Maxwell–Schrödinger System: Well-Posedness and Standing Waves

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1 Introduction

Die Beherrschung der Arithmetik, Herr Kollege, ist keine Frage der Überheblichkeit, hätte ich gedacht.

Alexander van der Bellen,
Rede im Nationalrat am 21. April 2009

The theories of electromagnetism and quantum mechanics are cornerstones in our current understanding of the universe as observed by scientists. Within both theories, wave phenomena occur in an essential way. The description and analysis of wave phenomena in a mathematical language quite often leads to the field of dispersive partial differential equations. This active field of research is powered both by the wealth of applications coming from physics and by the mathematical curiosity seeking to apply and develop analytical tools for solving challenging problems. Accordingly, progress in the field can be as diverse as mastering a particular difficulty in a toy problem by new analytical techniques or developing and analyzing new algorithms for numerical simulations of a dispersive equation on a computer. In this field, the wave- and the Schrödinger equation are of fundamental importance and thus have attracted a lot of attention in the literature. They can be considered as model problems, and our understanding of them is representative of the state of the art of the whole field. In this thesis, we contribute to the study of a dispersive system that couples the wave- and the Schrödinger equation together.

The Maxwell–Schrödinger system

The *Maxwell–Schrödinger system* describes a charged quantum mechanical particle interacting with its self-generated electromagnetic field. Coupling the linear Schrödinger equation for a free particle with the Maxwell equation for the electromagnetic potential, one obtains the system

$$\begin{aligned} \mathrm{i}\partial_t u(t,x) + \Delta_{A(t,x)}\, u(t,x) &= \phi(t,x)u(t,x), \\ -\Delta\phi(t,x) - \partial_t \operatorname{div} A(t,x) &= \rho(t,x), \\ \partial_t^2 A(t,x) - \Delta A(t,x) + \nabla(\partial_t\phi(t,x) + \operatorname{div} A(t,x)) &= J(u(t,x), A(t,x)), \end{aligned} \tag{1.1}$$

where the time variable t belongs to some interval $I \subseteq \mathbb{R}$ and the space variable x to the full space $\mathbb{R}^3$. We are looking for solutions of the system (1.1) in terms of the *wave function* $u\colon I \times \mathbb{R}^3 \to \mathbb{C}$ and the *electromagnetic potential* $(\phi, A)\colon I \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$. The coupling between the Schrödinger and the Maxwell part of the system is realized by the *magnetic Schrödinger operator* $\Delta_A := \nabla_A^2$ (with the magnetic derivative $\nabla_A \coloneqq \nabla - \mathrm{i}A$), on the one hand, and via the *charge density* ρ given by $\rho(t,x) = |u(t,x)|^2$ and the *current density* J given by

$$J(u(t,x), A(t,x)) = 2\,\mathrm{Im}\,\bar{u}(t,x)\,\nabla_{A(t,x)}\,u(t,x),$$

on the other hand.

The Maxwell–Schrödinger system in the formulation of (1.1) is not well-posed as it does not determine solutions uniquely. The reason is that the physically observable electromagnetic field does not determine the electromagnetic potential but leaves freedom to choose a *gauge*. In fact, if (u, ϕ, A) is a solution of the system (1.1) and $\lambda\colon I \times \mathbb{R}^3 \to \mathbb{R}$ is any sufficiently regular function, then applying the *gauge transform*

$$(\tilde{u}, \tilde{\phi}, \tilde{A}) = (\mathrm{e}^{\mathrm{i}\lambda} u, \phi - \partial_t \lambda, A + \nabla\lambda)$$

yields another solution $(\tilde{u}, \tilde{\phi}, \tilde{A})$. By specifying an additional requirement for the solution, the so-called *fixing the gauge*, one can remove this ambiguity. There are many possible choices to do this. Here we choose the *Coulomb gauge* which requires that the magnetic vector potential is divergence free, i.e.

$$\operatorname{div} A(t,x) = 0, \quad t \in I, x \in \mathbb{R}^3. \tag{1.2}$$

With this choice, by the second line of (1.1), the electric potential has to satisfy the Poisson equation

$$-\Delta\phi(t,x) = \rho(t,x), \quad t \in I, x \in \mathbb{R}^3,$$

which can be solved with the fundamental solution of the Laplace equation as

$$\phi(u) = (-\Delta)^{-1}\rho = \gamma * |u|^2\,,$$

where $\gamma(x) = \frac{1}{4\pi}|x|^{-1}$. To the last equation in the system (1.1) we apply the Helmholtz projection P to obtain the *Maxwell–Schrödinger system in Coulomb gauge*

$$\begin{aligned} \mathrm{i}\partial_t u(t,x) + \Delta_{A(t,x)}\,u(t,x) &= \phi(u)(t,x)u(t,x), \\ \partial_t^2 A(t,x) - \Delta A(t,x) &= PJ(u(t,x), A(t,x)), \\ \operatorname{div} A(t,x) &= 0, \end{aligned} \tag{1.3}$$

for $t \in I$ and $x \in \mathbb{R}^3$. For $p \in (1,\infty)$, the Helmholtz projection P is a bounded linear operator on $L^p(\mathbb{R}^3, \mathbb{R}^3)$ given as a Fourier multiplier $P \coloneqq 1 - \mathcal{F}^{-1}\frac{\xi\xi^\top}{|\xi|^2}\mathcal{F}$. Its range are the divergence free vector fields in L^p. We aim to solve the Cauchy problem (1.3) in the natural state space $X^{s,\sigma}$ defined by

$$X^{s,\sigma} = \left\{(u, A, B) \in H^s(\mathbb{R}^3, \mathbb{C}) \times H^\sigma(\mathbb{R}^3, \mathbb{R}^3) \times H^{\sigma-1}(\mathbb{R}^3, \mathbb{R}^3) : \operatorname{div} A = \operatorname{div} B = 0\right\}$$

for some values $s, \sigma \in \mathbb{R}$. Here B stands for the time derivative $\partial_t A$ of the magnetic potential.

The following variant of the Maxwell–Schrödinger system (1.1) is central to our work. Introducing a *charge* $e \in \mathbb{R}$ and adding a focusing power nonlinearity $-|u|^{p-1} u$ on the right side of the magnetic Schrödinger equation with a parameter $p > 1$, we investigate

$$\begin{aligned} \mathrm{i}\partial_t u(t,x) + \Delta_{eA(t,x)} u(t,x) - e\phi(t,x)u(t,x) &= -|u(t,x)|^{p-1} u(t,x), \\ -\Delta\phi - \partial_t \operatorname{div} A(t,x) &= \frac{e}{2} |u(t,x)|^2, \\ \partial_t^2 A(t,x) - \Delta A(t,x) + \nabla(\partial_t\phi(t,x) + \operatorname{div} A(t,x)) &= \frac{e}{2} J(u(t,x), eA(t,x)), \end{aligned} \tag{1.4}$$

for $t \in I$ and $x \in \mathbb{R}^3$. As for the original Maxwell–Schrödinger system (1.1), the same remarks about gauge invariance apply to the system (1.4). In Coulomb gauge the system (1.4) becomes

$$\begin{aligned} \mathrm{i}\partial_t u(t,x) + \Delta_{eA(t,x)} u(t,x) &= e\phi(u)(t,x)u(t,x) - |u(t,x)|^p u(t,x), \\ \partial_t^2 A(t,x) - \Delta A(t,x) &= \frac{e}{2} PJ(u(t,x), A(t,x)), \\ \operatorname{div} A(t,x) &= 0, \end{aligned} \tag{1.5}$$

for $t \in I$ and $x \in \mathbb{R}^3$, where $\phi(u) = \frac{e}{2}(-\Delta)^{-1} |u|^2$. The task is to solve the Cauchy problem for the system (1.5) with given initial data (u_0, A_0, A_1) in $X^{s,\sigma}$. We refer to the article [AdM17] for a discussion about the relevance of this system as a model problem in physics.

In the special case of vanishing charge, i.e. $e = 0$, the system (2.1) is no longer coupled. If we choose $(A_0, A_1) = 0$, then Maxwell's equations admit the trivial solution $A = 0$ and $\phi = 0$. In this case, the whole problem reduces to the study of the nonlinear Schrödinger equation

$$\mathrm{i}\partial_t u + \Delta u + |u|^{p-1} u = 0, \quad u(0) = u_0. \tag{1.6}$$

The Cauchy problem for the nonlinear Schrödinger equation (1.6) has been extensively studied, and it is much better understood than the Maxwell–Schrödinger system. Many results on nonlinear Schrödinger equations are collected in the monograph [Caz03]. A particularly interesting case occurs if the nonlinearity in equation (1.6) satisfies $p \in \left(1, \frac{7}{3}\right)$. Then there exists special solutions of (1.6) which have the form of a standing wave given by

$$u(t,x) = \mathrm{e}^{\mathrm{i}t\omega}\varphi(x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3, \tag{1.7}$$

for some frequency ω and a fixed profile φ which decays at infinity. A classical result by Thierry Cazenave and Pierre-Louis Lions in [CL82] shows that these standing waves are *orbitally stable*. Informally, this means the following: Any solution whose initial value is close to a standing wave profile will stay close in a suitable topology to a standing wave for all times, at least modulo a translation and a phase shift, which are the symmetries inherent in the nonlinear Schrödinger equation (1.6). The importance of Cazenave's and Lions's seminal article [CL82] lies not in its particular application to the nonlinear Schrödinger equation. Rather, their article outlines a general method to prove orbital stability that is applicable whenever certain properties of a partial differential equation are satisfied and fit together in the correct way. These properties are global well-posedness, conservation laws, and a certain variational structure from which the wave profile φ arises.

It is natural to ask if standing wave solutions of the form (1.7) with $A = 0$ also exist in Maxwell–Schrödinger systems. With this ansatz in the system (1.3), we obtain the nonlinear elliptic problem

$$-\Delta\varphi(x) + \phi(\varphi)\varphi(x) = \omega\varphi(x), \quad x \in \mathbb{R}^3. \tag{1.8}$$

On the other hand, starting from the Maxwell–Schrödinger system (1.5) with additional power nonlinearity, we obtain

$$-\Delta\varphi(x) + e\phi(\varphi)\varphi(x) = \omega\varphi(x) + |\varphi(x)|^{p-1}\varphi(x), \quad x \in \mathbb{R}^3. \tag{1.9}$$

Related work

The Maxwell–Schrödinger system has been the subject of several works, the first of which date back more than thirty years ago. The first local well-posedness theory appears in [NT86]. Their result shows local well-posedness in the space $X^{s,\sigma}$ for $s = \sigma$, where s and σ are integers larger than 3. (We remark that they treat the Cauchy problem in the Lorentz gauge, not in the Coulomb gauge.) In [GNS95] weak solutions in the energy space $X^{1,1}$ are constructed which exist globally in time. However, their approach lacks uniqueness. We also mention the articles [Tsu93], [Shi03], [GV03], [GV06], [GV07], [GV08a], and [GV08b] in which scattering theory for the Maxwell–Schrödinger system is developed.

Two articles by Makoto Nakamura and Takeshi Wada ([NW05] and [NW07]) and one by Ioan Bejenaru and Daniel Tataru ([BT09]) give comprehensive local and global well-posedness results for (1.3). In [NW07] local well-posedness is established in the space $X^{s,\sigma}$, where the lowest regularity which can be treated by their method requires $s \geq \frac{11}{8}$ and $\sigma > 1$. Even though this misses the space $X^{1,1}$ which is naturally associated to the conserved energy of the system, they can conclude that solutions exist globally in time. The article [BT09] introduces nonstandard function spaces which are adapted to the system and proves novel Strichartz estimates for the magnetic Schrödinger equations. With these tools, the authors achieve global well-posedness in the energy space $X^{1,1}$. These results provide a satisfactory well-posedness theory for the Maxwell–Schrödinger system (1.3). Concerning the question of existence of standing waves for (1.3), it is known from the article [CG04] that there are no nontrivial solutions of (1.8) which are radially symmetric. The nonexistence of standing waves without the assumption of radial symmetry is further investigated in Chapter 5 of this thesis.

The Maxwell–Schrödinger system (1.4) with an additional power nonlinearity in the Schrödinger part is much less investigated. It is treated in the recent article [AdM17] (but with a defocusing nonlinearity, i.e $+\left|u\right|^{p-1}u$ on the right side) and in the series of related articles [CW17], [CW19a] and [CW19b] by Mathieu Colin and Tatsuya Watanabe. In [AdM17], the study of the system (1.4) is motivated by physical considerations and a connection to quantum magneto-hydrodynamic systems arising in the description of quantum plasmas in astrophysics or semiconductor devices. This is elaborated in Section 5 of the article [AdM17]. We shall report on the results achieved in [CW19b] and [CW19a] since they motivate our work and we can compare our well-posedness results with theirs. We start with the article [CW19b] which improves upon the results from [CW17]. The main concern of these articles is to solve the elliptic problem (1.9) through variational methods and to conclude orbital stability of standing waves in the spirit of [CL82]. We summarize a part of the main result from Theorems 1.1 and 1.2 in [CW19b], where we only focus on the particular case $p = 2$ which is currently best understood.

Theorem 1.1 (Orbitally stability by Colin and Watanabe)
Let $p = 2$. Let $\omega \in (0, \infty)$.

(1) There exists a constant $e_0 \in \left(0, \frac{2}{3}\right]$ such that for all $e \in (0, e_0)$ there exists a ground state φ of (1.9) which is unique up to phase shift and translations. Moreover, φ can be chosen real-valued and radially symmetric.

(2) Assume that the Cauchy problem for the Maxwell–Schrödinger system (1.5) is globally well-posed in $X^{1,1}$. Then the standing wave $(u(t), A(t)) = \left(\mathrm{e}^{\mathrm{i}\omega t}\varphi, 0\right)$, $t \in \mathbb{R}$, is orbitally stable in the following sense: For all $\varepsilon \in (0, \infty)$, there exists $\delta \in (0, \infty)$ such that the global solution $(u, A, \partial_t A) \in C(\mathbb{R}, X^{1,1})$ of (1.5) with initial values $\big(u(0), A(0), \partial_t A(0)\big) = (u_0, A_0, A_1) \in X^{1,1}$ and

$$\left\|\nabla u_0 - \mathrm{i}eA_0u_0 - \nabla\varphi\right\|_{L^2} + \left\|u_0 - \varphi\right\|_{L^2} + \left\|\operatorname{rot} A_0\right\|_{L^2} + \left\|A_1\right\|_{L^2} < \delta$$

satisfies

$$\begin{aligned}\sup_{t>0}\Bigg\{ &\inf_{y\in\mathbb{R}^3,\chi\in C^2} \left\|\nabla u(t) - \mathrm{i}eA(t)u(t) - \mathrm{e}^{\mathrm{i}e\chi(t)}\nabla\varphi(\cdot + y)\right\|_{L^2} \\ &+ \inf_{y\in\mathbb{R}^3,\chi\in C^2} \left\|u(t) - \mathrm{e}^{\mathrm{i}e\chi(t)}\varphi(\cdot + y)\right\|_{L^2} \\ &+ \inf_{\chi\in C^2} \left\|\operatorname{rot} A(t) - \nabla\chi(t)\right\|_{L^2} + \left\|\partial_t A(t) - \partial_t\nabla\chi(t)\right\|_{L^2}\Bigg\} < \varepsilon.\end{aligned}$$

It is an open problem whether the Cauchy problem for the Maxwell–Schrödinger system (1.5) is indeed globally well-posed in $X^{1,1}$, and Theorem 1.1 motivates research in this direction. As a first step, the following result is obtained in [CW19b].

Theorem 1.2 (Local well-posedness by Colin and Watanabe)
Let $s \in \left(\frac{3}{2}, \infty\right) \cap \mathbb{N}$. Let $p \in \left[7, \infty\right)$. For every $(u_0, A_0, A_1) \in X^{s+2,s+2}$, there exists $T^* > 0$ and a unique solution of the Maxwell–Schrödinger system (1.5) on $[0, T^*]$ satisfying the initial conditions and

$$u \in C\big([0,T^*], H^{s+2}(\mathbb{R}^3)\big) \cap C^1\big([0,T^*], H^{s}(\mathbb{R}^3)\big),$$
$$A \in C\big([0,T^*], H^{s+2}(\mathbb{R}^3)\big) \cap C^1\big([0,T^*], H^{s+1}(\mathbb{R}^3)\big).$$

In the introduction to [CW19b], the authors remark that “it is not clear” whether the arguments from [NW07] or [BT09] can be applied to the system (1.5). Much of the present work is motivated by the desire to investigate this particular problem.

Content and contributions of this thesis

In Chapter 2, we develop a comprehensive local well-posedness theory for the Maxwell–Schrödinger system (1.5) with power nonlinearity. Our approach is based on the methods from [NW05] and [NW07]. Our main result is stated in Theorem 2.1, and we shall compare it with Theorem 1.2. First, our result shows local well-posedness of (1.5) in $X^{s,\sigma}$ for $s \geq \frac{11}{8}$ and $\sigma > 1$. The number $\frac{11}{8}$ is the optimum which can be expected from the method of [NW07] and we emphasize that $\frac{11}{8} < \frac{3}{2}$. Therefore, the Sobolev embedding $H^s(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ which only holds true if $s > \frac{3}{2}$ is not available in our setting. This causes several complications in finding good enough estimates for nonlinear terms. We overcome them by a sophisticated use of Lemma 2.9, in which we interpolate between $H^s(\mathbb{R}^3)$ and the auxiliary space $H^{s-1/2,6}(\mathbb{R}^3)$, throughout our calculations. We remark that the desire to obtain solutions in spaces with as little regularity as possible is directly related to the quality of the result. Our method requires a lower bound $p > s$ for the exponent of the power nonlinearity which is explained in Remark 2.23. The treatment of small powers p thus requires a local well-posedness theory at low regularity. If $s < \frac{3}{2}$, we also face an upper bound on the range of p, which seems to be optimal in view of the loss of derivatives in the magnetic Schrödinger equation, see again Remark 2.23. Lemma 2.20 is a typical result showing the persistence of regularity through an argument involving Gronwall’s inequality. Second, in comparison to Theorem 1.2, we also tackle the delicate problem of continuous dependence in detail. We show with some additional effort that the involved arguments from [NW07] are robust enough to be adapted to incorporate the power nonlinearity. For this part, additional restrictions on p apply.

Chapter 3 does not contain original research but is of expository character. Its aim is to give a self-contained introduction to the recent theory of U^p- and V^p-spaces which are developed to tackle difficult problems in the field of dispersive equations. Being a rather new theory, a definitive set-in-stone textbook version does not seem available yet. We therefore explore some variants of the proofs for the results established in [HHK09] and [Koc14] and add illustrating examples. The motivation to learn this theory stems from the desire to understand [BT09] and to apply its ideas to the Maxwell–Schrödinger system in Chapter 2. This aim is pursued in Chapter 4.

In Chapter 4, we apply the theory of U^p- and V^p-spaces to several dispersive equations. We discuss three examples. The first example deals with the existence of global solutions of a 2-dimensional critical nonlinear Schrödinger equation with small initial data in $L^2(\mathbb{R}^2)$. While our result is not new, we use this as a rather simple example to demonstrate that the theory developed in Chapter 3 yields efficient and elegant proofs. In the second example, we treat a critical nonlinear Schrödinger equation with a derivative nonlinearity. This is an artificial example to highlight the power of the present method for it cannot be treated in the framework of more classical function spaces. Again, our result is not new but it was first obtained in Tobias Schottdorf's PhD-thesis [Sch13]. We present a variant of his proof following more closely the presentation of [HHK09] to get acquainted with the method. The third example discusses a nonlinear magnetic Schrödinger equation. The setting of this example is close to the Maxwell–Schrödinger system. Instead of the coupled system we only discuss the Schrödinger part where we provide a fixed (but time-dependent) magnetic field. We then proceed to show local existence of H^1-solutions by using the Strichartz estimates from [BT09]. This can be seen as a first step in the attempt to extend the method from [BT09] to the full Maxwell–Schrödinger system with additional nonlinearity.

Chapter 5 is of different flavour. Here we come back to the original Maxwell–Schrödinger system (1.3). From the article [CG04], it is known that there are no standing wave solutions of type (1.7) with a radially symmetric profile. It is an open problem, whether this holds true in full generality. Our result, however, shows the nonexistence of standing waves in a more general class than [CG04]. Instead of radial symmetry we have to assume a mild decay property. Under this assumption, we rule out the existence of standing wave solutions by making use of the method from the article [FHHH82] which was designed to show nonexistence of eigenvalues for Schrödinger operators. In our reasoning we also benefit a lot from methods which were recently developed in [AHK19].

2 A version of the Maxwell–Schrödinger system with additional nonlinearity

> These are the only ones of which the news has come to Harvard,
> And there may be many others, but they haven't been discovered.
>
> Tom Lehrer, *The Elements*

In the notation of the article [CW17], the Maxwell–Schrödinger system with an additional power nonlinearity is given by the system

$$\begin{aligned} \mathrm{i}\partial_t u + \Delta_{eA}\, u &= e\phi u - |u|^{p-1}\, u, && \text{in } I \times \mathbb{R}^3, \\ -\Delta\phi &= \frac{e}{2}\,|u|^2 + \partial_t \operatorname{div} A, && \text{in } I \times \mathbb{R}^3, \\ \partial_t^2 A - \Delta A + \nabla(\partial_t\phi + \operatorname{div} A) &= \frac{e}{2} J(u, eA), && \text{in } I \times \mathbb{R}^3. \end{aligned} \tag{2.1}$$

Here, the time variable t belongs to some interval $I \subseteq \mathbb{R}$ containing 0 and the space variable x belongs to the full space $\mathbb{R}^3$. We are looking for solutions of the Cauchy problem for the system (2.1) in terms of the *wave function* $u\colon I \times \mathbb{R}^3 \to \mathbb{C}$ and the *electromagnetic potential* $(\phi, A)\colon I \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$, where initial values $u(0) = u_0$, $A(0) = A_0$ and $\partial_t A(0) = A_1$ are prescribed. It is part of the problem to find large enough function spaces from which the initial values can be drawn to obtain a proper solution theory. The coupling between the Schrödinger and the Maxwell part of the system is realized by the *magnetic Schrödinger operator* $\Delta_A = \nabla_A^2$, where the magnetic derivative is given as $\nabla_A = \nabla - \mathrm{i}A$, on the one hand, and via the *charge density* ρ given by $\rho(t,x) = \frac{e}{2}\,|u(t,x)|^2$ and the *current density* J given by

$$J\big(u(t,x), A(t,x)\big) = 2\operatorname{Im} \bar{u}(t,x)\, \nabla_{A(t,x)}\, u(t,x),$$

on the other hand. In comparison with the Maxwell–Schrödinger system (1.1), an additional parameter, the *charge*, $e \in \mathbb{R}$ is introduced and there is a focusing power nonlinearity $-\,|u|^{p-1}\, u$ on the right side of the magnetic Schrödinger equation with a parameter $p > 1$. In this section, we develop a local well-posedness theory for the Cauchy problem associated to the system (2.1) in the Coulomb gauge.

Statement of the main result

We shall construct solutions of the Maxwell–Schrödinger system on an interval I, where $I = [0, T]$ for some $T > 0$. The second order Maxwell part of the system is treated as a first order system where we solve for the field A as well as its time derivative $\partial_t A$. We are looking for solutions which have at each moment in time values in the state space

$$X^{s,\sigma} := \big\{(u,A,B) \in H^s(\mathbb{R}^3,\mathbb{C}) \times H^\sigma(\mathbb{R}^3,\mathbb{R}^3) \times H^{\sigma-1}(\mathbb{R}^3,\mathbb{R}^3) : \operatorname{div} A = \operatorname{div} B = 0\big\},$$

where $s, \sigma \in \mathbb{R}$ are specified below. The local solution is constructed in vector-valued Lebesgue- and Sobolev spaces on the interval I. For any Banach space X, and any numbers $q \in [1,\infty]$ and $k \in \mathbb{N}$, we use the shorthand notation

$$L^q_T X := L^q(I,X) \quad \text{and} \quad W^{k,q}_T X := W^{k,q}(I,X)$$

to denote such function spaces. As in [NW07], we also introduce the spaces

$$M^{1,\sigma}_T := L^\infty_T H^\sigma(\mathbb{R}^3) \cap W^{1,\infty}_T H^{\sigma-1}(\mathbb{R}^3)$$

and

$$M^{2,\sigma}_T := L^\infty_T H^\sigma(\mathbb{R}^3) \cap W^{1,\infty}_T H^{\sigma-1}(\mathbb{R}^3) \cap W^{2,\infty}_T H^{\sigma-2}(\mathbb{R}^3),$$

in which we study solutions to the Maxwell part of the system. We collect some formulas which are helpful in the calculations. Assume in the following that $u\colon \mathbb{R}^3 \to \mathbb{C}$ and $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ are sufficiently regular and vanish at infinity sufficiently fast so that the formulas make sense and always assume that A is divergence-free, i.e. $\operatorname{div} A = 0$. We use the symbol $(\cdot,\cdot)_{L^2}$ to denote the inner product both in $L^2(\mathbb{R}^3,\mathbb{C})$, $L^2(\mathbb{R}^3,\mathbb{C}^3)$ (linear in the first, conjugate-linear in the second component) and also in $L^2(\mathbb{R}^3,\mathbb{R}^3)$. We use a dot "$\cdot$" to write a bilinear form on $\mathbb{R}^3$ or $\mathbb{C}^3$, such as in $\nabla u \cdot A = \sum_{j=1}^3 \partial_j u A_j$. Moreover, when we apply a differential operator such as ∂_j to the vector field A componentwise, we simply write $\partial_j A$ to mean $(\partial_j A_1, \partial_j A_2, \partial_j A_3)$. We have

$$\Delta_A u = (\nabla_A)^2 u = \Delta u - 2\mathrm{i} A \cdot \nabla u - (A \cdot A)u. \tag{2.2}$$

Note that $\nabla_A u$ is a skew-symmetric and Δ_A is a symmetric operator, since

$$(\nabla_A u, v)_{L^2} = \int \nabla u \bar{v} - \mathrm{i} A u \bar{v} \,\mathrm{d}x = -\int u \nabla \bar{v} - u \overline{\mathrm{i} A v} \,\mathrm{d}x = -(u, \nabla_A v)_{L^2}, \tag{2.3}$$

$$(\Delta_A u, v)_{L^2} = (u, \Delta_A v)_{L^2}. \tag{2.4}$$

There are commutator relations

$$[\partial_j, \nabla_A]u = -\mathrm{i} u (\partial_j A), \tag{2.5}$$

$$[\partial_j, \Delta_A]u = -2\mathrm{i} \nabla_A u \cdot \partial_j A. \tag{2.6}$$

If the functions $u\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ and $A\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ are time-dependent, we also obtain

$$[\partial_t, \nabla_A]u = -\mathrm{i} u (\partial_t A), \tag{2.7}$$

$$[\partial_t, \Delta_A]u = -2\mathrm{i} \nabla_A u \cdot \partial_t A. \tag{2.8}$$

Let $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a divergence free vector field, i.e. $\operatorname{div} F = 0$ and hence $PF = F$. Then the following formula involving the nonlinearity holds

$$(PJ(u,A),F)_{L^2} = 2\mathrm{i}\,(u, \nabla_A u \cdot F)_{L^2} \in \mathbb{R}. \tag{2.9}$$

In fact, we compute

$$\begin{aligned}(PJ(u,A),F)_{L^2} &= (J(u,A),PF)_{L^2} = (J(u,A),F)_{L^2} = 2\operatorname{Im}(\bar{u}\,\nabla_A u, F)_{L^2}\\ &= 2\operatorname{Im}(\nabla_A u, uF)_{L^2} = -2\operatorname{Im}(u,\nabla_A u\cdot F)_{L^2}.\end{aligned}$$

Since $\overline{(u,\nabla_A u\cdot F)_{L^2}} = (\nabla_A u \cdot F, u)_{L^2} = -(u,\nabla_A u\cdot F)_{L^2}$ is an imaginary number, the desired formula follows from $-2\operatorname{Im}(u,\nabla_A u\cdot F)_{L^2} = 2\mathrm{i}\,(u,\nabla_A u\cdot F)_{L^2}$.

As for the Maxwell–Schrödinger system (1.1), also the system (2.1) is not well-posed as it does not determine solutions uniquely. The physically observable electromagnetic field does not determine the electromagnetic potential but leaves freedom to choose a *gauge*. In fact, if (u,ϕ,A) is a solution of the system (2.1) and $\lambda\colon I\times\mathbb{R}^3 \to \mathbb{R}$ is any sufficiently regular function, then applying the *gauge transform*

$$\big(\tilde{u},\tilde{\phi},\tilde{A}\big) = (\mathrm{e}^{\mathrm{i}\lambda}u, \phi - \partial_t\lambda, A + \nabla\lambda)$$

yields another solution $(\tilde{u},\tilde{\phi},\tilde{A})$ of the system (2.1). To remedy this ambiguity, we choose the *Coulomb gauge* which requires that the magnetic vector potential is divergence free, i.e.

$$\operatorname{div} A(t,x) = 0, \quad t\in I,\ x\in\mathbb{R}^3. \tag{2.10}$$

With this choice, by the second line of (2.1), the electric potential has to satisfy the Poisson equation

$$-\Delta\phi(t,x) = \rho(t,x), \quad t\in I,\ x\in\mathbb{R}^3,$$

which can be solved with a Green's function as

$$\phi(u) = \gamma * \rho = \gamma * \tfrac{e}{2}\,|u|^2,$$

where $\gamma(x) = \frac{1}{4\pi}|x|^{-1}$ for $x\in\mathbb{R}^3\setminus\{0\}$. We apply the Helmholtz projection P to the last equation in the system (2.1) and obtain

$$\begin{aligned}\mathrm{i}\partial_t u + \Delta_{eA} u &= e\phi u - |u|^{p-1}u, && \text{in } I\times\mathbb{R}^3,\\ \partial_t^2 A - \Delta A &= \frac{e}{2}PJ(u,eA), && \text{in } I\times\mathbb{R}^3,\end{aligned}$$

where $\phi = \frac{e}{2}(-\Delta)^{-1}|u|^2$. For the following analysis, it is more convenient to multiply the second equation with e and replace eA with A. Moreover, we set $\lambda = \frac{e^2}{2}$ and we define $\phi = (-\Delta)^{-1}|u|^2$. Hence, we study

$$\begin{aligned}\mathrm{i}\partial_t u + \Delta_A u &= \lambda\phi u - |u|^{p-1}u, && \text{in } I\times\mathbb{R}^3,\\ \partial_t^2 A - \Delta A &= \lambda PJ(u,A), && \text{in } I\times\mathbb{R}^3,\end{aligned} \tag{2.11}$$

with initial values

$$u(0) = u_0, \quad A(0) = eA_0, \quad \partial_t A(0) = eA_1,$$

where A_0 and A_1 are divergence free.

The main theorem requires that the parameters s and σ which characterize the state space $X^{s,\sigma}$ fulfil some restrictions. We recall from [NW07] the notation

$$\begin{aligned} \mathcal{R}_* &= \left\{(s,\sigma) \in \mathbb{R}^2 : \sigma \geq \max\{1, s-2, \tfrac{2s-1}{4}\}, (s,\sigma) \neq (\tfrac{7}{2}, \tfrac{3}{2})\right\}, \\ \mathcal{R}^* &= \left\{(s,\sigma) \in \mathbb{R}^2 : \sigma \leq \min\{s+1, \tfrac{3}{2}s, 2s-\tfrac{3}{4}\}, (s,\sigma) \neq (2,3)\right\} \end{aligned} \tag{2.12}$$

and $\mathcal{R} = \mathcal{R}_* \cap \mathcal{R}^*$. These restrictions arise from the coupled nature of the system. When dealing with the Schrödinger part of the system in H^s then one encounters nonlinear terms such as $A \cdot \nabla u$ which can only be controlled if the parameter σ is not too small relative to s. Conversely, estimates of the nonlinear parts of the Maxwell equation in H^σ often require that s is not too small in comparison with σ. Thus, the conditions $\mathcal{R}_*$ appear in conjunction with the Schrödinger- and $\mathcal{R}^*$ with the Maxwell part.

If we study the Schrödinger part at regularity below $H^{3/2}$, we need to impose restrictions on the growth of the power nonlinearity. To formulate these restrictions, we introduce

$$p^*(s) := \frac{5-2s}{3-2s} \quad \text{and} \quad \tilde{p}^*(s) := \frac{4-2s}{3-2s} \quad \text{for } s < \frac{3}{2} \tag{2.13}$$

and we set $p^*(s) := \infty$ and $\tilde{p}^*(s) := \infty$ for all $s \geq \frac{3}{2}$.

Theorem 2.1 (Local well-posedness of the Maxwell–Schrödinger system)
Let $s \in \left[\frac{11}{8}, 2\right]$ and $\sigma \in (1,\infty)$ such that $(s,\sigma) \in \mathcal{R}$. Let $p \in \big(s, p^*(s)\big)$.

(1) For every $(u_0, A_0, A_1) \in X^{s,\sigma}$, the Maxwell–Schrödinger system (2.11) has a unique maximal solution in $C_T X^{s,\sigma}$.

(2) If additionally $s > \frac{11}{8}$, $(s+1,\sigma) \in \mathcal{R}_*$, and $p \in \big(s+1, \tilde{p}^*(s)\big)$, then the solution map $(u_0, A_0, A_1) \to (u, A, \partial_t A)$ is continuous from a neighbourhood of (u_0, A_0, A_1) in $X^{s,\sigma}$ to $C_{\tilde{T}} X^{s,\sigma}$, where $\tilde{T}$ is less than the maximal existence time of the solution $(u, A, \partial_t A)$.

In Figure 1 the parameters s, σ and p for which the statements of Theorem 2.1 hold are illustrated. The proof of Theorem 2.1 requires several preparations.

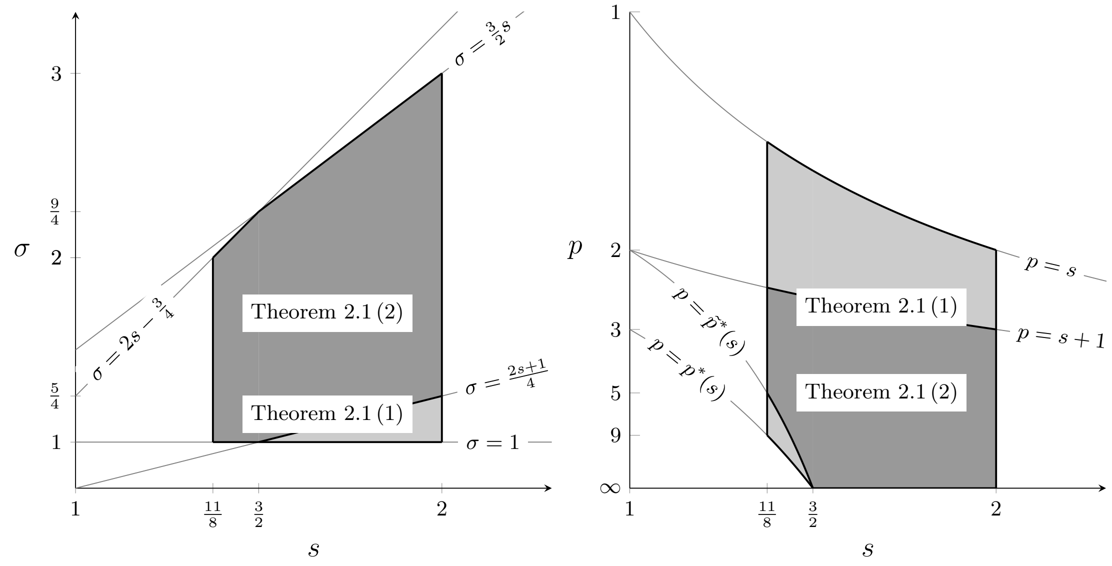

Figure 1 – The parameters s, σ and p as in Theorem 2.1.

Preliminaries

We first remark in Lemma 2.2 that the Maxwell–Schrödinger system (2.11) has an associated energy, which is conserved at least for smooth solutions. A key tool in the proof of the well-posedness result are Strichartz estimates. These estimates capture the dispersive nature of both Schrödinger and wave equations and are essential to treat nonlinear terms in the equation by providing improved integrability properties of solutions to these equations. We recall standard Strichartz estimates for the Klein–Gordon equation in Lemma 2.4 and for the Schrödinger equation in Lemma 2.6. We also prove Strichartz estimates with a so-called loss for the Schrödinger equation in Lemma 2.7, extending the ones previously given in [NW07]. These estimates allow us to work with an auxiliary function space which plays an essential role with most nonlinear estimates at low regularity. We devote Lemma 2.9 to state the range of available estimates using this auxiliary space and we conclude in Corollary 2.10 bounds for the power nonlinearity $|u|^{p-1}\,u$. Through the Lemmata 2.11 and 2.12 we establish bounds for the other nonlinear terms $PJ(u, A)$ and $\phi(u)u$ in fractional Sobolev spaces by using the fractional Leibniz rule and interpolation. Finally, we recall in Lemma 2.14 an application of Strichartz estimates with loss and in Lemma 2.15 the construction of an evolution family for the magnetic Schrödinger equation. Both of these results are contained in [NW07].

The square of the L^2-norm of the wave function is commonly referred to as the *mass* of the system. The *energy functional* associated to the system (2.11) is given by

$$\begin{aligned} E(u,A,B) = \|\nabla_A u\|_{L^2}^2 + \frac{\lambda}{2}\|\nabla\phi(u)\|_{L^2}^2 \\ + \frac{1}{2\lambda}\|\nabla A\|_{L^2}^2 + \frac{1}{2\lambda}\|B\|_{L^2}^2 - \frac{2}{p+1}\|u\|_{L^{p+1}}^{p+1}. \end{aligned} \tag{2.14}$$

We demonstrate that both quantities are constant along solution trajectories.

Lemma 2.2 (Conservation of mass and energy for regular solutions)
Let $T \in (0,\infty)$. Let $u \in C_T^1 L^2(\mathbb{R}^3) \cap C_T H^2(\mathbb{R}^3)$ and $A \in C_T^2 L^2(\mathbb{R}^3) \cap C_T^1 H^1(\mathbb{R}^3) \cap C_T H^2(\mathbb{R}^3)$ be a solution of the Maxwell–Schrödinger system (2.11) on the interval $[0,T]$. Then mass and energy are conserved quantities, that is for all $t \in [0,T]$ we have

$$\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2$$

and

$$E\big(u(t), A(t), \partial_t A(t)\big) = E(u_0, A_0, A_1).$$

Proof. Using integration by parts, the conservation of mass follows from

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|u\|_{L^2}^2 &= \operatorname{Re}(\partial_t u, u) \\ &= \operatorname{Re}\big(\mathrm{i}\,\Delta_A u - \mathrm{i}\lambda\phi(u)u + \mathrm{i}\,|u|^{p-1}u, u\big) \\ &= \operatorname{Re}\mathrm{i}\,\big(\|\nabla_A u\|_{L^2}^2 - \lambda\|\nabla\phi(u)\|_{L^2}^2 + \|u\|_{L^{p+1}}^{p+1}\big) = 0. \end{aligned}$$

Using formula (2.9) with "$F = \partial_t A$" and the commutator relation (2.7), we obtain

$$\begin{aligned} \big(PJ(u,A), \partial_t A\big)_{L^2} &= -2\operatorname{Im}(u, \nabla_A u \cdot \partial_t A)_{L^2} \\ &= -2\operatorname{Re}(-\mathrm{i}u\,\partial_t A, \nabla_A u)_{L^2} \\ &= -2\operatorname{Re}([\partial_t, \nabla_A]u, \nabla_A u)_{L^2}. \end{aligned}$$

We also compute

$$\begin{aligned} \operatorname{Re}\lambda(\partial_t\nabla\phi(u), \nabla\phi(u))_{L^2} &= -\lambda\operatorname{Re}(\partial_t\Delta\phi(u), \lambda\phi(u))_{L^2} \\ &= \lambda\operatorname{Re}\big(\partial_t|u|^2, \lambda\phi(u)\big)_{L^2} \\ &= 2\operatorname{Re}(\lambda\phi(u)u, \partial_t u)_{L^2}. \end{aligned}$$

With the help of these two preliminary computations we now deduce

$$
\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}E(u,A,\partial_t A) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\nabla_A u, \nabla_A u\right)_{L^2} + \tfrac{\lambda}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\nabla\phi(u), \nabla\phi(u)\right)_{L^2} - \tfrac{2}{p+1}\tfrac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^{p+1}}^{p+1} \\
&\quad + \tfrac{1}{2\lambda}\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\nabla A, \nabla A\right)_{L^2} + \tfrac{1}{2\lambda}\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\partial_t A, \partial_t A\right)_{L^2} \\
&= 2\operatorname{Re}\left(\partial_t \nabla_A u, \nabla_A u\right)_{L^2} + \operatorname{Re}\lambda\left(\partial_t \nabla\phi(u), \nabla\phi(u)\right)_{L^2} \\
&\quad - 2\operatorname{Re}\big(|u|^{p-1}u, \partial_t u\big)_{L^2} + \tfrac{1}{\lambda}\left(\partial_t^2 A - \Delta A, \partial_t A\right)_{L^2} \\
&= 2\operatorname{Re}\big(-\Delta_A u + \lambda\phi(u)u - |u|^{p-1}u, \partial_t u\big)_{L^2} \\
&\quad + 2\operatorname{Re}\left([\partial_t, \nabla_A]u, \nabla_A u\right)_{L^2} + \big(PJ(u,A), \partial_t A\big)_{L^2} \\
&= 2\operatorname{Re}\mathrm{i}\,\|\partial_t u\|_{L^2}^2 = 0. \qquad \square
\end{aligned}
$$

In the next lemmata we state various types of Strichartz estimates which are an important tool in the study of dispersive equations such as Klein–Gordon, Schrödinger and wave equations. Such kind of estimates were first established by Robert S. Strichartz in [Str77b], where he interpreted estimates on the restriction of the Fourier transform to curved surfaces as space-time decay (which means lying in $L^p(\mathbb{R}^{d+1})$) of solutions to linear dispersive equations posed on $\mathbb{R}\times\mathbb{R}^d$. These type of estimates were subsequently generalized in many directions and all have in common that they show improved integrability properties of solutions to linear dispersive equations. We start by taking a short look at the Klein–Gordon equation. Let $\mathcal{X} = H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$ and consider the Klein–Gordon operator

$$\mathcal{A}(A,B) = (B, \Delta A - A)$$

with domain

$$\operatorname{dom}(\mathcal{A}) = \left\{(A,B)\in\mathcal{X} : \Delta A \in L^2(\mathbb{R}^3), B\in H^1(\mathbb{R}^3)\right\}.$$

The operator $\mathcal{A}$ is skew-adjoint, see e.g. Proposition 2.6.9 in [CH98], and hence generates a unitary C_0-group $\mathcal{T}$ on $\mathcal{X}$. Let $(A_0, A_1)\in\mathcal{X}$ and let $A\colon \mathbb{R}\to H^1(\mathbb{R}^3)$ be the first component of $t\mapsto \mathcal{T}(t)(A_0,A_1)$. Then $A\in C\big(\mathbb{R}, H^1(\mathbb{R}^3)\big)\cap C^1\big(\mathbb{R}, L^2(\mathbb{R}^3)\big)\cap C^2\big(\mathbb{R}, H^{-1}(\mathbb{R}^3)\big)$ is a solution of the homogeneous Klein–Gordon equation

$$\partial_t^2 A - \Delta A + A = 0$$

with initial values $A(0) = A_0$ and $\partial_t A(0) = A_1$, see Proposition 3.5.11 in [CH98]. The parameters in the following Strichartz estimates for the Klein–Gordon equation satisfy a certain relation which we introduce in the next definition.

Definition 2.3 (Klein–Gordon admissible pair)
A pair of numbers (q,r) is called *Klein–Gordon admissible* if

$$\frac{2}{q} = 1 - \frac{2}{r} \quad \text{and} \quad 2 < q \leq \infty.$$

For reference we quote the following Strichartz estimates for the Klein–Gordon equation from Lemma 2.3 in [NW07].

Lemma 2.4 (Strichartz estimates for the Klein–Gordon equation)
Let I be an interval with $0 \in I$ and let $\sigma \in \mathbb{R}$. Let (q, r) and $(\tilde{q}, \tilde{r})$ be Klein–Gordon admissible pairs. Let $(A_0, A_1) \in H^\sigma(\mathbb{R}^3) \times H^{\sigma-1}(\mathbb{R}^3)$ and $f \in L^{\tilde{q}'}\big(I, H^{\sigma-1+2/\tilde{q},\tilde{r}'}(\mathbb{R}^3)\big)$. Then a solution A of the Klein–Gordon equation

$$\partial_t^2 A - \Delta A + A = f$$

with $A(0) = A_0$ and $\partial_t A(0) = A_1$ belongs to $C\big(I, H^\sigma(\mathbb{R}^3)\big) \cap C^1\big(I, H^{\sigma-1}(\mathbb{R}^3)\big)$ and satisfies the estimate

$$\max_{k=0,1} \|\partial_t^k A\|_{L^q(I,H^{\sigma-k-2/q,r})} \leq C \left(\|(A_0, A_1)\|_{H^\sigma \times H^{\sigma-1}} + \|f\|_{L^{\tilde{q}'}(I,H^{\sigma-1+2/\tilde{q},\tilde{r}'})} \right).$$

Note that the pair $(\infty, 2)$ is admissible and that this is the only pair for which one does not lose "$-2/q$-derivatives" in the estimate.

We next state Strichartz estimates for the Schrödinger equation. We recall that the Schrödinger operator $\mathrm{i}\Delta$ on $\mathbb{R}^3$ turns into the multiplication operator $-\mathrm{i}\xi^2$ by applying the Fourier transform. Therefore, the free Schrödinger group S acting on $u_0 \in \mathcal{S}(\mathbb{R}^d)$ is given by the formula

$$\mathcal{F}(S(t)u_0)(\xi) = \mathrm{e}^{-\mathrm{i}t\xi^2} \mathcal{F}u_0(\xi), \quad \text{for all } t \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^3.$$

This formula extends to any Sobolev space $H^s(\mathbb{R}^3)$. The scaling property of the homogeneous Schrödinger equation gives rise to the following admissibility condition.

Definition 2.5 (Schrödinger admissible pair)
A pair of numbers (q, r) is called *Schrödinger admissible* if

$$\frac{2}{q} = \frac{3}{2} - \frac{3}{r} \quad \text{and} \quad 2 \leq q, r \leq \infty. \tag{2.15}$$

We reproduce the following Strichartz estimates in fractional Besov spaces stated in Corollary 2.3.9 in [Caz03].

Lemma 2.6 (Strichartz estimates for the Schrödinger equation)
Let I be an interval and let $t_0 \in \bar{I}$. Let $s \in \mathbb{R}$ and let (q, r) and $(\tilde{q}, \tilde{r})$ be two Schrödinger admissible pairs satisfying the admissibility condition (2.15). Moreover, let $u_0 \in H^s(\mathbb{R}^3)$ and $f \in L^{\tilde{q}'}\big(I, B^s_{\tilde{r}',2}(\mathbb{R}^3)\big)$. Define

$$u(t) := S(t)u_0 + \mathrm{i} \int_{t_0}^t S(t-\tau) f(\tau) \,\mathrm{d}\tau, \quad \text{for } t \in I.$$

Then the function u belongs to $L^q\big(I, B^s_{r,2}(\mathbb{R}^3)\big)$ and satisfies the Strichartz estimate

$$\|u\|_{L^q(I,B^s_{r,2})} \lesssim \|u(t_0)\|_{H^s} + \|f\|_{L^{\tilde{q}'}(I,B^s_{\tilde{r}',2})}\,, \tag{2.16}$$

where the constant in (2.16) is independent of the time interval I.

In the following lemma we prove another type of Strichartz estimates for the Schrödinger equation which feature a so-called "loss". These estimates are useful to treat less-regular inhomogeneous terms and will be in our case crucial to deal with the term $A \cdot \nabla u$ in the magnetic Laplacian regarded as a perturbation of the free Schrödinger equation. As a special case, the estimate in the form of (2.17) was proved in [NW07], Lemma 2.4. By extending the proof in [NW07], we are more flexible in treating different time and space regularities in the inhomogeneity. The parameter α in Lemma 2.7 describes the "loss" in the estimates, meaning that the spatial regularity which is controlled on the left side is less than the spatial regularity which enters on the right, see the Sobolev spaces $H^{s-\alpha,r}$ vs. H^s. The idea of the proof is to use a spectrally localized estimates which are obtained from a Littlewood–Paley decomposition and a partition of the time interval which is adapted to the frequency decomposition. Such an approach has been successfully used to prove Strichartz estimates with (fractional) loss of derivatives in various situations, notably in [BGT04] for nonlinear Schrödinger equations on manifolds and in [KT03] for the Benjamin–Ono equation.

Lemma 2.7 (Strichartz estimates with loss of derivatives)
Let $T \in (0,\infty)$, $s \in \mathbb{R}$, and $\alpha \in (0,\infty)$. Let (q,r), and $(\tilde{q},\tilde{r})$ be two Schrödinger admissible pairs satisfying the admissibility condition (2.15). Moreover, let $\beta \in [\tilde{q}', 2]$ and $f \in L^\beta\big([0,T], B^{s-(3-2/\tilde{q}-2\beta)\alpha}_{\tilde{r}',\beta}(\mathbb{R}^3)\big)$. Let

$$u \in L^\infty\big([0,T], H^s(\mathbb{R}^3)\big) \cap W^{1,\infty}\big([0,T], H^{s-2}(\mathbb{R}^3)\big)$$

be a solution of the inhomogeneous Schrödinger equation

$$\mathrm{i}\partial_t u(t) + \Delta u(t) = f(t), \quad t \in [0,T].$$

Then u belongs to $L^q\big([0,T], H^{s-\alpha,r}(\mathbb{R}^3)\big)$ and satisfies

$$\|u\|_{L^q_T H^{s-\alpha,r}} \le C\Big(\|u\|_{L^\infty_T H^s} + T^{1-1/\tilde{q}-1/\beta}\,\|f\|_{L^\beta_T B^{s-(3-2/\tilde{q}-2/\beta)\alpha}_{\tilde{r}',\beta}}\Big).$$

We point out two special cases of this estimate. In both cases we consider the Strichartz pair $(\tilde{q},\tilde{r}) = (\infty, 2)$. Thus, we may take $\beta \in [1,2]$. If $\beta = 2$, then we obtain

$$\|u\|_{L^q_T H^{s-\alpha,r}} \le C\Big(\|u\|_{L^\infty_T H^s} + T^{1/2}\,\|f\|_{L^2_T H^{s-2\alpha}}\Big), \tag{2.17}$$

since $B^{s-2\alpha}_{2,2}(\mathbb{R}^3) = H^{s-2\alpha}(\mathbb{R}^3)$. This is the estimate already obtained in [NW07], Lemma 2.4. On the other hand, if we take $\beta = 1$, then we obtain

$$\|u\|_{L^q_T H^{s-\alpha,r}} \leq C\Big(\|u\|_{L^\infty_T H^s} + \|f\|_{L^1_T B^{s-\alpha}_{2,1}}\Big). \tag{2.18}$$

Proof. We prove the case $s = 0$ only. This simplification is only for notational convenience to shorten some long terms and does not affect any of the arguments. Let $u = \sum_{j=0}^\infty u_j$ and $f = \sum_{j=0}^\infty f_j$ be Littlewood–Paley decompositions of u and f, where we define the Littlewood–Paley blocks $u_j = \Delta_j(u)$ and $f_j = \Delta_j(f)$ for $j \in \mathbb{N}_0$ as in Definition A.5. Fix $j \in \mathbb{N}_0$. Then u_j is a solution of the inhomogeneous Schrödinger equation

$$\mathrm{i}\partial_t u_j(t) + \Delta u_j(t) = f_j(t). \tag{2.19}$$

Let I denote the interval $[0, T]$. We choose a partition of I into disjoint intervals with the following properties. There is $m_j \in \mathbb{N}$ and there are m_j intervals I^j_k, such that $I = \bigcup_{k=1}^{m_j} I^j_k$ and the interval lengths satisfy

$$2^{-2\alpha j}T \leq \left|I^j_k\right| \leq 2^{-2\alpha j+1}T$$

for $k \in \{1, \dots, m_j\}$. Furthermore, for each $k \in \{1, \dots, m_j\}$ we take a point t^j_k such that t^j_k belongs to the closure of I^j_k and $\|u_j\|_{L^2}$ has a minimum on I^j_k in the point t^j_k. From the partition of I into smaller intervals, we obtain

$$\|u_j\|_{L^q(I,L^r)} = \left(\sum_{k=1}^{m_j} \int_{I^j_k} \|u_j(t)\|^q_{L^r}\, \mathrm{d}t\right)^{1/q} = \left(\sum_{k=1}^{m_j} \|u_j\|^q_{L^q(I^j_k,L^r)}\right)^{1/q}. \tag{2.20}$$

The standard Strichartz estimate applied to the Schrödinger equation (2.19), see Lemma 2.6, yields

$$\|u_j\|_{L^q(I^j_k,L^r)} \leq C\big\|u_j\big(t^j_k\big)\big\|_{L^2} + C\,\|f_j\|_{L^{\tilde{q}'}(I^j_k,L^{\tilde{r}'})}.$$

Applying this Strichartz estimate to each summand in (2.20), using the triangle inequality and the embeddings $\ell^\beta \hookrightarrow \ell^2 \hookrightarrow \ell^q$ due to $\beta \leq 2 \leq q$, we obtain

$$\begin{aligned}\|u_j\|_{L^q(I,L^r)} &\leq \left(\sum_{k=1}^{m_j} \|u_j\|^q_{L^q(I^j_k,L^r)}\right)^{1/q} \\ &\lesssim \left(\sum_{k=1}^{m_j} \big\|u_j(t^j_k)\big\|^2_{L^2}\right)^{1/2} + \left(\sum_{k=1}^{m_j} \|f_j\|^\beta_{L^{\tilde{q}'}(I^j_k,L^{\tilde{r}'})}\right)^{1/\beta}.\end{aligned}$$

We next use the specific construction of the partition to put the pieces back together and obtain an upper bound only involving the functions on the whole interval I. For the first part of the sum we use the lower bound $1 \le T^{-1} 2^{2\alpha j} \big|I_k^j\big|$ and the minimality of $\big\|u_j(t_k^j)\big\|_{L^2}$ to get

$$\left(\sum_{k=1}^{m_j} \big\|u_j(t_k^j)\big\|_{L^2}^2\right)^{1/2} \le \left(\sum_{k=1}^{m_j} T^{-1} 2^{2\alpha j} \big|I_k^j\big| \big\|u_j(t_k^j)\big\|_{L^2}^2\right)^{1/2} \le T^{-1/2} \big\|2^{\alpha j} u_j\big\|_{L^2(I,L^2)}.$$

For the second part we use Hölder's inequality, $\tilde{q}' \le \beta$ and the upper bound $\big|I_k^j\big| \le 2^{-2\alpha j+1} T$ to obtain

$$\begin{aligned}\left(\sum_{k=1}^{m_j} \|f_j\|_{L^{\tilde{q}'}(I_k^j, L^{\tilde{r}'})}^{\beta}\right)^{1/\beta} &\le \left(\sum_{k=1}^{m_j} \big|I_k^j\big|^{\beta(1/\tilde{q}'-1/\beta)} \|f_j\|_{L^\beta(I_k^j, L^{\tilde{r}'})}^{\beta}\right)^{1/\beta} \\ &\le (2T)^{1/\tilde{q}'-1/\beta} \big\|2^{-2\alpha j(1/\tilde{q}'-1/\beta)} f_j\big\|_{L^\beta(I,L^{\tilde{r}'})}.\end{aligned}$$

Multiplying with $2^{-\alpha j}$, we obtain altogether

$$\begin{aligned}\big\|2^{-\alpha j} u_j\big\|_{L^q(I,L^r)} \lesssim{}& T^{-1/2} \|u_j\|_{L^2(I,L^2)} \\ &+ T^{1-1/\tilde{q}-1/\beta} \big\|2^{-2\alpha j(1-1/\tilde{q}-1/\beta)-\alpha j} f_j\big\|_{L^\beta(I,L^{\tilde{r}'})}.\end{aligned} \tag{2.21}$$

To conclude the computation, it remains to relate the estimates (2.21) on each Littlewood–Paley block to the functions u and f. We need the following results from Littlewood–Paley theory. We use that the Sobolev space $H^{-\alpha,r}(\mathbb{R}^3)$ is isomorphic to the Triebel–Lizorkin space $F_{r,2}^{-\alpha}(\mathbb{R}^3)$, see Theorem A.8. Since $r \ge 2$ we next use the embedding $B_{r,2}^{-\alpha}(\mathbb{R}^3) \hookrightarrow F_{r,2}^{-\alpha}(\mathbb{R}^3)$, see Theorem A.9 (2). Since $q \ge 2$ we finally apply Minkowski's integral inequality and arrive at

$$\begin{aligned}\|u\|_{L_T^q H^{-\alpha,r}} &\lesssim \|u\|_{L_T^q F_{r,2}^{-\alpha}} \\ &\lesssim \|u\|_{L_T^q B_{r,2}^{-\alpha}} \\ &= \left(\int_I \left(\sum_{j=0}^{\infty} 2^{-2\alpha j} \|u_j(t)\|_{L^r}^2\right)^{q/2} \mathrm{d}t\right)^{1/q} \\ &\le \left(\sum_{j=0}^{\infty} \left(\int_I 2^{-q\alpha j} \|u_j(t)\|_{L^r}^q \,\mathrm{d}t\right)^{q/2}\right)^{1/2} \\ &= \left(\sum_{j=0}^{\infty} \big\|2^{-\alpha j} u_j\big\|_{L_T^q L^r}^2\right)^{1/2}.\end{aligned}$$

We now apply (2.21) for each summand and arrive at

$$\left(\sum_{j=0}^{\infty}\left\|2^{-\alpha j}u_j\right\|^2_{L^q_T L^r}\right)^{1/2} \lesssim T^{-1/2}\left(\sum_{j=0}^{\infty}\|u_j\|^2_{L^2_T L^2}\right)^{1/2} + T^{1-1/\tilde{q}-1/\beta}\left(\sum_{j=0}^{\infty}\left\|2^{-2\alpha j(1-1/\tilde{q}-1/\beta)-\alpha j}f_j\right\|^2_{L^\beta_T L^{\tilde{r}'}}\right)^{1/2}.$$

From these terms, we can conclude the desired assertion. Concerning the first term, we compute

$$\begin{aligned} T^{-1/2}\left(\sum_{j=0}^{\infty}\|u_j\|^2_{L^2_T L^2}\right)^{1/2} &= T^{-1/2}\left(\sum_{j=0}^{\infty}\int_I \|u_j(t)\|^2_{L^2}\,\mathrm{d}t\right)^{1/2} \\ &= T^{-1/2}\left(\int_I \|u(t)\|^2_{B^0_{2,2}}\,\mathrm{d}t\right)^{1/2} \\ &\lesssim \|u\|_{L^\infty_T L^2}, \end{aligned}$$

where, in the last step, we use Hölder's inequality and the mutual isomorphy of the spaces $B^0_{2,2}(\mathbb{R}^3)$, $F^0_{2,2}(\mathbb{R}^3)$ and $L^2(\mathbb{R}^3)$, see again Theorems A.8 and A.9 as above. We treat the second term similarly. By the embedding $\ell^\beta \hookrightarrow \ell^2$ due to $\beta \le 2$, the second term is less than

$$\begin{aligned} &T^{1-1/\tilde{q}-1/\beta}\left(\sum_{j=0}^{\infty}\left\|2^{-2\alpha j(1-1/\tilde{q}-1/\beta)-\alpha j}f_j\right\|^\beta_{L^\beta_T L^{\tilde{r}'}}\right)^{1/\beta} \\ &= T^{1-1/\tilde{q}-1/\beta}\left(\int_I \|f\|^\beta_{B^{-2\alpha(1-1/\tilde{q}-1/\beta)-\alpha}_{\tilde{r}',\beta}}\,\mathrm{d}t\right)^{1/\beta} \\ &= T^{1-1/\tilde{q}-1/\beta}\,\|f\|_{L^\beta_T B^{-2\alpha(1-1/\tilde{q}-1/\beta)-\alpha}_{\tilde{r}',\beta}}. \end{aligned}$$

This proves the assertion.

As we mentioned in the statement of the lemma, the special case $\beta = 2$ and the fact $\tilde{r}' \le 2$ allows to use the embedding $H^{s,\tilde{r}'}(\mathbb{R}^3) \cong F^s_{\tilde{r}',2}(\mathbb{R}^3) \hookrightarrow B^s_{\tilde{r}',2}(\mathbb{R}^3)$, for which we refer again to Theorems A.8 and A.9 cited above. This shows in particular (2.17). Otherwise, the final estimate involves some Besov space-norm of the function f. □

In the next lemmata we prepare the estimates which we need to treat the power nonlinearity. In Lemma 2.8 we investigate the smoothness properties of the map $z \mapsto |z|^{p-1} z$ and we calculate explicitly its derivatives in order to derive pointwise estimates.

Lemma 2.8 (Pointwise bounds for differences of the power nonlinearity)
Let $p \in [1, \infty)$. Let $u, \tilde{u} \in C_c^\infty(\mathbb{R} \times \mathbb{R}^3, \mathbb{C})$. Define $g \colon \mathbb{C} \to \mathbb{C}$, $g(z) = |z|^{p-1} z$.

(1) The function g is real differentiable. If $p > 2$, g is two times real differentiable.

(2) We have

$$\left| |u|^{p-1} u - |\tilde{u}|^{p-1} \tilde{u} \right| \lesssim \left(|u|^{p-1} + |\tilde{u}|^{p-1} \right) |u - \tilde{u}| . \tag{2.22}$$

(3) We have

$$\left| \partial_j |u|^{p-1} u \right| \lesssim |u|^{p-1} |\partial_j u| \tag{2.23}$$

and if $p > 2$, we even have

$$\begin{aligned} &\left| \partial_j \left(|u|^{p-1} u - |\tilde{u}|^{p-1} \tilde{u} \right) \right| \\ &\quad \lesssim |u|^{p-1} |\partial_j (u - \tilde{u})| + \left(|u|^{p-2} + |\tilde{u}|^{p-2} \right) |\partial_j \tilde{u}| \, |u - \tilde{u}| \end{aligned} \tag{2.24}$$

for $j \in \{0, 1, 2, 3\}$.

Proof. In the case $p = 1$ all assertions are evident, so we assume $p > 1$ throughout the proof. To compute the real derivatives of g most efficiently, we use the Wirtinger calculus for the operators $\partial_z = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$ and $\bar{\partial}_z = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$. In this way, we can use the product- and the chain rule and we note that for any function $h \colon \mathbb{R}^d \to \mathbb{C}$ we have $\partial_j h = 2\partial_{z_j} h$. For any $\alpha > 0$, it is convenient to write $|z|^\alpha = (z\bar{z})^{\alpha/2}$. By the chain rule, we obtain

$$\partial_z |z|^\alpha = \tfrac{\alpha}{4}(z\bar{z})^{\frac{\alpha-2}{2}} \bar{z} + \tfrac{\alpha}{4}(z\bar{z})^{\frac{\alpha-2}{2}} \bar{z} = \tfrac{\alpha}{2} |z|^{\alpha-2} \bar{z} \quad \text{and} \quad \bar{\partial}_z |z|^\alpha = \overline{\partial_z |z|^\alpha} = \tfrac{\alpha}{2} |z|^{\alpha-2} z.$$

Using the product rule, we infer that

$$\partial_z g(z) = \tfrac{p-1}{2} |z|^{p-3} |z|^2 + |z|^{p-1} = \tfrac{p+1}{2} |z|^{p-1} \quad \text{and} \quad \bar{\partial}_z g(z) = \tfrac{p-1}{2} |z|^{p-3} z^2. \tag{2.25}$$

This shows real differentiability of g on $\mathbb{C} \setminus \{0\}$. Looking at the difference quotient, we directly see that g is even complex differentiable in 0 with $g'(0) = 0$. Hence, the formulas in (2.25) also hold true for $z = 0$. We obtain the estimate

$$\max \left\{ |\partial_z g(z)| , |\bar{\partial}_z g(z)| \right\} \lesssim |z|^{p-1} \quad \text{for all } z \in \mathbb{C}. \tag{2.26}$$

With the formulas above, it is easy to compute the second derivatives of g. We obtain

$$\begin{aligned} \partial_z^2 g(z) &= \tfrac{p^2-1}{4} |z|^{p-3} \bar{z}, & \bar{\partial}_z \partial_z g(z) &= \tfrac{p^2-1}{4} |z|^{p-3} z \\ \partial_z \bar{\partial}_z g(z) &= \tfrac{(p-1)(p-3)}{2} |z|^{p-3} z + (p-1) |z|^{p-3} z, & \bar{\partial}_z^2 g(z) &= \tfrac{(p-1)(p-3)}{4} |z|^{p-5} z^3. \end{aligned}$$

Again, if $p > 2$ the difference quotient reveals that $\partial_z g$ and $\bar{\partial}_z g$ are complex differentiable in 0 with derivative 0. Here we obtain the estimate

$$\max \left\{ \left| \partial_z^2 g(z) \right| , |\bar{\partial}_z \partial_z g(z)|, |\partial_z \bar{\partial}_z g(z)|, |\bar{\partial}_z^2 g(z)| \right\} \lesssim |z|^{p-2} , \quad \text{for all } z \in \mathbb{C}. \tag{2.27}$$

Let $z, w \in \mathbb{C}$. Define $h\colon [0,1] \to \mathbb{C}$, $h(\tau) = \tau z + (1-\tau)w$. We note that $h'(\tau) = z - w$ and that $|h(\tau)|^\alpha \lesssim |z|^\alpha + |w|^\alpha$ for all $\tau \in [0,1]$ and any $\alpha > 0$. This, the fundamental theorem of calculus and (2.26) yield

$$\begin{aligned} |g(z) - g(w)| &= \left| \int_0^1 (\partial_z g)\big(h(\tau)\big)(z-w) + (\bar{\partial}_z g)\big(h(\tau)\big)(\bar{z} - \bar{w})\, \mathrm{d}\tau \right| \\ &\lesssim \left(|z|^{p-1} + |w|^{p-1}\right) |z - w| \,. \end{aligned} \tag{2.28}$$

Hence, estimate (2.22) is proved.

Let $j \in \{0,1,2,3\}$. The chain rule yields

$$\partial_j (g \circ u) = (\partial_z g \circ u) \partial_j u + (\bar{\partial}_z g \circ u) \overline{\partial_j u}.$$

Using (2.26), we obtain (2.23). We further obtain

$$\begin{aligned} |\partial_j (g \circ u) - \partial_j (g \circ \tilde{u})| \leq{}& |\partial_z g \circ u| \, |\partial_j u - \partial_j \tilde{u}| + |\partial_z g \circ u - \partial_z g \circ \tilde{u}| \, |\partial_j \tilde{u}| \\ &+ \left|\bar{\partial}_z g \circ u\right| \left|\overline{\partial_j u} - \overline{\partial_j \tilde{u}}\right| + \left|\bar{\partial}_z g \circ u - \bar{\partial}_z g \circ \tilde{u}\right| \left|\overline{\partial_j \tilde{u}}\right| . \end{aligned}$$

From this estimate, we can deduce the assertion by using again (2.26) and by applying (2.28) to the functions $\partial_z g$ and $\bar{\partial}_z g$ which yields together with (2.27) the estimate

$$\max\left\{ |\partial_z g \circ u - \partial_z g \circ \tilde{u}| \,, \left|\bar{\partial}_z g \circ u - \bar{\partial}_z g \circ \tilde{u}\right| \right\} \lesssim \left(|u|^{p-2} + |\tilde{u}|^{p-2}\right) |u - \tilde{u}| \,. \quad \square$$

The key problem we face with the nonlinearity is the following. If we start with a function $u \in H^s(\mathbb{R}^3)$, then we can only ensure that $|u|^{p-1} u$ belongs to the space $H^s(\mathbb{R}^3)$ if we can control the L^∞-norm of u, cf. estimate (2.33) below. If we had $s > \frac{3}{2}$, we could simply use the Sobolev embedding $H^s(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$. For the case $s \leq \frac{3}{2}$, we must use additional information on u. We shall later see that thanks to the Strichartz estimates with loss from the previous Lemma 2.7 we can work with the auxiliary space $L_T^2 H^{s-1/2,6}(\mathbb{R}^3)$. If $s > 1$, we then take advantage of the Sobolev embedding $H^{s-1/2,6}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$. However, we cannot simply estimate each L^∞-norm by the $H^{s-1/2,6}$-norm. The reason is that we must also take into account the time integrability. A typical term which we later encounter is $\left\| |u|^{p-1} \right\|_{L_T^1 L^\infty}$. The naive estimate gives

$$\left\| |u|^{p-1} \right\|_{L_T^1 L^\infty} \lesssim \int_0^T \|u(t)\|_{H^{s-1/2,6}}^{p-1} \,\mathrm{d}t \leq T^{(3-p)/2} \, \|u\|_{L_T^2 H^{s-1/2,6}}^{p-1} \,,$$

which requires that $p < 3$. This is an unacceptable low upper bound for p. By interpolation between the spaces $H^s(\mathbb{R}^3)$ and $H^{s-1/2,6}(\mathbb{R}^3)$, aiming to shift as much weight as possible on the former space, we can considerably increase the range of p which we can handle by this method. Since terms of this type appear in several variants below, we collect all the available estimates in Lemma 2.9.

Lemma 2.9 (Interpolation with the auxiliary space)
Let $s \in [0, \infty)$. Let $u \in H^s(\mathbb{R}^3) \cap H^{s-1/2,6}(\mathbb{R}^3)$. Let $\alpha \in (0, \infty)$ and $\beta \in [1, \infty]$ such that $\alpha\beta \geq 2$. If $\beta = \infty$, this condition is satisfied for any α, and one has to read $\frac{1}{\infty} = 0$ in the formulas below. Define

$$\gamma(\alpha, \beta, s) \coloneqq \alpha(3 - 2s) - \frac{6}{\beta}.$$

If $s > 1 - \frac{3}{\alpha\beta}$ and $s > \frac{3}{4} - \frac{3}{2\alpha\beta}$, then $|u|^\alpha$ belongs to $L^\beta(\mathbb{R}^3)$ and for every $\gamma \in [0, \alpha]$ which satisfies $\gamma(\alpha, \beta, s) < \gamma < \min\{2\alpha s, \frac{3}{2}\alpha - \frac{3}{\beta}\}$, we have the estimate

$$\left\| |u|^\alpha \right\|_{L^\beta} \lesssim \|u\|_{H^s}^{\alpha-\gamma} \|u\|_{H^{s-1/2,6}}^{\gamma} . \tag{2.29}$$

If $\beta \geq 2$, $s > 2 - \frac{3}{\beta}$, and $s > \frac{7}{4} - \frac{3}{2\beta}$, then ∇u belongs to $L^\beta(\mathbb{R}^3)$ and for every $\gamma \in [0, 1]$ which satisfies $\gamma(1, \beta, s-1) < \gamma < \min\{2(s-1), \frac{3}{2} - \frac{3}{\beta}\}$, we have

$$\|\nabla u\|_{L^\beta} \lesssim \|u\|_{H^s}^{1-\gamma} \|u\|_{H^{s-1/2,6}}^{\gamma} . \tag{2.30}$$

In the applications of the estimates above, we are usually interested to take γ as small as possible. If $s > \frac{3}{2}$, we observe that $\gamma(\alpha, \beta, s) < 0$ and we may choose $\gamma = 0$. If $s \leq \frac{3}{2}$, we note that

$$\gamma(\alpha, \beta, s) < 1 \quad \text{if} \quad \alpha < \frac{1 + 6/\beta}{3 - 2s} \quad \text{and} \quad \gamma(\alpha, \beta, s) < 2 \quad \text{if} \quad \alpha < \frac{2 + 6/\beta}{3 - 2s}. \tag{2.31}$$

Proof. Let $\theta \in [0, 1]$. We have the interpolation result

$$\left[H^{s-1/2,6}(\mathbb{R}^3), H^s(\mathbb{R}^3)\right]_\theta = H^{s-\theta/2, 6/(3-2\theta)}(\mathbb{R}^3).$$

If $2s > \theta$, $\frac{6}{3-2\theta} < \alpha\beta$, and $s + \frac{1}{2}\theta - \frac{3}{2} > -\frac{3}{\alpha\beta}$, we also have the Sobolev embedding

$$H^{s-\theta/2, 6/(3-2\theta)}(\mathbb{R}^3) \hookrightarrow L^{\alpha\beta}(\mathbb{R}^3).$$

To use both results, we must find $\theta \in [0, 1]$ which satisfies $\theta < \min\{2s, \frac{3}{2} - \frac{3}{\alpha\beta}\}$ and $\theta > 3 - 2s - \frac{6}{\alpha\beta}$. We can find such θ under the stated assumptions in the lemma and we thus obtain

$$\left\| |u|^\alpha \right\|_{L^\beta} = \|u\|_{L^{\alpha\beta}}^\alpha \lesssim \|u\|_{H^{s-\theta/2, 6/(3-2\theta)}}^\alpha \lesssim \|u\|_{H^s}^{\alpha - \alpha\theta} \|u\|_{H^{s-1/2,6}}^{\alpha\theta} .$$

By setting $\gamma = \alpha\theta$, we obtain (2.29). We deduce (2.30) by applying (2.29) with $\alpha = 1$ and $s - 1$ to the function ∇u. □

Corollary 2.10 (An upper bound for the power nonlinearity)
Let $p \in (1, \infty)$ and $s \in \big(1, p\big)$. Let $u \in H^s(\mathbb{R}^3) \cap H^{s-1/2,6}(\mathbb{R}^3)$. Then $|u|^{p-1} u$ belongs to $H^s(\mathbb{R}^3)$ with the bound

$$\big\| |u|^{p-1} u \big\|_{H^s} \lesssim \|u\|_{H^s}^{p-\gamma} \|u\|_{H^{s-1/2,6}}^{\gamma} , \tag{2.32}$$

where $\gamma \in \big(\gamma(p-1, \infty, s), p-1\big]$ as in Lemma 2.9. If $s > \frac{3}{2}$, we may choose $\gamma = 0$, and if $s \leq \frac{3}{2}$, we may choose

$$\gamma < 1 \quad \text{if} \quad p < \frac{4-2s}{3-2s} \qquad \text{and} \qquad \gamma < 2 \quad \text{if} \quad p < \frac{5-2s}{3-2s}.$$

Proof. By Theorem 5.4.3/1 from [RS96], we have

$$\big\| |u|^{p-1} u \big\|_{H^s} \lesssim \|u\|_{H^s} \|u\|_{L^\infty}^{p-1} . \tag{2.33}$$

The assertion thus follows from (2.29) and (2.31) in Lemma 2.9. □

The nonlinearity in the Maxwell part of the system is given by the current density J. In the next lemma, we repeat the statement and the proof of Lemma 2.6 in [NW07] which gives upper bounds on this term which are sufficient for the fixed-point argument. The conditions on the parameters s and σ are given in (2.12). Note that every condition stated in the set $\mathcal{R}^*$ enters into the proof of Lemma 2.11. In particular, the fact that $H^{1/2,6}(\mathbb{R}^3)$ is not embedded in $L^\infty(\mathbb{R}^3)$ is the reason why the point $(s, \sigma) = (2, 3)$ is excluded from the set $\mathcal{R}^*$.

Lemma 2.11 (An upper bound for the current density)
Let $s \in [\frac{11}{8}, \infty)$ and $\sigma \in [1, \infty)$ such that $(s, \sigma) \in \mathcal{R}^*$. Let $u \in L^\infty\big([0,T], H^s(\mathbb{R}^3)\big) \cap L^2\big([0,T], H^{s-1/2,6}(\mathbb{R}^3)\big)$ and $A \in L^\infty\big([0,T], H^\sigma(\mathbb{R}^3)\big)$. Then $PJ(u, A)$ belongs to the space $L^1\big([0,T], H^{\sigma-1}(\mathbb{R}^3)\big)$ and satisfies the bound

$$\|PJ(u,A)\|_{L^1_T H^{\sigma-1}} \lesssim T^{1/4} \langle T \rangle^{1/4} \big\langle \|A\|_{L^\infty_T H^\sigma} \big\rangle \|u\|^2_{L^\infty_T H^s \cap L^2_T H^{s-1/2,6}} . \tag{2.34}$$

Proof. In the first part of the proof we neglect the dependence on the time variable and prove estimates for $PJ(u, A)$ in $H^{\sigma-1}(\mathbb{R}^3)$ under the assumption that $u \in H^s(\mathbb{R}^3) \cap H^{s-1/2,6}(\mathbb{R}^3)$ and $A \in H^\sigma(\mathbb{R}^3)$. We recall that the definition of the current density J and the boundedness of the Helmholtz projection on $H^{\sigma-1}(\mathbb{R}^3)$ imply

$$\|PJ(u,A)\|_{H^{\sigma-1}} \leq 2 \|P\bar{u}\nabla u\|_{H^{\sigma-1}} + 2 \|\bar{u}Au\|_{H^{\sigma-1}} . \tag{2.35}$$

We start to estimate the first summand. Using the Kato–Ponce commutator estimate from the Appendix of [KP88] and that the Helmholtz projection vanishes on a gradient field, Lemma 2.5 in [NW07] shows that

$$\|P\bar{u}\nabla u\|_{H^{\sigma-1}} \lesssim \|u\|_{H^{\sigma-1,p_1}} \|\nabla u\|_{L^{p_2}} \tag{2.36}$$

for any $p_1 \in (1,\infty)$ and $p_2 \in (1,\infty]$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$. We choose p_1 and p_2 depending on s, where we distinguish three cases. We use Theorem A.10 for interpolation.

We first consider the case $s \in [\frac{11}{8}, \frac{3}{2})$. In this case, we set $p_1 = \frac{3}{2s-2}$ and $p_2 = \left(\frac{1}{2} - \frac{1}{p_1}\right)^{-1} = \frac{6}{7-4s}$. Note that $p_1 \in (3,4]$ and $p_2 \in [4,6)$. Interpolation with parameter $\theta_1 = 3 - 2s$, which belongs to $(\frac{1}{4}, 1]$, shows that

$$\left[H^{s-1}(\mathbb{R}^3), H^{s-3/2,6}(\mathbb{R}^3)\right]_{\theta_1} = L^{p_2}(\mathbb{R}^3).$$

Interpolation with parameter $\theta_2 = 2s - \frac{5}{2}$, which belongs to $[\frac{1}{4}, \frac{1}{2})$, also shows that

$$\left[H^{s}(\mathbb{R}^3), H^{s-1/2,6}(\mathbb{R}^3)\right]_{\theta_2} \hookrightarrow H^{\sigma-1,p_1}(\mathbb{R}^3).$$

Here we note that $\frac{\theta_2}{2} + \frac{1-\theta_2}{6} = \frac{2s-2}{3} = \frac{1}{p_1}$ and we use that the condition $2s - \frac{3}{4} \geq \sigma - 1$ is equivalent to $\theta_2 s + (1-\theta_2)(s - \frac{1}{2}) \geq \sigma - 1$. From (2.36), we thus obtain

$$\|P\bar{u}\nabla u\|_{H^{\sigma-1}} \lesssim \|u\|_{H^s}^{\theta_1+\theta_2} \|u\|_{H^{s-1/2,6}}^{2-\theta_1-\theta_2} = \|u\|_{H^s}^{1/2} \|u\|_{H^{s-1/2,6}}^{3/2}.$$

The next case is $s \in [\frac{3}{2}, 2)$. Here we set $p_1 = \frac{6}{2s-1}$ and $p_2 = \left(\frac{1}{2} - \frac{1}{p_1}\right)^{-1} = \frac{3}{2-s}$. Note that $p_1 \in (2,3]$ and $p_2 \in [6,\infty)$. Since $s - \frac{3}{2} - \frac{3}{6} = s - 2 = -\frac{3}{p_2}$, we have the Sobolev embedding $H^{s-3/2,6}(\mathbb{R}^3) \hookrightarrow L^{p_2}(\mathbb{R}^3)$. Interpolation with parameter $\theta_3 = s - 1$, which belongs to $[\frac{1}{2}, 1)$, further yields

$$\left[H^{s}(\mathbb{R}^3), H^{s-1/2,6}(\mathbb{R}^3)\right]_{\theta_3} \hookrightarrow H^{\sigma-1,p_1}(\mathbb{R}^3).$$

Here we note that $\frac{\theta_3}{2} + \frac{1-\theta_3}{6} = \frac{2s-1}{6} = \frac{1}{p_1}$ and we use that the condition $\frac{3}{2}s \geq \sigma$ is equivalent to $\theta_3 s + (1-\theta_3)(s - \frac{1}{2}) \geq \sigma - 1$. From (2.36), we next obtain

$$\|P\bar{u}\nabla u\|_{H^{\sigma-1}} \lesssim \|u\|_{H^s}^{\theta_3} \|u\|_{H^{s-1/2,6}}^{2-\theta_3} = \|u\|_{H^s}^{s-1} \|u\|_{H^{s-1/2,6}}^{3-s}.$$

We now discuss the case $s = 2$. Since $(s,\sigma) \in \mathcal{R}^*$ by assumption, we have $\sigma < 3$. Consequently, there exists $\varepsilon \in (0,1)$, such that $\sigma \leq 3 - \frac{3}{2}\varepsilon$. We set $p_1 = \frac{2}{1-\varepsilon}$ and $p_2 = (1 - \frac{1}{p_1})^{-1} = \frac{2}{\varepsilon}$. Using the Sobolev embeddings $H^s(\mathbb{R}^3) \hookrightarrow H^{\sigma-1,p_1}(\mathbb{R}^3)$ and $H^{s-3/2,6}(\mathbb{R}^3) \hookrightarrow L^{p_2}(\mathbb{R}^3)$, we deduce from (2.36) that

$$\|P\bar{u}\nabla u\|_{H^{\sigma-1}} \lesssim \|u\|_{H^{\sigma-1,p_1}} \|\nabla u\|_{L^{p_2}} \lesssim \|u\|_{H^s} \|u\|_{H^{s-1/2,6}}.$$

Finally, we consider the case $s \in (2,\infty)$. Here we set $p_1 = 2$ and $p_2 = \infty$. Then (2.36) and the Sobolev embedding $H^{s-3/2,6}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ yield

$$\|P\bar{u}\nabla u\|_{H^{\sigma-1}} \lesssim \|u\|_{H^{\sigma-1}} \|\nabla u\|_{L^\infty} \lesssim \|u\|_{H^s} \|u\|_{H^{s-1/2,6}},$$

where we use the condition $s \geq \sigma - 1$ in the second step. This finishes the discussion of the first summand in (2.35).

We consider the second summand in (2.35). Here we distinguish two cases. The first case is $\sigma \in [1, \frac{3}{2}]$. The fractional Leibniz rule from Theorem A.11 and the Sobolev embeddings $H^\sigma(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, $H^s(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, $H^{s-1/2,6}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ and $H^s(\mathbb{R}^3) \hookrightarrow H^{\sigma-1,3}(\mathbb{R}^3)$ show that

$$\begin{aligned}\|\bar{u}Au\|_{H^{\sigma-1}} &\lesssim \|A\|_{L^6} \|u\|_{H^{\sigma-1,3}} \|u\|_{L^\infty} + \|A\|_{H^{\sigma-1,6}} \|u\|_{L^6} \|u\|_{L^6} \\ &\lesssim \|A\|_{H^\sigma} \|u\|_{H^s} \|u\|_{H^{s-1/2,6}}\end{aligned}$$

In the other case we have $\sigma \in (\frac{3}{2}, \infty)$. Here we combine the fractional Leibniz rule with the Sobolev embedding $H^\sigma(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ to obtain

$$\begin{aligned}\|\bar{u}Au\|_{H^{\sigma-1}} &\lesssim \|A\|_{L^\infty} \|u\|_{H^{\sigma-1}} \|u\|_{L^\infty} + \|A\|_{H^{\sigma-1,6}} \|u\|_{L^6} \|u\|_{L^6} \\ &\lesssim \|A\|_{H^\sigma} \|u\|_{H^s} \|u\|_{H^{s-1/2,6}} ,\end{aligned}$$

which is, overall, the same estimate as in the first case. Here we use the condition $s \geq \sigma - 1$ in the second step.

To deduce assertion (2.34), we combine all the above estimates with Hölder's inequality applied to the time integration. As an example, we consider the case where $s \in [\frac{11}{8}, \frac{3}{2})$. We compute

$$\begin{aligned}&\|PJ(u,A)\|_{L^1_T H^{\sigma-1}} \\ &\quad = \int_0^T \|PJ\big(u(t), A(t)\big)\|_{H^{\sigma-1}} \,\mathrm{d}t \\ &\quad \lesssim \int_0^T \|P\bar{u}(t)\nabla u(t)\|_{H^{\sigma-1}} + \|\bar{u}(t)A(t)u(t)\|_{H^{\sigma-1}} \,\mathrm{d}t \\ &\quad \lesssim T^{1/4} \|u\|^{1/2}_{L^\infty_T H^s} \|u\|^{3/2}_{L^2_T H^{s-1/2,6}} + T^{1/2} \|A\|_{L^\infty_T H^\sigma} \|u\|_{L^\infty_T H^s} \|u\|_{L^2_T H^{s-1/2,6}} \\ &\quad \lesssim T^{1/4} \langle T\rangle^{1/4} \big\langle \|A\|_{L^\infty_T H^\sigma} \big\rangle \|u\|^2_{L^\infty_T H^s \cap L^2_T H^{s-1/2,6}} .\end{aligned}$$

The other cases are treated similarly. □

Yet another nonlinear term in the Maxwell–Schrödinger system is the electric potential ϕ given by $\phi = (-\Delta)^{-1} |u^2|$. To estimate this term in fractional Sobolev spaces, we use the following lemma which is based on the Hardy–Littlewood–Sobolev inequality combined with the fractional Leibniz rule and Sobolev embeddings.

Lemma 2.12 (An upper bound for the electric potential)

(1) Let $q \in (2,3)$ and set $r = \frac{3q}{6-2q}$. (Conversely $q = \frac{6r}{3+2r}$.) For every $u \in L^q(\mathbb{R}^3)$, the electric potential $\phi(u) = (-\Delta)^{-1}(|u|^2)$ belongs to $L^r(\mathbb{R}^3)$ and satisfies

$$\|\phi(u)\|_{L^r} \lesssim \|u\|^2_{L^q} .$$

(2) Let $q \in (2,6)$ and set $\tilde{r} = \frac{3q}{6-q}$. (Conversely $q = \frac{6r}{3+\tilde{r}}$.) For every $u \in L^q(\mathbb{R}^3)$, the electric potential $\phi(u) = (-\Delta)^{-1}(|u|^2)$ has a distributional derivative, $\nabla\phi(u)$ belongs to $L^{\tilde{r}}(\mathbb{R}^3)$ and satisfies

$$\|\nabla\phi(u)\|_{L^{\tilde{r}}} \lesssim \|u\|_{L^q}^2 .$$

(3) For every $u \in H^1(\mathbb{R}^3)$, the electric potential $\phi(u) = (-\Delta)^{-1}(|u|^2)$ is bounded and satisfies

$$\|\phi(u)\|_{L^\infty} \lesssim \|u\|_{H^1}^2 .$$

(4) Let $s, s_1, s_2, s_3 \in \mathbb{R}$ satisfy the conditions

(i) $0 \le s \le s_3$,
(ii) $\max\{s-2, 0\} \le \min\{s_1, s_2\}$,
(iii) $s_1 + s_2 > 0$,
(iv) $s + 1 \le \min\{s_1 + s_2 + s_3, \frac{3}{2}\}$.

If $s_j = \frac{3}{2}$ for some $j \in \{1,2,3\}$ or if $s = s_3 < \frac{3}{2}$, then condition (iv) should hold with strict inequality. Let $u_j \in H^{s_j}(\mathbb{R}^3)$ for $j \in \{1,2,3\}$. Then $(-\Delta)^{-1}(u_1 u_2)u_3$ belongs to the space $H^s(\mathbb{R}^3)$ and is bounded by

$$\|(-\Delta)^{-1}(u_1 u_2)u_3\|_{H^s} \lesssim \prod_{j=1}^{3} \|u_j\|_{H^{s_j}} .$$

Proof. For the first and second assertion, we note that the solution of the Poisson equation is given by

$$\phi(u)(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|} \, \mathrm{d}y, \quad x \in \mathbb{R}^3$$

and its gradient is given by

$$\nabla\phi(u)(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|u(y)|^2 (x-y)}{|x-y|^3} \, \mathrm{d}y, \quad x \in \mathbb{R}^3, \tag{2.37}$$

see Theorem 6.21 in [LL01]. We deduce both assertions from the Hardy–Littlewood–Sobolev or the weak Young inequality, see e.g. Theorem 4.3 in [LL01]. We recall that for any $p \in (1,\infty)$ the weak L^p-space $L^p_w(\mathbb{R}^3)$ consists of all measurable functions f such that $\sup_{t>0} t \, |\{x : |f(x)| > t\}|^{1/p} < \infty$. The function $x \mapsto |x|^{-1}$ belongs to $L^3_w(\mathbb{R}^3)$ and the function $x \mapsto x\,|x|^{-3}$ belongs to $L^{3/2}_w(\mathbb{R}^3)$. Since $1 + \frac{1}{r} = \frac{1}{3} + \frac{2}{q}$, we obtain

$$\|\phi(u)\|_{L^r} \lesssim \left\||u|^2\right\|_{L^{q/2}} \le \|u\|_{L^q}^2 .$$

Moreover, we have $1 + \frac{1}{\tilde{r}} = \frac{2}{3} + \frac{2}{q}$. Hence, we also obtain

$$\|\nabla \phi(u)\|_{L^{\tilde{r}}} \lesssim \left\| |u|^2 \right\|_{L^{q/2}} \leq \|u\|_{L^q}^2 .$$

The third assertion follows from the following direct computation. Using Hölder's inequality and the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, we obtain

$$\begin{aligned}
\|\phi(u)\|_{L^\infty} &\lesssim \left(\sup_{x\in\mathbb{R}^3} \int_{\{|y-x|<1\}} \frac{|u(y)|^2}{|x-y|} \,\mathrm{d}y + \int_{\{|y-x|\geq 1\}} \frac{|u(y)|^2}{|x-y|} \,\mathrm{d}y \right) \\
&\leq \sup_{x\in\mathbb{R}^3} \left(\int_{\{|y-x|<1\}} |x-y|^{-3/2} \,\mathrm{d}y \right)^{2/3} \|u\|_{L^6}^2 + \|u\|_{L^2}^2 \\
&\lesssim \|u\|_{H^1}^2 .
\end{aligned}$$

The fourth assertion is proved in Lemma 2.1 in [NW05]. The proof essentially uses the fractional Leibniz rule, Sobolev embeddings and again the Hardy–Littlewood–Sobolev inequality. □

We collect several facts about solutions of the magnetic Schrödinger equation. We study solutions of

$$\mathrm{i}\partial_t v + \Delta_A v = \lambda \phi(u) v + f, \tag{2.38}$$

with initial value $v(0) = u_0$, where the functions A, u and f are given. In the following we solve (2.38) under the following assumptions. For some $\sigma > 1$, the function $A\colon I \times \mathbb{R}^3 \to \mathbb{R}^3$ satisfies

$$A \in M_T^{1,\sigma} \cap L_T^2 L^\infty(\mathbb{R}^3), \quad \operatorname{div} A = 0,$$

and the potential $\phi(u) = (-\Delta)^{-1} |u|^2$ is defined for some function $u\colon I \times \mathbb{R}^3 \to \mathbb{C}$ with

$$u \in L_T^\infty H^1(\mathbb{R}^3).$$

For the inhomogeneity in the magnetic Schrödinger equation (2.38), we require at least that

$$f \in L_T^\infty H^{s-2}(\mathbb{R}^3).$$

We introduce the same solution concepts as in [NW07], namely we distinguish between weak and strong H^s-valued solutions of the magnetic Schrödinger equation (2.38).

Definition 2.13 (Weak and strong solutions of the magnetic Schrödinger equation) Let $s \in [0, 2]$. A function $v\colon I \times \mathbb{R}^3 \to \mathbb{C}$ is called a *weak H^s-solution* of (2.38), if $v \in L_T^\infty H^s(\mathbb{R}^3) \cap W_T^{1,\infty} H^{s-2}(\mathbb{R}^3)$ satisfies the magnetic Schrödinger equation (2.38). The function v is called a *strong H^s-solution*, if $v \in C_T H^s(\mathbb{R}^3) \cap C_T^1 H^{s-2}(\mathbb{R}^3)$ satisfies the magnetic Schrödinger equation (2.38).

If we regard the magnetic Schrödinger equation as a perturbation of the free Schrödinger equation, we encounter the term $-2\mathrm{i}A \cdot \nabla v$ which turns out to be the most challenging. The standard Strichartz estimates such as in Lemma 2.6 are not particularly useful to treat this term since they only yield upper bounds which require more regularity than the function to be estimated. This is an obstacle for any fixed-point argument. To overcome this difficulty, we next present an application of the Strichartz estimates with loss from Lemma 2.7 to the magnetic Schrödinger equation. This result was obtained in Lemma 3.1 of [NW07].

Lemma 2.14 (Application of Strichartz estimates with loss to magnetic Schrödinger equations)
Let $s \in [0,\infty)$, $\sigma \in (1,\infty)$ such that $\sigma \geq s-1$ and $T \in (0,\infty)$. Let $A \in L^\infty_T H^\sigma(\mathbb{R}^3) \cap L^2_T L^\infty(\mathbb{R}^3)$ be divergence free. Let $u \in L^\infty_T H^{\max\{s-1,1\}}(\mathbb{R}^3)$. Let $f \in L^2_T H^{s-1}(\mathbb{R}^3)$. Let v be a weak H^s-solution of the magnetic Schrödinger equation

$$\mathrm{i}\partial_t v + \Delta_A v = \lambda\phi(u)v + f$$

on the interval $[0,T]$. Then v belongs to the space $L^2_T H^{s-1/2,6}(\mathbb{R}^3)$ and there exists a constant $m \in (0,\infty)$ such that the function v satisfies the estimate

$$\begin{aligned}\|v\|_{L^2_T H^{s-1/2,6}} \lesssim \langle T\rangle^m \left\langle \max\left\{\|A\|_{L^\infty_T H^\sigma \cap L^2_T L^\infty}, \langle\lambda\rangle \|u\|_{L^\infty_T H^{\max\{s-1,1\}}}\right\}\right\rangle^m \|v\|_{L^\infty_T H^s} \\ + T^{1/2}\|f\|_{L^2_T H^{s-1}}.\end{aligned}$$

Proof. We apply the Strichartz estimate with loss (2.17) from Lemma 2.7 with the parameter $\alpha = \frac{1}{2}$ to the magnetic Schrödinger equation (2.38). We obtain

$$\|v\|_{L^\infty_T H^s \cap L^2_T H^{s-1/2,6}} \lesssim \|v\|_{L^\infty_T H^s} + T^{1/2}\|2\mathrm{i}A\cdot\nabla v + A\cdot Av + \lambda\phi(u)v + f\|_{L^2_T H^{s-1}}.$$

We start with the discussion of the term $A \cdot \nabla v$ for which we distinguish several cases. First, we treat the cases $s = 1$ and $s = 0$. In the former case we obtain from Hölder's inequality that

$$\|A\cdot\nabla v\|_{L^2_T L^2} \lesssim \|A\|_{L^2_T L^\infty}\|\nabla v\|_{L^\infty_T L^2} \leq \|A\|_{L^2_T L^\infty}\|v\|_{L^\infty_T H^1}.$$

In the latter case we use duality and integration by parts where we observe that $\operatorname{div} A = 0$ to obtain that

$$\begin{aligned}\|A\cdot\nabla v\|_{L^2_T H^{-1}} &\lesssim \left\|t \mapsto \sup_{\|\varphi\|_{H^1}=1}\left|\int_{\mathbb{R}^3} A(t,x)\cdot\nabla v(t,x)\varphi(x)\,\mathrm{d}x\right|\right\|_{L^2_T} \\ &= \left\|t \mapsto \sup_{\|\varphi\|_{H^1}=1}\left|\int_{\mathbb{R}^3} v(t,x)A(t,x)\cdot\nabla\varphi(x)\,\mathrm{d}x\right|\right\|_{L^2_T} \\ &\lesssim \|A\|_{L^2_T L^\infty}\|v\|_{L^\infty_T L^2}.\end{aligned}$$

Interpolation between these two estimates leads to

$$\|A \cdot \nabla v\|_{L^2_T H^{s-1}} \lesssim \|A\|_{L^2_T L^\infty} \|v\|_{L^\infty_T H^s}$$

for all $s \in [0,1]$. If $s \in (1,\infty)$, we use the fractional Leibniz rule which yields

$$\|A \cdot \nabla v\|_{L^2_T H^{s-1}} \lesssim \|A\|_{L^2_T L^\infty} \|v\|_{L^\infty_T H^s} + \|A\|_{L^{q_1}_T H^{s-1,r_1}} \|v\|_{L^{q_2}_T H^{1,r_2}},$$

where we can choose any $q_1, q_2, r_1, r_2 \in [1,\infty]$ such that $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{2}$ and $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{2}$. We claim that we can find $\theta \in \left(\frac{1}{2}, 1\right)$ and q_1, q_2, r_1 and r_2 for which the estimate

$$\|A\|_{L^{q_1}_T H^{s-1,r_1}} \|v\|_{L^{q_2}_T H^{1,r_2}} \lesssim T^{(1-\theta)/2} \|A\|_{L^\infty_T H^\sigma \cap L^2_T L^\infty} \|v\|^{1-\theta}_{L^\infty_T H^s} \|v\|^{\theta}_{L^2_T H^{s-1/2,6}} \quad (2.39)$$

holds true. Accepting (2.39), we obtain the desired result. Namely, by combining (2.39) with Young's inequality, we obtain for any $\varepsilon > 0$ that

$$\begin{aligned} &\|A \cdot \nabla v\|_{L^2_T H^{s-1}} \\ &\quad\lesssim \|A\|_{L^2_T L^\infty} \|v\|_{L^\infty_T H^s} + \|A\|_{L^{q_1}_T H^{s-1,r_1}} \|v\|_{L^{q_2}_T H^{1,r_2}} \\ &\quad\lesssim \|A\|_{L^2_T L^\infty} \|v\|_{L^\infty_T H^s} + T^{1/2} \varepsilon^{-\theta/(1-\theta)} \|A\|^{1/(1-\theta)}_{L^\infty_T H^\sigma \cap L^2_T L^\infty} \|v\|_{L^\infty_T H^s} + \varepsilon \|v\|_{L^2_T H^{s-1/2,6}}. \end{aligned}$$

By choosing ε small enough, we can absorb the last term in proving the final estimate.

We thus prove (2.39). We start with the most difficult case $s \in (1,2)$. We set $\theta = s-1$, $r_1 = \frac{2}{s-1}$, $r_2 = \frac{2}{2-s}$ and $q_1 = \frac{2}{2-s}$, $q_2 = \frac{2}{s-1}$. We use interpolation and the Sobolev embedding

$$\left[H^{s-1/2,6}(\mathbb{R}^3), H^s(\mathbb{R}^3)\right]_{s-1} = H^{(s+1)/2,6/(5-2s)}(\mathbb{R}^3) \hookrightarrow H^{1,2/(2-s)}(\mathbb{R}^3).$$

We also use the estimate

$$\|A\|_{L^{2/(2-s)}_T H^{s-1,2/(s-1)}} \lesssim \|A\|^{s-1}_{L^\infty_T H^\sigma} \|A\|^{2-s}_{L^2_T L^\infty},$$

which is proved in Lemma 3.1 of [NW07]. We thus obtain

$$\begin{aligned} &\|A\|_{L^{2/(2-s)}_T H^{2/s-1}} \|v\|_{L^{2/(s-1)}_T H^{1,2/(2-s)}} \\ &\qquad\lesssim T^{(2-s)/2} \|A\|_{L^\infty_T H^\sigma \cap L^2_T L^\infty} \|v\|^{2-s}_{L^\infty_T H^s} \|v\|^{s-1}_{L^2_T H^{s-1/2,6}}. \end{aligned}$$

If $s = 2$, we take $\theta = 1 - 3\delta$, where we set $\delta = \min\left\{\frac{\sigma-1}{2}, \frac{1}{4}\right\}$ as in (2.42). We set $q_1 = \infty$, $q_2 = 2$ and $\frac{1}{r_1} = \frac{1-\delta}{2}$, $\frac{1}{r_2} = \frac{\delta}{2}$. Using interpolation and the Sobolev embeddings

$$\left[H^{3/2,6}(\mathbb{R}^3), H^2(\mathbb{R}^3)\right]_{1-3\delta} = H^{(3+3\delta)/2,6/(1+6\delta)}(\mathbb{R}^3) \hookrightarrow H^{1,r_2}(\mathbb{R}^3)$$

and $H^\sigma(\mathbb{R}^3) \hookrightarrow H^{1,r_1}(\mathbb{R}^3)$, we arrive at

$$\|A\|_{L^\infty_T H^{1,r_1}} \|v\|_{L^2_T H^{1,r_2}} \lesssim T^{(1-\theta)/2} \|A\|_{L^\infty_T H^\sigma} \|v\|^{1-\theta}_{L^\infty_T H^s} \|v\|^{\theta}_{L^2_T H^{s-1/2,6}} .$$

If $s \in (2,\infty)$, we take $\theta \in \big(\max\{5-2s, \frac{1}{2}\}, 1\big)$. We set $r_1 = 2$, $r_2 = \infty$ and $q_1 = \infty$, $q_2 = 2$. Using interpolation and the Sobolev embeddings

$$\big[H^{s-1/2,6}(\mathbb{R}^3), H^s(\mathbb{R}^3)\big]_\theta = H^{s-\theta/2, 6/(3-2\theta)}(\mathbb{R}^3) \hookrightarrow H^{1,\infty}(\mathbb{R}^3)$$

and $H^\sigma(\mathbb{R}^3) \hookrightarrow H^1(\mathbb{R}^3)$, we arrive at

$$\|A\|_{L^\infty_T H^{s-1}} \|v\|_{L^2_T H^{1,\infty}} \lesssim T^{(1-\theta)/2} \|A\|_{L^\infty_T H^\sigma} \|v\|^{1-\theta}_{L^\infty_T H^s} \|v\|^{\theta}_{L^2_T H^{s-1/2,6}} .$$

The next term is $A \cdot Av$. If $s \in [0,1]$, we use the Sobolev embeddings $L^{6/(5-2s)}(\mathbb{R}^3) \hookrightarrow H^{s-1}(\mathbb{R}^3)$, $H^s(\mathbb{R}^3) \hookrightarrow L^{6/(3-2s)}(\mathbb{R}^3)$, $H^\sigma(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$ and Hölder's inequality to obtain

$$\begin{aligned} \|A \cdot Av\|_{L^2_T H^{s-1}} &\lesssim \|A \cdot Av\|_{L^2_T L^{6/(5-2s)}} \\ &\leq \|A\|_{L^2_T L^\infty} \|A\|_{L^\infty_T L^3} \|v\|_{L^\infty_T L^{6/(3-2s)}} \\ &\lesssim \|A\|_{L^2_T L^\infty} \|A\|_{L^\infty_T H^\sigma} \|v\|_{L^\infty_T H^s} . \end{aligned}$$

If, on the other hand, $s \in (1,\infty)$, then we use the fractional Leibniz rule and the Sobolev embeddings $H^s(\mathbb{R}^3) \hookrightarrow H^{s-1,6}(\mathbb{R}^3)$, $H^\sigma(R^3) \hookrightarrow L^6(\mathbb{R}^3)$ and Hölder's inequality to obtain that

$$\begin{aligned} \|A \cdot Av\|_{L^2_T H^{s-1}} &\lesssim \|A \cdot A\|_{L^2_T L^3} \|v\|_{L^\infty_T H^{s-1,6}} + \|A \cdot A\|_{L^2_T H^{s-1,r_s}} \|v\|_{L^\infty_T L^{q_s}} \\ &\lesssim \|A\|_{L^2_T L^\infty} \|A\|_{L^\infty_T H^\sigma} \|v\|_{L^\infty_T H^s} . \end{aligned}$$

Here we distinguish two subcases for the second term. If $s \in (1, \frac{3}{2}]$, we choose $q_s = 6$, $r_s = 3$, use the Sobolev embeddings $H^s(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, $H^\sigma(\mathbb{R}^3) \hookrightarrow H^{s-1,3}(\mathbb{R}^3)$ and the estimate $\|A \cdot A\|_{L^2_T H^{s-1,3}} \lesssim \|A\|_{L^2_T L^\infty} \|A\|_{L^\infty_T H^{s-1,3}}$ by the fractional Leibniz rule. If $s \in (\frac{3}{2}, \infty)$, we choose instead $q_s = \infty$ and $r_s = 2$ and argue similarly as above using the embeddings $H^\sigma(\mathbb{R}^3) \hookrightarrow H^{s-1}(\mathbb{R}^3)$ and $H^s(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$.

Finally, we consider the term $\phi(u)v$. If $s \in [0,1]$, we use the trivial embedding $L^2(\mathbb{R}^3) \hookrightarrow H^{s-1}(\mathbb{R}^3)$ Hölder's inequality and Lemma 2.12 (3) to estimate

$$\begin{aligned} \|\phi(u)v\|_{L^2_T H^{s-1}} &\lesssim T^{1/2} \|\phi(u)v\|_{L^\infty_T L^2} \\ &\lesssim T^{1/2} \|\phi(u)\|_{L^\infty_T L^\infty} \|v\|_{L^\infty_T L^2} \\ &\lesssim T^{1/2} \|u\|^2_{L^\infty_T H^1} \|v\|_{L^\infty_T H^s} . \end{aligned}$$

In the case $s > 1$, we use Lemma 2.12 (4) with the parameters $s_1 = s_2 = \max\{1, s-1\}$ and $s_3 = s$ to obtain that

$$\|\phi(u)v\|_{L^2_T H^{s-1}} \lesssim T^{1/2} \|u\|_{L^\infty_T H^{\max\{s-1,1\}}} \|v\|_{L^\infty_T H^s} .$$

From these estimates, we deduce the assertion. □

We continue to recall the theory for the linear magnetic Schrödinger equation (2.38) as presented in [NW07] where an evolution family of operators is constructed to solve the nonautonomous problem (2.38). Summarizing their Lemmata 3.2 and 3.3, the following facts are known.

Lemma 2.15 (The evolution family of the magnetic Schrödinger equation)
Let $s \in [-2, 2]$, $\sigma \in (1, \infty)$. Let $u \in L^\infty_T H^1(\mathbb{R}^3)$ and $A \in M^{1,\sigma}_T \cap L^2_T L^\infty(\mathbb{R}^3)$ with $\operatorname{div} A = 0$ be given.

(1) The homogeneous magnetic Schrödinger equation (2.38), i.e. we consider the case $f = 0$, with initial value $v_0 \in H^s(\mathbb{R}^3)$ has a unique weak H^s-solution.

(2) Part (1) induces an evolution family $U_s \colon [0, T]^2 \to \mathcal{L}\big(H^s(\mathbb{R}^3)\big)$ where the solution v is given by $v(t) = U(t, 0)v_0$, $t \in [0, T]$. For $s > s'$ and every $t, \tau \in [0, T]$, the operator $U_s(t, \tau)$ is the restriction of $U_{s'}(t, \tau)$ to $H^s(\mathbb{R}^3)$. Therefore, we use the notation U_A instead of U_s.

(3) If $u \in C^1_T H^s(\mathbb{R}^3)$, we even obtain strong H^s-solutions of (2.38). In particular, the map $t \mapsto U_A(t, \tau)v_0$ belongs to $C_T H^s(\mathbb{R}^3)$.

(4) Define
$$K_s := \sup_{0 \le t, \tau \le T} \|U_A(t, \tau)\|_{H^s \to H^s} .$$
We have the estimate $K_s \geq 1$ and more importantly there exists constants $c \in (0, \infty)$ and $l \in (0, \infty)$ such that
$$\begin{aligned} K_s \lesssim \big\langle \|A\|_{L^\infty_T \dot{H}^1} \big\rangle^{2|s|} \\ \times \exp\Big(c\, |s|\, T^{1/2} \langle T \rangle^l \big\langle \max\{\|A\|_{M^{1,\sigma}_T \cap L^2_T L^\infty}, \langle \lambda \rangle \|u\|_{L^\infty_T H^1}\}\big\rangle^l\Big) \end{aligned} \tag{2.40}$$
holds true.

(5) Let $f \in L^\infty_T H^{s-2}(\mathbb{R}^3)$. Let w be a weak L^2-solution of the inhomogeneous equation (2.38). Then w is given by Duhamel's formula, i.e.
$$w(t) = U_A(t, 0)w(0) - \mathrm{i} \int_0^t U_A(t, \tau) f(\tau) \,\mathrm{d}\tau. \tag{2.41}$$

Local well-posedness

The previous lemma finishes the presentation of the prerequisites for the proof of Theorem 2.1. We now establish Theorem 2.1 through a series of lemmata. Throughout the rest of this section, we fix the following notation. For given $\sigma > 1$, we define
$$\delta = \min\left\{\frac{\sigma - 1}{2}, \frac{1}{4}\right\}, \qquad \frac{1}{q} = \frac{1}{2} - \frac{2\delta}{3}, \qquad \frac{1}{r} = \frac{2\delta}{3}. \tag{2.42}$$

Note that this pair (q, r) is admissible for the Strichartz estimates of the Klein–Gordon equation, see Definition 2.3. Due to $\sigma - \frac{2}{q} - \delta - \frac{3}{r} \geq \frac{1}{3}\delta > 0$, we obtain that the Sobolev space $H^{\sigma-2/q-\delta,r}(\mathbb{R}^3)$ appearing in the Strichartz estimates of Lemma 2.4 embeds into $L^\infty(\mathbb{R}^3)$ by the Sobolev embedding A.9 (4). Since $\sigma - 1 - \frac{3}{2} \geq -\frac{3}{2} + 2\delta = -\frac{3}{q}$, we obtain the Sobolev embedding $H^{\sigma-1}(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$. Finally, since $s - \frac{1}{2} - \frac{3}{6} > s - 1 - 2\delta = s - 1 - \frac{3}{r}$, we also obtain the Sobolev embedding $H^{s-1/2,6}(\mathbb{R}^3) \hookrightarrow H^{s-1,r}(\mathbb{R}^3)$.

In many ways our strategy follows the arguments in [NW07]. In particular, we use the same fixed-point space as they do in order to construct a solution of the Maxwell–Schrödinger system. Let $T \in (0, 1)$ and take parameters $R_1, R_2, R_3 > 1$. Define the space

$$\mathcal{B} = \big\{(u, A) : \|u\|_{L^\infty_T H^s} \leq R_1, \|u\|_{L^2_T H^{s-1/2,6}} \leq R_2, \|A\|_{M^{1,\sigma}_T \cap L^2_T L^\infty} \leq R_3, \operatorname{div} A = 0\big\}$$

endowed with the metric induced by the norm

$$\|(u, A)\|_{\mathcal{B}} = \|u\|_{L^\infty_T L^2} + \|A\|_{L^\infty_T H^{1/2} \cap L^4_T L^4} .$$

The space $\mathcal{B}$ is complete. In the next lemma we use Banach's fixed-point theorem in the space $\mathcal{B}$ to construct a *weak $X^{s,\sigma}$-solution* of the Maxwell–Schrödinger system (2.11). By this, we mean that we construct functions $(u, A, \partial_t A) \in L^\infty_T X^{s,\sigma} \cap W^{1,\infty}_T X^{s-2,\sigma-1}$ such that u satisfies

$$u(t) = U_A(t, 0)u_0 + \mathrm{i} \int_0^t U_A(t, \tau) \, |u(\tau)|^{p-1} \, u(\tau) \, \mathrm{d}\tau$$

and A satisfies the wave equation

$$\partial_t^2 A(t) - \Delta A(t) = \lambda P J\big(u(t), A(t)\big)$$

for almost all $t \in [0, T]$. Besides existence of solutions, a proper well-posedness result should comprise uniqueness and continuous dependence of the solutions on the initial values. To some extent, these properties follow from the techniques used in the fixed-point argument in a straightforward way and they are included in the statement of the next lemma. We call this result our basic version of local well-posedness. However, Lemma 2.16 does not settle the problem completely. At least three issues need to be addressed separately. First, in the fixed-point argument below we can only close the estimates if we assume that the two solutions of the Schrödinger- and of the Maxwell part of the linearized system belong to some auxiliary function spaces. Therefore, we also obtain uniqueness only within this restricted class of solutions. Only later in Corollary 2.19 we show that in fact the Maxwell–Schrödinger system enjoys the property of *unconditional uniqueness*. Second, Lemma 2.16 states continuous dependence of solutions only in a topology which is strictly weaker than the one of the solution space. To obtain the full result requires substantial additional work which cumulates in Lemma 2.22. Third, we should also expect that solutions preserve mass and energy, cf. Lemma 2.2. This and further regularity properties of solutions are shown in Lemma 2.17.

Lemma 2.16 (Local well-posedness, basic version)
Let $s \in \left[\frac{11}{8}, 2\right]$ and $\sigma \in (1,\infty)$ such that $(s,\sigma) \in \mathcal{R}$. Let $p \in \big(s, p^*(s)\big)$.

For any $(u_0, A_0, A_1) \in X^{s,\sigma}$, there exists $T_{\max} \in (0,\infty]$ such that the Maxwell–Schrödinger system (2.11) has a unique, maximal solution (u, A) with

$$u \in L^\infty_T H^s(\mathbb{R}^3) \cap W^{1,\infty}_T H^{s-2}(\mathbb{R}^3) \cap L^2_T H^{s-1/2,6}(\mathbb{R}^3)$$

and

$$A \in L^\infty_T H^\sigma(\mathbb{R}^3) \cap W^{1,\infty}_T H^{\sigma-1}(\mathbb{R}^3) \cap L^2_T L^\infty(\mathbb{R}^3)$$

for every $T \in (0, T_{\max})$. We have the blowup alternative: If $T_{\max} < \infty$, then

$$\lim_{t\to T_{\max}} \|(u(t), A(t), \partial_t A(t))\|_{X^{s,\sigma}} = \infty.$$

The solution depends continuously on (u_0, A_0, A_1) in the following sense: Let $\big((u_0^n, A_0^n, A_1^n)\big)_n$ be a sequence in $X^{s,\sigma}$ which converges to (u_0, A_0, A_1). There exists $T \in (0, T_{\max})$ such that the solution $(u^n, A^n, \partial_t A^n)$ with initial value (u_0^n, A_0^n, A_1^n) exists on $[0,T]$ and is bounded in $L^\infty_T X^{s,\sigma}$ for all sufficiently large $n \in \mathbb{N}$. Moreover, for any $\varepsilon > 0$ we have $(u^n, A^n, \partial_t A^n) \to (u, A, \partial_t A)$ in $L^\infty_T X^{s-\varepsilon,\sigma-\varepsilon}$ as $n \to \infty$.

Proof. First step. Construction of a solution by a fixed-point argument. Let $T \in (0,1)$ be specified in (2.55) and (2.60), and let R_1, R_2, R_3 by given by the equations (2.54) below. Note that $T < 1$ and $R_j > 1$ so that $\langle T\rangle \le \sqrt{2}$ and $\langle R_j\rangle \le \sqrt{2}R_j$ for $j \in \{1,2,3\}$. To solve the Maxwell–Schrödinger system (2.11) on the interval $I = [0,T]$, we linearize these equations and study

$$\mathrm{i}\partial_t v + \Delta_A v = \lambda\phi(u)v - |u|^{p-1} u, \qquad \text{in } I \times \mathbb{R}^3, \tag{2.43}$$

$$\partial_t^2 B - \Delta B + B = \lambda P J(u, A) + A, \qquad \text{in } I \times \mathbb{R}^3, \tag{2.44}$$

with $\phi(u) = (-\Delta)^{-1} |u|^2$ as before. For given $(u, A) \in \mathcal{B}$, we obtain a solution v of the magnetic Schrödinger equation (2.43) and a solution B of the Klein–Gordon equation (2.44). We denote by $\Phi\colon (u, A) \mapsto (v, B)$ the joint solution map. We show that $\Phi\colon \mathcal{B} \to \mathcal{B}$ is a contraction. The weak solution of the inhomogeneous magnetic Schrödinger equation (2.43) can be expressed with the evolution family through the Duhamel formula (2.41)

$$v(t) = U_A(t,0)u_0 + \mathrm{i}\int_0^t U_A(t,\tau)\, |u(\tau)|^{p-1}\, u(\tau)\,\mathrm{d}\tau. \tag{2.45}$$

To estimate the nonlinearity, we fix a number $\gamma \in \big(\gamma(p-1,\infty,s), p-1\big]$ as required by Corollary 2.10. If $s > \frac{3}{2}$, we set $\gamma = 0$. Otherwise, we choose γ according to the conditions (2.31), i.e. we always have $\gamma < 2$. Using the boundedness of the evolution family on $H^s(\mathbb{R}^3)$ with constant K_s given in (2.40) and estimate (2.32) to control the nonlinearity, we get

$$\begin{aligned}
&\left\| \int_0^{\cdot} U_A(\cdot,\tau)\, |u(\tau)|^{p-1} u(\tau)\, \mathrm{d}\tau \right\|_{L_T^\infty H^s} \\
&\quad \le \operatorname*{esssup}_{t\in I} K_s \int_0^t \left\| |u(\tau)|^{p-1} u(\tau) \right\|_{H^s} \mathrm{d}\tau \\
&\quad \lesssim \operatorname*{esssup}_{t\in I} K_s \int_0^t \|u(\tau)\|_{H^s}^{p-\gamma} \|u(\tau)\|_{H^{s-1/2,6}}^{\gamma} \mathrm{d}\tau \\
&\quad \le K_s T^{1-\gamma/2} \|u\|_{L_T^\infty H^s}^{p-\gamma} \|u\|_{L_T^2 H^{s-1/2,6}}^{\gamma} \\
&\quad \le K_s T^{1-\gamma/2} R_1^{p-\gamma} R_2^{\gamma}.
\end{aligned} \tag{2.46}$$

Hence, our estimates of the right side of the Duhamel formula (2.45) yield

$$\|v\|_{L_T^\infty H^s} \le C K_s \left(\|u_0\|_{H^s} + T^{1-\gamma/2} R_1^{p-\gamma} R_2^{\gamma} \right). \tag{2.47}$$

As we see in (2.54) and (2.55), it is possible to choose R_1, R_2 and T only depending on $\|(u_0, A_0, A_1)\|_{X^{s,\sigma}}$ such that $\|v\|_{L_T^\infty H^s} \le R_1$.

We next consider the Schrödinger equation in the auxiliary space $L_T^2 H^{s-1/2,6}(\mathbb{R}^3)$. Here we use the Strichartz estimates with loss from Lemma 2.7. Since they are stated for the Schrödinger equation without magnetic fields, we change our point of view on the magnetic Schrödinger equation (2.43). First, we treat the magnetic Laplacian as a perturbation of the standard Laplacian and second, we split the equation in two parts with the aim to employ different Strichartz estimates to each subproblem. More concretely, we define $f_1 = 2\mathrm{i}A \cdot \nabla v + (A \cdot A)v + \lambda\phi(u)v$ and $f_2 = -|u|^{p-1} u$. Then we decompose $v = v_1 + v_2$, where v_1 is a solution of the Schrödinger equation

$$\mathrm{i}\partial_t v_1(t) + \Delta v_1(t) = f_1(t), \quad \text{in } I \times \mathbb{R}^3, \tag{2.48}$$

with initial value $v(0) = u_0$ and v_2 solves

$$\mathrm{i}\partial_t v_2(t) + \Delta v_2(t) = f_2(t), \quad \text{in } I \times \mathbb{R}^3, \tag{2.49}$$

with $v(0) = 0$. By applying Lemma 2.14, which is crucially based on Strichartz estimates with loss (2.17), to the equation (2.48), we obtain the estimate

$$\|v_1\|_{L_T^2 H^{s-1/2,6}} \lesssim \langle T\rangle^m \max\{\langle\lambda\rangle R_1, R_3\}^m \|v_1\|_{L_T^\infty H^s}.$$

We apply the standard Strichartz estimate, namely (2.16) from Lemma 2.6, to equation (2.49). The Strichartz pairs $(\infty, 2)$ and $(2, 6)$, the embeddings $B^s_{6,2}(\mathbb{R}^3) \hookrightarrow F^s_{6,2}(\mathbb{R}^3) \cong H^{s,6}(\mathbb{R}^3) \hookrightarrow H^{s-1/2,6}(\mathbb{R}^3)$, see Theorem A.9 (2) and Theorem A.8, and estimate (2.32) from Corollary 2.10 imply that

$$\|v_2\|_{L^\infty_T H^s \cap L^2_T H^{s-1/2,6}} \lesssim \left\| |u|^{p-1} u \right\|_{L^1_T H^s} \lesssim T^{1-\gamma/2} R_1^{p-\gamma} R_2^\gamma.$$

In this step, we estimate the inhomogeneity in the same way as in (2.46) above. Since $v_1 = v - v_2$, this further implies that

$$\|v_1\|_{L^\infty_T H^s} \leq \|v\|_{L^\infty_T H^s} + \|v_2\|_{L^\infty_T H^s} \lesssim R_1 + T^{1-\gamma/2} R_1^{p-\gamma} R_2^\gamma.$$

We thus obtain

$$\begin{aligned}
\|v\|_{L^2_T H^{s-1/2,6}} &\leq \|v_1\|_{L^2_T H^{s-1/2,6}} + \|v_2\|_{L^2_T H^{s-1/2,6}} \\
&\lesssim \langle T \rangle^m \max\{\langle \lambda \rangle R_1, R_3\}^m \|v_1\|_{L^\infty_T H^s} + T^{1/2} R_1^4 R_2 \\
&\leq \langle T \rangle^m \max\{\langle \lambda \rangle R_1, R_3\}^m \big(R_1 + T^{1-\gamma/2} R_1^{p-\gamma} R_2^\gamma\big) + T^{1-\gamma/2} R_1^{p-\gamma} R_2^\gamma.
\end{aligned}$$

Using $T < 1$ and $R_j > 1$ for $j \in \{1, 2, 3\}$, we conclude that there is a constant $C \in (0, \infty)$ such that

$$\begin{aligned}
\|v\|_{L^2_T H^{s-1/2,6}} \leq\, & C \max\{\langle \lambda \rangle R_1, R_3\}^m R_1 \\
&+ C T^{1-\gamma/2} \max\{\langle \lambda \rangle R_1, R_3\}^m R_1^{p-\gamma} R_2^\gamma.
\end{aligned} \tag{2.50}$$

Finally, the estimate for the Klein–Gordon equation (2.44) does not differ from the one obtained in [NW07]. Namely, from the estimate (2.34) for the current density given in Lemma 2.11, it follows that

$$\begin{aligned}
\|PJ(u, A)\|_{L^1_T H^{\sigma-1}} &\lesssim T^{1/4} \langle T \rangle^{1/4} \left\langle \|A\|_{L^\infty_T H^\sigma} \right\rangle \|u\|^2_{L^\infty_T H^s \cap L^2_T H^{s-1/2,6}} \\
&\lesssim T^{1/4} R_3 \max\{R_1, R_2\}^2.
\end{aligned} \tag{2.51}$$

The standard Strichartz estimate for the Klein–Gordon equation from Lemma 2.4 with the admissible pairs $(\infty, 2)$ and (q, r) further implies that

$$\|B\|_{M^{1,\sigma}_T \cap L^2_T L^\infty} \leq Ce\left(\|A_0\|_{H^\sigma} + \|A_1\|_{H^{\sigma-1}}\right) + C \langle \lambda \rangle T^{1/4} R_3 \max\{R_1, R_2\}^2, \tag{2.52}$$

where we use the estimate $\|B\|_{L^2_T L^\infty} \lesssim \|B\|_{L^q_T H^{\sigma-2/q,r}}$ which we deduce from the Sobolev embedding $H^{\sigma-2/q,r}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$.

The dependency of the operator norm K_s on the bounds R_1 and R_3 is stated in Lemma 2.15, which, as we recall here, yields

$$K_s \leq C R_3^{2s} \exp\left(\tilde{c} T^{1/2} \max\{\langle \lambda \rangle R_1, R_3\}^l\right), \tag{2.53}$$

where $\tilde{c}$ is a constant. The estimates above remain true if we enlarge the constant C. We therefore fix the same constant C in the estimates (2.47), (2.50), (2.52) and (2.53) as well as in the estimates (2.58) and (2.59) below. To ensure that Φ maps $\mathcal{B}$ into itself, we must select the parameters $T \in (0,1)$ and $R_j \in (1,\infty)$ for $j \in \{1,2,3\}$ appropriately. We now choose, in the order given below,

$$\begin{aligned} &R_3 \geq 2Ce\left(\|A_0\|_{H^\sigma} + \|A_1\|_{H^{\sigma-1}}\right), \\ &R_1 \geq 4C\,\|u_0\|_{H^s} R_3^{2s}, \\ &R_2 \geq 2C \max\{\langle\lambda\rangle R_1, R_3\}^m R_1. \end{aligned} \tag{2.54}$$

Moreover we choose $T \in (0,1)$ which satisfies

$$\begin{aligned} C\exp(\tilde{c}T^{1/2}\max\{\langle\lambda\rangle R_1, R_3\}^l) &\leq 2, \\ CT^{1-\gamma/2}R_1^{p-\gamma-1}R_2^{\gamma}R_3^{2s} &\leq \tfrac{1}{4}, \\ CT^{1-\gamma/2}\max\{\langle\lambda\rangle R_1, R_2\}^m R_1^{p-\gamma}R_2^{\gamma} &\leq \tfrac{1}{2}, \\ C\,\langle\lambda\rangle\, T^{1/4}\max\{R_1, R_2\}^2 &\leq \tfrac{1}{2}. \end{aligned} \tag{2.55}$$

With these choices, we obtain $K_s \leq 2R_3^{2s}$ from (2.53) and the terms on the right sides of (2.47), (2.50) and (2.52) can be bounded by R_1, R_2 and R_3, respectively. Hence, the image $\Phi(\mathcal{B})$ is contained in $\mathcal{B}$. Note that (2.60) lists one more restriction on T, which ensures that Φ is also a contraction.

We therefore estimate the difference of two solutions $(v,B) = \Phi(u,A)$ and $(\tilde{v},\tilde{B}) = \Phi(\tilde{u},\tilde{A})$, starting from $(u,A),(\tilde{u},\tilde{A}) \in \mathcal{B}$. We define $w = v - \tilde{v}$. Then w satisfies

$$\begin{aligned} w(t) = &-\mathrm{i}\int_0^t U_A(t,\tau)\left[2\mathrm{i}\big(A(\tau)-\tilde{A}(\tau)\big)\cdot\nabla\tilde{v}(\tau)\right]\,\mathrm{d}\tau \\ &-\mathrm{i}\int_0^t U_A(t,\tau)\left[\big(A(\tau)-\tilde{A}(\tau)\big)\cdot\big(A(\tau)+\tilde{A}(\tau)\big)\tilde{v}(\tau)\right]\,\mathrm{d}\tau \\ &-\mathrm{i}\lambda\int_0^t U_A(t,\tau)\left[\big(\phi(u)(\tau)-\phi(\tilde{u})(\tau)\big)\tilde{v}(\tau)\right]\,\mathrm{d}\tau \\ &+\mathrm{i}\int_0^t U_A(t,\tau)\left[|v(\tau)|^{p-1}v(\tau)-|\tilde{v}(\tau)|^{p-1}\tilde{v}(\tau)\right]\mathrm{d}\tau. \end{aligned} \tag{2.56}$$

We bound the difference w in the space $L_T^\infty L^2(\mathbb{R}^3)$. In each case, we use the unitarity of $U_A(t,\tau)$ and it thus remains to prove bounds on the terms in brackets in the space $L_T^1 L^2(\mathbb{R}^3)$. The first three integrals on the right side of (2.56) are estimated as in [NW07] and we just repeat the arguments from the proof of Proposition 4.1. Using Hölder's inequality and the interpolation result

$$L_T^{8/3}H^{s-3/8,4}(\mathbb{R}^3) = \left[L_T^\infty H^s(\mathbb{R}^3), L_T^2 H^{s-1/2,6}(\mathbb{R}^3)\right]_{3/4},$$

see Theorem 1 and Remark 3 in Section 1.18.4 of [Tri95], we first obtain

$$\begin{aligned}\|(A-\tilde{A})\nabla\tilde{v}\|_{L^1_T L^2} &\le T^{3/8}\|A-\tilde{A}\|_{L^4_T L^4}\,\|\nabla\tilde{v}\|_{L^{8/3}_T L^4}\\ &\lesssim T^{3/8}\|A-\tilde{A}\|_{L^4_T L^4}\,\|\tilde{v}\|_{L^\infty_T H^s\cap L^2_T H^{s-1/2,6}}\\ &\lesssim T^{3/8}\max\{R_1,R_2\}\|(u-\tilde{u},A-\tilde{A})\|_{\mathcal{B}}.\end{aligned} \tag{2.57}$$

We point out, that in this estimate the condition $s\ge\frac{11}{8}$ is crucial to bound $\tilde{v}$ in the space $L^{8/3}_T H^{1,4}(\mathbb{R}^3)$. Due to the Sobolev embeddings $H^{1/2}(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3)$, $H^s(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$ and Lemma 2.12 (4), the estimates

$$\begin{aligned}\|(A-\tilde{A})(A+\tilde{A})\tilde{v}\|_{L^1_T L^2} &\le T^{1/2}\|A-\tilde{A}\|_{L^\infty_T L^3}\|A+\tilde{A}\|_{L^2_T L^\infty}\,\|\tilde{v}\|_{L^\infty_T L^6}\\ &\lesssim T^{1/2}R_1R_3\|(u-\tilde{u},A-\tilde{A})\|_{\mathcal{B}}\end{aligned}$$

and

$$\begin{aligned}\left\|\big(\phi(u)-\phi(\tilde{u})\big)\tilde{v}\right\|_{L^1_T L^2} &\lesssim T\,\|u-\tilde{u}\|_{L^\infty_T L^2}\,\|u+\tilde{u}\|_{L^\infty_T H^1}\,\|\tilde{v}\|_{L^\infty_T H^1}\\ &\lesssim TR_1^2\|(u-\tilde{u},A-\tilde{A})\|_{\mathcal{B}}\end{aligned}$$

are less delicate. To estimate the last integral, we use (2.22) to compute

$$\left\||v|^{p-1}\,v-|\tilde{v}|^{p-1}\,\tilde{v}\right\|_{L^1_T L^2}\lesssim\left(\int_0^T\left\||v(\tau)|^{p-1}\right\|_{L^\infty}+\left\||\tilde{v}(\tau)|^{p-1}\right\|_{L^\infty}\,\mathrm{d}\tau\right)\|v-\tilde{v}\|_{L^\infty_T L^2}\,.$$

Since we have $\gamma>\gamma(p-1,\infty,s)$ above, we can use this number also with (2.29) to further estimate

$$\int_0^T\left\||v(\tau)|^{p-1}\right\|_{L^\infty}\,\mathrm{d}\tau\lesssim\int_0^T\|v(\tau)\|^{p-1-\gamma}_{H^s}\,\|v(\tau)\|^{\gamma}_{H^{s-1/2,6}}\,\mathrm{d}\tau\le T^{1-\gamma/2}R_1^{p-\gamma-1}R_2^{\gamma}.$$

Of course, the same also holds true for $\tilde{v}$. Altogether we obtain the estimate

$$\begin{aligned}&\|v-\tilde{v}\|_{L^\infty_T L^2}\\ &\le C\,\langle\lambda\rangle\left(T^{3/8}\max\{R_1,R_2,R_3\}^2+T^{1-\gamma/2}R_1^{p-\gamma-1}R_2^{\gamma}\right)\left\|(u-\tilde{u},A-\tilde{A})\right\|_{\mathcal{B}}.\end{aligned} \tag{2.58}$$

We finally estimate the difference $B-\tilde{B}$ in the same way as in the proof of Proposition 4.1 in [NW07]. We apply the Strichartz estimates from Lemma 2.4 with regularity $H^{1/2}$ and admissible pairs $(\infty,2)$ and $(4,4)$ to the difference of the Klein–Gordon equations satisfied by B and $\tilde{B}$. This leads to

$$\begin{aligned}&\|B-\tilde{B}\|_{L^\infty_T H^{1/2}\cap L^4_T L^4}+\left\|\partial_t(B-\tilde{B})\right\|_{L^\infty_T H^{-1/2}}\\ &\qquad\lesssim T\|A-\tilde{A}\|_{L^\infty_T H^{-1/2}}+\lambda\|PJ(u,A)-PJ(\tilde{u},\tilde{A})\|_{L^{4/3}_T L^{4/3}}.\end{aligned}$$

We expand

$$J(u,A) - J(\tilde u,\tilde A) = 2\,\mathrm{Im}\left(\bar u \nabla u - \bar{\tilde u}\nabla \tilde u - \mathrm{i}A\,|u|^2 + \mathrm{i}\tilde A\,|\tilde u|^2\right)$$

and we note that at this point the Coulomb gauge is crucial. Namely, we use that the Helmholtz projection P vanishes on gradient fields and obtain the identity

$$\begin{aligned} P\,\mathrm{Im}(\bar u\nabla u - \bar{\tilde u}\nabla\tilde u) &= P\,\mathrm{Im}\big((\bar u - \bar{\tilde u})(\nabla u + \nabla \tilde u) - \bar u \nabla \tilde u + \bar{\tilde u}\nabla u\big) \\ &= P\,\mathrm{Im}\big((\bar u - \bar{\tilde u})(\nabla u + \nabla \tilde u) - \nabla(\bar u \tilde u)\big) \\ &= P\,\mathrm{Im}\big((\bar u - \bar{\tilde u})(\nabla u + \nabla \tilde u)\big). \end{aligned}$$

Therefore, no term of the form $\nabla(u-\tilde u)$ appears in the estimate. We further have

$$\begin{aligned} \|P\,\mathrm{Im}(\bar u\nabla u - \bar{\tilde u}\nabla\tilde u)\|_{L_T^{4/3}L^{4/3}} &\lesssim T^{3/8}\,\|\nabla(u+\tilde u)\|_{L_T^{8/3}L^4}\,\|u-\tilde u\|_{L_T^\infty L^2} \\ &\lesssim T^{3/8}\max\{R_1,R_2\}\|(u-\tilde u, A-\tilde A)\|_{\mathcal B}, \end{aligned}$$

where we use the same interpolation inequality as in (2.57). The other terms in the difference of the current densities are handled by the estimate

$$\begin{aligned} \big\|A\,|u|^2 - \tilde A\,|\tilde u|^2\big\|_{L_T^{4/3}L^{4/3}} &= \big\|(A-\tilde A)\,|u|^2 + \tilde A u(\bar u - \bar{\tilde u}) + \tilde A(u-\tilde u)\bar{\tilde u}\big\|_{L_T^{4/3}L^{4/3}} \\ &\lesssim T^{1/2}\|A-\tilde A\|_{L_T^4L^4}\,\|u\|^2_{L_T^\infty L^4} \\ &\quad + T^{1/2}\|\tilde A\|_{L_T^\infty L^6}\,\|u+\tilde u\|_{L_T^\infty L^{12}}\,\|u-\tilde u\|_{L_T^\infty L^2} \\ &\lesssim T^{1/2}\max\{R_1^2,R_3\}\|(u-\tilde u,A-\tilde A)\|_{\mathcal B}. \end{aligned}$$

We obtain that

$$\begin{aligned} &\|B-\tilde B\|_{L_T^\infty H^{1/2}\cap L_T^4L^4} + \big\|\partial_t(B-\tilde B)\big\|_{L_T^\infty H^{-1/2}} \\ &\quad \le C\,\langle\lambda\rangle\,T^{3/8}\max\{R_1,R_2,R_3\}^2\|(u-\tilde u,A-\tilde A)\|_{\mathcal B}. \end{aligned} \tag{2.59}$$

Thus, in addition to (2.55), we require that T is chosen such that

$$C\,\langle\lambda\rangle\left(T^{3/8}\max\{R_1,R_2,R_3\}^2 + T^{1-\gamma/2}R_1^{p-\gamma-1}R_2^{\gamma}\right) \le \tfrac14. \tag{2.60}$$

We conclude that Φ is a contraction in the space $\mathcal B$ and hence it has a unique fixed point (u,A) in $\mathcal B$ which is the desired weak solution of the Maxwell–Schrödinger system on the interval $[0,T]$.

Second step. Uniqueness. In the given class, uniqueness is implicitly contained in the estimates required for the contraction argument in the first step. To make this more explicit, we argue as follows. Let (u,A) and $(\tilde u,\tilde A)$ be solutions of the Maxwell–Schrödinger system (2.11) on an arbitrary interval $[0,T]$ as in the statement of the lemma. In all the calculation leading to (2.58) and (2.59) we have some wiggle room when we apply Hölder's inequality in the time variable. Therefore, we can also show that there exists $C \in (0,\infty)$, $\alpha_1, \alpha_2 \in (1,\infty)$ and $\alpha_3 \in (1,4)$ such that

$$\begin{aligned} &\|u-\tilde{u}\|_{L^\infty(J,L^2)} + \|A-\tilde{A}\|_{L^\infty(J,H^{1/2})} + \|A-\tilde{A}\|_{L^4(J,L^4)} \\ &\quad \le C\Big(\|u-\tilde{u}\|_{L^{\alpha_1}(J,L^2)} + \|A-\tilde{A}\|_{L^{\alpha_2}(J,H^{1/2})} + \|A-\tilde{A}\|_{L^{\alpha_3}(J,L^4)}\Big) \end{aligned} \tag{2.61}$$

holds true for every interval $J \subseteq [0,T]$ with $0 \in J$. Here the constant C only depends on T and the norms of the functions $u, \tilde{u}$ in $L^\infty_T H^s(\mathbb{R}^3) \cap L^2_T H^{s-1/2,6}(\mathbb{R}^3)$ and $A, \tilde{A}$ in $L^\infty_T H^\sigma(\mathbb{R}^3) \cap W^{1,\infty}_T H^{\sigma-1}(\mathbb{R}^3) \cap L^2_T L^\infty(\mathbb{R}^3)$. The elementary Lemma 4.2.2 in [Caz03] shows that (2.61) implies that $u = \tilde{u}$ and $A = \tilde{A}$ almost everywhere on $[0,T]$.

Third step. Maximal solution and the blowup alternative. We define

$$T_{\max} = \sup\{T \in (0,\infty) : \text{there exists a solution } (u,A) \text{ of } (2.11)\}.$$

It follows from the first two steps that a solution exists on the interval $[0, T_{\max})$. Assume that $T_{\max} < \infty$ and there exists $M \in (0,\infty)$ with

$$\limsup_{t\to T_{\max}} \|(u(t), A(t), \partial_t A(t))\|_{X^{s,\sigma}} \le M.$$

According to (2.54) we choose $R_3(M) = 4CeM$, $R_1(M) = 4CMR_3^{2s}$, $R_2(M) = 2C\max\{\langle\lambda\rangle R_1, R_3\}^m R_1$ and $T(M) \in (0,1)$ satisfying (2.55) and (2.60). Starting at time $t_0 = T_{\max} - \frac{1}{2}T(M)$ with initial value $(u(t_0), A(t_0), \partial_t A(t_0))$, we can construct a solution of (2.11) on the interval $[t_0, t_0 + T(M)]$ by the first step, thereby extending a solution beyond $T_{\max}$. Since this is impossible, the blowup alternative is established.

Fourth step. Continuous dependence. The last assertion is also a consequence of the contraction argument in the first step. Fix $n_0 \in \mathbb{N}$ such that

$$\|(u_0^n, A_0^n, A_1^n)\|_{X^{s,\sigma}} \le 2\|(u_0, A_0, A_1)\|_{X^{s,\sigma}}$$

for all $n \ge n_0$. We define $R_3 = 8Ce\|(u_0, A_0, A_1)\|_{X^{s,\sigma}}$, $R_1 = 8CR_3^{2s}\|(u_0, A_0, A_1)\|_{X^{s,\sigma}}$ and $R_2 = 2C\max\{\langle\lambda\rangle R_1, R_3\}^m R_1$ and we choose $T \in (0,1)$ satisfying (2.55) and (2.60). From (2.54), it follows that the solutions $(u^n, A^n, \partial_t A^n)$ constructed in the first step belong to the same set $\mathcal{B}$ for all $n \ge n_0$. The choice of T, estimates (2.58) and (2.59), the unitarity of U_A and Lemma 2.4 imply that

$$\begin{aligned} \|(u^n - u, A^n - A)\|_{\mathcal{B}} &\le C\,\|u_0^n - u_0\|_{L^2} + C\,\|(A_0^n - A_0, A_1^n - A_1)\|_{H^{1/2}\times H^{-1/2}} \\ &\quad + \frac{1}{2}\,\|(u^n - u, A^n - A)\|_{\mathcal{B}} \end{aligned}$$

for all $n \ge n_0$. In particular, we conclude that $(u^n, A^n, \partial_t A^n)$ converges to $(u, A, \partial_t A)$ in $L^\infty_T X^{0,1/2}$ as $n \to \infty$ and that $(u^n, A^n, \partial_t A^n)$ is bounded in $L^\infty_T X^{s,\sigma}$. Interpolation finally yields convergence of the solutions in $L^\infty_T X^{s-\varepsilon,\sigma-\varepsilon}$ for every $\varepsilon > 0$. □

In Lemma 2.17 we show that this weak solution is in fact a *strong $X^{s,\sigma}$-solution* which even belongs to $C_T X^{s,\sigma} \cap C_T^1 X^{s-2,\sigma-1}$.

Lemma 2.17 (Regularity and conservation of mass and energy)
Let $s \in \left[\frac{11}{8}, 2\right]$ and $\sigma \in (1,\infty)$ such that $(s,\sigma) \in \mathcal{R}$. Let $p \in \big(s, p^*(s)\big)$.

Let (u, A) be the weak solution obtained in Lemma 2.16 on the time interval $[0,T]$ for some $T > 0$. Then (u, A) is a strong solution lying in the space $C_T X^{s,\sigma}$. Moreover, if $p > 2$, the mass $\|u(t)\|_{L^2}^2$ and the energy (2.14) are conserved, i.e. for all $t \in [0,T]$ we have

$$\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2$$

and

$$E\big(u(t), A(t), \partial_t A(t)\big) = E(u_0, eA_0, eA_1).$$

Note that the energy (2.14) of the Maxwell–Schrödinger system (2.11) yields the expression

$$E(u_0, eA_0, eA_1) = \|\nabla_A u_0\|_{L^2}^2 + \frac{\lambda}{2} \|\nabla \phi(u_0)\|_{L^2}^2 + \|\nabla A_0\|_{L^2}^2 + \|A_1\|_{L^2}^2 - \frac{2}{p+1} \|u_0\|_{L^{p+1}}^{p+1}.$$

It is only the second term which actually depends on the parameter $\lambda = \frac{e^2}{2}$.

Proof. We first discuss continuity of the solution (u, A) constructed in Lemma 2.16. In the proof of Lemma 2.16 we show in estimate (2.51) that $PJ(u, A) \in L_T^1 H^{\sigma-1}(\mathbb{R}^3)$. Hence, it follows from the Strichartz estimates stated in Lemma 2.4 that $A \in C_T H^\sigma(\mathbb{R}^3) \cap C_T^1 H^{\sigma-1}(\mathbb{R}^3)$. We next note that the function u solves the Schrödinger equation such that

$$u(t) = S(t)u_0 - \mathrm{i} \int_0^t S(t-\tau) \left[2\mathrm{i}A \cdot \nabla u + A \cdot Au + \lambda \phi u - |u|^{p-1} u\right] \mathrm{d}\tau \qquad (2.62)$$

for every $t \in [0,T]$. Since $u_0 \in H^s(\mathbb{R}^3)$, we note that the map $t \mapsto S(t)u_0$ belongs to $C_T H^s(\mathbb{R}^3)$. The term in brackets can be easily bounded in $L_T^1 L^2(\mathbb{R}^3)$ by the following computation

$$\begin{aligned} &\left\|2\mathrm{i}A \cdot \nabla u + A \cdot Au + \lambda \phi u - |u|^{p-1} u\right\|_{L_T^1 L^2} \\ &\lesssim T^{1/2} \|A\|_{L_T^2 L^\infty} \|\nabla u\|_{L_T^\infty L^2} + \|A\|_{L_T^2 L^\infty}^2 \|u\|_{L_T^\infty L^2} + \lambda T \|\phi(u) u\|_{L_T^\infty L^2} + T \|u\|_{L_T^\infty L^{2p}}^{2p} \\ &\lesssim T^{1/2} \|A\|_{L_T^2 L^\infty} \|u\|_{L_T^\infty H^s} + \|A\|_{L_T^2 L^\infty}^2 \|u\|_{L_T^\infty H^s} + \lambda T \|u\|_{L_T^\infty H^s}^3 + T \|u\|_{L_T^\infty H^s}^{2p}, \end{aligned}$$

where we use the Sobolev embedding $H^s(\mathbb{R}^3) \hookrightarrow L^{2p}(\mathbb{R}^3)$ which holds in particular whenever $p < p^*(s)$ and the estimate

$$\|\phi(u) u\|_{L_T^\infty L^2} \le \|\phi(u)\|_{L_T^\infty L^4} \|u\|_{L_T^\infty L^4} \lesssim \|u\|_{L_T^\infty L^{24/11}}^2 \|u\|_{L_T^\infty H^s} \lesssim \|u\|_{L_T^\infty H^s}^3,$$

which follows from Lemma 2.12 (1). We thus obtain that u belongs to $C_T L^2(\mathbb{R}^3)$ from formula (2.62). This and the boundedness of u in $L^\infty_T H^s(\mathbb{R}^3)$ imply by interpolation that $u \in C_T H^{\tilde{s}}$ for every $\tilde{s} \in [0, s)$. Using (2.29), we further obtain

$$\|u(t) - u(\tau)\|_{L^\infty} \leq \|u(t) - u(\tau)\|_{H^{5/4}}^{1/4} \|u(t) - u(\tau)\|_{H^{3/4,6}}^{3/4}, \quad t, \tau \in [0, T],$$

and we deduce in particular that $u \in C_T H^1(\mathbb{R}^3) \cap C_T L^\infty(\mathbb{R}^3)$. This implies that $U_A(\cdot, 0)u_0 \in C_T H^s(\mathbb{R}^3)$ by using Lemma 2.15 (3) and it also shows that the map $t \mapsto |u(t)|^{p-1} u(t)$ belongs to $L^\infty_T H^s(\mathbb{R}^3) \cap C_T H^{s-2}(\mathbb{R}^3)$. Hence, the Duhamel formula from Lemma 2.15 (5) yields that the map $t \mapsto \int_0^t U_A(t, \tau) |u(\tau)|^{p-1} u(\tau) \, \mathrm{d}\tau$ belongs to the space $C_T H^s(\mathbb{R}^3) \cap C^1_T H^{s-2}(\mathbb{R}^3)$. This shows that u is a strong H^s-solution.

Finally, we prove conservation of mass and energy. For strong H^2-solutions, this follows by direct computation, see Lemma 2.2. Consider a sequence (u_0^j, A_0^j) in $H^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ which converges to (u_0, A_0) in $H^s \times H^\sigma$. By the continuous dependence result in Lemma 2.16, we obtain that the solutions $(u_j, A_j)_j$ converge to (u, A) in $L^\infty_T X^{1,1}$. Combining Lemma 2.18 and Lemma 2.20, we show persistence of regularity for solutions of the magnetic Schrödinger equations, i.e. taking $\tilde{s} = 2$ in Lemma 2.20, we see that the H^2-norm of u does not blow-up on $[0, T]$. For this reason, we assume $p > 2$ in this step. From Lemma 2.4 and Lemma 2.11, it also follows that the H^2-norm of A does not blow-up on $[0, T]$. Therefore, the approximate solutions conserve mass and energy on the full interval $[0, T]$. Hence, we obtain that

$$E\big(u^j(t), A^j(t), \partial_t A^j(t)\big) \to E\big(u(t), A(t), \partial_t A(t)\big)$$

for $j \to \infty$ and every $t \in [0, T]$. $\square$

It is crucial for the fixed-point argument in Lemma 2.16 to assume that the functions u and A lie in certain auxiliary spaces which provide more integrability than $L^\infty_T X^{s,\sigma}$. In the next lemma we show that for any solution we can recover this extra integrability by using Strichartz estimates. This also allows us to deduce unconditional uniqueness of solutions. The main difficulty in the following proof is that we have to proceed in several small steps: Gaining a bit of integrability in the Maxwell part, we can use this to make progress on the Schrödinger part which in turn allows to further improve the Maxwell part and so on. The basic idea of this lemma is contained in Lemma 5.1 of [NW07] but the presence of our nonlinearity forces us to insert some extra steps in between.

Lemma 2.18 (Additional integrability of solutions)
Let $s \in \left[\frac{11}{8}, 2\right]$, $\sigma \in \left(1, \frac{10}{9}\right]$. Let $T > 0$ and let $(u, A, \partial_t A) \in C_T X^{s,\sigma}$ be a solution of the Maxwell–Schrödinger system (2.11). Recall from (2.42) the definitions of the positive number δ and of the Klein–Gordon admissible pair (q, r). Let $\tilde{s} \in [1, s]$ and let $R_{\tilde{s}} \in (0, \infty)$ such that

$$\|(u, A, \partial_t A)\|_{L^\infty_T X^{\tilde{s},1}} \leq R_{\tilde{s}}. \tag{2.63}$$

Then we have the following estimate

$$\|A\|_{L^q_T L^r} \leq C(R_1) \langle T \rangle . \tag{2.64}$$

Moreover, if $p \leq \frac{15-4\tilde{s}+8\delta}{9-6\tilde{s}}$ where $\tilde{s} \in \left[1, \min\{\frac{3}{2}-\delta, s\}\right]$ then we also have

$$\|u\|_{L^2_T H^{1/2-\delta,6}} \leq C(R_{\tilde{s}}) \langle T \rangle^3 , \tag{2.65}$$

$$\|A\|_{M^{1,\sigma}_T \cap L^2_T L^\infty} \leq C(R_{\tilde{s}}) \langle T \rangle^4 . \tag{2.66}$$

We note that $p^*(\tilde{s}) < \frac{15-4\tilde{s}+8\delta}{9-6\tilde{s}}$. If $p \leq p^*(s)$, we also have

$$\|u\|_{L^2_T H^{s-1/2,6}} \leq C(R_s) \langle T \rangle^m \tag{2.67}$$

where $m \in (0, \infty)$ is a constant.

Proof. To show (2.64), we apply Strichartz estimates from Lemma 2.4 with the admissible pairs $(\infty, 2)$ and (q, r) at the regularity level $H^{2/q}$ to the Klein–Gordon equation, i.e. to the second equation of the system (2.11), and obtain

$$\|A\|_{L^q_T L^r} \leq \|(A_0, A_1)\|_{H^{2/q} \times H^{2/q-1}} + \|A\|_{L^1_T H^{2/q-1,2}} + \|PJ(u, A)\|_{L^{q'}_T H^{4/q-1,r'}} .$$

Since $q > 2$, it follows that $\|A\|_{L^1_T H^{2/q-1,2}} \leq T \|A\|_{L^\infty_T L^2} \leq C(R_1)T$ by (2.63). For the last term above, we use Hölder's inequality in the time variable, Lemma 2.5 from [NW07], the fractional Leibniz rule, and the Sobolev embeddings $H^1(\mathbb{R}^3) \hookrightarrow H^{4/q-1,q}(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3)$ to obtain

$$\begin{aligned}
\|PJ(u, A)\|_{L^{q'}_T H^{4/q-1,r'}} &\lesssim T^{1/q'} \|Pu\nabla u\|_{L^\infty_T H^{4/q-1,r'}} + T^{1/q'} \left\|A \,|u|^2\right\|_{L^\infty_T H^{4/q-1,r'}} \\
&\lesssim T^{1/q'} \|u\|_{L^\infty_T H^{4/q-1,q}} \|\nabla u\|_{L^\infty_T L^2} \\
&\quad + T^{1/q'} \|A\|_{L^\infty_T H^{4/q-1,q}} \|u\|^2_{L^\infty_T L^4} \\
&\quad + T^{1/q'} \|u\|_{L^\infty_T H^{4/q-1,q}} \|u\|_{L^\infty_T L^4} \|A\|_{L^\infty_T L^4} \\
&\lesssim T^{1/q'} \|u\|^2_{L^\infty_T H^1} \left\langle \|A\|_{L^\infty_T H^1} \right\rangle \\
&\leq C(R_1) \langle T \rangle .
\end{aligned}$$

To show (2.65), we use Lemma 2.7 twice. We first apply the Strichartz estimate with loss (2.17) at the regularity level H^1 to the Schrödinger part where we use the loss $\alpha = 1 - \frac{1}{3}\tilde{s} + \frac{2}{3}\delta$ and the admissible pairs $(2, 6)$ and $(\infty, 2)$. Here we obtain

$$\begin{aligned}
&\|u\|_{L^2_T H^{(\tilde{s}-2\delta)/3,6}} \\
&\qquad \lesssim \|u\|_{L^\infty_T H^1} + T^{1/2} \left\|2\mathrm{i}A \cdot \nabla u + |A|^2 u + \phi(u) u + |u|^{p-1} u\right\|_{L^2_T H^{-(3-2\tilde{s}+4\delta)/3}} .
\end{aligned}$$

We note that $-\frac{3-2\tilde{s}+4\delta}{3} \le -2\delta$ due to the assumption $\tilde{s} \le \frac{3}{2} - \delta$. The Sobolev embedding $L^{6/(3+4\delta)}(\mathbb{R}^3) \hookrightarrow H^{-2\delta}(\mathbb{R}^3)$ and Hölder's inequality yield

$$\|A \cdot \nabla u\|_{L_T^2 H^{-2\delta}} \lesssim \|A \cdot \nabla u\|_{L_T^2 L^{6/(3+4\delta)}} \le T^{(q-2)/(2q)} \|A\|_{L_T^q L^r} \|\nabla u\|_{L_T^\infty L^2} .$$

Even simpler, just by using $L^2(\mathbb{R}^3) \hookrightarrow H^{-2\delta}(\mathbb{R}^3)$, the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, Hölder's inequality and Lemma 2.12 (1) we obtain

$$\left\||A|^2 u\right\|_{L_T^2 H^{-2\delta}} \lesssim \left\||A|^2 u\right\|_{L_T^2 L^2} \le T^{1/2} \|A\|_{L_T^\infty L^6}^2 \|u\|_{L_T^\infty L^6} \le T^{1/2} \|A\|_{L_T^\infty H^1}^2 \|u\|_{L_T^\infty H^1}$$

and

$$\|\phi(u)u\|_{L_T^2 H^{-2\delta}} \lesssim \|\phi(u)u\|_{L_T^2 L^2} \le T^{1/2} \|\phi(u)\|_{L_T^\infty L^4} \|u\|_{L_T^\infty L^4} \le T^{1/2} \|u\|_{L_T^\infty H^1}^3 .$$

From the Sobolev embedding $L^{18/(15-4\tilde{s}+8\delta)}(\mathbb{R}^3) \hookrightarrow H^{-(3-2\tilde{s}+4\delta)/3}(\mathbb{R}^3)$ and the embedding $H^{\tilde{s}}(\mathbb{R}^3) \hookrightarrow L^{18p/(15-4\tilde{s}+8\delta)}(\mathbb{R}^3)$, where we use the assumption $p \le \frac{15-4\tilde{s}+8\delta}{9-6\tilde{s}}$ for the latter embedding, we obtain

$$\left\||u|^{p-1} u\right\|_{L_T^2 H^{-(3-2\tilde{s}+4\delta)/3}} \lesssim T^{1/2} \|u\|_{L_T^\infty L^{18p/(15-4\tilde{s}+8\delta)}}^p \lesssim T^{1/2} \|u\|_{L_T^\infty H^{\tilde{s}}}^p .$$

We therefore conclude that

$$\|u\|_{L_T^2 H^{(\tilde{s}-2\delta)/3,6}} \lesssim C(R_{\tilde{s}}) \langle T \rangle^2 . \tag{2.68}$$

We apply again the Strichartz estimate with loss (2.17) at the regularity level H^1, here using the loss $\alpha = \frac{1}{2} + \delta$ and the admissible pairs $(2, 6)$ and $(\infty, 2)$. We obtain

$$\|u\|_{L_T^2 H^{1/2-\delta,6}} \lesssim \|u\|_{L_T^\infty H^1} + T^{1/2} \left\|2\mathrm{i}A \cdot \nabla u + |A|^2 u + \phi(u)u + |u|^{p-1} u\right\|_{L_T^2 H^{-2\delta}}.$$

We know from the step before that

$$\left\|2\mathrm{i}A \cdot \nabla u + |A|^2 u + \phi(u)u\right\|_{L_T^2 H^{-2\delta}} \lesssim C(R_{\tilde{s}}) \langle T \rangle^2 .$$

If $p \in [1, 2]$, we use the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^{2p}(\mathbb{R}^3)$ and obtain that

$$\left\||u|^{p-1} u\right\|_{L_T^2 H^{-2\delta}} \le T^{1/2} \|u\|_{L_T^\infty L^{2p}}^p \lesssim T^{1/2} \|u\|_{L_T^\infty H^1}^p \le C(R_{\tilde{s}}) \langle T \rangle . \tag{2.69}$$

Otherwise, if $p > 2$, we next use the Sobolev embeddings $L^{6/(3+4\delta)}(\mathbb{R}^3) \hookrightarrow H^{-2\delta}(\mathbb{R}^3)$, $H^{(\tilde{s}-2\delta)/3,6}(\mathbb{R}^3) \hookrightarrow L^{18/(3-2\tilde{s}+4\delta)}$ and $H^{\tilde{s}}(\mathbb{R}^3) \hookrightarrow L^{18(p-1)/(6+2\tilde{s}+8\delta)}(\mathbb{R}^3)$, the latter embedding uses the assumption $p \le \frac{15-4\tilde{s}+8\delta}{9-6\tilde{s}}$, and the interpolation

$$\left[L_T^2 L^{18/(3-2\tilde{s}+4\delta)}(\mathbb{R}^3), L_T^\infty L^{18(p-1)/(6+2\tilde{s}+8\delta)}(\mathbb{R}^3)\right]_{1/p} = L_T^{2p} L^{6p/(3+4\delta)}(\mathbb{R}^3)$$

to estimate

$$\begin{aligned}\left\||u|^{p-1}u\right\|_{L_T^2H^{-2\delta}} &\lesssim \|u\|^p_{L_T^{2p}L^{6p/(3+4\delta)}}\\ &\lesssim \|u\|_{L_T^2L^{18/(3-2\tilde{s}+4\delta)}}\|u\|^{p-1}_{L_T^\infty L^{18(p-1)/(6+2\tilde{s}+8\delta)}}\\ &\lesssim \|u\|_{L_T^2H^{(\tilde{s}-2\delta)/3,6}}\|u\|^{p-1}_{L_T^\infty H^{\tilde{s}}}\,.\end{aligned}$$

Due to (2.68), the estimates above yield (2.65).

With this additional integrability of u we can go back to the Maxwell part and improve the estimates for A. Applying once again Strichartz estimates from Lemma 2.4 with the admissible pairs $(\infty, 2)$ and $(6, 3)$, we obtain

$$\|A\|_{M_T^{1,\sigma}\cap L_T^qH^{\sigma-2/q,r}} \lesssim \|(A_0,A_1)\|_{H^\sigma\times H^{\sigma-1}}+T\|A\|_{L_T^\infty H^{\sigma-1}}+\|PJ(u,A)\|_{L_T^{6/5}H^{\sigma-2/3,3/2}}\,.$$

Using Hölder's inequality in the time variable, Lemma 2.5 from [NW07], the fractional Leibniz rule and the Sobolev embeddings $H^{1/2,6}(\mathbb{R}^3)\hookrightarrow H^{\sigma-2/3,6}(\mathbb{R}^3)$, $H^1(\mathbb{R}^3)\hookrightarrow H^{\sigma-2/3,18/7}(\mathbb{R}^3)$ and $H^{1/2-\delta,6}(\mathbb{R}^3)\hookrightarrow L^9(\mathbb{R}^3)$ which make use of the assumption that $\sigma\le\frac{10}{9}$, we deduce

$$\begin{aligned}\|PJ(u,A)\|_{L_T^{6/5}H^{\sigma-2/3,3/2}} &\lesssim T^{1/3}\|Pu\nabla u\|_{L_T^2H^{\sigma-2/3,3/2}}+T^{1/3}\left\|A|u|^2\right\|_{L_T^2H^{\sigma-2/3,3/2}}\\ &\lesssim T^{1/3}\|u\|_{L_T^2H^{\sigma-2/3,6}}\|\nabla u\|_{L_T^\infty L^2}\\ &\quad+T^{1/3}\|A\|_{L_T^\infty H^{\sigma-2/3,18/7}}\|u\|_{L_T^2L^9}\|u\|_{L_T^\infty L^6}\\ &\quad+T^{1/3}\|u\|_{L_T^2H^{\sigma-2/3,6}}\|u\|_{L_T^\infty L^3}\|A\|_{L_T^\infty L^6}\\ &\lesssim T^{1/3}\|u\|_{L_T^2H^{1/2-\delta,6}}\|u\|_{L_T^\infty H^1}\left\langle\|A\|_{L_T^\infty H^1}\right\rangle\\ &\le C(R_{\tilde{s}})\langle T\rangle^{3+1/3}\,.\end{aligned}$$

Since $\sigma-\frac{2}{q}-\frac{3}{r}=\frac{2(\sigma-1)}{3}>0$, the Sobolev embedding $H^{\sigma-2/q,r}(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$ finally shows that

$$\|A\|_{L_T^2L^\infty}\lesssim T^{1/r}\|A\|_{L_T^qH^{\sigma-2/q,r}}\le C(R_{\tilde{s}})\langle T\rangle^4$$

and so (2.66) is proved.

We proceed to the proof of (2.67). As before, we apply Lemma 2.7 twice. We first use (2.17) at the regularity level H^s with loss $\alpha=\frac{s}{2}$. Here we obtain

$$\|u\|_{L_T^2H^{s/2,6}}\lesssim\|u\|_{L_T^\infty H^s}+T^{1/2}\left\|2\mathrm{i}A\cdot\nabla u+|A|^2u+\phi(u)u+|u|^{p-1}u\right\|_{L_T^2L^2}.$$

The proof of Lemma 2.14 shows that

$$\left\|2\mathrm{i}A\cdot\nabla u+|A|^2u+\phi(u)u\right\|_{L_T^2L^2}\le C(R_s)\langle T\rangle^m\,.$$

If $p \in [1,2]$, we can also use (2.69). If $p > 2$, we furthermore estimate

$$\left\| |u|^{p-1} u \right\|_{L^2_T L^2} \lesssim \|u\|^p_{L^{2p}_T L^{2p}} \lesssim \langle T \rangle \|u\|_{L^2_T H^{1/2-\delta,6}} \|u\|^{p-1}_{L^\infty_T H^s} \lesssim C(R_s) \langle T \rangle^4 ,$$

where we use

$$\left[H^{1/2-\delta,6}(\mathbb{R}^3), H^s(\mathbb{R}^3)\right]_{1/p} = H^{(1+2(p-1)s-2\delta)/(2p),6p/(3p-2)}(\mathbb{R}^3) \hookrightarrow L^{2p}(\mathbb{R}^3),$$

by interpolation and a Sobolev embedding which holds if $p \leq \frac{6-2s-2\delta}{3-2s}$. Hence, we can use

$$\|u\|_{L^2_T H^{s/2,6}} \leq C(R_s) \langle T \rangle^m$$

to apply Lemma 2.7 a last time. We apply (2.17) at the regularity level H^s with loss $\alpha = \frac{1}{2}$ and arrive at

$$\|u\|_{L^2_T H^{s-1/2,6}} \lesssim \|u\|_{L^\infty_T H^s} + T^{1/2} \left\| 2\mathrm{i} A \cdot \nabla u + |A|^2 u + \phi(u) u + |u|^{p-1} u \right\|_{L^2_T H^{s-1}}.$$

As above, we know from the proof of Lemma 2.14 that

$$\left\| 2\mathrm{i} A \cdot \nabla u + |A|^2 u + \phi(u) u \right\|_{L^2_T H^{s-1}} \leq C(R_s) \langle T \rangle^m .$$

In the case where $p \in [1,2]$, we use (2.33) and we obtain with the Sobolev embedding $H^{s/2,6}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ that

$$\left\| |u|^{p-1} u \right\|_{L^2_T H^{s-1}} \lesssim \|u\|^{p-1}_{L^2_T L^\infty} \|u\|_{L^\infty_T H^{s-1}} \lesssim C(R_s) \|u\|^{p-1}_{L^2_T H^{s/2,6}} \leq C(R_s) \langle T \rangle^m .$$

Let $p > 2$ in the remaining part of the proof. Due to the Sobolev embedding $H^{1,6/(7-2s)}(\mathbb{R}^3) \hookrightarrow H^{s-1}(\mathbb{R}^3)$ and (2.23), we consider

$$\begin{aligned}
\left\| |u|^{p-1} u \right\|_{L^2_T H^{s-1}} &\lesssim \left\| |u|^{p-1} u \right\|_{L^2_T H^{1,6/(7-2s)}} \\
&\lesssim \left\| |u|^{p-1} u \right\|_{L^2_T L^{6/(7-2s)}} + \left\| |u|^{p-1} \nabla u \right\|_{L^2_T L^{6/(7-2s)}} .
\end{aligned}$$

The first term can be essentially treated as in the step before. From the Sobolev embedding $H^s(\mathbb{R}^3) \hookrightarrow H^{1,6/(5-2s)}(\mathbb{R}^3)$ and

$$\left[H^{s/2,6}(\mathbb{R}^3), H^s(\mathbb{R}^3)\right]_{1/(p-1)} = H^{s(2p-3)/2(p-1),6(p-1)/(3p-5)}(\mathbb{R}^3) \hookrightarrow L^{3(p-1)}(\mathbb{R}^3),$$

which holds for every $p \leq \frac{7-3s}{(3-2s)_+}$, we deduce for every $t \in [0,T]$ that

$$\begin{aligned}
\left\| |u(t)|^{p-1} \nabla u(t) \right\|_{L^{6/(7-2s)}} &\lesssim \|u(t)\|^{p-1}_{L^{3(p-1)}} \|\nabla u(t)\|_{L^{6/(5-2s)}} \\
&\lesssim \|u(t)\|_{H^{s/2,6}} \|u(t)\|^{p-2}_{H^s} \|\nabla u(t)\|_{H^{s-1}} .
\end{aligned}$$

We thus conclude that

$$\left\| |u|^{p-1} u \right\|_{L^2_T H^{s-1}} \lesssim C(R_s) \langle T \rangle^m ,$$

and this finishes the proof of (2.67). □

Corollary 2.19 (Unconditional uniqueness of solutions)
Let $s \in \left[\frac{11}{8}, 2\right]$ and $\sigma \in (1,\infty)$ such that $(s,\sigma) \in \mathcal{R}$. Let $p \in \big(s, p^*(s)\big)$. For each $(u_0, A_0, A_1) \in X^{s,\sigma}$, the Maxwell–Schrödinger system (2.11) has at most one solution (u, A) belonging to $C_T X^{s,\sigma}$ with initial value (u_0, A_0, A_1).

Proof. By Lemma 2.18, we find that u belongs to $L^2_T H^{s-1/2,6}(\mathbb{R}^3)$ and A belongs to $L^2_T L^\infty(\mathbb{R}^3)$. Uniqueness of solutions in this class is guaranteed by Lemma 2.16. □

Continuous dependence

The next lemmata contain the technical estimates which are necessary to show the continuous dependence of the solution on the initial data. They serve as a preparation for Lemma 2.22. To this end, we have to control the difference of two solutions starting from different initial values. In the fixed-point argument in Lemma 2.16, we study the difference of two solutions to the linearized equations but only at the regularity level L^2 for the Schrödinger- and $H^{1/2}$ for the Maxwell part. Here we are aiming to obtain a result in the topology of the solution space $X^{s,\sigma}$. Therefore, we need to compare two solutions in higher order norms and this poses additional difficulties. As we see in (2.75), Lemma 2.21, our estimates go through with reduced regularity $X^{s-1,\sigma-\delta}$. However, when working with full regularity $X^{s,\sigma}$ we cannot treat some terms appearing in the estimates unless we assume that for one of the two solutions the Schrödinger part lies in H^{s+1}. We obtain a bound for the H^{s+1}-norm of the Schrödinger part in Lemma 2.20. Here we make use of Lemma 3.4 in [NW07], whose proof is rather involved, and we apply Gronwall's inequality.

Lemma 2.20 (Persistence of higher regularity)
Let $s \in (1,3)$ and $\sigma \in (1,\infty)$ such that $(s+1,\sigma) \in \mathcal{R}_*$. Let $p \in \big(s+1, p^*(s)\big)$. Let $T \in (0,\infty)$. Let $(u, A) \in L^\infty_T X^{s+1,\sigma}$ be a solution of the Maxwell–Schrödinger system (2.11). Let $R \geq 0$ be a bound such that

$$\|u\|_{L^\infty_T H^s \cap W^{1,\infty}_T H^{s-2} \cap L^2_T H^{s-1/2,6}} + \|A\|_{M^{2,\sigma}_T \cap L^2_T L^\infty} \leq R, \tag{2.70}$$

Then there exists a constant $C(R) \in (0,\infty)$ for which we have

$$\|u\|_{L^\infty_T H^{s+1} \cap W^{1,\infty}_T H^{s-1}} \leq C(R) \|u_0\|_{H^{s+1}}.$$

We may replace $s+1$ by any $\tilde{s} \in (s, 4)$ in the assumptions of this lemma and we still obtain that

$$\|u\|_{L^\infty_T H^{\tilde{s}}} \leq C(R) \|u_0\|_{H^{\tilde{s}}}.$$

Proof. In this proof, we denote by $C(R)$ any constant which only depends on the quantities in (2.70). Thus, $C(R)$ may change from line to line. From Lemma 3.4 in

[NW07], we deduce that

$$\sup_{0\le t,\tau\le T} \|U_A(t,\tau)\|_{H^{s+1}\to H^{s+1}} \le C(R).$$

Since $p < p^*(s)$, we can choose $\gamma \in \big(\gamma(p-1,\infty,s), p-1\big)$ with $\gamma < 2$. Let $t \in [0,T]$. Starting from

$$u(t) = U_A(t,0)u_0 + \mathrm{i}\int_0^t U_A(t,\tau)\,|u(\tau)|^{p-1}\,u(\tau)\,\mathrm{d}\tau,$$

we compute

$$\begin{aligned}
\|u(t)\|_{H^{s+1}} &\le \|U_A(t,0)u_0\|_{H^{s+1}} + \int_0^t \left\|U_A(t,\tau)\,|u(\tau)|^{p-1}\,u(\tau)\right\|_{H^{s+1}}\,\mathrm{d}\tau \\
&\le C(R)\,\|u_0\|_{H^{s+1}} + C(R)\int_0^t \left\||u(\tau)|^{p-1}\,u(\tau)\right\|_{H^{s+1}}\,\mathrm{d}\tau \\
&\lesssim C(R)\,\|u_0\|_{H^{s+1}} + C(R)\int_0^t \left\||u(\tau)|^{p-1}\right\|_{L^\infty}\|u(\tau)\|_{H^{s+1}}\,\mathrm{d}\tau \\
&\lesssim C(R)\,\|u_0\|_{H^{s+1}} + C(R)\int_0^t \|u(\tau)\|_{H^s}^{p-1-\gamma}\,\|u(\tau)\|_{H^{s-1/2,6}}^{\gamma}\,\|u(\tau)\|_{H^{s+1}}\,\mathrm{d}\tau \\
&\le C(R)\,\|u_0\|_{H^{s+1}} + C(R)\int_0^t \|u(\tau)\|_{H^{s-1/2,6}}^{\gamma}\,\|u(\tau)\|_{H^{s+1}}\,\mathrm{d}\tau.
\end{aligned}$$

where we use (2.33) and Lemma 2.9. With the inequalities of Gronwall and Hölder we obtain the desired estimate

$$\begin{aligned}
\|u(t)\|_{H^{s+1}} &\le C(R)\,\|u_0\|_{H^{s+1}}\exp\left(C(R)\int_0^t \|u(\tau)\|_{H^{s-1/2,6}}^{\gamma}\,\mathrm{d}\tau\right) \\
&\le C(R)\,\|u_0\|_{H^{s+1}}\exp\left(C(R)t^{(2-\gamma)/2}\,\|u\|_{L_T^2 H^{s-1/2,6}}\right) \\
&\le C(R)\,\|u_0\|_{H^{s+1}}
\end{aligned}$$

for every $t \in [0,T]$. The addendum is proved analogously. From the equation, it further follows that

$$\begin{aligned}
\|\partial_t u\|_{L_T^\infty H^{s-1}} &\lesssim \|\Delta_A\,u\|_{L_T^\infty H^{s-1}} + \|\lambda\phi(u)u\|_{L_T^\infty H^{s-1}} + \left\||u|^{p-1}\,u\right\|_{L_T^\infty H^{s-1}} \\
&\lesssim C(R)\,\|u\|_{L_T^\infty H^{s+1}} + \|u\|_{L_T^\infty L^\infty}^{p-1}\,\|u\|_{L_T^\infty H^{s-1}} \\
&\lesssim C(R)\,\|u\|_{L_T^\infty H^{s+1}}
\end{aligned}$$

from which we deduce the assertion. Here we use (3.13) from [NW07] and the estimate

$$\|u\|_{L_T^\infty L^\infty}^{p-1} \lesssim \|u\|_{L_T^\infty H^{s+1}}^{\theta(p-1)}\,\|u\|_{L_T^\infty H^s}^{(1-\theta)(p-1)}$$

for some $\theta \in [0,1]$ satisfying $\theta > \frac{3}{2} - s$ and $\theta(p-1) \leq 1$. We also rely on $p < p^*(s)$ in this last step. $\square$

Lemma 2.21 (Comparing two solutions, cf. Lemma 6.1, [NW07])
Let $s \in \left[\frac{11}{8}, 2\right]$ and $\sigma \in (1,\infty)$ such that $(s,\sigma) \in \mathcal{R}$ and $(s+1,\sigma) \in \mathcal{R}_*$. Let $p \in \big(2, \tilde{p}^*(s)\big)$. Let $T \in (0,\infty)$. Let $(u,A) \in C([0,T], X^{s,\sigma})$ and $(\tilde{u},\tilde{A}) \in C([0,T], X^{s+1,\sigma})$ be solutions of the Maxwell–Schrödinger system (2.11) on $[0,T]$ with initial functions $(u_0, A_0, A_1) \in X^{s,\sigma}$ and $(\tilde{u}_0, \tilde{A}_0, \tilde{A}_1) \in X^{s+1,\sigma}$, respectively. Recall from (2.42) the definitions of the positive number δ and of the Klein–Gordon admissible pair (q,r). Let $R \geq 0$ be a bound such that

$$\|u\|_{L^\infty_T H^s \cap W^{1,\infty}_T H^{s-2} \cap L^2_T H^{s-1/2,6}} + \|A\|_{M^{1,\sigma}_T \cap L^q_T H^{\sigma-2/q,r}} \leq R, \tag{2.71}$$

$$\|\tilde{u}\|_{L^\infty_T H^s \cap W^{1,\infty}_T H^{s-2} \cap L^2_T H^{s-1/2,6}} + \|\tilde{A}\|_{M^{1,\sigma}_T \cap L^q_T H^{\sigma-2/q,r}} \leq R. \tag{2.72}$$

Then there exists a constant $C(R) \in (0,\infty)$ and a number $\theta_s \in (0,1]$ such that the estimates

$$\begin{aligned}
&\|u-\tilde{u}\|_{L^\infty_T H^s \cap L^2_T H^{s-1/2,6}} + \|A-\tilde{A}\|_{M^{1,\sigma}_T \cap L^q_T H^{\sigma-2/q,r}} \\
&\quad \leq C(R)\|(u_0,A_0,A_1) - (\tilde{u}_0,\tilde{A}_0,\tilde{A}_1)\|_{X^{s,\sigma}} \\
&\qquad + C(R)\|A-\tilde{A}\|_{L^\infty_T H^1 \cap L^q_T H^{\sigma-2/q-\delta,r}} \|\tilde{u}\|_{W^{1,\infty}_T H^{s-1}} \\
&\qquad + C(R)\,\|u-\tilde{u}\|^{\theta_s}_{L^\infty_T H^{s-1}} \|\tilde{u}\|^{\theta_s}_{W^{1,\infty}_T H^{s-1}}\,.
\end{aligned} \tag{2.73}$$

and

$$\begin{aligned}
\|A-\tilde{A}\|_{M^{1,\sigma}_T \cap L^q_T H^{\sigma-2/q,r}} &\leq C(R)\|(A_0,A_1) - (\tilde{A}_0,\tilde{A}_1)\|_{H^\sigma \times H^{\sigma-1}} \\
&\quad + \langle\lambda\rangle\, C(R)\,\|u-\tilde{u}\|_{L^\infty_T H^s \cap L^2_T H^{s-1/2,6}}
\end{aligned} \tag{2.74}$$

hold true. Moreover, if additionally $s > \frac{11}{8}$ and $\sigma \leq 2s - \frac{7}{4}$ we have

$$\begin{aligned}
&\|u-\tilde{u}\|_{L^\infty_T H^{s-1}} + \|A-\tilde{A}\|_{M^{1,\sigma-\delta}_T \cap L^q_T H^{\sigma-2/q-\delta,r}} \\
&\quad \leq C(R)\|(u_0,A_0,A_1) - (\tilde{u}_0,\tilde{A}_0,\tilde{A}_1)\|_{X^{s-1,\sigma-\delta}}.
\end{aligned} \tag{2.75}$$

Proof. Assume that $T \in (0,1)$. We start with the Schrödinger equation and take the difference between the two solutions, resulting in

$$\begin{aligned}
\mathrm{i}\partial_t(u-\tilde{u}) &= -\Delta_A(u-\tilde{u}) + \lambda\phi(u)(u-\tilde{u}) + 2\mathrm{i}(A-\tilde{A})\cdot\nabla\tilde{u} \\
&\qquad + (A-\tilde{A})(A+\tilde{A})\tilde{u} + \lambda\big(\phi(u)-\phi(\tilde{u})\big)\tilde{u} + |u|^{p-1}u - |\tilde{u}|^{p-1}\tilde{u} \\
&= \big(-\Delta_A + \lambda\phi(u)\big)(u-\tilde{u}) + F_1,
\end{aligned} \tag{2.76}$$

where we introduce the abbreviation F_1 for all remaining terms. If we formulate equation (2.76) with Duhamel's formula and the evolution family U_A, we get equation (2.56) plus the term $U_A(\cdot,0)(u_0-\tilde{u}_0)$. The time derivative of equation (2.76) is given by

$$\begin{aligned}
\mathrm{i}\partial_t^2(u-\tilde{u}) = &-\Delta_A\,\partial_t(u-\tilde{u}) + \lambda\phi(u)\partial_t(u-\tilde{u}) \\
&+ 2\mathrm{i}\partial_t A\cdot\nabla(u-\tilde{u}) + 2\partial_t A\cdot A(u-\tilde{u}) + \partial_t\lambda\phi(u)(u-\tilde{u}) \\
&+ 2\mathrm{i}\partial_t(A-\tilde{A})\cdot\nabla\tilde{u} + 2\mathrm{i}(A-\tilde{A})\cdot\nabla\partial_t\tilde{u} + \partial_t(A-\tilde{A})\cdot(A+\tilde{A})\tilde{u} \\
&+ (A-\tilde{A})\cdot\partial_t(A+\tilde{A})\tilde{u} + (A-\tilde{A})\cdot(A+\tilde{A})\partial_t\tilde{u} + \lambda\partial_t\big(\phi(u)-\phi(\tilde{u})\big)\tilde{u} \\
&+ \lambda\big(\phi(u)-\phi(\tilde{u})\big)\partial_t\tilde{u} + \partial_t(|u|^{p-1}\,u - |\tilde{u}|^{p-1}\,\tilde{u}) \\
=& \big(-\Delta_A + \lambda\phi(u)\big)\partial_t(u-\tilde{u}) + F_2,
\end{aligned}$$

where we write F_2 for all remaining terms.

Throughout this proof we use the expression $C(R)$ to denote a constant that arises from any estimate using (2.71) or (2.72). Thus, $C(R)$ may change from line to line. We use equation (2.76) to estimate the difference $u-\tilde{u}$ in the space $L_T^2 H^{s-1/2,6}(\mathbb{R}^3)$ by applying Lemma 2.14. The terms in F_1, except the last one, are amenable to the same methods as in the proof of Lemma 2.14. Treating $(A-\tilde{A})\cdot\nabla\tilde{u}$ in the same way as $A\cdot\nabla v$ in Lemma 2.14, we obtain

$$\begin{aligned}
\|(A-\tilde{A})\cdot\nabla\tilde{u}\|_{L_T^2 H^{s-1}} &\lesssim \|A-\tilde{A}\|_{L_T^\infty H^\sigma\cap L_T^2 L^\infty}\,\|\tilde{u}\|_{L_T^\infty H^s}^{1-\theta}\,\|\tilde{u}\|_{L_T^2 H^{s-1/2,6}}^{\theta} \\
&\lesssim T^{1-q/2}C(R)\|A-\tilde{A}\|_{M_T^{1,\sigma}\cap L_T^q H^{\sigma-2/q,r}},
\end{aligned}$$

where we use the Sobolev embedding $L_T^q H^{\sigma-2/q,r}(\mathbb{R}^3)\hookrightarrow L_T^2 L^\infty(\mathbb{R}^3)$. The term $(A-\tilde{A})\cdot(A+\tilde{A})\tilde{u}$ is basically the same as $A\cdot Av$ in Lemma 2.14 so that we similarly obtain

$$\|(A-\tilde{A})\cdot(A+\tilde{A})\tilde{u}\|_{L_T^2 H^{s-1}} \lesssim T^{1-q/2}C(R)\|A-\tilde{A}\|_{M_T^{1,\sigma}\cap L_T^q H^{\sigma-2/q,r}}.$$

For the next term $\big(\phi(u)-\phi(\tilde{u})\big)\tilde{u}$, we obtain

$$\big\|\big(\phi(u)-\phi(\tilde{u})\big)\tilde{u}\big\|_{L_T^2 H^{s-1}} \lesssim T^{1/2}C(R)\,\|u-\tilde{u}\|_{L_T^\infty H^1} \lesssim T^{1/2}C(R)\,\|u-\tilde{u}\|_{L_T^\infty H^s}$$

by following the reasoning in the proof of Lemma 2.14 and observing that

$$\phi(u)-\phi(\tilde{u}) = -\Delta^{-1}\big(u(\bar{u}-\bar{\tilde{u}})\big) - \Delta^{-1}\big((u-\tilde{u})\bar{\tilde{u}}\big).$$

Therefore, Lemma 2.14 yields

$$\begin{aligned}
\|u-\tilde{u}\|_{L_T^2 H^{s-1/2,6}} &\le C(R)\,\|u-\tilde{u}\|_{L_T^\infty H^s} + T^{1/2}\,\|F_1\|_{L_T^2 H^{s-1}} \\
&\le C(R)\big(\|u-\tilde{u}\|_{L_T^\infty H^s} + \|A-\tilde{A}\|_{L_T^\infty H^\sigma\cap L_T^q H^{\sigma-2/q,r}}\big) \qquad (2.77)\\
&\quad + T^{1/2}\big\||u|^{p-1}\,u - |\tilde{u}|^{p-1}\,\tilde{u}\big\|_{L_T^2 H^{s-1}}.
\end{aligned}$$

Furthermore, we estimate

$$\left\| |u|^{p-1} u - |\tilde{u}|^{p-1} \tilde{u} \right\|_{L^2_T L^2} \le \left(\left\| |u|^{p-1} \right\|_{L^2_T L^3} + \left\| |\tilde{u}|^{p-1} \right\|_{L^2_T L^3} \right) \|u - \tilde{u}\|_{L^\infty_T L^6} ,$$

employing (2.22) and Hölder's inequality. Using also (2.24), we further obtain

$$\begin{aligned} &\left\| |u|^{p-1} u - |\tilde{u}|^{p-1} \tilde{u} \right\|_{L^2_T H^1} \\ &\quad \lesssim \left\| |u|^{p-1} u - |\tilde{u}|^{p-1} \tilde{u} \right\|_{L^2_T L^2} + \left\| |u|^{p-1} \right\|_{L^2_T L^3} \|\nabla u - \nabla \tilde{u}\|_{L^\infty_T L^6} \\ &\qquad + \Big(\left\| |u|^{p-2} \right\|_{L^2_T L^\infty} + \left\| |\tilde{u}|^{p-2} \right\|_{L^2_T L^\infty} \Big) \|\nabla \tilde{u}\|_{L^\infty_T L^2} \|u - \tilde{u}\|_{L^\infty_T L^\infty} . \end{aligned}$$

We observe from (2.31) that $\gamma(p-1,3,s) < 1$ if $p < \frac{6-2s}{3-2s}$ and that $\gamma(p-2,\infty,s) < 1$ if $p < \frac{4-2s}{3-2s} = \tilde{p}^*(s)$. Therefore, there exists $\gamma < 1$ such that (2.29) implies the estimates

$$\left\| |u|^{p-1} \right\|_{L^2_T L^3} \lesssim T^{(1-\gamma)/2} \|u\|^{p-1-\gamma}_{L^\infty_T H^s} \|u\|^{\gamma}_{L^2_T H^{s-1/2,6}} \lesssim T^{(1-\gamma)/2} C(R)$$

and

$$\left\| |u|^{p-2} \right\|_{L^2_T L^\infty} \lesssim T^{(1-\gamma)/2} \|u\|^{p-2-\gamma}_{L^\infty_T H^s} \|u\|^{\gamma}_{L^2_T H^{s-1/2,6}} \lesssim T^{(1-\gamma)/2} C(R).$$

We also have the same estimates for the function $\tilde{u}$. By using in addition the Sobolev embeddings $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, we thus obtain

$$\left\| |u|^{p-1} u - |\tilde{u}|^{p-1} \tilde{u} \right\|_{L^2_T L^2} \le T^{(1-\gamma)/2} C(R) \|u - \tilde{u}\|_{L^\infty_T H^1} , \tag{2.78}$$

and

$$\left\| |u|^{p-1} u - |\tilde{u}|^{p-1} \tilde{u} \right\|_{L^2_T H^1} \le T^{(1-\gamma)/2} C(R) \|u - \tilde{u}\|_{L^\infty_T H^2} . \tag{2.79}$$

Similarly as in (2.28) we write

$$|u|^{p-1} u - |\tilde{u}|^{p-1} \tilde{u} = \int_0^1 g'\big(\tau u + (1-\tau)\tilde{u}\big) \,\mathrm{d}\tau \,(u - \tilde{u}). \tag{2.80}$$

Freezing the terms under the integral and regarding the right side of (2.80) as a linear map Ψ, the calculations leading to the estimates (2.78) and (2.79) imply that Ψ belongs to $\mathcal{L}(H^1(\mathbb{R}^3), L^2(\mathbb{R}^3))$ and $\mathcal{L}(H^2(\mathbb{R}^3), H^1(\mathbb{R}^3))$. Interpolation yields

$$\left\| |u|^{p-1} u - |\tilde{u}|^{p-1} \tilde{u} \right\|_{L^2_T H^{s-1}} \le T^{(1-\gamma)/2} C(R) \|u - \tilde{u}\|_{L^\infty_T H^s} . \tag{2.81}$$

For small T, this term can absorbed in the final estimate.

To estimate the difference $u - \tilde{u}$ in the space $L^\infty_T H^s(\mathbb{R}^3)$, we use the equation above for the second derivative. We rewrite this equation in integral form using the evolution family to obtain

$$\partial_t (u - \tilde{u})(t) = U_A(t,0) \partial_t (u - \tilde{u})(0) - \mathrm{i} \int_0^t U_A(t,\tau) F_2(\tau) \,\mathrm{d}\tau. \tag{2.82}$$

By (6.6) in the proof of Lemma 6.1 in [NW07], one can trade one time derivative for two spatial derivatives, meaning that one obtains the estimates

$$\begin{aligned}&\|u-\tilde{u}\|_{H^s}+C(R)\big(\|u-\tilde{u}\|_{L^2}+\|A-\tilde{A}\|_{H^\sigma}\big)\\&\quad\lesssim\|\partial_t(u-\tilde{u})\|_{H^{s-2}}+C(R)\big(\|u-\tilde{u}\|_{L^2}+\|A-\tilde{A}\|_{H^\sigma}\big)\\&\quad\lesssim\|u-\tilde{u}\|_{H^s}+C(R)\big(\|u-\tilde{u}\|_{L^2}+\|A-\tilde{A}\|_{H^\sigma}\big).\end{aligned}\tag{2.83}$$

We estimate (2.82) in the usual way and apply (2.83) to obtain

$$\begin{aligned}&\|u-\tilde{u}\|_{L^\infty_T H^s\cap W^{1,\infty}_T H^{s-2}}\\&\quad\le C(R)\|(u_0,A_0,A_1)-(\tilde{u}_0,\tilde{A}_0,\tilde{A}_1)\|_{X^{s,\sigma}}\\&\qquad+C(R)\left(\|u-\tilde{u}\|_{L^\infty_T L^2}+\|A-\tilde{A}\|_{L^\infty_T H^\sigma}+\|F_2\|_{L^1_T H^{s-2}}\right).\end{aligned}\tag{2.84}$$

In this step, we also use that $K_{s-2}\lesssim C(R)$. Next, we estimate each of the terms in F_2 in the same way as in Lemma 6.1, [NW07], yielding

$$\|2\mathrm{i}\partial_t A\cdot\nabla(u-\tilde{u})\|_{L^1_T H^{s-2}}\lesssim T^{1/2}\,\|\partial_t A\|_{L^\infty_T H^{\sigma-1}}\,\|u-\tilde{u}\|_{L^2_T H^{s-1/2,6}}\,,\tag{2.85}$$

$$\begin{aligned}&\|2\partial_t A\cdot A(u-\tilde{u})\|_{L^1_T H^{s-2}}\\&\quad\lesssim T^{1-1/q}\,\|\partial_t A\|_{L^\infty_T H^{\sigma-1}}\,\|A\|_{L^q_T H^{\sigma-2/q,r}}\,\|u-\tilde{u}\|_{L^\infty_T H^s}\,,\end{aligned}\tag{2.86}$$

$$\|\partial_t\phi(u)(u-\tilde{u})\|_{L^1_T H^{s-2}}\lesssim T\,\|u\|_{L^\infty_T H^s\cap W^{1,\infty}_T H^{s-2}}\,\|u-\tilde{u}\|_{L^\infty_T H^s}\,,\tag{2.87}$$

$$\|2\mathrm{i}\partial_t(A-\tilde{A})\cdot\nabla\tilde{u}\|_{L^1_T H^{s-2}}\lesssim T^{1/2}\|\partial_t(A-\tilde{A})\|_{L^\infty_T H^{\sigma-1}}\,\|\tilde{u}\|_{L^2_T H^{s-1/2,6}}\,,\tag{2.88}$$

$$\|\mathrm{i}(A-\tilde{A})\cdot\nabla\partial_t\tilde{u}\|_{L^1_T H^{s-2}}\lesssim T^{1-1/q}\|A-\tilde{A}\|_{L^q_T H^{\sigma-2/q-\delta,r}}\,\|\partial_t\tilde{u}\|_{L^\infty_T H^{s-1}}\,,\tag{2.89}$$

$$\begin{aligned}&\|\partial_t(A-\tilde{A})\cdot(A+\tilde{A})\tilde{u}\|_{L^1_T H^{s-2}}\\&\quad\lesssim T^{1-1/q}\|\partial_t(A-\tilde{A})\|_{L^\infty_T H^{\sigma-1}}\|A+\tilde{A}\|_{L^q_T H^{\sigma-2/q,r}}\,\|\tilde{u}\|_{L^\infty_T H^s}\,,\end{aligned}\tag{2.90}$$

$$\begin{aligned}&\|(A-\tilde{A})\cdot\partial_t(A+\tilde{A})\tilde{u}\|_{L^1_T H^{s-2}}\\&\quad\lesssim T^{1-1/q}\|A-\tilde{A}\|_{L^q_T H^{\sigma-2/q,r}}\|\partial_t(A+\tilde{A})\|_{L^\infty_T H^{\sigma-1}}\,\|\tilde{u}\|_{L^\infty_T H^s}\,,\end{aligned}\tag{2.91}$$

$$\begin{aligned}&\|(A-\tilde{A})\cdot(A+\tilde{A})\partial_t\tilde{u}\|_{L^1_T H^{s-2}}\\&\quad\lesssim T^{1-2/q}\|A-\tilde{A}\|_{L^q_T H^{\sigma-2/q-\delta,r}\cap L^\infty_T H^1}\|A+\tilde{A}\|_{L^q_T H^{\sigma-2/q}\cap L^\infty_T H^1}\,\|\partial_t\tilde{u}\|_{L^\infty_T H^{s-1}}\,,\end{aligned}\tag{2.92}$$

$$\begin{aligned}&\big\|\partial_t\big(\phi(u)-\phi(\tilde{u})\big)\tilde{u}\big\|_{L^1_T H^{s-2}}\\&\quad\lesssim T\,\|u-\tilde{u}\|_{L^\infty_T H^s\cap W^{1,\infty}_T H^{s-2}}\,\|u+\tilde{u}\|_{L^\infty_T H^s\cap W^{1,\infty}_T H^{s-2}}\,\|\tilde{u}\|_{L^\infty_T H^s}\,,\end{aligned}\tag{2.93}$$

$$
\begin{aligned}
&\left\| \big(\phi(u) - \phi(\tilde u)\big)\partial_t \tilde u \right\|_{L^1_T H^{s-2}} \\
&\quad \lesssim T \, \|u - \tilde u\|_{L^\infty_T H^s} \, \|u + \tilde u\|_{L^\infty_T H^s} \, \|\tilde u\|_{L^\infty_T H^s \cap W^{1,\infty}_T H^{s-2}} \, .
\end{aligned} \tag{2.94}
$$

We give examples how we obtain these estimates. Consider (2.85). If $s = 2$, we have

$$
\begin{aligned}
\|2\mathrm{i}\partial_t A \cdot \nabla(u - \tilde u)\|_{L^1_T L^2} &\lesssim T^{1/2} \, \|\partial_t A\|_{L^\infty_T L^q} \, \|\nabla(u - \tilde u)\|_{L^2_T L^r} \\
&\lesssim T^{1/2} \, \|\partial_t A\|_{L^\infty_T H^{\sigma-1}} \, \|u - \tilde u\|_{L^2_T H^{3/2,6}} \, ,
\end{aligned}
$$

where we use Hölder's inequality and the Sobolev embeddings $H^{\sigma-1}(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ and $H^{3/2,6}(\mathbb{R}^3) \hookrightarrow H^{1,r}(\mathbb{R}^3)$ which are mentioned in the discussion of the choice of the parameters q, r in (2.42). In the case $s = 1$, we use duality, integrate by parts and observe that $\operatorname{div} \partial_t A = 0$, use Hölder's inequality and the same arguments as in the case $s = 2$ above, to deduce

$$
\begin{aligned}
&\|2\mathrm{i}\partial_t A \cdot \nabla(u - \tilde u)\|_{L^1_T H^{-1}} \\
&\quad \lesssim \left\| t \mapsto \sup_{\|\varphi\|_{H^1} = 1} |\langle \partial_t A(t) \cdot \nabla(u - \tilde u)(t), \varphi \rangle| \right\|_{L^1_T} \\
&\quad = \left\| t \mapsto \sup_{\|\varphi\|_{H^1} = 1} \left| \int_{\mathbb{R}^3} \partial_t A(t,x)(u - \tilde u)(t,x) \nabla \varphi(x) \, \mathrm{d}x \right| \right\|_{L^1_T} \\
&\quad \le \|\partial_t A(u - \tilde u)\|_{L^1_T L^2} \\
&\quad \lesssim T^{1/2} \, \|\partial_t A\|_{L^\infty_T H^{\sigma-1}} \, \|u - \tilde u\|_{L^2_T H^{1/2,6}} \, .
\end{aligned}
$$

Interpolation between the estimates for $s = 1$ and $s = 2$ shows that (2.85) holds true for $s \in [1, 2]$. As another example we consider (2.89). Here we have

$$
\begin{aligned}
\|\mathrm{i}(A - \tilde A) \cdot \nabla \partial_t \tilde u\|_{L^1_T L^2} &\lesssim T^{1-1/q} \|A - \tilde A\|_{L^q_T L^\infty} \, \|\nabla \partial_t \tilde u\|_{L^\infty_T L^2} \\
&\lesssim T^{1-1/q} \|A - \tilde A\|_{L^q_T H^{\sigma-2/q-\delta,r}} \, \|\partial_t \tilde u\|_{L^\infty_T H^1} \, ,
\end{aligned}
$$

in the case $s = 2$, where we use the Sobolev embedding $H^{\sigma-2/q-\delta,r}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$. For $s = 1$, we compute with duality

$$
\begin{aligned}
\|\mathrm{i}(A - \tilde A) \cdot \nabla \partial_t \tilde u\|_{L^1_T H^{-1}} &= \left\| t \mapsto \sup_{\|\varphi\|_{H^1} = 1} \left| \big\langle (A - \tilde A)(t) \cdot \nabla \partial_t \tilde u(t), \varphi \big\rangle \right| \right\|_{L^1_T} \\
&\le \left\| t \mapsto \sup_{\|\varphi\|_{H^1} = 1} \left| \int_{\mathbb{R}^3} (A - \tilde A)(t,x) \partial_t \tilde u(t,x) \varphi(x) \, \mathrm{d}x \right| \right\|_{L^1_T} \\
&\lesssim \left\| (A - \tilde A) \partial_t \tilde u \right\|_{L^1_T L^2} \\
&\lesssim T^{1-1/q} \|A - \tilde A\|_{L^q_T H^{\sigma-2/q-\delta,r}} \, \|\partial_t \tilde u\|_{L^\infty_T L^2} \, ,
\end{aligned}
$$

where we use again integration by parts and that A and $\tilde{A}$ are divergence free. Hence, (2.89) follows again by interpolation. All the other estimates claimed in (2.85)–(2.94) are handled by the same methods.

Finally, we consider the last term in F_2, which is $\partial_t(|u|^{p-1}\, u - |\tilde{u}|^{p-1}\, \tilde{u})$. It arises from the power nonlinearity and is thus not treated in [NW07]. In view of (2.24), we only have to consider the terms

$$\left\| |u|^{p-1}\, \partial_t(u-\tilde{u}) \right\|_{L^1_T H^{s-2}} \quad \text{and} \quad \left\| \left(|u|^{p-2} + |\tilde{u}|^{p-2}\right) \partial_t \tilde{u} (u-\tilde{u}) \right\|_{L^1_T H^{s-2}}. \tag{2.95}$$

We use a different method for each of the two. We start with the first term and argue similarly as above through interpolation. Using Hölder's inequality, we obtain

$$\begin{aligned} \left\| |u|^{p-1}\, \partial_t(u-\tilde{u}) \right\|_{L^1_T L^2} &\le \left\| |u|^{p-1} \right\|_{L^1_T L^\infty} \left\| \partial_t(u-\tilde{u}) \right\|_{L^\infty_T L^2} \\ &\lesssim T^{1-\gamma/2} C(R) \left\| \partial_t(u-\tilde{u}) \right\|_{L^\infty_T L^2}. \end{aligned}$$

Here we use that $\gamma(p-1,\infty,s) = (p-1)(3-2s) < 2$ for all $p < p^*(s)$ which implies that there exists $\gamma < 2$ such that

$$\left\| |u|^{p-1} \right\|_{L^1_T L^\infty} \lesssim T^{1-\gamma/2} \left\| u \right\|^{p-1-\gamma}_{L^\infty_T H^s} \left\| u \right\|^{\gamma}_{L^2_T H^{s-1/2,6}} \le T^{1-\gamma/2} C(R).$$

To arrive at the corresponding estimate in $H^{-1}(\mathbb{R}^3)$, we use duality to obtain

$$\begin{aligned} \left\| |u|^{p-1}\, \partial_t(u-\tilde{u}) \right\|_{L^1_T H^{-1}} &= \left\| t \mapsto \sup_{\|\varphi\|_{H^1}=1} \left| \left\langle |u(t)|^{p-1}\, \partial_t(u-\tilde{u})(t), \varphi \right\rangle \right| \right\|_{L^1_T} \\ &\le \left\| t \mapsto \sup_{\|\varphi\|_{H^1}=1} \left\| \varphi\, |u(t)|^{p-1} \right\|_{H^1} \right\|_{L^1_T} \left\| \partial_t(u-\tilde{u}) \right\|_{L^\infty_T H^{-1}}. \end{aligned}$$

Using the fractional Leibniz rule, the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and (2.23), we further estimate

$$\begin{aligned} &\left\| t \mapsto \sup_{\|\varphi\|_{H^1}=1} \left\| \varphi\, |u(t)|^{p-1} \right\|_{H^1} \right\|_{L^1_T} \\ &\quad \lesssim \left\| t \mapsto \sup_{\|\varphi\|_{H^1}=1} \|\varphi\|_{H^1} \left\| |u(t)|^{p-1} \right\|_{L^\infty} + \|\varphi\|_{L^6} \left\| |u(t)|^{p-1} \right\|_{H^{1,3}} \right\|_{L^1_T} \\ &\quad \lesssim \left\| t \mapsto \left\| |u(t)|^{p-1} \right\|_{L^\infty} + \left\| |u(t)|^{p-1} \right\|_{L^3} + \left\| |u(t)|^{p-2} \right\|_{L^\infty} \left\| \nabla u(t) \right\|_{L^3} \right\|_{L^1_T} \\ &\quad \lesssim T^{1-\gamma/2} C(R). \end{aligned}$$

In the last step, we use that $\gamma(p-2,\infty,s)+\gamma(1,3,s-1)=(p-1)(3-2s)<2$ for all $p<p^*(s)$. Hence, there exist γ_1, $\gamma_2>0$ and $\gamma=\gamma_1+\gamma_2<2$ such that

$$\begin{aligned} &\left\| t \mapsto \left\| |u(t)|^{p-2} \right\|_{L^\infty} \|\nabla u(t)\|_{L^3} \right\|_{L^1_T} \\ &\lesssim \int_0^T \|u(t)\|_{H^s}^{p-2-\gamma_1} \|u(t)\|_{H^{s-1/2,6}}^{\gamma_1} \|u(t)\|_{H^s}^{1-\gamma_2} \|u(t)\|_{H^{s-1/2,6}}^{\gamma_2} \, \mathrm{d}t \\ &\leq T^{1-\gamma/2} \|u\|_{L^\infty_T H^s}^{p-1-\gamma} \|u\|_{L^2_T H^{s-1/2,6}}^{\gamma} = T^{1-\gamma/2} C(R). \end{aligned} \tag{2.96}$$

By interpolation, we conclude that

$$\left\| |u|^{p-1} \partial_t (u-\tilde{u}) \right\|_{L^1_T H^{s-2}} \lesssim T^{1-\gamma/2} C(R) \|\partial_t (u-\tilde{u})\|_{L^\infty_T H^{s-2}}. \tag{2.97}$$

For the second term in (2.95), we distinguish two cases. If $s \in \left[\frac{3}{2}, 2\right]$, we use the Sobolev embeddings $L^{6/(7-2s)}(\mathbb{R}^3) \hookrightarrow H^{s-2}(\mathbb{R}^3)$ and $H^{s-1}(\mathbb{R}^3) \hookrightarrow L^{12/(7-2s)}(\mathbb{R}^3)$ to obtain that

$$\begin{aligned} &\left\| \left(|u|^{p-2} + |\tilde{u}|^{p-2}\right) \partial_t \tilde{u} (u-\tilde{u}) \right\|_{L^1_T H^{s-2}} \\ &\lesssim \left\| \left(|u|^{p-2} + |\tilde{u}|^{p-2}\right) \partial_t \tilde{u} (u-\tilde{u}) \right\|_{L^1_T L^{6/(7-2s)}} \\ &\lesssim \left(\left\| |u|^{p-2} \right\|_{L^1_T L^\infty} + \left\| |\tilde{u}|^{p-2} \right\|_{L^1_T L^\infty} \right) \|\partial_t \tilde{u}\|_{L^\infty_T L^{12/(7-2s)}} \|u-\tilde{u}\|_{L^\infty_T L^{12/(7-2s)}} \\ &\lesssim C(R) \|\partial_t \tilde{u}\|_{L^\infty_T H^{s-1}} \|u-\tilde{u}\|_{L^\infty_T H^{s-1}}. \end{aligned}$$

We set $\theta_s = 1$ in this case. If $s \in \left[\frac{11}{8}, \frac{3}{2}\right)$, we set $\theta_s = 2-s$. We note $\theta_s \in \left[\frac{1}{2}, \frac{5}{8}\right]$, $1-\theta_s = s-1$ and that we have the interpolation results

$$\left[H^{s-1}(\mathbb{R}^3), H^{s-2}(\mathbb{R}^3)\right]_{\theta_s} = L^2(\mathbb{R}^3)$$

and

$$\left[H^{s-1}(\mathbb{R}^3), H^{s-1/2,6}(\mathbb{R}^3)\right]_{\theta_s} = H^{3(s-1)/2, 6/(5-2s)}(\mathbb{R}^3) \hookrightarrow L^{3/(2-s)}(\mathbb{R}^3).$$

We start the estimate as above by using the Sobolev embedding $L^{6/(7-2s)}(\mathbb{R}^3) \hookrightarrow H^{s-2}(\mathbb{R}^3)$, but then we employ Hölder's inequality with $\frac{7-2s}{6} = \frac{1}{2} + \frac{2-s}{3}$ to obtain

$$\begin{aligned} &\left\| \left(|u|^{p-2} + |\tilde{u}|^{p-2}\right) \partial_t \tilde{u} (u-\tilde{u}) \right\|_{L^1_T H^{s-2}} \\ &\lesssim \left\| \left(|u|^{p-2} + |\tilde{u}|^{p-2}\right) \partial_t \tilde{u} (u-\tilde{u}) \right\|_{L^1_T L^{6/(7-2s)}} \\ &\lesssim \left\| t \mapsto \left(\left\| |u(t)|^{p-2} \right\|_{L^\infty} + \left\| |\tilde{u}(t)|^{p-2} \right\|_{L^\infty} \right) \|(u-\tilde{u})(t)\|_{L^{3/(2-s)}} \right\|_{L^1_T} \|\partial_t \tilde{u}\|_{L^\infty_T L^2}. \end{aligned}$$

With the interpolation results given above, we obtain

$$\|\partial_t \tilde{u}\|_{L^\infty_T L^2} \lesssim \|\partial_t \tilde{u}\|_{L^\infty_T H^{s-1}}^{\theta_s} \|\partial_t \tilde{u}\|_{L^\infty_T H^{s-2}}^{1-\theta_s} \lesssim C(R) \|\partial_t \tilde{u}\|_{L^\infty_T H^{s-1}}^{\theta_s}$$

and

$$\begin{aligned}\|(u-\tilde{u})(t)\|_{L^{3/(2-s)}} &\lesssim \|u-\tilde{u}\|^{\theta_s}_{L^\infty_T H^{s-1}} \|(u-\tilde{u})(t)\|^{1-\theta_s}_{H^{s-1/2,6}} \\ &\lesssim \left(\|u(t)\|^{1-\theta_s}_{H^{s-1/2,6}} + \|\tilde{u}(t)\|^{1-\theta_s}_{H^{s-1/2,6}}\right) \|u-\tilde{u}\|^{\theta_s}_{L^\infty_T H^{s-1}} .\end{aligned}$$

Using that $\gamma(p-2,\infty,s)+1-\theta_s = p(3-2s)-7+5s < 2$ for all $p<\tilde{p}^*(s)$ if $s \in \left[\frac{11}{8},\frac{3}{2}\right)$, we conclude that

$$\left\|\left(|u|^{p-2}+|\tilde{u}|^{p-2}\right)\partial_t\tilde{u}(u-\tilde{u})\right\|_{L^1_T H^{s-2}} \lesssim C(R)\,\|\partial_t\tilde{u}\|^{\theta_s}_{L^\infty_T H^{s-1}} \|u-\tilde{u}\|^{\theta_s}_{L^\infty_T H^{s-1}} . \tag{2.98}$$

According to (2.84) we also have to estimate $u-\tilde{u}$ in $L^\infty_T L^2(\mathbb{R}^3)$. From (2.76), we obtain that

$$u(t)-\tilde{u}(t) = U_A(t,0)(u_0-\tilde{u}_0) - \mathrm{i}\int_0^t U_A(t,\tau)F_1(\tau)\,\mathrm{d}\tau.$$

Since $U_A(t,0)$ is a unitary operator on L^2, we obtain

$$\|U_A(\cdot,0)(u_0-\tilde{u}_0)\|_{L^\infty_T L^2} \leq \|u-\tilde{u}_0\|_{L^2} .$$

The second term involving F_1 can be treated exactly as in the proof of the contraction property in Lemma 2.16 where we obtain the estimates (2.58) and (2.59). We note that the terms which are controlled by the constants R_1, R_2 and R_3 in (2.58) and (2.59) can also be bounded by $C(R)$, since we have the embedding $L^q_T H^{\sigma-2/q,r}(\mathbb{R}^3) \hookrightarrow L^2_T L^\infty(\mathbb{R}^3)$. Taking into account that here we also must add the difference of the initial values (A_0,A_1) and $(\tilde{A}_0,\tilde{A}_1)$ to the estimate (2.59), we obtain that

$$\begin{aligned}&\|u-\tilde{u}\|_{L^\infty_T L^2} + \left\|A-\tilde{A}\right\|_{L^\infty_T H^{1/2}\cap L^4_T L^4} \\ &\quad\leq C(R)\|(u_0,A_0,A_1)-(\tilde{u}_0,\tilde{A}_0,\tilde{A}_1)\|_{X^{0,1/2}} \\ &\qquad + T^\alpha C(R)\left(\|u-\tilde{u}\|_{L^\infty_T L^2} + \left\|A-\tilde{A}\right\|_{L^\infty_T H^{1/2}\cap L^4_T L^4}\right),\end{aligned}$$

for some number $\alpha>0$. Hence, for T small enough we obtain

$$\begin{aligned}&\|u-\tilde{u}\|_{L^\infty_T L^2} + \left\|A-\tilde{A}\right\|_{L^\infty_T H^{1/2}\cap L^4_T L^4} \\ &\quad\leq C(R)\|(u_0,A_0,A_1)-(\tilde{u}_0,\tilde{A}_0,\tilde{A}_1)\|_{X^{0,1/2}}.\end{aligned} \tag{2.99}$$

Now we can control all terms on the right side of equation (2.84) appropriately. Choosing $T>0$ small enough, we absorb the terms in (2.85), (2.86), (2.87), (2.88), (2.90), (2.91), (2.93), (2.94) and (2.97) on the left side and by (2.89), (2.92), (2.98) and (2.99) we conclude

$$
\begin{aligned}
\|u-\tilde u\|_{L^\infty_T H^s} \le C(R)\Big(&\|(u_0,A_0,A_1)-(\tilde u_0,\tilde A_0,\tilde A_1)\|_{X^{s,\sigma}} \\
&+\|A-\tilde A\|_{M^{1,\sigma}_T\cap L^q_T H^{\sigma-2/q,r}} \\
&+\|A-\tilde A\|_{L^\infty_T H^1\cap L^q_T H^{\sigma-2/q-\delta,r}}\,\|\tilde u\|_{L^\infty_T H^{s+1}\cap W^{1,\infty}_T H^{s-1}} \\
&+\|u-\tilde u\|^{\theta_s}_{L^\infty_T H^{s-1}}\,\|\tilde u\|^{\theta_s}_{W^{1,\infty}_T H^{s-1}}\Big).
\end{aligned}
\tag{2.100}
$$

For the Maxwell part, the desired estimate (2.74) is proved in the same way as in [NW07]. By inserting (2.74) into (2.100) above, we also obtain (2.73).

Finally, we prove (2.75). Starting from equation (2.76) in the integral form as in (2.56), we directly obtain the estimate

$$
\begin{aligned}
\|u-\tilde u\|_{L^\infty_T H^{s-1}} &\lesssim K_{s-1}\left(\|u_0-\tilde u_0\|_{H^{s-1}}+\|F_1\|_{L^1_T H^{s-1}}\right) \\
&\le C(R)\left(\|u_0-\tilde u_0\|_{H^{s-1}}+\|F_1\|_{L^1_T H^{s-1}}\right).
\end{aligned}
\tag{2.101}
$$

The first three summands in F_1 can be estimated similarly as in the proof of Lemma 2.14. Compare with the discussion at the very beginning of this proof and note that we here we can use Hölder's inequality for the time integration more generously. We obtain the estimates

$$
\|\mathrm{i}(A-\tilde A)\nabla\tilde u\|_{L^1_T H^{s-1}} \lesssim T^{1/2}\|A-\tilde A\|_{L^\infty_T H^{\sigma-\delta}\cap L^q_T H^{\sigma-2/q-\delta,r}}\,\|\tilde u\|_{L^\infty_T H^s\cap L^2_T H^{s-1/2,6}}\,,
$$

$$
\begin{aligned}
&\|(A-\tilde A)(A+\tilde A)\tilde u\|_{L^1_T H^{s-1}} \\
&\quad\lesssim T^{1/2}\|A-\tilde A\|_{L^\infty_T H^{\sigma-\delta}\cap L^q_T H^{\sigma-2/q-\delta,r}}\|A+\tilde A\|_{L^\infty_T H^{\sigma-\delta}\cap L^q_T H^{\sigma-2/q-\delta,r}}\,\|\tilde u\|_{L^\infty_T H^s}\,,
\end{aligned}
$$

and

$$
\big\|\big(\phi(u)-\phi(\tilde u)\big)\tilde u\big\|_{L^1_T H^{s-1}} \lesssim T\,\|u-\tilde u\|_{L^\infty_T H^{s-1}}\,\|u+\tilde u\|_{L^\infty_T H^s}\,\|\tilde u\|_{L^\infty_T H^s}\,.
$$

The estimates for the last term proceed rather similarly as the computations leading to (2.78) and (2.79). More precisely, we obtain

$$
\begin{aligned}
\big\||u|^{p-1}u-|\tilde u|^{p-1}\tilde u\big\|_{L^1_T L^2} &\le \left(\big\||u|^{p-1}\big\|_{L^1_T L^\infty}+\big\||\tilde u|^{p-1}\big\|_{L^1_T L^\infty}\right)\|u-\tilde u\|_{L^\infty_T L^2} \\
&\lesssim T^{1-\gamma/2}C(R)\,\|u-\tilde u\|_{L^\infty_T L^2}\,,
\end{aligned}
$$

where we use that $\gamma(p-1,\infty,s)<2$ for $p<p^*(s)$. We also have

$$
\begin{aligned}
&\big\||u|^{p-1}u-|\tilde u|^{p-1}\tilde u\big\|_{L^1_T H^1} \\
&\quad\lesssim \big\||u|^{p-1}u-|\tilde u|^{p-1}\tilde u\big\|_{L^1_T L^2}+\big\||u|^{p-1}\big\|_{L^1_T L^\infty}\|\nabla u-\nabla\tilde u\|_{L^\infty_T L^2} \\
&\qquad+\Big\|t\mapsto\left(\big\||u(t)|^{p-2}\big\|_{L^\infty}+\big\||\tilde u(t)|^{p-2}\big\|_{L^\infty}\right)\|\nabla\tilde u(t)\|_{L^3}\Big\|_{L^1_T}\,\|u-\tilde u\|_{L^\infty_T L^6}\,.
\end{aligned}
$$

Since $\gamma(p-2,\infty,s)+\gamma(1,3,s-1)=(p-1)(3-2s)<2$ for $p<p^*(s)$, we argue as in (2.96) to obtain that

$$\left\| t\mapsto \left(\left\| |u(t)|^{p-2} \right\|_{L^\infty} + \left\| |\tilde{u}(t)|^{p-2} \right\|_{L^\infty} \right) \|\nabla \tilde{u}(t)\|_{L^3} \right\|_{L^1_T} \le T^{1-\gamma/2} C(R).$$

Using the Sobolev embedding $H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$ and an interpolation argument as the one leading to (2.81), we thus find that

$$\left\| |u|^{p-1} u - |\tilde{u}|^{p-1} \tilde{u} \right\|_{L^1_T H^{s-1}} \le T^{1-\gamma/2} C(R) \|u-\tilde{u}\|_{L^\infty_T H^{s-1}}$$

for all $s\in[1,2]$. Altogether, by choosing T small enough, we conclude from (2.101) that

$$\|u-\tilde{u}\|_{L^\infty_T H^{s-1}} \le C(R) \left(\|u_0-\tilde{u}_0\|_{H^{s-1}} + \|A-\tilde{A}\|_{M^{1,\sigma-\delta}_T \cap L^q_T H^{\sigma-2/q-\delta,r}} \right). \quad (2.102)$$

The Maxwell part can be treated as in Lemma 6.1, [NW07]. Here we obtain

$$\begin{aligned} \|A-\tilde{A}\|_{M^{1,\sigma-\delta}_T \cap L^q_T H^{\sigma-2/q-\delta,r}} &\le C(R) \|(A_0,A_1)-(\tilde{A}_0,\tilde{A}_1)\|_{H^{\sigma-\delta}\times H^{\sigma-1-\delta}} \\ &\quad + T^{1/r} C(R) \|u-\tilde{u}\|_{L^\infty_T H^{s-1}\cap L^2_T H^{s-1/2,6}}. \end{aligned} \quad (2.103)$$

Combining (2.103) with (2.102), we arrive at (2.75). □

Lemma 2.22 (Continuous dependence)
Let $s\in\left(\frac{11}{8},2\right]$ and $\sigma\in(1,\infty)$ such that $(s,\sigma)\in\mathcal{R}$ and $(s+1,\sigma)\in\mathcal{R}_*$. Let $p\in\left(s+1,\tilde{p}^*(s)\right)$. Let $R>0$. Then the solution map

$$\Phi\colon \{x\in X^{s,\sigma} : \|x\|_{X^{s,\sigma}} \le R\} \to C_{T(R)} X^{s,\sigma}, \quad \Phi(u_0,A_0,A_1)=(u,A,\partial_t A)$$

of the Maxwell–Schrödinger system (2.11) is continuous. Here $T(R)$ is a positive time such that all solutions starting from a ball of radius R exist at least up to $T(R)$ by Lemma 2.16.

Proof. We fix $(u_0,A_0,A_1)\in X^{s,\sigma}$ and $R\in(0,\infty)$ such that $R\ge 2\|(u_0,A_0,A_1)\|_{X^{s,\sigma}}$. We write T instead of $T(R)$ throughout the proof.

First, we consider smooth initial values. Namely, for $\varepsilon>0$ we define regularized initial values $(u_0^\varepsilon,A_0^\varepsilon,A_1^\varepsilon)$ by setting $u_0^\varepsilon=\eta_\varepsilon * u_0$, $A_0^\varepsilon=\eta_{\varepsilon^{1/\delta}} * A_0$ and $A_1^\varepsilon=\eta_{\varepsilon^{1/\delta}} * A_1$, where we use a mollifier η_ε as in Lemma A.4. This lemma yields the bounds

$$\|(u_0^\varepsilon,A_0^\varepsilon,A_1^\varepsilon)\|_{X^{s,\sigma}} \le C, \qquad \|u_0^\varepsilon\|_{H^{s+1}} \le C\varepsilon^{-1} \quad (2.104)$$

and for $j\in\{0,1\}$ the convergence

$$\varepsilon^{-j} \|(u_0-u_0^\varepsilon, A_0-A_0^\varepsilon, A_1-A_1^\varepsilon)\|_{X^{s-j,\sigma-j\delta}} \to 0 \quad \text{as } \varepsilon\to 0. \quad (2.105)$$

Let $(u^\varepsilon, A^\varepsilon, \partial_t A^\varepsilon) = \Phi(u_0^\varepsilon, A_0^\varepsilon, A_1^\varepsilon)$ be the corresponding solution of the Maxwell–Schrödinger system (2.11) on the interval $[0,T]$. Then Lemma 2.20 shows that

$$\|u^\varepsilon\|_{L_T^\infty H^{s+1} \cap W_T^{1,\infty} H^{s-1}} \lesssim \|u_0^\varepsilon\|_{H^{s+1}} . \tag{2.106}$$

Note that at this point we require the smoothness $p > s+1$ of the nonlinear term. Next, we estimate the difference between u and u^ε. Choose $\tilde{\sigma} \in \big(1, \min\{\sigma, 2s - \frac{7}{4}\}\big]$. With this choice we can apply (2.75) below. Associated to $\tilde{\sigma}$ are the numbers $\tilde{\delta}$, $\tilde{q}$ and $\tilde{r}$ which are chosen according to (2.42). From (2.73), we deduce

$$\begin{aligned} \|u - u^\varepsilon\|_{L_T^\infty H^s \cap L_T^2 H^{s-1/2,6}} \lesssim\ & \|(u_0, A_0, A_1) - (u_0^\varepsilon, A_0^\varepsilon, A_1^\varepsilon)\|_{X^{s,\tilde{\sigma}}} \\ & + \|A - A^\varepsilon\|_{L_T^\infty H^1 \cap L_T^q H^{\tilde{\sigma}-2/\tilde{q}-\tilde{\delta},\tilde{r}}} \|u^\varepsilon\|_{W_T^{1,\infty} H^{s-1}} \\ & + \|u - u^\varepsilon\|_{L_T^\infty H^{s-1}}^{\theta_s} \|u^\varepsilon\|_{W_T^{1,\infty} H^{s-1}}^{\theta_s} . \end{aligned} \tag{2.107}$$

Using (2.75), (2.106) and (2.104), we obtain

$$\begin{aligned} & \|A - A^\varepsilon\|_{L_T^\infty H^1 \cap L_T^q H^{\tilde{\sigma}-2/\tilde{q}-\tilde{\delta},\tilde{r}}} \|u^\varepsilon\|_{W_T^{1,\infty} H^{s-1}} \\ & \quad \lesssim \|(u_0, A_0, A_1) - (u_0^\varepsilon, A_0^\varepsilon, A_1^\varepsilon)\|_{X^{s-1,\tilde{\sigma}-\tilde{\delta}}} \|u_0^\varepsilon\|_{H^{s+1}} \\ & \quad \lesssim \varepsilon^{-1} \|(u_0, A_0, A_1) - (u_0^\varepsilon, A_0^\varepsilon, A_1^\varepsilon)\|_{X^{s-1,\tilde{\sigma}-\tilde{\delta}}} . \end{aligned}$$

Analogously we find that

$$\begin{aligned} & \|u - u^\varepsilon\|_{L_T^\infty H^{s-1}}^{\theta_s} \|u^\varepsilon\|_{W_T^{1,\infty} H^{s-1}}^{\theta_s} \\ & \quad \lesssim \varepsilon^{-\theta_s} \|(u_0, A_0, A_1) - (u_0^\varepsilon, A_0^\varepsilon, A_1^\varepsilon)\|_{X^{s-1,\tilde{\sigma}-\tilde{\delta}}}^{\theta_s} . \end{aligned}$$

Hence, (2.105) yields that u^ε converges to the solution u in $C_T H^s(\mathbb{R}^3) \cap L_T^2 H^{s-1/2,6}$ as $\varepsilon \to 0$. Next we estimate the difference between A and A^ε. From (2.74), we obtain

$$\begin{aligned} & \|A - A^\varepsilon\|_{L_T^\infty H^\sigma \cap W_T^{1,\infty} H^{\sigma-1}} \\ & \quad \lesssim \|(A_0, A_1) - (A_0^\varepsilon, A_1^\varepsilon)\|_{H^\sigma \times H^{\sigma-1}} + \|u - u^\varepsilon\|_{L_T^\infty H^s \cap L_T^2 H^{s-1/2,6}} . \end{aligned}$$

Formula (2.105) and the result for the Schrödinger part imply that $(A^\varepsilon, \partial_t A^\varepsilon)$ converges to the solution $(A, \partial_t A)$ in $C_T(H^\sigma \times H^{\sigma-1})$ as $\varepsilon \to 0$.

With this preparation we can now prove continuity of the solution map. Namely, let $\big((u_0^n, A_0^n, A_1^n)\big)_n$ be a sequence in $X^{s,\sigma}$ which tends to (u_0, A_0, A_1). We set

$$(u^n, A^n, \partial_t A^n) = \Phi(u_0^n, A_0^n, A_1^n)$$

and consider regularized initial function $u_0^{n\varepsilon}$, etc. as before. By the argument above, the functions $(u^\varepsilon, A^\varepsilon, \partial_t A^\varepsilon)$ and $(u^{n\varepsilon}, A^{n\varepsilon}, \partial_t A^{n\varepsilon})$ converge to $(u, A, \partial_t A)$ respectively

$(u^n, A^n, \partial_t A^n)$ in the space $C_T X^{s,\sigma}$ as $\varepsilon \to 0$ uniformly in $n \in \mathbb{N}$. Moreover, (2.75) shows that $(u^{n\varepsilon}, A^{n\varepsilon}, \partial_t A^{n\varepsilon})$ converges to $(u^\varepsilon, A^\varepsilon, \partial_t A^\varepsilon)$ for each fixed $\varepsilon > 0$ in the space $C_T X^{s,\sigma}$ as $n \to \infty$. Therefore, the assertion follows from the simple decomposition

$$
\begin{aligned}
&\|(u^n, A^n, \partial_t A^n) - (u, A, \partial_t A)\|_{L_T^\infty X^{s,\sigma}} \\
&\quad \le \|(u^n, A^n, \partial_t A^n) - (u^{n\varepsilon}, A^{n\varepsilon}, \partial_t A^{n\varepsilon})\|_{L_T^\infty X^{s,\sigma}} \\
&\qquad + \|(u^{n\varepsilon}, A^{n\varepsilon}, \partial_t A^{n\varepsilon}) - (u^\varepsilon, A^\varepsilon, \partial_t A^\varepsilon)\|_{L_T^\infty X^{s,\sigma}} \\
&\qquad + \|(u^\varepsilon, A^\varepsilon, \partial_t A^\varepsilon) - (u, A, \partial_t A)\|_{L_T^\infty X^{s,\sigma}} .
\end{aligned}
$$

□

Remark 2.23 (The role of regularity of the power nonlinearity)
The statement of Theorem 2.1 excludes small values of p, i.e. we require that $p > s$ for the basic well-posedness result and even $p > s + 1$ for the full result including continuous dependence in the topology of the solution space. In particular, we do not achieve any result if $p < \frac{11}{8}$. The only reason for this regrettable restriction is that $|u|^{p-1} u$ is not smooth enough to be estimated in $H^s(\mathbb{R}^3)$ via (2.33). We expect that there would be no obstacle to prove well-posedness results for nonlinearities of the form $g(|u|^2)u$ where $g \colon \mathbb{R} \to \mathbb{R}$ is a smooth function and satisfies appropriate growth conditions.

We shall compare our assumptions on p with known results about the nonlinear Schrödinger equation (1.6). First we remark that (1.6) has a scaling property which determines the critical exponent $p^*_{\text{NLS}}(s) = \frac{7-2s}{3-2s}$. Therefore, $p \le p^*_{\text{NLS}}(s)$ is a natural upper bound for a local well-posedness theory in $H^s(\mathbb{R}^3)$ and little is understood about the supercritical case $p > p^*_{\text{NLS}}(s)$. Such scaling considerations do not apply to the Maxwell–Schrödinger system (2.11) since the Schrödinger- and the Maxwell part scale differently in the time variable. However, we remark that our critical exponent $p^*(s)$, which is less than $p^*_{\text{NLS}}(s)$, seems to be directly related to the loss of one half derivative in the Strichartz estimates of Lemma 2.14. If we had the estimates in $H^{s,6}(\mathbb{R}^3)$ instead of $H^{s-1/2,6}(\mathbb{R}^3)$, we would also arrive at $p^*_{\text{NLS}}(s)$ for the Maxwell–Schrödinger system.

We also comment on the lower bound on p. At least in the case $s \le 2$, there is a standard procedure for the nonlinear Schrödinger equation (1.6) to avoid any technical restriction on p from below by differentiating the equation in time. Since morally one time derivative is equivalent to two spatial derivatives in the nonlinear Schrödinger equation, one then obtains local well-posedness in $H^s(\mathbb{R}^3)$ even for $p > 1$, see e.g. [Kat87]. It is worthwhile to investigate whether a similar approach would also work for the Maxwell–Schrödinger system. If s is larger than 2, then restrictions on p from below are apparent in all known results. We refer to [UW12] for the current state of the art. We also stress that this lower bounds in the local well-posedness theory are not a purely technical artefact but that the regularity of the nonlinearity can pose a genuine obstacle to well-posedness. The first result from this very interesting line of research is given in [CDW17].

We conclude this section with a remark about the obstacles to prove global existence of solutions.

Remark 2.24 (On global existence)
Assume that s, σ and p are as in the statement of Theorem 2.1 (1) and that $p > 2$. By Lemma 2.17, mass and energy of the system are conserved quantities. For the nonlinear Schrödinger equation (1.6), it is well known that in the case of a focusing power nonlinearity with $p \in \left(1, \frac{7}{3}\right)$ the conservation of mass and energy implies the boundedness of the H^1-norm of solutions. Unfortunately, we cannot prove an analogous result for the Maxwell–Schrödinger system (2.11).

For the rest of the discussion, we assume that instead of a focusing nonlinearity as in (2.11) we deal with a defocusing nonlinearity $+\left|u\right|^{p-1} u$. The local well-posedness theory developed in this Chapter is insensitive to the sign of the nonlinearity and Theorem 2.1 also holds true in this case. It is then straightforward to see that the conserved quantities implies that the solution (u, A) remains bounded in $X^{1,1}$ on its maximal existence interval. From Lemma 2.18, we infer that A remains bounded in $L^\infty H^\sigma(\mathbb{R}^3) \cap W^{1,\infty} H^{\sigma-1}(\mathbb{R}^3)$ and u remains bounded in $L^2 H^{1/2-\delta,6}$ as long as the solution exists. Therefore, if blow-up occurs it is the H^s-norm of u which becomes infinite. From Lemma 2.20, we see that we could control the H^s-norm of u if we had information about the L^∞-norm of u. Regrettably our bound in $L^2 H^{1/2-\delta,6}$ is not enough to achieve this task. Nevertheless, we can formulate an improved blow-up criterion for the defocusing case. If $T_{\max} < \infty$, then $\lim_{t \to T_{\max}} \left\|u(t)\right\|_{L^\infty} = \infty$.

3 Spaces of bounded variation

"Nuc-u-lar". It's pronounced, "nuc-u-lar".

Homer Simpson, *Simpson Tide*

In this section, we introduce the space V^p of functions of bounded p-variation and the closely related atomic spaces U^p. All main statements of this section can be found in Section 2 of [HHK09] (also note the erratum [HHK10]). Our proofs of the basic properties of these spaces are mostly paraphrases of the proofs in [HHK09], sometimes furnished with additional details. However, the duality introduced in Proposition 3.10 rather follows the approach of Chapter 4 in [Koc14], which differs from the presentation in [HHK09]. We believe that the proofs of [HHK09] allow getting to the point more directly, while in [Koc14] the role of Lemma 3.6 is emphasized: Any computation for U^p-functions should be based on a computation with U^p-atoms. While we still follow many arguments taken from [HHK09], we present all our proofs in such a way that they directly relate to Lemma 3.6. The lack of density of step functions in the space V^p is a drawback of the presentation in [Koc14]. This issue has been pointed out in [CH18]. For our purpose, however, this does not present an obstacle, since step functions still form a norming set for the space U^p, see Lemma 3.11.

The spaces V^p and U^p were introduced in the context of dispersive equations by Herbert Koch and Daniel Tataru, see the articles [KT05] and [KT07]. These spaces are used in the study of critical problems for dispersive equations and play the role of Jean Bourgain's $X^{s,b}$-spaces, which were introduced in the articles [Bou93b] and [Bou93a] for the study of nonlinear Schrödinger- and Korteweg–de Vries equations. They have been successfully used in the study of several equations, such as the Kadomptsev–Petviashvili equation ([Had08]), the Maxwell–Schrödinger system ([BT09]), a supercritical generalized Korteweg–de Vries equation ([Str14]), the massive Dirac–Klein–Gordon system ([BH17]), the Klein–Gordon–Zakharov system ([KK18]), and others.

Throughout this section we are studying various spaces of functions defined on the real line and taking values in a Hilbert space. The main aim of this section is to identify the preduals of the spaces of bounded p-variation in terms of atomic spaces (cf. Proposition 3.10) and the characterization of the duality pairing in Proposition 3.17.

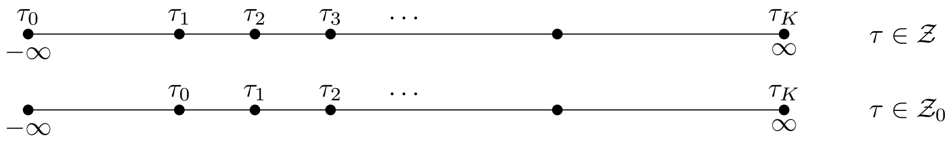

Figure 2 – The different types of partitions from Definition 3.1.

Definitions and basic properties

In the following, we have to be careful how our function should behave at $\pm\infty$. Therefore, we start by introducing two classes of partitions of $\mathbb{R}$ on which the subsequent definitions of various function spaces rely.

Definition 3.1 (Notations for finite partitions of $\mathbb{R}$)
A *finite partition of* $\mathbb{R}$ is a finite sequence $\tau = (\tau_k)_{k=0}^K$ in $\mathbb{R} \cup \{\pm\infty\}$ such that

$$-\infty \leq \tau_0 < \tau_1 < \ldots < \tau_K \leq \infty.$$

The *number of subintervals* in $[\tau_0, \tau_K]$ induced by the partition τ is denoted by $|\tau| \coloneqq |\tau^{-1}([-\infty,\infty])| - 1 = K$. We define the set of finite partitions of $\mathbb{R}$ including both endpoints

$$\mathcal{Z} \coloneqq \left\{\tau \text{ finite partition} : -\infty = \tau_0 < \tau_1 < \ldots < \tau_{|\tau|} = \infty\right\},$$

and the set of finite partitions of $\mathbb{R}$ excluding $-\infty$ by

$$\mathcal{Z}_0 \coloneqq \left\{\tau \text{ finite partition} : -\infty < \tau_0 < \tau_1 < \ldots < \tau_{|\tau|} = \infty\right\}.$$

Let H be a Hilbert space. The first class of functions we consider are H-valued step functions. For our purpose, it is enough to treat right-continuous step functions which vanish at $-\infty$.

Definition 3.2 (Right-continuous step functions)
We say that a function $s\colon \mathbb{R} \to H$ is a *right-continuous step function* if we can write

$$s = \sum_{k=1}^{|\tau|} \mathbb{1}_{[\tau_{k-1},\tau_k)} \phi_k$$

for some partition $\tau \in \mathcal{Z}_0$ and some sequence $(\phi_k)_{k=1}^{|\tau|}$ in H. We denote the set of right-continuous step functions by $\mathcal{S}_{\text{rc}}$. Note that any $s \in \mathcal{S}_{\text{rc}}$ satisfies $\lim_{t\to-\infty} s(t) = 0$ since its construction is based on a partition belonging to $\mathcal{Z}_0$.

The following class of functions was introduced in the book [Aum54] by the German name “Regelfunktionen”. In the book [Die60], the name was translated as “regulated functions”. Sometimes these functions are also called “ruled functions” in English.

Definition 3.3 (Regulated functions and the space of right-continuous functions)
A function $f\colon \mathbb{R} \to H$ is called *regulated* if left limits exists in every point of $(-\infty, \infty]$ and right limits exists in every point of $[-\infty, \infty)$. The space of regulated functions is denoted by $\mathcal{R}$. We also define the subspace $\mathcal{R}_{\mathrm{rc}} \subseteq \mathcal{R}$ of all functions $f \in \mathcal{R}$, which are right-continuous and satisfy $\lim\limits_{t\to-\infty} f(t) = 0$.

The following basic properties of regulated functions are stated in Proposition 7.3.2.1 in the book [Aum54]. A function is regulated if and only if it is the uniform limit of a sequence of step functions. Endowed with the supremum norm, $\mathcal{R}$ and $\mathcal{R}_{\mathrm{rc}}$ are Banach spaces. Moreover, the set of discontinuities of a regulated function is at most countable.

Next, we define the atomic space U^p. The so-called U^p-atoms are the building blocks of the U^p-space, as every function can be written as a weighted series of U^p-atoms. From a practical point of view, calculations in U^p-spaces are convenient, in the sense that many properties only have to be checked to hold for atoms and then automatically follow for the full space. Note in particular that in the following the value of p only enters the definition of a U^p-atom, but not the way in which the space U^p is constructed from its atoms. We also remark that our notation for U^p-atoms is slightly different to the one used in Definition 2.1 of [HHK09].

Definition 3.4 (U^p-atoms and the space U^p)
Let $p \in [1, \infty)$. Let $\tau \in \mathcal{Z}_0$ be a partition, and consider a sequence $(\phi_k)_{k=1}^{|\tau|}$ in H with the property that

$$\sum_{k=1}^{|\tau|} \|\phi_k\|_H^p \leq 1.$$

A *U^p-atom* is a right-continuous step function $a\colon \mathbb{R} \to H$ given by

$$a = \sum_{k=1}^{|\tau|} \mathbb{1}_{[\tau_{k-1},\tau_k)}\phi_k.$$

We define the atomic space

$$U^p := \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j : a_j \text{ is a } U^p\text{-atom},\ \lambda_j \in \mathbb{C},\ \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\}$$

endowed with the norm

$$\|u\|_{U^p} := \inf\left\{\sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j,\ \lambda_j \in \mathbb{C},\ a_j \text{ is a } U^p\text{-atom}\right\}.$$

We denote by

$$U_c^p := U^p \cap C(\mathbb{R}, H)$$

the subspace of continuous U^p-functions. Note that the series appearing above are well-defined as they converge in the space $\mathcal{R}_{\mathrm{rc}}$, see the arguments below in the beginning of the proof of Proposition 3.5.

In the following proposition, we collect basic properties of the U^p-spaces which follow from the definition in a straightforward way.

Proposition 3.5 (Elementary properties of the U^p-spaces)
Let $p, q \in [1, \infty)$ with $p < q$, and let $u \in U^p$. The following assertions are true.

(1) The space U^p is a Banach space.

(2) The set of right-continuous step functions $\mathcal{S}_{\mathrm{rc}}$ is dense in U^p.

(3) The function u is right-continuous, i.e. for every $t_0 \in \mathbb{R}$, we have

$$\lim_{t \to t_0+} \|u(t) - u(t_0)\|_H = 0.$$

(4) The function u vanishes at $-\infty$, and the limit $\lim_{t\to\infty} u(t)$ exists. In the following, we use the notation $u_\infty := \lim_{t\to\infty} u(t)$.

(5) The embeddings $U^p \hookrightarrow U^q \hookrightarrow \mathcal{R}_{\mathrm{rc}}$ are continuous.

(6) The space U_c^p is a Banach space.

Proof. We start by showing that the series in the definition of the space U^p converges uniformly. For any U^p-atom a, we have

$$\|a\|_\infty = \max\left\{\|\phi_k\|_H : k = 1, \ldots, |\tau|\right\} \leq 1.$$

Let $u = \sum_{j=1}^{\infty} \lambda_j a_j$ be a function in U^p written as a series with U^p-atoms a_j and complex coefficients λ_j for $j \in \mathbb{N}$. We thus obtain the inequality

$$\|u\|_\infty \leq \sum_{j=1}^{\infty} |\lambda_j| \, \|a_j\|_\infty \leq \sum_{j=1}^{\infty} |\lambda_j| \,.$$

Hence, the series $\sum_{j=1}^{\infty} \lambda_j a_j$ converges uniformly to u. By taking the infimum over all possible representations of u, we obtain

$$\|u\|_{\infty} \leq \|u\|_{U^p} . \tag{3.1}$$

We next show that U^p is a Banach space. It is clear that U^p is a vector space and that the expression $\|\cdot\|_{U^p}$ is positive definite and positive homogeneous. To show the triangle inequality, let us fix $\varepsilon > 0$ and consider $u = \sum_{j=1}^{\infty} \lambda_j a_j$ and $v = \sum_{j=1}^{\infty} \mu_j b_j$ in U^p where we have chosen representations satisfying $\sum_{j=1}^{\infty} |\lambda_j| < \|u\|_{U^p} + \varepsilon$ and $\sum_{j=1}^{\infty} |\mu_j| < \|v\|_{U^p} + \varepsilon$. By defining

$$\nu_j \coloneqq \begin{cases} \lambda_j, & j \text{ odd}, \\ \mu_j, & j \text{ even}, \end{cases} \quad \text{and} \quad c_j \coloneqq \begin{cases} a_j, & j \text{ odd}, \\ b_j, & j \text{ even}, \end{cases}$$

we obtain $u + v = \sum_{j=1}^{\infty} \nu_j c_j$ and hence

$$\|u+v\|_{U^p} \leq \sum_{j=1}^{\infty} |\nu_j| = \sum_{j=1}^{\infty} |\lambda_j| + \sum_{j=1}^{\infty} |\mu_j| \leq \|u\|_{U^p} + \|v\|_{U^p} + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the triangle inequality is proved. For completeness of U^p, we show that every absolutely convergent series converges. To this end, consider a sequence $(u_n)_{n\in\mathbb{N}}$ in U^p which satisfies $\sum_{n=1}^{\infty} \|u_n\|_{U^p} < \infty$. For $j, n \in \mathbb{N}$, we choose U^p-atoms $a_j^{(n)}$ and $\lambda_j^{(n)} \in \mathbb{C}$ such that for each $n \in \mathbb{N}$ we have $u_n = \sum_{j=1}^{\infty} \lambda_j^{(n)} a_j^{(n)}$ and $\sum_{j=1}^{\infty} \big|\lambda_j^{(n)}\big| \leq 2 \|u_n\|_{U^p}$. By relabelling the countable set of the above U^p-atoms, we can find a sequence of U^p-atoms $(b_j)_{j\in\mathbb{N}}$ such that

$$\big\{a_j^{(n)} : j, n \in \mathbb{N}\big\} = \{b_j : j \in \mathbb{N}\} .$$

We can now write $u_n = \sum_{j=1}^{\infty} \mu_j^{(n)} b_j$ for suitable coefficients $\mu_j^{(n)}$, $j, n \in \mathbb{N}$. For $j \in \mathbb{N}$, we define $\mu_j \coloneqq \sum_{n=1}^{\infty} \mu_j^{(n)}$. We have

$$\sum_{j=1}^{\infty} |\mu_j| \leq \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \big|\mu_j^{(n)}\big| \leq 2 \sum_{n=1}^{\infty} \|u_n\|_{U^p} < \infty. \tag{3.2}$$

Hence, the function u defined by $u \coloneqq \sum_{j=1}^{\infty} \mu_j b_j$ is an element of U^p. The series $\sum_{n=1}^{\infty} u_n$ indeed converges to u, since

$$\begin{aligned} \left\| u - \sum_{n=1}^{N} u_n \right\|_{U^p} &= \left\| \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \mu_j^{(n)} b_j - \sum_{n=1}^{N} \sum_{j=1}^{\infty} \mu_j^{(n)} b_j \right\|_{U^p} \\ &= \left\| \sum_{n=N+1}^{\infty} \sum_{j=1}^{\infty} \mu_j^{(n)} b_j \right\|_{U^p} \\ &\leq \sum_{n=N+1}^{\infty} \sum_{j=1}^{\infty} \big|\mu_j^{(n)}\big| \to 0 \quad \text{as } N \to \infty, \end{aligned}$$

where we used (3.2) in the last step.

We next show density of the set $\mathcal{S}_{\mathrm{rc}}$. Let $u \in U^p$ and $\varepsilon > 0$. Take any representation $u = \sum_{j=1}^{\infty} \lambda_j a_j$ with U^p-atoms a_j and coefficients $\lambda_j \in \mathbb{C}$. Choose an index $n \in \mathbb{N}$ such that $\sum_{j=n+1}^{\infty} |\lambda_j| < \varepsilon$. Then the function $u_n \coloneqq \sum_{j=1}^{n} \lambda_j a_j$ belongs to $\mathcal{S}_{\mathrm{rc}}$ and satisfies

$$\|u - u_n\|_{U^p} \leq \sum_{j=n+1}^{\infty} |\lambda_j| < \varepsilon.$$

From estimating the supremum norm in (3.1), we already know that convergence in U^p implies uniform convergence. The density of $\mathcal{S}_{\mathrm{rc}}$ in U^p thus yields that every function $u \in U^p$ is right-continuous and vanishes at $-\infty$. Moreover, every function in $\mathcal{S}_{\mathrm{rc}}$ has a limit at $+\infty$. Again, this property is preserved by uniform convergence and assertions (3) and (4) follow.

The next step is to show the claimed embeddings. Since the sequence spaces l^r are increasing in $r \in [1, \infty]$, we see that every U^p-atom is a U^q-atom, too. Thus, we deduce the embedding $U^p \hookrightarrow U^q$ together with the norm inequality

$$\|u\|_{U^q} \leq \|u\|_{U^p}\,, \tag{3.3}$$

since the infimum on the left side of (3.3) is taken over a larger set than on the right side. The embedding of U^q into the space $\mathcal{R}_{\mathrm{rc}}$ follows from the estimate (3.1) in the beginning of the proof.

Finally, we show that U^p_c is a Banach space. In fact, U^p_c is a closed subspace of the space U^p, since convergence in U^p implies uniform convergence by assertion (5). □

The following lemma shows that every linear operator acting on right-continuous step functions which is uniformly bounded on U^p-atoms can be uniquely extended to the full space U^p.

Lemma 3.6 (Extension from $\mathcal{S}_{\mathrm{rc}}$)
Let X be an arbitrary Banach space. Let $T\colon \mathcal{S}_{\mathrm{rc}} \to X$ be a linear operator. Assume that there is a constant $C > 0$ such that the operator T satisfies

$$\|Ta\|_X \leq C$$

for all U^p-atoms a.

(1) Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}_{\mathrm{rc}}$, which converges in U^p to some $u \in U^p$. Then the sequence $(Tu_n)_{n\in\mathbb{N}}$ converges in X.

(2) There exists a unique extension of T to a linear operator $T\colon U^p \to X$ satisfying

$$\|Tu\|_X \leq C\,\|u\|_{U^p} \quad \text{for all } u \in U^p.$$

Proof. To prove the first assertion, we note that by assumption $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in U^p. Thus, there are sequences $\lambda_j^{(n,m)} \in \mathbb{C}^{\mathbb{N}}$ and U^p-atoms $a_j^{(n,m)}$ $(j \in \mathbb{N})$ such that $u_n - u_m = \sum_{j=1}^{\infty} \lambda_j^{(n,m)} a_j^{(n,m)}$ and $\sum_{j=1}^{\infty} \big|\lambda_j^{(n,m)}\big| \to 0$ as $n, m \to \infty$. It follows that

$$\|Tu_n - Tu_m\|_X = \left\| \sum_{j=1}^{\infty} \lambda_j^{(n,m)} T a_j^{(n,m)} \right\|_X \leq \sum_{j=1}^{\infty} \big|\lambda_j^{(n,m)}\big| \big\|T a_j^{(n,m)}\big\|_X \leq C \sum_{j=1}^{\infty} \big|\lambda_j^{(n,m)}\big|,$$

and hence $(Tu_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X.

Next, we show that T has a unique extension from $\mathcal{S}_{\mathrm{rc}}$ to U^p. Let $u \in U^p$. Take a representation $u = \sum_{j=1}^{\infty} \lambda_j a_j$ with U^p-atoms a_j and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. Since the series converges to u in U^p, we can apply part (1) to define $Tu := \sum_{j=1}^{\infty} \lambda_j T a_j$. A standard mixing argument shows that this definition is independent of the choice of the representation. Clearly, T is hereby extended to a linear operator on U^p and we immediately obtain

$$\|Tu\|_X \leq C \sum_{j=1}^{\infty} |\lambda_j| \,.$$

The desired estimate is obtained by taking the infimum over all atomic representation of u. □

Definition 3.7 (Spaces of bounded p-variation)
Let $p \in [1, \infty)$. For a function $v \colon \mathbb{R} \to H$ for which the limit $\lim\limits_{t\to-\infty} v(t)$ exists, we define $v(-\infty) := \lim\limits_{t\to-\infty} v(t)$, and we always set $v(\infty) := 0$. We call

$$\|v\|_{V^p} := \sup_{\tau\in\mathcal{Z}} \left(\sum_{k=1}^{|\tau|} \|v(\tau_k) - v(\tau_{k-1})\|_H^p \right)^{1/p} \tag{3.4}$$

the *p-variation* of v. We define the space V^p *of bounded p-variation* by

$$V^p := \Big\{ v \colon \mathbb{R} \to H : \lim_{t\to\pm\infty} v(t) \text{ exists and } \|v\|_{V^p} < \infty \Big\}.$$

The subspace of V^p-functions vanishing at $-\infty$ is denoted by

$$V^p_- := \Big\{ v \colon \mathbb{R} \to H : v \in V^p \text{ and } \lim_{t\to-\infty} v(t) = 0 \Big\}.$$

If we further restrict to right-continuous functions, we use the notation

$$V^p_{-,\mathrm{rc}} := V^p_- \cap \mathcal{R}_{\mathrm{rc}}.$$

In the next proposition, we give an equivalent norm on V^p and state the obvious embeddings of the spaces in the V^p-scale.

Proposition 3.8 (Elementary properties of the V^p-spaces)
Let $p, q \in [1, \infty)$ with $p < q$. The following assertions are true.

(1) Let $v \colon \mathbb{R} \to H$ be a function satisfying

$$\|v\|_{V_0^p} := \sup_{\tau \in \mathcal{Z}_0} \left(\sum_{k=1}^{|\tau|} \|v(\tau_k) - v(\tau_{k-1})\|_H^p \right)^{1/p} < \infty.$$

Then v is a regulated function. Moreover, v belongs to V^p and $\|v\|_{V^p} = \|v\|_{V_0^p}$.

(2) The embeddings $V^p \hookrightarrow V^q \hookrightarrow \mathcal{R}$ and $V_-^p \hookrightarrow V_-^q$ are continuous and for all $v \in V^p$ we have

$$\|v\|_\infty \leq \|v\|_{V^q} \leq \|v\|_{V^p}.$$

(3) The spaces V^p and V_-^p are Banach spaces.

Proof. We start to prove the first statement. Let $v \colon \mathbb{R} \to H$ be a function with $\|v\|_{V_0^p} < \infty$. Fix $t_0 \in (-\infty, \infty)$ and assume that $\lim_{t \to t_0+} v(t)$ does not exist. Then there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ we can find $t_1^{(n)}, t_2^{(n)} \in (t_0, t_0 + \frac{1}{n})$ such that

$$\left\|v(t_2^{(n)}) - v(t_1^{(n)})\right\|_H > \varepsilon.$$

By choosing partitions containing sufficiently many of the points $t_1^{(n)}, t_2^{(n)}$, we thus see that $\|v\|_{V_0^p} = \infty$, a contradiction. The case $t_0 = -\infty$ and the argument for the left limits on $(-\infty, \infty]$ are similar. It is clear that $\|v\|_{V_0^p} \leq \|v\|_{V^p}$. For the converse inequality, we already know that $v(-\infty) = \lim_{t \to -\infty} v(t)$ exists. Let $\varepsilon > 0$. Let $\tau \in \mathcal{Z}$. Take $\tilde{\tau} \in (-\infty, \tau_1)$ such that $\|v(t) - v(-\infty)\|_H \leq \varepsilon$ for all $t \in (-\infty, \tilde{\tau}]$. Define the partition $\sigma = (\tilde{\tau}, \tau_1, \ldots, \tau_{|\tau|}) \in \mathcal{Z}_0$. We estimate

$$\begin{aligned}
&\left(\sum_{k=1}^{|\tau|} \|v(\tau_k) - v(\tau_{k-1})\|_H^p \right)^{1/p} \\
&\quad \leq \left(\Big(\|v(\tilde{\tau}) - v(-\infty)\|_H + \|v(\tau_1) - v(\tilde{\tau})\|_H \Big)^p + \sum_{k=2}^{|\tau|} \|v(\tau_k) - v(\tau_{k-1})\|_H^p \right)^{1/p} \\
&\quad \leq \varepsilon + \left(\sum_{k=1}^{|\sigma|} \|v(\sigma_k) - v(\sigma_{k-1})\|_H^p \right)^{1/p} \\
&\quad \leq \varepsilon + \|v\|_{V_0^p},
\end{aligned}$$

which yields the desired estimate $\|v\|_{V^p} \leq \|v\|_{V_0^p}$ by taking the supremum over all partitions $\tau \in \mathcal{Z}$ and since $\varepsilon > 0$ was chosen arbitrary.

To prove the second statement, we note that the embeddings $V^p \hookrightarrow V^q$ and $V^p_- \hookrightarrow V^q_-$ directly follow from the embedding $l^p(\mathbb{N}) \hookrightarrow l^q(\mathbb{N})$.

To prove that $V^q \hookrightarrow \mathcal{R}$, take $v \in V^q$ and fix $\varepsilon > 0$. Since v is regulated, there is $t \in \mathbb{R}$ such that $\|v(t)\|_H > \|v\|_\infty - \varepsilon$. The q-variation of v with respect to the partition $\tau = (t, \infty)$ is given by $\|v(t) - v(\infty)\|_H = \|v(t)\|_H > \|v\|_\infty - \varepsilon$. Hence, we obtain $\|v\|_\infty \leq \|v\|_{V^q}$.

As a last step, we prove that V^p and V^p_- are indeed Banach spaces. If $\|v\|_p = 0$, then $v = 0$ since the term $v(\infty) = 0$ appears in definition (3.4). It is straightforward to check that the map $\|\cdot\|_{V^p}$ satisfies absolute homogeneity and the triangle inequality. It remains to prove that the normed vector space V^p is complete. Let $(v_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in V^p. By the previous steps, each v_n is a regulated function and the sequence converges uniformly to some $v \in \mathcal{R}$. We have to show that the function v belongs to V^p and is the limit of the sequence $(v_n)_{n\in\mathbb{N}}$ in V^p. Let $\tau \in \mathcal{Z}$ and $n \in \mathbb{N}$. Using the triangle inequality for the p-norm, we compute

$$
\begin{aligned}
&\left(\sum_{k=1}^{|\tau|} \|v(\tau_k) - v(\tau_{k-1})\|_H^p\right)^{1/p} \\
&\leq \left(\sum_{k=1}^{|\tau|} \Big[\|v(\tau_k) - v_n(\tau_k)\|_H + \|v_n(\tau_k) - v_n(\tau_{k-1})\|_H + \|v_n(\tau_{k-1}) - v(\tau_{k-1})\|_H\Big]^p\right)^{1/p} \\
&\leq 2\left(\sum_{k=0}^{|\tau|} \|v(\tau_k) - v_n(\tau_k)\|_H^p\right)^{1/p} + \|v_n\|_{V^p}.
\end{aligned}
$$

Letting n tend to infinity, we thus obtain that

$$
\left(\sum_{k=1}^{|\tau|} \|v(\tau_k) - v(\tau_{k-1})\|_H^p\right)^{1/p} \leq \limsup_{n\to\infty} \|v_n\|_{V^p}.
$$

Taking the supremum over all partitions $\tau \in \mathcal{Z}$, we conclude that v belongs to V^p and that $\|v\|_{V^p} \leq \limsup_{n\to\infty} \|v_n\|_{V^p}$. Repeating the same computation as above for the p-variation of $v - v_n$ instead of v yields

$$
\|v - v_n\|_{V^p} \leq \limsup_{m\to\infty} \|v_m - v_n\|_{V^p}.
$$

This estimate implies that the sequence $(v_n)_{n\in\mathbb{N}}$ converges to v in V^p.

Finally, V^p_- is a closed subspace of V^p since convergence in V^p implies uniform convergence, see step (2). □

Next, we want to clarify the relations between the U- and V-spaces. Duality is the most important thing, but before, we start with an embedding between U^p-spaces and the right-continuous subspaces of V^p.

Proposition 3.9 (Embeddings, cf. Proposition 2.4, Corollary 2.6 in [HHK09]) Let $p, q \in [1, \infty)$ with $p < q$. The embeddings

$$U^p \hookrightarrow V^p_{-,\mathrm{rc}} \hookrightarrow U^q$$

are continuous.

Proof. We first show that $U^p \hookrightarrow V^p_{-,\mathrm{rc}}$. By Proposition 3.5 parts (3) and (4), every element of U^p is right-continuous, has a limit at $+\infty$, and vanishes at $-\infty$. In view of Lemma 3.6 it remains to check the norm estimate for atoms. Let $a = \sum_{k=1}^{|\tau|} \mathbb{1}_{[\tau_{k-1},\tau_k)}\phi_k$ be a U^p-atom and $\sigma \in \mathcal{Z}$ a partition. If there is an index $k \in \{1, \ldots, |\sigma|\}$ for which $a(\sigma_k) - a(\sigma_{k-1}) = 0$, we can remove the point σ_k from the partition σ without altering the p-variation of a with respect to the partition σ. After removing all superfluous points from σ, we obtain the partition $\tilde{\sigma}$ satisfying $a(\tilde{\sigma}_j) - a(\tilde{\sigma}_{j-1}) = \phi_{k_j} - \phi_{k_{j-1}}$ for some subsequence $(\phi_{k_j})_{j=0}^{|\tilde{\sigma}|}$.

We now compute

$$\begin{aligned}
\sum_{j=1}^{|\sigma|} \|a(\sigma_j) - a(\sigma_{j-1})\|_H^p &= \sum_{j=1}^{|\tilde{\sigma}|} \|a(\tilde{\sigma}_j) - a(\tilde{\sigma}_{j-1})\|_H^p \\
&= \sum_{j=1}^{|\tilde{\sigma}|} \left\|\phi_{k_j} - \phi_{k_{j-1}}\right\|_H^p \\
&\leq 2^{p-1} \sum_{j=1}^{|\tilde{\sigma}|} \left(\left\|\phi_{k_j}\right\|_H^p + \left\|\phi_{k_{j-1}}\right\|_H^p\right) \leq 2^p.
\end{aligned}$$

Therefore, $\|a\|_{V^p}^p \leq 2^p$. Lemma 3.6 thus implies that $\|u\|_{V^p} \leq 2\,\|u\|_{U^p}$ for all $u \in U^p$.

Next we show that $V^p_{-,\mathrm{rc}} \hookrightarrow U^q$. Let $v \in V^p_{-,\mathrm{rc}}$ and assume that $\|v\|_{V^p} = 1$. By Proposition 2.5 in [HHK09], for each $N \in \mathbb{N}$ there exists the following decomposition

$$v(t) = \sum_{n=0}^{N} u_n(t) + v_N(t),$$

which for all $n \in \mathbb{N}$ possesses the following properties.

- u_n is a right-continuous step function with partition $\tau^{(n)} \subseteq \tau^{(n+1)}$,
- $\left|\tau^{(n)}\right| \leq 2^{1+np}$,
- $\|u_n\|_\infty \leq 2^{-n+1}$,
- $\|v_n\|_\infty \leq 2^{-n}$.

This decomposition implies the claim. In fact, we have $u_n = \sum_{k=1}^{|\tau^{(n)}|} \mathbb{1}_{[\tau_{k-1}^{(n)}, \tau_k^{(n)})} \phi_k$ for some $\phi_k \in H$ with $\|\phi_k\|_H \leq 2^{1-n}$ for $k = 1, \dots, |\tau^{(n)}|$. In particular, u_n is a multiple of a U^q-atom with

$$\|u_n\|_{U^q} \leq \left(\sum_{k=1}^{|\tau^{(n)}|} \|\phi_k\|_H^q \right)^{1/q} \leq |\tau^{(n)}|^{1/q} 2^{1-n} \leq 2^{1+1/q} \cdot 2^{n\left(\frac{p}{q}-1\right)}.$$

Since U^q is a Banach space, every absolutely converging series in U^q is also convergent. Hence, we obtain

$$\|v\|_{U^q} \leq \sum_{n=1}^{\infty} \|u_n\|_{U^q} \leq 2^{1+1/q} \left(1 - 2^{\frac{p}{q}-1}\right)^{-1}.$$

This estimate implies the claimed embedding.

For completeness of this exposition, we repeat the inductive proof of Proposition 2.5 in [HHK09]. For $n = 0$, we define the partition $\tau_0 = \{-\infty, \infty\}$ and we set $u_0 = 0$. Then $v_0 = v$ and all required properties are satisfied. Now we assume that for some $N \in \mathbb{N}$ the functions $u_0, \dots, u_N$ and $v_N = v - \sum_{n=0}^N u_n$ have been constructed as requested. Let $k \in \{0, \dots, |\tau^{(n)}|\}$. For $j = 0$, we set $\tau_k^{(n+1,0)} = \tau_k^{(n)}$ and inductively for $j \geq 1$

$$\tau_k^{(n+1,j)} := \inf \left\{ t \in \left(\tau_k^{(n+1,j-1)}, \tau_{k+1}^{(n+1)}\right] : \left\|v(t) - v\left(\tau_k^{(n+1,j-1)}\right)\right\| > 2^{-n-1} \right\},$$

as long as the set on the right side is nonempty. The partition $\tau^{(n+1)}$ is defined to contain all the points in $\bigcup_{j=1}^{\infty} \{\tau_k^{(n+1,j)}\}$. Next, we define

$$u_{n+1} = \sum_{k=1}^{n+1} \mathbb{1}_{[\tau_{k-1}^{(n+1)}, \tau_k^{(n+1)})} v_n(\tau_{k-1}^{n+1}).$$

This construction yields the inequalities

$$\|u_{n+1}\|_\infty \leq \|v_n\|_\infty \leq 2^{-n},$$

as well as

$$\begin{aligned} \|v_{n+1}\|_\infty &= \|v_n - u_{n+1}\|_\infty \\ &\leq \sup_{k \in \{1, \dots, |\tau^{(n+1)}|\}} \sup_{t \in [\tau_{k-1}^{(n+1)}, \tau_k^{(n+1)})} \left\|v_n(t) - v_n\left(\tau_{k-1}^{(n+1)}\right)\right\|_H \\ &\leq 2^{-n-1}. \end{aligned}$$

Finally, each time a new partition point is inserted to $\tau^{(n+1)}$, the function v varies on the new segment at least by $2^{-(n+1)}$. Therefore,

$$\left(|\tau^{(n+1)}| - |\tau^{(n)}|\right) 2^{-(n+1)p} \leq \|v\|_{V^p}^p = 1,$$

which implies $|\tau^{(n+1)}| \leq |\tau^{(n)}| + 2^{(n+1)p} \leq 2^{1+np} + 2^{(n+1)p} \leq 2^{1+(n+1)p}$. □

Duality

We now arrive at the main result of this section, the duality between U- and V-spaces. We first define the duality pairing for step functions and then extend to the full space U^p by the abstract result from Lemma 3.6. We also give a more explicit formula for the duality pairing in Proposition 3.17 which is reminiscent of a Stieltjes integral. To motivate the duality pairing, recall the Riesz representation theorem stated in [Rie09].

> Étant donnée l'opération linéaire $A[f(x)]$, on peut déterminer la fonction à variation bornée $\alpha(x)$, telle que, quelle que soit la fonction continue $f(x)$ on ait
> $$A[f(x)] = \int_0^1 f(x)\,\mathrm{d}\alpha(x).$$

This means that every bounded linear functional on $C([0,1],\mathbb{R})$ is represented by a function of bounded 1-variation via a Stieltjes integral. For suitably regular functions f and α, the above duality statement takes the form

$$\int f\,\mathrm{d}\alpha = -\int \alpha\,\mathrm{d}f = -\int \alpha f'\,\mathrm{d}t.$$

Compare this with the statement of Proposition 3.17.

Proposition 3.10 (Duality between $U^{p'}$ and V^p, cf. [HHK09, Thm 2.8])
Let $p \in (1,\infty)$. We denote the dual exponent of p by p' satisfying $\frac{1}{p}+\frac{1}{p'}=1$. Let $\tau \in \mathcal{Z}_0$. For $u = \sum_{k=1}^{|\tau|} \mathbb{1}_{[\tau_{k-1},\tau_k)}\phi_k \in \mathcal{S}_{\mathrm{rc}}$ and $v \in V^p$, we define

$$B(u,v) := \sum_{k=1}^{|\tau|} \big(u(\tau_{k-1}), v(\tau_k) - v(\tau_{k-1})\big)_H. \tag{3.5}$$

Recall that by convention $v(\tau_{|\tau|}) = v(\infty) = 0$ in the above formula. Definition (3.5) gives rise to a sesquilinear form $B\colon \mathcal{S}_{\mathrm{rc}} \times V^p \to \mathbb{C}$. It can be uniquely extended to a sesquilinear form $B\colon U^{p'} \times V^p \to \mathbb{C}$ which satisfies the estimate

$$|B(u,v)| \le \|u\|_{U^{p'}} \|v\|_{V^p} \quad \text{for all } u \in U^{p'} \text{ and } v \in V^p.$$

In the following sense the space $U^{p'}$ is the predual of V^p. The map

$$T\colon V^p \to (U^{p'})^*, \quad T(v) := B(\cdot, v)$$

is an isometric isomorphism. In particular, for $u \in U^{p'}$ it holds

$$\|u\|_{U^{p'}} = \sup_{v\in V^p,\ \|v\|_{V^p}\le 1} |B(u,v)| \tag{3.6}$$

and for $v \in V^p$ we have

$$\|v\|_{V^p} = \sup_{a\ U^{p'}\text{-atom}} |B(a,v)|\,. \tag{3.7}$$

Proof. Note that representing the same function u in a different form by refining the partition τ does not alter formula (3.5). Hence, the mapping $B\colon \mathcal{S}_{\mathrm{rc}} \times V^p \to \mathbb{C}$ is well-defined and it is clearly a sesquilinear form. In the first step, we show that B extends to $U^{p'} \times V^p$. Fix $v \in V^p$. For any $U^{p'}$-atom a with corresponding partition $\tau \in \mathcal{Z}_0$, we can estimate with Cauchy–Schwarz' and Hölder's inequalities that

$$|B(a,v)| \leq \left(\sum_{k=1}^{|\tau|} \|a(\tau_{k-1})\|_H^{p'}\right)^{1/p'} \left(\sum_{k=1}^{|\tau|} \|v(\tau_k) - v(\tau_{k-1})\|_H^p\right)^{1/p} \leq \|v\|_{V^p}.$$

The linear operator $B(\cdot, v)\colon \mathcal{S}_{\mathrm{rc}} \to \mathbb{C}$ thus satisfies the assumptions of Lemma 3.6 and there exists a unique extension $B(\cdot, v)\colon U^{p'} \to \mathbb{C}$ which satisfies

$$|B(u,v)| \leq \|u\|_{U^{p'}} \|v\|_{V^p} \quad \text{for all } u \in U^{p'}.$$

We show next that T is an isometry. Let $\varepsilon > 0$ and $v \in V^p \setminus \{0\}$. We choose a partition $\tau \in \mathcal{Z}_0$ such that

$$\|v\|_{V^p}^p \leq \sum_{k=1}^{|\tau|} \|v(\tau_k) - v(\tau_{k-1})\|_H^p + \varepsilon.$$

Note that by Proposition 3.8 (1) we can actually achieve this with a $\mathcal{Z}_0$-partition. For each $k = 1, \dots, |\tau|$, we choose a vector $x_k \in H$ with $\|x_k\|_H = 1$ and

$$(x_k, v(\tau_k) - v(\tau_{k-1}))_H \geq (1-\varepsilon) \|v(\tau_k) - v(\tau_{k-1})\|_H$$

and we define $\phi_k := \|v\|_{V^p}^{1-p} \|v(\tau_k) - v(\tau_{k-1})\|_H^{p-1} x_k$. We define $a := \sum_{k=1}^{|\tau|} \mathbb{1}_{[\tau_{k-1},\tau_k)} \phi_k$, and we note that a is a $U^{p'}$-atom since

$$\sum_{k=1}^{|\tau|} \|\phi_k\|_H^{p'} \leq \|v\|_{V^p}^{-p} \sum_{k=1}^{|\tau|} \|v(\tau_k) - v(\tau_{k-1})\|_H^p \leq 1.$$

It follows that

$$\begin{aligned} B(a,v) &= \sum_{k=1}^{|\tau|} (\phi_k, v(\tau_k) - v(\tau_{k-1}))_H \\ &\geq \|v\|_{V^p}^{1-p} \sum_{k=1}^{|\tau|} (1-\varepsilon) \|v(\tau_k) - v(\tau_{k-1})\|_H^p \\ &\geq (1-\varepsilon) \|v\|_{V^p} - \varepsilon(1-\varepsilon) \|v\|_{V^p}^{1-p}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary and since we have seen in the beginning of the proof that $|B(a,v)| \leq \|v\|_{V^p}$, we obtain that $\|T(v)\|_{U^{p'}} = \|v\|_{V^p}$.

Since T is an isometry, it is injective. It remains to prove surjectivity of T. Let $L \in (U^{p'})^* \setminus \{0\}$. For each $t \in \mathbb{R}$, by the Riesz representation theorem, there exists a vector $v(t) \in H$ such that

$$(\phi, v(t))_H = -L(\mathbb{1}_{[t,\infty)}\phi) \quad \text{for all } \phi \in H.$$

We show that $v \in V^p$ and that $B(u,v) = L(u)$ for all $u \in U^{p'}$. Note that $v \neq 0$ since $L \neq 0$. Consider a partition $\tau \in \mathcal{Z}_0$. Define the number

$$\gamma = \sum_{j=1}^{|\tau|} \|v(\tau_j) - v(\tau_{j-1})\|_H^p .$$

For $k = 1, \ldots, |\tau|$, we define

$$\phi_k := \gamma^{-1/p'} \|v(\tau_k) - v(\tau_{k-1})\|_H^{p-2} \big(v(\tau_k) - v(\tau_{k-1})\big).$$

We set $a := \sum_{k=1}^{|\tau|} \mathbb{1}_{[\tau_{k-1},\tau_k)}\phi_k$ and we see that a is a $U^{p'}$-atom since

$$\sum_{k=1}^{|\tau|} \|\phi_k\|^{p'} = \gamma^{-1} \sum_{k=1}^{|\tau|} \|v(\tau_k) - v(\tau_{k-1})\|_H^{(p-1)p'} = 1.$$

We obtain

$$\begin{aligned}
\|L\| &\geq |L(a)| \\
&= \left| \sum_{k=1}^{|\tau|} L(\mathbb{1}_{[\tau_{k-1},\infty)}\phi_k) - L(\mathbb{1}_{[\tau_k,\infty)}\phi_k) \right| \\
&= \left| \sum_{k=1}^{|\tau|} \big(\phi_k, v(\tau_k) - v(\tau_{k-1})\big)_H \right| \\
&= \left(\sum_{k=1}^{|\tau|} \|v(\tau_k) - v(\tau_{k-1})\|_H^p \right)^{1/p} .
\end{aligned}$$

By Proposition 3.8 (1), this estimate implies that $v \in V^p$. Finally, let $\tau \in \mathcal{Z}_0$ be a partition and a a $U^{p'}$-atom given by $a = \sum_{k=1}^{|\tau|} \mathbb{1}_{[\tau_{k-1},\tau_k)}\phi_k$. We then compute

$$T(v)(a) = B(a,v) = \sum_{k=1}^{|\tau|} \big(\phi_k, v(\tau_k) - v(\tau_{k-1})\big)_H = \sum_{k=1}^{|\tau|} L(\mathbb{1}_{[\tau_{k-1},\tau_k)}\phi_k) = L(a).$$

Since this equality holds for all atoms, we conclude $T(v) = L$.

Formula (3.6) is clear by duality and so is

$$\|v\|_{V^p} = \sup_{u\in U^{p'},\ \|u\|_{U^p}\le 1} |B(u,v)| \geq \sup_{a\ U^{p'}\text{-atom}} |B(a,v)|\,,$$

for every $v \in V^p$. The extension via Lemma 3.6 applied to $T(v)\colon U^{p'} \to \mathbb{C}$ shows that we have equality in the line above and thus (3.7) follows. □

The definition of the space U^p through atoms makes it difficult to actually compute the U^p-norm of a function or even of an atom. Another strategy to compute the U^p-norm is to use the duality pairing. The following lemma says that the U^p-norm can be computed by evaluating the duality pairing solely on right-continuous step functions. Note that this statement is not trivial since the space $\mathcal{S}_{\mathrm{rc}}$ is not a dense subspace of $V^{p'}$. The observation that $\mathcal{S}_{\mathrm{rc}}$ is norming is contained in Theorem 4.4 of the article [CH18].

Lemma 3.11 ($\mathcal{S}_{\mathrm{rc}}$ norms U^p)
Let $p \in (1,\infty)$. Denote the set of right continuous step functions in the unit ball of $V^{p'}$ by $\mathcal{B}^{p'}_{\mathrm{rc}} := \{v \in \mathcal{S}_{\mathrm{rc}} : \|v\|_{V^{p'}} \le 1\}$. Step functions are norming for U^p, i.e. for every $u \in U^p$, we have

$$\|u\|_{U^p} = \sup_{v\in\mathcal{B}^{p'}_{\mathrm{rc}}} |B(u,v)|\,. \tag{3.8}$$

Proof. We first consider step functions. Let $u = \sum_{k=1}^{|\tau|} \mathbb{1}_{[\tau_{k-1},\tau_k)}\phi_k \in \mathcal{S}_{\mathrm{rc}}$ for some partition $\tau \in \mathcal{Z}_0$ and a sequence $(\phi_k)_{k=1}^{|\tau|}$ in H. By the previous Proposition 3.10 and the Hahn–Banach theorem, there exists $v \in V^{p'}$ with $\|v\|_{V^{p'}} \le 1$ such that

$$\|u\|_{U^p} = |B(u,v)| = \left|\sum_{k=1}^{|\tau|}\big(u(\tau_{k-1}), v(\tau_k) - v(\tau_{k-1})\big)_H\right|.$$

Define a step function $\tilde{v} \in \mathcal{S}_{\mathrm{rc}}$ by setting $\tilde{v} = \sum_{k=1}^{|\tau|} \mathbb{1}_{[\tau_{k-1},\tau_k)} v(\tau_{k-1})$. Since $\tilde{v}(\tau_k) = v(\tau_k)$ for $k = 0,\dots,|\tau|$, we have $B(u,\tilde{v}) = B(u,v)$. We also have

$$\|\tilde{v}\|_{V^{p'}} = \left(\sum_{k=1}^{|\tau|} \|v(\tau_k) - v(\tau_{k-1})\|^{p'}\right)^{1/p'} \le \|v\|_{V^{p'}} \le 1,$$

and hence we have shown that

$$\|u\|_{U^p} = \sup_{v\in\mathcal{B}^{p'}_{\mathrm{rc}}} |B(u,v)|\,.$$

Finally, let $u \in U^p$. By Proposition 3.5 (2), there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{S}_{\mathrm{rc}}$ which converges to u. By the first step, there exists $v_n \in \mathcal{S}_{\mathrm{rc}}$ with $\|v_n\|_{V^{p'}} \leq 1$ such that $\|u_n\|_{U^p} = |B(u_n, v_n)|$ for every $n \in \mathbb{N}$. Hence, we conclude

$$\begin{aligned}\|u\|_{U^p} &= \lim_{n\to\infty} \|u_n\|_{U^p} \\ &= \lim_{n\to\infty} |B(u_n, v_n)| \\ &\leq \limsup_{n\to\infty} |B(u_n - u, v_n)| + |B(u, v_n)| \\ &\leq \sup_{v\in\mathcal{B}^{p'}_{\mathrm{rc}}} |B(u, v)| .\end{aligned}$$ □

The duality pairing $B(u, v)$ was initially defined by formula (3.5) for $u \in \mathcal{S}_{\mathrm{rc}}$ and $v \in V^p$. In the following examples, we show how to calculate $B(u, v)$ for arbitrary $u \in U^{p'}$ in two easy cases: first, if v is a constant function and second, if v is supported in one point.

Example 3.12 (Duality pairing with a constant function)
Let $p \in (1,\infty)$. Let $x \in H$ and consider the constant function $v(t) = x$ for $t \in \mathbb{R}$. Then $v \in V^p$ and $\|v\|_{V^p} = \|x\|_H$. We want to compute $B(u, v)$ for any $u \in U^{p'}$. First, let $\tau \in \mathcal{Z}_0$ and let $w = \sum_{k=1}^{|\tau|} \mathbb{1}_{[\tau_{k-1},\tau_k)}\phi_k \in \mathcal{S}_{\mathrm{rc}}$ be a right-continuous step function. By Definition (3.5), we obtain

$$B(w, v) = \sum_{k=1}^{|\tau|} \big(w(\tau_{k-1}), v(\tau_k) - v(\tau_{k-1})\big)_H = -\big(w(\tau_{|\tau|-1}), x\big)_H ,$$

since $v(\tau_{|\tau|}) = v(\infty) = 0$ by convention.

Next, take an arbitrary function $u \in U^{p'}$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}_{\mathrm{rc}}$ which converges to u in U^p as in Proposition 3.5 (2). Uniform convergence of $(u_n)_{n\in\mathbb{N}}$ yields

$$u_n(\tau_{|\tau|-1}) = \lim_{t\to\infty} u_n(t) \longrightarrow \lim_{t\to\infty} u(t) = u_\infty \quad \text{as } n \to \infty.$$

Therefore, we obtain the formula

$$B(u, v) = \lim_{n\to\infty} B(u_n, v) = \lim_{n\to\infty} -\big(u_n(\tau_{|\tau|-1}), x\big)_H = -(u_\infty, x)_H . \tag{3.9}$$

Example 3.13 (Duality pairing with a function supported in one point)
Let $p \in (1,\infty)$. Let $x \in H$, $s \in \mathbb{R}$ and consider the function $v(t) = \mathbb{1}_{\{s\}}(t)x$ for $t \in \mathbb{R}$. Then $v \in V^p$ and $\|v\|_{V^p} = 2^{1/p}\|x\|_H$. Let $\tau \in \mathcal{Z}_0$ and let $w = \sum_{k=1}^{|\tau|} \mathbb{1}_{[\tau_{k-1},\tau_k)}\phi_k \in \mathcal{S}_{\mathrm{rc}}$ be a right-continuous step function. Let $\delta > 0$ be a positive number so small that $\delta < \min\{\tau_k - \tau_{k-1} : k = 1,\ldots,|\tau|\}$. By adding the points $s - \delta$ and s to the partition τ, we obtain the partition $\sigma = \tau \cup \{s - \delta, s\} \in \mathcal{Z}_0$. We then compute

$$B(w, v) = \sum_{k=1}^{|\sigma|} \big(w(\sigma_{k-1}), v(\sigma_k) - v(\sigma_{k-1})\big)_H = \big(w(s - \delta) - w(s), x\big)_H. \tag{3.10}$$

Next, let $u \in U^{p'}$. By Proposition 3.5 (2), there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{S}_{\mathrm{rc}}$ which converges to u in $U^{p'}$. For each $n \in \mathbb{N}$, we take $\delta_n > 0$ such that formula (3.10) is satisfied for u_n and such that $\delta_n \to 0$ as $n \to \infty$. Since the sequence $(u_n)_{n\in\mathbb{N}}$ converges to u uniformly, we obtain

$$\begin{aligned} B(u,v) &= \lim_{n\to\infty} B(u_n, v) \\ &= \lim_{n\to\infty} \big(u_n(s-\delta_n) - u(s-\delta_n) + u(s-\delta_n) - u_n(s), x\big)_H \\ &= \Big(\lim_{t\to s-} u(s) - u(s), x\Big)_H. \end{aligned}$$

In particular, if u is continuous at s then $B(u,v) = 0$.

In the following lemma, we obtain an explicit formula for $B(u,v)$ where we now take arbitrary $u \in U^{p'}$ and a left-continuous step function v, which vanishes at $-\infty$. We use this representation in Proposition 3.17 to derive yet another formula for the duality pairing.

Lemma 3.14 ($B(u,\cdot)$ on left-continuous step functions)
Let $p \in (1,\infty)$, $u \in U^{p'}$, $\tau \in \mathcal{Z}_0$, $(\psi_k)_{k=1}^{|\tau|} \subseteq H$, and $v = \sum_{k=1}^{|\tau|} \mathbb{1}_{(\tau_{k-1},\tau_k]}\psi_k$. Then

$$B(u,v) = -\sum_{k=1}^{|\tau|-1} \big(u(\tau_k) - u(\tau_{k-1}), \psi_k\big)_H - \big(u_\infty - u(\tau_{|\tau|-1}), \psi_{|\tau|}\big)_H. \tag{3.11}$$

Proof. For every $u \in U^{p'}$ we define

$$F_v(u) := -\sum_{k=1}^{|\tau|-1} \big(u(\tau_k) - u(\tau_{k-1}), \psi_k\big)_H - \big(u_\infty - u(\tau_{|\tau|-1}), \psi_{|\tau|}\big)_H.$$

Then $F_v \colon U^{p'} \to \mathbb{C}$ is a linear map. Let $a = \sum_{k=1}^{|\sigma|} \mathbb{1}_{[\sigma_{k-1},\sigma_k)}\phi_k$ be a $U^{p'}$-atom with partition $\sigma \in \mathcal{Z}_0$. We check below that $F_v(a) = B(a,v)$. The uniqueness of the extension provided by Lemma 3.6 then implies the claimed equation (3.11).

Define a new partition $\rho = \tau \cup \sigma$. Note that $\rho_0 = \min\{\tau_0, \sigma_0\} \le \tau_0$ and that $\rho_{|\rho|} = \tau_{|\tau|} = \sigma_{|\sigma|} = \infty$. By possibly adding one point to the partition σ, we may and will assume that $\rho_{|\rho|-1} = \sigma_{|\sigma|-1} > \tau_{|\tau|-1}$. Since $v(\rho_0) = 0$ and $v(\rho_{|\rho|}) = v(\infty) = 0$, summation by parts yields the identity

$$\begin{aligned} \sum_{k=1}^{|\rho|} \big(a(\rho_{k-1}), v(\rho_k) - v(\rho_{k-1})\big)_H &= \sum_{k=1}^{|\rho|} \big(a(\rho_{k-1}), v(\rho_k)\big)_H - \sum_{k=1}^{|\rho|} \big(a(\rho_{k-1}), v(\rho_{k-1})\big)_H \\ &= \sum_{k=1}^{|\rho|-1} \big(a(\rho_{k-1}), v(\rho_k)\big)_H - \sum_{k=1}^{|\rho|-1} \big(a(\rho_k), v(\rho_k)\big)_H \\ &= -\sum_{k=1}^{|\rho|-1} \big(a(\rho_k) - a(\rho_{k-1}), v(\rho_k)\big)_H. \end{aligned}$$

Since a is constant on every interval $[\sigma_{k-1}, \sigma_k)$ for $k \in \{1, \ldots, |\sigma|\}$, we obtain from Definition (3.5) and by a telescoping sum argument that

$$\begin{aligned} B(a, v) &= \sum_{k=1}^{|\sigma|} \big(a(\sigma_{k-1}), v(\sigma_k) - v(\sigma_{k-1})\big)_H \\ &= \sum_{k=1}^{|\rho|} \big(a(\rho_{k-1}), v(\rho_k) - v(\rho_{k-1})\big)_H. \end{aligned}$$

Since v is constant on every interval $(\tau_{k-1}, \tau_k]$ for $k \in \{1, \ldots, |\tau| - 1\}$ and $v(t) = \psi_{|\tau|}$ for $t \in (\tau_{|\tau|-1}, \tau_{|\tau|}) = (\tau_{|\tau|-1}, \infty)$, we also obtain

$$\begin{aligned} F_v(a) &= -\sum_{k=1}^{|\tau|-1} \big(a(\tau_k) - a(\tau_{k-1}), \psi_k\big)_H - \big(a_\infty - a(\tau_{|\tau|-1}), \psi_{|\tau|}\big) \\ &= -\sum_{k=1}^{|\rho|-1} \big(a(\rho_k) - a(\rho_{k-1}), v(\rho_k)\big)_H. \end{aligned}$$

Note that in this calculation we have used that $\rho_{|\rho|-1} > \tau_{|\tau|-1}$ to obtain that $v(\rho_{|\rho|-1}) = \psi_{|\rho|}$ and $a(\rho_{|\rho|-1}) = a_\infty$. Comparing the last two results to the left and right side of the identity above, we have thus proven that $F_v(a) = B(a, v)$ for every $U^{p'}$-atom a. As noted in the beginning, the proof of the claimed formula (3.11) is complete. □

The following lemma contains another useful identity for B. If the first argument is a continuous function, then we may change the second function on at most countably many points. Since V^p-functions are regulated functions, the lemma allows us to replace them in the above situation by their right- or left-continuous variant.

Lemma 3.15 (Countable perturbations)
Let $p \in (1, \infty)$. Let $D \subseteq \mathbb{R}$ be a countable set. Let $u \in U^{p'}$ be continuous and let $v, \tilde{v} \in V^p$ such that $v(t) = \tilde{v}(t)$ all $t \in \mathbb{R} \backslash D$. Then

$$B(u, v) = B(u, \tilde{v}).$$

Proof. Let $\varepsilon > 0$, and fix $u \in U_c^{p'}$. Define $w := v - \tilde{v} \in V^p$. We show that $B(u, w) = 0$ which implies the assertion by linearity. By assumption, we can write $w = \sum_{k=1}^\infty \mathbb{1}_{\{s_k\}} x_k$ for some $s_k \in \mathbb{R}$ and $x_k \in H$. Moreover

$$\|w\|_{V^p} \le \sum_{k=1}^\infty \big\|\mathbb{1}_{\{s_k\}} x_k\big\|_{V^p} = 2^{1/p} \sum_{k=1}^\infty \|x_k\|_H < \infty$$

and the finite sums $\sum_{k=1}^{n} \mathbb{1}_{\{s_k\}} x_k$ converge to w in V^p as $n \to \infty$. Since u is continuous, the result of Example 3.13 yields

$$B(u,w) = \lim_{n\to\infty} B\left(u, \sum_{k=1}^{n} \mathbb{1}_{\{s_k\}} x_k\right) = \lim_{n\to\infty} \sum_{k=1}^{n} B\left(u, \mathbb{1}_{\{s_k\}} x_k\right) = 0. \qquad \square$$

Corollary 3.16 ($B(u,\cdot)$ on right-continuous step functions)
Let $p \in (1,\infty)$. Let $u \in U^{p'}$ be continuous, $\tau \in \mathcal{Z}_0$, and $(\psi_k)_{k=1}^{|\tau|} \subseteq H$. Let $v \in \mathcal{S}_{\mathrm{rc}}$ be given by $v = \sum_{k=1}^{|\tau|} \mathbb{1}_{[\tau_{k-1},\tau_k)} \psi_k$. Then

$$B(u,v) = -\sum_{k=1}^{|\tau|-1} \big(u(\tau_k) - u(\tau_{k-1}), v(\tau_{k-1})\big)_H - \big(u_\infty - u(\tau_{|\tau|-1}), v(\tau_{|\tau|-1})\big)_H. \quad (3.12)$$

Proof. Let $\tilde{v} = \sum_{k=1}^{|\tau|} \mathbb{1}_{(\tau_{k-1},\tau_k]} \psi_k$. Then the functions v and $\tilde{v}$ agree everywhere except on a finite set. Therefore, $B(u,v) = B(u,\tilde{v})$ by Lemma 3.15, and the assertion follows from formula (3.11) in Lemma 3.14. $\square$

We are now able to derive the announced representation of the duality pairing as an integral. The proof uses formula (3.12) together with the fundamental theorem of calculus. The result turns out to be quite useful in handling the Duhamel term of an inhomogeneous equation, we refer to Section 4 for applications.

Proposition 3.17 (The duality pairing as an integral, cf. [HHK09, Prop 2.10])
Let $p \in (1,\infty)$. Let $u \in V_-^1$ be a function which is absolutely continuous on compact intervals, and let $v \in \mathcal{S}_{\mathrm{rc}}$. Then the duality pairing B can be written as the following integral

$$B(u,v) = -\int_{-\infty}^{\infty} (u'(t), v(t))_H \, \mathrm{d}t. \quad (3.13)$$

Proof. By the embedding $V_{-,\mathrm{rc}}^1 \hookrightarrow U^q$ for any $q \in (1,\infty)$ proved in Proposition 3.9, u belongs to $U^{p'}$ so that the expression $B(u,v)$ makes sense. Since the Hilbert space H has the Radon–Nikodym property, the assumptions on u furthermore imply that $u' \in L^1(\mathbb{R},H)$ and that the fundamental theorem of calculus holds true, see e.g. Propositions 1.2.3, 1.2.4, Definition 1.2.5 and Corollary 1.2.7 in the book [ABHN11].

Let $\tau \in \mathcal{Z}_0$ be the partition associated to v such that $v = \sum_{k=1}^{|\tau|} \mathbb{1}_{[\tau_{k-1},\tau_k} v(\tau_{k-1})$. Using formula (3.12) and the fundamental theorem of calculus, we compute

$$\begin{aligned}
B(u,v) &= -\sum_{k=1}^{|\tau|-1} \big(u(\tau_k) - u(\tau_{k-1}), v(\tau_{k-1})\big)_H - \big(u_\infty - u(\tau_{|\tau|-1}), v(\tau_{|\tau|-1})\big)_H \\
&= -\sum_{k=1}^{|\tau|} \int_{\tau_{k-1}}^{\tau_k} \big(u'(s), v(s)\big)_H \,\mathrm{d}s \\
&= -\int_{-\infty}^{\infty} \big(u'(s), v(s)\big)_H \,\mathrm{d}s. \qquad \square
\end{aligned}$$

Remark 3.18 (Generalizations of the integral formula)
Formula (3.13) can be extended to a larger class of functions v. Example 3.12 allows treating functions v which do not vanish at $-\infty$, and Lemma 3.15 allows treating functions v which are not right-continuous. By approximation, one can also take v from the closure $\mathcal{S}_{\mathrm{rc}}$ in $V^p_{-,\mathrm{rc}}$. However, $\mathcal{S}_{\mathrm{rc}}$ is not dense in $V^p_{-,\mathrm{rc}}$ and thus our proof does not yield formula (3.13) for *all* $v \in V^p$. Nevertheless, Proposition 2.10 in [HHK09] proves the validity of formula (3.13) for all $v \in V^p$ by a different argument. Since we use formula (3.13) mainly to compute U^p-norms, see e.g. the proof of Lemma 3.20, and since $\mathcal{S}_{\mathrm{rc}}$ is norming by Lemma 3.11, we are content with the assumptions of Proposition 3.17.

Next, we introduce a Littlewood–Paley decomposition which is beneficial for more complicated nonlinearities by treating high and low frequencies differently.

Definition 3.19 (Littlewood–Paley decomposition)
A *dyadic number* N is a number of the form $N = 2^n$ for some $n \in \mathbb{Z}$. For *dyadic sums*, we use the notation

$$\sum_N a_N \coloneqq \sum_{n\in\mathbb{Z}} a_{2^n} \quad \text{and} \quad \sum_{N\geq M} a_N \coloneqq \sum_{n\in\mathbb{Z},\, 2^n\geq M} a_{2^n}.$$

We fix a function $\chi \in C_0^\infty\big((-2,2)\big)$ which is even, nonnegative, and $\chi(t) = 1$ for all $t \in [-1,1]$. We define the cut-off function $\psi(t) \coloneqq \chi(t) - \chi(2t)$ for $t \in \mathbb{R}$ which is supported in the annulus $\{\frac{1}{2} \leq |t| \leq 2\}$. We scale ψ to dyadic intervals

$$\psi_N \coloneqq \psi(N^{-1}\cdot) \quad \text{for each dyadic number } N.$$

Let N, M be dyadic numbers. Since $\operatorname{supp}\psi_N \subseteq \{N \leq |t| \leq 2N\}$, we observe that the supports of ψ_N and ψ_M are disjoint unless N and M are neighbours. This motivates to use the notation $N \sim M$ to express that $|\log_2 N - \log_2 M| \leq 1$. We also write $N \ll M$ to say that $N \leq M$ but *not* $N \sim M$. Let $u \in \mathcal{S}'(\mathbb{R}^{1+d})$ a tempered distribution and N a dyadic number. We define the *temporal Littlewood–Paley projection*

$$\big(\mathcal{F}_t Q_N u\big)(\tau, x) \coloneqq \psi_N(\tau)\big(\mathcal{F}_t u\big)(\tau, x).$$

We also use the operators

$$Q_{\geq M} u \coloneqq \sum_{N \geq M} Q_N u \quad \text{and} \quad Q_{<M} u \coloneqq I - Q_{\geq M} u.$$

Lemma 3.20 (Properties of the temporal Littlewood–Paley projections)
Let $p \in (1, \infty)$ and M a dyadic number.

(1) The projection onto high frequencies act favourably on $L^2(\mathbb{R}, H)$, i.e. for all $v \in V^2$

$$\|Q_M v\|_{L^2(\mathbb{R},H)} \leq C M^{-1/2} \|v\|_{V^2} \tag{3.14}$$

and

$$\|Q_{\geq M} v\|_{L^2(\mathbb{R},H)} \leq C M^{-1/2} \|v\|_{V^2} . \tag{3.15}$$

(2) The operators $Q_{<M}$ and $Q_{\geq M}$ are bounded on U^p and V^p, i.e. for $u \in U^p$ and $v \in V^p$ we have the estimates

$$\begin{aligned} \|Q_{<M} u\|_{U^p} &\leq C \|u\|_{U^p} , & \|Q_{\geq M} u\|_{U^p} &\leq C \|u\|_{U^p} , \\ \|Q_{<M} v\|_{V^p} &\leq C \|v\|_{V^p} , & \|Q_{\geq M} v\|_{V^p} &\leq C \|v\|_{V^p} . \end{aligned}$$

Proof. Let $v \in V^2$. The first formula can be reformulated as the statement

$$\sup_M M^{1/2} \|Q_M v\|_{L^2} \leq C \|v\|_{V^2} .$$

By Definition 2.15 in [BCD11], the expression on the left is the norm of v in the homogeneous Besov space $\dot{B}^{1/2}_{2,\infty}(\mathbb{R}, H)$. So in fact, the proof below shows the embedding $V^2 \hookrightarrow \dot{B}^{1/2}_{2,\infty}(\mathbb{R}, H)$. To do this, it is more convenient to use Theorem 2.36 in [BCD11] which provides the equivalent norm $\sup_{h>0} h^{-1/2} \|v(\cdot + h) - v\|_{L^2(\mathbb{R},H)}$ on $\dot{B}^{1/2}_{2,\infty}(\mathbb{R}, H)$. We compute

$$\begin{aligned} &\sup_{h>0} h^{-1/2} \|v(\cdot + h) - v\|_{L^2(\mathbb{R},H)} \\ &\quad = \sup_{h>0} h^{-1/2} \left(\int_{\mathbb{R}} \|v(t+h) - v(t)\|_H^2 \,\mathrm{d}t \right)^{1/2} \\ &\quad = \sup_{h>0} h^{-1/2} \left(\sum_{n \in \mathbb{Z}} \int_{nh}^{(n+1)h} \|v(t+h) - v(t)\|_H^2 \,\mathrm{d}t \right)^{1/2} \\ &\quad \leq \sup_{h>0} h^{-1/2} \left(\sum_{n \in \mathbb{Z}} h \sup_{t \in [nh,(n+1)h]} \|v(t+h) - v(t)\|_H^2 \,\mathrm{d}t \right)^{1/2} . \end{aligned}$$

Let $\varepsilon > 0$. Choose $t_n \in [nh, (n+1)h]$ such that

$$\sup_{t \in [nh,(n+1)h]} \|v(t+h) - v(t)\|_H^2 \leq (1+\varepsilon) \|v(t_n + h) - v(t_n)\|_H^2 .$$

The term on the right is less or equal to $(1+\varepsilon) \|v\|_{V^2}$, and thus the assertion (3.14) follows.

From (3.14), we conclude the second estimate (3.15). Indeed, (3.14) yields

$$\|Q_{\geq M} v\|_{L^2(\mathbb{R},H)} \leq \sum_{N \geq M} \|Q_N v\|_{L^2(\mathbb{R},H)} \leq C \sum_{N \geq M} N^{-1/2} \|v\|_{V^2} ,$$

and we can easily compute the dyadic sum

$$\sum_{N \geq M} N^{-1/2} = M^{-1/2} \sum_{N \geq 1} N^{-1/2} = \frac{1}{1 - 2^{-1/2}} M^{-1/2}.$$

Fix a dyadic number M. For the second part, it suffices to prove boundedness of the operators $Q_{<M}$ since $Q_{\geq M} = I - Q_{<M}$ by definition. Let $v \in V^p$. Write the projection $Q_{<M}$ as a convolution via

$$Q_{<M} v = \sum_{N<M} Q_N v = \sum_{N<M} \mathcal{F}_t^{-1} \left(\psi_N \mathcal{F}_t u \right) = \left(\sum_{N<M} \mathcal{F}_t^{-1} \psi_N \right) * v = \phi_M * v,$$

where we define the Schwartz function $\phi_M = \sum_{N<M} \mathcal{F}_t^{-1} \psi_N$. Note that by scaling ϕ_M has the L^1-norm

$$\begin{aligned} \|\phi_M\|_{L^1} &= \int_{\mathbb{R}} |\phi_M(t)| \, \mathrm{d}t \\ &= \int_{\mathbb{R}} \frac{1}{M} \left| \sum_{N<M} \mathcal{F}_t^{-1} \psi_N \left(\tfrac{t}{M} \right) \right| \mathrm{d}t \\ &= \int_{\mathbb{R}} \left| \sum_{N<M} \mathcal{F}_t^{-1} \psi_{N/M}(t) \right| \mathrm{d}t = \|\phi_1\|_{L^1} \end{aligned}$$

which is independent of M. Take a partition $\tau \in \mathcal{Z}_0$. Minkowski's integral inequality in the measure space $\mathbb{R}^{|\tau|} \times L^2(\mathbb{R}, H)$ and the translation invariance of the p-variation yield that

$$
\begin{aligned}
&\left(\sum_{k=1}^{|\tau|} \|Q_{<M}v(\tau_k) - Q_{<M}v(\tau_{k-1})\|_H^p\right)^{1/p} \\
&\quad = \left(\sum_{k=1}^{|\tau|} \left\| \int_{\mathbb{R}} \phi_M(s)\big(v(\tau_k - s) - v(\tau_{k-1} - s)\big)\,\mathrm{d}s \right\|_H^p\right)^{1/p} \\
&\quad \le \left(\sum_{k=1}^{|\tau|} \left(\int_{\mathbb{R}} |\phi_M(s)|\, \|v(\tau_k - s) - v(\tau_{k-1} - s)\|_H \,\mathrm{d}s\right)^p\right)^{1/p} \\
&\quad \le \int_{\mathbb{R}} |\phi_M(s)| \left(\sum_{k=1}^{|\tau|} \|v(\tau_k - s) - v(\tau_{k-1} - s)\|_H^p\right)^{1/p} \mathrm{d}s \\
&\quad \le \int_{\mathbb{R}} |\phi_M(s)|\, \|v\|_{V^p} \,\mathrm{d}s = \|\phi_1\|_{L^1} \|v\|_{V^p} .
\end{aligned}
$$

Hence, the operator $Q_{<M}$ is bounded on V^p and its norm is uniformly bounded with respect to M.

Let $u \in U^p$. By duality (3.8), we can compute $\|Q_{<M}u\|_{U^p}$ by

$$
\|Q_{<M}u\|_{U^p} = \sup_{v \in \mathcal{B}^{p'}_{\mathrm{rc}}} |B(\phi_M * u, v)| .
$$

Let $v \in \mathcal{S}_{\mathrm{rc}}$ with $\|v\|_{V^{p'}} \le 1$. Using formula (3.12), we obtain

$$
\begin{aligned}
|B(\phi_M * u, v)| &= \Bigg| \sum_{k=1}^{|\tau|-1} \big(\phi_M * u(\tau_k) - \phi_M * u(\tau_{k-1}), v(\tau_{k-1})\big)_H \\
&\qquad + \big((\phi_M * u)_\infty - \phi_M * u(\tau_{|\tau|-1}), v(\tau_{|\tau|-1})\big)_H \Bigg| \\
&\le \int_{\mathbb{R}} |\phi_M(s)| \Bigg| \sum_{k=1}^{|\tau|-1} (u(\tau_k - s) - u(\tau_{k-1} - s), v(\tau_{k-1}))_H \\
&\qquad + \big(u_\infty - u(\tau_{|\tau|-1} - s), v(\tau_{|\tau|-1})\big)_H \Bigg| \,\mathrm{d}s \\
&= \int_{\mathbb{R}} |\phi_M(s)|\, |B(u(\cdot - s), v)| \,\mathrm{d}s \\
&= \int_{\mathbb{R}} |\phi_M(s)|\, |B(u, v)| \,\mathrm{d}s \\
&\le \|\phi_1\|_{L^1} \|u\|_{U^p} .
\end{aligned}
$$

Thus, the operator $Q_{<M}$ is also bounded on the space U^p and its norm is uniformly bounded with respect to M. □

Adapted spaces

In the end, we introduce the so-called adapted function spaces. These are spaces of U-/V-type which are related to the linear evolution of a Cauchy problem.

Definition 3.21 (Adapted function spaces)
Let $S(\cdot)$ be a C_0-group on H, and let $p \in [1, \infty)$. We define the *adapted spaces* $U^p_S := S(\cdot)U^p$, and $V^p_S := S(\cdot)V^p$ with norms given by $\|u\|_{U^p_S} := \|S(-\cdot)u\|_{U^p}$, and $\|u\|_{V^p_S} := \|S(-\cdot)u\|_{V^p}$.

The embeddings of Proposition 3.9 have an immediate counterpart for the adapted spaces. We also note the following relations with the temporal Littlewood–Paley projections.

Corollary 3.22 (Temporal Littlewood–Paley projections in adapted spaces)
Let $S(\cdot)$ be a C_0-group on H, and let $p \in [1, \infty)$. Let M by a dyadic number. We define the adapted Littlewood–Paley projections

$$Q^S_M := S(\cdot)Q_M S(-\cdot), \quad Q^S_{<M} := S(\cdot)Q_{<M} S(-\cdot) \quad \text{and} \quad Q^S_{\geq M} := S(\cdot)Q_{\geq M} S(-\cdot).$$

Then the estimates of Lemma 3.20 have a counterpart in adapted spaces.

(1) For all $v \in V^2_S$, we have

$$\left\|Q^S_M v\right\|_{L^2(\mathbb{R},H)} \leq CM^{-1/2} \|v\|_{V^2_S} \tag{3.16}$$

and

$$\left\|Q^2_{\geq M} v\right\|_{L^2(\mathbb{R},H)} \leq CM^{-1/2} \|v\|_{V^2_S} . \tag{3.17}$$

(2) The operators $Q^S_{<M}$ and $Q^S_{\geq M}$ are bounded on U^p_S and V^p_S.

Proof. The assertions follow directly from Lemma 3.20 and Definition 3.21. □

A crucial part in the theory of adapted spaces is the following *transfer principle*. Recall that we regard the C_0-group which enters the definition of the adapted spaces as an object which we understand well. The next proposition tells us how we can transfer estimates on the C_0-group to estimates on suitable adapted U^p-spaces. For the proof, it suffices again to check the assertion on atoms.

Proposition 3.23 (Transfer principle, [HHK09, Proposition 2.19])
Let $n \in \mathbb{N}$, let $T_0\colon L^2(\mathbb{R}^d) \times \cdots \times L^2(\mathbb{R}^d) \to L^1_{\mathrm{loc}}(\mathbb{R}^d)$ be an n-linear operator, and let $q, r \in [1, \infty)$. Assume that there exists $C \geq 0$ such that

$$\big\| T_0\big(S(\cdot)\psi_1, \ldots, S(\cdot)\psi_n\big) \big\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^d))} \leq C \prod_{j=1}^{n} \|\psi_j\|_{L^2(\mathbb{R}^d)}$$

for all $\psi_j \in L^2(\mathbb{R}^d)$. Then there exists an extension $T\colon U^q_S \times \cdots \times U^q_S \to L^q(\mathbb{R}, L^r(\mathbb{R}^d))$ satisfying

$$\big\| T\big(u_1, \ldots, u_n\big) \big\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^d))} \leq C \prod_{j=1}^{n} \|u_j\|_{U^q_S}$$

for all $u_j \in U^q_S$ where $\big(T(u_1, \ldots, u_n)\big)(t)(x) = \big(T_0(u_1(t), \ldots, u_n(t))\big)(x)$ for a.e. $x \in \mathbb{R}^d$.

Proof. For notational simplicity, we prove the case $n = 1$ below. The multilinear case is a straightforward adaption of this proof.

Let $a = \sum_{k=1}^{|\tau|} \mathbb{1}_{[\tau_{k-1}, \tau_k)} S(\cdot)\phi_k$ be a U^q_S-atom, $\tau \in \mathscr{Z}_0$ is the corresponding partition, and $(\phi_k)_{k=1}^{|\tau|}$ satisfies $\sum_{k=1}^{|\tau|} \|\phi_k\|_{L^2}^q \leq 1$. Then

$$\begin{aligned}
\|Ta\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^d))} &\leq \left\| \sum_{k=1}^{|\tau|} \mathbb{1}_{[\tau_{k-1}, \tau_k)} \|T_0 S(\cdot)\phi_k\|_{L^r} \right\|_{L^q} \\
&= \left(\sum_{k=1}^{|\tau|} \|T_0 S(\cdot)\phi_k\|_{L^q(\mathbb{R}, L^r)}^q \right)^{1/q} \\
&\leq C \left(\sum_{k=1}^{|\tau|} \|\phi_k\|_{L^2}^q \right)^{1/q} \leq C.
\end{aligned}$$

The claim now follows from Lemma 3.6. □

4 Well-posedness results via adapted spaces

Il est souvent question de « donner du sens » au calcul mais il ne faut pas oublier que le calcul est porteur de sens en lui-même.

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A mass-critical nonlinear Schrödinger equation

We start with a rather simple problem. In some sense the following problem is actually too simple, since our main tools, the U^2- and V^2-spaces are not really needed to achieve the desired result. However, we think the example is still a nice illustration of how the method of adapted U^2-spaces works in principle without the burden of too many technical complications.

Example 4.1 (The cubic nonlinear Schrödinger equation in 2 dimensions)
We consider the nonlinear Schrödinger equation with cubic nonlinearity in two space dimensions

$$\mathrm{i}\partial_t u(t,x) + \Delta u(t,x) + \mu \left|u(t,x)\right|^2 u(t,x) = 0, \quad (t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2. \tag{4.1}$$

Here the parameter μ belongs to $\{1,-1\}$, we refer to $\mu = 1$ as the *focusing* and to $\mu = -1$ as the *defocusing* case. In addition, we impose the initial condition $u(0,x) = u_0(x)$, $x \in \mathbb{R}^2$, for some $u_0 \in L^2(\mathbb{R}^2)$. Observe that equation (4.1) is invariant with respect to the scaling $\lambda u(\lambda^2 t, \lambda x)$. Since this scaling leaves the L^2-norm invariant, the equation is called *mass-critical*. It is well known that (4.1) is locally well-posed in the space $C([0,t_1), L^2(\mathbb{R}^2))$, where the maximal time of existence t_1 depends on the initial function u_0, see e.g. Theorem 5.3 in the monograph [LP09]. For initial data with sufficiently small L^2-norm, the solution even exists globally by Corollary 5.2 in [LP09].

Our aim is to reprove the global small data well-posedness result of Example 4.1 in the framework of adapted U^2- and V^2-spaces. Recall that the operator $\mathrm{i}\Delta$ with domain $H^2(\mathbb{R}^2)$ is a self-adjoint operator on $L^2(\mathbb{R}^2)$ and thus generates the unitary free Schrödinger group $S(\cdot)$ on $L^2(\mathbb{R}^2)$. Instead of the differential equation (4.1) we solve the integral equation

$$u(t) = S(t)u_0 + \mathrm{i}\mu \int_0^t S(t-\tau) \left|u(\tau)\right|^2 u(\tau)\,\mathrm{d}\tau, \quad t \in \mathbb{R}_{\geq 0}, \tag{4.2}$$

by applying Banach's fixed-point theorem in a suitable Banach space.

In the following, the Hilbert space H on which all constructions in Section 3 are based is chosen as $H = L^2(\mathbb{R}^2)$. For $p \in (1, \infty)$, we use the corresponding adapted spaces U^p_S and V^p_S as introduced in Definition 3.21. Note that these spaces are defined in Section 3 only for functions with domain $\mathbb{R}$. By the following remark, we can consider these spaces also on subintervals of $\mathbb{R}$.

Remark 4.2 (Restriction to intervals)
Let $X \subseteq C(\mathbb{R}, L^2(\mathbb{R}^2))$ be a Banach space, and let J be an interval. Then the *restriction space*

$$X(J) := \left\{u \in C(J, L^2(\mathbb{R}^2)) : \exists \tilde{u} \in X \text{ with } \tilde{u}|_J = u\right\}$$

endowed with the norm $\|u\|_{X(I)} := \inf\left\{\|\tilde{u}\|_X : \tilde{u} \in X \text{ with } \tilde{u}|_J = u\right\}$ is a Banach space.

With these preparations, we can formulate the following well-posedness result for the cubic nonlinear Schrödinger equation (4.1) in the critical space $L^2(\mathbb{R}^2)$.

Theorem 4.3 (Global well-posedness for small initial L^2-data)
There exists $\delta > 0$ such that for all $u_0 \in L^2(\mathbb{R}^2)$ with $\|u_0\|_{L^2} < \delta$ the cubic nonlinear Schrödinger equation (4.2) has a global solution $u \in U^2_{c,S}([0, \infty))$.

A key element of the proof is the following multilinear estimate, which, in this case, is a consequence of a standard Strichartz estimate.

Lemma 4.4 (A simple multilinear estimate)
Let $u_1, \ldots u_4 \in V^2_{-,S}$. Then the following estimate holds

$$\left|\int_{\mathbb{R}} \int_{\mathbb{R}^2} \prod_{j=1}^4 u_j(t, x)\, \mathrm{d}x\, \mathrm{d}t\right| \leq C \prod_{j=1}^4 \|u_j\|_{V^2_S}.$$

Proof. From Hölder's inequality, we deduce that

$$\left|\int_{\mathbb{R}} \int_{\mathbb{R}^2} \prod_{j=1}^4 u_j(t, x)\, \mathrm{d}x\, \mathrm{d}t\right| \leq \prod_{j=1}^4 \|u_j\|_{L^4(\mathbb{R}\times\mathbb{R}^2)}.$$

By Theorem 4.2 in [LP09], the pair $(4, 4)$ is admissible for the Strichartz estimates of equation (4.1). The homogeneous Strichartz estimate yields $\|S(\cdot)\psi\|_{L^4(\mathbb{R},L^4)} \leq C\|\psi\|_{L^2}$ for all $\psi \in L^2(\mathbb{R}^2)$. By applying the transfer principle from Proposition 3.23, this Strichartz estimate corresponds to the estimate

$$\|u\|_{L^4(\mathbb{R},L^4)} \leq C\|u\|_{U^4_S} \tag{4.3}$$

for all $u \in U_S^4$. If u belongs to $V_{-,S}^2$, we can change u at countably many points so that the function u becomes right-continuous. This step does not alter the L^4-norm. Thus, the embedding $V_{-,\mathrm{rc}}^2 \hookrightarrow U^4$ stated in Proposition 3.9 implies that

$$\|u\|_{L^4(\mathbb{R},L^4)} \leq C\,\|u\|_{V_S^2}\,,$$

and the assertion is proved. □

In order to prove Theorem 4.3, we have to solve the integral equation (4.2). The space for the fixed-point argument is defined as follows. Define the function space X as the closure of $C(\mathbb{R}, H^2(\mathbb{R}^2)) \cap U_S^2$ in U_S^2 and the space Y as the closure of $C(\mathbb{R}, H^2(\mathbb{R}^2)) \cap V_{-,\mathrm{rc},S}^2$ in V_S^2. The embedding $U_S^2 \hookrightarrow V_{-,\mathrm{rc},S}^2$, cf. Proposition 3.9, implies that $X \hookrightarrow Y$.

The required estimates to control the Duhamel integral in the space U_S^2 are provided in the next lemma. For the computation of the U^2-norm, we use the particular integral form of the duality pairing in Proposition 3.17 whose structure fits well to the Duhamel integral.

Lemma 4.5 (Mapping properties of the Duhamel integral)
For arbitrary functions $u_1, u_2, u_3 \in C(\mathbb{R}, H^2(\mathbb{R}^2)) \cap V_{-,\mathrm{rc},S}^2$, we define

$$I(u_1,u_2,u_3)(t) \coloneqq \mathrm{i}\mu \mathbb{1}_{[0,\infty)}(t) \int_0^t S(t-\tau)u_1(\tau)\bar{u}_2(\tau)u_3(\tau)\,\mathrm{d}\tau.$$

Then $I(u_1,u_2,u_3) \in X$ and we have the estimate

$$\|I(u_1,u_2,u_3)\|_{U_S^2} \leq C \prod_{j=1}^{3} \|u_j\|_{V_S^2}\,.$$

Moreover, the functional I continuously extends to a 3-linear operator

$$I\colon Y \times Y \times Y \to X,$$

as well as

$$I\colon X \times X \times X \to X. \tag{4.4}$$

Proof. First, we note that $H^2(\mathbb{R}^2)$ is an algebra and that the Schrödinger group is also a strongly continuous group on $H^2(\mathbb{R}^2)$. This implies that the integrand $\tau \mapsto S(-\tau)u_1(\tau)\bar{u}_2(\tau)u_3(\tau)$ belongs to the space $C(\mathbb{R}, H^2(\mathbb{R}^2))$. By taking the integral, the whole expression is now even continuously differentiable with values in $H^2(\mathbb{R}^2)$. We thus obtain $I(u_1,u_2,u_3) \in C(\mathbb{R}, H^2(\mathbb{R}^2))$. To show that $I(u_1,u_2,u_3) \in X$, we compute the U_S^2-norm of I via the duality pairing (3.8) by using the integral formula from Proposition 3.17. This, in turn, allows us to apply the multilinear estimates obtained in Lemma 4.4. In detail, we compute

$$
\begin{aligned}
\|I(u_1,u_2,u_3)\|_{U^2_S} &= \left\| \mathbb{1}_{[0,\infty)} \int_0^{\cdot} S(-\tau)\big(u_1(\tau)\bar{u}_2(\tau)u_3(\tau)\big)\,\mathrm{d}\tau \right\|_{U^2} \\
&= \sup_{v\in\mathcal{B}^2_{\mathrm{rc}}} \left| B\left(\mathbb{1}_{[0,\infty)} \int_0^{\cdot} S(-\tau)\big(u_1(\tau)\bar{u}_2(\tau)u_3(\tau)\big)\,\mathrm{d}\tau, v\right)\right| \\
&= \sup_{v\in\mathcal{B}^2_{\mathrm{rc}}} \left| \int_0^\infty \int_{\mathbb{R}^2} u_1(\tau,x)\bar{u}_2(\tau,x)u_3(\tau,x)S(-\tau)v(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau \right| \\
&\le \sup_{\|\tilde{v}\|_{V^2_{-,S}}\le 1} \left| \int_0^\infty \int_{\mathbb{R}^2} u_1(\tau,x)\bar{u}_2(\tau,x)u_3(\tau,x)\tilde{v}(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau \right| \\
&\le C\prod_{j=1}^3 \|u_j\|_{V^2_S}.
\end{aligned}
$$

The mapping property (4.4) finally follows from the embedding $X \hookrightarrow Y$. □

Having good estimates of the Duhamel integral, we can now proceed to the proof of the main theorem. Note that in adapted spaces the handling of the linear evolution is immediate.

Proof of Theorem 4.3. Fix $C > 0$ as in the statement of Lemma 4.5. Choose $r = (7C+1)^{-1/2} < 1$ and $\delta = r^3$. Our goal is to show that the map

$$\Phi(u)(t) \coloneqq S(t)u_0 + I(u,u,u)(t), \quad t \ge 0,$$

has a unique fixed point in the set

$$B_r \coloneqq \big\{u \in X\big([0,\infty)\big) : \|u\|_{U^2_S} \le r\big\}.$$

We first show that the solution of the homogeneous equation $S(\cdot)u_0$ belongs to $X\big([0,\infty)\big)$. It is clear that $S(\cdot)u_0$ belongs to $U^2_{c,S}([0,\infty))$ and satisfies

$$\|S(\cdot)u_0\|_{U^2_S} \le \|u_0\|_{L^2} \le \delta.$$

Take a sequence $(u_n)_n$ in $H^2(\mathbb{R}^2)$ which converges to u_0 in $L^2(\mathbb{R}^2)$. Then $S(\cdot)u_n$ belongs to $C\big([0,\infty), H^2(\mathbb{R}^2)\big)$ and

$$\|S(\cdot)u_n - S(\cdot)u_0\|_{U^2_S} \le \|u_n - u_0\|_{L^2} \to 0 \quad \text{as } n \to \infty.$$

Therefore, $S(\cdot)u_0 \in X\big([0,\infty)\big)$. Let $u \in B_r$. Using Lemma 4.5 to estimate the Duhamel integral, we now obtain

$$\|\Phi(u)\|_{U^2_S} = \|S(\cdot)u_0 + I(u,u,u)\|_{U^2_S} \le \delta + C\,\|u\|^3_{U^2_S} \le \delta + Cr^3 \le (C+1)r^3 \le r.$$

To show that Φ is a contraction on B_r, take $u, v \in B_r$. Since the nonlinearity is a polynomial, we note the following identity

$$I(u,u,u) - I(v,v,v) = I(u,u,u-v) + I(u,u-v,v) + I(u-v,v,v).$$

Thus, we obtain again by Lemma 4.5

$$\begin{aligned}
\|\Phi(u) - \Phi(v)\|_{U^2_S} &= \|I(u,u,u) - I(v,v,v)\|_{U^2_S} \\
&\leq 3C\big(\|u\|^2_{U^2_S} + \|v\|^2_{U^2_S}\big)\,\|u-v\|_{U^2_S} \\
&\leq 6Cr^2\,\|u-v\|_{U^2_S} \\
&\leq \tfrac{6}{7}\,\|u-v\|_{U^2_S},
\end{aligned}$$

and the proof is finished. □

A derivative nonlinear Schrödinger equation

We now turn our attention to a more difficult problem, a derivative nonlinear Schrödinger equation. This equation was studied in the context of U/V-spaces in Tobias Schottdorf's PhD-thesis [Sch13], the method is also sketched in Section 6.3 of the book [Koc14]. We replace the nonlinearity $|u|^2 u$ in Example 4.1 by the expression $\bar{u}\partial_{x_1}\bar{u}$ involving a derivative. Of course, the main difficulty is how to control this derivative. We use the framework of U/V-spaces as in the preceding example but in addition we use a Littlewood–Paley decomposition which allows us to treat high- and low-frequency parts separately.

Example 4.6 (The derivative nonlinear Schrödinger equation in 2 dimensions) We consider a nonlinear Schrödinger equation with derivative nonlinearity in two space dimensions

$$\mathrm{i}\partial_t u(t,x) + \Delta u(t,x) + \mu\bar{u}(t,x)\partial_{x_1}\bar{u}(t,x) = 0, \quad (t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2. \tag{4.5}$$

As in Example 4.1, the nonlinear Schrödinger equation (4.5) is invariant under the scaling $\lambda u(\lambda^2 t, \lambda x)$ so that the equation is again L^2-critical. The small data global well-posedness is proved in Theorem 5.1 in the PhD-thesis [Sch13].

We keep the notation from Example 4.1 for the free Schrödinger group $S(\cdot)$. As before, the main object of interest is the Duhamel formula

$$u(t) = S(t)u_0 + \mathrm{i}\mu \int_0^t S(t-\tau)\bar{u}(\tau)\partial_{x_1}\bar{u}(\tau)\,\mathrm{d}\tau, \quad t \in \mathbb{R}_{\geq 0}, \tag{4.6}$$

to which we apply a fixed-point argument. In order to treat the derivative ∂_{x_1} in the nonlinearity we have to use finer arguments than in Example 4.1. In particular, we need to introduce the following Littlewood–Paley decomposition in the first space variable. Using the same notation as in Definition 3.19, we define the *spatial Littlewood–Paley projection*

$$\big(\mathcal{F}_x P_N u\big)(t,\xi,\eta) \coloneqq \psi_N(|\xi|)\big(\mathcal{F}_x u\big)(t,\xi,\eta)$$

for $u \in \mathcal{S}'(\mathbb{R}\times\mathbb{R}^2)$. For later use in the proofs, we mention the following almost orthogonality property of Littlewood–Paley projections. From the support of the functions ψ_N it follows that $(P_M u, P_N u)_{L^2} = 0$ unless $M \sim N$. A consequence of this property is that the norms

$$\|u\|_{L^2} = \Big\|\sum\nolimits_N P_n u\Big\|_{L^2} \quad \text{and} \quad \left(\sum_N \|P_N u\|_{L^2}^2\right)^{1/2} \tag{4.7}$$

are equivalent. This property is e.g. discussed in the beginning of Section 2.3 of the book [BCD11]. We next introduce the function space X in which we solve (4.6).

Definition 4.7 (Function spaces)
Define X as the closure of $C(\mathbb{R}, H^{1,4}(\mathbb{R}^2)) \cap U^2_{c,S}$ with respect to the norm

$$\|u\|_X \coloneqq \left(\sum_N \|P_N u\|_{U^2_S}^2\right)^{1/2}$$

in the space $C(\mathbb{R}, L^2(\mathbb{R}^2))$ and Y as the closure of $C(\mathbb{R}, H^{1,4}(\mathbb{R}^2)) \cap V^2_{-,\mathrm{rc},S}$ with respect to the norm

$$\|v\|_Y \coloneqq \left(\sum_N \|P_N v\|_{V^2_S}^2\right)^{1/2}$$

in the space $C(\mathbb{R}, L^2(\mathbb{R}^2))$.

The main result is analogous to Theorem 4.3.

Theorem 4.8 (Global well-posedness of (4.5) for small L^2-initial data)
There exists $\delta > 0$ such that for all $u_0 \in L^2(\mathbb{R}^2)$ with $\|u_0\|_{L^2} < \delta$ the derivative nonlinear Schrödinger equation (4.6) has a global solution $u \in X$.

We repeat a consequence of the Strichartz estimates which was the key ingredient in the discussion of the cubic nonlinear Schrödinger equation (4.1).

Lemma 4.9 (Linear Strichartz estimate)
Let $u \in V^2_{-,S}$. Then it holds

$$\|u\|_{L^4(\mathbb{R},L^4)} \le C\,\|u\|_{V^2_S}. \tag{4.8}$$

Proof. This estimate follows from a standard Strichartz estimate for the Schrödinger group and the transfer principle from Proposition 3.23. Its proof is already contained in the proof of Lemma 4.4. □

In addition to linear Strichartz estimates we also use the following bilinear estimates, which are a refinement of (4.8) to treat products of two Littlewood–Paley projections. The first estimate is concerned with functions from the space U^2_S and follows from well-known bilinear Strichartz estimates established by Bourgain and the transfer principle. For later applications, this first estimate is not good enough, since in the computation of the U^2_S-norm via duality at least one of the terms only belongs to the larger space V^2_S. The estimate for V^2_S-functions is established by an interpolation argument which looses a logarithmic factor.

Lemma 4.10 (Bilinear Strichartz estimates)
Let $u_1, u_2 \in U^2_S$ and let N_1, N_2 be dyadic numbers with $N_2 \leq N_1$. Then we have

$$\|P_{N_1} u_1 P_{N_2} u_2\|_{L^2(\mathbb{R},L^2)} \leq C \left(\frac{N_2}{N_1}\right)^{1/2} \|u_1\|_{U^2_S} \|u_2\|_{U^2_S}. \tag{4.9}$$

If $u_1, u_2 \in V^2_{-,S}$, then it also holds

$$\|P_{N_1} u_1 P_{N_2} u_2\|_{L^2(\mathbb{R},L^2)} \leq C \left(\frac{N_2}{N_1}\right)^{1/2} \left(\log\left(\frac{N_1}{N_2}\right) + 1\right)^2 \|u_1\|_{V^2_S} \|u_2\|_{V^2_S}. \tag{4.10}$$

Proof. According to Lemma 111 in Bourgain's article [Bou98], the bilinear estimate

$$\|P_{N_1} \psi_1 P_{N_2} \psi_2\|_{L^2(\mathbb{R},L^2)} \leq C \left(\frac{N_2}{N_1}\right)^{1/2} \|\psi_1\|_{L^2} \|\psi_2\|_{L^2}$$

holds for all $\psi_1, \psi_2 \in L^2(\mathbb{R}^2)$. Thus, estimate (4.9) is a direct consequence of the transfer principle in Proposition 3.23. Note that in the case $N_1 = N_2$ Bourgain's estimate is just the same as the linear Strichartz estimate (4.8) in the U^2-space. To obtain the improved estimate (4.10) in the V^2-space, we interpolate (4.3) and (4.9) by using Proposition 2.20 in [HHK09]. To repeat the arguments from Corollary 2.21, we define $\tilde{P}_N = \sum_{M \sim N} P_M$ for a dyadic number N. By this construction $\tilde{P}_N P_N = P_N$. Consider the bilinear operator $T_1 \colon u_2 \mapsto \tilde{P}_{N_1} u_1 \tilde{P}_{N_2} u_2$. From estimate (4.3) and the embedding $U^2_S \hookrightarrow U^4_S$, we deduce that

$$\|T_1 u_2\|_{L^2(\mathbb{R},L^2)} \leq \left\|\tilde{P}_{N_1} u_1\right\|_{L^4(\mathbb{R},L^4)} \left\|\tilde{P}_{N_2} u_2\right\|_{L^4(\mathbb{R},L^4)} \leq C \left\|\tilde{P}_{N_1} u_1\right\|_{U^2_S} \left\|\tilde{P}_{N_2} u_2\right\|_{U^4_S}.$$

Hence, T_1 is bounded linear operator from U^4_S to $L^2(\mathbb{R}, L^2)$ with norm less than $C \|u_1\|_{U^2_S}$. With the bilinear estimate (4.9) we see that

$$\|T_1 u_2\|_{L^2(\mathbb{R},L^2)} \leq C \left(\tfrac{N_1}{N_2}\right)^{1/2} \left\|\tilde{P}_{N_1} u_1\right\|_{U^2_S} \left\|\tilde{P}_{N_2} u_2\right\|_{U^2_S}$$

and hence T_1 is bounded from U_S^2 to $L^2(\mathbb{R}, L^2)$ with norm less than $C\left(\frac{N_1}{N_2}\right)^{1/2} \|u_1\|_{U_S^2}$. Proposition 2.20 in [HHK09] allows to interpolate these two estimates with the result that T_1 is a bounded linear operator from $V^2_{-,\mathrm{rc},S}$ to $L^2(\mathbb{R}, L^2)$ with

$$\|T_1 u_2\|_{L^2(\mathbb{R},L^2)} \le C\left(\tfrac{N_1}{N_2}\right)^{1/2} \left(\log\left(\tfrac{N_2}{N_1}\right) + 1\right) \|u_1\|_{U_S^2} \|u_2\|_{V_S^2}. \tag{4.11}$$

Next, define $T_2\colon u_1 \mapsto \tilde{P}_{N_1} u_1 \tilde{P}_{N_2} u_2$. As for the operator T_1, using (4.8), we see that T_2 is a bounded operator from U_S^4 to $L^2(\mathbb{R}, L^2)$ whose norm is less than $C\|u_2\|_{V_S^2}$. From (4.11), we obtain that

$$\|T_2 u_1\|_{L^2(\mathbb{R},L^2)} \le C\left(\tfrac{N_1}{N_2}\right)^{1/2} \left(\log\left(\tfrac{N_2}{N_1}\right) + 1\right) \|u_1\|_{U_S^2} \|u_2\|_{V_S^2}$$

and hence T_2 is a bounded linear operator from U_S^2 to $L^2(\mathbb{R}, L^2)$ with norm bounded by $C\left(\frac{N_1}{N_2}\right)^{1/2} \left(\log\left(\frac{N_2}{N_1}\right) + 1\right) \|u_2\|_{V_S^2}$. Interpolating again with Proposition 2.20 from [HHK09], we arrive at

$$\left\|\tilde{P}_{N_1} u_1 \tilde{P}_{N_2} u_2\right\|_{L^2(\mathbb{R},L^2)} \le C\left(\tfrac{N_1}{N_2}\right)^{1/2} \left(\log\left(\tfrac{N_2}{N_1}\right) + 1\right)^2 \|u_1\|_{V_S^2} \|u_2\|_{V_S^2} \tag{4.12}$$

for all $u_1, u_2 \in V^2_{-,\mathrm{rc},S}$. To deduce (4.10), take $u_1, u_2 \in V^2_{-,S}$. By changing u_1 and u_2 at countable many points, these functions become right continuous and neither the L^2-norm nor the V^2-norm changes. By applying (4.12) to $P_{N_1} u_1$ and $P_{N_2} u_2$, (4.10) follows. □

The next lemma contains the essential technical calculations and is the counterpart to the much easier Lemma 4.4. It shows how to combine the linear and bilinear Strichartz estimates from Lemma 4.9 and 4.10 to obtain multilinear estimates on dyadic blocks which allow to control the derivative in the nonlinearity.

Lemma 4.11 (Estimates on dyadic blocks)
Let N_1, N_2, N_3 be dyadic numbers. Let $u_{N_1}, u_{N_2}, u_{N_3} \in V^2_{-,S}$ be functions such that the support of $\mathcal{F}_{t,x} u_{N_j}$ is contained in the set

$$A_{N_j} \coloneqq \left\{(\tau, \xi, \eta) \in \mathbb{R} \times \mathbb{R}^2 : \tfrac{1}{2} N_j \le |\xi| \le 2N_j\right\} \quad \text{for } j \in \{1, 2, 3\}.$$

(1) Assume that $N_1 \sim N_3$. Then

$$\begin{aligned} &\left|\sum_{N_2 \ll N_1} \int_0^\infty \int_{\mathbb{R}^2} u_{N_1}(t,x) u_{N_2}(t,x) u_{N_3}(t,x)\, \mathrm{d}x\, \mathrm{d}t\right| \\ &\qquad \le C \|u_{N_1}\|_{V_S^2} \left(\sum_{N_2 \ll N_1} N_2^{-2} \|u_{N_2}\|_{V_S^2}^2\right)^{1/2} \|u_{N_3}\|_{V_S^2}. \end{aligned} \tag{4.13}$$

(2) Assume that $N_2 \sim N_3$. Then

$$\begin{aligned}\left| \sum_{N_1 \ll N_2} \int_0^\infty \int_{\mathbb{R}^2} u_{N_1}(t,x) u_{N_2}(t,x) u_{N_3}(t,x) \,\mathrm{d}x \,\mathrm{d}t \right| \\ \leq C \left(\sum_{N_1 \ll N_2} \|u_{N_1}\|_{V_S^2}^2 \right)^{1/2} N_2^{-1} \|u_{N_2}\|_{V_S^2} \|u_{N_3}\|_{V_S^2} .\end{aligned} \tag{4.14}$$

(3) Assume that $N_1 \sim N_2$. Then

$$\begin{aligned}\left(\sum_{N_3 \lesssim N_1} \sup_{\|v\|_{V_S^2} \leq 1} \left| \int_0^\infty \int_{\mathbb{R}^2} u_{N_1}(t,x) u_{N_2}(t,x) P_{N_3} v(t,x) \,\mathrm{d}x \,\mathrm{d}t \right|^2 \right)^{1/2} \\ \leq C \|u_{N_1}\|_{V_S^2} N_2^{-1} \|u_{N_2}\|_{V_S^2} .\end{aligned} \tag{4.15}$$

Proof. Let M be a dyadic number which is chosen below. We adapt the temporal Littlewood–Paley projections from Definition 3.19 to the Schrödinger equation. Let $u \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^2)$ be a Schwartz function. Recall from Corollary 3.22 the definition

$$Q_M^S u := S(\cdot) Q_M S(-\cdot) u.$$

For each $t \in \mathbb{R}$, the free Schrödinger group $S(t)$ is given as an $\mathcal{F}_x$-Fourier multiplier with symbol $\mathrm{e}^{-\mathrm{i}t|\cdot|^2}$. The projection Q_M is an $\mathcal{F}_t$-Fourier multiplier with symbol ψ_M. Also note for every $(\xi,\eta) \in \mathbb{R}^2$, the $\mathcal{F}_t$-transform of the function $t \mapsto \mathrm{e}^{-\mathrm{i}t(\xi^2+\eta^2)}$ is the Dirac delta distribution $\delta_{\xi^2+\eta^2}$. We these formulas in mind, we obtain

$$\begin{aligned}\big(\mathcal{F}_{t,x} Q_M^S u\big)(\tau,\xi,\eta) &= \big(\mathcal{F}_{t,x} S(\cdot) Q_M S(-\cdot) u\big)(\tau,\xi,\eta) \\ &= \mathcal{F}_t \left(t \mapsto \mathrm{e}^{-\mathrm{i}t|\cdot|^2} Q_M \mathrm{e}^{\mathrm{i}t|\cdot|^2} \mathcal{F}_x u \right)(\tau,\xi,\eta) \\ &= \left(\delta_{\xi^2+\eta^2} *_\tau \left(\psi_M \mathcal{F}_{t,x} u(\cdot + \xi^2 + \eta^2, \xi, \eta) \right) \right)(\tau) \\ &= \psi_M(\tau - \xi^2 - \eta^2) \mathcal{F}_{t,x} u(\tau,\xi,\eta).\end{aligned}$$

We have thus shown that

$$\mathcal{F}_{t,x} Q_M^S u(\tau,\xi,\eta) = \psi_M(\tau - \xi^2 - \eta^2) \mathcal{F}_{t,x} u(\tau,\xi,\eta). \tag{4.16}$$

There are analogous formulas for $Q_{<M}^S$ and $Q_{\geq M}^S$.

Next, we decompose the integral into high and low frequencies of each term obtaining eight pieces of the form

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} Q_1^S u_{N_1} Q_2^S u_{N_2} Q_3^S u_{N_3} \,\mathrm{d}x \,\mathrm{d}t,$$

where each Q_i^S is either a projection $Q_{<M}^S$ or $Q_{\geq M}^S$ for $i = 1, 2, 3$. Here we integrate over $\mathbb{R}$ and extend the functions by 0 to the whole real line. This does not cause any problems since for all $u \in V_S^2$ the restriction satisfies $\|\mathbb{1}_{[0,\infty)} u\|_{V_S^2} \leq 2 \|u\|_{V_S^2}$.

The remaining plan of the proof is as follows. If the low frequency projection falls on all three terms at the same time, we show that the integral vanishes. If the high frequency is at least on the second term, then we can use Strichartz estimates (4.8) and the high frequency estimate (3.17) to gain one derivative in the estimate. Finally, if the high frequency term is not on the second term, then we need in addition the bilinear Strichartz estimates from Lemma 4.10.

We start by considering the interaction of three low frequency terms. By low frequency, we mean the projection $Q_{<M}^S$, where we set $M := \frac{1}{12} N_2^2$. We obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} Q_{<M}^S u_{N_1} Q_{<M}^S u_{N_2} Q_{<M}^S u_{N_3} \, \mathrm{d}(x,t) \\
&\qquad = c \mathcal{F}_{t,x}\big(Q_{<M}^S u_{N_1} Q_{<M}^S u_{N_2} Q_{<M}^S u_{N_3}\big)(0) \\
&\qquad = c \left(\mathcal{F}_{t,x} Q_{<M}^S u_{N_1} * \mathcal{F}_{t,x} Q_{<M}^S u_{N_2} * \mathcal{F}_{t,x} Q_{<M}^S u_{N_3}\right)(0).
\end{aligned}$$

Let $\mu_j := (\tau_j, \xi_j, \eta_j) \in \operatorname{supp} \mathcal{F}_{t,x} Q_{<M}^S u_{N_j}$ for $j = 1, 2, 3$. The above integral vanishes if we cannot satisfy the equation $\mu_1 + \mu_2 + \mu_3 = 0$. So assume that $\mu_1 + \mu_2 + \mu_3 = 0$. Define $\lambda_j := \tau_j - \xi_j^2 - \eta_j^2$ for $j \in \{1, 2, 3\}$. Since

$$\mathcal{F}_{t,x} Q_{<M}^S u_{N_j} = \psi_{<M}(\lambda_j) \psi_{N_j}(|\xi_j|) \mathcal{F}_{t,x} u,$$

we obtain the bounds $N_j/2 \leq |\xi_j| \leq 2N_j$ and $|\lambda_j| < M$ for $j \in \{1, 2, 3\}$. Observe that $\tau_1 + \tau_2 + \tau_3 = 0$. We thus obtain

$$\tfrac{1}{4} N_2^2 \leq \xi_2^2 \leq \left|\xi_1^2 + \xi_2^2 + \xi_3^2 + \eta_1^2 + \eta_2^2 + \eta_3^2\right| = |\lambda_1 + \lambda_2 + \lambda_3| < 3M.$$

Since we have defined $M = \frac{1}{12} N_2^2$, this inequality cannot be true and thus the above integral vanishes.

Now we turn to the proof of (4.13). The case of three low frequency interactions is already settled. To discuss four of the remaining seven cases, we first assume that $Q_2^S = Q_{\geq M}^S$. For the Q_1- and Q_3-terms, we use the L^4-Strichartz estimate (4.8), and for the sum over N_2, we use that the Littlewood–Paley blocks are almost orthogonal in the sense of (4.7) to obtain that

$$\begin{aligned}
&\left| \sum_{N_2 \ll N_1} \int_{\mathbb{R}^3} Q_1^S u_{N_1} Q_{\geq M}^S u_{N_2} Q_3^S u_{N_3} \, \mathrm{d}x \, \mathrm{d}t \right| \\
&\qquad \leq \left\| Q_1^S u_{N_1} \right\|_{L^4} \left\| \sum_{N_2 \ll N_1} Q_{\geq M}^S u_{N_2} \right\|_{L^2} \left\| Q_3^S u_{N_3} \right\|_{L^4} \\
&\qquad \leq C \left\| u_{N_1} \right\|_{V_S^2} \left(\sum_{N_2 \ll N_1} \left\| Q_{\geq M}^S u_{N_2} \right\|_{L^2}^2 \right)^{1/2} \left\| u_{N_3} \right\|_{V_S^2}.
\end{aligned}$$

In the last estimate, also the boundedness of the projections Q_j^S on V_S^2 is used, see Corollary 3.22. By the crucial estimate for the high frequencies (3.17), we have

$$\left\|Q_{\geq M}^S u_{N_j}\right\|_{L^2} \leq CM^{-1/2}\left\|u_{N_j}\right\|_{V_S^2} \leq CN_2^{-1}\left\|u_{N_j}\right\|_{V_S^2} \quad \text{for } j \in \{1,2,3\}. \tag{4.17}$$

Inserting (4.17) for $j=2$ in the estimate above, we obtain (4.13) in these cases.

For the next two cases, we assume that the high frequency falls on the first term, i.e. $Q_1^S = Q_{\geq M}^S$. We use again (4.17) for the high frequency term and the bilinear estimate (4.10) for the other two terms. The logarithmic factor in (4.10) can be roughly estimated by

$$\left(\frac{N_2}{N_3}\right)^{1/2}\left(\log\left(\frac{N_3}{N_2}\right)+1\right)^2 \lesssim \left(\frac{N_2}{N_1}\right)^{1/4},$$

where we use the assumption that $N_1 \sim N_3$. Together with Hölder's inequality and the boundedness of Q_j^S on V_S^2 by Corollary 3.22, we obtain the bound

$$\begin{aligned}
&\left|\sum_{N_2 \ll N_1} \int_{\mathbb{R}^3} Q_{\geq M}^S u_{N_1} Q_2^S u_{N_2} Q_3^S u_{N_3}\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad \leq \sum_{N_2 \ll N_1}\left\|Q_{\geq M}^S u_{N_1}\right\|_{L^2}\left\|Q_2^S u_{N_2} Q_3^S u_{N_3}\right\|_{L^2} \\
&\quad \leq C\sum_{N_2 \ll N_1} N_2^{-1}\left\|u_{N_1}\right\|_{V_S^2}\left(\tfrac{N_2}{N_1}\right)^{1/4}\left\|u_{N_2}\right\|_{V_S^2}\left\|u_{N_3}\right\|_{V_S^2} \\
&\quad \leq C\left\|u_{N_1}\right\|_{V_S^2}\left(\sum_{N_2 \ll N_1}\left(\tfrac{N_2}{N_1}\right)^{1/2}\right)^{1/2}\left(\sum_{N_2 \ll N_1} N_2^{-2}\left\|u_{N_2}\right\|_{V_S^2}^2\right)^{1/2}\left\|u_{N_3}\right\|_{V_S^2}.
\end{aligned}$$

The dyadic sum can be easily computed and gives the constant

$$\sum_{N_2 \ll N_1}\left(\tfrac{N_2}{N_1}\right)^{1/2} \leq \sum_{N \geq 1} N^{-1/2} = \sum_{n=0}^{\infty} 2^{-n/2} = \frac{1}{1-2^{-1/2}}.$$

Therefore, the desired bound (4.13) is proved in this case. Finally, the remaining case is $Q_3^S = Q_{\geq M}^S$. Here we repeat the same calculation that we just performed in the cases $Q_1^S = Q_{\geq M}^S$ just by swapping the roles of Q_1^S with Q_3^S. Consequently, the proof of (4.13) is finished.

In the next step, we prove (4.14). We start again with the case $Q_2^S = Q_{\geq M}^S$. The arguments are similar as before. We use estimate (4.17) once more for the high frequency term and the bilinear Strichartz estimate (4.10) for the other terms. As above, the assumption $N_2 \sim N_3$ yields the rough estimate

$$\left(\frac{N_1}{N_3}\right)^{1/2}\left(\log\left(\frac{N_1}{N_3}\right)+1\right)^2 \lesssim \left(\frac{N_1}{N_2}\right)^{1/4}.$$

We compute

$$\begin{aligned}
&\left|\sum_{N_1 \ll N_2} \int_{\mathbb{R}^3} Q_1^S u_{N_1} Q_{\geq M}^S u_{N_2} Q_3^S u_{N_3} \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad \leq \sum_{N_1 \ll N_2} \left\|Q_1^S u_{N_1} Q_3^S u_{N_3}\right\|_{L^2} \left\|Q_{\geq M}^S u_{N_2}\right\|_{L^2} \\
&\quad \leq C \sum_{N_1 \ll N_2} \big(\tfrac{N_1}{N_2}\big)^{1/4} \|u_{N_1}\|_{V_S^2} N_2^{-1} \|u_{N_3}\|_{V_S^2} \|u_{N_2}\|_{V_S^2} \\
&\quad \leq C \left(\sum_{N_1 \ll N_2} \big(\tfrac{N_1}{N_2}\big)^{1/2}\right)^{1/2} \left(\sum_{N_1 \ll N_2} \|u_{N_1}\|_{V_S^2}^2\right)^{1/2} N_2^{-1} \|u_{N_2}\|_{V_S^2} \|u_{N_3}\|_{V_S^2} .
\end{aligned}$$

This is the claimed estimate in this case. To deal with $Q_3^S = Q_{\geq M}^S$, we repeat this computation only changing the roles of u_{N_2} and u_{N_3}. We obtain the same result, and hence (4.14) is proved in these cases.

It remains to consider the case $Q_1^S = Q_{\geq M}^S$. The ingredients are as before, only the high frequency estimate (4.17) is now applied to the first term and we use the simpler linear Strichartz estimate (4.8) for the other terms. The sum poses no additional problem and we get

$$\begin{aligned}
&\left|\sum_{N_1 \ll N_2} \int_{\mathbb{R}^3} Q_{\geq M}^S u_{N_1} Q_2^S u_{N_2} Q_3^S u_{N_3} \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad \leq \left\|\sum_{N_1 \ll N_2} Q_{\geq M}^S u_{N_1}\right\|_{L^2} \left\|Q_{\geq M}^S u_{N_2}\right\|_{L^4} \left\|Q_3^S u_{N_3}\right\|_{L^4} \\
&\quad \leq C \left(\sum_{N_1 \ll N_2} \left\|Q_{\geq M}^S u_{N_1}\right\|_{L^2}^2\right)^{1/2} \|u_{N_2}\|_{V_S^2} \|u_{N_3}\|_{V_S^2} \\
&\quad \leq C \left(\sum_{N_1 \ll N_2} \|u_{N_1}\|_{V_S^2}^2\right)^{1/2} N_2^{-1} \|u_{N_2}\|_{V_S^2} \|u_{N_3}\|_{V_S^2} .
\end{aligned}$$

This finishes the proof of (4.14).

We turn to the proof of (4.15). The cases where $Q_i^S = Q_{\geq M}^S$ for $i = 1, 2$ are similar as in the proof of the first estimates (4.13) and (4.14) above. We demonstrate the case $Q_1^S = Q_{\geq M}^S$. Encore, using (3.17) on the high frequency term and the bilinear estimate (4.10) for the other terms, we obtain

$$\begin{aligned}
&\left(\sum_{N_3\lesssim N_1}\sup_{\|v\|_{V_S^2}\le 1}\left|\int_0^\infty\int_{\mathbb{R}^2} Q_{\ge M}^S u_{N_1}(t,x)Q_2^S u_{N_2}(t,x)Q_3^S P_{N_3}v(t,x)\,\mathrm{d}x\,\mathrm{d}t\right|^2\right)^{1/2}\\
&\quad\le\left(\sum_{N_3\lesssim N_1}\sup_{\|v\|_{V_S^2}\le 1}\left\|Q_{\ge M}^S u_{N_1}\right\|_{L^2}^2\left\|Q_2^S u_{N_2}Q_3^S P_{N_3}v\right\|_{L^2}^2\right)^{1/2}\\
&\quad\le C\left(\sum_{N_3\lesssim N_1} N_2^{-2}\|u_{N_1}\|_{V_S^2}^2\left(\tfrac{N_3}{N_2}\right)^{1/2}\|u_{N_2}\|_{V_S^2}^2\sup_{\|v\|_{V_S^2}\le 1}\|P_{N_3}v\|_{V_S^2}^2\right)^{1/2}\\
&\quad\le C\left(\sum_{N_3\lesssim N_1}\left(\tfrac{N_3}{N_2}\right)^{1/2}\right)^{1/2}\|u_{N_1}\|_{V_S^2}N_2^{-1}\|u_{N_2}\|_{V_S^2}.
\end{aligned}$$

As before, using the assumption $N_1\sim N_2$, the dyadic sum gives a constant independent of N_1 and N_2. The case $Q_2^S=Q_{\ge M}^S$ can be proved in the same way by interchanging the roles of u_{N_1} with u_{N_2}.

The remaining case is $Q_3^S=Q_{\ge M}^S$ which we treat as follows. We use again (3.17) on the high frequency term and the L^4-Strichartz estimate (4.8) and the boundedness of Q_j^S on V_S^2 for the other terms. We also have to use the symmetry of the projection P_{N_3} and the almost orthogonality (4.7) of the Littlewood–Paley blocks. In detail

$$\begin{aligned}
&\left(\sum_{N_3\lesssim N_1}\sup_{\|v\|_{V_S^2}\le 1}\left|\int_0^\infty\int_{\mathbb{R}^2} Q_1^S u_{N_1}(t,x)Q_2^S u_{N_2}(t,x)Q_{\ge M}^S P_{N_3}v(t,x)\,\mathrm{d}x\,\mathrm{d}t\right|^2\right)^{1/2}\\
&\quad\le\left(\sum_{N_3\lesssim N_1}\sup_{\|v\|_{V_S^2}\le 1}\left\|P_{N_3}Q_1^S u_{N_1}Q_2^S u_{N_2}\right\|_{L^2}^2\left\|Q_{\ge M}^S P_{N_3}v\right\|_{L^2}^2\right)^{1/2}\\
&\quad\le C\left(\sum_{N_3\lesssim N_1}\sup_{\|v\|_{V_S^2}\le 1}\left\|P_{N_3}Q_1^S u_{N_1}Q_2^S u_{N_2}\right\|_{L^2}^2 N_2^{-2}\|P_{N_3}v\|_{V_S^2}^2\right)^{1/2}\\
&\quad\le CN_2^{-1}\left\|Q_1^S u_{N_1}Q_2^S u_{N_2}\right\|_2\\
&\quad\le C\|u_{N_1}\|_{V_S^2}N_2^{-1}\|u_{N_2}\|_{V_S^2}.
\end{aligned}$$

This finishes the proof. □

Lemma 4.12 (The Duhamel integral)
We define the Duhamel integral for $u_1,u_2\in X\cap C(\mathbb{R},H^{1,4}(\mathbb{R}^2))$ by

$$I(u_1,u_2)(t):=\mathrm{i}\mu\mathbb{1}_{[0,\infty)}(t)\int_0^t S(t-\tau)\bar{u}_1(\tau)\partial_{x_1}\bar{u}_2(\tau)\,\mathrm{d}\tau,\quad t\in\mathbb{R}.$$

For $u_1, u_2 \in X \cap C(\mathbb{R}, H^{1,4}(\mathbb{R}^2))$, we have

$$\|I(u_1, u_2)\|_X \le C \|u_1\|_Y \|u_2\|_Y .$$

Hence, I extends to a bilinear form $I \colon Y \times Y \to X$ and, by the embedding $X \hookrightarrow Y$, also to $I \colon X \times X \to X$.

Proof. We imitate the proof of Lemma 4.5. The assumption $u_1, u_2 \in C(\mathbb{R}, H^{1,4}(\mathbb{R}^2))$ implies again that $\tau \mapsto S(-\tau)\bar{u}_1(\tau)\partial_{x_1}\bar{u}_2(\tau)$ belongs to $C(\mathbb{R}, L^2(\mathbb{R}^2))$ and we can differentiate the Duhamel integral. We use the duality pairing (3.8) and the integral formula from Proposition 3.17 to obtain that

$$\begin{aligned}
\|I(u_1, u_2)\|_X^2 &= \sum_{N_3} \|P_{N_3} I(u_1, u_2)\|_{U_S^2}^2 \\
&= \sum_{N_3} \sup_{v \in \mathcal{B}_{\mathrm{rc}}^2} \big| B\big(S(-\cdot)P_{N_3} I(u_1, u_2), v\big)\big|^2 \\
&= \sum_{N_3} \sup_{v \in \mathcal{B}_{\mathrm{rc}}^2} \left| \int_0^\infty \int_{\mathbb{R}^2} \bar{u}_1(\tau, x)\partial_{x_1}\bar{u}_2(\tau, x) P_{N_3} S(-\tau) v(\tau, x) \,\mathrm{d}x \,\mathrm{d}\tau \right|^2 \\
&\le \sum_{N_3} \sup_{\|\tilde{v}\|_{V_{-,S}^2} \le 1} \left| \int_0^\infty \int_{\mathbb{R}^2} \bar{u}_1(\tau, x)\partial_{x_1}\bar{u}_2(\tau, x) P_{N_3} \tilde{v}(\tau, x) \,\mathrm{d}x \,\mathrm{d}\tau \right|^2 .
\end{aligned}$$

We further decompose each summand by

$$\sum_{N_1, N_2} \int_0^\infty \int_{\mathbb{R}^2} P_{N_1}\bar{u}_1(\tau, x) P_{N_2}\partial_{x_1}\bar{u}_2(\tau, x) P_{N_3}\tilde{v}(\tau, x) \,\mathrm{d}x \,\mathrm{d}\tau$$

and we distinguish the three cases

$$\sum_{N_1, N_2} = \sum_{N_1} \sum_{N_2 \ll N_1} + \sum_{N_1} \sum_{N_2 \sim N_1} + \sum_{N_1} \sum_{N_2 \gg N_1} .$$

Recall from the proof of Lemma 4.11 that not all frequencies contribute to the integral in the last line. Namely, the integral is nontrivial, only if there exists $\xi_i \in A_{N_i}$ for $i \in \{1, 2, 3\}$ such that $\xi_1 + \xi_2 + \xi_3 = 0$. Therefore, it is impossible that only one of the three frequencies is significantly larger than the other two. In the first case, the condition $N_2 \ll N_1$ implies that $N_3 \sim N_1$. In the second case, $N_2 \sim N_1$ implies that $N_3 \lesssim N_1$. In the third case, $N_1 \ll N_2$ implies that $N_3 \sim N_2$. We treat the dyadic pieces in each of these cases with the estimates from Lemma 4.11.

First, we consider the case $N_3 \sim N_1$ and we estimate the summands with $N_2 \ll N_1$ by using (4.13). The factor N_2^{-1} compensates the derivative and we obtain that

$$\sum_{N_3} \sup_{\|v\|_{V^2_{-,S}}\le 1} \left| \sum_{N_1\sim N_3} \sum_{N_2\ll N_1} \int_0^\infty \int_{\mathbb{R}^2} P_{N_1}\bar{u}_1 P_{N_2}\partial_{x_1}\bar{u}_2 P_{N_3} v \,\mathrm{d}x\,\mathrm{d}t \right|^2$$
$$\le C \sum_{N_3} \sum_{N_1\sim N_3} \|P_{N_1}u_1\|^2_{V^2_S} \left(\sum_{N_2\ll N_1} N_2^{-2} \|P_{N_2}\partial_{x_1}u_2\|^2_{V^2_S} \right) \sup_{\|v\|_{V^2_{-,S}}\le 1} \|P_{N_3}v\|^2_{V^2_S}$$
$$\le C \sum_{N_1} \|P_{N_1}u_1\|^2_{V^2_S} \sum_{N_2} \|P_{N_2}u_2\|^2_{V^2_S}$$
$$\le C \|u_1\|^2_Y \|u_2\|^2_Y .$$

In the second case, we consider $N_3 \lesssim N_1$ and sum all the integrals where $N_1 \sim N_2$. We first estimate by the triangle inequality

$$\left\| \sum_{N_1} \sum_{N_2\sim N_1} I(P_{N_1}u_1, P_{N_2}u_2) \right\|_X \le \sum_{N_1} \sum_{N_2\sim N_1} \|I(P_{N_1}u_1, P_{N_2}u_2)\|_X$$
$$\le \sum_{N_1} \sum_{N_2\sim N_1} \left(\sum_{N_3\lesssim N_1} \sup_{\|v\|_{V^2_S}\le 1} \left| \int_0^\infty \int_{\mathbb{R}^2} P_{N_1}\bar{u}_1 P_{N_2}\partial_x \bar{u}_2 P_{N_3} v \,\mathrm{d}x\,\mathrm{d}t \right|^2 \right)^{1/2} .$$

Using estimate (4.15), we further compute

$$\sum_{N_1} \sum_{N_2\sim N_1} \left(\sum_{N_3\lesssim N_1} \sup_{\|v\|_{V^2_S}\le 1} \left| \int_0^\infty \int_{\mathbb{R}^2} P_{N_1}u_1 P_{N_2}\partial_x u_2 P_{N_3} v \,\mathrm{d}x\,\mathrm{d}t \right|^2 \right)^{1/2}$$
$$\le C \sum_{N_1} \sum_{N_2\sim N_1} \|P_{N_1}u_1\|_{V^2_S} N_2^{-1} \|P_{N_2}\partial_{x_1}u_2\|_{V^2_S}$$
$$\le C \left(\sum_{N_1} \|P_{N_1}u_1\|^2_{V^2_S} \right)^{1/2} \left(\sum_{N_2} \|P_{N_2}u_2\|^2_{V^2_S} \right)^{1/2}$$
$$= C \|u_1\|_Y \|u_2\|_Y .$$

In the last case, we have $N_2 \sim N_3$ and we note that $\sum_{N_1} \sum_{N_2 \gg N_1} = \sum_{N_2} \sum_{N_1 \ll N_2}$. Using (4.14), we obtain

$$\begin{aligned}
&\sum_{N_3} \sup_{\|v\|_{V^2_{-,S}} \le 1} \left| \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_2} \int_0^\infty \int_{\mathbb{R}^2} P_{N_1} \bar{u}_1 P_{N_2} \partial_{x_1} \bar{u}_2 P_{N_3} v \,\mathrm{d}x \,\mathrm{d}t \right|^2 \\
&\quad \le C \sum_{N_3} \sum_{N_2 \sim N_3} \sum_{N_1 \ll N_2} \|P_{N_1} u\|^2_{V^2_S} N_2^{-2} \|P_{N_2} \partial_{x_1} u_2\|^2_{V^2_S} \sup_{\|v\|_{V^2_{-,S}} \le 1} \|P_{N_3} v\|^2_{V^2_S} \\
&\quad \le C \sum_{N_1} \|P_{N_1} u_1\|^2_{V^2_S} \sum_{N_2} \|P_{N_2} u_2\|^2_{V^2_S} \\
&\quad \le C \|u_1\|^2_Y \|u_2\|^2_Y . \qquad \square
\end{aligned}$$

Proof of Theorem 4.8. Thanks to Lemma 4.12, the proof of Theorem 4.8 works very similar as the proof of Theorem 4.3.

Fix $C > 0$ as in the statement of Lemma 4.12. Choose $r = (5C+1)^{-1} < 1$ and $\delta = r^2$. We show that the map

$$\Phi(u)(t) := S(t)u_0 + I(u,u)(t), \quad t \ge 0,$$

has a unique fixed point in the set

$$B_r := \big\{ u \in X\big([0,\infty)\big) : \|u\|_X \le r \big\}.$$

It is clear that $S(\cdot)u_0$ belongs to $U^2_{c,S}([0,\infty))$ and satisfies

$$\|S(\cdot)u_0\|_X \le \|u_0\|_{L^2} \le \delta.$$

Let $u \in B_r$. Using Lemma 4.12 to estimate the Duhamel integral, we obtain

$$\|\Phi(u)\|_X = \|S(\cdot)u_0 + I(u,u)\|_X \le \delta + C\|u\|^2_X \le \delta + Cr^2 \le (C+1)r^2 \le r.$$

To show that Φ is a contraction on B_r, take $u, v \in B_r$. Since the nonlinearity is a polynomial, we note the following identity

$$I(u,u) - I(v,v) = I(u, u-v) + I(u-v, v).$$

Thus, we obtain again by Lemma 4.12

$$\begin{aligned}
\|\Phi(u) - \Phi(v)\|_X &= \|I(u,u) - I(v,v)\|_X \\
&\le 2C\big(\|u\|_X + \|v\|_X\big) \|u-v\|_X \\
&\le 4Cr \|u-v\|_X \le \tfrac{4}{5} \|u-v\|_X ,
\end{aligned}$$

and the proof is finished. $\square$

A magnetic nonlinear Schrödinger equation

The greatest difficulty in the investigation of the Maxwell–Schrödinger systems (1.3) or (1.5) arises from the magnetic parts of the Schrödinger equation. In Chapter 2 these parts are treated as a perturbation of the free Schrödinger equation with the help of Strichartz estimates with loss. There also exists Strichartz estimates for the magnetic Schrödinger equation, we refer e.g. to the articles [DF08], [FV09] and [DFVV10]. Given a fixed magnetic potential A with suitable properties, the above results are good enough to establish local and global well-posedness results for nonlinear magnetic Schrödinger equations. As an example, we mention the article [CCL14] in which the global well-posedness in H^1 of a magnetic nonlinear Schrödinger equation with defocusing cubic nonlinearity is proved. However, in this result the potential A is assumed to be time-independent and to satisfy certain decay estimates which are required by the Strichartz estimates in [DFVV10]. Hence, this approach is unsuitable to deal with Maxwell–Schrödinger systems where the magnetic potential A satisfies another partial differential equation.

Another form of Strichartz estimates for the magnetic Schrödinger equation is discovered in [BT09]. They show spectrally localized Strichartz estimates, make use of the U/V-spaces and finally solve the Maxwell–Schrödinger system (1.3) in the energy space. In this Section, we make a first step to see whether their methods also apply to the Maxwell–Schrödinger system (1.5) with additional power nonlinearity. Instead of the full system, we only investigate the Schrödinger part where we regard the potential A as given, but we aim to pose only assumptions on A which are not too far away from the situation which one would encounter in the full system. Due to technical problems, we also discuss the case of an artificial quadratic nonlinearity only. We expect that the case of a cubic nonlinearity $|u|^2 u$ or of $|u| u$ can be treated with a refinement of our method.

Example 4.13 (A magnetic nonlinear Schrödinger equation)
Let $\sigma \in (1, \infty)$. Let $T \in (0, \infty)$. Let $A \in L_T^\infty H^\sigma \cap W_T^{1,\infty} H^{\sigma-1} \cap L_T^2 L^\infty$ with $\operatorname{div} A = 0$ be given. We study a magnetic nonlinear Schrödinger equation with quadratic nonlinearity

$$\mathrm{i}\partial_t u + \Delta_A u = u^2. \tag{4.18}$$

Under the given assumptions on A, the method from [NW07] (without the electric potential ϕ) shows that there exists an evolution family $S_A \colon [0, T]^2 \to \mathcal{L}(H^s)$ for $s \in [-2, 2]$ which solves the homogeneous equation, compare with Lemma 2.15. We stress that we cannot solve the nonlinear equation (4.18) in $H^1(\mathbb{R}^3)$ with the techniques from Chapter 2. Setting up a fixed-point argument leads to the term

$$\int_0^t \left\| S_A(t,\tau) u^2(\tau) \right\|_{H^1} \mathrm{d}\tau \lesssim \int_0^t \|u(\tau)\|_{L^\infty} \|u(\tau)\|_{H^1} \mathrm{d}\tau$$

in the Duhamel formula. The auxiliary space $L^2_T H^{1/2,6}(\mathbb{R}^3)$, in which the solution can be controlled through Strichartz estimates with loss, just fails to be embedded in $L^2_T L^\infty(\mathbb{R}^3)$ and we cannot close the estimates for a fixed-point argument. Therefore, more sophisticated arguments are needed to construct an H^1-solution of (4.18). To do this, we introduce a few tools from [BT09]. First, we assume in addition to the above that $A \in U^2_W H^1$. Here $U^2_W H^1$ is a U^2-space adapted to the evolution of the wave equation, see the definition in Section 3 of [BT09]. Second, we take $H^1(\mathbb{R}^3)$ as the Hilbert space on which we base the constructions in Chapter 3. With the evolution family S_A we define the corresponding adapted space

$$U^2_A H^1 := \left\{ S_A(0,\cdot)u : u \in U^2 H^1 \right\}.$$

The duality relation is now $(U^2_A H^1)^* = V^2_A H^{-1}$, see Proposition 23 c) in [BT09]. We denote the space $U^2_A H^1$ on the interval $[0,T]$ as X in the following.

Theorem 4.14 (Local well-posedness of (4.18) in H^1)
Let $A \in U^2_W H^1 \cap L^\infty_T H^\sigma \cap W^{1,\infty}_T H^{\sigma-1} \cap L^2_T L^\infty$ with $\operatorname{div} A = 0$ be given. Let $u_0 \in H^1$. Then there exists $T \in (0,\infty)$ such that the magnetic Schrödinger equation (4.18) has a unique solution in X with initial value $u(0) = u_0$. Additionally, the solution map $u_0 \mapsto u$ from H^1 to X is Lipschitz continuous on bounded sets.

The proof of Theorem 4.14 relies crucially on the following Strichartz estimate for the magnetic Schrödinger equation from Proposition 24 in [BT09] which is the major achievement of that article.

Lemma 4.15 (Magnetic Strichartz estimate)
Let (q,r) be a Schrödinger admissible pair, see (2.15) in Definition 2.5. Let $A \in U^2_W H^1$ with $\operatorname{div} A = 0$. Let N be a dyadic integer. There exists $\delta \in (0,1)$ for which the following spectrally localized Strichartz estimate

$$\|P_N u\|_{L^q_T L^r} \lesssim T^{\delta/q} N^{1/q-1} \|u\|_{U^2_A H^1} \tag{4.19}$$

holds true.

Lemma 4.16 (The Duhamel integral)
Let $T \in (0,1)$. We define the Duhamel integral for $u_1, u_2 \in X$ by

$$I(u_1,u_2)(t) := \mathrm{i}\mathbb{1}_{[0,T]}(t) \int_0^t S_A(t,\tau) u_1(\tau) u_2(\tau)\,\mathrm{d}\tau, \quad t \in [0,T].$$

For $u_1, u_2 \in X$, we have

$$\|I(u_1,u_2)\|_X \le C T^{1/4} \|u_1\|_X \|u_2\|_X .$$

Proof. To evaluate the norm of the Duhamel integral in X, we use the duality $(U_A^2 H^1)^* = V_A^2 H^{-1}$, see Proposition 23 c) in [BT09]. Similarly as in the proof of Lemma 4.12, we use the duality pairing (3.8) and the integral formula from Proposition 3.17 to obtain that

$$\begin{aligned}\|I(u_1,u_2)\|_X &= \sup_{v\in\mathcal{B}^2_{\mathrm{rc}}} \big|B\big(S_A(0,\cdot)I(u_1,u_2),v\big)\big| \\ &= \sup_{v\in\mathcal{B}^2_{\mathrm{rc}}} \left|\int_0^T \int_{\mathbb{R}^3} u_1(\tau,x)u_2(\tau,x)S_A(0,\tau)v(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau\right| \\ &\leq \sup_{\|\tilde v\|_{V^2_{-,A}H^{-1}}\leq 1} \left|\int_0^T \int_{\mathbb{R}^3} u_1(\tau,x)u_2(\tau,x)\tilde v(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau\right|.\end{aligned}$$

We next use the Littlewood–Paley decomposition

$$\sum_{N_1}\sum_{N_2}\sum_{N_3}\int_0^T\int_{\mathbb{R}^3} P_{N_1}u_1(\tau,x)P_{N_2}u_2(\tau,x)P_{N_3}\tilde v(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau.$$

As in the proof of Lemma 4.12, we only get contributions from the integral, if the two largest frequencies are comparable. Moreover, the expression is symmetric in u_1 and u_2 and it is hence enough to treat the two cases

$$\sum_N \sum_{N_1\gtrsim N}\sum_{N_2\sim N_1} \quad\text{and}\quad \sum_N\sum_{N_2\lesssim N}\sum_{N_1\sim N}.$$

We start with the first case where the last term has low frequency. Take $\varepsilon\in(0,\frac14)$. With the Strichartz estimate (4.19) we compute

$$\begin{aligned}&\sup_{\|v\|_{V^2_{-,A}H^{-1}}\leq 1}\sum_{N_1}\sum_{N\lesssim N_1}\left|\int_0^T\int_{\mathbb{R}^3} P_{N_1}u_1(\tau,x)P_{N_1}u_2(\tau,x)P_N v(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau\right| \\ &\lesssim \sup_{\|v\|_{V^2_{-,A}H^{-1}}\leq 1}\sum_{N_1}\|P_{N_1}u_1\|_{L_T^{8/3}L^4}\|P_{N_1}u_2\|_{L_T^{8/3}L^4}\sum_{N\lesssim N_1} N^{1+\varepsilon}\|P_N v\|_{L_T^4L^2}N^{-1-\varepsilon} \\ &\lesssim T^{3\delta/4}\sup_{\|v\|_{V^2_{-,A}H^{-1}}\leq 1}\sum_{N_1} N_1^{-5/8}\|u_1\|_X N_1^{-5/8}\|u_2\|_X N_1^{1+\varepsilon}\left(\sum_{N\lesssim N_1}\|P_Nv\|^4_{L^4_TH^{-1-\varepsilon}}\right)^{1/4}.\end{aligned}$$

Using that

$$\left(\sum_N \|P_Nv\|^4_{L^4_TH^{-1-\varepsilon}}\right)^{1/4} \lesssim T^{1/4}\|v\|_{L^\infty_T H^{-1}} \lesssim T^{1/4}\|v\|_{V^2_AH^{-1}}, \tag{4.20}$$

we obtain the desired estimate in the first case. In the second case, we compute similarly

$$\begin{aligned}
&\sup_{\|v\|_{V^2_{-,A}H^{-1}}\le 1} \sum_N \sum_{N_2\lesssim N} \left| \int_0^T \int_{\mathbb{R}^3} P_N u_1(\tau,x) P_{N_2} u_2(\tau,x) P_N v(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau \right| \\
&\lesssim \sup_{\|v\|_{V^2_{-,A}H^{-1}}\le 1} \sum_N \|P_N u_1\|_{L^{8/3}_T L^4} \|P_N v\|_{L^4_T L^2} \sum_{N_2\lesssim N} \|P_{N_2} u_2\|_{L^{8/3}_T L^4} \\
&\lesssim T^{3\delta/4} \sup_{\|v\|_{V^2_{-,A}H^{-1}}\le 1} \sum_N N^{-5/8} \|u_1\|_X N^{-5/8} \|u_2\|_X N^{1+\varepsilon} \|P_N v\|_{L^4_T H^{-1-\varepsilon}} \\
&\lesssim T^{3\delta/4} \sup_{\|v\|_{V^2_{-,A}H^{-1}}\le 1} \|u_1\|_X \|u_2\|_X \left(\sum_N \|P_N v\|^4_{L^4_T H^{-1-\varepsilon}} \right)^{1/4}.
\end{aligned}$$

Employing (4.20) again, we deduce the assertion. □

Proof of Theorem 4.14. As in the previous examples, the main work is contained in Lemma 4.16.

Fix $C > 0$ as in the statement of Lemma 4.16. Choose $R \ge 2\|u_0\|_{H^1}$ and $T \in (0,1)$ with $T \le \frac{1}{256C^4R^4}$. We show that the map

$$\Phi(u)(t) := S_A(t)u_0 + I(u,u)(t), \quad t \ge 0,$$

has a unique fixed point in the ball $B_R := \{u \in X : \|u\|_X \le R\}$. It is clear that $S_A(\cdot,0)u_0$ belongs to X and satisfies

$$\|S_A(\cdot,0)u_0\|_X \le \|u_0\|_{H^1}.$$

Let $u \in B_R$. Using Lemma 4.16 to estimate the Duhamel integral, we obtain

$$\|\Phi(u)\|_X = \|S_A(\cdot,0)u_0 + I(u,u)\|_X \le \|u_0\|_{H^1} + CT^{1/2}\|u\|_X^2 \le \frac{1}{2}R + CT^{1/4}R^2 \le R.$$

To show that Φ is a contraction on B_R, take $u, v \in B_R$. Since the nonlinearity is a polynomial, we note the following identity

$$I(u,u) - I(v,v) = I(u+v, u-v).$$

Thus, we obtain again by Lemma 4.16

$$\begin{aligned}
\|\Phi(u)-\Phi(v)\|_X &= \|I(u,u)-I(v,v)\|_X \\
&\le CT^{1/4}\big(\|u\|_X + \|v\|_X\big)\|u-v\|_X \\
&\le 2CT^{1/4}R\|u-v\|_X \le \tfrac{1}{2}\|u-v\|_X.
\end{aligned} \tag{4.21}$$

Hence, $\Phi = \Phi_{u_0}$ has a unique fixed-point u in B_R. Let also $v_0 \in H^1$ with $\|v_0\|_{H^1} \leq \frac{R}{2}$. We then have the solution $v = \Phi_{v_0}(v)$. Using (4.21), we derive

$$\begin{aligned}
\|u - v\|_X &= \|\Phi_{u_0}(u) - \Phi_{v_0}(v)\|_X \\
&\leq \|S_A(\cdot, 0)(u_0 - v_0)\|_X + \|I(u, , u) - I(v, v)\|_X \\
&\leq \|u_0 - v_0\|_{H^1} + \frac{1}{2} \|u - v\|_X .
\end{aligned}$$

We conclude that $\|u - v\|_X \leq 2 \|u_0 - v_0\|_{H^1}$. □

f

5 Nonexistence of standing waves

Two scientists walk into a bar.
First scientist: "I'll take a glass of H_2O."
Second scientist: "I'll take a glass of H_2O, too."

Folklore

In this section, we come back to study the original Maxwell–Schrödinger system (1.1) without any additional nonlinearity. We recall that this system only becomes a reasonable Cauchy problem if we choose a gauge and that the system in Coulomb gauge is given by

$$\begin{aligned} \mathrm{i}\partial_t u + \Delta_A u &= \phi(u)u, \quad \text{in } I \times \mathbb{R}^3, \\ \partial_t^2 A - \Delta A &= PJ(u,A), \quad \text{in } I \times \mathbb{R}^3, \\ \operatorname{div} A &= 0, \quad \text{in } I \times \mathbb{R}^3. \end{aligned} \tag{5.1}$$

The main motivation in Section 2 for the investigations of the Maxwell–Schrödinger system (2.11) with additional power nonlinearity is the existence of standing waves of the form

$$\big(u(t,x), A(t,x)\big) = (\mathrm{e}^{-\mathrm{i}\omega t}\varphi(x), 0) \tag{5.2}$$

for $t \in \mathbb{R}$ and $x \in \mathbb{R}^3$ as special solutions of the system. Using the same standing wave ansatz (5.2) for the system (5.1), we obtain the elliptic problem

$$-\Delta\varphi(x) + \phi(\varphi)\varphi(x) = \omega\varphi(x), \quad x \in \mathbb{R}^3. \tag{5.3}$$

We start with the basic observation about equation (5.3) that nontrivial solutions can only exist if ω is positive.

Lemma 5.1
Let $\varphi \in H^1(\mathbb{R}^3)$. Let $\omega \in \mathbb{R}$. Assume that (φ, ω) is a distributional solution of equation (5.3). If $\varphi \neq 0$, then $\omega > 0$.

Proof. Let $\varphi \neq 0$ be a solution of (5.3). By Lemma 2.12/Lemma 5.4, we have $\phi \in L^\infty(\mathbb{R}^3) \cap H^{1,p}(\mathbb{R}^3)$ for all $p \in [3,\infty)$ and that

$$\left(\int_{\mathbb{R}^3} |\nabla\phi(x)|^2 \,\mathrm{d}x\right)^{1/2} \lesssim \|\varphi\|_{H^1}^2 < \infty.$$

From the fractional Leibniz rule, we obtain that $\omega\varphi + \phi\varphi$ belongs to $H^1(\mathbb{R}^3)$, and elliptic regularity implies that $\varphi \in H^3(\mathbb{R}^3)$. Hence, $\varphi \in H^{1,p}(\mathbb{R}^3)$ for all $p \in [2, \infty)$, and we even infer that $\varphi \in H^{3,p}(\mathbb{R}^3)$ for all $p \in [2, \infty)$. In particular, we conclude that $\varphi \in C^2(\mathbb{R}^3)$. Multiplying (5.3) with φ, using $|\varphi|^2 = -\Delta\phi$, and integrating by parts, we obtain

$$\begin{aligned}0 &= \int_{\mathbb{R}^3} \omega\, |\varphi(x)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} \Delta\varphi(x)\varphi(x)\,\mathrm{d}x - \int_{\mathbb{R}^3} \phi(x)\, |\varphi(x)|^2 \,\mathrm{d}x \\ &= \omega\, \|\varphi\|_{L^2}^2 - \int_{\mathbb{R}^3} |\nabla\varphi(x)|^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} |\nabla\phi(x)|^2 \,\mathrm{d}x.\end{aligned}$$

Hence, $\omega > 0$. □

We observe that if $\varphi \in H^1(\mathbb{R}^3)$ is a solution of (5.3), then ω is a positive eigenvalue of the Schrödinger operator $H_\phi = -\Delta + \phi$. This suggests the following strategy to prove that (5.3) has no nontrivial solution at all. First, find suitable conditions on the potential ϕ such that the Schrödinger operator H_ϕ has no positive eigenvalues. Then it remains to show that if $\varphi \in H^1(\mathbb{R}^3)$ is a solution of (5.3) the electric potential ϕ given by the formula

$$\phi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|\varphi(y)|^2}{|x-y|}\,\mathrm{d}y, \quad x \in \mathbb{R}^3, \tag{5.4}$$

satisfies the conditions under which the absence of positive eigenvalues of H_ϕ can be proved.

This approach was successfully used in the article [CG04] in the radially symmetric case. We briefly recall the argument from [CG04] in Theorem 5.8 below, showing that (5.3) does not possess any nontrivial radially symmetric solution. The proof in [CG04] for the absence of positive eigenvalues with radially symmetric eigenfunctions is based on Agmon's Lemma, see Lemma 5.5.

We then proceed to prove a more general nonexistence result for (5.3) which excludes every nontrivial H^1-solution of (5.3) which satisfy a mild decay property, see Assumption 5.2. We point out that by Remark 5.3 every radial H^1-function satisfies our decay assumption so that our result also includes the nonexistence result from [CG04]. To exclude positive eigenvalues of the Schrödinger operator H_ϕ, we use the method from the article [FHHH82]. With the setup and some technicalities, in particular with the convenient choice of the weight functions, we also follow the presentation in [AHK19], where the method from [FHHH82] is extended to the study of magnetic Schrödinger operators.

Although our method of proof studies the Schrödinger operator H_ϕ and is thus concerned with properties of the function ϕ, we impose the following condition on φ rather than on ϕ.

Assumption 5.2 (Decay)
Let $\varphi \in H^1(\mathbb{R}^3)$. If we assume mild decay of φ, we mean that the function $x \mapsto |x|^{1/2}\,\varphi(x)$ belongs to $L^4(\mathbb{R}^3)$.

Remark 5.3 (Radial H^1-functions decay)
Let $\varphi \in H^1(\mathbb{R}^3)$ be a radial function. The Radial Lemma of Walter Strauss established in the article [Str77a] shows that φ decays at infinity such that the estimate $|\varphi(x)| \lesssim |x|^{-1}\,\|\varphi\|_{H^1}$ is satisfied for almost all $x \in \{|x| \geq 1\}$. It follows that

$$\int_{\{|x|\geq 1\}} |x|^2\,|\varphi(x)|^4\,\mathrm{d}x \lesssim \|\varphi\|_{H^1}^2 \int_{\{|x|\geq 1\}} |\varphi(x)|^2\,\mathrm{d}x \leq \|\varphi\|_{H^1}^4 < \infty.$$

Hence, Assumption 5.2 is fulfilled for every radial H^1-function. This implies that the assertion of Theorem 5.8 is contained in Corollary 5.15.

In the next lemma, we repeat the known regularity properties of the electric potential which are stated in Lemma 2.12 and we add an additional property if the charge density satisfies Assumption 5.2.

Lemma 5.4 (Properties of the electric potential)
Let $\varphi \in H^1(\mathbb{R}^3)$. Let ϕ be given by (5.4).

(1) Then $\phi \in L^r(\mathbb{R}^3)$ for all $r \in (3,\infty]$ and $\nabla\phi \in L^r(\mathbb{R}^3)$ for all $r \in (\frac{3}{2},\infty)$. We have the estimates

$$\|\phi\|_{L^r} \lesssim \|\varphi\|_{H^1}^2 \quad \text{and} \quad \|\nabla\phi\|_{L^r} \lesssim \|\varphi\|_{H^1}^2.$$

(2) Assume in addition that Assumption 5.2 is satisfied. Then the function $x \mapsto x \cdot \nabla\phi(x)$ belongs to $L^6(\mathbb{R}^3)$ and satisfies

$$\|x \cdot \nabla\phi\|_{L^6} \lesssim \|\varphi\|_{H^1}^2 + \left\||x|^{1/2}\,\varphi\right\|_{L^4}^2.$$

Proof. We know that the function φ belongs to $L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$ by the Sobolev embedding. Lemma 2.12 thus yields the first assertion.

To prove the second assertion, we recall formula (2.37) and compute

$$\begin{aligned} x \cdot \nabla\phi(x) &= \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x\cdot(x-y)\,|\varphi(y)|^2}{|x-y|^3}\mathrm{d}y \\ &= \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{|\varphi(y)|^2}{|x-y|}\mathrm{d}y + \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{y\cdot(x-y)\,|\varphi(y)|^2}{|x-y|^3}\mathrm{d}y. \end{aligned}$$

By assumption, the function $y \mapsto |y|\,|\varphi(y)|^2$ belongs to $L^2(\mathbb{R}^3)$. We conclude the proof with the weak Young inequality in the same way as in Lemma 2.12. $\square$

Review of the radial case

To give a quick overview over the radial case, we need the following lemma which was proved by Shmuel Agmon in Theorem 3 of the article [Agm70]. It generalizes previous results of Franz Rellich and Tosio Kato and its main application is to show the nonexistence of positive eigenvalues for Schrödinger operators. The statement below is a special case of Agmon's result if we take "$p_1 = 0, \eta_0, \eta_1 = 0$".

Lemma 5.5 (Agmon's Lemma)
Let $R_0 \in (0, \infty)$, and let $\Omega \subseteq \mathbb{R}^3$ be an open set such that $\{|x| \geq R_0\} \subseteq \Omega$. Let $\varphi \in C^2(\Omega) \setminus \{0\}$ be a solution of

$$-\Delta\varphi(x) = p(x)\varphi(x), \qquad x \in \Omega.$$

Assume that $p \in C(\Omega)$ is a function which is nonnegative on $\{|x| \geq R_0\}$, possesses a continuous radial derivative and there exists $\alpha \in (0, 1)$ such that

$$\frac{\partial p}{\partial r}(x) + \frac{2(1-\alpha)}{|x|} p(x) \geq 0 \quad \text{for all } x \in \{|x| \geq R_0\}.$$

Then

$$\liminf_{R\to\infty} \frac{1}{R^\alpha} \int_{\{R_0 \leq |x| \leq R\}} p(x)\varphi^2(x)\,\mathrm{d}x > 0.$$

In addition to Agmon's Lemma, the proof of the radial result is based on a particular representation formula which is stated in Lemma 5.7. To derive this formula, we first compute spherical means of the Green's function in Lemma 5.6. These two lemmata are contained in Lemma 9.1, 9.2 and 9.3 of [CG04].

Lemma 5.6 (Spherical means of the Green's function)
Let $x \in \mathbb{R}^3 \setminus \{0\}$ and $\rho \in (0, \infty)$. Then

$$\frac{1}{4\pi} \int_{\{|\omega|=1\}} \frac{1}{|x + \rho\omega|}\,\mathrm{d}\sigma(\omega) = \frac{1}{\max\{|x|, \rho\}}.$$

Proof. First, observe that $|x + \rho\omega| = \sqrt{|x|^2 + \rho^2 + 2\rho x \cdot \omega}$. By applying a suitable rotation, we assume without loss of generality that $x = (0, 0, |x|)$. We now compute in polar coordinates

$$\begin{aligned}
\frac{1}{4\pi} \int_{\{|\omega|=1\}} \frac{1}{|x + \rho\omega|}\,\mathrm{d}\sigma(\omega) &= \frac{1}{4\pi} \int_{\{|\omega|=1\}} \frac{1}{\sqrt{|x|^2 + \rho^2 + 2\rho x \cdot \omega}}\,\mathrm{d}\sigma(\omega) \\
&= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{\sqrt{|x|^2 + \rho^2 + 2\rho\,|x| \sin\theta}}\,\mathrm{d}\theta.
\end{aligned}$$

If $|x| = \rho$, we directly conclude that

$$\frac{1}{4\pi}\int_{\{|\omega|=1\}} \frac{1}{|x+\rho\omega|}\,\mathrm{d}\sigma(\omega) = \frac{1}{2\sqrt{2}\rho}\int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{\sqrt{1+\sin\theta}}\,\mathrm{d}\theta = \frac{1}{\sqrt{2}\rho}\Big[\sqrt{1+\sin(\theta)}\Big]_{-\pi/2}^{\pi/2} = \frac{1}{\max\{|x|\,,\rho\}}.$$

If $|x| \neq \rho$, we use the transformation $t(\theta) = \sqrt{|x|^2 + \rho^2 + 2\rho\,|x|\sin\theta}$ which implies $t'(\theta) = \frac{\rho|x|\cos\theta}{t(\theta)}$ and $t\big(\pm\frac{\pi}{2}\big) = \big||x| \pm r\big|$. In this case, we also obtain

$$\frac{1}{4\pi}\int_{\{|\omega|=1\}} \frac{1}{|x+\rho\omega|}\,\mathrm{d}\sigma(\omega) = \int_{||x|-r|}^{|x|+r} \frac{1}{2\,|x|\,\rho}\,\mathrm{d}t = \frac{1}{\max\{|x|\,,\rho\}}.$$ □

Lemma 5.7 (Solution formula in the radially symmetric case)
Let $f \in C_0^\infty(\mathbb{R})$. The solution of the Poisson equation $-\Delta\varphi(x) = f(|x|)$, $x \in \mathbb{R}^3$, has the representation

$$\varphi(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(|y|)}{\max\{|x|\,,|y|\}}\,\mathrm{d}y, \quad x \in \mathbb{R}^3.$$

Moreover the radial derivative of the function u at $r = |x|$ is given by

$$\frac{\partial\varphi}{\partial r}(x) = -\frac{1}{4\pi r^2}\int_{\{|y|<r\}} f(|y|)\,\mathrm{d}y, \quad x \in \mathbb{R}^3 \setminus \{0\}.$$

Proof. We start from the solution formula with Green's function, transform to polar coordinates, and then apply Lemma 5.6 obtaining

$$\begin{aligned}\varphi(x) &= \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(|y|)}{|x-y|}\,\mathrm{d}y \\ &= \frac{1}{4\pi}\int_0^\infty f(\rho)\rho^2 \int_{\{|\omega|=1\}} \frac{1}{|x-\rho\omega|}\,\mathrm{d}\sigma(\omega)\,\mathrm{d}\rho \\ &= \int_0^\infty f(\rho)\rho^2 \frac{1}{\max\{|x|\,,\rho\}}\,\mathrm{d}\rho \\ &= \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(|y|)}{\max\{|x|\,,|y|\}}\,\mathrm{d}y.\end{aligned}$$

Hence, the first assertion is proved. For the second assertion, we recall the penultimate line in the above calculation. For $r = |x|$, we obtain

$$\varphi(x) = \int_0^\infty f(\rho)\rho^2 \frac{1}{\max\{|x|\,,\rho\}}\,\mathrm{d}\rho = \int_0^r \frac{f(\rho)\rho^2}{r}\,\mathrm{d}\rho + \int_r^\infty f(\rho)\rho\,\mathrm{d}\rho.$$

The second claim follows by differentiation. □

Theorem 5.8 (Nonexistence of radially symmetric standing waves)
Let $\omega > 0$. Let $\varphi \in H^1(\mathbb{R}^3)$. Assume that φ is radially symmetric and that (φ, ω) is a distributional solution of equation (5.3). Then $\varphi = 0$.

Proof. We aim to apply the result of Lemma 5.5. Let $\alpha \in (0,1)$. Define the function

$$p := -\phi + \omega.$$

Let $x \in \mathbb{R}^3 \setminus \{0\}$ and $r = |x|$. By the formula in Lemma 5.7 and the estimate $\frac{1}{\max\{r,|y|\}} \leq \frac{1}{r}$ for all $y \in \mathbb{R}^3$, we obtain

$$p(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|\varphi(y)|^2}{\max\{|x|,|y|\}} \, \mathrm{d}y + \omega \geq -\frac{1}{4\pi r} \|\varphi\|_{L^2}^2 + \omega.$$

Similarly, also using that $-\frac{\partial \phi}{\partial r} \geq 0$ by Lemma 5.7, we estimate

$$\begin{aligned}
\frac{\partial p}{\partial r}(x) + \frac{2(1-\alpha)}{|x|} p(x) &= \left(-\frac{\partial \phi}{\partial r}(x) - \frac{2(1-\alpha)}{|x|} \phi(x) \right) + \frac{2(1-\alpha)\omega}{|x|} \\
&\geq -\frac{2(1-\alpha)}{4\pi r} \int_{\mathbb{R}^3} \frac{|\varphi(y)|^2}{\max\{|x|,|y|\}} \, \mathrm{d}y + \frac{2(1-\alpha)}{r} \omega \\
&\geq -\frac{2(1-\alpha)}{4\pi r^2} \|\varphi\|_{L^2}^2 + \frac{2(1-\alpha)}{r} \omega.
\end{aligned}$$

Hence, there exists $R_0 > 0$ such that

$$p(x) \geq 0 \quad \text{and} \quad \frac{\partial p}{\partial r}(x) + \frac{2(1-\alpha)}{|x|} p(x) \geq 0$$

for all $|x| \geq R_0$. Supposing that $\varphi \neq 0$, we can therefore apply Agmon's Lemma 5.5. On the other hand, we obtain the bound

$$\begin{aligned}
0 &\leq \int_{\{R_0 \leq |x| \leq R\}} p(x) \varphi^2(x) \, \mathrm{d}x \\
&\leq \int_{\mathbb{R}^3} p(x) \varphi^2(x) \, \mathrm{d}x \\
&= \int_{\mathbb{R}^3} -\phi(x) \varphi(x)^2 \, \mathrm{d}x + \omega \int_{\mathbb{R}^3} \varphi(x)^2 \, \mathrm{d}x \\
&\leq \|\phi\|_{L^6} \|\varphi^2\|_{L^{6/5}} + \omega \|\varphi\|_{L^2}^2 \\
&\lesssim \|\varphi\|_{H^1}^4 + \omega \|\varphi\|_{L^2}^2.
\end{aligned}$$

This yields

$$\lim_{R \to \infty} \frac{1}{R^\alpha} \int_{\{R_0 \leq |x| \leq R\}} p(x) \varphi^2(x) \, \mathrm{d}x = 0,$$

which is in direct contradiction to Lemma 5.5. □

The general case

After this short review of the radially symmetric case, we investigate the general case. The method from [FHHH82] tackles the problem of absent eigenvalues in two steps. In the first step, rapid decay of eigenfunctions is shown, see Lemma 5.13. In the second step, the nonexistence of eigenfunctions with rapid decay is established, see Theorem 5.14. The first step relies on a proof by contradiction, excluding the possibility of a slowly decaying eigenfunction. Two ingredients are necessary. We compute in Lemma 5.11 the commutator of the Hamiltonian H_ϕ with the generator of the unitary dilation group introduced in Lemma 5.9. The contradiction obtained in Lemma 5.13 then relies on employing this commutator on eigenfunctions with suitably chosen weight functions. Preliminary computations of lower order terms arising from this weight functions are performed in Lemma 5.12.

Lemma 5.9 (Dilation group)
For every $t \in \mathbb{R}$ and every $f \in L^2(\mathbb{R}^3)$, we define

$$U(t)f(x) = \mathrm{e}^{\frac{3}{2}t} f(\mathrm{e}^t x) \quad \text{for all } x \in \mathbb{R}^3.$$

The map

$$U\colon \mathbb{R} \to \mathcal{L}(L^2(\mathbb{R}^3))$$

is a unitary C_0-group, which has a skew-adjoint generator $D\colon \operatorname{dom}(D) \to L^2(\mathbb{R}^3)$. For every $f \in C_c^\infty(\mathbb{R}^3)$, we have

$$Df(x) = \tfrac{1}{2}\big(\nabla(xf(x)) + x \cdot \nabla f(x)\big) = \tfrac{3}{2}f(x) + x \cdot \nabla f(x) \quad \text{for all } x \in \mathbb{R}^3,$$

and the space $C_c^\infty(\mathbb{R}^3)$ is a core for D.

Proof. It is clear that U is a group. Let $t \in \mathbb{R}$. For every $f, g \in L^2(\mathbb{R}^3)$, we compute

$$\|U(t)f\|_{L^2}^2 = \int_{\mathbb{R}^3} \mathrm{e}^{3t} |f(\mathrm{e}^t x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^3} |f(y)|^2 \,\mathrm{d}y = \|f\|_{L^2}^2$$

and

$$(U(t)f, g) = \int_{\mathbb{R}^3} \mathrm{e}^{\frac{3}{2}t} f(\mathrm{e}^t x)\overline{g(x)} \,\mathrm{d}x = \int_{\mathbb{R}^3} f(y)\overline{\mathrm{e}^{-\frac{3}{2}t} g(\mathrm{e}^{-t} y)} \,\mathrm{d}y = (f, U(-t)g)\,.$$

Hence, the operator $U(t)$ is unitary. Let $f \in C_c^\infty(\mathbb{R}^3)$. Since f and ∇f are uniformly continuous, we obtain

$$\|U(t)f - f\|_{L^2} \to 0$$

and

$$\left\| \tfrac{1}{t}\left(U(t)f - f\right) - Df \right\|_{L^2} \to 0$$

as $t \to 0$. Finally, it is also clear that each operator $U(t)$ leaves the space $C_c^\infty(\mathbb{R}^3)$ invariant. Since $C_c^\infty(\mathbb{R}^3)$ is dense in $L^2(\mathbb{R}^3)$, it follows that U is strongly continuous and that $C_c^\infty(\mathbb{R}^3)$ is a core for D. □

Let $\varphi \in H^1(\mathbb{R}^3)$ and let ϕ be given by (5.4). We consider the sesquilinear forms

$$q_1\colon H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \to \mathbb{C}, \quad q_1(u,v) = (\nabla u, \nabla v)\,,$$
$$q_2\colon H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \to \mathbb{C}, \quad q_2(u,v) = (\phi u, v)$$

and $q = q_1 + q_2$. The operator $H_\phi = -\Delta + \phi$ is associated to the form q.

A key step in the method from [FHHH82] is the computation of the commutator $[H_\phi, D]$. To perform this calculation, we approximate the operator D by the difference quotient

$$D_t \coloneqq \frac{1}{t}\big(U(t) - I\big), \quad t \in \mathbb{R} \setminus \{0\}.$$

Since $U(t)^* = U(-t)$, we note that $D_t^* = -D_{-t}$ for every $t \in \mathbb{R}$. For every $u \in C_c^\infty(\mathbb{R}^3)$, we have

$$([H_\phi, D]u, u) = q(Du, u) - q(u, D^*u) = \lim_{t\to 0} q(D_t u, u) - q(u, D_t^* u). \tag{5.5}$$

We show in Lemma 5.11 that under Assumption 5.2 the limit on the right in (5.5) even exists for all $u \in H^1(\mathbb{R}^3)$. We then take (5.5) as the definition of the expression $([H_\phi, D]u, u)$ for $u \in H^1(\mathbb{R}^3)$.

Lemma 5.10 (The commutator of the potential with the generator of the dilation group)
Let $p, q \in [3, \infty]$. Let f be a function such that $f \in L^p(\mathbb{R}^3)$ and $x \cdot \nabla f \in L^q(\mathbb{R}^3)$. For every $u \in H^1(\mathbb{R}^3)$, we have

$$\lim_{t\to 0} ([f, D_t]u, u) = -\big((x \cdot \nabla f)u, u\big). \tag{5.6}$$

Proof. First, let $u \in C_c^\infty(\mathbb{R}^3)$. For every $x \in \mathbb{R}^3$, we obtain

$$\begin{aligned} [f, D_t]u(x) &= \frac{1}{t} f(x) \left(\mathrm{e}^{\frac{3}{2}t} u(\mathrm{e}^t x) - u(x)\right) - \frac{1}{t}\left(\mathrm{e}^{\frac{3}{2}t} f(\mathrm{e}^t x) u(\mathrm{e}^t x) - f(x)u(x)\right) \\ &= \frac{1}{t}\mathrm{e}^{\frac{3}{2}t} u(\mathrm{e}^t x)\big(f(x) - f(\mathrm{e}^t x)\big) \\ &= -\frac{1}{t}\mathrm{e}^{\frac{3}{2}t} u(\mathrm{e}^t x) \int_1^{\mathrm{e}^t} x \cdot \nabla f(sx)\,\mathrm{d}s. \end{aligned}$$

Fubini and the transformation $y = sx$ further imply

$$\begin{aligned} ([f, D_t]u, u) &= \int_{\mathbb{R}^3} \frac{1}{t}\mathrm{e}^{\frac{3}{2}t} u(\mathrm{e}^t x) \int_1^{\mathrm{e}^t} x \cdot \nabla f(sx)\,\mathrm{d}s\, \bar{u}(x)\,\mathrm{d}x \\ &= \int_1^{\mathrm{e}^t} \frac{1}{t}\mathrm{e}^{\frac{3}{2}t} \int_{\mathbb{R}^3} u(\mathrm{e}^t x) x \cdot \nabla f(sx) \bar{u}(x)\,\mathrm{d}x\,\mathrm{d}s \\ &= \int_1^{\mathrm{e}^t} \frac{1}{t}\mathrm{e}^{\frac{3}{2}t} \int_{\mathbb{R}^3} u(\mathrm{e}^t s^{-1} y)(s^{-1}y) \cdot \nabla f(y) \bar{u}(s^{-1}y) s^{-3}\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_{\mathbb{R}^3} y \cdot \nabla f(y) \frac{1}{t} \int_1^{\mathrm{e}^t} u(\mathrm{e}^t s^{-1} y) \bar{u}(s^{-1}y) s^{-4}\,\mathrm{d}s\,\mathrm{d}y. \end{aligned} \tag{5.7}$$

We obtain that

$$\lim_{t\to 0}\frac{1}{t}\int_1^{\mathrm{e}^t} u(\mathrm{e}^t s^{-1}y)\bar{u}(s^{-1}y)s^{-4}\,\mathrm{d}s = u(y)\bar{u}(y)$$

for all $y \in \mathbb{R}^3$ since $u \in C_c^\infty(\mathbb{R}^3)$. With dominated convergence it follows from calculation (5.7) that (5.6) holds true for all $u \in C_c^\infty(\mathbb{R}^3)$.

We show that the left side of (5.6) extends to $H^1(\mathbb{R}^3)$. To this end, let $u \in H^1(\mathbb{R}^3)$. Let $\varepsilon > 0$ and let $u_\varepsilon \in C_c^\infty(\mathbb{R}^3)$ such that $\|u - u_\varepsilon\|_{H^1} \le \varepsilon$. We have

$$([f, D_t]u, u) - ([f, D_t]u_\varepsilon, u_\varepsilon) = ([f, D_t](u - u_\varepsilon), u) + ([f, D_t]u_\varepsilon, u - u_\varepsilon)\,.$$

With the second line of (5.7) and the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^{2q/(q-2)}(\mathbb{R}^3)$, we obtain that

$$\begin{aligned}|([f, D_t](u - u_\varepsilon), u)| &\le \frac{1}{t}\mathrm{e}^{\frac{3}{2}t}\int_1^{e^t} s^{-1-3/q}\,\mathrm{d}s\,\left\|(u - u_\varepsilon)(\mathrm{e}^t\cdot)\right\|_{H^1}\|x\cdot\nabla f\|_{L^q}\|u\|_{L^2}\\ &\le C_t\varepsilon\,\|x\cdot\nabla f\|_{L^q}\|u\|_{L^2}\end{aligned}$$

and similarly

$$|([f, D_t](u_\varepsilon), u - u_\varepsilon)| \le C_t\varepsilon\,\|x\cdot\nabla f\|_{L^q}\|u_\varepsilon\|_{H^1}\,,$$

where the constant C_t is bounded for t in a bounded interval. Taking the limit as $t \to 0$, we obtain (5.6) for all $u \in H^1(\mathbb{R}^3)$. $\square$

Lemma 5.11 (The commutator of the Hamiltonian with the generator of the dilation group)
Let $\varphi \in H^1(\mathbb{R}^3)$ satisfy Assumption 5.2, and let ϕ be given by (5.4). For every $u \in H^1(\mathbb{R}^3)$, we have

$$([H_\phi, D]u, u) = 2\|\nabla u\|_{L^2}^2 - \big((x\cdot\nabla\phi)u, u\big).$$

Note that $x\cdot\nabla\phi \in L^6(\mathbb{R}^3)$ by Lemma 5.4 (2) so that $(x\cdot\nabla\phi)u$ belongs to $L^2(\mathbb{R}^3)$.

Proof. Let $u \in H^1(\mathbb{R}^3)$, and let $t \in \mathbb{R}\setminus\{0\}$. We first consider the contribution of q_1. Using $D_t^* = -D_{-t}$, we compute

$$\begin{aligned}q_1(D_t u, u) - q_1(u, D_t^* u) &= (\nabla D_t u, \nabla u) - (\nabla u, \nabla D_t^* u)\\ &= ([\nabla, D_t]u + D_t\nabla u, \nabla u) - (\nabla u, [\nabla, D_t^*]u + D_t^*\nabla u)\\ &= ([\nabla, D_t]u, \nabla u) + (\nabla u, [\nabla, D_{-t}]u)\,.\end{aligned}$$

We further obtain

$$[\nabla, D_t]u = \frac{1}{t}\nabla\big(U(t)u - u\big) - \frac{1}{t}\big(U(t)\nabla u - \nabla u\big) = \frac{\mathrm{e}^t - 1}{t}U(t)\nabla u.$$

Since U is strongly continuous, $[\nabla, D_t]u$ converges to ∇u in $L^2(\mathbb{R}^3)$ as $t \to 0$. Hence, we arrive at

$$\lim_{t\to 0} q_1(D_t u, u) - q_1(u, D_t^* u) = 2 \|\nabla u\|_{L^2}^2 .$$

Concerning q_2, we compute

$$\begin{aligned} q_2(D_t u, u) - q_2(u, D_t^* u) &= (\phi D_t u, u) - (\phi u, D_t^* u) \\ &= ([\phi, D_t] + D_t \phi u, u) - (\phi u, D_t^* u) \\ &= ([\phi, D_t]u, u) . \end{aligned}$$

Due to Lemma 5.4, the potential ϕ satisfies the assumptions of Lemma 5.10 which yields

$$\lim_{t\to 0} q_2(D_t u, u) - q_2(u, D_t^* u) = \lim_{t\to 0} ([\phi, D_t]u, u) = -\big((x\cdot\nabla\phi)u, u\big). \qquad \square$$

Lemma 5.12 (The commutator applied to weighted eigenfunctions)
Let $\omega > 0$. Let $\varphi \in H^1(\mathbb{R}^3)$ satisfy Assumption 5.2, and let ϕ be given by (5.4). Let $\psi \in H^1(\mathbb{R}^3)$, and assume that $H_\phi \psi = \omega\psi$. Let $F \in C^\infty(\mathbb{R}^3, \mathbb{R}) \cap L^\infty(\mathbb{R}^3)$ satisfy $\nabla F(x) = x g(x)$ for some positive function $g \in C^\infty(\mathbb{R}^3, \mathbb{R})$ such that $\partial^\alpha g(x)$ decay exponentially as $|x| \to \infty$ for all multi-indices $\alpha \in \mathbb{N}^3$ with order $|\alpha| \leq 2$. Define $\psi_F = \mathrm{e}^F \psi$. Note that ψ_F and $\sqrt{g} D\psi_F$ belong to $L^2(\mathbb{R}^3)$. Then we have the identities

$$([H_\phi, D]\psi_F, \psi_F) = 2\left((\omega + |\nabla F|^2)\psi_F, \psi_F\right) - \left((2\phi + x\cdot\nabla\phi)\psi_F, \psi_F\right) \tag{5.8}$$

and

$$([H_\phi, D]\psi_F, \psi_F) = -4 \|\sqrt{g} D\psi_F\|_{L^2}^2 + \left(\left((x\cdot\nabla)^2 g - x\cdot\nabla|\nabla F|^2\right)\psi_F, \psi_F\right). \tag{5.9}$$

Proof. Lemma 5.11 shows that

$$\begin{aligned} &([H_\phi, D]\psi_F, \psi_F) \\ &\quad = 2\|\nabla\psi_F\|_{L^2}^2 - (x\cdot\nabla\phi\psi_F, \psi_F) \\ &\quad = 2(\nabla\psi_F, \nabla\psi_F) + 2(\phi\psi_F, \psi_F) - \big((2\phi + x\cdot\nabla\phi)\psi_F, \psi_F\big). \end{aligned} \tag{5.10}$$

We note that the assumption $H_\phi\psi = \omega\psi$ means

$$q(\psi, v) = (\nabla\psi, \nabla v) + (\phi\psi, v) = (\omega\psi, v)$$

for all $v \in H^1(\mathbb{R}^3)$. Taking $v = \mathrm{e}^F \psi_F$, we thus get the identity

$$\big(\nabla\psi, \nabla(\mathrm{e}^F\psi_F)\big) = (\omega\psi_F, \psi_F) - (\phi\psi_F, \psi_F) . \tag{5.11}$$

We compute

$$
\begin{aligned}
&(\nabla\psi_F, \nabla\psi_F)\\
&\quad= \big(\nabla(\mathrm{e}^F\psi), \nabla\psi_F\big)\\
&\quad= \big((\nabla F)\psi_F, \nabla\psi_F\big) + \big(\mathrm{e}^F\nabla\psi, \nabla\psi_F\big)\\
&\quad= \big((\nabla F)\psi_F, \nabla\psi_F\big) + \big(\nabla\psi, \nabla(\mathrm{e}^F\psi_F)\big) - \big(\mathrm{e}^F\nabla\psi, (\nabla F)\psi_F\big)\\
&\quad= \big(\nabla\psi, \nabla(\mathrm{e}^F\psi_F)\big) + \big(|\nabla F|^2\,\psi_F, \psi_F\big) + 2\mathrm{i}\,\mathrm{Im}\big((\nabla F)\psi_F, \nabla\psi_F\big)\\
&\quad= (\omega\psi_F, \psi_F) - (\phi\psi_F, \psi_F) + \big(|\nabla F|^2\,\psi_F, \psi_F\big)\,,
\end{aligned}
\tag{5.12}
$$

where we use (5.11) in the last step and note that the expression in every line of the computation is real. Inserting this result into (5.10), we obtain (5.8).

To derive (5.9), we use the same approach as in Lemma 5.11, i.e., we evaluate $\lim_{t\to 0} q(D_t\psi_F, \psi_F) - q(\psi_F, D_t^*\psi_F)$. First, let $u, v \in H^1(\mathbb{R}^3)$. We compute

$$
\begin{aligned}
q(\mathrm{e}^F u, \mathrm{e}^{-F} v) &= -\big((\nabla F)\mathrm{e}^F u, (\nabla F)\mathrm{e}^{-F} v\big) + \big((\nabla F)\mathrm{e}^F u, \mathrm{e}^{-F}\nabla v\big)\\
&\quad - \big(\mathrm{e}^F\nabla u, (\nabla F)\mathrm{e}^{-F} v\big) + \big(\mathrm{e}^F\nabla u, \mathrm{e}^{-F}\nabla v\big) + \big(\phi\mathrm{e}^F u, \mathrm{e}^{-F} v\big)\\
&= -\big(|\nabla F|^2\,u, v\big) + \big((\nabla F)u, \nabla v\big) - \big(\nabla u, (\nabla F)v\big) + q(u, v)
\end{aligned}
$$

Using that $\nabla F(x) = xg(x)$ for every $x \in \mathbb{R}^3$, we integrate by parts and infer

$$
\begin{aligned}
\big((\nabla F)u, \nabla v\big) - \big(\nabla u, (\nabla F)v\big) &= \big(u, g(x\cdot\nabla v)\big) + \big(u, 3gv + (x\cdot\nabla g)v + g(x\cdot\nabla v)\big)\\
&= \big(u, 3gv + 2g(x\cdot\nabla v) + (x\cdot\nabla g)v\big)\\
&= 2\,(u, gDv) + \big(u, (x\cdot\nabla g)v\big).
\end{aligned}
$$

In the same way, we can alternatively obtain

$$
-\big((\nabla F)u, \nabla v\big) + \big(\nabla u, (\nabla F)v\big) = 2\,(gDu, v) + \big((x\cdot\nabla g)u, v\big).
$$

We thus deduce

$$
q(u, v) = q(\mathrm{e}^F u, \mathrm{e}^{-F} v) - 2\,(u, gDv) - \big(u, (x\cdot\nabla g)v\big) + \big(|\nabla F|^2\,u, v\big) \tag{5.13}
$$

and analogously

$$
q(u, v) = q(\mathrm{e}^{-F} u, \mathrm{e}^{F} v) - 2\,(gDu, v) - \big((x\cdot\nabla g)u, v\big) + \big(|\nabla F|^2\,u, v\big)\,. \tag{5.14}
$$

Setting $u = D_t\psi_F$ and $v = \psi_F$ in (5.13) and using that $H_\phi\psi = \omega\psi$, we compute

$$
\begin{aligned}
q(D_t\psi_F, \psi_F) &= q(\mathrm{e}^F D_t\psi_F, \psi) - 2\,(D_t\psi_F, gD\psi_F)\\
&\quad - \big(D_t\psi_F, (x\cdot\nabla g)\psi_F\big) + \big(|\nabla F|^2\,D_t\psi_F, \psi_F\big)\\
&= \omega\,(D_t\psi_F, \psi_F) - 2\,(D_t\psi_F, gD\psi_F)\\
&\quad - \big(D_t\psi_F, (x\cdot\nabla g)\psi_F\big) + \big(|\nabla F|^2\,D_t\psi_F, \psi_F\big)\,.
\end{aligned}
$$

Similarly with $u = \psi_F$ and $v = D_t^* \psi_F$ in (5.14), we obtain

$$\begin{aligned} q(\psi_F, D_t^*\psi_F) &= q(\psi, \mathrm{e}^F D_t^*\psi_F) - 2\,(gD\psi_F, D_t^*\psi_F) \\ &\quad - \big((x\cdot\nabla g)\psi_F, D_t^*\psi_F\big) + \big(\psi_F, |\nabla F|^2\, D_t^*\psi_F\big) \\ &= \omega\,(\psi_F, D_t^*\psi_F) - 2\,(gD\psi_F, D_t^*\psi_F) \\ &\quad - \big((x\cdot\nabla g)\psi_F, D_t^*\psi_F\big) + \big(|\nabla F|^2\,\psi_F, D_t^*\psi_F\big)\,. \end{aligned}$$

Altogether, this gives

$$\begin{aligned} q(D_t\psi_F, \psi_F) - q(\psi_F, D_t^*\psi_F) &= -2\,(D_t\psi_F, gD\psi_F) - 2\,(gD\psi_F, D_{-t}\psi_F) \\ &\quad - \big([(x\cdot\nabla g), D_t]\psi_F, \psi_F\big) + \big([|\nabla F|^2, D_t]\psi_F, \psi_F\big)\,. \end{aligned}$$

Since $\sqrt{g}D\psi_F$ belongs to $L^2(\mathbb{R}^3)$, we obtain

$$\lim_{t\to 0} -2\,(D_t\psi_F, gD\psi_F) - 2\,(gD\psi_F, D_{-t}\psi_F) = -4\,\|\sqrt{g}D\psi_F\|_{L^2}^2\,.$$

From Lemma 5.10, we further deduce

$$\lim_{t\to 0}\big([(x\cdot\nabla g), D_t]\psi_F, \psi_F\big) = -\,\big(((x\cdot\nabla)^2 g)\psi_F, \psi_F\big)$$

and

$$\lim_{t\to 0}\big([|\nabla F|^2, D_t]\psi_F, \psi_F\big) = -\,\big((x\cdot\nabla\,|\nabla F|^2)\psi_F, \psi_F\big)\,.$$

This concludes the proof of identity (5.9). □

In the next lemma, we show that an eigenfunction of the operator H_ϕ for a positive eigenvalue has to decay faster than an exponential function. The proof relies on the identities derived in Lemma 5.12 where the weight functions F have to be chosen in a clever way. To prepare the proof, we first write down some elementary computations. For $\mu, \varepsilon, \lambda \in (0,\infty)$, we define the bounded weight function

$$F_{\mu,\varepsilon,\lambda}\colon \mathbb{R}^3 \to \mathbb{R}, \quad x \mapsto \mu\varepsilon^{-1}\big(1 - \mathrm{e}^{-\varepsilon\langle x\rangle_\lambda}\big). \tag{5.15}$$

Here we use the notation $\langle x\rangle_\lambda := (\lambda + |x|^2)^{1/2}$ for every $x \in \mathbb{R}^3$. If $\lambda = 1$, this expression coincides with the Japanese bracket $\langle\cdot\rangle$ which is our preferred notation in this case. It is useful to note the formulas

$$\nabla\,|x|^2 = 2x, \quad \nabla\,\langle x\rangle_\lambda^{-n} = -nx\,\langle x\rangle_\lambda^{-n-2} \quad \text{and} \quad \nabla\mathrm{e}^{-\varepsilon\langle x\rangle_\lambda} = -\varepsilon x\,\langle x\rangle_\lambda^{-1}\,\mathrm{e}^{-\varepsilon\langle x\rangle_\lambda}.$$

We note the following properties of the functions $F_{\mu,\varepsilon,\lambda}$. For every $x \in \mathbb{R}^3$, we have

$$\lim_{\varepsilon\to 0+} F_{\mu,\varepsilon,\lambda}(x) = \mu\,\langle x\rangle_\lambda\,. \tag{5.16}$$

Setting $g_{\mu,\varepsilon,\lambda}(x) = \mu \langle x\rangle_\lambda^{-1} \mathrm{e}^{-\varepsilon\langle x\rangle_\lambda}$ for $x \in \mathbb{R}^3$, we obtain $\nabla F_{\mu,\varepsilon,\lambda}(x) = x g_{\mu,\varepsilon,\lambda}(x)$ and the function $g_{\mu,\varepsilon,\lambda}$ satisfies the assumptions imposed in Lemma 5.12. Moreover, we have the estimates

$$F_{\mu,\varepsilon,\lambda}(x) \le \mu \langle x\rangle_\lambda \quad \text{and} \quad |\nabla F_{\mu,\varepsilon,\lambda}| \le \mu |x| \langle x\rangle_\lambda^{-1} \mathrm{e}^{-\varepsilon\langle x\rangle_\lambda} \le \mu. \tag{5.17}$$

In view of (5.9), we further compute the following terms. Since $|x|^2 = \langle x\rangle_\lambda^2 - \lambda$, we have

$$|\nabla F_{\mu,\varepsilon,\lambda}(x)|^2 = \mu^2 |x|^2 \langle x\rangle_\lambda^{-2} \mathrm{e}^{-2\varepsilon\langle x\rangle_\lambda} = \mu^2 \mathrm{e}^{-2\varepsilon\langle x\rangle_\lambda} - \mu^2\lambda \langle x\rangle_\lambda^{-2} \mathrm{e}^{-2\varepsilon\langle x\rangle_\lambda}$$

and hence

$$\begin{aligned} &- x \cdot \nabla |\nabla F_{\mu,\varepsilon,\lambda}(x)|^2 \\ &\quad = -x \cdot \nabla \left(\mu^2 \mathrm{e}^{-2\varepsilon\langle x\rangle_\lambda} - \mu^2\lambda \langle x\rangle_\lambda^{-2} \mathrm{e}^{-2\varepsilon\langle x\rangle_\lambda}\right) \\ &\quad = -x \cdot \left(-2\mu^2\varepsilon x \langle x\rangle_\lambda^{-1} + 2\mu^2\lambda x \langle x\rangle_\lambda^{-4} + 2\mu^2\varepsilon\lambda x \langle x\rangle_\lambda^{-3}\right) \mathrm{e}^{-2\varepsilon\langle x\rangle_\lambda} \\ &\quad = \left(2\mu^2\varepsilon |x|^2 \langle x\rangle_\lambda^{-1} - 2\mu^2\lambda |x|^2 \langle x\rangle_\lambda^{-4} - 2\mu^2\varepsilon\lambda |x|^2 \langle x\rangle_\lambda^{-3}\right) \mathrm{e}^{-2\varepsilon\langle x\rangle_\lambda}. \end{aligned} \tag{5.18}$$

We also compute

$$\begin{aligned} (x\cdot\nabla) g_{\mu,\varepsilon,\lambda}(x) &= x \cdot \nabla \left(\mu \langle x\rangle_\lambda^{-1} \mathrm{e}^{-\varepsilon\langle x\rangle_\lambda}\right) \\ &= x \cdot \left(-\mu x \langle x\rangle_\lambda^{-3} \mathrm{e}^{-\varepsilon\langle x\rangle_\lambda} - \mu\varepsilon x \langle x\rangle_\lambda^{-2} \mathrm{e}^{-\varepsilon\langle x\rangle_\lambda}\right) \\ &= -\mu |x|^2 \langle x\rangle_\lambda^{-3} \mathrm{e}^{-\varepsilon\langle x\rangle_\lambda} - \mu\varepsilon |x|^2 \langle x\rangle_\lambda^{-2} \mathrm{e}^{-\varepsilon\langle x\rangle_\lambda} \end{aligned}$$

and thus we obtain

$$\begin{aligned} &(x\cdot\nabla)^2 g_{\mu,\varepsilon,\lambda}(x) \\ &\quad = x \cdot \nabla \left(-\mu |x|^2 \langle x\rangle_\lambda^{-3} \mathrm{e}^{-\varepsilon\langle x\rangle_\lambda} - \mu\varepsilon |x|^2 \langle x\rangle_\lambda^{-2} \mathrm{e}^{-\varepsilon\langle x\rangle_\lambda}\right) \\ &\quad = x \cdot \left(-2\mu x \langle x\rangle_\lambda^{-3} + 3\mu x |x|^2 \langle x\rangle_\lambda^{-5} + \mu\varepsilon x |x|^2 \langle x\rangle_\lambda^{-4}\right) \mathrm{e}^{-\varepsilon\langle x\rangle_\lambda} \\ &\qquad + x \cdot \left(-2\mu\varepsilon x \langle x\rangle_\lambda^{-2} + 2\mu\varepsilon x |x|^2 \langle x\rangle_\lambda^{-4} + \mu\varepsilon^2 x |x|^2 \langle x\rangle_\lambda^{-3}\right) \mathrm{e}^{-\varepsilon\langle x\rangle_\lambda} \\ &\quad = \left(-2\mu |x|^2 \langle x\rangle_\lambda^{-3} + 3\mu |x|^4 \langle x\rangle_\lambda^{-5} + \mu\varepsilon |x|^4 \langle x\rangle_\lambda^{-4}\right) \mathrm{e}^{-\varepsilon\langle x\rangle_\lambda} \\ &\qquad + \left(-2\mu\varepsilon |x|^2 \langle x\rangle_\lambda^{-2} + 2\mu\varepsilon |x|^4 \langle x\rangle_\lambda^{-4} + \mu\varepsilon^2 |x|^4 \langle x\rangle_\lambda^{-3}\right) \mathrm{e}^{-\varepsilon\langle x\rangle_\lambda}. \end{aligned} \tag{5.19}$$

Lemma 5.13 (Rapid decay of eigenfunctions)
Let $\omega > 0$. Let $\varphi \in H^1(\mathbb{R}^3)$ satisfy Assumption 5.2, and let ϕ be given by (5.4). Let $\psi \in H^1(\mathbb{R}^3)$ satisfy $H_\phi\psi = \omega\psi$. Then

$$\mathrm{e}^{\mu\langle x\rangle}\psi \in L^2(\mathbb{R}^3) \quad \text{for all } \mu > 0.$$

Proof. Define $\mu_* = \sup\left\{\mu \in [0,\infty) : \mathrm{e}^{\mu\langle x\rangle}\psi \in L^2(\mathbb{R}^3)\right\}$. We prove that $\mu_* = \infty$ in two steps. In the first step, we show that the assumption $\mu_* = 0$ leads to a contradiction, and in the second step, we refute the assumption $\mu_* < \infty$. Both steps are based on the identities derived in Lemma 5.12 with suitably chosen weight functions.

First step. Assume that $\mu_* = 0$. Taking (5.16) into account, there exists positive null sequences (μ_n) and (ε_n) such that

$$\|\mathrm{e}^{F_n}\psi\|_{L^2} \to \infty \quad \text{as } n \to \infty,$$

where we use the abbreviation $F_n = F_{\mu_n,\varepsilon_n,1}$. We next define

$$\varphi_n = \|\mathrm{e}^{F_n}\psi\|_{L^2}^{-1}\mathrm{e}^{F_n}\psi, \quad n \in \mathbb{N}. \tag{5.20}$$

We note that $\|\varphi_n\|_{L^2} = 1$ and that $|\varphi_n(x)| \to 0$ as $n \to \infty$ for every $x \in \mathbb{R}^3$. The Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ for $p \in [2,6]$ and dominated convergence imply that $\varphi_n \to 0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^3)$ as $n \to \infty$ for all $p \in [2,6]$. We also note that

$$|\nabla\varphi_n(x)| = \|\mathrm{e}^{F_n}\psi\|_{L^2}^{-1}\left|\nabla F_n(x)\mathrm{e}^{F_n(x)}\psi(x) + \mathrm{e}^{F_n(x)}\nabla\psi(x)\right| \to 0 \quad \text{as } n \to \infty,$$

for every $x \in \mathbb{R}^3$, so that we also have $\nabla\varphi_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$. Using (5.12), we obtain

$$(\nabla\varphi_n, \nabla\varphi_n) = \omega - (\phi\varphi_n, \varphi_n) + \left(|\nabla F_n|^2\varphi_n, \varphi_n\right)$$

and with (5.17) we infer that

$$\|\varphi_n\|_{H^1}^2 = \|\varphi_n\|_{L^2}^2 + \|\nabla\varphi_n\|_{L^2}^2 \le 1 + \omega + \mu_n^2 + |(\phi\varphi_n, \varphi_n)|\,. \tag{5.21}$$

We now apply Lemma 5.12 with the functions φ_n. Identity (5.8) gives

$$([H_\phi, D]\varphi_n, \varphi_n) = 2\left((\omega + |\nabla F|^2)\varphi_n, \varphi_n\right) - ((2\phi + x\cdot\nabla\phi)\varphi_n, \varphi_n)\,.$$

Using Hölder's inequality and the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^{12/5}(\mathbb{R}^3)$, we compute

$$\begin{aligned}
|(\phi\varphi_n, \varphi_n)| &\le \int_{\{|x|<R\}} |\phi(x)|\,|\varphi_n(x)|^2\,\mathrm{d}x + \int_{\{|x|\ge R\}} |\phi(x)|\,|\varphi_n(x)|^2\,\mathrm{d}x \\
&\le \|\phi\|_{L^\infty}\|\varphi_n\|_{L^2(\{|x|<R\})}^2 + \|\phi\|_{L^6(\{|x|\ge R\})}\left\||\varphi_n|^2\right\|_{L^{6/5}} \\
&\lesssim \|\phi\|_{L^\infty}\|\varphi_n\|_{L^2(\{|x|<R\})}^2 + \|\phi\|_{L^6(\{|x|\ge R\})}\|\varphi_n\|_{H^1}^2\,.
\end{aligned}$$

For any $\varepsilon > 0$, we may choose $R > 0$ so large that $\|\phi\|_{L^6(\{|x|\ge R\})} \le \varepsilon$. Due to (5.21), we conclude that $\lim_{n\to\infty} |(\phi\varphi_n, \varphi_n)| = 0$ and $\limsup_{n\to\infty} \|\varphi_n\|_{H^1} < \infty$. We further compute

$$\begin{aligned}
|(x\cdot\nabla\phi\,\varphi_n, \varphi_n)| &\le \int_{\{|x|<R\}} |x\cdot\nabla\phi(x)|\,|\varphi_n(x)|^2\,\mathrm{d}x + \int_{\{|x|\ge R\}} |x\cdot\nabla\phi(x)|\,|\varphi_n(x)|^2\,\mathrm{d}x \\
&\lesssim \|x\cdot\nabla\phi\|_{L^6}\|\varphi_n\|_{L^{12/5}(\{|x|<R\})}^2 + \|x\cdot\nabla\phi\|_{L^6(\{|x|\ge R\})}\|\varphi_n\|_{H^1}^2\,.
\end{aligned}$$

With the same argument as above, we obtain $\lim_{n\to\infty} |((x\cdot\nabla\phi)\varphi_n,\varphi_n)| = 0$. We conclude that

$$\liminf_{n\to\infty} ([H_\phi, D]\varphi_n, \varphi_n) \geq 2\omega. \tag{5.22}$$

On the other hand, equation (5.9) yields

$$([H_\phi, D]\varphi_n, \varphi_n) = -4\left\|\sqrt{g}D\varphi_n\right\|_{L^2}^2 + \left(\left((x\cdot\nabla)^2 g_n - x\cdot\nabla\left|\nabla F_n\right|^2\right)\varphi_n, \varphi_n\right).$$

Using that $\sup_{x\in\mathbb{R}^3} |x| \langle x\rangle^{-1} = 1$ and $\sup_{x\in\mathbb{R}^3} \varepsilon |x| \mathrm{e}^{-\varepsilon\langle x\rangle} \leq 1$, we deduce from equations (5.18) and (5.19) the estimate

$$\left((x\cdot\nabla)^2 g_n - x\cdot\nabla\left|\nabla F_n\right|^2\right)_+ \leq 2\mu_n^2 + 3\mu_n + 4\mu_n\varepsilon_n. \tag{5.23}$$

This implies

$$\limsup_{n\to\infty} ([H_\phi, D]\varphi_n, \varphi_n) \leq \limsup_{n\to\infty} \left(2\mu_n^2 + 3\mu_n + 4\mu_n\varepsilon_n\right) \|\varphi_n\|_{L^2}^2 \leq 0,$$

which is in contradiction with (5.22). Hence, $\mu_* > 0$.

Second step. Assume that $\mu_* < \infty$. Then there exist sequences $(\mu_n)_n$ and $(\varepsilon_n)_n$ which converge to μ_* respectively 0 from above such that

$$\|\mathrm{e}^{F_n}\psi\|_{L^2} \to \infty \quad \text{as } n\to\infty.$$

Here F_n has the same meaning as in the first step and we also define φ_n in the same way as in (5.20). The arguments presented in the first step show that (5.22) holds true. To derive a contradiction, we have to estimate finer than in (5.23). We now use that

$$\left((x\cdot\nabla)^2 g_n - x\cdot\nabla\left|\nabla F_n\right|^2\right)_+ \leq 2\mu_n^2\varepsilon_n \langle x\rangle \mathrm{e}^{-2\varepsilon\langle x\rangle} + 4\mu_n\varepsilon_n + 3\mu_n \langle x\rangle^{-1}. \tag{5.24}$$

The contributions of the last two terms are straightforward, since

$$(4\mu_n\varepsilon_n\varphi_n, \varphi_n) \leq 4\mu_n\varepsilon_n \|\varphi_n\|_{L^2}^2 \to 0 \quad \text{as } n\to\infty$$

and, for any $R > 0$,

$$\left(3\mu_n \langle x\rangle^{-1}\varphi_n, \varphi_n\right) \leq 3\mu_n \|\varphi_n\|_{L^2(\{|x|<R\})}^2 + 3\mu_n \sup_{|x|\geq R} \langle x\rangle^{-1} \|\varphi_n\|_{L^2}^2 \to 3\mu_* \langle R\rangle^{-1}$$

as $n\to\infty$. The first term is more delicate. We fix $\delta > 0$. For each $n\in\mathbb{N}$, we define $R_n \in [0,\infty)$ by $\langle R_n\rangle = \max\{\delta\varepsilon_n^{-1}, 1\}$. The key observation is that for all $x\in\mathbb{R}^3$ with $|x| \geq R_n$ we obtain $\varepsilon_n\langle x\rangle \geq \delta$ and hence the weight functions satisfy

$$F_n(x) = \mu_n(\varepsilon_n\langle x\rangle)^{-1}\left(1 - \mathrm{e}^{-\varepsilon_n\langle x\rangle}\right)\langle x\rangle \leq \mu_n\delta^{-1}(1-\mathrm{e}^{-\delta})\langle x\rangle.$$

Since $\mu_* > 0$ by step 1, there exists $\nu \in (0, \mu_*)$ such that $\mathrm{e}^{F_n(x)} \le \mathrm{e}^{\nu\langle x\rangle}$ for all $x \in \{|x| \ge R_n\}$ and almost all $n \in \mathbb{N}$. Since $\mathrm{e}^{\nu\langle x\rangle}\psi$ belongs to $L^2(\mathbb{R}^3)$ by definition of μ_*, it follows that

$$\begin{aligned}\left(2\mu_n^2\varepsilon_n \langle x\rangle \mathrm{e}^{-2\varepsilon\langle x\rangle}\varphi_n, \varphi_n\right) &= \int_{\mathbb{R}^3} 2\mu_n^2\varepsilon_n \langle x\rangle \mathrm{e}^{-2\varepsilon\langle x\rangle} |\varphi_n(x)|^2 \,\mathrm{d}x \\ &\le 2\mu_n^2\delta \|\varphi_n\|_{L^2}^2 + \|\mathrm{e}^{F_n}\psi\|_{L^2}^{-2} \int_{\{|x|\ge R_n\}} 2\mu_n^2 \left|\mathrm{e}^{\nu\langle x\rangle}\psi(x)\right|^2 \mathrm{d}x \\ &\to 2\mu_*^2\delta \quad \text{as } n\to\infty.\end{aligned}$$

Since $\delta > 0$ can be chosen arbitrarily, we obtain also in this case that

$$\limsup_{n\to\infty} \left([H_\phi, D]\varphi_n, \varphi_n\right) \le 0.$$

This contradiction with (5.22) implies the assertion $\mu_* = \infty$. □

Theorem 5.14 (Nonexistence of positive eigenvalues)
Let $\omega > 0$. Let $\varphi \in H^1(\mathbb{R}^3)$ satisfy Assumption 5.2 and let ϕ be given by (5.4). Let $\psi \in H^1(\mathbb{R}^3)$ satisfy $H_\phi\psi = \omega\psi$. Then $\psi = 0$.

Proof. Let F be a weight function satisfying the assumptions of Lemma 5.12. Set $\psi_F = \mathrm{e}^F\psi$. The computations in the proof of Lemma 5.12 show that instead of (5.8) we might as well obtain the identity

$$([H_\phi, D]\psi_F, \psi_F) = \left((\omega + |\nabla F|^2)\psi_F, \psi_F\right) + \|\nabla\psi_F\|_{L^2}^2 - ((\phi + x\cdot\nabla\phi)\psi_F, \psi_F).$$

For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that with the inequalities of Hölder, Gagliardo–Nirenberg and Young, we obtain that

$$\begin{aligned}|((\phi + x\cdot\nabla\phi)\psi_F, \psi_F)| &\le \|\phi + x\cdot\nabla\phi\|_{L^6} \|\psi_F\|_{L^{12/5}}^2 \\ &\lesssim \|\phi + x\cdot\nabla\phi\|_{L^6} \|\psi_F\|_{L^2}^{3/2} \|\nabla\psi_F\|_{L^2}^{1/2} \\ &\lesssim \|\phi + x\cdot\nabla\phi\|_{L^6} \left(\varepsilon \|\nabla\psi_F\|_{L^2}^2 + C_\varepsilon \|\psi_F\|_{L^2}^2\right).\end{aligned}$$

By choosing $\varepsilon > 0$ appropriately, we can fix $C > 0$ such that the estimate from below

$$([H_\phi, D]\psi_F, \psi_F) \ge \left(|\nabla F|^2 \psi_F, \psi_F\right) - C \|\psi_F\|_{L^2}^2$$

holds true. Using (5.9), we also estimate the commutator from above so that we obtain

$$\left(|\nabla F|^2 \psi_F, \psi_F\right) \le C \|\psi_F\|_{L^2}^2 + \left(\left((x\cdot\nabla)^2 g - x\cdot\nabla |\nabla F|^2\right)\psi_F, \psi_F\right). \tag{5.25}$$

We choose $F = F_{\mu,\varepsilon,\lambda}$ as in (5.15) and aim to show that (5.25) cannot hold true unless $\psi = 0$. Taking the limit $\varepsilon \to 0$ in (5.18) and (5.19), we obtain

$$\lim_{\varepsilon\to 0+} -x\cdot\nabla |\nabla F_{\mu,\varepsilon,\lambda}(x)|^2 = -2\mu^2\lambda |x|^2 \langle x\rangle_\lambda^{-4}$$

and

$$\lim_{\varepsilon\to 0+}(x\cdot\nabla)^2 g_{\mu,\varepsilon,\lambda}(x) = -2\mu\,|x|^2\,\langle x\rangle_\lambda^{-3} + 3\mu\,|x|^4\,\langle x\rangle_\lambda^{-5}\,.$$

We set $F_{\mu,\lambda}(x) = \lim_{\varepsilon\to 0} F_{\mu,\varepsilon,\lambda}(x) = \mu\,\langle x\rangle_\lambda$ and $\psi_{\mu,\lambda} = \mathrm{e}^{F_{\mu,\lambda}}\psi$. Since ψ is rapidly decaying by Lemma 5.13, we obtain from (5.25) that

$$\mu^2\left(|x|^2\,\langle x\rangle_\lambda^{-2}\,\psi_{\mu,\lambda},\psi_{\mu,\lambda}\right) \le C\,\|\psi_{\mu,\lambda}\|_{L^2}^2 + 3\mu\,\|\psi_{\mu,\lambda}\|_{L^2}^2\,.$$

Taking also the limit $\lambda\to 0$, we arrive at

$$(\mu^2 - 3\mu)\,\|\psi_\mu\|_{L^2}^2 \le C\,\|\psi_\mu\|_{L^2}^2\,.$$

where $\psi_\mu(x) = \mathrm{e}^{\mu|x|}\psi(x)$. Since $\mu^2 - 3\mu\to\infty$ as $\mu\to\infty$, we conclude that $\psi = 0$. □

Corollary 5.15 (Nonexistence of standing waves with mild decay)
No solution (φ,ω) of (5.3) exists where $\varphi\in H^1(\mathbb{R}^3)\setminus\{0\}$ satisfies Assumption 5.2.

A Function spaces

> It was frequently necessary to raise my head in order to see better, and sometimes I had to work entirely be feel.
>
> Leonid Ivanovich Rogozov, *Self Operation*

Fix a positive integer d. For $x \in \mathbb{R}^d$, we use the notation of the *Japanese bracket* $\langle x \rangle = (1 + |x^2|)^{1/2}$. In this appendix, we recall the definitions of various function spaces on $\mathbb{R}^d$ and discuss some of their properties.

We start by introducing the Schwartz space consisting of smooth functions of which all derivatives are rapidly decaying. Its dual space is the space of tempered distributions which is such a large class of generalized functions that it encompasses all subsequently defined function spaces.

Definition A.1 (Schwartz functions and tempered distributions)
For a smooth function $f \in C^\infty(\mathbb{R}^d)$ and multi-indices $\alpha, \beta \in \mathbb{N}^d$, the *Schwartz seminorms* are given by

$$\rho_{\alpha,\beta}(f) \coloneqq \sup_{x \in \mathbb{R}^d} \left| x^\alpha \partial^\beta f(x) \right| .$$

The *Schwartz space* $\mathcal{S}(\mathbb{R}^d)$ is defined by

$$\mathcal{S}(\mathbb{R}^d) \coloneqq \left\{ f \in C^\infty(\mathbb{R}^d) : \rho_{\alpha,\beta}(f) < \infty \text{ for all } \alpha, \beta \in \mathbb{N}^d \right\} .$$

The Schwartz space with its family of seminorms is a complete metrizable locally convex space. Its dual space is called the space of *tempered distributions* $\mathcal{S}'(\mathbb{R}^d)$.

Harmonic analysis is a broad field in which powerful quantitative estimates applying to large classes of generic functions are obtained. The most fundamental tool in harmonic analysis is the Fourier transform.

Definition A.2 (Fourier transform)
For a Schwartz function $f \in \mathcal{S}(\mathbb{R}^d)$, its *Fourier transform* $\mathcal{F}f \colon \mathbb{R}^d \to \mathbb{C}$ is given by

$$\mathcal{F}f(\xi) \coloneqq (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x) \, \mathrm{d}x, \quad \xi \in \mathbb{R}^d.$$

We also use the common notation $\hat{f}$ for $\mathcal{F}f$. The Fourier transform is an isomorphic mapping from $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{S}(\mathbb{R}^d)$ and its inverse is given by

$$\mathcal{F}^{-1}f(\xi) = \mathcal{F}f(-\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\xi} f(x) \, \mathrm{d}x, \quad \xi \in \mathbb{R}^d.$$

The operators $\mathcal{F}$ and $\mathcal{F}^{-1}$ extend to operators from $\mathcal{S}'(\mathbb{R}^d)$ onto $\mathcal{S}'(\mathbb{R}^d)$.

We remark that there are commonly used alternative definitions of the Fourier transform which differ in the choice of scaling factors from the one given here. With the choice above, $\mathcal{F}$ is an isometry on $L^2(\mathbb{R}^d)$. Moreover, for all $f, g \in \mathcal{S}(\mathbb{R}^d)$ the Fourier transform of a convolution is given by the formula

$$\mathcal{F}(f * g) = (2\pi)^{d/2} (\mathcal{F}f)(\mathcal{F}g)$$

and for every multi-index $\alpha \in \mathbb{N}_0^d$ we obtain the formula

$$\mathcal{F}(\partial^\alpha f)(\xi) = \mathrm{i}^{|\alpha|} \prod_{k=1}^{d} \xi_k^{\alpha_k} \mathcal{F}f(\xi) \quad \text{for every } \xi \in \mathbb{R}^d.$$

Definition A.2 coincides with the definitions given in the textbooks [Tri83], [Tri95] and [AF03]. The books [BL76], [Gra14a], [Gra14b] use different conventions.

Smoothness of a function corresponds to decay of its Fourier transform. This fact motivates the following definition of fractional order Sobolev spaces.

Definition A.3 (Fractional Sobolev spaces)
Let $s \in \mathbb{R}$ and $p \in (1, \infty)$. For $f \in \mathcal{S}'(\mathbb{R}^d)$, the fractional Sobolev norm is defined as

$$\|f\|_{H^{s,p}} := \left\| \mathcal{F}^{-1} \langle \cdot \rangle^s \mathcal{F}f \right\|_{L^p}.$$

The *inhomogeneous Sobolev space* $H^{s,p}(\mathbb{R}^d)$ is defined by

$$H^{s,p}(\mathbb{R}^d) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{H^{s,p}} < \infty \right\}.$$

If $p = 2$, we use the shorter notation $H^s(\mathbb{R}^d) := H^{s,2}(\mathbb{R}^d)$.

The space $H^{0,p}(\mathbb{R}^d)$ coincides with $L^p(\mathbb{R}^d)$. For any positive integer k, the space $H^{k,p}(\mathbb{R}^d)$ coincides with the classical Sobolev space $W^{k,p}(\mathbb{R}^d)$ of all functions which are k-times weakly differentiable with derivatives belonging to $L^p(\mathbb{R}^d)$, see [Tri95, Theorem 2.3.3 b)].

In the next lemma, we show how standard mollifiers, i.e., convolution with smooth functions, can be used to approximate functions in $H^s(\mathbb{R}^d)$ by more regular functions. The point of the proof is to work out the relation between the speed of approximation and the regularity of the metric in which the approximation is measured.

Lemma A.4 (Mollifier and fractional Sobolev spaces)
Let $s, r \in \mathbb{R}$ and $\varepsilon \in (0, 1)$. Let $\eta \in \mathcal{S}(\mathbb{R}^d)$ be a Schwartz function with $\int_{\mathbb{R}^3} \eta(x)\,\mathrm{d}x = 1$. Define $\eta_\varepsilon := \varepsilon^{-d} \eta(\cdot/\varepsilon)$. For every $f \in H^s(\mathbb{R}^d)$, we define $f^\varepsilon := \eta_\varepsilon * f$.

(1) There exists a constant $C_r > 0$ such that

$$\|f^\varepsilon\|_{H^{s+r}} \leq C_r \varepsilon^{-r} \|f\|_{H^s}.$$

(2) The regularized functions f^ε converge to f in $H^{s-r}(\mathbb{R}^d)$ with order $o(\varepsilon^r)$, i.e.,

$$\varepsilon^{-r} \|f - f^\varepsilon\|_{H^{s-r}} \to 0 \quad \text{as } \varepsilon \to 0.$$

This convergence is uniform on compact subsets of $H^{s-r}(\mathbb{R}^d)$.

Proof. In this proof, we use the simple estimates

$$\langle \xi/\varepsilon \rangle^r \leq \varepsilon^{-r} \langle \xi \rangle^r , \qquad \varepsilon^{-r} \langle \xi \rangle^{-r} \leq \langle \varepsilon\xi \rangle^{-r} ,$$

and the identity $\widehat{\eta_\varepsilon}(\xi) = \hat{\eta}(\varepsilon\xi)$ for all $\xi \in \mathbb{R}^d$.

First, let $f \in H^s(\mathbb{R}^d) \cap \mathcal{S}(\mathbb{R}^d)$. We compute

$$\begin{aligned} \|f^\varepsilon\|_{H^{s+r}}^2 &= \int_{\mathbb{R}^d} \langle \xi \rangle^{2(s+r)} \left|\widehat{f^\varepsilon}(\xi)\right|^2 \mathrm{d}\xi \\ &= (2\pi)^d \int_{\mathbb{R}^d} \langle \xi \rangle^{2r} \left|\widehat{\eta_\varepsilon}(\xi)\right|^2 \langle \xi \rangle^{2s} \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi \\ &\leq (2\pi)^d \left\|\langle \cdot \rangle^r \widehat{\eta_\varepsilon}\right\|_{L^\infty}^2 \|f\|_{H^s}^2 . \end{aligned}$$

Hence, the first assertion follows from the estimate

$$\left\|\langle \cdot \rangle^r \widehat{\eta_\varepsilon}\right\|_{L^\infty} = \sup_{\xi \in \mathbb{R}^d} \left|\langle \xi \rangle^r \hat{\eta}(\varepsilon\xi)\right| = \sup_{\zeta \in \mathbb{R}^d} \left|\langle \zeta/\varepsilon \rangle^r \hat{\eta}(\zeta)\right| \leq C_r (2\pi)^{-d/2} \varepsilon^{-r},$$

where we set $C_r \coloneqq (2\pi)^{d/2} \left\|\langle \cdot \rangle^r \hat{\eta}\right\|_\infty$.

Next, let $R > 0$. By splitting the domain $\mathbb{R}^d$ into a ball of radius R and its complement, we compute

$$\begin{aligned} \varepsilon^{-2r} \|f - f^\varepsilon\|_{H^{s-r}}^2 &= \int_{\mathbb{R}^d} \varepsilon^{-2r} \langle \xi \rangle^{2(s-r)} \left|\mathcal{F}(f - f^\varepsilon)(\xi)\right|^2 \mathrm{d}\xi \\ &\leq \int_{\mathbb{R}^d} \langle \varepsilon\xi \rangle^{-2r} \left|\mathbb{1} - (2\pi)^{d/2}\hat{\eta}(\varepsilon\xi)\right|^2 \langle \xi \rangle^{2s} \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi \\ &\leq \left\|\langle \cdot \rangle^{-r} (\mathbb{1} - (2\pi)^{d/2}\hat{\eta})\right\|_{L^\infty(\{|\xi| < \varepsilon R\})}^2 \|f\|_{H^s}^2 \\ &\qquad + \left\|\langle \cdot \rangle^{-r} (\mathbb{1} - (2\pi)^{d/2}\hat{\eta})\right\|_{L^\infty}^2 \int_{\{|\xi| \geq R\}} \langle \xi \rangle^{2s} \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi. \end{aligned}$$

Let $\delta > 0$. Choose $R > 0$ so large that the second summand is less than δ. Since $(2\pi)^{d/2}\hat{\eta}(0) = 1$ and $\hat{\eta}$ is continuous, there exists $\tilde{\varepsilon} \in (0, 1)$ such that

$$\left\|\langle \cdot \rangle^{-r} (\mathbb{1} - (2\pi)^{d/2}\hat{\eta})\right\|_{L^\infty(\{|\xi| < \varepsilon R\})}^2 \|f\|_{H^s}^2 < \delta$$

for all $\varepsilon \in (0, \tilde{\varepsilon})$. This proves the second assertion. The addendum follows from a standard covering argument in which we use that

$$\|f^\varepsilon - g^\varepsilon\|_{H^{s-r}} = \|(f - g)^\varepsilon\|_{H^{s-r}} \lesssim \|f - g\|_{H^{s-r}}$$

for any $g \in H^{s-r}$ due to part (1). □

For some applications, Definition A.3 does not provide a fine enough description of a function. In such cases, a more refined analysis can be based on decomposing a function into blocks in the frequency domain. From this decomposition, one can proceed in two different ways. Either one can first take the L^p-norm of each block and then study a weighted sum in ℓ^r, or the other way round. This two approaches lead to Besov- and Triebel–Lizorkin spaces, respectively.

Definition A.5 (Littlewood–Paley decomposition)
Let $\psi \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ be a positive function which is supported in the ball $\{\xi : |\xi| \le 2\}$ and which satisfies $\psi(\xi) = 1$ for all ξ with $|\xi| \le 1$. Define

$$\varphi(\xi) \coloneqq \psi(\xi/2) - \psi(\xi), \quad \text{for all } \xi \in \mathbb{R}^d$$

and suppose that $\varphi \ge 0$. For every $j \in \mathbb{Z}$, we define the dilation $\varphi_j \coloneqq \varphi(2^{-j+1}\cdot)$. By construction, the function φ_j is supported in the annulus $\{\xi : 2^{j-1} \le |\xi| \le 2^{j+1}\}$. For $u \in \mathcal{S}'(\mathbb{R}^d)$, we define

$$\Delta_0(u) \coloneqq \mathcal{F}^{-1}\psi\mathcal{F}u$$

and

$$\Delta_j(u) \coloneqq \mathcal{F}^{-1}\varphi_j\mathcal{F}u$$

for every $j \in \mathbb{N}$. The formal identity

$$I = \sum_{j=0}^{\infty} \Delta_j$$

is called *Littlewood–Paley decomposition* .

Definition A.6 (Besov spaces)
Let $s \in \mathbb{R}$ and $p, r \in (1, \infty)$. For $f \in \mathcal{S}'(\mathbb{R}^d)$, the Besov norm is defined as

$$\|f\|_{B^s_{p,r}} \coloneqq \left(\sum_{j=0}^{\infty} \left(2^{js} \|\Delta_j(f)\|_{L^p}\right)^r \right)^{1/r}.$$

The *inhomogeneous Besov space* $B^s_{p,r}(\mathbb{R}^d)$ is defined by

$$B^s_{p,r}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{B^s_{p,r}} < \infty \right\}.$$

Definition A.7 (Triebel–Lizorkin spaces)
Let $s \in \mathbb{R}$ and $p, r \in (1, \infty)$. For $f \in \mathcal{S}'(\mathbb{R}^d)$, the Triebel–Lizorkin norm is defined as

$$\|f\|_{F^s_{p,r}} \coloneqq \left\| \left(\sum_{j=0}^{\infty} \left(2^{js} |\Delta_j(f)|\right)^r \right)^{1/r} \right\|_{L^p}.$$

The *inhomogeneous Triebel–Lizorkin space* $F^s_{p,r}(\mathbb{R}^d)$ is defined by

$$F^s_{p,r}(\mathbb{R}^d) \coloneqq \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{F^s_{p,r}} < \infty \right\}.$$

Theorem A.8 (Equivalent norms)
Let $s \in \mathbb{R}$ and $p \in (1, \infty)$. We have $H^{s,p}(\mathbb{R}^d) \cong F^s_{p,2}(\mathbb{R}^d)$.

References in the literature. See [Gra14b, Theorem 1.3.6], [Tri83, Theorem 2.5.6 (i)] or [Tri95, Theorem 2.3.3 a)]. See also [AF03, § 7.57, §§ 7.62–7.65], where the Sobolev spaces $H^{s,p}$ are defined via complex interpolation as in Theorem A.10 (2). □

The next theorem describes the relations between Besov- and Triebel–Lizorkin spaces with different regularity and integrability parameters. In the first two parts, the regularity of the spaces is the same. The last two parts state Sobolev-type embeddings which generalize the observation that integrability properties of the derivative of a function imply improved integrability properties of the function itself. Through Theorem A.8, the classical Sobolev embeddings for the spaces $W^{k,p}(\mathbb{R}^d)$, see [AF03, Theorem 4.12], are included in the statements below.

Theorem A.9 (Embeddings)
Let $s, t \in \mathbb{R}$, $p, q \in (1, \infty)$ and $r, r_1, r_2 \in [1, \infty]$.

(1) If $t > s$ and $r_1 \leq r_2$, then

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow B^t_{p,\infty}(\mathbb{R}^d) \hookrightarrow B^s_{p,r_1}(\mathbb{R}^d) \hookrightarrow B^s_{p,r_2}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$$

and

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow F^t_{p,\infty}(\mathbb{R}^d) \hookrightarrow F^s_{p,r_1}(\mathbb{R}^d) \hookrightarrow F^s_{p,r_2}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d).$$

(2) If $1 < r \leq p < \infty$, then

$$B^s_{p,r}(\mathbb{R}^d) \hookrightarrow F^s_{p,r}(\mathbb{R}^d) \hookrightarrow B^s_{p,p}(\mathbb{R}^d).$$

If $1 < p \leq r < \infty$, then

$$B^s_{p,p}(\mathbb{R}^d) \hookrightarrow F^s_{p,r}(\mathbb{R}^d) \hookrightarrow B^s_{p,r}(\mathbb{R}^d).$$

(3) If $t \leq s$, $p \leq q$ and $s - \frac{d}{p} = t - \frac{d}{q}$, then

$$B^s_{p,r}(\mathbb{R}^d) \hookrightarrow B^t_{q,r}(\mathbb{R}^d)$$

and if additionally $1 < r < \infty$, then

$$F^s_{p,r}(\mathbb{R}^d) \hookrightarrow F^t_{q,r}(\mathbb{R}^d).$$

(4) Define $\alpha = s - \frac{d}{p}$. If $\alpha > 0$ and $\alpha \notin \mathbb{N}$, then

$$B^s_{p,r}(\mathbb{R}^d) \hookrightarrow C^\alpha(\mathbb{R}^d)$$

and if additionally $1 < r < \infty$, then

$$F^s_{p,r}(\mathbb{R}^d) \hookrightarrow C^\alpha(\mathbb{R}^d).$$

References in the literature.

(1) [Tri83, Proposition 2.3.2/2 (i), (ii)] or [Tri95, Theorem 2.3.3 c)]

(2) [Tri83, Proposition 2.3.2/2 (iii)] or [Tri95, Theorem 2.3.3 d)]

(3) [Tri83, Theorem 2.7.1 (i), (ii)] or [Tri95, Theorem 2.8.1 a), b)]

(4) [Tri95, Theorem 2.8.1 c), d), e)] □

Theorem A.10 (Interpolation)
Let $s_0, s_1 \in \mathbb{R}$, $p_0, p_1 \in (1, \infty)$, $r_0, r_1 \in [1, \infty]$ and $\theta \in (0, 1)$. Define

$$s := (1-\theta)s_0 + \theta s_1, \quad \tfrac{1}{p} := \tfrac{1-\theta}{p_0} + \tfrac{\theta}{p_1}, \quad \tfrac{1}{r} := \tfrac{1-\theta}{r_0} + \tfrac{\theta}{r_1}.$$

(1) The complex interpolation method yields

$$B^s_{p,r} = \left[B^{s_0}_{p_0,r_0}, B^{s_1}_{p_1,r_1}\right]_\theta.$$

(2) If $1 < r_0, r_1 < \infty$, then the complex interpolation method yields

$$F^s_{p,r} = \left[F^{s_0}_{p_0,r_0}, F^{s_1}_{p_1,r_1}\right]_\theta.$$

References in the literature.

(1) See [BL76, Theorem 6.4.5 (6)] or [Tri95, Theorem 2.4.1 d)]. See [Tri83, Theorem 2.4.7 (i) and Remark 2.4.7/2] for a different method yielding the same result.

(2) See [Tri95, Theorem 2.4.2 d)]. See [Tri83, Theorem 2.4.7 (ii) and Remark 2.4.7/2] for a different method yielding the same result. □

In the study of nonlinear equations, one often has to deal with products of functions. By the classical product or Leibniz rule, the derivative of a product decomposes into a sum of two products in which the derivative either acts only on the first or only on the second function. The next theorem states estimates which operationalize this principle in fractional Sobolev spaces.

Theorem A.11 (Fractional Leibniz rule)
Let $s \in [0, \infty)$, $r \in [1, \infty]$ and $p_1, p_2, q_1, q_2 \in (1, \infty]$ such that $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1}$ and $\frac{1}{r} = \frac{1}{p_2} + \frac{1}{q_2}$. Let $f \in H^{s,p_1}(\mathbb{R}^d) \cap L^{p_2}(\mathbb{R}^d)$ and $g \in L^{q_1}(\mathbb{R}^d) \cap H^{s,q_2}(\mathbb{R}^d)$. Then the product fg belongs to $H^{s,r}(\mathbb{R}^d)$ and satisfies the estimate

$$\|fg\|_{H^{s,r}} \lesssim \|f\|_{H^{s,p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|g\|_{H^{s,q_2}}.$$

References in the literature.
The case $r = p_1 = q_2$ and $q_1 = p_2 = \infty$ is proved in [KP88, Lemma X4]. See [Pon91, Lemma 2.2] for the extension to the case $p_1, p_2, q_1, q_2 \in (1, \infty)$. The general case is proved in [GO14, Theorem 1]. □

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