_{ij}, r^+)}{d^+(r_{ij}, r^+) + d^-(r_{ij}, r^-)} w_j$$

A1=0.9029

A2=0.8950

A3=0.9337

A4=0.1080

Therefore the ranking of alternative A4 > A2 > A1 > A3 (lower the Ai value higher the rank)

Case: 2 partially known weights from decision makers'

$$0.18 \le w_1 \le 0.20$$
$$0.15 \le w_2 \le 0.25$$
$$0.30 \le w_3 \le 0.35$$
$$0.30 \le w_4 \le 0.40$$
$$\sum_{j=1}^{n} w_j = 1$$

Step: 5 Priorities of attributes obtain after solving MGLP with unknown weights using NSAH are

$$w_1 = 0.2291, w_2 = 0.2126, w_3 = 0.1996, w_4 = 0.3587$$

Step: 6 Ranking based on weighted correlation coefficients of each alternatives

A1=0.9047

A2=0.8948

A3=0.9333

A4=0.1034

Therefore the ranking of alternative A4 > A2 > A1 > A3 (lower the Ai value higher the rank)

Example: 2 The decision making problem is adapted from [47]. Suppose that an organization plans to implement ERP system. The first step is to format project team that consists of CIO and two senior representatives from user departments. By collecting all information about ERP vendors and systems, project team chooses four potential ERP systems Ai (i = 1, 2, 3, 4) as candidates. The company employs some external professional organizations (experts) to aid this decision making. The project team selects four attributes to evaluate the alternatives: (1) $C1$: function and technology, (2) $C2$: strategic fitness, (3) $C3$: vendors' ability, and (4) $C4$: vendor's reputation. Suppose that there are three decision makers, denoted by $D1$, $D2$, $D3$, whose corresponding weight vector is λ = (1/3, 1/3, 1/3). The four possible alternatives are to be evaluated under these four attributes and are in the form of IVNNs for each decision maker, as shown in the following:

Interval valued neutrosophic decision matrix:

D1=

$$\begin{bmatrix} \{\{[0.4, 0.5], [0.2, 0.3], [0.3, 0.5]\} & \{[0.3, 0.4], [0.3, 0.6], [0.2, 0.4]\} & \{[0.2, 0.5], [0.2, 0.6], [0.3, 0.5]\} & \{[0.5, 0.6], [0.3, 0.5], [0.2, 0.5]\}\} \\ \{\{[0.6, 0.7], [0.1, 0.2], [0.2, 0.3]\} & \{[0.1, 0.3], [0.1, 0.4], [0.2, 0.5]\} & \{[0.4, 0.5], [0.2, 0.5], [0.3, 0.7]\} & \{[0.2, 0.4], [0.1, 0.4], [0.3, 0.3]\}\} \\ \{\{[0.3, 0.4], [0.2, 0.3], [0.3, 0.4]\} & \{[0.3, 0.6], [0.2, 0.3], [0.2, 0.5]\} & \{[0.2, 0.7], [0.2, 0.4], [0.3, 0.6]\} & \{[0.2, 0.6], [0.4, 0.7], [0.2, 0.7]\}\} \\ \{\{[0.2, 0.6], [0.1, 0.2], [0.1, 0.2]\} & \{[0.2, 0.5], [0.4, 0.5], [0.1, 0.6]\} & \{[0.3, 0.5], [0.1, 0.3], [0.2, 0.2]\} & \{[0.4, 0.4], [0.1, 0.6], [0.1, 0.5]\}\} \end{bmatrix}$$

D2=

$$\begin{bmatrix} \{\{[0.4, 0.6], [0.1, 0.3], [0.2, 0.4]\} & \{[0.3, 0.5], [0.1, 0.4], [0.3, 0.4]\} & \{[0.4, 0.5], [0.2, 0.4], [0.1, 0.3]\} & \{[0.3, 0.6], [0.3, 0.6], [0.3, 0.6]\}\} \\ \{\{[0.3, 0.5], [0.1, 0.2], [0.2, 0.3]\} & \{[0.3, 0.4], [0.2, 0.2], [0.1, 0.3]\} & \{[0.2, 0.7], [0.3, 0.5], [0.3, 0.6]\} & \{[0.2, 0.5], [0.2, 0.7], [0.1, 0.2]\}\} \\ \{\{[0.5, 0.6], [0.2, 0.3], [0.3, 0.4]\} & \{[0.1, 0.4], [0.1, 0.3], [0.3, 0.5]\} & \{[0.5, 0.5], [0.4, 0.6], [0.3, 0.4]\} & \{[0.1, 0.2], [0.1, 0.4], [0.5, 0.6]\}\} \\ \{\{[0.3, 0.4], [0.1, 0.2], [0.1, 0.3]\} & \{[0.3, 0.3], [0.1, 0.5], [0.2, 0.4]\} & \{[0.2, 0.3], [0.4, 0.5], [0.5, 0.6]\} & \{[0.3, 0.3], [0.2, 0.3], [0.1, 0.4]\}\} \end{bmatrix}$$

D3=

$$\begin{bmatrix} \{\{[0.1, 0.3], [0.2, 0.3], [0.4, 0.5]\} & \{[0.3, 0.3], [0.1, 0.3], [0.3, 0.4]\} & \{[0.2, 0.6], [0.3, 0.5], [0.3, 0.5]\} & \{[0.4, 0.6], [0.3, 0.4], [0.2, 0.3]\}\} \\ \{\{[0.3, 0.6], [0.3, 0.5], [0.3, 0.5]\} & \{[0.3, 0.4], [0.3, 0.4], [0.3, 0.5]\} & \{[0.3, 0.5], [0.2, 0.4], [0.1, 0.5]\} & \{[0.1, 0.2], [0.3, 0.5], [0.3, 0.4]\}\} \\ \{\{[0.4, 0.5], [0.2, 0.4], [0.2, 0.4]\} & \{[0.2, 0.3], [0.1, 0.1], [0.3, 0.4]\} & \{[0.1, 0.4], [0.2, 0.6], [0.3, 0.6]\} & \{[0.4, 0.5], [0.2, 0.6], [0.1, 0.3]\}\} \\ \{\{[0.2, 0.4], [0.3, 0.4], [0.1, 0.3]\} & \{[0.1, 0.4], [0.2, 0.5], [0.1, 0.5]\} & \{[0.3, 0.6], [0.2, 0.4], [0.2, 0.2]\} & \{[0.2, 0.4], [0.3, 0.3], [0.2, 0.6]\}\} \end{bmatrix}$$

Step: 2 Group Decision Matrix after aggregation with decision maker's creditability

$$\begin{bmatrix} \{0.2213,\ 0.9906,\ 0.9953\} & \{0.3663,\ 0.9790,\ 0.9972\} & \{0.1861,\ 0.9923,\ 0.9995\} & \{0.3869,\ 0.9875,\ 0.9987\} \\ \{0.4120,\ 0.9867,\ 0.9954\} & \{0.5045,\ 0.9952,\ 0.9955\} & \{0.2659,\ 0.9942,\ 0.9835\} & \{0.3976,\ 0.9835,\ 0.9964\} \\ \{0.3409,\ 0.9955,\ 0.9984\} & \{0.2991,\ 0.9926,\ 0.9964\} & \{0.2659,\ 0.9972,\ 0.9955\} & \{0.1968,\ 0.9987,\ 0.9940\} \\ \{0.3350,\ 0.9964,0.9722\} & \{0.2783,\ 0.9782,\ 0.9942\} & \{0.2213,\ 0.9972,\ 0.9973\} & \{0.3936,\ 0.9953,\ 0.9924 \end{bmatrix}$$

Step: 3 Normalized group decision matrix (criteria 4 is cost type attribute) apply Equation: 3 to step 2 to normalize so that all attributes are in benefit type.

$$\begin{bmatrix} \{0.2520\ 0.4481]\ [0.1680\ 0.3000]\ [0.3048\ 0.4687]\} & \{[0.3000\ 0.3915][0.1723\ 0.4482][0.2681\ 0.4000]\} \\ \{0.3780\ 0.5944]\ [0.1723\ 0.3160]\ [0.2348\ 0.3743]\} & \{[0.2080\ 0.3634\ [0.2042\ 0.3396]\ [0.2042\ 0.4407]\} \\ [\{0.3915\ 0.4932][0.2000\ 0.3351][0.2681\ 0.4000]\} & \{[0.1817\ 0.4160][0.1347\ 0.2388][0.2681\ 0.4687]\} \\ [\{0.2289\ 0.4579][0.1723\ 0.2732][0.1000\ 0.2681]\} & \{[0.1817\ 0.4160][0.1347\ 0.2388][0.2681\ 0.4687]\} \end{bmatrix}$$

$$\begin{bmatrix} \{[0.2520\ 0.5313]\ [0.2348\ 0.5068]\ [0.2388\ 0.4407]\} & \{[0.3915\ 0.6000]\ [0.3000\ 0.5068]\ [0.2348\ 0.4808]\} \\ \{[0.2884\ 0.5593]\ [0.2348\ 0.4687]\ [0.2388\ 0.6085]\} & \{[0.1587\ 0.3420]\ [0.2042\ 0.5519]\ [0.2388\ 0.3048]\} \\ \{[0.2154\ 0.5192]\ [0.2732\ 0.5421]\ [0.3000\ 0.5421]\} & \{[0.2000\ 0.3915]\ [0.2440\ 0.5840]\ [0.2886\ 0.5620]\} \\ \{[0.2621\ 0.4481]\ [0.2440\ 0.4056]\ [0.3160\ 0.3650]\} & \{[0.2884\ 0.3634]\ [0.2042\ 0.4191]\ [0.1347\ 0.5068]\} \end{bmatrix}$$

Step: 4 Multi Goal Linear Equation formed as

$$\min 1.19451\acute{\omega}_1 + 1.4945\acute{\omega}_2 + 1.6462\acute{\omega}_3 + 1.6798\acute{\omega}_4$$

Subjected to

Case: 1 completely unknown weights $\sum_{i=1}^{n} w_i = 1$ and $w_j \in [0, 1]$ here j=1, 2, 3, 4

Step: 5 Priorities of attributes obtain after solving MGLP with unknown weights using NSAH are

$$w_1 = 0.18, w_2 = 0.1211, w_3 = 0.4378, w_4 = 0.2611$$

Step: 6 Ranking based on weighted correlation coefficients of each alternatives

A1=0.3831
A2=0.3830
A3=0.4238
A4=0.3623

Therefore the ranking of alternative A4 > A2 > A1 > A3 (lower the Ai value higher the rank)

Case: 2 partially known weights from decision makers'

$$\begin{aligned} 0.18 \leq\ & w_1 \leq 0.20 \\ 0.15 \leq\ & w_2 \leq 0.25 \\ 0.30 \leq\ & w_3 \leq 0.35 \\ 0.30 \leq\ & w_4 \leq 0.40 \end{aligned}$$

$$\sum_{j=1}^{n} w_j = 1|$$

Step: 5 Priorities of attributes obtain after solving MGLP with unknown weights using NSAH are

$$w_1 = 0.1856, w_2 = 0.1939, w_3 = 0.3138, w_4 = 0.3067$$

Step: 6 Ranking based on weighted correlation coefficients of each alternatives

A1=0.3803
A2=0.3811
A3=0.4177
A4=0.3641

Therefore the ranking of alternative A4 > A1 > A2 > A3 (lower the Ai value higher the rank)

6. Comparative Analysis and Discussion

The results obtain from two examples with partially known and completely unknown weights are compared to Sahin and Liu [44] and Liu and Luo [45] methods.

1. Sahin and Liu [44] developed score and accuracy discrimination functions for MCGDM problem after proposing two aggregation operators. The unknown weights of attributes are determined by constructing linear equation based on maximizing deviation method. The attribute weights are obtained by solving linear equation using Lagrange technique. Then individual decision matrixes are grouped with aid of geometric weighted aggregation operator. For each alternative weighted aggregated neutrosophic values are calculated using obtained attribute weights to aggregated group decision matrix. Therefore the ranking of each alternative is based on score and accuracy functions applied to alternative weighted aggregated neutrosophic values.

2. Liu and Luo [45] proposed weighted distance from positive ideal solution to each alternative based linear equation for determining unknown weights of attributes after observing some drawback in [27] for MAGDM under SVNS. The linear function aims to minimize overall weighted distance from PIS where attribute weights are unknown. The partially known or unknown conditions are subjected to proposed linear equation and solved using any linear programming technique results weights of attributes. Then ranking of alternatives given based on weighted hamming distance from PIS. The proposed model also extended to IVNS.

3. Proposed method aimed to enhance results accuracy, flexible to operate and effectiveness. In table 2 two examples are evaluated with two cases. Then the proposed method given similar results to [44] and [45] except for example 2 case 2. Liu method and proposed method ranked first as A4 but sachin method ranks A2 as first. The successive ranks for Liu are A2, A1 and A3 but in case of present method A1, A2, and A3 respectively because present method considers weighted positive and negative symmetric deviation from PIS and NIS. Therefore the proposed method is accurate, flexible and effective.

Table: 2 Comparisons of Methods

Type of Problem	Sachin and Liu [44]		Liu and Luo [45]		Proposed Method	
	Example 1	**Example 2**	**Example 1**	**Example 2**	**Example 1**	**Example 2**
Completely Unknown weights **(case 1)**	$A2 \succ A4 \succ A1 \succ A3$	$A2 \succ A4 \succ A1 \succ A3$	$A2 \succ A4 \succ A1 \succ A3$	$A2 \succ A4 \succ A1 \succ A3$	$A4 \succ A2 \succ A1 \succ A3$	$A4 \succ A2 \succ A1 \succ A3$
Partially Unknown Weights **(case 2)**	$A2 \succ A4 \succ A1 \succ A3$	$A2 \succ A4 \succ A1 \succ A3$	$A2 \succ A4 \succ A1 \succ A3$	$A4 \succ A2 \succ A1 \succ A3$	$A4 \succ A2 \succ A1 \succ A3$	$A4 \succ A1 \succ A2 \succ A3$

7. Conclusion

Real world problems involved inconsistent, indeterminate and imprecise information therefore present method represents decision makers' expression in Neutrosophic Sets (SVNS/INS). Group Decision makers' creditability weights are considered to aggregate their expressions to overcome partial or incomplete knowledge of decision makers in the respective attributes to alternatives. Partially known or completely unknown priorities of MCGDM problem is solved by establishing MGLP based on symmetric discrimination measure from each alternative to PIS and NIS then solved using NSAH algorithm. Ranks of alternatives are given based on weighted correlation coefficients of each alternative lower the value higher the rank. Illustrative examples are demonstrated its effectiveness, accuracy and flexibility by compared with two recent methods. The proposed technique can be applied to scientific and engineering problems such as project evaluation, supplier selection, manufacturing system, data mining, and medical diagnosis and management decisions.

References

1. N Caterino, I Iervolino (2008), A Comparative Analysis Of Decision Making Methods For The Seismic Retrofit Of Rc Buildings, The 14thWorld Conference on Earthquake Engineering October 12-17, 2008, Beijing, China.
2. Milan Janic and Aura Reggiani (2002), An Application of the Multiple Criteria Decision Making (MCDM) Analysis to the Selection of a New Hub Airport, OTB Research institute Delft University of Technology Delft The Netherlands.
3. Aarushi Singh, Sanjay Kumar Malik (2014), Major MCDM Techniques and their application-A Review, M. Tech. Student, Hindu College of Engineering, Sonepat, IOSR Journal of Engineering (IOSRJEN) www.iosrjen.org ISSN (e): 2250-3021, ISSN (p): 2278-8719 Vol. 04.
4. Rohan K Gavade, Multi-Criteria Decision Making: An overview of different selection problems and methods, International Journal of Computer Science and Information Technologies, Vol. 5 (4) , 2014, 5643-5646.
5. P.O. Box 64732, Virginia Beach, VA, 23467, An Analysis of Multi-Criteria Decision Making Methods, International Journal of Operations Research Vol. 10, No. 2, 56-66 (2013).
6. E Triantaphyllou, Multi-Criteria Decision Making: An Operations Research Approach, Encyclopedia of Electrical and Electronics Engineering, (J.G. Webster, Ed.), John Wiley & Sons, New York, NY, Vol. 15, pp. 175-186, (1998).
7. L.A. Zadeh(1965), Fuzzy sets, Information and control 8 (3) (1965) 338–353.
8. R.E. Bellman, L.A. Zadeh, Decision making in a fuzzy environment, Management Science 17 (4) (1970) 141–164.
9. K Atanassov (1994), Operators over interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems, 64(2), 159–174.
10. K Atanassov and G.Gargov, (1989).Interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems, 31(3), 343–349.
11. F Smarandache, (1999).A unifying field in logics. Neutrosophy: Neutrosophic probability, American Research Press, Rehoboth
12. F.Smarandache (2005), A generalization of the intuitionistic fuzzy set.International journal of Pure and Applied Mathematics, 24, 287-297.
13. H Wang, F Smarandache, and R.Sunderraman, (2010). Single valued neutrosophic sets, Multispace and Multistructure (4) 410-413.
14. J Ye, (2013).Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment, International Journal of General Systems 42(4) 386-394.
15. [H Wang and F. Smarandache, Y. Q. Zhang, (2005). Interval neutrosophic sets and logic: Theory and applications in computing, Hexis, Phoenix, AZ.
16. J Ye, (2014). Similarity measures between interval neutrosophic sets and their applications in Multi-criteria decision-making. Journal of Intelligent and Fuzzy Systems 26 165-172
17. J Ye (2014). A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets, J. Intell. Fuzzy Syst. 26 (5) 2459–2466.
18. Jun Ye (2014), A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. J. Intell. Fuzzy Syst. 26, 2459–2466.
19. Z.P. Tian, J. Wang, H.Y. Zhang, X.H. Chen, J.Q. Wang (2015), Simplified neutrosophic linguistic normalized weighted Bonferron mean operator and its application to multi-criteria decision making problems. FILOMAT. doi:10.2298/FIL1508576F

20. Jun Ye (2015), Interval Neutrosophic Multiple Attribute Decision-Making Method with Credibility Information. International Journal of Fuzzy Systems ISSN 1562-2479 Int. J. Fuzzy Syst. DOI 10.1007/s40815-015-0122-4.
21. Y e, J. Single Valued Neutrosophic Cross-Entropy for Multicriteria Decision Making Problems. –Applied Mathematical Modelling, Vol. 38, 2014, pp. 1170-1175.
22. Jun Ye (2015), Improved Cross Entropy Measures of Single Valued Neutrosophic Sets and Interval Neutrosophic Sets and Their Multi criteria Decision Making Methods. Cybernetics and Information Technologies • January 2015, DOI: 10.1515/cait-2015-0051
23. Jun Ye (2013), Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment. International Journal of General Systems, 2013 Vol. 42, No. 4, 386–394, http://dx.doi.org/10.1080/03081079.2012.761609.
24. Surapati et al (2015), TOPSIS for Single Valued Neutrosophic Soft Expert Set Based Multi-attribute Decision Making Problems. Neutrosophic Sets and Systems, Vol. 10, 2015.
25. Said Broumi et al (2015), An Extended TOPSIS Method for Multiple Attribute Decision Making based on Interval Neutrosophic Uncertain Linguistic Variables. Neutrosophic Sets and Systems, Vol. 8, 2015.
26. Pranab Biswas et al (2016), Aggregation of triangular fuzzy neutrosophic set information and its application to multi-attribute decision making. Neutrosophic Sets and Systems, Vol. 12, 2016.
27. Pranab Biswas et al (2016), Value and ambiguity index based ranking method of single-valued trapezoidal neutrosophic numbers and its application to multi-attribute decision making. Neutrosophic Sets and Systems, Vol. 12, 2016.
28. Pranab Biswas et al (2014), Cosine Similarity Measure Based Multi-attribute Decision-making with Trapezoidal Fuzzy Neutrosophic Numbers. Neutrosophic Sets and Systems, Vol. 8, 2014.
29. Jun Ye (2015), Simplified neutrosophic harmonic averaging projection-based method for multiple attribute decision-making problems. Int. J. Mach. Learn. & Cyber. DOI 10.1007/s13042-015-0456-0.
30. Xu ZS, Da QL (2004) Projection method for uncertain multiattribute decision making with preference information on alternatives. Int J Inf Technol Decis Mak 3(3):429–434
31. Xu Z, Hu H (2010) Projection models for intuitionistic fuzzy multiple attribute decision making. Int J Inf Technol Decis Mak 9(2):267–280.
32. Jun Ye (2016), Bidirectional projection method for multiple attribute group decision making with neutrosophic number. Neural Comput & Applic, DOI 10.1007/s00521-015-2123-5.
33. Surapati and Kalyan (2015), Interval Neutrosophic Multi-Attribute Decision-Making Based on Grey Relational Analysis. Neutrosophic Sets and Systems, Vol. 9, 2015.
34. Rıdvan Şahin (2015), Cross-entropy measure on interval neutrosophic sets and its applications in Multicriteria decision making. Bayburt University, Faculty of Education, Bayburt, 69000, Turkey.
35. Kalyan and Surapati (2015), Neutrosophic Decision Making Model for Clay-Brick Selection in Construction Field Based on Grey Relational Analysis. Neutrosophic Sets and Systems, Vol. 9, 2015.
36. Jun Ye (2016), Fault diagnoses of steam turbine using the exponential similarity measure of neutrosophic numbers. Journal of Intelligent & Fuzzy Systems 30 (2016) 1927–1934 DOI: 10.3233/IFS-151903 IOS Press.
37. Tian et al (2016), Multi-criteria decision-making based on generalized prioritized aggregation operators under simplified neutrosophic uncertain linguistic environment. Int. J. Mach. Learn. & Cyber. DOI 10.1007/s13042-016-0552-9.
38. Irfan Deli et al (2015), Bipolar Neutrosophic Sets and Their Application Based on Multi-Criteria Decision Making Problems. Proceedings of the 2015 International Conference on Advanced Mechatronic Systems, Beijing, China, August, 22-24, 2015.
39. Roy and Dos (2016), Neutrosophic Goal Programming applied to Bank: Three Investment Problem. Neutrosophic Sets and Systems, Vol. 12, 2016.
40. Feng Li (2011), Closeness coefficient based nonlinear programming method for interval-valued intuitionistic fuzzy multiattribute decision making with incomplete preference information. Applied Soft Computing 11 (2011) 3402–3418.
41. Tian et al (2015), Multi-criteria decision-making method based on a cross-entropy with interval neutrosophic sets. International Journal of Systems Science, Taylor & Francis.
42. P. Bosc, O. Pivert, More on a fuzzy bipolar relational algebra, Information Sciences, 219 (2013) 1–16.
43. Thamaraiselvi and Santhi (2016), A New Approach for Optimization of Real Life Transportation Problem in Neutrosophic Environment. Hindawi Publishing Corporation Mathematical Problems in Engineering Volume 2016, Article ID 5950747, 9 pages http://dx.doi.org/10.1155/2016/5950747.
44. RidvanSahin and Peide Liu (2015), "Maximizing deviation method for neutrosophic multiple attribute decision making with incomplete weight information," Neural Computing and Applications, pp. 1–13, 2015.
45. Liu and Luo (2016), The Weighted Distance Measure Based Method to Neutrosophic Multiattribute Group Decision Making. Hindawi Publishing Corporation Mathematical Problems in Engineering Volume 2016, Article ID 3145341, 8 pages http://dx.doi.org/10.1155/2016/3145341.
46. Luo (2013), A Novel Self-Adaptive Harmony Search Algorithm. Hindawi Publishing Corporation Journal of Applied Mathematics Volume 2013, Article ID 653749, 16 pages http://dx.doi.org/10.1155/2013/653749

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Rough Standard Neutrosophic Sets:

An Application on Standard Neutrosophic Information Systems

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Abstract: A rough fuzzy set is the result of the approximation of a fuzzy set with respect to a crisp approximation space. It is a mathematical tool for the knowledge discovery in the fuzzy information systems. In this paper, we introduce the concepts of rough standard neutrosophic sets and standard neutrosophic information system, and give some results of the knowledge discovery on standard neutrosophic information system based on rough standard neutrosophic sets.

Keywords: rough set, standard neutrosophic set, rough standard neutrosophic set, standard neutrosophic information systems

1 Introduction

Rough set theory was introduced by Z. Pawlak in 1980s [1]. It became a useful mathematical tool for data mining, especially for redundant and uncertain data. At first, the establishment of the rough set theory is based on the equivalence relation. The set of equivalence classes of the universal set, obtained by an equivalence relation, is the basis for the construction of upper and lower approximation of the subset of universal set.

Fuzzy set theory was introduced by L. Zadeh since 1965 [2]. Immediately, it became a useful method to study in the problems of imprecision and uncertainty. Ever since, a lot of new theories treating imprecision and uncertainty have been introduced. For instance, intuitionistic fuzzy sets were introduced in1986, by K. Atanassov [3], which is a generalization of the notion of a fuzzy set. While the fuzzy set gives the degree of membership of an element in a given set, intuitionistic fuzzy set gives a degree of membership and a degree of non-membership of an element in a given set. In 1999 [17], F. Smarandache introduced the concept of neutrosophic set which generalized fuzzy set and intuitionistic fuzzy set. It is a set in which each proposition is estimated to have a degree of truth (T), a degree of indeterminacy (I) and a degree of falsity (F). After a while, the subclass of neutrosophic sets was proposed. They are more advantageous in the practical application. Wang et al. [18] proposed the interval neutrosophic sets, and some of their operators. Smarandache [17] and Wang et al. [19] introduced a single valued neutrosophic set as an instance of the neutrosophic set accompanied with various set theoretic operators and properties. Ye [20] defined the concept of simplified neutrosophic set. It is a set where each element of the universe has a degree of truth, indeterminacy and falsity respectively, stretching between [0, 1]. Ye also suggested some operational laws for simplified neutrosophic sets, and two aggregation operators, including a simplified neutrosophic weighted arithmetic average operator and a simplified neutrosophic weighted geometric average operator.

In 2013, B.C. Cuong and V. Kreinovich introduced the concept of picture fuzzy set [4, 5], in which a given set has three memberships: a degree of positive membership, a degree of negative membership, and a degree of neutral membership of an element in this set. After that, L. H. Son gave the application of the picture fuzzy set in the clustering problems [7, 8]. We regard picture fuzzy sets as particular cases of the standard neutrosophic sets [6].

In addition, combining rough set and fuzzy set enhanced many interesting results. The approximation of rough (or fuzzy) sets in fuzzy approximation space give us the fuzzy rough set [9,10,11]; and the approximation of fuzzy sets in crisp approximation space give us the rough fuzzy set [9,10]. W. Z. Wu et al. [11] presented a general framework for the study of the fuzzy rough sets in both constructive and axiomatic approaches. Moreover, W. Z. Wu and Y. H. Xu investigated the fuzzy topological structures on the rough fuzzy sets [12], in which both constructive and axiomatic approaches are used. In 2012, Y. H. Xu and W. Z. Wu investigated the rough intuitionistic fuzzy set and the intuitionistic fuzzy topologies in crisp approximation spaces [13]. In 2013, B. Davvaz and M. Jafarzadeh studied the rough intuitionistic fuzzy information system [14]. In 2014, X. T. Nguyen introduced the rough picture fuzzy sets. It is the result of approximation of a picture fuzzy set with respect to a crisp approximation space [15].

In this paper, we introduce the concept of standard neutrosophic information system, and study some problems of the knowledge discovery of standard neutrosophic information system based on rough standard neutrosophic sets. The remaining part of this paper is organized as follows: we recall the basic notions of rough set, standard neutrosophic set and rough standard neutrosophic set on the crisp approximation space, respectively, in Sections 2 and 3. In Section 4, we introduce the basic concepts of standard neutrosophic information system. Finally, we investigate some problems of the knowledge discovery of standard neutrosophic information system: the knowledge reduction and extension of the standard neutrosophic information system, in Section 5 and Section 6, respectively.

2 Basic notions of standard neutrosophic set and rough set

In this paper, we denote by U a nonempty set called the universe of discourse. The class of all subsets of U will be denoted by P(U) and the class of all fuzzy subsets of U will be denoted by F(U).

Definition 1. [6]. A standard neutrosophic (PF) set A on the universe U is an object of the form

$$A=\{(x,\mu_A(x),\eta_A(x),\ \gamma_A(x))\mid x\in U\}$$

where $\mu_A(x)(\in[0,1])$ is called the "degree of positive membership of x in A", $\eta_A(x)(\in[0,1])$ is called the "degree of neutral membership of x in A" and $\gamma_A(x)(\in[0,1])\gamma_A(x)(\in[0,1])$ is called the "degree of negative mem-bership of x in A", where μ_A, η_A $\mu_A,\gamma_A$and γ_A $\eta_A$satisfy the following condition:

$\mu_A(x)+\eta_A(x)+\ \gamma_A(x)\le 1,\ (\forall\ x\in X)$ $\mu_A(x)+\gamma_A(x)+\eta_A(x))\le 1,(\forall x\in X)$.

The family of all standard neutrosophic set in U is denoted by PFS(U). The complement of a picture fuzzy set A is

$$\sim A=\{(x,\ \gamma_A(x),\ \eta_A(x),\ \mu_A(x))\mid \forall x\in U\}.$$

Obviously, any intuitionistic fuzzy set:

$$A=\{(x,\mu_A(x),\gamma_A(x))\}$$

may be identified with the standard neutrosophic set in the form

$$A=\{(x,\mu_A(x),0,\ \gamma_A(x)X)\mid x\in U\}$$

$A=\{(x,\mu_A(x),\gamma_A(x),0)|x\in U\}$.

The operators on PFS(U): $A\subseteq B$, $A\cap B$, $A\cup B$ were introduced in [4].

Now we define some special PF sets: a constant PF set is the PF set $\widehat{(\alpha,\beta,\theta)}=\{(x,\alpha,\beta,\theta)|x\in U\}$; the PF universe set is $U=1_U=\widehat{(1,0,0)}=\{(x,1,0,0)|x\in U\}$ and the PF empty set is $\emptyset=0_U=\widehat{(0,0,1)}=\{(x,0,0,1)|x\in U\}\emptyset=0_U=\widehat{(0,1,0)}=\{(x,0,1,0)|x\in U\}$.

For any $x\in U$, standard neutrosophic set 1_x and $1_{U\text{-}\{x\}}$ are, respectively, defined by: for all $y\in U$

$$\mu_{1_x}(y)=\begin{cases}1 \text{ if } y=x\\ 0 \text{ if } y\neq x\end{cases},\ \eta_{1_x}(y)=\begin{cases}0 \text{ if } y=x\\ 0 \text{ if } y\neq x\end{cases},$$

$$\gamma_{1_x}(y)=\begin{cases}0 \text{ if } y=x\\ 1 \text{ if } y\neq x\end{cases};\ \mu_{1_{U-\{x\}}}(y)=\begin{cases}0 \text{ if } y=x\\ 1 \text{ if } y\neq x\end{cases},$$

$$\eta_{1_{U-\{x\}}}(y)=\begin{cases}0 \text{ if } y=x\\ 0 \text{ if } y\neq x\end{cases},\ \gamma_{1_{U-\{x\}}}(y)=\begin{cases}1 \text{ if } y=x\\ 0 \text{ if } y\neq x\end{cases}$$

Definition 2. (Lattice $(D^*,\le_{D^*})$). Let
$D^*=\{(x_1,x_2,x_3)\in[0,1]^3: x_1+x_2+x_3\le 1\}$.
We define a relation $\le_{D^*}$ on D^* as follows:
$\forall(x_1,x_2,x_3),(y_1,y_2,y_3)\in D^*$
then
$(x_1,x_2,x_3)\le_{D^*}(y_1,y_2,y_3)(x_1,x_2,x_3)\le_{D^*}(y_1,y_2,y_3)$
if only if
(or $(x_1<y_1,\ x_3\ge y_3)$ $(x_1<y_1,x_3\ge y_3)$ or $(x_1=y_1,x_3>y_3)(x=x',y>y')$
or $(x_1=y_1,x_3=y_3,x_2\le y_2)(x=x',y=y',z\le z'))$
and $(x_1,x_2,x_3)=_{D^*}(y_1,y_2,y_3)\Leftrightarrow(x_1=y_1,x_2=y_2,x_3=y_3)$.

We have $(D^*,\le_{D^*})$ is a lattice. Denote $0_{D^*}=(0,0,1)$, $1_{D^*}=(1,0,0)$ Now, we define some operators on D^*.

Definition 3.

(i) Negative of $x=(x_1,x_2,x_3)\in D^*$ is $\overline{x}=(x_3,x_2,x_1)$

(ii) For all $x=(x_1,x_2,x_3)\in D^*$ we have
$$x\wedge y=(x_1\wedge y_1,x_2\wedge y_2,x_3\vee y_3)$$
$$x\vee y=(x_1\vee y_1,x_2\wedge y_2,x_3\wedge y_3).$$

We have some properties of those operators.

Lemma 1.

(a) For all $x = (x_1, x_2, x_3) \in D^*$ we have

(b1) $\overline{x \wedge y} = \bar{x} \vee \bar{y}\, \overline{x \wedge y} = \bar{x} \vee \bar{y}$

(b2) $\overline{x \vee y} = \bar{x} \wedge \bar{y}\, \overline{x \vee y} = \bar{x} \wedge \bar{y}$

(b) For all $x, y, u, v \in D^*$ and $x \leq_{D^*} u, y \leq_{D^*} v$ we have

(c1) $x \wedge y \leq_{D^*} u \wedge v$

(c2) $x \vee y \leq_{D^*} u \vee v$

Proof.

(a) We have $\overline{x \wedge y} = (x_3 \vee y_3, x_2 \wedge y_2, x_1 \wedge y_1) = (x_3, x_2, x_1) \vee (y_3, y_2, y_1) = \bar{x} \vee \bar{y}$

Similary $\overline{x \vee y} = (x_3 \wedge y_3, x_2 \wedge y_2, x_1 \vee y_1) = (x_3, x_2, x_1) \vee (y_3, y_2, y_1) = \bar{x} \vee \bar{y}$

(b) For $a, b, c, d \in [0,1]$, if $a \leq b, c \leq d$ then $a \wedge c \leq b \wedge d$ and. From definitions 2 and 3, we have the result to prove. □

Now, we mention the level sets of the standard neutrosophic sets, where $(\alpha, \beta, \theta) \in D^*$; we define:

- (α, β, θ)- level cut set of the standard neutrosophic set $A = \{(x, \mu_A(x), \eta_A(x), \gamma_A(x)) \mid x \in U\}$

$A = \{(x, \mu_A(x), \gamma_A(x), \eta_A(x)) | x \in U\}$as follows:

$A_\theta^{\alpha,\beta} = \{x \in U | (\mu_A(x), \eta_A(x), \gamma_A(x)) \geq (\alpha, \beta, \theta)\}$ $= \{x \in U | (\mu_A(x), \eta_A(x), \gamma_A(x)) \geq (\alpha, \beta, \theta)\}$

- strong (α, β, θ)- level cut set of the standard neutrosophic set A as follows:

$$A_{\theta^+}^{\alpha^+,\beta^+} = \{x \in U | (\mu_A(x), \eta_A(x), \gamma_A(x)) > (\alpha, \beta, \theta)\}$$

- $(\alpha^+, \beta, \theta)$-- level cut set of the standard neutrosophic set A as

$$A_\theta^{\alpha^+,\beta} = \{x \in U | \mu_A(x) > \alpha, \gamma_A(x) \leq \theta\}$$

- $(\alpha, \beta, \theta^+) -$ level cut set of the standard neutrosophic set A as

$$A_{\theta^+}^{\alpha,\beta} = \{x \in U | \mu_A(x) \geq \alpha, \ \gamma_A(x) < \theta\}$$

By $\beta = 0$ we denoted

$A_\theta^\alpha = A_\theta^{\alpha,0}$

- (α^+, θ^+)- level cut set of the standard neutrosophic set A as

$$A_{\theta^+}^{\alpha^+} = \{x \in U | \mu_A(x) > \alpha, \gamma_A(x) < \theta\}$$

- α- level cut set of the degree of positive membership of x in A as

$$A^\alpha = \{x \in U | \mu_A(x) \geq \alpha\}$$

the strong α- level cut set of the degree of positive membership of x in A as

$$A^{\alpha^+} = \{x \in U | \mu_A(x) > \alpha\}$$

- θ- level low cut set of the degree of negative membership of x in A as

$$A_\theta = \{x \in U | \gamma_A(x) \leq \theta\}$$

the strong θ- level low cut set of the degree of negative membership of x in A as

$$A_{\theta^+} = \{x \in U | \gamma_A(x) < \theta\}$$

Example 1. Given the universe $U = \{u_1, u_2, u_3\}$. Then $A = ((u_1, 0.8, 0.05, 0.1), (u_2, 0.7, 0.1, 0.2), (u_3, 0.5, 0.01, 0.4))$ is a standard neutrosophic set on U. Then $A_{0.1}^{0.7,0.2} = \{u_1, u_2\}$ but $A_{0.1}^{0.7,0.1} = \{u_1\}$ and $A_{0.1^+}^{0.7,0.2} = \{u_1\}$, $A_{0.1}^{0.7} = \{u_1\}$, $A_{0.1^+}^{0.7} = \emptyset$, $A^{0.5} = \{u_1, u_2, u_3\}$, $A^{0.5^+} = \{u_1, u_2\}$, $A_{0.2^+} = \{u_1\}$, $A_{0.2} = \{u_1, u_2\}$.

Definition 3. Let U be a nonempty universe of discourse which may be infinite. A subset $R \in P(U \times U)$ is referred to as a (crisp) binary relation on U. The relation R is referred to as:

- Reflexive: if for all $x \in U, (x, x) \in R$.
- Symmetric: if for all $x, y \in U, (x, y) \in R$ $x, y \in U, (x, y) \in R$ then $(y, x) \in R$.
- Transitive: if for all $x, y, z \in U, (x, y) \in R, (y, z) \in R$ $x, y, z \in U, (x, y) \in R, (y, z) \in R$ then $(x, z) \in R$
- Similarity: if R is reflexive and symmetric
- Preorder: if R is reflexive and transitive
- Equivalence: if R is reflexive and symmetric, transitive.

A crisp approximation space is a pair (U, R). For an arbitrary crisp relation R on U, we can define a set-valued mapping $R_s : U \to P(U)$ by:

$$R_s(x) = \{y \in U | (x, y) \in R\},\ x \in U.$$

Then, $R_s(x)$ is called the successor neighborhood of x x with respect to (w.r.t) R .

Definition 4.[9]. Let (U, R) be a crisp approximation space. For each crisp set $A \subseteq U$, we define the upper and lower approximations of A (w.r.t) (U, R) denoted by $\overline{R}(A)$ and $\underline{R}(A)$, respectively, are defined as follows:

$\overline{R}(A) = \{x \in U: R_s(x) \cap A \neq \emptyset\}$,

$\underline{R}(A) = \{x \in U:\ R_s(x) \subseteq A\}$ $\underline{R}(A) = \{x \in U: R_s(x) \subseteq A\}$.

Remark 2.1. Let (U, R) be a Pawlak approximation space, i.e. R is an equivalence relation. Then $R_s(x) = [x]_R$ holds. For each crisp set $A \subseteq U$, the upper and lower approximations of A (w.r.t) (U, R) denoted by $\overline{R}(A)$ and $\underline{R}(A)$, respectively, are defined as follows:

$\overline{R}(A) = \{x \in U: [x]_R \cap A \neq \emptyset\}\underline{R}(A) = \{x \in U: [x]_R \subseteq A\}$

Definition 5. [16] Let (U, R) be a crisp approximation space. For each fuzzy set $A \subseteq U$, we define the upper and lower approximations of A (w.r.t) (U, R) denoted by $\overline{R}(A)$ and $\underline{R}(A)$, respectively, are defined as follows:

$\overline{R}(A) = \{x \in U: R_s(x) \cap A \neq \emptyset\}$,

$\underline{R}(A) = \{x \in U:\ R_s(x) \subseteq A\}$

where

$\mu_{\overline{R}(A)}(x) = \max\{\mu_A(y) | y \in R_s(x)\}$,

$\mu_{\underline{R}A}(x) = min\{\mu_A(y) \mid y \in R_s(x)\}$

Remark 2.2. Let (U, R) be a Pawlak approximation space, i.e. *R* is an equivalence relation. Then $R_s(x) = [x]_R$ holds. For each fuzzy set $A \subseteq U$, the upper and lower approximations of A (w.r.t) (U, R) denoted by $\overline{R}(A)$ and $\underline{R}(A)$, respectively, are defined as follows:

$\overline{R}(A) = \{x \in U: [x]_R \cap A \neq \emptyset\}$,
$\underline{R}(A) = \{x \in U: [x]_R \subseteq A\}$

This is the rough fuzzy set in [6].

3. Rough standard neutrosophic set

A rough standard neutrosophic set is the approximation of a standard neutrosophic set w. r. t a crisp approximation space. Here, we consider the upper and lower approximations of a standard neutrosophic set in the crisp approximation spaces together with their membership functions, respectively.

Definition 5: Let (U, R) be a crisp approximation space. For $A \in PFS(U)$, the upper and lower approximations of A (w.r.t) (U, R) denoted by $\overline{RP}(A)\overline{RP}(A)$ and $\underline{RP}(A)$, respectively, are defined as follows:

$\overline{RP}(A) = \{(x, \mu_{\overline{RP}(A)}(x), \eta_{\overline{RP}(A)}(x), \gamma_{\overline{RP}(A)}(x)) | x \in U\}$

$\underline{RP}(A) = \{(x,\ \mu_{\underline{RP}(A)}(x), \eta_{\underline{RP}(A)}(x), \gamma_{\underline{RP}(A)}(x)) \mid x \in U\}$

where

$\mu_{\overline{RP}(A)}(x) = \vee_{y \in R_s(x)} \mu_A(y)$, $\eta_{\overline{RP}(A)}(x) = \wedge_{y \in R_s(x)} \eta_A(y)$, $\underline{RP}(A) = \{(x, \mu_{\underline{RP}(A)}(x), \gamma_{\underline{RP}(A)}(x), \eta_{\underline{RP}(A)}(x)) | x \in U\}$;

and

$\underline{RP}(A) = \{(x, \mu_{\underline{RP}(A)}(x), \gamma_{\underline{RP}(A)}(x), \eta_{\underline{RP}(A)}(x)) | x \in U\}$, $\eta_{\underline{RP}(A)}(x) = \wedge_{y \in R_s(x)} \eta_A(y)$, $\gamma_{\underline{RP}(A)}(x) = \vee_{y \in R_s(x)} \gamma_A(y)$.

$\underline{RP}(A) = \{(x, \mu_{\underline{RP}(A)}(x), \gamma_{\underline{RP}(A)}(x), \eta_{\underline{RP}(A)}(x)) | x \in U\}$

We have $\overline{RP}(A)$ and $\underline{RP}(A)$, two standard neutrosophic sets in U. Indeed, for each $x \in U$, for all $\epsilon > 0$, it exists $y_0 \in U$ $y_0 \in U$ such that $\mu_{\overline{RP}(A)}(x)$-$\epsilon \leq \mu_A(y_0) \leq \mu_{\overline{RP}(A)}(x)$, $\eta_{\overline{RP}(A)}(x) \leq \eta_A(y_0)$, $\gamma_{\overline{RP}(A)}(x) \leq \gamma_A(y_0)$

so that $\mu_{\overline{RP}(A)}(x) - \epsilon + \eta_{\overline{RP}(A)}(x) + \gamma_{\overline{RP}(A)}(x)$

$$\leq \mu_A(y_0) + \eta_A(y_0) + \gamma_A(y_0) \leq 1$$

$\mu_{\overline{RP}(A)}(x)$-$\epsilon + \eta_{\overline{RP}(A)}(x) + \gamma_{\overline{RP}(A)}(x) \leq$.

Hence $\mu_{\overline{RP}(A)}(x) + \eta_{\overline{RP}(A)}(x) + \gamma_{\overline{RP}(A)}(x) \leq 1 + \epsilon$, for all $\epsilon > 0$. It means that $\overline{RP}(A)$ is a standard neutrosophic set. By the same way, we obtain $\underline{RP}(A)$ a standard neutrosophic set. Moreover, $\underline{RP}(A) \subset \overline{RP}(A)$.

Thus, the standard neutrosophic mappings $\overline{RP}$, $\underline{RP}$: $PFS(U) \to PFS(U)$are referred to as the upper and lower PF approximation operators, respectively, and the pair $PR(A) = (\underline{PR}(A), \overline{RP}(A))$ is called the rough standard neutrosophic set of A w.r.t the approximation space. The picture fuzzy set denoted by $\sim\mathrm{RP}(\mathrm{A})$ and is defined by $\sim PR(A) = (\sim \underline{PR}(A), \sim \overline{RP}(\mathrm{A}))$ $\sim\mathrm{RP}(\mathrm{A}) = (\sim\underline{\mathrm{RP}}(\mathrm{A}), \sim\overline{\mathrm{RP}}(\mathrm{A}))$ where $\sim\underline{\mathrm{RP}}(\mathrm{A})$ and $\sim\overline{\mathrm{RP}}(\mathrm{A})$ are the complements of the PF sets $\overline{\mathrm{RP}}(\mathrm{A})$ and $\underline{\mathrm{RP}}(\mathrm{A})$ respectively.

Example 2. We consider the universe set $\mathrm{U} = \{\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4, \mathrm{u}_5\}$ and a binary relation R on U in Table 1. Here, if $\mathrm{u}_i\mathrm{R}\mathrm{u}_j$ then cell (i, j) takes a value of 1, cell (i, j) takes a value of 0 (i, j = 1, 2, 3, 4, 5). A standard neutrosophic

$$A = \{(u_1, 0.7, 0.1, 0.2), (u_2, 0.6, 0.2, 0.1), (u_3, 0.6, 0.2, 0.05), (u_2, 0.6, 0.2, 0.1), (u_3, 0.6, 0.2, 0.05)\}$$

Table 1: *Binary relation R on U*

R	u_1	u_2	u_3	u_4	u_5
u_1	1	0	1	0	0
u_2	0	1	0	1	1
u_3	1	0	1	0	1
u_4	0	1	0	1	0
u_5	0	0	1	1	1

We have $\mathrm{R_s}(\mathrm{u}_1) = \{\mathrm{u}_1, \mathrm{u}_3\}$, $\mathrm{R_s}(\mathrm{u}_2) = \{\mathrm{u}_2, \mathrm{u}_4, \mathrm{u}_5\}$,

$\mathrm{R_s}(\mathrm{u}_3) = \{\mathrm{u}_1, \mathrm{u}_3, \mathrm{u}_5\}$, $\mathrm{R_s}(\mathrm{u}_4) = \{\mathrm{u}_2, \mathrm{u}_4\}$,
$\mathrm{R_s}(\mathrm{u}_5) = \{u_3, u_4, u_5\}$ $\mathrm{R_s}(\mathrm{u}_5) = \{\mathrm{u}_3, \mathrm{u}_4, \mathrm{u}_5\}$.

Therefore, we obtain the results

$$\mu_{\overline{\mathrm{RP}}(\mathrm{A})}(\mathrm{u}_1) = \vee_{\mathrm{y} \in \mathrm{R_s}(\mathrm{u}_1)} \mu_\mathrm{A}(\mathrm{y})$$

$$\mu_{\overline{\mathrm{RP}}(\mathrm{A})}(\mathrm{u}_1) = \vee_{\mathrm{y} \in \mathrm{R_s}(\mathrm{u}_1)} \mu_\mathrm{A}(\mathrm{y}) = \max\{\mu_\mathrm{A}(\mathrm{u}_1), \mu_\mathrm{A}(\mathrm{u}_3)\}$$

$$= \max\{0.7, 0.6\} = 0.7,$$

$$\eta_{\underline{\mathrm{RP}}(\mathrm{A})}(\mathrm{u}_1) = \wedge_{\mathrm{y} \in \mathrm{R_s}(\mathrm{u}_1)} \eta_\mathrm{A}(\mathrm{y}) = \min\{\eta_A(u_1), \eta_A(u_3)\}$$

$$= \max\{0.7, 0.6\} = 0.7,$$

$$\gamma_{\underline{\mathrm{RP}}(\mathrm{A})}(\mathrm{u}_1) = \wedge_{\mathrm{y} \in \mathrm{R_s}(\mathrm{u}_1)} \gamma_\mathrm{A}(\mathrm{y}) = \min\{\gamma_A(u_1), \gamma_A(u_3)\}$$

$$\gamma_{\underline{\mathrm{RP}}(\mathrm{A})}(\mathrm{u}_1) = \wedge_{\mathrm{y} \in \mathrm{R_s}(\mathrm{u}_1)} \gamma_\mathrm{A}(\mathrm{y}) = \min\{\gamma_\mathrm{A}(\mathrm{u}_1), \gamma_\mathrm{A}(\mathrm{u}_3)\} = \max\{0.7, 0.6\} = 0.7 \min\{0.2, 0.05\} = 0.05$$

Similar calculations for other elements of U, we have upper approximations of A

$$\overline{\mathrm{RP}}(\mathrm{A}) = \{(u_1, 0.7, 0.1, 0.05), (u_2, 0.6, 0.2, 0.1), (u_3, 0.7, 0.1, 0.05), (u_4, 0.6, 0.2, 0.1), (u_5, 0.6, 0.2, 0.05)\}$$

and lower approximations of A is

$$\underline{RP}(\mathrm{A}) = \{(u_1, 0.6, 0.1, 0.2), (u_2, 0.4, 0.2, 0.2), (u_3, 0.4, 0.1, 0.2), (u_4, 0.5, 0.2, 0.15), (u_5, 0.4, 0.2, 0.2)\}.$$

Some basic properties of rough standard neutrosophic set operators are presented in the following theorem:

Theorem 1. Let (U, R) be a crisp approximation space, then the upper and lower rough standard neutrosophic approximation operators satisfy the following properties: $\forall \mathrm{A}, \mathrm{B}, \mathrm{A}_j \in \mathrm{PFS}(\mathrm{U})$, $j \in J$, J is an index set,

(PL1) $\underline{PR}(\sim A) = \sim \overline{RP}(\mathrm{A})$

(PL2) $\underline{\mathrm{RP}}(\mathrm{A} \cup (\alpha, \beta, \theta)) = \underline{\mathrm{RP}}(\mathrm{A}) \cup (\alpha, \beta, \theta)$
$\underline{\mathrm{RP}}(\mathrm{A} \cup (\widehat{\alpha, \beta, \theta})) = \underline{\mathrm{RP}}(\mathrm{A}) \cup (\widehat{\alpha, \beta, \theta})$

(PL3) $\underline{\mathrm{RP}}(\mathrm{U}) = \mathrm{U}\,\underline{\mathrm{RP}}(\mathrm{U}) = \mathrm{U}$

$\eta_{\underline{\mathrm{RP}}(\mathrm{A})}(\mathrm{x}) = \wedge_{\mathrm{y} \in \mathrm{R_s}(\mathrm{x})} \eta_\mathrm{A}(\mathrm{y})$

(PL5) $\underline{\mathrm{RP}}(\mathrm{A} \cup \mathrm{B}) \supseteq \underline{\mathrm{RP}}(\mathrm{A}) \cup \underline{\mathrm{RP}}(\mathrm{B})$

(PL6) $\mathrm{A} \subseteq \mathrm{B} \Rightarrow \underline{\mathrm{RP}}(\mathrm{A}) \subseteq \underline{\mathrm{RP}}(\mathrm{B})$

(PU1) $\overline{\mathrm{RP}}(\sim\mathrm{A}) = \sim\underline{\mathrm{RP}}(\mathrm{A})\,\overline{RP}(\sim \mathrm{A})$
$= \sim \underline{PR}(A)$

(PU2) $\overline{\mathrm{PR}}(\mathrm{A} \cap (\widehat{\alpha, \beta, \theta})) = \overline{\mathrm{PR}}(\mathrm{A}) \cap (\widehat{\alpha, \beta, \theta})$

(PU3) $\overline{\mathrm{PR}}(\emptyset) = \emptyset$

(PU4) $\overline{\mathrm{RP}}(\cup_{j \in J} \mathrm{A}_j) = \cup_{j \in J} \overline{\mathrm{RP}}(\mathrm{A}_j)$

(PU5) $\overline{\mathrm{RP}}(\mathrm{A} \cap \mathrm{B}) \subseteq \overline{\mathrm{RP}}(\mathrm{A}) \cap \overline{\mathrm{RP}}(\mathrm{B})$

(PU6) $\mathrm{A} \subseteq \mathrm{B} \Rightarrow \overline{\mathrm{RP}}(\mathrm{A}) \subseteq \overline{\mathrm{RP}}(\mathrm{B})$

Proof.

(PL1).

$\underline{RP}(\sim A)=\{(x,\ \mu_{\underline{RP}(\sim A)}(x),\eta_{\underline{RP}(\sim A)}(x),\ \gamma_{\underline{RP}(\sim A)}(x))\,|\,x\in U\}$

in which,

$\mu_{\underline{RP}(\sim A)}(x)=\vee_{y\in R_s(x)}\mu_{\sim A}(y)=\vee_{y\in R_s(x)}\gamma_A(y)=$

$\gamma_{\overline{RP}(A)}(x)$;

$\eta_{\underline{RP}(\sim A)}(x)=\wedge_{y\in R_s(x)}\eta_{\sim A}(y)=\wedge_{y\in R_s(x)}\eta_A(y)=$

$\eta_{\overline{RP}(A)}(x)$

$\gamma_{\underline{RP}(\sim A)}(x)=\wedge_{y\in R_s(x)}\gamma_{\sim A}(y)=\wedge_{y\in R_s(x)}\mu_A(y)=$

$\mu_{\overline{RP}(A)}(x)$

From that and lemma 1, we have $\underline{PR}(\sim A)=\sim\overline{RP}(A)$.

(PL2) Because $\widehat{(\alpha,\beta,\theta)}=\{(x,\alpha,\beta,\theta)|x\in U\}$, we have

$\mu_{\underline{RP}(A\cup(\alpha,\beta,\theta))}(x)=\bigvee_{y\in R_s(x)}\mu_{\underline{RP}(A\cup(\alpha,\beta,\theta))}(y)$

$\bigvee_{y\in R_s(x)}\mu_{\underline{RP}(A\cup\widehat{(\alpha,\beta,\theta)})}(y)=\bigvee_{y\in R_s(x)}\max\{\mu_{\underline{RP}(A)}(y),\alpha\}$

$=\max\{\bigvee_{y\in R_s(x)}\mu_{\underline{RP}(A)}(y),\bigvee_{y\in R_s(x)}\alpha\}$

$=\max\{\bigvee_{y\in R_s(x)}\mu_{\underline{RP}(A)}(y),\bigvee_{y\in R_s(x)}\alpha\}$

$max\{\mu_{\underline{RP}(A)}(x),\mu_{((\alpha,\beta,\theta))}(x)\}=\mu_{\underline{RP}(A)\cup(\alpha,\beta,\theta)}(x)$.

By the same way, we have

$\eta_{\underline{RP}(A\cup(\alpha,\beta,\theta))}(x)=\eta_{\underline{RP}A\cup(\alpha,\beta,\theta)}(x)$

and

$\gamma_{\underline{RP}(A\cup(\alpha,\beta,\theta))}(x)=\gamma_{\underline{RP}A\cup(\alpha,\beta,\theta)}(x)$.

It means $\underline{RP}(A\cup\widehat{(\alpha,\beta,\theta)})=\underline{RP}(A)\cup\widehat{(\alpha,\beta,\theta)}$.

(PL3) Since $U=1_U=\widehat{(1,0,0)}=\{(x,1,0,0)|x\in U\}$, then we can obtain (PL3) $\underline{RP}(U)=U$ by using definition 5.

The results (PL4), (PL5), (PL6) were proved by using the definition of lower and upper approximation spaces (definition 5) and lemma 1. $\mu_{\mu_{\underline{RP}(\widehat{(\alpha,\beta,\theta)})}}(x)$

Similarly, we have (PU1), (PU2), (PU3), (PU4), (PU5), PU(6). □

Theorem 2. Let (U,R) be a crisp approximation space. Then

a) $\underline{RP}(U)=U=\overline{RP}(U)$ and $\underline{RP}(\varnothing)=\varnothing=\overline{RP}(\varnothing)$ $\underline{RP}(\emptyset)=\emptyset=\overline{RP}(\emptyset)$.

b) $\underline{RP}(A)\subseteq\overline{RP}(A)$ forall $A\in PFS(U)$.□

Proof.

(a) Using (PL3), (PL6), (PU3), (PU6), we easy prove $\underline{RP}(U)=U=\overline{RP}(U)$ and $\underline{RP}(\emptyset)=\emptyset=\overline{RP}(\emptyset)$.

(b) Based on definition 5, we have

$\mu_{\underline{RP}(A)}(x)=\bigwedge_{y\in R_s(x)}\mu_A(y)$

$\leq\ \mu_{\overline{RP}(A)}(x)=\bigvee_{y\in R_s(x)}\mu_A(y)$,

$\eta_{\underline{RP}(A)}(x)=\bigwedge_{y\in R_s(x)}\mu_A(y)=\eta_{\overline{RP}(A)}(x)$,

and

$\gamma_{\underline{RP}(A)}(x)=\bigvee_{y\in R_s(x)}\gamma_A(y)\geq$

$\bigwedge_{y\in R_s(x)}\gamma_A(y)=\gamma_{\overline{RP}(A)}(x)$

So $\underline{RP}(A)\subseteq\overline{RP}(A)$ for all $A\in PFS(U)$.□

In the case of connections between special types of crisp relation on U, and properties of rough standard neutrosophic approximation operators, we have the following:

Lemma 2. If R is a symmetric crisp binary relation on U, then for all $A,B\in PFS(U)$,

$$\overline{RP}(A)\subseteq B\Leftrightarrow A\subseteq\underline{RP}(B)$$

Proof.

Let R be a symmetric crisp binary relation on U, i.e. $y\in R_s(x)\Leftrightarrow x\in R_s(y)$, $\forall x,y\in U$. We assume contradiction that $\overline{RP}(A)\subseteq B$ but $A\not\subset\underline{RP}(B)$.

For each $x\in U$, we consider all the cases:

+ if $\mu_A(x) > \mu_{\underline{RP}(B)}(x) = \wedge_{y\in R_s(x)}\mu_B(y)$ then it exists $y_0 \in R_s(x)$ such that $\mu_A(x) > \mu_B(y_0) \geq \mu_{\overline{RP}(A)}(y_0) = \vee_{z\in R_s(y_0)}\mu_A(z) \geq \mu_A(x)$ (because $y_0 \in R_s(x)$ then $x \in R_s(y_0)$. This is not true.

+ the cases $\gamma_A(x) < \gamma_{\overline{RP}(B)}(x)$ or $\eta_A(x) > \eta_{\overline{RP}(B)}(x)$ are also not true. □

Theorem 3. Let (U, R) be a crisp approximation space, and $\overline{RP}$, the upper and lower PF approximation operators.

Then:

(a) R is reflexive if and only if at least one of the following conditions are satisfied

(a1) (PLR)$\underline{RP}(A) \subseteq A \forall A \in PFS(U)$

(a2) (PUR)$A \subseteq \overline{RP}(A) \forall A \in PFS(U)$

(b) R is symmetric if and only if at least one of the following conditions are satisfied

(b1) (PLR)$\overline{RP}(\underline{RP}(A)) \subseteq A \forall A \in PFS(U)$

(b2) (PUR)$A \subseteq \underline{RP}\left(\overline{RP}(A)\right) \forall A \in PFS(U)$

(c) R is transitive if and only if at least one of the following conditions are satisfied

(c1) (PLT)$\underline{RP}(A) \subseteq \underline{RP}(\underline{RP}(A)) \forall A \in PFS(U)$

(c2) (PUT)$\overline{RP}(A) \subseteq \overline{RP}\left(\overline{RP}(A)\right) \forall A \in PFS(U)$

Proof.

(a). We assume that R is reflexive, i.e., $x \in R_S(x)$, so that $\forall A \in PFS(U)$ we have $\mu_{\underline{RP}(A)}(x) = \wedge_{y\in R_s(x)}\mu_A(y) \leq \mu_A(x)$, $\eta_{\underline{RP}(A)}(x) = \wedge_{y\in R_s(x)}\mu_A(y) \leq \eta_A(x)$,

and $\gamma_{\underline{RP}(A)}(x) = \vee_{y\in R_s(x)}\gamma_A(y) \geq \gamma_A(x)$. It means that $\underline{RP}(A) \subseteq A$, $\forall A \in PFS(U)$, i.e. (a1) was verified.

Similarly, we consider upper approximation of:

$\mu_{\overline{RP}(A)}(x) = \vee_{y\in R_s(x)}\mu_A(y) \geq \mu_A(x)$, $\eta_{\overline{RP}(A)}(x) = \wedge_{y\in R_s(x)}\mu_A(y) = \eta_A(x)$, and $\gamma_{\overline{RP}(A)}(x) = \wedge_{y\in R_s(x)}\gamma_A(y) \leq \gamma_A(x)$.

It means $A \subseteq \overline{RP}(A)$, $\forall A \in PFS(U)$, i.e. (a2) is satisfied.

Now, assume that (a1) $\underline{RP}(A) \subseteq A$, $\forall A \in PFS(U)$; we show that R is reflexive. Indeed, we assume contradiction that R is not reflexive, i.e. $x \notin R_S(x)$.

We consider $A = 1_{U-\{x\}}$, i.e. $\mu_{1_{U-\{x\}}}(y) = \begin{cases} 0 \text{ if } y = x \\ 1 \text{ if } y \neq x \end{cases}$,

$\eta_{1_{U-\{x\}}}(y) = \begin{cases} 0 \text{ if } y = x \\ 0 \text{ if } y \neq x \end{cases}$, $\gamma_{1_{U-\{x\}}}(y) = \begin{cases} 1 \text{ if } y = x \\ 0 \text{ if } y \neq x \end{cases}$.

Then $\gamma_{\underline{RP}(A)}(x) = \vee_{y\in R_s(x)}\gamma_A(y) = 0 \geq \gamma_A(x) = 1$.

This is not true. It implies R is reflexive.

Similarly, we assume that (a2) $A \subseteq \overline{RP}(A)$, $\forall A \in PFS(U)$; we show that R is reflexive. Indeed, we assume contradiction that R is not reflexive, i.e., $x \notin R_S(x)$.

We consider $A = 1_x$, i.e., $\mu_{1_x}(y) = \begin{cases} 1 \text{ if } y = x \\ 0 \text{ if } y \neq x \end{cases}$,

$\eta_{1_x}(y) = \begin{cases} 0 \text{ if } y = x \\ 0 \text{ if } y \neq x \end{cases}$, $\gamma_{1_x}(y) = \begin{cases} 0 \text{ if } y = x \\ 1 \text{ if } y \neq x \end{cases}$.

Then $\mu_{\overline{RP}(A)}(x) = \vee_{y\in R_s(x)}\mu_A(y) = 0 \geq \mu_A(x) = 1$.

This is not true. It implies R is reflexive.

(b).

We verify case (b1).

We assume that R is symmetric, i.e., if $x \in R_S(y)$ then $y \in R_S(x)$. For all $A \in PFS(U)$, because $x \in R_S(y)$ then $\wedge_{z\in R_s(y)}\mu_A(z) \leq \mu_A(x)$, $\wedge_{z\in R_s(y)}\mu_A(z) \leq \mu_A(x)$, $\vee_{z\in R_s(y)}\gamma_A(z) \geq \gamma_A(x)$ for all $y \in R_S(x)$, we have

$\mu_{\overline{RP}(\underline{RP}(A))}(x) = \vee_{y\in R_s(x)}(\wedge_{z\in R_s(y)}\mu_A(z)) \leq \mu_A(x)$,

$\eta_{\overline{RP}(\underline{RP}(A))}(x) = \vee_{y\in R_s(x)}(\wedge_{z\in R_s(y)}\eta_A(z)) \leq \eta_A(x)$; and

$\gamma_{\overline{RP}(\underline{RP}(A))}(x) = \wedge_{y\in R_s(x)}(\vee_{z\in R_s(y)}\gamma_A(z)) \geq \gamma_A(x)$.

It means that $\overline{RP}\left(\underline{RP}(A)\right) \subseteq A \quad \forall A \in PFS(U)$.

We assume contradiction that $\overline{RP}\left(\underline{RP}(A)\right) \subseteq A \quad \forall A \in PFS(U)$ but R is not symmetric, i.e., if $x \in R_S(y)$ then $y \notin R_S(x)$ and if $y \in R_S(x)$ then $x \notin R_S(y)$.

We consider $A = 1_{U-\{x\}}$. Then, $\mu_{\overline{RP}(\underline{RP}(A))}(x) = \vee_{y \in R_s(x)}(\wedge_{z \in R_s(y)} \mu_A(z)) = 1 > \mu_A(x) = 0$. It is not true, because $\mu_{\overline{RP}(\underline{RP}(A))}(x) \le \mu_A(x)$, for all $x \in U$. So that R is symmetric.

By the same way, it yields (b2).

(c). R is transitive, i.e., if for all $x, y, z \in U$: $z \in R_S(y), y \in R_S(x)$ then $z \in R_S(x)$. It means that $R_S(y) \subseteq R_S(x)$, so that for all $A \in PFS(U)$ we have $\wedge_{z \in R_s(x)} \mu_A(z) \le \wedge_{z \in R_s(y)} \mu_A(z)$.

Hence

$\wedge_{y \in R_s(x)}(\wedge_{z \in R_s(x)} \mu_A(z)) \le \wedge_{y \in R_s(x)}(\wedge_{z \in R_s(y)} \mu_A(z))$.

Because $\mu_{\underline{RP}(A)}(x) = \wedge_{y \in R_s(x)}(\wedge_{z \in R_s(x)} \mu_A(z))$

and $\mu_{\underline{RP}(\underline{RP}(A))}(x) = \wedge_{y \in R_s(x)}(\wedge_{z \in R_s(y)} \mu_A(z))$.

So $\mu_{\underline{RP}(A)}(x) \le \mu_{\underline{RP}(\underline{RP}(A))}(x)$, for all $x \in U, A \in PFS(U)$.

It mean that (c1) was varified. Now, we assume contradiction that (c1): $\overline{RP}(A) \subseteq \overline{RP}\left(\overline{RP}(A)\right) \forall A \in PFS(U)$, but R is not transitive, i.e., $x, y, z \in U$: $z \in R_S(y), y \in R_S(x)$ then $z \notin R_S(x)$. We consider $A = 1_{U-\{x\}}$, then $\mu_{\underline{RP}(A)}(x) = \wedge_{z \in R_s(x)} \mu_A(z) = 1$, but $\mu_{\underline{RP}(\underline{RP}(A))}(x) = \wedge_{y \in R_s(x)}(\wedge_{z \in R_s(y)} \mu_A(z)) = 0$.

It is false. By same way, we show that (c2) is true. Hence, (c) was verified.□

Now, according to Theorem 1, Lemma 1 and Theorem 3, we obtain the following results:

Theorem 4. Let R be a similarity crisp binary relation on U and $\overline{RP}$, $\underline{RP}$: $PFS(U) \to PFS(U)$ the upper and lower PF approximation operators. Then, for all $A \in PFS(U)$

$A = \underline{RP}(A) - \quad \overline{RP}(A) = A$

$- \quad \sim A = \underline{RP}(\sim A) \Leftrightarrow \overline{RP}(\sim A) = \sim A$.

4. The standard neutrosophic information systems

In this section, we introduce a new concept: standard neutrosophic information system.

Let (U, A, F) be a classical information system. Here U is the (nonempty) set of objects, i.e. $U = \{u_1, u_2, \dots, u_n\}$, $A = \{a_1, a_2, \dots, a_m\}$ is the attribute set, and F is the relation set of U and A, i.e. $F = \{f_j: U \to V_j, j = 1,2, \dots, m\}$, where V_j is the domain of the attribute $a_j, j = 1, 2, \dots, m$.

We call (U, A, F, D, G) an information system or decision table, where $U, A, F)$ is the classical information system, A is the condition attribute set and D is the decision attribute set, i.e. $D = \{d_1, d_2, \dots, d_p\}$ and G is the relation set of U an D, i.e. $G = \{g_j: U \to V_j', j = 1,2, \dots, p\}$ where V_j' is the domain of the attribute $d_j, j = 1, 2, \dots, p$.

Let (U, A, F, D, G) be the information system. For $B \subseteq A \cup D$, we define a relation, denoted $R_B = IND(B)$, as follows, $\forall x, y \in U$:

$xIND(B)y \Leftrightarrow f_j(x) = f_j(y)$ for all $j \in \{j: a_j \in B\}$.

The equivalence class of $x \in U$ based on R_B is $[x]_B = \{y \in U: yR_Bx\}$.

Here, we consider $R_A = IND(A), R_D = IND(D)$. If $R_A \subseteq R_D$ $R_A \subseteq R_D$, i.e., for any $[x]_A, x \in U$ there exists $[x]_D$ such that $[x]_A \subseteq [x]_D$, then the information system is called a consistent information system, other called an inconsistent information system.

Let (U, A, F, D, G) be the information system, where (U, A, F) is a classical information system.

If $D = \{D_k | k = 1,2, \dots, q\}$, where D_k is a fuzzy subset of U, then (U, A, F, D, G) is the fuzzy information system.

If $D = \{D_k | k = 1,2, \dots, q\}$ where D_k is an intutionistic fuzzy subset of U, then (U, A, F, D, G) is an intuitionistic fuzzy information system.

Definition 6. Let (U, A, F, D, G) be the information system or decision table, where (U, A, F) is a classical information system. If $D = \{D_k | k = 1,2, \dots, q\}$, where D_k is a standard neutrosophic subset of U, and G is the relation set of U and D, then (U, A, F, D, G) is called a standard neutrosophic information system.

Example 2. The following Table 2 gives a standard neutrosophic information system, where the objects set $U = \{u_1, u_2, \dots, u_{10}\}$, , the condition attribute set is $A = \{a_1, a_2, a_3\}$, and the decision attribute set is $D = \{D_1, D_2, D_3\}$, where $D_k (k = 1,2,3)$ is the standard neutrosophic subsets of U.

Table 2: *A standard neutrosophic information system*

U	a_1	a_2	a_3	D_1	D_2	D_3
u_1	3	2	1	(0.2,0,3,0.5)	(0.15,0.6,0.2)	(0.4,0.05,0.5)
u_2	1	3	2	(0.3,0.1,0.5)	(0.3,0.3,0.3)	(0.35,0.1,0.4)
u_3	3	2	1	(0.6,0,0.4)	(0.3,0.05,0.6)	(0.1,0.45,0.4)
u_4	3	3	1	(0.15,0.1,0.7)	(0.1,0.05,0.8)	(0.2,0.4,0.3)
u_5	2	2	4	(0.05,0,2,0.7)	(0.2,0.4,0.3)	(0.05,0.4,0.5)
u_6	2	3	4	(0.1,0.3,0.5)	(0.2,0.3,0.4)	(1,0,0)
u_7	1	3	2	(0.25,0.3,0.4)	(1,0,0)	(0.3,0.3,0.4)
u_8	2	2	4	(0.1,0.6,0.2)	(0.25,0.3,0.4)	(0.4,0,0.6)
u_9	3	2	1	(0.45,0,1,0.45)	(0.25,0.4,0.3)	(0.2,0.5,0.3)
u_{10}	1	3	2	(0.05,0.05,0.9)	(0.4,0.2,0.3)	(0.05,0.7,0.2)

5. The knowledge discovery in the standard neutrosophic information systems

In this section, we will give some results about the knowledge discovery for a standard neutrosophic information systems by using the basic theory of rough standard neutrosophic set in Section 3. Throughout this paper, let (U, A, F, D, G) be the standard neutrosophic information system and by $B \subseteq A$, we denote $\underline{RP}_B(D_j)$ the lower rough standard neutrosophic approximation of $D_j \in PFS(U)$ on approximation space (U, R_B).

Theorem 5. Let (U, A, F, D, G) be the standard neutrosophic information system and $B \subseteq A$. If for any $x \in U$:

$$\left(\mu_{D_i}(x), \eta_{D_i}(x), \gamma_{D_i}(x)\right) \geq \left(\alpha(x), \beta(x), \theta(x)\right)$$

$$= \underline{RP}_B(D_i)(x) > \underline{RP}_B(D_j)(x) (i \neq j),$$

then $[x]_B \cap (\sim D_j)^{\beta(x),0}_{\alpha(x)} \neq \emptyset$ $[x]_B \cap (\sim D_j)^{\beta(x),0}_{\alpha(x)} \neq \emptyset$ $[x]_B \cap \left(\sim D_j\right)^{\theta(x),0}_{\alpha(x)} \neq \varnothing$ $[x]_B \cap (\sim D_j)^{\theta(x),0}_{\alpha(x)} \neq \emptyset$ and $[x]_B \subseteq (D_i)^{\alpha(x),\beta(x)}_{\theta(x)}$ $[x]_B \cap (\sim D_j)^{\beta(x),0}_{\alpha(x)} \neq \emptyset [x]_B \subseteq (D_i)^{\alpha(x),\theta(x)}_{\beta(x)}$ $[x]_B \cap (\sim D_j)^{\beta(x),0}_{\alpha(x)} \neq \emptyset$

where $\left(\alpha(x), \beta(x), \theta(x)\right) \in D^*$.

Proof.

We have

$$\left(D_i\right)^{\alpha(x),\beta(x)}_{\theta(x)} = \{y \in U : \left(\mu_{D_i}(y), \eta_{D_i}(y), \gamma_{D_i}(y)\right)$$

$$\geq \left(\alpha(x), \beta(x), \theta(x)\right)\}.$$

Since $\left(\alpha(x), \beta(x), \theta(x)\right) = \underline{RP}_B(D_i)(x)$,

we have $\alpha(x) = \wedge_{y \in [x]_B} \mu_{D_i}(y)$, $\beta(x) = \wedge_{y \in [x]_B} \eta_{D_i}(y)$, and $\theta(x) = \vee_{y \in [x]_B} \gamma_{D_i}(y)$. So that, for any $x \in U, y \in [x]_B$ then $\mu_{D_i}(y) \geq \alpha(x)$, $\eta_{D_i}(y) \geq \theta(x)$ $\gamma_{D_i}(y) \leq \theta(x)$ and $\eta_{D_i}(y) \geq \theta(x)$. It means that $y \in \left(D_i\right)^{\alpha(x),\beta(x)}_{\theta(x)}$, i.e.,

$$[x]_B \subseteq \left(D_i\right)^{\alpha(x),\beta(x)}_{\theta(x)} [x]_B \subseteq (D_i)^{\alpha(x),\beta(x)}_{\theta(x)}$$

Now, since

$$\left(\alpha(x), \beta(x), \theta(x)\right) = \underline{RP}_B\left(D_i\right)(x) > \underline{RP}_B\left(D_j\right)(x)\left(i \neq j\right)$$

then there exists $y \in [x]_B$ such that

$$\left(\mu_{D_i}(y), \eta_{D_i}(y), \gamma_{D_i}(y)\right) < \left(\alpha(x), \beta(x), \theta(x)\right)$$

$\left(\mu_{D_i}(y), \eta_{D_i}(y), \gamma_{D_i}(y)\right) < \left(\alpha(x), \beta(x), \theta(x)\right)$,i.e., or $(\mu_{D_i}(y) < \alpha(x)$, $\gamma_{D_i}(y) \geq \theta(x))$ or $(\mu_{D_i}(y) = \alpha(x)$, $\gamma_{D_i}(y) > \theta(x))$ or $(\mu_{D_i}(y) = \alpha(x)$, $\gamma_{D_i}(y) > \theta(x))$ and $\eta_{D_i}(y) < \beta(x))$. It means that here exists $y \in [x]_B$ such that $\left(\gamma_{D_i}(y), \eta_{D_i}(y), \mu_{D_i}(y)\right) \geq \left(\theta(x), 0, \alpha(x)\right)$, i.e. $y \in (\sim D_j)^{\theta(x),0}_{\alpha(x)}$. So that $[x]_B \cap (\sim D_j)^{\theta(x),0}_{\alpha(x)} \neq \emptyset$.□

Let (U, A, F, D, G) be the standard neutrosophic information system, R_A the equivalence classes which are induced by the condition attribute set A, and the universe is divided by R_A as following: $U/R_A = \{X_1, X_2 \dots, X_k\}$. Then the approximation of the standard neutrosophic decision denoted as, for all $i = 1,2, \dots, k$

$\underline{RP}_A(D(X_i)) = (\underline{RP}_A(D_1(X_i)), \underline{RP}_A(D_2(X_i)), \ldots, \underline{RP}_A(D_q(X_i)))$

Example 3. We consider the standard neutrosophic information system in Table 2. The equivalent classes

$U/R_A = \{X_1 = \{u_1, u_3, u_9\}, X_2 = \{u_2, u_7, u_{10}\},$

$X_3 = \{u_4\}, X_4 = \{u_5, u_8\}, X_5 = \{u_6\}\}$

The approximation of the standard neutrosophic decision is as follows:

Table 3: *The approximation of the picture fuzzy decision*

U/R_A	$\underline{RP}_A(D_1(X_i))$	$\underline{RP}_A(D_2(X_i))$	$\underline{RP}_A(D_3(X_i))$
X_1	(0.2,0,0.5)	(0.15,0.05,0.6)	(0.1,0.05,0.5)
X_2	(0.05,0.05,0.9)	(0.3,0.1,0.3)	(0.05,0.1,0.4)
X_3	(0.15, 0.1,0.7)	(0.1,0.05,0.8)	(0.2,0.4,0.3)
X_4	(0.05,0.2,0.7)	(0.2,0.3,0.4)	(0.05,0,0.6)
X_5	(0.1,0.3,0.5)	(0.2,0.3,0.4)	(1,0,0)

Indeed, for $X_1 = \{u_1, u_3, u_9\}$.

We have $\forall x \in X_1$,

$\mu_{\underline{RP}_A(D_1)}(x) = \wedge_{y \in X_1} \mu_{D_1}(y) = \min\{0.2, 0.6, 0.45\} = 0.2$,

$\eta_{\underline{RP}_A(D_1)}(x) = \wedge_{y \in X_1} \eta_{D_1}(y) = \min\{0.3, 0, 0.1\} = 0$

$\gamma_{\underline{RP}_A(D_1)}(x) = \vee_{y \in X_1} \gamma_{D_1}(y) = \max\{0.5, 0.4, 0.45\} = 0.5$,

$y \in (\sim D_j)_{\alpha(x)}^{\beta(x),0}$, so that $\underline{RP}_A(D_1)(x) = (0.2, 0.5, 0)$. And

$\mu_{\underline{RP}_A(D_2)}(x) = \wedge_{y \in X_1} \mu_{D_2}(y) = \min\{0.15, 0.3, 0.25\} = 0.15$,

$\eta_{\underline{RP}_A(D_2)}(x) = \wedge_{y \in X_1} \eta_{D_2}(y) = \min\{0.6, 0.05, 0.4\} = 0.05$,

$\gamma_{\underline{RP}_A(D_2)}(x) = \vee_{y \in X_1} \gamma_{D_2}(y) = \max\{0.2, 0.6, 0.3\} = 0.6$

so $\underline{RP}_A(D_2)(x) = (0.15, 0.6, 0.05)$ and

$\mu_{\underline{RP}_A(D_3)}(x) = \wedge_{y \in X_1} \mu_{D_3}(y) = \min\{0.4, 0.1, 0.2\} = 0.1$,

$\eta_{\underline{RP}_A(D_3)}(x) = \wedge_{y \in X_1} \eta_{D_3}(y) = \min\{0.05, 0.45, 0.5\} = 0.05$,

$\gamma_{\underline{RP}_A(D_3)}(x) = \vee_{y \in X_1} \gamma_{D_3}(y) = \max\{0.5, 0.2, 03\} = 0.5$

so that $\underline{RP}_A(D_3)(x) = (0.1, 0.5, 0.05)$.

Hence, for $X_1 = \{u_1, u_3, u_9\}$, $\forall x \in X_2$, $max_{i=\{1,2,3\}} \underline{RP}_A(D_i)(x) =$

$\underline{RP}_A(D_1)(x) = (0.2, 0.5, 0)$, $\max_{i=\{1,2,3\}} \underline{RP}_A(D_i)(x) =$

and $X_1 = \{u_1, u_3, u_9\} \subseteq (D_1)_{0.5}^{0.2,0} = \{u_1, u_2, u_3, u_7, u_9\}$;

For $X_2 = \{u_2, u_7, u_{10}\}$. We have $\forall x \in X_2$,

$\max_{i=\{1,2,3\}} \underline{RP}_A(D_i)(x) = \underline{RP}_A(D_2)(x) = (0.3, 0.3, 0.1)$,

and $X_2 = \{u_2, u_7, u_{10}\} \subseteq (D_2)_{0.3}^{0.3,0.1} = \{u_2, u_7, u_{10}\}$.

For $X_3 = \{u_4\}$, we have $\forall x \in X_2$,

$\max_{i=\{1,2,3\}} \underline{RP}_A(D_i)(x) = \underline{RP}_A(D_3)(x) = (0.2, 0.3, 0.4)$,

and $X_3 = \{u_4\} \subseteq (D_2)_{0.3}^{0.3,0.1} = \{u_4, u_6, u_9\}$ $X_3 = \{u_4\} \subseteq (D_2)_{0.3}^{0.3,0.1} = \{u_4, u_6, u_9\}$.

For $X_3 = \{u_4\}$, we have $\forall x \in X_2$

$\max_{i=\{1,2,3\}} \underline{RP}_A(D_i)(x) = \underline{RP}_A(D_3)(x) = (0.2, 0.3, 0.4)$

and $X_4 = \{u_5, u_8\} \subseteq (D_2)_{0.4}^{0.2,0.3} = \{u_2, u_5, u_8, u_9, u_{10}\}$

$X_4 = \{u_5, u_8\} \subseteq (D_2)_{0.4}^{0.2,0.3} = \{u_2, u_5, u_8, u_9, u_{10}\}$.

For $X_3 = \{u_4\}$, we have $\forall x \in X_2$,

$\max_{i=\{1,2,3\}} \underline{RP}_A(D_i)(x) = \underline{RP}_A(D_3)(x) = (0.2, 0.3, 0.4)$, and $X_5 = \{u_6\} \subseteq (D_2)_0^{1,0} = \{u_6\}$.

6 The knowledge reduction and extension of standard neutrosophic information systems

Definition 7.

(i) Let (U, A, F) (U, A, F) be the classical information system and $B \subseteq A$. B is called the standard neutrosophic reduction of the classical information system (U, A, F), if B is the minimum set which satisfies the following relations: for any $X \in PFS(U), x \in U$.

$$\underline{RP}_A(X) = \underline{RP}_B(X),\ \overline{RP}_A(X) = \overline{RP}_B(X)$$

(ii) B is called the standard neutrosophic lower approximation reduction of the classical information system (U, A, F), if B is the minimum set which satisfies the following relations: for any $X \in PFS(U), x \in U$

$\underline{RP}_A(X) = \underline{RP}_B(X)$,

(iii) B is called the standard neutrosophic upper approximation reduction of the classical information system (U, A, F), if B is the minimum set which satisfies the following relations: for any $X \in PFS(U), x \in U$

$\overline{RP}_A(X) = \overline{RP}_B(X)$

where $\underline{RP}_A(X), \underline{RP}_B(X), \overline{RP}_A(X), \overline{RP}_B(X)$ $\underline{RP}_A(X), \underline{RP}_B(X), \overline{RP}_A(X), \overline{RP}_B(X)$ are standard neutrosophic lower and standard neutrosophic upper approximation sets of standard neutrosophic set $X \in PFS(U)$ based on R_A, R_B R_A, R_B, respectively.

Now, we express the knowledge of the reduction of standard neutrosophic information system by introducing the discernibility matrix.

Definition 8. Let (U, A, F, D, G) be the standard neutrosophic information system. Then $M = [D_{ij}]_{k\times k}$ where

$$D_{ij} = \begin{cases} \{a_l \in A : f_l(X_i) \neq f_l(X_j)\}; & g_{X_i}(D_t) \neq g_{X_j}(D_t) \\ A & ; g_{X_i}(D_t) = g_{X_j}(D_t) \end{cases}$$

is called the discernibility matrix of (U, A, F, D, G) (where $g_{X_i}(D_k)$ is the maximum of $\underline{RP}_A(D(X_i))$ obtained at D_t D_k, i.e., $g_{X_i}(D_t) = \underline{RP}_A(D_t(X_i))$

$= \max\{\underline{RP}_A(D_z(X_i)), z = 1,2,\ldots,q\})$ $g_{X_i}(D_k) = \underline{RP}_A(D_k(X_i)) = \max\{\underline{RP}_A(D_t(X_i)), t = 1,2,\ldots,q\})$.

Definition 9. Let (U, A, F, D, G) be the standard neutrosophic information system, for any $B \subseteq A$, if the following relations holds, for any $x \in U$:

$\underline{RP}_B(D_i)(x) > \underline{RP}_B(D_j)(x) - \underline{RP}_A(D_i)(x) > \underline{RP}_A(D_j)(x)(i \neq j)$

then B is called the consistent set of A.

Theorem 6. Let (U, A, F, D, G) be the standard neutrosophic information system. If there exists a subset $B \subseteq A$ such that $B \cap D_{ij} \neq \emptyset$, then B is the consistent set of A.

Definition 10. Let (U, A, F, D, G) be the standard neutrosophic information system

$$D_{ij}^C = \begin{cases} \{a_l \in A : f_l(X_i) = f_l(X_j)\}; & g_{X_i}(D_t) \neq g_{X_j}(D_t) \\ \emptyset & ; g_{X_i}(D_t) = g_{X_j}(D_t) \end{cases}$$

is called the discernibility matrix of (U, A, F, D, G) (where $g_{X_i}(D_k)$ is the maximum of $\underline{RP}_A(D(X_i))$ obtained at D_k, i.e.

$g_{X_i}(D_t) = \underline{RP}_A(D_t(X_i)) = \max\{\underline{RP}_A(D_z(X_i)), z = 1,2,\ldots,q\})$.

$g_{X_i}(D_k) = \underline{RP}_A(D_k(X_i)) = \max\{\underline{RP}_A(D_t(X_i)), t = 1,2,\ldots,q\})$.

Theorem 7. Let (U, A, F, D, G) be the standard neutrosophic information system. If there exists a subset $B \subseteq A$ such that $B \cap D_{ij}^C = \emptyset$, then B is the consistent set of A.

Proof. If $B \cap D_{ij}^C = \emptyset$, then $B \subseteq D_{ij}$. According to Theorem 6, B is the consistent set of A.□

The extension of a standard neutrosophic information system suggested the following definition:

Definition 11.

(i) Let (U, A, F) be the classical information system and $A \subseteq B$. B is called the standard neutrosophic extension of the classical information system (U, A, F), if B satisfies the following relations:

for any $X \in PFS(U), x \in U$

$\underline{RP}_A(X) = \underline{RP}_B(X)$, $\overline{RP}_A(X) = \overline{RP}_B(X)$

(ii) B is called the standard neutrosophic lower approximation extension of the classical information system (U, A, F), if B B satisfies the following relations:

for any $X \in PFS(U), x \in U$

$\underline{RP}_A(X) = \underline{RP}_B(X)$

(iii) B is called the standard neutrosophic upper approximation extension of the classical information system (U, A, F), if B satisfies the following relations:

for any $X \in PFS(U), x \in U$

$\overline{RP}_A(X) = \overline{RP}_B(X)$

where $\underline{RP}_A(X), \underline{RP}_B(X)$, $\overline{RP}_A(X), \overline{RP}_B(X)$ are picture fuzzy lower and upper approximation sets of standard neutrosophic set $X \in PFS(U)$ based on R_A, R_B, respectively.

We can easily obtain the following results:

Definition 12. Let (U, A, F) be the classical information system, for any hyper set B, such that $A \subseteq B$, if A is the standard neutrosophic reduction of the classical information system (U, B, F), then (U, B, F) is the standard neutrosophic extension of (U, A, F), but not conversely necessary.

Example 4. In the approximation of the standard neutrosophic decision in Table 2, Table 3. Let $B = \{a_1, a_2\}$, then we obtain the family of all equivalent classes of U based on the equivalent relation $R_B = IND(B)$ as follows:

$$U/R_B = \{X_1 = \{u_1, u_3, u_9\}, X_2 = \{u_2, u_7, u_{10}\}, X_3 = \{u_4\}, X_4 = \{u_5, u_8\}, X_5 = \{u_6\}\}$$

We can get the approximation value given in Table 4.

Table 4: *The approximation of the standard neutrosophic decision*

U / R_B	$\underline{RP}_B(D_1(X_i))$	$\underline{RP}_B(D_2(X_i))$	$\underline{RP}_B(D_3(X_i))$
X_1	(0.2,0,0.5)	(0.15,0.05,0.6)	(0.1,0.05,0.5)
X_2	(0.05,0.05,0.9)	(0.3,0.1,0.3)	(0.05,0.1,0.4)
X_3	(0.15, 0.1,0.7)	(0.1,0.05,0.8)	(0.2,0.4,0.3)
X_4	(0.05,0.2,0.7)	(0.2,0.3,0.4)	(0.05,0,0.6)
X_5	(0.1,0.3,0.5)	(0.2,0.3,0.4)	(1,0,0)

It is easy to see that B satisfies Definition 7 (ii), i.e., B is the standard neutrosophic lower reduction of the classical information system (U, A, F).

The discernibility matrix of the standard neutrosophic information system (U, A, F, D, G) will be presented in Table 5.

Table 5: *The discernibility matrix of the standard neutrosophic information system*

U/R_B	X_1	X_2	X_3	X_4	X_5
X_1	A				
X_2	A	A			
X_3	$\{a_2\}$	$\{a_1, a_3\}$	A		
X_4	$\{a_1, a_3\}$	A	A	A	
X_5	$\{a_1, a_3\}$	A	A	$\{a_2\}$	A

7 Conclusion

In this paper, we introduced the concept of standard neutrosophic information system, and studied the knowledge discovery of standard neutrosophic information system based on rough standard neutrosophic sets. We investigated some problems of the knowledge discovery of standard neutrosophic information system: the knowledge reduction and extension of the standard neutrosophic information systems.

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References

[1] Z. Pawlak, *Rough sets*, International Journal of Computer and Information Sciences, vol. 11, no.5 , pp 341 – 356, 1982.

[2] L. A. Zadeh, *Fuzzy Sets*, Information and Control, Vol. 8, No. 3 (1965), p 338-353

[3] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy set and systems, vol.20, pp.87-96, 1986.

[4] B.C. Cuong, V. Kreinovick, *Picture fuzzy sets – a new concept for computational intelligence problems*, in the Proceedings of the 3rd World congress on information and communication technologies WICT'2013, Hanoi, Vietnam, December 15-18, pp 1-6, 2013.

[5] B.C. Cuong, *Picture Fuzzy Sets*, Journal of Computer Science and Cybernetics, Vol.30, n.4, 2014, 409-420.

[6] B.C. Cuong, P.H.Phong and F. Smarandache, *Standard Neutrosophic Soft Theory: Some First Results*, Neutrosophic Sets and Systems, vol.12, 2016, pp.80-91.

[7] L.H. Son, *DPFCM: A novel distributed picture fuzzy clustering method on picture fuzzy sets*, Expert systems with applications 42, pp 51-66, 2015.

[8] P.H. Thong and L.H.Son, *Picture Fuzzy Clustering : A New Computational Intelligence Method*, Soft Computing, v.20 (9) 3544-3562, 2016.

[9] D. Dubois, H. Prade, *Rough fuzzy sets and fuzzy rough sets*, International journal of general systems, Vol. 17, p 191-209, 1990.

[10] Y.Y. Yao, *Combination of rough and fuzzy sets based on α-level sets*, Rough sets and Data mining: analysis for imprecise data, Kluwer Academic Publisher, Boston, p 301 – 321, 1997.

[11] W. Z. Wu, J. S. Mi, W. X. Zhang, *Generalized fuzzy rough sets*, Information Sciences 151, p. 263-282, 2003.

[12] W. Z. Wu, Y. H. Xu, *On fuzzy topological structures of rough fuzzy sets*, Transactions on rough sets XVI, LNCS 7736, Springer – Verlag Berlin Heidelberg, p 125-143, 2013.

[13] Y.H. Xu, W.Z. Wu, *Intuitionistic fuzzy topologies in crisp approximation spaces*, RSKT 2012, LNAI 7414, © Springer – Verlag Berlin Heidelberg, pp 496-503, 2012.

[14] B. Davvaz, M. Jafarzadeh, *Rough intuitionistic fuzzy information systems*, Fuzzy Inform., vol.4, pp 445-458, 2013.

[15] N.X. Thao, N.V. Dinh, *Rough picture fuzzy set and picture fuzzy topologies*, Science computer and Cybernetics, *Vol 31, No 3 (2015), pp 245-254.*

[16] B. Sun, Z. Gong, *Rough fuzzy set in generalized approximation space*, Fifth Int. Conf. on Fuzzy Systems and Knowledge Discovery, IEEE computer society 2008, pp 416-420.

[17] F. Smarandache, *A unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic*, ARP, Rehoboth 1999.

[18] H. Wang, F. Smarandache, Y.Q. Zhang et al., *Interval neutrosophic sets and logic: Theory and applications in computing*, Hexis, Phoenix, AZ 2005.

[19] H. Wang, F. Smarandache, Y.Q. Zhang, et al., *Single valued neutrosophic sets*, Multispace & Multistructure 4 (2010), 410-413.

[20] J. Ye, *A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets*, Journal of Intelligent & Fuzzy Systems 26 (2014) 2459-2466.
[21] P. Majumdar, *Neutrosophic sets and its applications to decision making*, Computation intelligentce for big data analysis (2015), V.19, pp 97-115.
[22] J. Peng, J. Q. Wang, J. Wang, H. Zhang, X. Chen, *Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems*, International journal of systems science (2016), V.47, issue 10, pp 2342-2358.

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Optimal Design of Truss Structures Using a Neutrosophic Number Optimization Model under an Indeterminate Environment

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Abstract. This paper defines basic operations of neutrosophic numbers and neutrosophic number functions for objective functions and constraints in optimization models. Then, we propose a general neutrosophic number optimization model for the optimal design of truss structures. The application and effectiveness of the neutrosophic number optimization method are demonstrated through the design example of a two-bar truss structure under indeterminate environment to achieve the minimum weight objective under stress and stability constraints. The comparison of the neutrosophic number optimal design method with traditional optimal design methods proves the usability and suitability of the presented neutrosophic number optimization design method under an indeterminate/neutrosophic number environment.

Keywords: Neutrosophic number, neutrosophic number function, neutrosophic number optimization model, neutrosophic number optimal solution, truss structure design.

1 Introduction

In the real-world, there is incomplete, unknown, and indeterminate information. How to express incomplete, unknown, and indeterminate information is an important problem. Hence, Smarandache [1-3] firstly introduced a concept of indeterminacy, which is denoted by the symbol "I" as the imaginary value, and defined a neutrosophic number as $N = a + bI$ for $a, b \in R$ (all real numbers), which consists of both the determinate part a and the indeter-minate part bI. So it can express determinate and/or inde-terminate information in incomplete, uncertain, and inde-terminate problems. After that, Ye [4, 5] applied neutro-sophic numbers to decision making problems. Then, Kong et al. [6] and Ye [7] applied neutrosophic numbers to fault diagnosis problems under indeterminate environments. Further, Smarandache [8] introduced an interval function (so-called neutrosophic function/thick function $g(x) = [g_1(x), g_2(x)]$ for $x \in R$) to describe indeterminate problems by the interval functions. And also, Ye et al. [9] introduced neutrosophic/interval functions of the joint roughness coef-ficient and the shear strength in rock mechanics under in-determinate environments. It is obvious that neutrosophic numbers are very suitable for the expression of determinate and/or indeterminate information. Unfortunately, existing optimization design methods [10-13] cannot express and deal with indeterminate optimization design problems of engineering structures under neutrosophic number environments. Furthermore, the Smarandache's neutrosophic function [8] cannot also express such an indeterminate function involving neutrosophic numbers. Till now, there are no concepts of neutrosophic number functions and neutrosophic number optimization designs in all existing literature. Therefore, one has to define new functions containing NNs to handle indeterminate optimization problems of engineering designs under a neutrosophic number environment. To handle this issue, this paper firstly defines a new concept of neutrosophic number functions for the neutrosophic number objective functions and constraints in engineering optimization design problems with determinate and indeterminate information, and then proposes a general neutrosophic number optimization model and a solution method to realize neutrosophic number optimization problems of truss structure design, where the obtained neutrosophic number optimal solution can satisfy the design requirements in indeterminate situations.

The remainder of this paper is structured as follows. Section 2 defines some new concepts of neutrosophic number functions to establish the neutrosophic number objective functions and constraints in indeterminate optimization design problems, and proposes a general neutrosophic number optimization model for truss structure designs. In Section 3, the neutrosophic number optimal design of a two-bar truss structure is presented under a neutrosophic number environment to illustrate the application and effectiveness of the proposed neutrosophic number optimization design method. Section 4 contains some conclusions and future research directions.

2 Neutrosophic numbers and optimization models

2.1 Some basic operations of neutrosophic numbers

It is well known that there are some indeterminate design parameters and applied forces in engineering design problems. For example, the allowable compressive stress of some metal material is given in design handbooks by a possible range between 420 MPa and 460 MPa, denoted by σ_p = [420, 460], which reveals the value of σ_p is an indeterminate range within the interval [420, 460]. Then a neutrosophic number $N = a + bI$ for $a, b \in R$ (all real numbers) can effectively express the determinate and/or indeterminate information as $N = 420 + 40I$ for $I \in [0, 1]$, where its determinate part is $a = 420$, its indeterminate part $bI = 40I$, and the symbol "I" denotes indeterminacy and belongs to the indeterminate interval [inf I, sup I] = [0, 1]. For another example, if some external force is within [2000, 2500] kN, then it can be expressed as the neutrosophic number $N = 2000 + 50I$ kN for $I\in [0, 10]$ or $N = 2000 + 5I$ kN for $I \in [0, 100]$ corresponding to some actual requirement.

It is noteworthy that there are $N = a$ for $bI = 0$ and $N = bI$ for $a = 0$ in two special cases. Clearly, the neutrosophic number can easily express its determinate and/or indeterminate information, where I is usually specified as a possible interval range [inf I, sup I] in actual applications. Therefore, neutrosophic numbers can easily and effectively express determinate and/or indeterminate information under indeterminate environments.

For convenience, let Z be all neutrosophic numbers (Z domain), then a neutrosophic number is denoted by $N = a + bI = [a + b(\inf I), a + b(\sup I)]$ for $I \in [\inf I, \sup I]$ and $N \in Z$. For any two neutrosophic numbers $N_1, N_2 \in Z$, we can define the following operations:

$$N_1 + N_2 = a_1 + a_2 + (b_1 + b_2)I = [a_1 + a_2 + b_1(\inf I) + b_2(\inf I), a_1 + a_2 + b_1(\sup I) + b_2(\sup I)]; \tag{1}$$

$$N_1 - N_2 = a_1 - a_2 + (b_1 - b_2)I = [a_1 - a_2 + b_1(\inf I) - b_2(\inf I), a_1 - a_2 + b_1(\sup I) - b_2(\sup I)]; \tag{2}$$

$$N_1 \times N_2 = a_1a_2 + (a_1b_2 + a_2b_1)I + b_1b_2I^2 = \left[\begin{array}{l} \min\begin{pmatrix}(a_1+b_1(\inf I))(a_2+b_2(\inf I)),\\ (a_1+b_1(\inf I))(a_2+b_2(\sup I)),\\ (a_1+b_1(\sup I))(a_2+b_2(\inf I)),\\ (a_1+b_1(\sup I))(a_2+b_2(\sup I))\end{pmatrix}, \\ \max\begin{pmatrix}(a_1+b_1(\inf I))(a_2+b_2(\inf I)),\\ (a_1+b_1(\inf I))(a_2+b_2(\sup I)),\\ (a_1+b_1(\sup I))(a_2+b_2(\inf I)),\\ (a_1+b_1(\sup I))(a_2+b_2(\sup I))\end{pmatrix}\end{array}\right]; \tag{3}$$

$$\frac{N_1}{N_2} = \frac{a_1+b_1I}{a_2+b_2I} = \frac{[a_1+b_1(\inf I), a_1+b_1(\sup I)]}{[a_2+b_2(\inf I), a_2+b_2(\sup I)]} = \left[\begin{array}{l} \min\begin{pmatrix}\frac{a_1+b_1(\inf I)}{a_2+b_2(\sup I)}, \frac{a_1+b_1(\inf I)}{a_2+b_2(\inf I)},\\ \frac{a_1+b_1(\sup I)}{a_2+b_2(\sup I)}, \frac{a_1+b_1(\sup I)}{a_2+b_2(\inf I)}\end{pmatrix}, \\ \max\begin{pmatrix}\frac{a_1+b_1(\inf I)}{a_2+b_2(\sup I)}, \frac{a_1+b_1(\inf I)}{a_2+b_2(\inf I)},\\ \frac{a_1+b_1(\sup I)}{a_2+b_2(\sup I)}, \frac{a_1+b_1(\sup I)}{a_2+b_2(\inf I)}\end{pmatrix}\end{array}\right]. \tag{4}$$

2.2 Neutrosophic number functions and neutrosophic number optimization model

In engineering optimal design problems, a general optimization model consists of the objective function and constrained functions. In indeterminate optimization problems of engineering designs, then, objective functions and constrained functions may contain indeterminate information. To establish an indeterminate optimization model in a neutrosophic number environment, we need to define neutrosophic number functions in Z domain.

Definition 1. A neutrosophic number function with n design variables in Z domain is defined as

$$F(\boldsymbol{X}, \boldsymbol{I}): Z^n \to Z. \tag{1}$$

where $\boldsymbol{X} = [x_1, x_2, \ldots, x_n]^T$ for $\boldsymbol{X} \in Z^n$ is a n-dimensional vector and $F(\boldsymbol{X}, I)$ is either a neutrosophic number linear function or a neutrosophic number nonlinear function.

For example, $F_1(\boldsymbol{X}, I) = (1+2I)x_1 + x_2 + (2+3)I$ for $\boldsymbol{X} = [x_1, x_2]^T \in Z^2$ is a neutrosophic number linear function, then $F_2(\boldsymbol{X}, I) = Ix_1^2 + (3+I)x_2^2$ for $\boldsymbol{X} = [x_1, x_2]^T\in Z^2$ is a neutrosophic number nonlinear function.

2.3 General neutrosophic number optimization model

Generally speaking, neutrosophic number optimization design problems with n design variables in Z domain can be defined as the general form of a neutrosophic number optimization model:

$$\begin{aligned} &\min F(\boldsymbol{X}, I) \\ &\text{s.t. } G_k(\boldsymbol{X}, I) \le 0,\ k = 1, 2, \dots, m \\ &H_j(\boldsymbol{X}, I)=0,\ j = 1, 2, \dots, s \\ &\boldsymbol{X}\in Z^n,\ I \in [\inf I, \sup I], \end{aligned} \tag{2}$$

where $F(\boldsymbol{X}, I)$ is a neutrosophic number objective function and $G_1(\boldsymbol{x})$, $G_2(\boldsymbol{x})$, …, $G_m(\boldsymbol{x})$ and $H_1(\boldsymbol{x})$, $H_2(\boldsymbol{x})$, …, $H_s(\boldsymbol{x})$: $Z^n \to Z$ are neutrosophic number inequality constraints and neutrosophic number equality constraints, respectively, for $\boldsymbol{X} \in Z^n$ and $I \in [\inf I, \sup I]$.

However, if the neutrosophic number optimal solution of design variables satisfies all these constrained conditions in a neutrosophic number optimization model, the optimal solution is feasible and otherwise is unfeasible. Generally speaking, the optimal solution of design variables and the value of the neutrosophic number objective function usually are neutrosophic numbers/interval ranges (but not always).

To solve the neutrosophic number optimization model (2), we use the Lagrangian multipliers for the neutrosophic number optimization model. Then the Lagrangian function that one minimizes is structured as the following form:

$$\begin{aligned} &L(\boldsymbol{X}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = F(\boldsymbol{X}, I) + \\ &\sum_{k=1}^{m} \mu_k G_k(\boldsymbol{X}, I) + \sum_{j=1}^{s} \lambda_j H_j(\boldsymbol{X}, I), \\ &\boldsymbol{\mu}\in Z^m,\ \boldsymbol{\lambda}\in Z^s,\ \boldsymbol{X}\in Z^n,\ I\in[\inf I, \sup I]. \end{aligned} \tag{3}$$

The common Karush-Kuhn-Tucker (KKT) necessary conditions are introduced as follows:

$$\nabla F(\boldsymbol{X}, I) + \sum_{k=1}^{m} \{\mu_k \nabla G_k(\boldsymbol{X}, I)\} + \sum_{j=1}^{s} \{\lambda_j \nabla H_j(\boldsymbol{X}, I)\} = 0 \tag{4}$$

combined with the original constraints, complementary slackness for the inequality constraints and $\mu_k \ge 0$ for k = 1, 2, …, m.

However, it may be difficult to solve neutrosophic nonlinear optimization models in indeterminate nonlinear optimization design problems, such as multiple-bar truss structure designs under neutrosophic number environments, by the Karush-Kuhn-Tucker (KKT) necessary conditions. Hence, this paper will research on the neutrosophic number optimization design problem of a simple two-bar truss structure in the following section to realize the primal investigation of the truss structure optimal design in a neutrosophic number environment.

3 Optimal design of a two-bar truss structure under a neutrosophic number environment

To demonstrate the neutrosophic number optimal design of a truss structure in an indeterminate environment, a simply two-bar truss structure is considered as an illustrative design example and showed in Fig.1. In this example, the two bars use two steel tubes with the length L, in which the wall thick is T=25mm. The optimal design is performed in a vertically external loading case. The vertical applied force is $2F = (3+0.4I)\times10^5$N, the material Young's modulus and density $E=2.1\times10^5$ MPa and ρ = 7800 kg/m^3, respectively, and the allowable compressive stress is σ_p = 420 + 40I.

The optimal design objective of the truss structure is to minimize the weight of the truss structure in satisfying the constraints of stress and stability. In this class of optimization problems, the average diameter D of the tube and the truss height H are taken into account as two design variables, denoted by the design vector $\boldsymbol{X} = [x_1, x_2]^T = [D, H]$.

Due to the geometric structure symmetry of the two-bar truss, we only consider the optimal model of one bar of both.

First, the total weight of the tube is expressed by the following formula:

$$M = 2\rho AL = 2\pi\rho T x_1 (\mathrm{B}^2 + x_2^2)^{1/2},$$

where A is the cross-sectional area $A = \pi T x_1$ and $2B$ is the distance between two supporting points.

Then, the compressive force of the steel tube is

$$F_1 = \frac{FL}{x_2} = \frac{F(B^2 + x_2^2)^{1/2}}{x_2},$$

where L is the length of the tube and F_1 is the compressive force of the tube. Thus, the compressive stress of the tube is represented as the following form:

$$\sigma = \frac{F_1}{A} = \frac{F(B^2 + x_2^2)^{1/2}}{\pi T x_1 x_2}.$$

Hence, the constrained condition of the strength for the tube is written as

$$\frac{F(B^2 + x_2^2)^{1/2}}{\pi T x_1 x_2} \le \sigma_p.$$

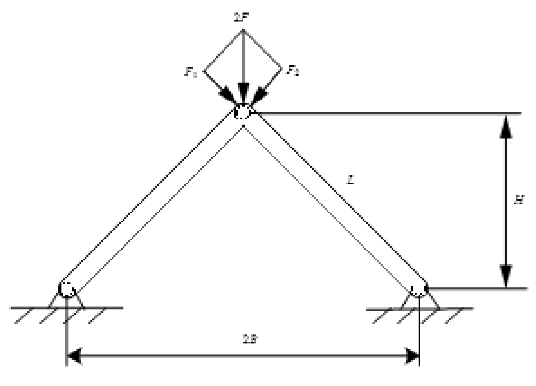

Fig. 1 Two-bar truss structure

For the stability of the compressive bar, the critical force of the tube is given as follows:

$$F_c = \frac{\pi^2 E W_I}{L^2} = \frac{\pi^2 EA(T^2 + x_1^2)}{8(B^2 + x_2^2)} ,$$

where W_I is the inertia moment of the cross-section of the tube.

The critical stress of the tube is given as

$$\sigma_c = \frac{F_c}{A} = \frac{\pi^2 E(T^2 + x_1^2)}{8(B^2 + x_2^2)} .$$

Thus, the constrained condition of the stability for the tube is written as

$$\frac{F(B^2 + x_2^2)^{1/2}}{\pi T x_1 x_2} \le \frac{\pi^2 E(T^2 + x_1^2)}{8(B^2 + x_2^2)} .$$

Finally, the neutrosophic optimization model of the truss structure can be formulated as:

$$\begin{aligned} &\min M(\boldsymbol{X}, I) = 2\pi\rho T x_1 (B^2 + x_2^2)^{1/2} \\ &s.t.\quad G_1(\boldsymbol{X}, I) = \frac{F(B^2 + x_2^2)^{1/2}}{\pi T x_1 x_2} - \sigma_p \le 0 \\ &\qquad G_2(\boldsymbol{X}, I) = \frac{F(B^2 + x_2^2)^{1/2}}{\pi T x_1 x_2} - \frac{\pi^2 E(T^2 + x_1^2)}{8(B^2 + x_2^2)} \le 0 \end{aligned} .$$

By solving the neutrosophic optimization model, the neutrosophic number optimal solution of the two design variables is given as follows:

$$X^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} \dfrac{\sqrt{2}F}{\pi T(420 + 40I)} \\ B \end{bmatrix} , = \begin{bmatrix} \dfrac{1.414(1.5 + 0.2I) \times 10^5}{7.85(420 + 40I)} \\ 760 \end{bmatrix}$$

In this case, the neutrosophic number optimal value of the objective function is obtained as follows:

$$M(X^*, I) = \frac{4\rho FB}{\sigma_p} = \frac{2371.2(1.5 + 0.2I)}{(420 + 40I)} .$$

Since there exists the indeterminacy I in these neutrosophic number optimal values, it is necessary that we discuss them when the indeterminacy I is specified as possible ranges according to actual indeterminate requirements in the actual application.

Obviously, the neutrosophic number optimization problem reveals indeterminate optimal results (usually neutrosophic number optimal solutions, but not always). If the indeterminacy I is specified as different possible ranges of I =0, $I \in [0, 1]$, $I \in [1, 3]$, $I \in [3, 5]$, $I \in [5, 7]$, and $I \in [7, 10]$ for convenient analyses, then all the results are shown in Table 1.

Table 1. Optimal results of two-bar truss structure design in different specified ranges of $I \in$ [inf I, sup I]

$I \in$ [inf I, sup I]	$D = x_1^*$ (mm)	$H = x_2^*$ (mm)	$M(\boldsymbol{X}^*, I)$ (kg)
$I = 0$	64.3312	760	8.4686
$I \in [0, 1]$	[58.7372, 72.9087]	760	[7.7322, 9.5977]
$I \in [1, 3]$	[56.7068, 82.2321]	760	[7.4649, 10.8250]
$I \in [3, 5]$	[61.0109, 83.3923]	760	[8.0315, 10.9778]
$I \in [5, 7]$	[64.3312, 84.2531]	760	[8.4686, 11.0911]
$I \in [7, 10]$	[63.7036, 90.0637]	760	[8.3860, 11.8560]

In Table 1, if I = 0, it is clear that the neutrosophic number optimization problem is degenerated to the crisp optimization problem (i.e., traditional determinate optimization problem). Then under a neutrosophic number environment, neutrosophic number optimal results are changed as the indeterminate ranges are changed. Therefore, one will take some interval range of the indeterminacy I in actual applications to satisfy actual indeterminate requirements of the truss structure design. For example, if we take the indeterminate range of $I \in [0, 1]$, then the neutrosophic number optimal solution is $D = x_1^*$ = [58.7372, 72.9087] mm and $H = x_2^*$ = 760mm. In actual design, we need the de-neutrosophication in the neutrosophic optimal solution to determinate the suitable optimal design values of the design variables to satisfy some indeterminate requirement. For example, if we take the maximum values of the optimal solution for $I \in [0, 1]$, we can obtain D = 73mm and H = 760mm for the two-bar truss structure design to satisfy this indeterminate requirement.

However, traditional optimization design methods [10-13] cannot express and handle the optimization design problems with neutrosophic number information and are special cases of the neutrosophic number optimization design method in some cases. The comparison of the proposed neutrosophic number optimization design method with traditional optimization design methods demonstrates the usability and suitability of this neutrosophic number optimization design method under a neutrosophic number environment.

4 Conclusion

Based on the concepts of neutrosophic numbers, this paper defined the operations of neutrosophic numbers and neutrosophic number functions to establish the neutrosophic number objective function and constraints in neutrosophic number optimization design problems. Then, we proposed a general neutrosophic number optimization model with constrained optimizations for truss structure design problems. Next, a two-bar truss structure design example was provided to illustrate the application and effectiveness of the proposed neutrosophic number optimization design method.

However, the indeterminate (neutrosophic number) optimization problems may contain indeterminate (neutrosophic number) optimal solutions (usually neutrosophic numbers, but not always), which can indicate possible optimal ranges of the design variables and objective function when indeterminacy *I* is specified as a possible interval ranges in actual applications.

In general, indeterminate designs usually imply indeterminate optimal solutions from an indeterminate viewpoint. Then in the de-neutrosophication satisfying actual engineering design requirements we can determinate the suitable optimal design values of design variables in the obtained optimal interval solution corresponding to designers' attitudes and/or some risk situations to be suitable for actual indeterminate requirements.

It is obvious that the neutrosophic number optimization design method in a neutrosophic number environment is more useful and more suitable than existing optimization design methods of truss structures since the traditional determinate/indeterminate optimization design methods cannot express and handle the neutrosophic number optimization design problems under an indeterminate environment. Therefore, the neutrosophic number optimization design method provides a new effective way for the optimal design of truss structures under indeterminate/neutrosophic number environments.

Nonetheless, due to existing indeterminacy "*I*" in the neutrosophic number optimization model, it may be difficult to solve complex neutrosophic number optimization models. In the future, therefore, we shall further study solving algorithms/methods for neutrosophic number optimization design problems and apply them to mechanical and civil engineering designs under indeterminate / neutrosophic number environments.

References

[1] F. Smarandache. Neutrosophy: Neutrosophic probability, set, and logic. American Research Press, Rehoboth, USA, 1998.

[2] F. Smarandache. Introduction to neutrosophic measure, neutrosophic integral, and neutrosophic probability. Sitech & Education Publisher, Craiova – Columbus, 2013.

[3] F. Smarandache. Introduction to neutrosophic statistics. Sitech & Education Publishing, 2014.

[4] J. Ye. Bidirectional projection method for multiple attribute group decision making with neutrosophic numbers. Neural Computing and Applications, (2015), DOI: 10.1007/s00521-015-2123-5.

[5] J. Ye. Multiple-attribute group decision-making method under a neutrosophic number environment. Journal of Intelligent Systems, 25(3) (2016), 377-386.

[6] L.W. Kong, Y.F. Wu, J. Ye. Misfire fault diagnosis method of gasoline engines using the cosine similarity measure of neutrosophic numbers. Neutrosophic Sets and Systems, 8 (2015), 43-46.

[7] J. Ye. Fault diagnoses of steam turbine using the exponential similarity measure of neutrosophic numbers. Journal of Intelligent & Fuzzy Systems, 30 (2016), 1927–1934.

[8] F. Smarandache. Neutrosophic precalculus and neutrosophic calculus. EuropaNova, Brussels, 2015.

[9] J. Ye, R. Yong, Q.F. Liang, M. Huang, S.G. Du. Neutrosophic functions of the joint roughness coefficient (JRC) and the shear strength: A case study from the pyroclastic rock mass in Shaoxing City, China. Mathematical Problems in Engineering, Volume 2016, (2016), 9 pages. http://dx.doi.org/10.1155/2016/4825709.

[10] M. Sonmez. Artificial bee colony algorithm for optimization of truss optimization. Applied Soft Computing, 11 (2011), 2406–2418.

[11] S.O. Degertekin. An improved harmony search algorithms for sizing optimization of truss structures. Computers and Structures, 92–93 (2012), 229–241.

[12] S. Jalili, Y. Hosseinzadeh. A cultural algorithm for optimal design of truss structures. Latin American Journal of Solids and Structures, 12(9) (2015), 1721-1747.

[13] S. Wang, X. Qing. A mixed interval arithmetic/affine arithmetic approach for robust design optimization with interval uncertainty. Journal of Mechanical Design, 138(4) (2016), 041403 -041403-10.

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