PERRON-FROBENIUS THEOREM FOR MULTI-HOMOGENEOUS MAPPINGS WITH APPLICATIONS TO NONNEGATIVE TENSORS

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Abstract

The Perron-Frobenius theorem for nonnegative matrices has been generalized to order-preserving homogeneous mappings on a cone and more recently to nonnegative tensors. We unify both approaches by introducing the concept of order-preserving multi-homogeneous mappings defined on a product of cones and their associated eigenvectors. By considering a vector valued version of the Hilbert metric, we prove several Perron-Frobenius type results for these mappings. We discuss the existence, the uniqueness and the maximality of nonnegative and positive eigenvectors of multi-homogeneous mappings. We prove a Collatz-Wielandt formula and a multi-linear Birkhoff-Hopf theorem. We study the convergence of the normalized iterates of multi-homogeneous mappings and prove convergence rates. Applications of our main results include the study of the $\ell^{p,q}$-singular vectors of nonnegative matrices, the ℓ^p-eigenvectors, rectangular $\ell^{p,q}$-singular vectors and $\ell^{p_1,\ldots,p_d}$-singular vectors of nonnegative tensors, the generalized DAD problem and the discrete generalized Schrödinger equation arising in multi-marginal optimal transport. We recast these problems in the multi-homogeneous framework and explain how our theorems can be used to refine, improve and offer a new point of view on previous results of the literature.

Zusammenfassung

Das Perron-Frobenius Theorem für nichtnegative Matrizen wurde auf homogene, ordnungserhaltende Abbildungen auf einem Kegel erweitert und, in letzter Zeit, auf nichtnegative Tensoren. Wir vereinheitlichen beide Ansätze, indem wir das Konzept der ordnungserhaltenden, multi-homogenen Abbildungen, die auf einem Produkt von Kegeln definiert sind, sowie deren zugehörige Eigenvektoren einführen. Indem wir eine vektorisierte Version der Hilbert-Metrik in Betracht ziehen, beweisen wir für diese Abbildungen mehrere Perron-Frobenius-Typ Ergebnisse. Wir diskutieren die Existenz, die Einzigartigkeit und die Maximalität nichtnegativer und positiver Eigenvektoren multihomogener Abbildungen. Wir beweisen eine Collatz-Wielandt-Formel und einen multi-linearen Birkhoff-Hopf Satz. Wir untersuchen die Konvergenz der normierten Iterationen von multi-homogenen Abbildungen und beweisen Konvergenzraten. Anwendungen unserer Hauptergebnisse umfassen die Untersuchung der $\ell^{p,q}$-singulären Vektoren nichtnegativer Matrizen, der ℓ^p-Eigenvektoren, rechteckiger $\ell^{p,q}$-singulärer Vektoren und $\ell^{p_1,\ldots,p_d}$-singulärer Vektoren nichtnegativer Tensoren, das generalisierte DAD-Problem und die diskrete generalisierte Schrödinger Gleichung, die im Zusammenhang mit multi-marginalem optimalen Transport auftritt. Wir übertragen diese Probleme in den multi-homogenen Rahmen und erklären, wie unsere Theoreme verwendet werden können, um frühere Ergebnisse der Literatur zu verfeinern, zu verbessern und eine neue Sichtweise auf diese zu bieten.

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Contents

1 Introduction

The Perron-Frobenius theorem [79, 33] implies that an irreducible matrix $A \in \mathbb{R}^{n\times n}$ with nonnegative entries has a unique eigenvector u up to scale in the nonnegative orthant $\mathbb{R}^n_+$. Moreover, u has positive entries, i.e. $u \in \mathbb{R}^n_{++}$, u corresponds to the spectral radius $\rho(A)$ of A and $\rho(A)$ is a geometrically simple eigenvalue. Furthermore, the spectral radius of A has a min-max and a max-min characterization given by the Collatz-Wielandt formula [27, 96]. If additionally, A is primitive then $\rho(A)$ is an algebraically simple eigenvalue of A and the power method [70] always converges towards an eigenvector of A corresponding to $\rho(A)$ when started in $\mathbb{R}^n_+ \setminus \{0\}$. The many implications of the Perron-Frobenius theorem has strongly motivated the study of nonnegative matrices and their applications in the past century [11, 8].

It is noted in [14] that, when considered as a mapping from $\mathbb{R}^n_+$ to $\mathbb{R}^n_+$, matrices with nonnegative entries are non-expansive with respect to the Hilbert (projective) metric and matrices with positive entries are strict contractions. The smallest Lipschitz constant of a linear mapping leaving a cone invariant is characterized in the Birkhoff-Hopf theorem [14, 47]. This observation allows to prove the Perron-Frobenius theorem using tools of fixed point theory. Remarkably, the Hilbert metric can be used to study the eigenvectors of nonlinear mappings as well. This has been noted in [22] where it is proved that a mapping leaving invariant a cone which is positively p-homogeneous and order-preserving, i.e. it preserves the partial ordering induced by the cone, has p as Lipschitz constant. In particular, if $p \leq 1$, then the mapping is non-expansive with respect to the Hilbert metric. Non-expansive mappings leaving invariant a cone in a Banach space were extensively studied in [75]. A recent exposition of the nonlinear Perron-Frobenius theory on cones in finite dimensional vector spaces can be found in the excellent monograph [60].

In the past two decades, the Perron-Frobenius theorem has been extended for tensors with nonnegative entries [64, 24, 26], where tensors are understood as matrices with two or more indexes. Likewise square matrices have eigenvectors and rectangular matrices have singular vectors, various eigenvector equations can be associated to a given tensor depending on its shape. For instance, if $T \in \mathbb{R}^{n\times n\times n}$ is a third order tensor with nonnegative entries, three different eigenvector equations involving T have been introduced in the literature and a Perron-Frobenius type theorem has been proved for each of these problems. The solutions to these equations are referred to as the ℓ^p-eigenvectors [64, 24], the rectangular $\ell^{p,q}$-singular vectors [26] and the $\ell^{p,q,r}$-singular vectors of T [31]. These problems, recalled with details in Section 4.2 below, are all particular cases of the following more general formulation: For $i = 1,\ldots,d$, let C_i be a cone in a finite dimensional real vector space, let $\mathcal{F}\colon C_1 \times \ldots \times C_d \to C_1 \times \ldots \times C_d$ be a mapping and consider the following system:

$$\begin{cases} \mathcal{F}(x_1,\ldots,x_d)_1 &= \lambda_1\, x_1 \\ \mathcal{F}(x_1,\ldots,x_d)_2 &= \lambda_2\, x_2 \\ & \vdots \\ \mathcal{F}(x_1,\ldots,x_d)_d &= \lambda_d\, x_d \end{cases} \tag{1.1}$$

with $(x_1,\ldots,x_d) \in (C_1\setminus\{0\})\times\ldots\times(C_d\setminus\{0\})$, $\lambda_1,\ldots,\lambda_d \geq 0$ and where $\mathcal{F}(x_1,\ldots,x_d)_i$

denotes the canonical projection of $\mathcal{F}(x_1, \ldots, x_d)$ onto C_i for $i = 1, \ldots, d$. Typically, the components of $\mathcal{F}$ are positively homogeneous in each variable but the homogeneity degree may differ from one variable to another and $\mathcal{F}$ preserves the ordering induced by the cone $\mathcal{C} = C_1 \times \ldots \times C_d$. When $d = 1$, Equation (1.1) reduces to the equation characterizing the eigenvectors of a homogeneous mapping on a cone [60].

This thesis is about proving a generalization of the Perron-Frobenius theorem for eigenvector equations of the type described in (1.1). This work includes a generalization of the Perron-Frobenius theorem, the Collatz-Wielandt formula, the Birkhoff-Hopf theorem and the power method for such problems. The advantage of these generalizations is that they unify results of the Perron-Frobenius theory of nonnegative tensors. Furthermore, as discussed in Section 11.2, the main results of this thesis (summarized in Section 11.1) allow to improve previous results in that they either hold for a larger class of problems or require weaker assumptions. Key ingredients for the study of the solutions to (1.1) are the introduction of multi-homogeneous mappings, defined in Section 3, together with the consideration of a vector valued version of the Hilbert projective metric, discussed in Section 6. The generality of (1.1) allows to derive results for other problems such as the study of the solution to the discrete generalized Schrödinger equation arising in multi-marginal optimal transport [10, 85], the study of quantum copulas [67] and the study of the solutions to the generalized DAD problem [16, 60].

This thesis is cumulative in that it builds on, and reproduce, results of a subset of the ten papers written during the graduation period, namely [36, 37, 38, 39, 40, 69, 72, 73, 74, 92]. In order to emphasize the scientific contribution of this thesis without disrupting the reading, we add the superscript $^\diamond$ when referring to one of these manuscripts.

2 Preliminaries

We recall here concepts of geometry, functional analysis, multi-linear algebra, and set basic notation that will be used throughout this thesis.

Let $(V, \|\cdot\|_V)$ be a finite dimensional real normed vector spaces. Let $S \subset V$ be a set. We denote by $\operatorname{int}(S)$, $\operatorname{cl}(S)$ and ∂S, the *interior*, the *closure* and respectively the *boundary* of S with respect to the norm topology. The *linear span* of S, the *affine hull* and the *conical hull* of S are respectively denoted by $\operatorname{span}(S)$, $\operatorname{aff}(S)$ and $\operatorname{cone}(S)$. If S is convex, we denote its *relative interior* by $\operatorname{relint}(S)$. For every $a \in \mathbb{R}$, we let $aS = \{as \mid s \in S\}$ and if $S' \subset V$, then we let $S + S' = \{s + s' \mid s \in S, s' \in S'\}$. The *dual space of* V is the real vector space of linear functions from V to $\mathbb{R}$ and is denoted V^*.

Let $U \subset V$ be an open set and let $(W, \|\cdot\|_W)$ be a finite dimensional real vector space. A mapping $f \colon U \to W$ is Fréchet differentiable at $x \in U$ if there exists a linear mapping $Df(x) \colon V \to W$ such that

$$\lim_{\|h\|_V \to 0} \frac{\|f(x+h) - f(x) - Df(x)h\|_W}{\|h\|_V} = 0.$$

In this case we say that f *is differentiable at* x and refer to $Df(x)$ as the derivative of f at x. Given a set $S \subset U$, if f is differentiable at every $x \in S$, then we say that f is differentiable on S and if f is differentiable on U, then we simply say that f is differentiable. In the particular case where $W = \mathbb{R}$, we denote the differential operator D by ∇, in other words, for $\Phi\colon U \to \mathbb{R}$ we denote the gradient of Φ by $\nabla\Phi(x)$.

Let $f\colon U \to W$ with $U \subset V$ and $p \in \mathbb{R}$. If for every $x \in U$ and $\alpha > 0$ such that $\alpha x \in U$ it holds $f(\alpha x) = \alpha^p f(x)$, then we say that f is *homogeneous of degree* p on U, or simply that f is p-homogeneous on U. We say that a mapping $f\colon U \to W$ is homogeneous, if there exists $p \in \mathbb{R}$ such that f is p-homogeneous on U. We note that, in the literature, such a mapping is sometimes referred to as a positively homogeneous mapping of degree p, however as we mostly work with homogeneous mappings on cones, this precision is unnecessary in our case. Note that every linear mapping is homogeneous of degree 1. Differentiable homogeneous mappings can be characterized in terms of their derivatives. This result is known as Euler's theorem for homogeneous functions.

Theorem 2.0.1 (Euler)**.** Let $U \subset V$ be an open convex subset of V, $p \in \mathbb{R}$ and $f\colon U \to W$ a differentiable mapping. Then, f is homogeneous of degree p if and only if $Df(x)x = pf(x)$ for every $x \in U$.

Proof. Suppose that f is p-homogeneous and let $x \in U$, then

$$Df(x)x = \lim_{t\to 0} \frac{f(x+tx) - f(x)}{t} = \lim_{t\to 0} f(x)\frac{(1+t)^p - 1}{t} = pf(x).$$

Suppose that $Df(x)x = pf(x)$ for all $x \in U$. Let $x \in U$ and $\alpha > 0$ such that $\alpha x \in U$. We prove that $f(\alpha x) = \alpha^p f(x)$. If $\alpha = 1$, there is nothing to prove. Suppose that $\alpha \neq 1$, the segment between x and αx is entirely contained in U as U is convex. Furthermore, as U is open, there exists $\epsilon > 0$ such that $(1+t)x \in U$ and $(\alpha+t)x \in U$ for all $t \in \mathbb{R}$ with $|t| < \epsilon$. If $\alpha < 1$ set $J = (\alpha - \epsilon, 1 + \epsilon)$ and if $\alpha > 1$ set $J = (1-\epsilon, \alpha+\epsilon)$. Define $\phi\colon J \to W$ as $\phi(t) = f(tx) - t^p f(x)$, then ϕ is differentiable and it holds $t\phi'(t) = Df(tx)(tx) - pt^p f(x) = p\phi(t)$. The only differentiable solution $\phi\colon J \to W$ to the differential equation $\phi'(t) - \frac{p}{t}\phi(t) = 0$ satisfying $\phi(1) = 0$ is $\phi(t) = 0$ for all $t \in J$. Hence, $\phi(\alpha) = 0$ and thus $f(\alpha x) = \alpha^p f(x)$. □

2.1 Vectors, matrices and tensors

We introduce here notation for the particular case where $V = \mathbb{R}^n, \mathbb{R}^{m\times n}, \mathbb{R}^{l\times m\times n}$, etc.

2.1.1 Vectors and matrices

For $x \in \mathbb{R}^n$, we denote by x_i the component of x in the canonical basis, so that $x = (x_1, \ldots, x_n)$. If a vector already has a subscript, then we simply append another subscript to denote its components, so that, for instance, if $(x_k)_{k=1}^{\infty} \subset \mathbb{R}^n$ is a sequence, then the i-th component of $x_k \in \mathbb{R}^n$ is denoted $x_{k,i}$. The constant vector of all ones is denoted by $\mathbf{1}$, i.e. $\mathbf{1} = (1, \dots, 1)^\top$. The set of vectors with nonnegative components is denote by $\mathbb{R}^n_+$ and its interior by $\mathbb{R}^n_{++}$, i.e. we let

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,|\, x_i \geq 0,\, i = 1, \dots, n\},$$
$$\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n \,|\, x_i > 0,\, i = 1, \dots, n\}.$$

We refer to the elements of $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$, as nonnegative and positive vectors, respectively. For $\alpha > 0$ and $x \in \mathbb{R}^n_+$, or $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n_{++}$, we let

$$x^\alpha = (x_1^\alpha, \dots, x_n^\alpha),$$

be the component-wise power of x. Furthermore, for $x, y \in \mathbb{R}^n$, we denote the component-wise product of the entries of x and y by $x \circ y$, i.e.

$$x \circ y = (x_1 y_1, \dots, x_n y_n).$$

Similarly, if $M \in \mathbb{R}^{m\times n}$, then we denote by $M_{i,j}$ the (i,j)-th component of M in the canonical basis of $\mathbb{R}^{m\times n}$. The set of nonnegative matrices is denote by $\mathbb{R}^{m\times n}_+$ and the set of positive matrices denoted by $\mathbb{R}^{m\times n}_{++}$, where

$$\mathbb{R}^{m\times n}_+ = \{M \in \mathbb{R}^{m\times n} \,|\, M_{i,j} \geq 0,\, i = 1, \dots, m, j = 1, \dots, n\},$$
$$\mathbb{R}^{m\times n}_{++} = \{M \in \mathbb{R}^{m\times n} \,|\, M_{i,j} > 0,\, i = 1, \dots, m, j = 1, \dots, n\}.$$

2.1.2 Third order tensors

For a third order tensor $T \in \mathbb{R}^{l\times m\times n}$, we denote by $T_{i,j,k}$ the components of T in the canonical basis of $\mathbb{R}^{l\times m\times n}$. Furthermore, we let

$$\mathbb{R}^{l\times m\times n}_+ = \{T \in \mathbb{R}^{l\times m\times n} \,|\, T_{i,j,k} \geq 0,\, i = 1, \dots, l, j = 1, \dots, m, k = 1, \dots, n\},$$
$$\mathbb{R}^{l\times m\times n}_{++} = \{T \in \mathbb{R}^{l\times m\times n} \,|\, T_{i,j,k} > 0,\, i = 1, \dots, l, j = 1, \dots, m, k = 1, \dots, n\},$$

be respectively the set of nonnegative and positive tensors. Given $T \in \mathbb{R}^{l\times m\times n}$, we denote by $f_T \colon \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ the multi-linear form induced by T, i.e.

$$f_T(x, y, z) = \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} T_{i,j,k} x_i y_j z_k. \tag{2.1}$$

For $x \in \mathbb{R}^l, y \in \mathbb{R}^m, z \in \mathbb{R}^n$, we let $T(\cdot, y, z) \in \mathbb{R}^l$, $T(x, \cdot, z) \in \mathbb{R}^m$ and $T(x, y, \cdot) \in \mathbb{R}^n$ be defined as

$$T(\cdot, y, z)_i = \sum_{j=1}^{m}\sum_{k=1}^{n} T_{i,j,k} y_j z_k \qquad \forall i = 1, \dots, l,$$
$$T(x, \cdot, z)_j = \sum_{i=1}^{l}\sum_{k=1}^{n} T_{i,j,k} x_i z_k \qquad \forall j = 1, \dots, m,$$
$$T(x, y, \cdot)_k = \sum_{i=1}^{l}\sum_{j=1}^{m} T_{i,j,k} x_i y_j \qquad \forall k = 1, \dots, n.$$

In particular, note that with this notation it holds

$$\nabla f_T(x,y,z) = \big(T(\cdot,y,z), T(x,\cdot,z), T(x,y,\cdot)\big).$$

The tensor $T \in \mathbb{R}^{n\times n\times n}$ is said to be a *symmetric tensor* if for every $i,j,k = 1,\ldots,n$ it holds

$$T_{i,j,k} = T_{i,k,j} = T_{j,i,k} = T_{k,i,j} = T_{j,k,i} = T_{k,j,i}.$$

If $T \in \mathbb{R}^{n\times n\times n}$ is symmetric, then for every $x \in \mathbb{R}^n$, it holds

$$\nabla f_T(x,x,x) = 3T(\cdot,x,x) = 3T(x,\cdot,x) = 3T(x,x,\cdot).$$

Furthermore, note that if $T \in \mathbb{R}^{n\times n\times n}$ is any tensor (not necessarily symmetric) and $\hat{T} \in \mathbb{R}^{n\times n\times n}$ is defined as

$$\hat{T}_{i,j,k} = \tfrac{1}{6}(T_{i,j,k} + T_{i,k,j} + T_{j,i,k} + T_{k,i,j} + T_{j,k,i} + T_{k,j,i}) \qquad \forall i,j,k = 1,\ldots,n, \quad (2.2)$$

then $\hat{T}$ is symmetric and it holds

$$f_T(x,x,x) = f_{\hat{T}}(x,x,x) \qquad \forall x \in \mathbb{R}^n.$$

If $T \in \mathbb{R}^{m\times n\times n}$, then T is said to be a *partially symmetric tensor* if for every $i = 1,\ldots,m$ and $j,k = 1,\ldots,n$, it holds $T_{i,j,k} = T_{i,k,j}$. If $T \in \mathbb{R}^{m\times n\times n}$ is partially symmetric, then for every $x \in \mathbb{R}^m, y \in \mathbb{R}^n$, it holds

$$\nabla f_T(x,y,y) = \big(T(\cdot,y,y), 2T(x,\cdot,y)\big) = \big(T(\cdot,y,y), 2T(x,y,\cdot)\big).$$

Furthermore, note that if $T \in \mathbb{R}^{m\times n\times n}$ is any tensor (not necessarily partially symmetric) and $\hat{T} \in \mathbb{R}^{m\times n\times n}$ is defined as

$$\hat{T}_{i,j,k} = \tfrac{1}{2}(T_{i,j,k} + T_{i,k,j}) \qquad \forall i = 1,\ldots,m,\ j,k = 1,\ldots,n, \quad (2.3)$$

then it holds

$$f_T(x,y,y) = f_{\hat{T}}(x,y,y) \qquad \forall x \in \mathbb{R}^m, y \in \mathbb{R}^n.$$

2.1.3 Tensors of any order

The above notation can be extended to tensors of any order. Namely, for $T \in \mathbb{R}^{n_1\times\ldots\times n_d}$, we say that T is a d-th order tensor and denote by $T_{j_1,\ldots,j_d}$ the components of T in the canonical basis. When dealing with tensors of general order, it is convenient to consider the index sets

$$[n_i] = \{1,\ldots,n_i\}, \qquad \forall i = 1,\ldots,d.$$

We let

$$\mathbb{R}_{+}^{n_1\times\ldots\times n_d} = \big\{T \in \mathbb{R}^{n_1\times\ldots\times n_d} \,\big|\, T_{j_1,\ldots,j_d} \geq 0, \forall j_1 \in [n_1],\ldots,j_d \in [n_d]\big\},$$

$$\mathbb{R}_{++}^{n_1\times\ldots\times n_d} = \big\{T \in \mathbb{R}^{n_1\times\ldots\times n_d} \,\big|\, T_{j_1,\ldots,j_d} > 0, \forall j_1 \in [n_1],\ldots,j_d \in [n_d]\big\}.$$

Given a tensor $T \in \mathbb{R}^{n_1 \times \ldots \times n_d}$, the multi-linear form induced by T is the function $f_T \colon \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_d} \to \mathbb{R}$ defined for every $(x_1, \ldots, x_d) \in \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_d}$ as

$$f_T(x_1, \ldots, x_d) = \sum_{j_1 \in [n_1], \ldots, j_d \in [n_d]} T_{j_1, \ldots, j_d} x_{1,j_1} \cdots x_{d,j_d}.$$

With this notation, if $e_1, \ldots, e_{n_i}$ is the canonical basis of $\mathbb{R}^{n_i}$, then $T_{j_1,\ldots,j_d} = f_T(e_{j_1}, \ldots, e_{j_d})$. For $(x_1, \ldots, x_d) \in \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_d}$ we let $T(\cdot, x_2, \ldots, x_d) \in \mathbb{R}^{n_1}$ be defined as

$$T(\cdot, x_2, \ldots, x_d)_{j_1} = \sum_{j_2 \in [n_2], \ldots, j_d \in [n_d]} T_{j_1, \ldots, j_d} x_{2,j_2} \cdots x_{d,j_d} \qquad \forall j_1 \in [n_1].$$

The vectors $T(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_d) \in \mathbb{R}^{n_i}$ for $i = 2, \ldots, d$ are defined analogously. In particular, we then have

$$\nabla f_T(x_1, \ldots, x_d) = \big(T(\cdot, x_2 \ldots, x_d), \ldots, T(x_1, \ldots, x_{d-1}, \cdot)\big).$$

A d-th order tensor $T \in \mathbb{R}^{n \times \ldots \times n}$ is said be to be *symmetric* if for every $j_1 \in [n_1], \ldots, j_d \in [n_d]$ it holds

$$T_{j_1,\ldots,j_d} = T_{j_{\sigma(1)},\ldots,j_{\sigma(d)}} \qquad \forall \sigma \in \mathfrak{S}(\{1, \ldots, d\}),$$

where $\mathfrak{S}(\{1, \ldots, d\})$ is the set of all permutations of $\{1, \ldots, d\}$, i.e.

$$\mathfrak{S}(\{1, \ldots, d\}) = \big\{\sigma \colon \{1, \ldots, d\} \to \{1, \ldots, d\} \,|\, \sigma \text{ is bijective}\big\}.$$

Similar to the third order case, it can be noted that if $T \in \mathbb{R}^{n \times \ldots \times n}$ is a d-th order tensor and $\hat{T} \in \mathbb{R}^{n \times \ldots \times n}$ is defined for all $j_1, \ldots, j_d$ as

$$\hat{T}_{j_1,\ldots,j_d} = \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}(\{1,\ldots,d\})} T_{j_{\sigma(1)},\ldots,j_{\sigma(d)}}, \tag{2.4}$$

then

$$f_T(x, \ldots, x) = f_{\hat{T}}(x, \ldots, x) \qquad \forall x \in \mathbb{R}^n.$$

If $T \in \mathbb{R}^{m \times \ldots \times m \times n \times \ldots \times n}$ where m appears a_1 times and n appears a_2 times with $a_1 + a_2 = d$, we say that T is *partially symmetric* if

$$T_{j_1,\ldots,j_d} = T_{j_{\sigma(1)},\ldots,j_{\sigma(a_1)},j_{\sigma'(a_1+1)},\ldots,j_{\sigma'(d)}}, \ \forall \sigma \in \mathfrak{S}(\{1, \ldots, a_1\}), \sigma' \in \mathfrak{S}(\{a_1 + 1, \ldots, d\}).$$

Furthermore, if $T \in \mathbb{R}^{m \times \ldots \times m \times n \times \ldots \times n}$ and $\hat{T} \in \mathbb{R}^{m \times \ldots \times m \times n \times \ldots \times n}$ is defined for all $j_1, \ldots, j_d$ as

$$\hat{T}_{j_1,\ldots,j_d} = \frac{1}{a_1! a_2!} \sum_{\substack{\sigma \in \mathfrak{S}(\{1,\ldots,a_1\}) \\ \sigma' \in \mathfrak{S}(\{a_1+1,\ldots,d\})}} T_{j_{\sigma(1)},\ldots,j_{\sigma(a_1)},j_{\sigma'(a_1+1)},\ldots,j_{\sigma'(d)}}, \tag{2.5}$$

then

$$f_T(x, \ldots, x, y, \ldots, y) = f_{\hat{T}}(x, \ldots, x, y, \ldots, y) \qquad \forall x \in \mathbb{R}^m, y \in \mathbb{R}^n.$$

There is a more general notion of partial symmetry of a tensor where the symmetry is considered among groups of indexes and we refer to [29] for a detailed discussion on symmetric and partially symmetric tensors.

2.2 Cones and partial orderings

Let V be a finite dimensional real normed vector spaces. A set $C \subset V$ is called a *cone*, if C is convex, $\alpha C \subset C$ for all $\alpha > 0$ and $C \cap (-C) = \{0\}$. The set C is a *closed cone*, if C is a closed subset of V. The cone C is a *solid cone* if it has non-empty interior. The cone $C \subset V$ induces a *partial ordering* on V defined as $x \preceq_C y$ if $y - x \in C$. If $x \preceq_C y$ and $x \neq y$, we write $x \precneq_C y$. If C is solid and $y - x \in \operatorname{int}(C)$, we write $x \prec_C y$. Let $x, y \in V$, then y *dominates* x if there exists $\alpha, \beta \in \mathbb{R}$ such that $\alpha y \preceq_C x \preceq_C \beta y$. If x dominates y and y dominates x then we say that x and y are *comparable* and write $x \sim_C y$. Note that if $x \sim_C y$, then there exists $\alpha, \beta > 0$ such that $\alpha y \preceq_C x \preceq_C \beta y$. The relation $\sim_C$ is an equivalence relation on C and the equivalence classes of $\sim_C$ are called the *parts* of C. By Lemma 1.2.2 of [60], we know that if C is a closed cone, then the parts of C are precisely the relative interior of the faces of C, where $F \subset C$ is a *face* if for all $x, y \in C$, the existence of $t \in (0, 1)$ such that $tx + (1 - t)y \in F$ implies that $x, y \in F$. Equivalently, by Exercise 2.16 of [11, Chapter 1], F is a face of C if for any $x \in F$ and $y \in C$, $y \preceq_C x$ implies $y \in F$. The sets $\{0\}$ and C are always faces of C and they are called *improper faces*. The other faces are said to be *proper*.

Given a norm $\|\cdot\|$ on V, we say that C is *normal* if there exists a constant $\delta > 0$ such that for all $x, y \in C$, $x \preceq_C y$ implies $\|x\| \leq \delta \|y\|$. The smallest such δ is called the *normality constant* of C. As V is finite dimensional, it follows from Lemma 1.2.5 of [60] that every closed cone is normal. In Lemma 2.1 of [83], it is proved that the normality constant of a cone is always greater or equal to 1. If $C = \mathbb{R}^n_+$ and for all $x, y \in \mathbb{R}^n$ such that $|x_i| \leq |y_i|$ for all $i = 1, \ldots, n$, it holds $\|x\| \leq \|y\|$, then we say that $\|\cdot\|$ is *monotonic with respect to* $\mathbb{R}^n_+$.

The *dual cone* of $C \subset V$ is defined as

$$C^* = \{w \in V^* \,|\, w(x) \geq 0, \forall x \in C\}.$$

Using the Hahn-Banach separation theorem 11.4 of [84], it can be shown that if C is a closed cone and $x \in C \setminus \{0\}$, then there exists $w \in V^*$ such that $w(x) > 0$. Indeed, the latter theorem implies the existence of a hyperplane $w \in V^*$ which (strongly) separates $-x \notin C$ and the convex set C. By changing the sign of w if necessary, we have $w(C) \subset [0, \infty)$ and thus $w \in C^*$. By using a similar argument, it is proved in [60, Lemma 1.2.1] that if C is a closed cone and $x, y \in V$, then $x \preceq_C y$ if and only if $w(x) \leq w(y)$ for all $w \in C^*$. In particular, it follows that if $x, y \in V$ satisfy $x \precneq_C y$ then there exists $w \in C^*$ such that $w(x) < w(y)$. While C^* may not be a cone in general, it follows from [60, Lemma 1.2.4] that if C is a solid closed cone, then C^* is a solid closed cone and if $w \in \operatorname{int}(C^*)$, then $w(x) > 0$ for all $x \in C \setminus \{0\}$. Furthermore, the set $\Sigma_w = \{x \in C \,|\, w(x) = 1\}$ is a compact convex subset of V. A cone C is *self-dual* if there exists an inner product $\langle \cdot\,, \cdot\rangle \colon V \times V \to \mathbb{R}$ such that $C^* = \{v \mapsto \langle v\,, x\rangle \,|\, x \in C\}$. When C is self-dual, we usually identify C^* with C.

Example 2.2.1. Let $V = \mathbb{R}^n$ and $C = \mathbb{R}^n_+$. Then, C is a solid closed cone and its interior is $\mathbb{R}^n_{++}$. Furthermore, it holds $x \preceq_{\mathbb{R}^n_+} y$ or $x \prec_{\mathbb{R}^n_+} y$, if $x_i \leq y_i$ or $x_i < y_i$ for all $i = 1, \ldots, n$, respectively. The cone $\mathbb{R}^n_+$ has 2^n parts and each part is characterized by a zero pattern. For instance $\{x \in \mathbb{R}^n_+ \,|\, x_1 = 0,\, x_2, \ldots, x_n > 0\}$ is a part of $\mathbb{R}^n_+$ as well as $\{x \in \mathbb{R}^n_+ \,|\, x_1 = x_2 = 0,\, x_3, \ldots, x_n > 0\}$ or $\{x \in \mathbb{R}^n_+ \,|\, x_2 = 0,\, x_1, x_3, \ldots, x_n > 0\}$. If we consider the usual Euclidean inner product on $\mathbb{R}^n$, then the cone $\mathbb{R}^n_+$ is self-dual. The ℓ^p-norm defined as

$$\|x\|_p = \Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p} \qquad \forall x \in \mathbb{R}^n,$$

is monotonic with respect to $\mathbb{R}^n_+$. Properties of norms on $\mathbb{R}^n$ which are monotonic with respect to $\mathbb{R}^n_+$ are discussed in [51] and we recall characterizations of such norms in the following.

Theorem 2.2.2. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, and for all $x \in \mathbb{R}^n$ let $|x| = (|x_1|, \ldots, |x_n|)$. Then, the following are equivalent:

a) $\|\cdot\|$ is monotonic with respect to $\mathbb{R}^n_+$, i.e. $\|x\| \leq \|y\|$ for all $x, y \in \mathbb{R}^n$ such that $|x| \preceq_{\mathbb{R}^n_+} |y|$.

b) For every $x \in \mathbb{R}^n$, it holds $\||x|\| = \|x\|$.

c) The dual norm induced by $\|\cdot\|$ is monotonic with respect to $\mathbb{R}^n_+$.

Proof. See Theorem 1 of [51]. □

Example 2.2.3. Let $V = \{M \in \mathbb{R}^{n \times n} \,|\, M^\top = M\}$ be the vector space of real symmetric matrices and let $C = \{M \in V \,|\, x^\top M x \geq 0, \forall x \in \mathbb{R}^n\}$ be the cone of positive semi-definite matrices. Then, $\mathrm{int}(C)$ is the cone of positive definite matrices, i.e. $\mathrm{int}(C) = \{M \in C \,|\, \det(M) > 0\}$. Furthermore, for $M, Q \in V$ it holds $M \preceq_C Q$ if and only if $x^\top M x \leq x^\top Q x$ for all $x \in \mathbb{R}^n$. If we consider the Hilbert-Schmidt inner-product $\langle \cdot \,,\, \cdot \rangle$ on V defined as

$$\langle M \,,\, Q \rangle = \sum_{i,j=1}^n M_{i,j} Q_{i,j},$$

then C is self-dual (see Example 2.24 in [19]). For any $p \in (1, \infty)$ the Schatten p-norm defined as

$$\|M\|_p = \Big(\sum_{i=1}^n \sigma_i(M)^p\Big)^{1/p},$$

where $\sigma_1(M), \ldots, \sigma_d(M)$ denote the singular values of M. The normality constant of $\|\cdot\|_p$ with respect to C is 1. As the singular values of a symmetric positive semi-definite matrix coincide with its eigenvalues, we note that for $M \in C$, it holds $\|M\|_p = \mathrm{Tr}(M^p)^{1/p}$ where, $\mathrm{Tr}(M^p)$ is the trace of $M^p \in C$ and M^α is the α-th power of M defined as $M^\alpha = UD^\alpha U^\top$ with $UDU^\top$ being a unitary diagonalization of M. We refer to [44] for the numerical aspect of M^α.

A cone $C \subset V$ is a *polyhedral cone* if there exists $w_1, \ldots, w_m \in V^*$ such that

$$C = \{x \in V \,|\, w_i(x) \geq 0, i = 1, \ldots, m\}.$$

A face F of a polyhedral cone C is called a *facet* if $\dim(\mathrm{span}(F)) = \dim(\mathrm{span}(C)) - 1$. Lemma 1.1.3 of [60] implies that if C is a polyhedral cone with N facets, then there exists N linear functionals $w_1, \ldots, w_N \in V^*$ such that $C = \{x \in V \,|\, w_i(x) \geq 0, i = 1, \ldots, N\}$ and each w_i corresponds to a unique facet of K. The functionals $w_1, \ldots, w_N$ are called the *facet defining functionals* of C. Theorem 2.5 of [11, Chapter 1] implies that C is polyhedral if and only if C^* is polyhedral. Nevertheless, there are polyhedral cones such that C and C^* don't have the same number of facets. Such an example is discussed in the lecture notes of K.C. Border [15] and we recall it here for completeness:

Example 2.2.4. Let $w_1, \ldots, w_5 \in \mathbb{R}^4$ be defined as $w_k = (1, k, k^2, k^3)^\top$ for $k = 1, \ldots, 5$. Let $v_1, \ldots, v_6$ be respectively defined as

$$\begin{pmatrix}-60\\47\\-12\\1\end{pmatrix}, \begin{pmatrix}-30\\31\\-10\\1\end{pmatrix}, \begin{pmatrix}-10\\17\\-8\\1\end{pmatrix}, \begin{pmatrix}6\\-11\\6\\-1\end{pmatrix}, \begin{pmatrix}12\\-19\\8\\-1\end{pmatrix}, \begin{pmatrix}20\\-29\\10\\-1\end{pmatrix}.$$

Then, the polyhedral cone $C = \{x \in \mathbb{R}^4 \,|\, \langle x\,,\, w_i\rangle \geq 0, i = 1, \ldots, 5\}$ is such that $C^* = \{w \in (\mathbb{R}^4)^* \,|\, w(v_i) \geq 0, i = 1, \ldots, 6\}$. Furthermore, C has 5 facets and C^* have 6 facets. A "p-norm" on $\mathbb{R}^4$ with normality constant 1 is given by

$$\|x\|_p = \Big(\sum_{k=1}^{5} |\langle w_k\,,\, x\rangle|^p\Big)^{1/p} \qquad \forall x \in \mathbb{R}^4,$$

where $\langle \cdot\,,\,\cdot \rangle$ is the Euclidean inner product on $\mathbb{R}^4$.

The parts of a polyhedral cone can be expressed in terms of its facet defining functionals. Let $w_1, \ldots, w_N \in V^*$ be the facet defining functionals of the polyhedral cone $C \subset V$ and for $x \in C$ and every part P of C, let

$$\mathcal{I}_x = \{i \,|\, w_i(x) > 0\} \quad \text{and} \quad \mathcal{I}(P) = \{i \,|\, w_i(x) > 0 \text{ for some } x \in P\}. \tag{2.6}$$

Then, Lemma 1.2.3 of [60] implies that $\mathcal{I}_x = \mathcal{I}_y$ if and only if $x \sim_C y$ and

$$P = \{x \in C \,|\, w_i(x) > 0 \text{ if and only if } i \in \mathcal{I}(P)\}.$$

Hence, a polyhedral cone has at most 2^N parts. Furthermore, there is a partial ordering on the set of parts defined for every parts P, Q of C as $P \trianglelefteq Q$ if there exists $x \in P$ and $y \in Q$ such that y dominates x or, equivalently, $P \trianglelefteq Q$ if $\mathcal{I}(P) \subset \mathcal{I}(Q)$.

The partial ordering induced by a cone can be used to meaningfully extend the definitions of infimum and supremum. Indeed, let $S \subset C$ where $C \subset V$ is a closed cone. Then $z \in C$ is an *upper bound* of S if $x \preceq_C z$ for all $x \in S$. If $z \in C$ is an upper bound of S and for every upper bound y of S, it holds $z \preceq_C y$, then z is the least upper bound, i.e. the *supremum*, of S and we denote it by $z = \sup\, S = \sup(S)$. Similarly $z' \in C$ is a lower bound of S, if $z' \preceq_C x$ for all $x \in S$ and z' is the *infimum* of S, denoted by $z' = \inf\, S = \inf(S)$, if $y \preceq_C z'$ for every lower bound y of S. A cone for which $\sup\{x,y\}$ and $\inf\{x,y\}$ exist for all $x,y \in C$ is called *minihedral*. Minihedral cones will play an important role in our computations because we will consider vector valued metrics. The ordering induced by a minihedral cone $C \subset V$ turns $(V, \prec_C)$ into a vector lattice [87, Chapter 2]. The *modulus* induced by a minihedral cone C, is defined as $|x| = \sup\{x, -x\} \in C$. In particular, it holds $|x| = x^+ + x^-$ and $x = x^+ - x^-$ with $x^+ = \sup\{x, 0\} \in C$ and $x^- = \sup\{-x, 0\} \in C$. It has been shown by Krein and Rutman that a solid cone is simplicial if and only if it is minihedral [57]. A solid polyhedral cone $C \subset V$ is *simplicial* if there exists $n = \dim(V)$ linearly independent vectors $v_1, \ldots, v_n \in V$ such that $C = \operatorname{cone}(\{v_1, \ldots, v_n\})$, i.e.

$$C = \Big\{ x \in V \,\Big|\, x = \sum_{i=1}^{n} \alpha_i v_i, \quad \text{with } \alpha_i \geq 0 \text{ for all } i = 1, \ldots, n \Big\}.$$

Simplicial cones C are essentially equal to $\mathbb{R}^n_+$ in the sense that if $C \subset V$ is a solid simplicial cone in the n dimensional real vector space V, then there exists a linear bijection $L\colon \mathbb{R}^n \to V$ such that $C = L(\mathbb{R}^n_+)$. In particular, as linear mappings between finite dimensional vector spaces are always continuous, this implies that results such as the sandwich theorem and the convergence of bounded monotonic sequences hold with respect to the partial ordering induced by C. Formally, if $(x_k)_{k=1}^\infty, (y_k)_{k=1}^\infty, (z_k)_{k=1}^\infty \subset C$ are sequences such that $x_k \preceq_C y_k \preceq_C z_k$ for all k and there exists $x \in C$ such that $\lim_{k\to\infty} y_k = \lim_{k\to\infty} z_k = x$, then $\lim_{k\to\infty} x_k = x$. Furthermore, if $(x_k)_{k=1}^\infty \subset C$ is a bounded sequence such that either $x_k \preceq_C x_{k+1}$ for all k or $x_{k+1} \preceq_C x_k$ for all k, then $(x_k)_{k=1}^\infty$ converges in C. The limit superior and inferior with respect to $\preceq_C$ are defined as follows: For every $(x_k)_{k=1}^\infty \subset C$, let

$$\limsup_{k\to\infty} x_k = \lim_{k\to\infty} \sup_{m\geq k} x_m \qquad \text{and} \qquad \liminf_{k\to\infty} x_k = \lim_{k\to\infty} \inf_{m\geq k} x_m.$$

In particular, note that if $(x_k)_{k=1}^\infty$ is bounded, then the above limits exist.

Example 2.2.5. Let $V = \mathbb{R}^{m\times n}$ and $C = \mathbb{R}^{m\times n}_+$. Then, C is simplicial and for every set $S \subset V$, it hold $\sup(S)_{i,j} = \sup\{M_{i,j} \mid M \in S\}$ and $\inf(S)_{i,j} = \sup\{M_{i,j} \mid M \in S\}$. Similarly, the modulus of $M \in V$ satisfies $|M|_{i,j} = |M_{i,j}|$, and the limit superior and inferior can be computed component-wise.

2.3 Order-preserving and monotonic mapping

Let $C \subset V$, $K \subset W$ be cones in the finite dimensional normed real vector spaces $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ and let $U \subset V$. A mapping $f\colon U \to W$ is said to be *order-preserving* (with respect to C and K) if $x \preceq_C y$ implies $f(x) \preceq_K f(y)$ for all $x, y \in U$. Analogously, f is *order-reversing* if $x \preceq_C y$ implies $f(y) \preceq_K f(x)$ for all $x, y \in U$. If C is a solid cone, the mapping f is *strongly order-preserving* if $x \precneqq_C y$ implies $f(x) \prec_K f(y)$ for all $x, y \in U$ and *strongly order-reversing* if $x \precneqq_C y$ implies $f(y) \prec_K f(x)$ for all $x, y \in U$. It is easily verified that the sum and composition of order-preserving mappings is again order-preserving. The sum of order-reversing mappings is order-reversing but the composition of two order-reversing mappings is order-preserving. Finally, we note that the composition of an order-preserving with an order-reversing mapping is order-reversing. A linear mapping $L\colon V \to V$ such that $L(C) \subset K$ is always order-preserving since $x - y \in C$ implies $L(x) - L(y) \in K$. A characterization of order-preserving mappings leaving C invariant is given by Theorem 1.3.1 in [60] where it is proved that a locally Lipschitz map $f\colon \operatorname{int}(C) \to C$ is order-preserving if, and only if, $Df(x)(C) \subset C$ for all $x \in C$ at which f is differentiable. Small changes in the proof of the latter theorem leads to the following result which characterizes order-preserving and order-reversing mappings between cones. We restrict ourselves to differentiable mappings for simplicity but note that a similar result holds if f is locally Lipschitz on $\operatorname{int}(C)$. Indeed, in this case, by Rademacher's theorem, f is differentiable almost everywhere on $\operatorname{int}(C)$.

Theorem 2.3.1. Let $C \subset V$ be a solid closed cone and $K \subset W$ a cone. Let $U \subset \operatorname{int}(C)$ be convex and open. If $f\colon U \to K$ is differentiable, then f is order-preserving, resp. order-reversing, with respect to C and K if and only if $Df(x)(C) \subset K$, resp. $-Df(x)(C) \subset K$, for all $x \in U$.

Proof. Let $x \in U$ and $y \in C$. There exists $\delta > 0$ such that $x + ty \in U$ for all $t \in (0, \delta)$. Furthermore, $x \preceq_C x + ty$ for $t > 0$ and

$$\lim_{t \to 0} \frac{f(x + ty) - f(x)}{t} = Df(x)y.$$

If f is order-preserving, then $f(x + ty) - f(x) \in K$ for all $t \in (0, \delta)$ and thus $Df(x)y \in K$. If f is order-reversing, then $f(x) - f(x + ty) \in K$ for all $t \in (0, \delta)$ and thus $-Df(x)y \in K$. If follows that $Df(x)(C) \subset K$ if f is order-preserving and $-Df(x)(C) \subset K$ if f is order-reversing. Now, we prove the reverse direction. Let $x, y \in U$ be such that $x \precsim_C y$. Define $\phi\colon [0, 1] \to K$ as $\phi(t) = f((1 - t)x + ty)$. Then ϕ is differentiable and

$$f(y) - f(x) = \phi(1) - \phi(0) = \int_0^1 \phi'(t)dt = \int_0^1 Df((1 - t)x + ty)(y - x)dt.$$

If $Df(u)(C) \subset K$ for all $u \in U$, then $f(y) - f(x) \in K$ and if $-Df(u)(C) \subset K$ for all $u \in U$, then $f(x) - f(y) \in K$. ☐

If $U \subset C$, $K = \mathbb{R}_+$ and $f\colon U \to K$ is order-preserving, i.e. $x \preceq_C y$ implies $f(x) \leq f(y)$ for all $x, y \in U$, then we say that f is a *monotonic function*. Note that if $\|\cdot\|$ is a norm on $\mathbb{R}^n$ which is monotonic with respect to $\mathbb{R}^n_+$, then $f(x) = \|x\|$ is a monotonic function with $U = \mathbb{R}^n_+$. We note however that there exist norms on $\mathbb{R}^n$ which induce monotonic functions but are not monotonic with respect to $\mathbb{R}^n_+$. An example is the operator norm discussed in Example 2.3.7. Monotonic functions can be used to build examples of order-preserving mappings as shown in the following example:

Example 2.3.2. Let $C \subset V$, $K \subset W$ be cones and let $U \subset C$. Let $f_1, \ldots, f_m\colon U \to [0, \infty)$ be monotonic functions, $v_1, \ldots, v_n \in K$ and $A \in \mathbb{R}^{m \times n}_+$ a nonnegative matrix. Then, the mapping $f\colon U \to K$ defined as

$$f(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{i,j} f_j(x) v_i \qquad \forall x \in U,$$

is order-preserving. If f_j is homogeneous of degree p for all j, then f is homogeneous of degree p. If $V = \mathbb{R}^n, W = \mathbb{R}^m$, $v_1, \ldots, v_m$ is the canonical basis of $\mathbb{R}^m$ and $f_j(x) = \langle x , e_j \rangle$ for all $j = 1, \ldots, n$, where $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{R}^n$, then $f(x) = Ax$. An order-reversing function can be obtained by assuming $f_j(x) > 0$ for $x \in U$ and replacing $f_j(x)$ by $f_j(x)^{-1}$ in the definition of f. If we let $C = \mathbb{R}^n_+$, $K = \mathbb{R}^m_+$ and $f_j(x) = x_j^{\alpha_j}$ for some $\alpha_j \geq 0$, then f is a posynomial mapping [18].

Next we show that if a mapping is homogeneous of degree 0 and either order-preserving or order-reversing, then it is constant on each part of the cone.

Lemma 2.3.3. Let $C \subset V$ and $K \subset W$ be cones. Suppose that $f\colon C \to K$ is homogeneous of degree 0 and either order-preserving or order-reversing. Then f is constant on each part of C and if f is continuous, then f is constant on C.

Proof. Let $x, y \in C$ be such that $x \sim_C y$. Then, there exists $\alpha, \beta > 0$ such that $\alpha x \preceq_C y \preceq_C \beta x$. As f is homogeneous of degree 0, we have $f(\alpha x) = f(\beta x) = f(x)$. It follows that if f is either order-preserving or order-reversing, we have $f(x) \preceq_C f(y) \preceq_C f(x)$ which implies that $f(y) = f(x)$. Now, suppose that f is continuous. Then, for every $\epsilon > 0$ we have $\epsilon x \sim_C x$ and by continuity of f, it follows that $f(x) = \lim_{\epsilon \to 0} f(\epsilon x) = f(0)$, which concludes the proof. ☐

We give examples of order-preserving homogeneous and monotonic functions which will be useful for later discussion.

Example 2.3.4. Let $V = \mathbb{R}^n$, $C = \mathbb{R}^n_+$ and $\alpha > 0$. The mapping $x \mapsto x^\alpha$ introduced in Section 2.1 is homogeneous of degree α and order-preserving. Note that the definition of $x \mapsto x^\alpha$ can be extended for $\alpha < 0$ if we restrict the domain to $\mathbb{R}^n_{++}$ and in this case, $x \mapsto x^\alpha$ is order-reversing.

A similar example exist for the cone of positive semi-definite matrices:

Example 2.3.5. Let V be the symmetric matrices in $\mathbb{R}^{n \times n}$ and C the cone of positive semi-definite matrices of Example 2.2.3. Let $\alpha > 0$ and define $f_\alpha\colon C \to C$ as

$$f_\alpha(M) = M^\alpha \qquad \forall M \in C,$$

Note that f_α is homogeneous of degree α. Furthermore, if $\alpha \leq 1$, then Theorem 1.4.1 of [60] implies that f_α is order-preserving. However, if $\alpha > 1$ then f_α is neither order-preserving nor order-reversing. Again, we can extend the definition of f_α for $\alpha < 0$ by restricting its domain to $\operatorname{int}(C)$, the positive definite matrices. Then, f_α is order-reversing for $\alpha \in [-1, 0)$ and neither order-preserving nor order-reversing for $\alpha < -1$.

In the next example, we recall that the spectral radius is a monotonic function with respect to the cone of nonnegative matrices.

Example 2.3.6. Let $V = \mathbb{R}^{n \times n}$ and $C = \mathbb{R}^{n \times n}_+$, define $f\colon C \to [0, \infty)$ as $f(M) = \rho(M)$ where $\rho(M)$ is the spectral radius of M. Then f is a monotonic function, i.e. $M \preceq_{\mathbb{R}^{d \times d}_+} Q$ implies $\rho(M) \leq \rho(Q)$. Furthermore, if $M \preceq_{\mathbb{R}^{d \times d}_+} Q$, $M \neq Q$ and $M + Q$ is irreducible, then $\rho(M) < \rho(Q)$. We refer to Corollary 1.5 of [11, Chapter 2] for a proof of these facts.

In the next example, we prove that the operator norm of nonnegative matrices induced by monotonic norms is a monotonic function.

Example 2.3.7. Let $V=\mathbb{R}^{m\times n}$, $C=\mathbb{R}^{m\times n}_+$ and let $\|\cdot\|_\alpha$, $\|\cdot\|_\beta$ be norms on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Suppose that $\|\cdot\|_\alpha$ is monotonic with respect to $\mathbb{R}^m_+$ and $\|\cdot\|_\beta$ is monotonic with respect to $\mathbb{R}^n_+$. Let $\|\cdot\|_{\beta\to\alpha}$ be the operator norm on V defined as

$$\|M\|_{\beta\to\alpha}=\max_{x\in\mathbb{R}^n\setminus\{0\}}\frac{\|Mx\|_\alpha}{\|x\|_\beta}\qquad\forall M\in\mathbb{R}^{m\times n}.$$

We prove that $f\colon\mathbb{R}^{m\times n}_+\to[0,\infty)$ defined as $f(M)=\|M\|_{\beta\to\alpha}$ is a monotonic function: First, note that for every $M\in\mathbb{R}^{m\times n}_+$ and $x\in\mathbb{R}^n$, by the triangle inequality, it holds $|Mx|\preceq_{\mathbb{R}^n_+}M|x|$. As $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are monotonic, by Theorem 2.2.2, for every $x\in\mathbb{R}^n$ and $y\in\mathbb{R}^m$ it holds $\|x\|_\beta=\||x|\|_\beta$ and $\|y\|_\alpha=\||y|\|_\alpha$. It follows that

$$\max_{x\in\mathbb{R}^n\setminus\{0\}}\frac{\|Mx\|_\alpha}{\|x\|_\beta}=\max_{x\in\mathbb{R}^n\setminus\{0\}}\frac{\||Mx|\|_\alpha}{\||x|\|_\beta}\le\max_{x\in\mathbb{R}^n\setminus\{0\}}\frac{\|M|x|\|_\alpha}{\||x|\|_\beta}=\max_{x\in\mathbb{R}^n_+\setminus\{0\}}\frac{\|Mx\|_\alpha}{\|x\|_\beta}$$

and thus

$$\|M\|_{\beta\to\alpha}=\max_{x\in\mathbb{R}^n_+\setminus\{0\}}\frac{\|Mx\|_\alpha}{\|x\|_\beta}\qquad\forall M\in\mathbb{R}^{m\times n}_+.\tag{2.7}$$

Now, suppose that $M,Q\in\mathbb{R}^{m\times n}_+$ satisfy $M\preceq_{\mathbb{R}^{m\times n}_+}Q$. Then for every $x\in\mathbb{R}^d_+$ it holds $Mx\preceq_{\mathbb{R}^m_+}Qx$ and thus $\|Mx\|_\alpha\le\|Qx\|_\alpha$. Hence, (2.7) implies that $\|M\|_{\beta\to\alpha}\le\|Q\|_{\beta\to\alpha}$ which proves the claim. We point out that while f is a monotonic function, it is not true in general that $\|\cdot\|_{\alpha\to\beta}$ is monotonic with respect to $\mathbb{R}^{m\times n}_+$. This can already be seen by considering $m=n\geq 2$ and $\|\cdot\|_\alpha=\|\cdot\|_\beta=\|\cdot\|_2$ where $\|\cdot\|_2$ is the ℓ^2-norm on $\mathbb{R}^n$. In this case, for any symmetric matrix $M\in\mathbb{R}^{n\times n}$, we have $\|M\|_{2\to2}=\rho(M)$. In particular, if $\|\cdot\|_{2\to2}$ is monotonic with respect to $\mathbb{R}^{n\times n}_+$, then we would have $\rho(|M|)=\rho(M)$ for all symmetric $M\in\mathbb{R}^{n\times n}$ which is absurd.

We conclude by giving examples of monotonic functions on the cone of positive definite matrices.

Example 2.3.8. Let V be the symmetric matrices in $\mathbb{R}^{n\times n}$ and C the cone of positive semi-definite matrices of Example 2.2.3. Let $x\in\mathbb{R}^n$, $\gamma\colon[0,1]\to\mathbb{R}^n$ a continuous curve and define $f,g,h\colon C\to[0,\infty)$ as

$$f(M)=\mathrm{Tr}(M),\quad g(M)=\langle x\,,\,Mx\rangle,\quad h(M)=\int_0^1\langle\gamma(t)\,,\,M\gamma(t)\rangle dt,$$

where $\mathrm{Tr}(M)$ denotes the trace of M and $\langle\cdot\,,\,\cdot\rangle$ is the Euclidean inner product on $\mathbb{R}^n$. Then, f,g and h are all monotonic functions.

3 Mappings on the product of cones

First we discuss products of cones. Then, we define multi-homogeneous mappings, give examples of such mappings and discuss their properties.

3.1 Products of cones and multi-normalizations

We start by introducing notation for the product of vector spaces. Let d be a positive integer and let $V_1, \ldots, V_d$ be finite dimensional vector spaces. Define $\mathcal{V} = V_1 \times \ldots \times V_d$. For $x \in \mathcal{V}$, we denote by x_i its canonical projection onto V_i so that $x = (x_1, \ldots, x_d)$ with $x_i \in V_i$ for all i. If $V_i = \mathbb{R}^{n_i}$ for $i = 1, \ldots, d$, then the components of $x_i \in \mathbb{R}^{n_i}$ are denoted by x_{i,j_i} for $j_i = 1, \ldots, n_i$. As the dual $\mathcal{V}^*$ of $\mathcal{V}$ is isomorphic to the product $V_1^* \times \ldots \times V_d^*$, we can use the same notation for elements in $\mathcal{V}^*$, i.e. for all $w \in \mathcal{V}^*$, we write $w = (w_1, \ldots, w_d)$ where $w_i \in V_i^*$ for $i = 1, \ldots, d$. In particular, note that $w(x) = w_1(x_1) + \ldots + w_d(x_d)$ for all $x \in \mathcal{V}$ and $w \in \mathcal{V}^*$. We also use this notation for sets, that is, if $\mathcal{S} \subset \mathcal{V}$ then we let $S_i = \{x_i \in V_i \,|\, x = (x_1, \ldots, x_d) \in \mathcal{S}\}$ so that $\mathcal{S} = S_1 \times \ldots \times S_d$.

Let $\mathcal{C} \subset \mathcal{V}$ be a cone, then $x \preceq_{\mathcal{C}} y$ if and only if $x_i \preceq_{C_i} y_i$ for all $i = 1, \ldots, d$. Furthermore, it holds $x \npreceq_{\mathcal{C}} y$ if and only if there exists $i \in \{1, \ldots, d\}$ such that $x_i \npreceq_{C_i} y_i$. Note that $\mathcal{C}$ is closed, solid, polyhedral respectively simplicial if, and only if, C_i is closed, solid, polyhedral, simplicial for $i = 1, \ldots, d$. Furthermore, if $\mathcal{P} \subset \mathcal{C}$ is a part of $\mathcal{C}$, then P_i is a part of C_i for all $i = 1, \ldots, d$ and if $\mathcal{Q} \subset \mathcal{C}$ is a face of $\mathcal{C}$, then Q_i is a face of C_i for all $i = 1, \ldots, d$. If $\mathcal{C}$ is polyhedral and C_i has N_i facets for $i = 1, \ldots, d$, then $\mathcal{C}$ has $\prod_{i=1}^d N_i$ facets. Furthermore, we note that $\mathcal{C}^* = C_1^* \times \ldots \times C_d^*$.

Finally, we introduce the following definition which allows to easily construct products of unit spheres and unit balls on $\mathcal{C}$.

Definition 3.1.1. Let $\mathcal{C} \subset \mathcal{V}$ be a cone. We say that $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d_+$ is a *multi-normalization* if it satisfies the following properties:

i) $\boldsymbol{\nu}$ is continuous,

ii) $\boldsymbol{\nu}$ is 1-homogeneous in each variable, i.e. for every $x \in \mathcal{C}$ and $\alpha_1, \ldots, \alpha_d \geq 0$ it holds $\boldsymbol{\nu}(\alpha_1 x_1, \ldots, \alpha_d x_d) = (\alpha_1 \boldsymbol{\nu}(x)_1, \ldots, \alpha_d \boldsymbol{\nu}(x)_d)$.

iii) For every $x \in \mathcal{C}$ and $i \in \{1, \ldots, d\}$ such that $x_i \neq 0$ it holds $\boldsymbol{\nu}(x)_i > 0$.

We say that $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d_+$ is a *monotonic multi-normalization* if $\boldsymbol{\nu}$ is order-preserving, i.e. $x \preceq_{\mathcal{C}} y$ implies $\boldsymbol{\nu}(x) \preceq_{\mathbb{R}^d_+} \boldsymbol{\nu}(y)$.

Perhaps, the most relevant example of multi-normalization on $\mathcal{C}$ is

$$\boldsymbol{\nu}(x) = (\|x_1\|_1, \ldots, \|x_d\|_d) \qquad \forall x \in \mathcal{C},$$

where $\|\cdot\|_i$ is a norm on C_i for $i = 1, \ldots, d$. In particular, if $\|\cdot\|_i$ is a monotonic function on C_i for all i, then $\boldsymbol{\nu}$ is a monotonic multi-normalization. Another example of monotonic multi-normalization on a solid closed cone is

$$\boldsymbol{\nu}(x) = (w_1(x_1), \ldots, w_d(x_d)) \qquad \forall x \in \mathcal{C},$$

where $w_i \in \operatorname{int}(C_i^*)$ for all $i = 1, \ldots, d$.

3.2 Multi-homogeneous mappings

Next, we formulate the definition of multi-homogeneous mappings and discuss their properties. Let d, d' be positive integers and let $V_1, \ldots, V_d, W_1, \ldots, W_{d'}$ be finite dimensional real vector spaces. Define $\mathcal{V} = V_1 \times \ldots \times V_d$ and $\mathcal{W} = W_1 \times \ldots \times W_{d'}$. Furthermore, let $\mathcal{U} \subset \mathcal{V}$ be a nonempty set.

Definition 3.2.1. We say that $\mathcal{F}\colon \mathcal{U} \to \mathcal{W}$ is *multi-homogeneous* if there exists a matrix $A \in \mathbb{R}^{d' \times d}$ such that for every $i = 1, \ldots, d'$ and $j = 1, \ldots, d$ the following condition is satisfied: For every $x \in \mathcal{U}$ and $\alpha > 0$ such that $\alpha x_j \in U_j$, it holds

$$\alpha^{A_{i,j}}\, \mathcal{F}(x)_i = \mathcal{F}(x_1, \ldots, x_{j-1}, \alpha\, x_j, x_{j+1}, \ldots, x_d)_i.$$

We refer to A as the *homogeneity matrix* of $\mathcal{F}$ and say that $\mathcal{F}$ is multi-homogeneous of degree A.

The notion of multi-homogeneous polynomial was already considered in [95] where the homogeneity matrix is restricted to have integer coefficients. If $d = d' = 1$, then the definition of multi-homogeneous mapping reduces to that of homogeneous mapping. We illustrate the definition with four examples that will be reused all along the thesis.

3.2.1 $\ell^{p,q}$-norm of a nonnegative matrix

For $p, q \in (1, \infty)$, let $\|\cdot\|_p$ be the ℓ^p-norm on $\mathbb{R}^m$ and let $\|\cdot\|_q$ be the ℓ^q-norm on $\mathbb{R}^n$. Furthermore, let $\langle \cdot\,, \cdot\rangle$ be the Euclidean inner product on $\mathbb{R}^m$. On $\mathbb{R}^{m\times n}$, consider the norm $\|\cdot\|_{p,q}$ defined as

$$\|Q\|_{p,q} = \max_{x\in\mathbb{R}^m\setminus\{0\}, y\in\mathbb{R}^n\setminus\{0\}} \frac{\langle x\,, Qy\rangle}{\|x\|_p\|y\|_q} \qquad \forall Q \in \mathbb{R}^{m\times n}.$$

We refer to $\|Q\|_{p,q}$ as the *$\ell^{p,q}$-norm of the matrix Q*. The $\ell^{p,q}$-norm is closely related to the matrix operator norm $\|\cdot\|_{\beta\to\alpha}$ discussed in Example 2.3.7. Indeed, if $p' = p/(p-1)$, then it holds $\|Q\|_{p,q} = \|Q\|_{q\to p'}$, i.e.

$$\|Q\|_{p,q} = \max_{y\in\mathbb{R}^n\setminus\{0\}} \frac{\|Qy\|_{p'}}{\|y\|_q} \qquad \forall Q \in \mathbb{R}^{m\times n}. \tag{3.1}$$

The latter follows from the Hölder inequality: Let $y \in \mathbb{R}^n \setminus \{0\}$ then with $z = |Qy|^{p'-2} \circ Qy$, we have

$$\frac{\|Qy\|_{p'}}{\|y\|_q} = \frac{\langle z\,, Qy\rangle}{\|z\|_p\|y\|_q} \leq \max_{x\neq 0} \frac{\langle x\,, Qy\rangle}{\|x\|_p\|y\|_q} \leq \frac{\|Qy\|_{p'}}{\|y\|_q}.$$

By taking the maximum over y in the above inequality, we get (3.1). The $\ell^{p,q}$-norm of a matrix is studied in [13, 31, 43, 90] and its computation is generally NP-hard [42, 89, 9]. Now, suppose that $M \in \mathbb{R}^{m\times n}_+$, then by the triangle inequality, for all x, y we have $|\langle x\,, My\rangle| \leq \langle |x|\,, M|y|\rangle$ and thus

$$\|M\|_{p,q} = \max_{x\in\mathbb{R}^m_+\setminus\{0\}, y\in\mathbb{R}^n_+\setminus\{0\}} \omega(x,y) \qquad \text{with} \qquad \omega(x,y) = \frac{\langle x\,, My\rangle}{\|x\|_p\|y\|_q},$$

i.e. the maximum is attained at a pair (x, y) in the product of cones $\mathcal{C} = \mathbb{R}^m_+ \times \mathbb{R}^n_+$. Now, note that for $x \in \mathbb{R}^m \setminus \{0\}$ and $y \in \mathbb{R}^n \setminus \{0\}$ it holds

$$\nabla\omega(x,y) = \frac{(My, M^\top x)}{\|x\|_p\|y\|_q} - \frac{\langle x\,,\, My\rangle}{\|x\|_p\|y\|_q}\Big(\frac{|x|^{p-2}\circ x}{\|x\|_p^p\|y\|_q}, \frac{|y|^{p-2}\circ y}{\|x\|_p\|y\|_q^q}\Big).$$

If $x \in \mathbb{R}^m_+ \setminus \{0\}$ and $y \in \mathbb{R}^n_+ \setminus \{0\}$, then the equation $\nabla\omega(x,y) = 0$, is equivalent to

$$\begin{cases} (My)^{1/(p-1)} = \lambda x \\ (M^\top x)^{1/(q-1)} = \theta y \end{cases} \tag{3.2}$$

with $\lambda = \|x\|_p^{-1}\omega(x,y)^{1/(p-1)}$ and $\theta = \|y\|_q^{-1}\omega(x,y)^{1/(q-1)}$. The left hand side of equation (3.2) defines the multi-homogeneous mapping $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ given by

$$\mathcal{F}(x,y) = \big(\,(My)^{1/(p-1)}\,,\,(M^\top x)^{1/(q-1)}\,\big) \qquad \forall (x,y) \in \mathcal{C}, \tag{3.3}$$

with homogeneity matrix

$$A = \begin{pmatrix} 0 & 1/(p-1) \\ 1/(q-1) & 0 \end{pmatrix}.$$

Furthermore, note that $\mathcal{F}$ is order-preserving.

3.2.2 $\ell^{p,q,r}$-norm of a third order nonnegative tensor

For $p, q, r \in (1, \infty)$, let $\|\cdot\|_p$ be the ℓ^p-norm on $\mathbb{R}^l$, $\|\cdot\|_q$ the ℓ^q-norm on $\mathbb{R}^m$ and $\|\cdot\|_r$ the ℓ^r-norm on $\mathbb{R}^n$. On $\mathbb{R}^{l\times m\times n}$, consider the norm

$$\|T\|_{p,q,r} = \max_{x\in\mathbb{R}^l\setminus\{0\}, y\in\mathbb{R}^m\setminus\{0\}, z\in\mathbb{R}^n\setminus\{0\}} \frac{f_T(x,y,z)}{\|x\|_p\|y\|_q\|z\|_r} \qquad \forall T \in \mathbb{R}^{l\times m\times n}, \tag{3.4}$$

where we recall that f_T is the multi-linear form induced by T defined in (2.1). We refer to $\|T\|_{p,q,r}$ as the *$\ell^{p,q,r}$-norm of the tensor T*. The $\ell^{p,q,r}$-norm of a tensor has been studied in [64, 31], [36, 40]$^\diamond$ and its computation is generally NP-hard [45]. Note that if $T \in \mathbb{R}^{l\times m\times n}_+$, then for every $x \in \mathbb{R}^l, y \in \mathbb{R}^m, z \in \mathbb{R}^n$ it holds $|f_T(x,y,z)| \leq f_T(|x|,|y|,|z|)$ and thus

$$\|T\|_{p,q,r} = \max_{x\in\mathbb{R}^l_+\setminus\{0\}, y\in\mathbb{R}^m_+\setminus\{0\}, z\in\mathbb{R}^n_+\setminus\{0\}} \omega(x,y,z) \quad \text{with} \quad \omega(x,y,z) = \frac{f_T(x,y,z)}{\|x\|_p\|y\|_q\|z\|_r},$$

i.e. the maximum is attained in the product of cones $\mathcal{C} = \mathbb{R}^l_+ \times \mathbb{R}^m_+ \times \mathbb{R}^n_+$. Now, note that for $x \in \mathbb{R}^l \setminus \{0\}, y \in \mathbb{R}^m \setminus \{0\}, z \in \mathbb{R}^n \setminus \{0\}$, it holds

$$\begin{aligned}
\big(\nabla\omega(x,y,z)\big)_1 &= \frac{T(\cdot,y,z)}{\|x\|_p\|y\|_q\|z\|_r} - \omega(x,y,z)\frac{|x|^{p-2}\circ x}{\|x\|_p^p\|y\|_q\|z\|_r},\\
\big(\nabla\omega(x,y,z)\big)_2 &= \frac{T(x,\cdot,z)}{\|x\|_p\|y\|_q\|z\|_r} - \omega(x,y,z)\frac{|y|^{q-2}\circ y}{\|x\|_p\|y\|_q^q\|z\|_r},\\
\big(\nabla\omega(x,y,z)\big)_3 &= \frac{T(x,y,\cdot)}{\|x\|_p\|y\|_q\|z\|_r} - \omega(x,y,z)\frac{|z|^{r-2}\circ z}{\|x\|_p\|y\|_q\|z\|_r^r}.
\end{aligned}$$

Hence, if $x \in \mathbb{R}^l_+ \setminus \{0\}, y \in \mathbb{R}^m_+ \setminus \{0\}, z \in \mathbb{R}^n_+ \setminus \{0\}$, then $\nabla\omega(x, y, z) = 0$ holds if and only if

$$\begin{cases} T(\cdot, y, z)^{1/(p-1)} = \lambda x \\ T(x, \cdot, z)^{1/(q-1)} = \theta y \\ T(x, y, \cdot)^{1/(r-1)} = \vartheta z \end{cases} \tag{3.5}$$

with $\lambda, \theta, \vartheta$ satisfying $\lambda = \|x\|_p^{-1}\omega(x, y, z)^{1/(p-1)}$, $\theta = \|y\|_q^{-1}\omega(x, y, z)^{1/(q-1)}$ and $\vartheta = \|z\|_r^{-1}\omega(x, y, z)^{1/(r-1)}$. The left hand side of equation (3.5) defines the order-preserving multi-homogeneous mapping $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ given by

$$\mathcal{F}(x, y, z) = \big(T(\cdot, y, z)^{1/(p-1)}, T(x, \cdot, z)^{1/(q-1)}, T(x, y, \cdot)^{1/(r-1)}\big), \tag{3.6}$$

with homogeneity matrix

$$A = \begin{pmatrix} 0 & 1/(p-1) & 1/(p-1) \\ 1/(q-1) & 0 & 1/(q-1) \\ 1/(r-1) & 1/(r-1) & 0 \end{pmatrix}.$$

3.2.3 Rectangular $\ell^{p,q}$-norm of a third order nonnegative tensor

For $p, q \in (1, \infty)$, let $\|\cdot\|_p$ be the ℓ^p-norm on $\mathbb{R}^m$ and let $\|\cdot\|_q$ be the ℓ^q-norm on $\mathbb{R}^n$. Consider the norm $\|\cdot\|_{p,q}$ on $\mathbb{R}^{m\times n\times n}$ defined as

$$\|T\|_{p,q} = \max_{x\in\mathbb{R}^m\setminus\{0\}, y\in\mathbb{R}^n\setminus\{0\}} \frac{f_T(x, y, y)}{\|x\|_p\|y\|_q^2} \qquad \forall T \in \mathbb{R}^{m\times n\times n}.$$

We refer to $\|T\|_{p,q}$ as the *rectangular $\ell^{p,q}$-norm of the tensor T*. The $\ell^{p,q}$-norm of a tensor has been studied in [40]$^\diamond$. If $\hat{T} \in \mathbb{R}^{m\times n\times n}$ is the partial symmetrization of T defined in (2.3), then $f_T(x, y, y) = f_{\hat{T}}(x, y, y)$ for every $x \in \mathbb{R}^m, y \in \mathbb{R}^n$ and thus $\|T\|_{p,q} = \|\hat{T}\|_{p,q}$. Theorem 1 in [7], implies that $\|T\|_{p,q} = \|\hat{T}\|_{p,q} = \|\hat{T}\|_{p,q,q}$ where $\|\hat{T}\|_{p,q,q}$ is the $\ell^{p,q,q}$-norm of $\hat{T}$ defined in (3.4). Nevertheless, the critical point equation induced by $\|T\|_{p,q}$ is of general interest as discussed in Section 4.2.3 and the above variational characterization is an intuitive way to motivate it. If $T \in \mathbb{R}^{m\times n\times n}_+$, then for every $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ it holds $|f_T(x, y, y)| \le f_T(|x|, |y|, |y|)$ so that

$$\|T\|_{p,q} = \max_{x\in\mathbb{R}^m_+\setminus\{0\}, y\in\mathbb{R}^n_+\setminus\{0\}} \omega(x, y) \qquad \text{with} \qquad \omega(x, y) = \frac{f_T(x, y, y)}{\|x\|_p\|y\|_q^2},$$

i.e. the maximum is attained in the product of cones $\mathcal{C} = \mathbb{R}^m_+ \times \mathbb{R}^n_+$. As T can be assumed to be partially symmetric without loss of generality, we have

$$\big(\nabla\omega(x, y)\big)_1 = \frac{T(\cdot, y, y)}{\|x\|_p\|y\|_q^2} - \omega(x, y)\frac{|x|^{p-2} \circ x}{\|x\|_p^p\|y\|_q^2},$$
$$\big(\nabla\omega(x, y)\big)_2 = \frac{2T(x, \cdot, y)}{\|x\|_p\|y\|_q^2} - 2\omega(x, y)\frac{|y|^{p-2} \circ y}{\|x\|_p\|y\|_q^q}.$$

Therefore, if $x \in \mathbb{R}^m_+ \setminus \{0\}, y \in \mathbb{R}^n_+ \setminus \{0\}$, then it holds $\nabla\omega(x, y) = 0$ if and only if

$$\begin{cases} T(\cdot, y, y)^{1/(p-1)} = \lambda x \\ T(x, \cdot, y)^{1/(q-1)} = \theta y \end{cases} \tag{3.7}$$

with $\lambda = \|x\|_p^{-1}\omega(x, y)^{1/(p-1)}$ and $\theta = \|y\|_q^{-1}\omega(x, y)^{1/(q-1)}$. The left hand side of equation (3.7) defines the order-preserving multi-homogeneous mapping $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ given by

$$\mathcal{F}(x, y) = \big(\, T(\cdot, y, y)^{1/(p-1)}\,,\, T(x, \cdot, y)^{1/(q-1)}\,\big) \tag{3.8}$$

with homogeneity matrix

$$\begin{pmatrix} 0 & 2/(p-1) \\ 1/(q-1) & 1/(q-1) \end{pmatrix}.$$

3.2.4 ℓ^p-norm of a third order nonnegative tensor

For $p \in (1, \infty)$, let $\|\cdot\|_p$ be the ℓ^p-norm on $\mathbb{R}^n$. On $\mathbb{R}^{n\times n\times n}$, consider the norm

$$\|T\|_p = \max_{x\in\mathbb{R}^n\setminus\{0\}} \frac{f_T(x, x, x)}{\|x\|_p^3} \qquad \forall T \in \mathbb{R}^{n\times n\times n}.$$

We refer to $\|T\|_p$ as the *ℓ^p-norm of the tensor T*. The ℓ^p-norm of a tensor has been studied in [64, 31] and [40]$^\diamond$. The case $p = 2$ is known as the spectral norm of T and is of particular interest as the maximum is attained at a best rank one approximation of T [63]. The computation of $\|T\|_p$ is generally NP-hard [45]. If $\hat{T} \in \mathbb{R}^{n\times n\times n}$ is the symmetrization of T defined in (2.2), then it holds $f_T(x, x, x) = f_{\hat{T}}(x, x, x)$ for all $x \in \mathbb{R}^n$ and thus $\|T\|_p = \|\hat{T}\|_p$. Theorem 1 in [7], implies that $\|T\|_p = \|\hat{T}\|_p = \|\hat{T}\|_{p,p,p}$ where $\|\hat{T}\|_{p,p,p}$ is the $\ell^{p,p,p}$-norm of $\hat{T}$ defined in (3.4). Nevertheless, as for the rectangular $\ell^{p,q}$-singular vectors, the critical point equation induced by $\|T\|_p$ is of general interest as discussed in Section 4.2.4 and the above variational characterization is an intuitive way to motivate it. If $T \in \mathbb{R}^{n\times n\times n}_+$, then for every $x \in \mathbb{R}^n$ it holds $|f_T(x, x, x)| \le f_T(|x|, |x|, |x|)$ and thus

$$\|T\|_p = \max_{x\in\mathbb{R}^n_+\setminus\{0\}} \omega(x) \qquad \text{with} \qquad \omega(x) = \frac{f_T(x, x, x)}{\|x\|_p^3},$$

i.e. the maximum is attained in the cone $\mathcal{C} = \mathbb{R}^n_+$. As T can be assumed to be symmetric without loss of generality, we have

$$\nabla\omega(x) = \frac{3T(\cdot, x, x)}{\|x\|_p^3} - 3\omega(x)\frac{|x|^{p-2} \circ x}{\|x\|_p^p}.$$

Therefore, if $x \in \mathbb{R}^n_+ \setminus \{0\}$, then it holds $\nabla\omega(x) = 0$ if and only if

$$T(\cdot, x, x)^{1/(p-1)} = \lambda x \qquad \text{with} \qquad \lambda = \|x\|_p^{-1}\omega(x)^{1/(p-1)}. \tag{3.9}$$

The left hand side of equation (3.9) defines the order-preserving homogeneous mapping $F\colon C \to C$ given by

$$F(x) = T(\cdot, x, x)^{1/(p-1)} \tag{3.10}$$

with homogeneity degree $2/(p-1)$.

3.2.5 General example

Examples of multi-homogeneous mappings can be constructed as in Example 2.3.2. We show how to proceed in the case $d = 2$ and $d' = 1$, the general case can be done analogously. Let $\mathcal{V} = V_1 \times V_2$ be the product of finite dimensional real vector spaces and W a finite dimensional real vector space. Let $\mathcal{C} \subset \mathcal{V}$ and $K \subset W$ be cones. Let $v_1, \ldots, v_n \in K$, $f_1, \ldots, f_{n_1} \colon C_1 \to [0, \infty)$ and $g_1, \ldots, g_{n_2} \colon C_2 \to [0, \infty)$. Suppose that f_i is homogeneous of degree p for all i and suppose that g_j is homogeneous of degree q for all j. Let $T \in \mathbb{R}_+^{m \times n_1 \times n_2}$ and define $\mathcal{F} \colon \mathcal{C} \to \mathcal{K}$ as

$$\mathcal{F}(x, y) = \sum_{i=1}^{m} \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} T_{i,j_1,j_2} f_{j_1}(x) g_{j_2}(y) v_i.$$

Then, $\mathcal{F}$ is multi-homogeneous of degree $A = (p, q) \in \mathbb{R}^{1 \times 2}$.

3.3 First properties and multi-homogeneous notation

Let $\mathcal{V} = V_1 \times \ldots \times V_d$ and $\mathcal{W} = W_1 \times \ldots \times W_{d'}$ be products of finite dimensional real vector spaces. Furthermore, let $\mathcal{U} \subset \mathcal{V}$ be a nonempty set. Multi-homogeneous mappings from $\mathcal{U}$ to $\mathcal{W}$ are homogeneous when the homogeneity have a particular structure. This is discussed in the next lemma:

Lemma 3.3.1. Let $\mathcal{F} \colon \mathcal{U} \to \mathcal{W}$ be multi-homogeneous of degree $A \in \mathbb{R}^{d' \times d}$. If there exists $r \in \mathbb{R}$ such that $A\mathbf{1} = r\mathbf{1}$, then $\mathcal{F}$ is homogeneous of degree r.

Proof. Let $x \in U$ and $\alpha_1, \ldots, \alpha_d > 0$. If $(\alpha_1 x_1, \ldots, \alpha_d x_d) \in \mathcal{U}$, then it holds

$$\mathcal{F}(\alpha_1 x_1, \ldots, \alpha_d x_d)_i = \Big(\prod_{j=1}^{d} \alpha_i^{A_{i,j}}\Big)\mathcal{F}(x)_i \qquad \forall i = 1, \ldots, d'.$$

In particular, as $A\mathbf{1} = r\mathbf{1}$, for $x \in \mathcal{U}$ and $\alpha > 0$ such that $\alpha x \in \mathcal{U}$, by setting $\alpha_1 = \ldots = \alpha_d = \alpha$ above, we have

$$\mathcal{F}(\alpha x)_i = \Big(\prod_{j=1}^{d} \alpha^{A_{i,j}}\Big)\mathcal{F}(x)_i = \alpha^{(A\mathbf{1})_i}\mathcal{F}(x)_i = \alpha^r \mathcal{F}(x)_i \qquad \forall i = 1, \ldots, d',$$

which concludes the proof. □

Next we show that in most cases the homogeneity matrix is unique.

Lemma 3.3.2. Let $\mathcal{F} \colon \mathcal{U} \to \mathcal{W}$ be multi-homogeneous of degree $A \in \mathbb{R}^{d' \times d}$. Suppose that there exists $\bar{x} \in \mathcal{U}$ and $\alpha \neq 1$ such that $\alpha\bar{x} \in \mathcal{U}$ and $\bar{x}_j \neq 0$, $\mathcal{F}(\bar{x})_i \neq 0$ for all $i = 1, \ldots, d'$, $j = 1, \ldots, d$. Then, the homogeneity matrix of $\mathcal{F}$ is unique.

Proof. Let $B \in \mathbb{R}^{d' \times d}$ be such that $\mathcal{F}$ is multi-homogeneous of degree B. Let $i \in \{1, \ldots, d'\}$ and $j \in \{1, \ldots, d\}$. It holds

$$\alpha^{A_{i,j}}\mathcal{F}(\bar{x})_i = \mathcal{F}(\bar{x}_1, \ldots, \bar{x}_{j-1}, \alpha\, \bar{x}_j, \bar{x}_{j+1}, \ldots, \bar{x}_d)_i = \alpha^{B_{i,j}}\mathcal{F}(\bar{x})_i,$$

so that $(\alpha^{A_{i,j}} - \alpha^{B_{i,j}})\mathcal{F}(\bar{x})_i = 0$ which implies that $\alpha^{A_{i,j}} = \alpha^{B_{i,j}}$ as $\mathcal{F}(\bar{x})_i \neq 0$. Since $\alpha \neq 1$, it follows that $A_{i,j} = B_{i,j}$. □

We prove that if a multi-homogeneous mapping is order-preserving then its homogeneity matrix is nonnegative.

Lemma 3.3.3. Let $\mathcal{C} \subset \mathcal{V}$ be a cone and let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}$. If there exists $\bar{x} \in \mathcal{C}$ such that $\bar{x}_i \neq 0$ and $\mathcal{F}(\bar{x})_i \neq 0$ for all $i = 1, \ldots, d$, then $A \in \mathbb{R}^{d\times d}_+$. Furthermore, for every $i, j \in \{1, \ldots, d\}$ such that $A_{i,j} = 0$, the mapping $y_j \mapsto \mathcal{F}(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_d)_i$ is constant on each part of C_j and if $\mathcal{F}$ is continuous, then $y_j \mapsto \mathcal{F}(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_d)_i$ is constant on $\mathcal{C}$.

Proof. Suppose by contradiction that there exists $i, j \in \{1, \ldots, d\}$ such that $A_{i,j} < 0$. For $\alpha > 0$, let $\bar{y}(\alpha) = (\bar{x}_1, \ldots, \bar{x}_{j-1}, \alpha\bar{x}_j, \bar{x}_{j+1}, \ldots, \bar{x}_d) \in \mathcal{C}$. Then, we have $\mathcal{F}(\bar{y}(\alpha))_i = \alpha^{A_{i,j}}\mathcal{F}(\bar{x})_i$. If $\alpha > 1$, then $\bar{x} \preceq_{\mathcal{C}} \bar{y}(\alpha)$ and $\alpha^{A_{i,j}} < 1$. Thus, $\mathcal{F}(\bar{x})_i \preceq_{C_i} \mathcal{F}(\bar{y}(\alpha))_i$ and $\mathcal{F}(\bar{y}(\alpha))_i \prec_{C_i} \mathcal{F}(\bar{x})_i$ which is absurd. It follows that $A \in \mathbb{R}^{d\times d}_+$. Finally, if there exists $i, j \in \{1, \ldots, d\}$ such that $A_{i,j} = 0$, then for every $x \in C$, the mapping $y_j \mapsto \mathcal{F}(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_d)_i$ is order-preserving and homogeneous of degree 0. Hence, Lemma 2.3.3 implies that $y_j \mapsto \mathcal{F}(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_d)_i$ is constant on each part of C_j. If $\mathcal{F}$ is continuous, then $y_j \mapsto \mathcal{F}(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_d)_i$ is continuous, and thus, by Lemma 2.3.3, is a constant mapping on C_j. $\square$

Remark 3.3.4. Note that a similar argument as in the proof of Lemma 3.3.3 shows that if $\mathcal{F}$ is order-reversing and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}$ then the existence of $\bar{x}$ such that $\bar{x}_i \neq 0$ and $\mathcal{F}(\bar{x})_i \neq 0$ for all i, implies that $-A \in \mathbb{R}^{d\times d}_+$. Furthermore, $A_{i,j} = 0$ implies that $y_j \mapsto \mathcal{F}(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_d)_i$ is constant on each part of C_j and if $\mathcal{F}$ is continuous, then $y_j \mapsto \mathcal{F}(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_d)_i$ is constant.

We introduce further notation for the study of multi-homogeneous mappings. First, we write vectors in $\mathbb{R}^d$ with bold font and their components in the canonical basis with normal font. So, for instance we have $\boldsymbol{\alpha} \in \mathbb{R}^d$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$. We denote by $\mathbf{e}_1, \ldots, \mathbf{e}_d$ the canonical basis of $\mathbb{R}^d$ and by $\mathbf{1}$ the vector of all ones, i.e. $\mathbf{1} = (1, \ldots, 1)^\top$. We let $\mathbb{R}^d_+ = \{\boldsymbol{\alpha} \in \mathbb{R}^d \,|\, \alpha_1, \ldots, \alpha_d \geq 0\}$ and $\mathbb{R}^d_{++} = \{\boldsymbol{\alpha} \in \mathbb{R}^d \,|\, \alpha_1, \ldots, \alpha_d > 0\}$. We denote the partial ordering induced by the solid cone $\mathbb{R}^d_+$ as $\leq$, i.e. $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ if $\alpha_i \leq \beta_i$ for all $i = 1, \ldots, d$. Again, we also define $\lneq$ and $<$ as $\boldsymbol{\alpha} \lneq \boldsymbol{\beta}$ if $\alpha_i \leq \beta_i$ for all i and $\alpha_j < \beta_j$ for some j, and $\boldsymbol{\alpha} < \boldsymbol{\beta}$ if $\alpha_i < \beta_i$ for all i. For $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$ and $B \in \mathbb{R}^{d'\times d}$, or $\boldsymbol{\alpha} \in \mathbb{R}^d_+$ and $B \in \mathbb{R}^{d'\times d}_+$, we define $\boldsymbol{\alpha}^B \in \mathbb{R}^{d'}_+$ as

$$\boldsymbol{\alpha}^B = \Big(\prod_{k=1}^d \alpha_k^{B_{1,k}}, \ldots, \prod_{k=1}^d \alpha_k^{B_{d',k}}\Big),$$

where we use the convention that $0^0 = 1$. An equivalent way to define $\boldsymbol{\alpha}^B$ for $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$ and $B \in \mathbb{R}^{d'\times d}$ is by using the component-wise logarithm and exponential. Indeed, if we let

$$\ln(\boldsymbol{\beta}) = (\ln(\beta_1), \ldots, \ln(\beta_d)) \qquad \text{and} \qquad \exp(\boldsymbol{\beta}) = (e^{\beta_1}, \ldots, e^{\beta_d}) \qquad \forall \boldsymbol{\beta} \in \mathbb{R}^d_{++},$$

then for $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$ and $B \in \mathbb{R}^{d' \times d}$ it holds $\boldsymbol{\alpha}^B = \exp(B \ln(\boldsymbol{\alpha}))$. A direct computation shows that for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^d_{++}$ and $B, C \in \mathbb{R}^{d \times d}$, or $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^d_{+}$ and $B, C \in \mathbb{R}^{d \times d}_{+}$, the following identities hold

$$\boldsymbol{\alpha}^B \circ \boldsymbol{\alpha}^C = \boldsymbol{\alpha}^{B+C}, \qquad \big(\boldsymbol{\alpha}^C\big)^B = \boldsymbol{\alpha}^{BC}, \qquad \big(\boldsymbol{\alpha} \circ \boldsymbol{\beta}\big)^B = \boldsymbol{\alpha}^B \circ \boldsymbol{\beta}^B, \tag{3.11}$$

where $\circ$ denotes the component wise (Hadamard) product, i.e.

$$\boldsymbol{\alpha} \circ \boldsymbol{\beta} = (\alpha_1\beta_1, \ldots, \alpha_d\beta_d) \qquad \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^d.$$

Moreover, if $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$, $B \in \mathbb{R}^{d \times d}$, $\mathbf{a} \in \mathbb{R}^d$ and $\lambda > 0$, then

$$\prod_{i=1}^{d} \big(\boldsymbol{\alpha}^B\big)_i^{a_i} = \prod_{i=1}^{d} \alpha_i^{(B^\top \mathbf{a})_i} \quad \text{ and } \quad (\lambda^{a_1}, \ldots, \lambda^{a_d})^B = (\lambda^{(B\mathbf{a})_1}, \ldots, \lambda^{(B\mathbf{a})_d}).$$

We use the symbol $\otimes$ to denote the following operation

$$\boldsymbol{\alpha} \otimes \mathbf{x} = (\alpha_1 \mathbf{x}_1, \ldots, \alpha_d \mathbf{x}_d) \qquad \forall \boldsymbol{\alpha} \in \mathbb{R}^d, \mathbf{x} \in V.$$

We note that the operation $\otimes$ is closely related to the inflation product defined in [32]. With this notation, the definition of multi-homogeneous mappings on cones can be made more compact. Indeed, a mapping $\mathcal{F}\colon \mathcal{C} \to \mathcal{K}$ is multi-homogeneous of degree $A \in \mathbb{R}^{d' \times d}$ if

$$\mathcal{F}(\boldsymbol{\alpha} \otimes x) = \boldsymbol{\alpha}^A \otimes \mathcal{F}(x) \qquad \forall x \in \mathcal{C}, \boldsymbol{\alpha} \in \mathbb{R}^d_{++}.$$

With the above observation and the identities in (3.11), the proof of the following lemma is straightforward:

Lemma 3.3.5. Let $\mathcal{C} \subset \mathcal{V}$ be a cone and let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be multi-homogeneous of degree $A \in \mathbb{R}^{d \times d}$ and $\mathcal{G}\colon \mathcal{C} \to \mathcal{C}$ be multi-homogeneous of degree $B \in \mathbb{R}^{d \times d}$. Define $\mathcal{H}\colon \mathcal{C} \to \mathcal{C}$ as $\mathcal{H}(x) = \mathcal{F}(\mathcal{G}(x))$ for all $x \in \mathcal{C}$, then $\mathcal{H}$ is multi-homogeneous of degree $AB \in \mathbb{R}^{d \times d}$.

A consequence of Lemma 3.3.5 is that the k-th composition of a multi-homogeneous mapping $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ with itself is again multi-homogeneous. Indeed, if $\mathcal{C} \subset \mathcal{V}$ is a cone, $\mathcal{F}$ is multi-homogeneous of degree $A \in \mathbb{R}^{d \times d}$ and for $k \geq 1$, $\mathcal{F}^k\colon \mathcal{C} \to \mathcal{C}$ is defined as $\mathcal{F}^k(x) = \mathcal{F}(\mathcal{F}^{k-1}(x))$ with $\mathcal{F}^0$ being the identity mapping, then $\mathcal{F}^k$ is multi-homogeneous of degree A^k. Another property which can be easily observed with the above notation is that the inverse of a multi-homogeneous mapping is multi-homogeneous as well.

Lemma 3.3.6. Let $\mathcal{C} \subset \mathcal{V}$ be a cone and suppose that there exists $\bar{x} \in \mathcal{C}$ with $\bar{x}_i \neq 0$ for all $i = 1, \ldots, d$. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be multi-homogeneous of degree $A \in \mathbb{R}^{d \times d}$. Suppose that $\mathcal{F}$ has an inverse $\mathcal{F}^{-1}\colon \mathcal{C} \to \mathcal{C}$. Then A is nonsingular and the homogeneity matrix of $\mathcal{F}^{-1}$ is A^{-1}.

Proof. Suppose by contradiction that A is singular, then there exists $\mathbf{c} \in \mathbb{R}^d \setminus \{0\}$ such that $A\mathbf{c} = 0$. Let $t > 1$ and define $\boldsymbol{\alpha} = (t^{c_1}, \ldots, t^{c_d})$. Since $\mathbf{c} \neq 0$, it holds $\boldsymbol{\alpha} \neq \mathbf{1}$ and thus $\boldsymbol{\alpha} \otimes \bar{x} \neq \bar{x}$. Furthermore, it holds

$$\mathcal{F}(\boldsymbol{\alpha} \otimes \bar{x}) = \boldsymbol{\alpha}^A \otimes \mathcal{F}(x) = (t^{(A\mathbf{c})_1}, \ldots, t^{(A\mathbf{c})_d}) \otimes \mathcal{F}(x) = \mathcal{F}(x),$$

and thus $\mathcal{F}$ is not injective, a contradiction to the existence of $\mathcal{F}^{-1}$. Now, let $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$ and $x \in \mathcal{C}$. Let $y \in \mathcal{C}$ be such that $x = \mathcal{F}(y)$, then

$$\mathcal{F}^{-1}(\boldsymbol{\alpha} \otimes x) = \mathcal{F}^{-1}(\boldsymbol{\alpha} \otimes \mathcal{F}(y)) = \mathcal{F}^{-1}(\mathcal{F}(\boldsymbol{\alpha}^{A^{-1}} \otimes y)) = \boldsymbol{\alpha}^{A^{-1}} \otimes y = \boldsymbol{\alpha}^{A^{-1}} \otimes \mathcal{F}^{-1}(x),$$

which concludes the proof. □

We note that multi-homogeneous mappings which are either order-preserving or order-reversing, map parts of cones to parts of cones.

Lemma 3.3.7. Let $\mathcal{C} \subset \mathcal{V}$ and $\mathcal{K} \subset \mathcal{W}$ be cones and let $\mathcal{F} \colon \mathcal{C} \to \mathcal{K}$ be multi-homogeneous of degree $A \in \mathbb{R}^{d' \times d}$. If $\mathcal{F}$ is either order-preserving or order-reversing, then for every $x, y \in \mathcal{C}$ with $x \sim_{\mathcal{C}} y$ it holds $\mathcal{F}(x) \sim_{\mathcal{K}} \mathcal{F}(y)$.

Proof. Let $x, y \in \mathcal{C}$ be such that $x \sim_{\mathcal{C}} y$. Then, there exists $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^d_{++}$ such that $\boldsymbol{\alpha} \otimes x \preceq_{\mathcal{C}} y \preceq_{\mathcal{C}} \boldsymbol{\beta} \otimes x$. If $\mathcal{F}$ is order-preserving, then $\boldsymbol{\alpha}^A \otimes \mathcal{F}(x) \preceq_{\mathcal{K}} \mathcal{F}(y) \preceq_{\mathcal{K}} \boldsymbol{\beta}^A \otimes \mathcal{F}(x)$ and thus $\mathcal{F}(x) \sim_{\mathcal{K}} \mathcal{F}(y)$. If $\mathcal{F}$ is order-reversing, then $\boldsymbol{\beta}^A \otimes \mathcal{F}(x) \preceq_{\mathcal{K}} \mathcal{F}(y) \preceq_{\mathcal{K}} \boldsymbol{\alpha}^A \otimes \mathcal{F}(x)$ and thus $\mathcal{F}(x) \sim_{\mathcal{K}} \mathcal{F}(y)$. □

Finally, we conclude with a simple observation which will be useful in various places of the upcoming discussion.

Lemma 3.3.8. Let $\mathcal{C}$ be a cone in $\mathcal{V}$ and let $\mathcal{F} \colon \mathcal{C} \to \mathcal{C}$ be order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{d \times d}_{+}$. Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^d_{++}$ and $x, y \in \mathcal{C} \setminus \{0\}$ be such that

$$\boldsymbol{\alpha} \otimes x \preceq_{\mathcal{C}} \mathcal{F}(x) \qquad \text{and} \qquad \mathcal{F}(y) \preceq_{\mathcal{C}} \boldsymbol{\beta} \otimes y,$$

then for every integer $k \geq 1$ it holds

$$\boldsymbol{\alpha}^{\sum_{j=0}^{k-1} A^j} \otimes x \preceq_{\mathcal{C}} \mathcal{F}^k(x) \qquad \text{and} \qquad \mathcal{F}^k(y) \preceq_{\mathcal{C}} \boldsymbol{\beta}^{\sum_{j=0}^{k-1} A^j} \otimes y.$$

Proof. The proof is by induction over $k \geq 1$. The case $k = 1$ is true by assumption. Suppose the statement holds for some $k \geq 1$. Then, we have

$$\begin{aligned}\boldsymbol{\alpha}^{\sum_{j=0}^{k} A^j} \otimes x = \boldsymbol{\alpha}^{A\sum_{j=0}^{k-1} A^j} \otimes (\boldsymbol{\alpha} \otimes x) \preceq_{\mathcal{C}} \Big(\boldsymbol{\alpha}^{\sum_{j=0}^{k-1} A^j}\Big)^A \otimes \mathcal{F}(x) = \mathcal{F}\Big(\boldsymbol{\alpha}^{\sum_{j=0}^{k-1} A^j} \otimes x\Big) \\ \preceq_{\mathcal{C}} \mathcal{F}\big(\mathcal{F}^k(x)\big) = \mathcal{F}^{k+1}(x).\end{aligned}$$

This concludes the induction for the first inequality. Similarly,

$$\mathcal{F}^{k+1}(y) \preceq_{\mathcal{C}} \mathcal{F}\Big(\boldsymbol{\beta}^{\sum_{j=0}^{k-1} A^j} \otimes y\Big) \preceq_{\mathcal{C}} \Big(\boldsymbol{\beta}^{\sum_{j=0}^{k-1} A^j}\Big)^A \otimes (\boldsymbol{\beta} \otimes y) = \boldsymbol{\beta}^{\sum_{j=0}^{k-1} A^j} \otimes y$$

which concludes the proof. □

4 Eigenvectors and eigenvalues

We define eigenvectors of multi-homogeneous mappings leaving a product of cones invariant. We discuss their properties and relate them to problems of the literature. Then, we explain how certain eigenvector problems of multi-homogeneous mappings can be reduced to a classical eigenvector problem of homogeneous mapping. Finally, we explain why, for the study of eigenvectors of multi-homogeneous mappings, it is meaningful to assume that the homogeneity matrix is irreducible.

4.1 Eigenvectors and eigenvalues of multi-homogeneous mappings

Let $\mathcal{V} = V_1 \times \ldots \times V_d$ be a product of finite dimensional real vector spaces and $\mathcal{C} \subset \mathcal{V}$ a cone. The eigenvectors of a multi-homogeneous mapping are defined as follows:

Definition 4.1.1. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be a multi-homogeneous mapping. If $x \in \mathcal{C}$ satisfies $x_i \neq 0$ for all $i = 1, \ldots, d$ and there exists $\boldsymbol{\lambda} \in \mathbb{R}^d_+$ such that $\mathcal{F}(x) = \boldsymbol{\lambda} \otimes x$, then we say that x is an *eigenvector* of $\mathcal{F}$.

If $d = 1$, the above definition reduces to that of eigenvectors of homogeneous mapping. In Definition 4.1.1, the restriction $x_i \neq 0$ for all i, rather than $x \neq 0$, is mainly motivated by the applications. In particular, when the critical point equation of an objective function defined on the product of spheres is recasted as a mutli-homogeneous eigenvector problem, then $x_i \neq 0$ ensures that the eigenvector can be rescaled to be in the feasible set. The fact that $\boldsymbol{\lambda}$ should have nonnegative entries is not restrictive since $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ and thus $\mathcal{F}(x) = \boldsymbol{\lambda} \otimes x$ implies that $\boldsymbol{\lambda} \in \mathbb{R}^d_+$ otherwise we would have $-\lambda_i x_i \in C_i$ for some i which forces $x_i = 0$ as C_i is a cone. We discuss scaling properties of eigenvectors of multi-homogeneous mappings in the following lemma:

Lemma 4.1.2. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}$. Let $x \in \mathcal{C}$ be an eigenvector of $\mathcal{F}$ and let $\boldsymbol{\lambda} \in \mathbb{R}^d_+$ be such that $\mathcal{F}(x) = \boldsymbol{\lambda} \otimes x$.

a) For every $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$, $y = \boldsymbol{\alpha} \otimes x$ is an eigenvector of $\mathcal{F}$ and $\mathcal{F}(y) = \boldsymbol{\theta} \otimes y$ with $\boldsymbol{\theta} = \boldsymbol{\lambda} \circ \boldsymbol{\alpha}^{A-I}$.

b) If either $\boldsymbol{\lambda} \in \mathbb{R}^d_{++}$ or $\mathcal{F}$ is order-preserving, then for every positive integer k, x is an eigenvector of $\mathcal{F}^k$ and $\mathcal{F}^k(x) = \boldsymbol{\vartheta} \otimes x$ with $\boldsymbol{\vartheta} = \boldsymbol{\lambda}^{\sum_{j=0}^{k-1} A^j}$.

Proof. a) It holds $\mathcal{F}(\boldsymbol{\alpha} \otimes x) = \boldsymbol{\alpha}^A \otimes \mathcal{F}(x) = (\boldsymbol{\alpha}^A \circ \boldsymbol{\lambda}) \otimes x = (\boldsymbol{\alpha}^{A-I} \circ \boldsymbol{\lambda}) \otimes (\boldsymbol{\alpha} \otimes x)$.

b) Note that if $\boldsymbol{\lambda} \in \mathbb{R}^d_{++}$, then $\boldsymbol{\lambda}^B$ is well defined for every $B \in \mathbb{R}^{d\times d}$. If $\mathcal{F}$ is order-preserving, then $A \in \mathbb{R}^{d\times d}_+$ by Lemma 3.3.3 and thus $\sum_{j=0}^{k-1} A^j \in \mathbb{R}^{d\times d}_+$ for all k. In both cases, $\boldsymbol{\vartheta}$ is well defined. The proof that $\mathcal{F}^k(x) = \boldsymbol{\vartheta} \otimes x$ can be done with a similar inductive argument as the proof of Lemma 3.3.8. □

As can be seen from Lemma 4.1.2, defining eigenvalues of multi-homogeneous mappings is a delicate task. We note that this is already the case if $d = 1$ and the mapping is not homogeneous of degree 1. Indeed, if $f\colon C \to C$ is homogeneous of degree $p > 0$ and $x \in C \setminus \{0\}$ satisfies $f(x) = \lambda x$ with $\lambda > 0$, then for $\alpha > 0$ it holds $f(\alpha x) = \alpha^{p-1}\lambda(\alpha x)$. Hence, the eigenvalues of f depend on the scaling of their corresponding eigenvector when f is not homogeneous of degree 1. The result of Lemma 4.1.2, a) suggest that in order to obtain scale invariant eigenvalues, we should require that the homogeneity matrix of $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ is the identity. This is however a very restrictive condition, in particular when $\mathcal{F}$ is continuous and order-preserving, since in this case, by Lemma 3.3.3 we have $\mathcal{F}(x) = (f_1(x_1), \ldots, f_d(x_d))$ where $f_i\colon C_i \to C_i$ is an order-preserving 1-homogeneous mapping for all $i = 1, \ldots, d$. It follows that the eigenvectors of such mappings can be studied directly with the results of the nonlinear Perron-Frobenius theory of homogeneous mappings. These observations suggest that defining $\boldsymbol{\lambda}$ as an eigenvalue of $\mathcal{F}$ if there exists an eigenvector x such that $\mathcal{F}(x) = \boldsymbol{\lambda} \otimes x$ is naive. Moreover, with such a definition, eigenvalues cannot always be compared since the ordering induced by $\mathbb{R}^d_+$ is not total. These issues can be addressed by considering the Perron vector of $A^\top$ which, we recall, is defined as follows: Let $B \in \mathbb{R}^{d\times d}_+$ be an irreducible matrix, then $\mathbf{b} \in \mathbb{R}^d_{++}$ is the *Perron vector of* B if $B\mathbf{b} = \rho(B)\mathbf{b}$ and $\sum_{i=1}^d b_i = 1$. The Perron-Frobenius theorem implies that the Perron vector of an irreducible matrix always exists and is unique. We define eigenvalues of multi-homogeneous mappings with irreducible nonnegative homogeneity matrix as follows:

Definition 4.1.3. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}_+$. Suppose that A is irreducible and let $\mathbf{b} \in \mathbb{R}^d_{++}$ be the Perron vector of $A^\top$. We say that $\theta \in \mathbb{R}_+$ is an *eigenvalue* of $\mathcal{F}$, if there exists an eigenvector $x \in \mathcal{C}$ of $\mathcal{F}$ and $\boldsymbol{\lambda} \in \mathbb{R}^d_+$ such that $\mathcal{F}(x) = \boldsymbol{\lambda} \otimes x$ and $\theta = \prod_{i=1}^d \lambda_i^{b_i}$. In this case, we say that θ is the eigenvalue of $\mathcal{F}$ corresponding to the eigenvector x.

If $d = 1$, the above definition reduces to that of eigenvalues of homogeneous mapping [60]. Indeed, if $d = 1$ and $f\colon C \to C$ is homogeneous of degree $p > 0$, then $A = p \in \mathbb{R}^{1\times 1}_+$ is irreducible, $\rho(A) = p$, and $\mathbf{b} = 1 \in \mathbb{R}^1_{++}$ so that if $x \in C$ is an eigenvector of f in the classical sense, i.e. there exists $\lambda \in \mathbb{R}$ such that $f(x) = \lambda x$, then $\theta = \lambda^1 = \lambda$. Furthermore, we will see in Section 4.4 that the irreducibility assumption on A is not restrictive when studying eigenvectors of order-preserving multi-homogeneous mappings. We prove in the next lemma that eigenvalues of multi-homogeneous mappings defined as in Definition 4.1.3 have properties similar to that of homogeneous mappings.

Lemma 4.1.4. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}_+$. Suppose that A is irreducible and let $\mathbf{b} \in \mathbb{R}^d_{++}$ be the Perron vector of $A^\top$. Let $x \in \mathcal{C}$ be an eigenvector of $\mathcal{F}$ and $\theta \geq 0$ the corresponding eigenvalue.

a) For every $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$, the eigenvalue of $\mathcal{F}$ corresponding to $\boldsymbol{\alpha} \otimes x$ is given by $\vartheta = \big(\prod_{i=1}^d \alpha_i^{b_i}\big)^{\rho(A)-1}\theta$.

b) If y is an eigenvector of $\mathcal{F}$ with corresponding eigenvalues $\vartheta \geq 0$ and $\theta \leq \vartheta$, then for every $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$, the eigenvalues $\tilde{\theta}$, $\tilde{\vartheta}$ respectively corresponding to $\boldsymbol{\alpha} \otimes x$ and $\boldsymbol{\alpha} \otimes y$ satisfy $\tilde{\theta} \leq \tilde{\vartheta}$.

c) For every positive integer k, x is an eigenvector of $\mathcal{F}^k$ with corresponding eigenvalue $\theta^{\sum_{j=0}^{k-1}\rho(A)^j}$.

Proof. Let $\boldsymbol{\lambda}\in\mathbb{R}^d_+$ be such that $\mathcal{F}(x)=\boldsymbol{\lambda}\otimes x$, then we have $\theta=\prod_{i=1}^d\lambda_i^{b_i}$.

a) By Lemma 4.1.2, we have $\mathcal{F}(\boldsymbol{\alpha}\otimes x)=(\boldsymbol{\lambda}\circ\boldsymbol{\alpha}^{A-I})\otimes(\boldsymbol{\alpha}\otimes x)$. Hence, the eigenvalues corresponding to $\boldsymbol{\alpha}\otimes x$ is given by

$$\begin{aligned}\vartheta&=\prod_{i=1}^d(\boldsymbol{\lambda}\circ\boldsymbol{\alpha}^{A-I})_i^{b_i}=\theta\prod_{i=1}^d(\boldsymbol{\alpha}^{A-I})_i^{b_i}=\theta\prod_{i=1}^d(\boldsymbol{\alpha}^{((A-I)^\top\mathbf{b})_i})_i\\&=\theta\prod_{i=1}^d\alpha_i^{(\rho(A)-1)b_i}=\theta\,\Big(\prod_{i=1}^d\alpha_i^{b_i}\Big)^{\rho(A)-1}.\end{aligned}$$

b) Let $\tilde{\boldsymbol{\lambda}}\in\mathbb{R}^d_+$ be such that $\mathcal{F}(y)=\tilde{\boldsymbol{\lambda}}\otimes y$. Then, by a), we have $\tilde{\theta}=\big(\prod_{i=1}^d\alpha_i^{b_i}\big)^{\rho(A)-1}\theta$ and $\tilde{\vartheta}=\big(\prod_{i=1}^d\alpha_i^{b_i}\big)^{\rho(A)-1}\vartheta$ and thus $\theta\leq\vartheta$ implies $\tilde{\theta}\leq\tilde{\vartheta}$.

c) By Lemma 4.1.2, we have $\mathcal{F}^k(x)=\boldsymbol{\lambda}^{\sum_{j=0}^{k-1}A^j}\otimes x$, hence the eigenvalue of $\mathcal{F}^k$ corresponding to x is given by

$$\prod_{i=1}^d\big(\boldsymbol{\lambda}^{\sum_{j=0}^{k-1}A^j}\big)_i^{b_i}=\prod_{i=1}^d\lambda_i^{((\sum_{j=0}^{k-1}A^j)^\top\mathbf{b})_i}=\prod_{i=1}^d\lambda_i^{\sum_{j=0}^{k-1}\rho(A)^jb_i}=\theta^{\sum_{j=0}^{k-1}\rho(A)^j},$$

which concludes the proof. □

Note that if, in Lemma 4.1.4, it holds $\rho(A)=1$, then the eigenvalues of $\mathcal{F}$ are scale invariant. This suggest that order-preserving multi-homogeneous mappings with homogeneity matrices having spectral radius equal to 1 is the natural generalization of order-preserving 1-homogeneous mappings. We will see in Section 6.2.2 that there is another strong argument in favor of this observation: Likewise order-preserving 1-homogeneous mappings on cones are non-expansive with respect to the Hilbert metric, if the homogeneity matrix of an order-preserving multi-homogeneous mappings has spectral radius equals to 1, then there exists a Hilbert type metric on the product of cones for which the mapping is non-expansive.

We conclude this section with a result which uses the Brouwer fixed point theorem to prove the existence of eigenvectors.

Theorem 4.1.5. Let $\mathcal{C}$ be a solid closed cone and $\mathcal{F}\colon\mathcal{C}\to\mathcal{C}$ a continuous mapping. There exists $u\in\mathcal{C}$ and $\boldsymbol{\lambda}\in\mathbb{R}^d_+$ such that $u_1,\ldots,u_d\neq 0$ and $\mathcal{F}(u)=\boldsymbol{\lambda}\otimes u$.

For the proof we need the following lemma:

Lemma 4.1.6. Let $\mathcal{C}$ be a solid closed cone. Let $w\in\operatorname{int}(\mathcal{C}^*)$ and define $\boldsymbol{\xi}_w\colon\mathcal{C}\to\mathbb{R}^d_+$ as $\boldsymbol{\xi}_w(x)=(w_1(x_1),\ldots,w_d(x_d))$ for all $x\in\mathcal{C}$. Let $\mathcal{F}\colon\mathcal{C}\to\mathcal{C}$ be continuous and such that $\mathcal{F}(x)_1,\ldots,\mathcal{F}(x)_d\neq 0$ for all $x\in\mathcal{C}$ with $\boldsymbol{\xi}_w(x)=\mathbf{1}$. Then, there exists $u\in\mathcal{C}$ and $\boldsymbol{\lambda}\in\mathbb{R}^d_{++}$ such that $\boldsymbol{\xi}_w(u)=\mathbf{1}$ and $\mathcal{F}(u)=\boldsymbol{\lambda}\otimes u$.

Proof. Let $\boldsymbol{\Sigma}_w = \{x \in \mathcal{C} \,|\, \boldsymbol{\xi}_w(x) = \mathbf{1}\}$ and let $\mathcal{G}\colon \boldsymbol{\Sigma}_w \to \boldsymbol{\Sigma}_w$ be defined as $\mathcal{G}(x) = \boldsymbol{\xi}_w(\mathcal{F}(x))^{-I} \otimes \mathcal{F}(x)$ for every $x \in \boldsymbol{\Sigma}_w$. Then $\mathcal{G}$ is well defined, continuous and $\boldsymbol{\Sigma}_w$ is a convex compact subset of $\mathcal{V}$. Hence, the Brouwer fixed point theorem [52] implies that $\mathcal{G}$ has a fixed point $u \in \boldsymbol{\Sigma}_w$. It follows that $\mathcal{F}(u) = \boldsymbol{\lambda} \otimes u$ with $\boldsymbol{\lambda} = \boldsymbol{\xi}_w(\mathcal{F}(u))$. Finally, note that $\boldsymbol{\lambda} \in \mathbb{R}^d_{++}$ since $\mathcal{F}(u)_i \neq 0$ for $i = 1, \ldots, d$ and thus $\boldsymbol{\xi}_w(\mathcal{F}(u)) \in \mathbb{R}^d_{++}$. □

Proof of Theorem 4.1.5. Let $w \in \operatorname{int}(\mathcal{C}^*)$, $\boldsymbol{\xi}_w(x) = (w_1(x_1), \ldots, w_d(x_d))$ for all $x \in \mathcal{C}$ and $\boldsymbol{\Sigma}_w = \{x \in \mathcal{C} \,|\, \boldsymbol{\xi}_w(x) = \mathbf{1}\}$. As $\mathcal{F}$ is continuous on the compact set $\boldsymbol{\Sigma}_w$, there exists $\alpha > 0$ large enough such that $\boldsymbol{\xi}_w(\mathcal{F}(x)) \leq \alpha \mathbf{1}$ for all $x \in \boldsymbol{\Sigma}_w$. Let $v \in \operatorname{int}(\mathcal{C}) \cap \boldsymbol{\Sigma}_w$ and for every $\epsilon > 0$, define $\mathcal{F}_\epsilon\colon \mathcal{C} \to \mathcal{C}$ as $\mathcal{F}_\epsilon(x) = \mathcal{F}(x) + \epsilon v$. Then it holds $\mathcal{F}_\epsilon(x) \in \operatorname{int}(\mathcal{C})$ for all $x \in \mathcal{C}$, and thus, by Lemma 4.1.6, there exists $u_\epsilon \in \boldsymbol{\Sigma}_w$ and $\boldsymbol{\lambda}_\epsilon \in \mathbb{R}^d_{++}$ such that $\mathcal{F}_\epsilon(u_\epsilon) = \boldsymbol{\lambda}_\epsilon \otimes u_\epsilon$. Now, note that

$$\boldsymbol{\lambda}_\epsilon = \boldsymbol{\xi}_w(\boldsymbol{\lambda}_\epsilon \otimes u_\epsilon) = \boldsymbol{\xi}_w(\mathcal{F}_\epsilon(u_\epsilon)) = \boldsymbol{\xi}_w(\mathcal{F}(u_\epsilon) + \epsilon v) = \boldsymbol{\xi}_w(\mathcal{F}(u_\epsilon)) + \epsilon \mathbf{1} \leq (\alpha + \epsilon)\mathbf{1}.$$

It follows that $\{\boldsymbol{\lambda}_\epsilon \,|\, \epsilon \in (0,1)\}$ is bounded. Hence, there exists $\boldsymbol{\lambda} \in \mathbb{R}^d_+$ and a sequence $(\epsilon_k)_{k=1}^\infty \subset (0,1)$ such that $\lim_{k\to\infty} \epsilon_k = 0$ and $\lim_{k\to\infty} \boldsymbol{\lambda}_{\epsilon_k} = \boldsymbol{\lambda}$. Furthermore, for every $k \geq 1$, it holds $x_{\epsilon_k} \in \boldsymbol{\Sigma}_w$ and thus there exists $u \in \boldsymbol{\Sigma}_w$ and a subsequence $(\epsilon_{k_l})_{l=1}^\infty \subset (\epsilon_k)_{k=1}^\infty$ such that $\lim_{l\to\infty} x_{\epsilon_{k_l}} = u$. Finally, as $\mathcal{F}$ is continuous, we have

$$\begin{aligned}\mathcal{F}(u) = \lim_{l\to\infty} \mathcal{F}(x_{\epsilon_{k_l}}) &= \lim_{l\to\infty} \big(\mathcal{F}(x_{\epsilon_{k_l}}) - \mathcal{F}_{\epsilon_{k_l}}(x_{\epsilon_{k_l}})\big) + \mathcal{F}_{\epsilon_{k_l}}(x_{\epsilon_{k_l}}) \\ &= \lim_{l\to\infty} \epsilon_{k_l} v + \boldsymbol{\lambda}_{\epsilon_{k_l}} \otimes x_{\epsilon_{k_l}} = \boldsymbol{\lambda} \otimes u,\end{aligned}$$

which concludes the proof. □

4.2 Examples

We discuss examples of eigenvectors of multi-homogeneous mappings.

4.2.1 $\ell^{p_1,\ldots,p_d}$-singular vectors of a nonnegative tensor

Let $d \geq 2$, $T \in \mathbb{R}_+^{n_1 \times \ldots \times n_d}$, $p_1, \ldots, p_d \in (1, \infty)$ and let $\|\cdot\|_{p_i}$ be the ℓ^{p_i}-norm on $\mathbb{R}^{n_i}$. Furthermore, let

$$\mathcal{C} = \mathbb{R}^{n_1}_+ \times \ldots \times \mathbb{R}^{n_d}_+ \qquad \text{and} \qquad \mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \|x_1\|_{p_1} = \ldots = \|x_d\|_{p_d} = 1\}.$$

Note that $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$ where $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d_+$ is the monotonic multi-normalization of $\mathcal{C}$ defined as $\boldsymbol{\nu}(x) = (\|x_1\|_{p_1}, \ldots, \|x_d\|_{p_d})$ for all $x \in \mathcal{C}$. Consider the following system of equations:

$$\left\{\begin{array}{ccc} T(\cdot, x_2, \ldots, x_d) & = & \lambda\, x_1^{p_1-1} \\ T(x_1, \cdot, x_3, \ldots, x_d) & = & \lambda\, x_2^{p_2-1} \\ \vdots & \vdots & \vdots \\ T(x_1, \ldots, x_{d-1}, \cdot) & = & \lambda\, x_d^{p_d-1} \end{array}\right. \quad \text{and} \quad (\lambda, x) \in \mathbb{R}_+ \times \mathcal{S}_{\boldsymbol{\nu}}. \tag{4.1}$$

Let $(\lambda, x) \in \mathbb{R}_+ \times \mathcal{S}_{\boldsymbol{\nu}}$ be a solution to (4.1). Then, x is called an $\ell^{p_1,\ldots,p_d}$-*singular vector* of T and λ the corresponding $\ell^{p_1,\ldots,p_d}$-*singular value* of T. These objects are studied in [64, 31] and [36, 40]$^\diamond$. In the particular case $d = 2$, T is a matrix and the ℓ^{p_1,p_2}-singular vectors/values of matrices are studied in [17, 13, 89].

Now, let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be defined as

$$\mathcal{F}(x) = \big(T(\boldsymbol{\cdot}, x_2, \ldots, x_d)^{1/(p_1-1)}, \ldots, T(x_1, \ldots, x_{d-1}, \boldsymbol{\cdot})^{1/(p_d-1)}\big) \qquad \forall x \in \mathcal{C}. \quad (4.2)$$

Note that $\mathcal{F}$ is order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}$ with

$$A = \operatorname{diag}\big(\tfrac{1}{p_1-1}, \ldots, \tfrac{1}{p_d-1}\big)(\mathbf{1}\mathbf{1}^\top - I).$$

In the next result, we relate the eigenvectors and eigenvalues of $\mathcal{F}$ with the $\ell^{p_1,\ldots,p_d}$-singular vectors and $\ell^{p_1,\ldots,p_d}$-singular values of T.

Proposition 4.2.1. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be as in (4.2) and $x \in \mathcal{S}_{\boldsymbol{\nu}}$. Then there exists $\lambda \geq 0$ such that (λ, x) is a solution of (4.1) if, and only if, there exists $\boldsymbol{\vartheta} \in \mathbb{R}^d_+$ such that $\mathcal{F}(x) = \boldsymbol{\vartheta} \otimes x$. Moreover, in this case it holds $\lambda = f_T(x)$ and $\vartheta_i = f_T(x)^{1/(p_i-1)}$ for $i = 1, \ldots, d$ so that $\prod_{i=1}^d \vartheta_i^{b_i} = \lambda^\gamma$ with $\gamma = \sum_{i=1}^d \frac{b_i}{p_i-1}$ and $\mathbf{b} \in \mathbb{R}^d_{++}$ the Perron vector of $A^\top$.

Proof. If (λ, x) is a solution of (4.1), then x is an eigenvector of $\mathcal{F}$ and $\mathcal{F}(x) = \boldsymbol{\vartheta} \otimes x$ with $\boldsymbol{\vartheta} = (\lambda^{1/(p_1-1)}, \ldots, \lambda^{1/(p_d-1)})$. Moreover, it holds

$$\lambda = \lambda\|x_1\|_{p_1}^{p_1} = \langle \lambda x_1^{p_1-1}\,,\, x_1\rangle = \langle T(\boldsymbol{\cdot}, x_2, \ldots, x_d)\,,\, x_1\rangle = f_T(x).$$

Conversely, if $x \in \mathcal{S}_{\boldsymbol{\nu}}$ satisfies $\mathcal{F}(x) = \boldsymbol{\vartheta} \otimes x$, then we have

$$T(x_1, \ldots, x_{i-1}, \boldsymbol{\cdot}, x_{i+1}, \ldots, x_d) = \vartheta_i^{p_i-1} x_i^{p_i-1} \qquad \forall i = 1, \ldots, d.$$

Hence, for every $i = 1, \ldots, d$, it holds

$$f_T(x) = \langle T(x_1, \ldots, x_{i-1}, \boldsymbol{\cdot}, x_{i+1}, \ldots, x_d)\,,\, x_i\rangle = \vartheta_i^{p_i-1}\langle x_i^{p_i-1}\,,\, x_i\rangle = \vartheta_i^{p_i-1}\|x_i\|_{p_i}^{p_i},$$

which proves that $\vartheta_i = f_T(x)^{1/(p_i-1)}$ for all $i = 1, \ldots, d$ and thus x is an $\ell^{p_1,\ldots,p_d}$-singular vector of T with corresponding $\ell^{p_1,\ldots,p_d}$-singular value $\lambda = f_T(x)$. Note that A is an irreducible matrix. Let $\mathbf{b} \in \mathbb{R}^d_{++}$ be the Perron vector of $A^\top$. Then, the eigenvalue of $\mathcal{F}$ corresponding to x is given by

$$\theta = \prod_{i=1}^d \vartheta_i^{b_i} = \prod_{i=1}^d f_T(x)^{b_i/(p_i-1)} = f_T(x)^\gamma = \lambda^\gamma,$$

which concludes the proof. □

Remark 4.2.2. Note that if $d = 2$ in Proposition 4.2.1, it can be verified that

$$\mathbf{b} = \beta\,\big(\sqrt{p_2-1}, \sqrt{p_1-1}\big)^\top \qquad \text{and} \qquad \gamma = \beta\,\Big(\sqrt{\frac{p_2-1}{p_1-1}} + \sqrt{\frac{p_1-1}{p_2-1}} - 1\Big),$$

with $\beta = \sqrt{p_1-1}+\sqrt{p_2-1}$. We are however not aware of such simple expressions for $\mathbf{b}$ and γ when $d > 2$ and $p_1, \ldots, p_d \in (1, \infty)$. Nevertheless, if there exists $p \in (1, \infty)$ such that $p_1 = \ldots = p_d = p$, then $\mathbf{b} = \frac{1}{d}\mathbf{1}$ and $\gamma = \frac{1}{p-1}$.

Remark 4.2.3. Proposition 4.2.1 implies that if $x, y \in \mathcal{S}_{\boldsymbol{\nu}}$ are $\ell^{p_1,\ldots,p_d}$-singular vectors of T with corresponding $\ell^{p_1,\ldots,p_d}$-singular values λ_x, λ_y, and θ_x, θ_y are the eigenvalues of $\mathcal{F}$ respectively corresponding to x, y, then it holds $\lambda_x \leq \lambda_y$ if and only if $\theta_x \leq \theta_y$. In particular, the eigenvector of $\mathcal{F}$ on $\mathcal{S}_{\boldsymbol{\nu}}$ with eigenvalue of largest magnitude coincide with the $\ell^{p_1,\ldots,p_d}$-singular vector of T with $\ell^{p_1,\ldots,p_d}$-singular value of largest magnitude.

We have seen in Section 3.2.2 that if $d = 3$, then the ℓ^{p_1,p_2,p_3}-singular vector of T can be characterized as the critical points of the function

$$\Phi(x) = \frac{f_T(x)}{\|x_1\|_{p_1}\|x_2\|_{p_2}\|x_3\|_{p_3}}.$$

A similar observation holds for every d. Indeed, Proposition 1 of [64] implies that the $\ell^{p_1,\ldots,p_d}$-singular vector of T coincide with the critical points of

$$\Phi(x) = \frac{f_T(x)}{\prod_{i=1}^{d}\|x_i\|_{p_i}}, \tag{4.3}$$

and that

$$\max_{x_1,\ldots,x_d\neq 0} \Phi(x) = \sup\{\lambda \,|\, \lambda \text{ is an } \ell^{p_1,\ldots,p_d}\text{-singular value of } T\}.$$

4.2.2 Discrete generalized Schrödinger equation

Let $T \in \mathbb{R}_{++}^{n\times\ldots\times n}$ be a tensor of order $d \geq 2$. Let $\mathcal{C} = \mathbb{R}^n_+ \times \ldots \times \mathbb{R}^n_+$ and $\mathcal{C}_0 = \{x \in \mathcal{C} \,|\, x_1, \ldots, x_d \neq 0\}$. Consider the following system of equations:

$$\begin{cases} T(\boldsymbol{\cdot}, x_2, \ldots, x_d) & = & x_1^{-1} \\ T(x_1, \boldsymbol{\cdot}, x_3, \ldots, x_d) & = & x_2^{-1} \\ \vdots & \vdots & \vdots \\ T(x_1, \ldots, x_{d-1}, \boldsymbol{\cdot}) & = & x_d^{-1} \end{cases} \quad \text{and} \quad x \in \mathcal{C}_0. \tag{4.4}$$

The above system of equations is a particular case of the discrete generalized Schrödinger equation [85] which arise in the context of multi-marginal optimal transport in [10]. Now, let $\mathcal{F}\colon \mathcal{C}_0 \to \mathcal{C}_0$ be defined as

$$\mathcal{F}(x) = \big(T(\boldsymbol{\cdot}, x_2, \ldots, x_d)^{-1}, \ldots, T(x_1, \ldots, x_{d-1}, \boldsymbol{\cdot})^{-1}\big) \qquad \forall x \in \mathcal{C}_0. \tag{4.5}$$

Then, note that $\mathcal{F}(\mathcal{C}_0) \subset \operatorname{int}(\mathcal{C})$ since $T \in \mathbb{R}_{++}^{n\times\ldots\times n}$. Furthermore, $\mathcal{F}$ is order-reversing and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}$ with $A = (I - \mathbf{1}\mathbf{1}^\top)$. We relate the solutions of (4.4) with the eigenvectors of $\mathcal{F}$ in the following:

Proposition 4.2.4. Let $\mathcal{F}\colon \mathcal{C}_0 \to \mathcal{C}_0$ be defined as in (4.5). Every solutions of (4.4) is an eigenvector of $\mathcal{F}$. Conversely, if $x \in \mathcal{C}_0$ is an eigenvector of $\mathcal{F}$, then $x \in \operatorname{int}(\mathcal{C})$ and it holds $f_T(x)\mathcal{F}(x) = nx$. In particular, tx is a solution of (4.4) with $t = \big(\frac{n}{f_T(x)}\big)^{1/d}$.

Proof. Note that the solutions of (4.4) coincide with the fixed points of $\mathcal{F}$. Now, suppose that $x \in \mathcal{C}_0$ is an eigenvector of $\mathcal{F}$ and let $\boldsymbol{\lambda} \in \mathbb{R}^d_+$ such that $\mathcal{F}(x) = \boldsymbol{\lambda} \otimes x$. As $\mathcal{F}(\mathcal{C}_0) \subset \operatorname{int}(\mathcal{C})$, we necessarily have $\boldsymbol{\lambda} \in \mathbb{R}^d_{++}$ and $x \in \operatorname{int}(\mathcal{C})$. Moreover, note that for all $i = 1, \ldots, d$ it holds

$$\frac{n}{\lambda_i} = \frac{\langle x_i \,,\, x_i^{-1} \rangle}{\lambda_i} = \langle x_i \,,\, \mathcal{F}(x)_i^{-1} \rangle = \langle x_i \,,\, T(x_1, \ldots, x_{i-1}, \boldsymbol{\cdot}, x_{i+1}, \ldots, x_d) \rangle = f_T(x).$$

Hence, it holds $\lambda_i = \frac{n}{f_T(x)}$ for $i = 1, \ldots, d$. Furthermore, by the multi-homogeneity of $\mathcal{F}$, for all $t > 0$, we have

$$\mathcal{F}(tx) = t^{1-d}\mathcal{F}(x) = t^{1-d}\boldsymbol{\lambda} \otimes x = t^{-d}\big(\tfrac{n}{f_T(x)}\big)\, tx.$$

Hence, we can let $t = \big(\frac{n}{f_T(x)}\big)^{1/d}$ to obtain $\mathcal{F}(tx) = tx$ which shows that tx is a solution of (4.4). □

4.2.3 Rectangular $\ell^{p,q}$-singular vectors of a nonnegative tensor

Let $n_1, \ldots, n_d$ be such that $n_1 = \ldots = n_a$ and $n_{a+1} = \ldots = n_d$. Let $T \in \mathbb{R}_+^{n_1 \times \ldots \times n_d}$, $p, q \in (1, \infty)$, $\|\cdot\|_p$ the ℓ^p-norm on $\mathbb{R}^m$ and $\|\cdot\|_q$ the ℓ^q-norm on $\mathbb{R}^n$. Define

$$\mathcal{C} = \mathbb{R}^m_+ \times \mathbb{R}^n_+ \qquad \text{and} \qquad \mathcal{S}_{\boldsymbol{\nu}} = \{(x, y) \in \mathcal{C} \mid \|x\|_p = \|y\|_q = 1\}.$$

Note that $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \mid \boldsymbol{\nu}(x, y) = \mathbf{1}\}$, where $\boldsymbol{\nu} \colon \mathcal{C} \to \mathbb{R}^2_+$ is the monotonic multi-normalization of $\mathcal{C}$ defined as $\boldsymbol{\nu}(x, y) = (\|x\|_p, \|y\|_q)$ for all $x \in \mathcal{C}$. Consider the following system of equations:

$$\begin{cases} T(\boldsymbol{\cdot}, x, \ldots, x, y, \ldots, y) &= \lambda x^{p-1} \\ T(x, \ldots, x, \boldsymbol{\cdot}, y, \ldots, y) &= \lambda y^{q-1} \end{cases} \quad \text{and} \quad (\lambda, (x, y)) \in \mathbb{R}_+ \times \mathcal{S}_{\boldsymbol{\nu}}. \tag{4.6}$$

Let $(\lambda, (x, y)) \in \mathbb{R}_+ \times \mathcal{S}_{\boldsymbol{\nu}}$ be a solution to (4.6). Then, x is called a *rectangular $\ell^{p,q}$-singular vector* of T and λ the corresponding *rectangular $\ell^{p,q}$-singular value* of T. These objects were studied in [65, 26, 99] and [40]$^{\diamond}$. In the particular case where $d = 2$, T is a matrix and the rectangular $\ell^{p,q}$-singular vectors/values coincide with its $\ell^{p,q}$-singular vectors/values discussed in Sections 3.2.1 and 4.2.1. Now, let $\mathcal{F} \colon \mathcal{C} \to \mathcal{C}$ be given for every $(x, y) \in \mathcal{C}$ by

$$\mathcal{F}(x, y) = \big(T(\boldsymbol{\cdot}, x, \ldots, x, y, \ldots, y)^{1/(p-1)}, T(x, \ldots, x, \boldsymbol{\cdot}, y, \ldots, y)^{1/(q-1)}\big) \tag{4.7}$$

Note that $\mathcal{F}$ is order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{2\times 2}_+$ with

$$A = \begin{pmatrix} (a-1)/(p-1) & (d-a)/(p-1) \\ a/(q-1) & (d-a-1)/(q-1) \end{pmatrix}.$$

In the next result, we relate the eigenvectors and eigenvalues of $\mathcal{F}$ with the rectangular $\ell^{p,q}$-singular vectors and rectangular $\ell^{p,q}$-singular values of T.

Proposition 4.2.5. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be defined as in (4.7) and $x \in \mathcal{S}_{\boldsymbol{\nu}}$. There exists $\lambda \geq 0$ such that (λ, x) is a solution of (4.6) if, and only if, there exists $\boldsymbol{\vartheta} \in \mathbb{R}^2_+$ such that $\mathcal{F}(x) = \boldsymbol{\vartheta} \otimes x$. Moreover, in this case it holds $\lambda = f_T(x, \ldots, x, y, \ldots, y)$ and $\vartheta_1 = \lambda^{p-1}, \vartheta_2 = \lambda^{q-1}$ so that $\vartheta_1^{b_1}\vartheta_2^{b_2} = \lambda^\gamma$ with $\gamma = \frac{b_1}{p-1} + \frac{b_2}{q-1}$ and $\mathbf{b} \in \mathbb{R}^2_{++}$ the Perron vector of $A^\top$.

Proof. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be defined as in (4.7). Every rectangular $\ell^{p,q}$-singular vector $x \in \mathcal{C}$ of T with corresponding rectangular $\ell^{p,q}$-singular value λ is an eigenvector of T with eigenvalue $\boldsymbol{\lambda} = (\lambda^{1/(p-1)}, \lambda^{1/(q-1)})$. Moreover, it holds

$$\lambda = \lambda\|x\|_p^p = \langle \lambda x^{p-1}\,,\, x\rangle = \langle T(\cdot, x, \ldots, x, y, \ldots, y)\,,\, x\rangle = f_T(x, \ldots, x, y, \ldots, y).$$

Conversely, if $(x, y) \in \mathcal{S}_{\boldsymbol{\nu}}$ satisfies $\mathcal{F}(x, y) = (\vartheta_1 x, \vartheta_2 y)$, then we have

$$T(\cdot, x, \ldots, x, y, \ldots, y) = \vartheta_1^{p-1}x^{p-1} \quad \text{and} \quad T(x, \ldots, x, \cdot, y, \ldots, y) = \vartheta_2^{q-1}y^{q-1}.$$

Hence, if $f_T\colon \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_d} \to \mathbb{R}$ is the multi-linear form induced by T, then

$$f_T(x, \ldots, x, y, \ldots, y) = \langle x\,,\, T(\cdot, x, \ldots, x, y, \ldots, y)\rangle = \vartheta_1^{p-1}\langle x\,,\, x^{p-1}\rangle = \vartheta_1^{p-1}\|x\|_p^p$$

$$f_T(x, \ldots, x, y, \ldots, y) = \langle y\,,\, T(x, \ldots, x, \cdot, y, \ldots, y)\rangle = \vartheta_2^{q-1}\langle y\,,\, y^{q-1}\rangle = \vartheta_2^{q-1}\|y\|_q^q.$$

It follows that $\vartheta_1^{1/(p-1)} = \vartheta_2^{1/(q-1)} = f_T(x, \ldots, x, y, \ldots, y)$ and thus (x, y) is a rectangular $\ell^{p,q}$-singular vector of T with corresponding rectangular $\ell^{p,q}$-singular value $\lambda = f_T(x, \ldots, x, y, \ldots, y)$. Furthermore, A is irreducible and if $\mathbf{b} \in \mathbb{R}^2_{++}$ is the Perron vector of $A^\top$, then $\vartheta_1^{b_1}\vartheta_2^{b_2} = \lambda^\gamma$, which concludes the proof. □

Remark 4.2.6. Proposition 4.2.5 implies that if $x, y \in \mathcal{S}_{\boldsymbol{\nu}}$ are rectangular $\ell^{p,q}$-singular vectors of T with corresponding rectangular $\ell^{p,q}$-singular values λ_x, λ_y, and θ_x, θ_y are the eigenvalues of $\mathcal{F}$ respectively corresponding to x, y, then it holds $\lambda_x \leq \lambda_y$ if and only if $\theta_x \leq \theta_y$. In particular, the eigenvector of $\mathcal{F}$ on $\mathcal{S}_{\boldsymbol{\nu}}$ with eigenvalue of largest magnitude coincide with the rectangular $\ell^{p,q}$-singular vector of T with rectangular $\ell^{p,q}$-singular value of largest magnitude.

We have seen in Section 3.2.3 that if T is partially symmetric, $d = 3$ and $a = 1$, then the rectangular $\ell^{p,q}$-singular vector of T can be characterized as the critical points of the function

$$\Phi(x, y) = \frac{f_T(x, y, y)}{\|x\|_p\|y\|_q^2}. \tag{4.8}$$

The argument can be generalized to prove that for every d, the rectangular $\ell^{p,q}$-singular vector of a partially symmetric tensor T coincide with the critical points of

$$\Phi(x, y) = \frac{f_T(x, \ldots, x, y, \ldots, y)}{\|x\|_p^a\,\|y\|_q^{d-a}}$$

and it holds

$$\max_{x,y\neq 0} \Phi(x) = \sup\{\lambda \mid \lambda \text{ is a rectangular } \ell^{p,q}\text{-singular vector of } T\}.$$

Finally, if T is the adjacency tensor of a multiplex network, i.e. T is of order three and is a collection of m graphs with n nodes, then the eigenvectors in the product of simplexes $\mathbf{\Sigma} = \{(x,y) \in \mathcal{C} \mid \sum_{i=1}^m x_i = \sum_{j=1}^n y_i = 1\}$ of the mapping $\mathcal{F}$ defined in (4.7) are known as the f-eigenvectors centrality of the multiplex T [92]$^\diamond$.

4.2.4 ℓ^p-eigenvectors of a nonnegative tensor

Let $T \in \mathbb{R}_+^{n\times\cdots\times n}$ be a tensor of order d, $p \in (1,\infty)$ and let $\|\cdot\|_p$ be the ℓ^p-norm on $\mathbb{R}^n$. Furthermore, let

$$\mathcal{C} = \mathbb{R}^n_+ \qquad \text{and} \qquad \mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \mid \|x\|_p = 1\}.$$

Note that $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \mid \boldsymbol{\nu}(x) = 1\}$ where $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}_+$ is the monotonic multi-normalization of $\mathcal{C}$ defined as $\boldsymbol{\nu}(x) = \|x\|_p$ for all $x \in \mathcal{C}$. Consider the following system of equations:

$$T(\boldsymbol{\cdot}, x, \ldots, x) = \lambda\, x^p \quad \text{and} \quad (\lambda, x) \in \mathbb{R}_+ \times \mathcal{S}_{\boldsymbol{\nu}}. \tag{4.9}$$

Let $(\lambda, x) \in \mathbb{R}_+ \times \mathcal{S}_{\boldsymbol{\nu}}$, be a solution of (4.1). Then, x is called an ℓ^p*-eigenvector* of T and λ the corresponding ℓ^p*-singular value* of T. These objects were introduced in [64, 80] and have attracted a considerable attention in the past decade [81]. In the particular case $p = 2$, the ℓ^p-eigenvectors of T are called Z-eigenvectors [25] and if $p = m$, the ℓ^p-eigenvectors of T are called H-eigenvectors [98].

Now, let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be defined as

$$\mathcal{F}(x) = T(\boldsymbol{\cdot}, x, \ldots, x)^{1/(p-1)} \qquad \forall x \in \mathcal{C}.$$

Note that $\mathcal{F}$ is order-preserving and multi-homogeneous of degree $A = (d-1)/(p-1) \in \mathbb{R}^{1\times 1}$. The eigenvectors and eigenvalues of $\mathcal{F}$ are directly related with the ℓ^p-eigenvectors and ℓ^p-eigenvalues of T. Indeed, (λ, x) is a solution of (4.9) if and only if x is an eigenvector of $\mathcal{F}$ with eigenvalue $\lambda^{1/(p-1)}$. Moreover, note that if (λ, x) is a solution of (4.9) then

$$\lambda = \lambda\|x\|_p^p = \langle \lambda x^{p-1}\,,\, x\rangle = \langle T(\boldsymbol{\cdot}, x, \ldots, x)\,,\, x\rangle = f_T(x).$$

We have seen in Section 3.2.2 that if $d = 3$ and T is symmetric, then the ℓ^p-eigenvectors of T can be characterized as the critical points of the function

$$\Phi(x) = \frac{f_T(x,x,x)}{\|x\|_p^3}.$$

A similar observation holds for every d. Indeed, it is discussed in [64, Section 4] that the ℓ^p-singular vector of a symmetric tensor T coincide with the critical points of

$$\Phi(x) = \frac{f_T(x,\ldots,x)}{\|x\|_p^d},$$

and it holds

$$\max_{x\neq 0} \Phi(x) = \sup\{\lambda \mid \lambda \text{ is an } \ell^p\text{-eigenvector of } T\}.$$

4.2.5 Quantum copulas

Let V_1 be the space of symmetric matrices in $\mathbb{R}^{m\times m}$ and V_2 the space of symmetric matrices in $\mathbb{R}^{n\times n}$. Let $C_1 \subset V_1$ and $C_2 \subset V_2$ be the cones of positive semi-definite matrices, $\mathcal{V} = V_1 \times V_2$ and $\mathcal{C} = C_1 \times C_2$. Let $\Phi\colon V_1 \to V_2$ be a linear mapping and denote by Φ^* its adjoint with respect to $\langle \cdot\,,\,\cdot\rangle$, the Hilbert-Schmidt inner products on V_1 and V_2. Suppose that $\Phi(C_1 \setminus \{0\}) \subset \operatorname{int}(C_2)$, then by Lemma 2 of [67], it holds $\Phi^*(C_2 \setminus \{0\}) \subset \operatorname{int}(C_1)$. Let $\mathcal{C} = C_1 \times C_2$. The following equation is discussed in the context of quantum copulas in [67]:

$$\begin{cases}\Phi(Y^{-1}) = \frac{1}{m}X^{-1}\\ \Phi^*(X) = \frac{1}{n}Y\end{cases} \qquad \text{with} \qquad (X,Y) \in \operatorname{int}(\mathcal{C}). \tag{4.10}$$

Define $\mathcal{F}\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ as

$$\mathcal{F}(X,Y) = \big(\Phi(Y^{-1})^{-1}, \Phi^*(X)\big) \qquad \forall (X,Y) \in \operatorname{int}(\mathcal{C}). \tag{4.11}$$

Then, $\mathcal{F}$ is order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{2\times 2}$ with

$$A = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

Moreover, the eigenvectors of $\mathcal{F}$ are related to the solutions of (4.10) by the following:

Proposition 4.2.7. Let $\mathcal{F}\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ be as in (4.11). Then, every solution of (4.10) is an eigenvector of $\mathcal{F}$. Conversely, if $(X,Y) \in \operatorname{int}(\mathcal{C})$ is an eigenvector of $\mathcal{F}$, there exists $\theta > 0$ such that $\mathcal{F}(X,Y) = (\frac{m}{n}\theta^{-1}X, \theta Y)$ and for all $t > 0$, $t\,(X, n\theta\, Y) \in \operatorname{int}(\mathcal{C})$ is a solution of (4.10).

Proof. Note that (X,Y) is a solution of (4.10) if and only if $\mathcal{F}(X,Y) = (mX, \frac{1}{n}Y)$. Now, suppose that $(X,Y) \in \operatorname{int}(\mathcal{C})$ is an eigenvector of $\mathcal{F}$. As $\mathcal{F}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$, there exists $\lambda, \theta > 0$ such that $\mathcal{F}(X,Y) = (\lambda X, \theta Y)$. Moreover, if $\operatorname{Tr}(\cdot)$ denotes the trace of a matrix, then it holds

$$\begin{aligned} n\theta &= \theta \operatorname{Tr}(Y^{-1}Y) = \operatorname{Tr}(Y^{-1}\mathcal{F}(X,Y)_2) = \langle Y^{-1}\,,\,\Phi^*(X)\rangle = \langle \Phi(Y^{-1})\,,\,X\rangle\\ &= \langle \mathcal{F}(X,Y)_1^{-1}\,,\,X\rangle = \lambda^{-1}\langle X^{-1}\,,\,X\rangle = \lambda^{-1}\operatorname{Tr}(X^{-1}X) = m\lambda^{-1},\end{aligned}$$

which proves that $\lambda = \frac{m}{n}\theta^{-1}$. Now, by the multi-homogeneity of $\mathcal{F}$, for all $\alpha, \beta > 0$ we have

$$\begin{aligned}\mathcal{F}(\alpha X, \beta Y) &= (\beta \mathcal{F}(X,Y)_1, \alpha\mathcal{F}(X,Y)_2)\\ &= \big(\tfrac{\beta\lambda}{\alpha}(\alpha X), \tfrac{\alpha\theta}{\beta}(\beta Y)\big) = \big(\tfrac{m}{n}\tfrac{\beta}{\alpha\theta}(\alpha X), \tfrac{\alpha\theta}{\beta}(\beta Y)\big).\end{aligned}$$

In particular for any $\alpha > 0$, the equation $\frac{\alpha\theta}{\beta} = \frac{1}{n}$ in β, has the unique solution $\beta = \alpha\xi$ with $\xi = \theta n$. Now, if $t > 0$, then with $\alpha = t$ and $\beta = \alpha\xi$, we have $\mathcal{F}(\alpha X, \beta Y) = (m\alpha X, \frac{1}{n}\beta Y)$. Hence, $(\alpha X, \beta Y) = t(X, \xi Y) \in \operatorname{int}(\mathcal{C})$ is a solution of (4.10) for all $t > 0$. □

4.2.6 Generalized DAD problem

Let $d \geq 2$. For $i = 1, \ldots, d$, let $V_i = \mathbb{R}^{n_i}$, $C_i = \mathbb{R}^{n_i}_+$, $z_i \in \mathbb{R}^{n_i}_{++}$ and $M_i \in \mathbb{R}^{n_{i+1} \times n_i}_+$ with $n_{d+1} = n_1$. Suppose that M_i has at least one positive entry per row. Let $\mathcal{C} = C_1 \times \ldots \times C_d$ and consider the following system of equations

$$\begin{cases} x_2 \circ M_1 x_1 & = & z_2 \\ x_3 \circ M_2 x_2 & = & z_3 \\ \vdots & \vdots & \vdots \\ x_d \circ M_{d-1} x_{d-1} & = & z_d \\ x_1 \circ M_d x_d & = & \lambda z_1 \end{cases} \qquad \text{with} \qquad (\lambda, x) \in \mathbb{R}_+ \times \operatorname{int}(\mathcal{C}). \tag{4.12}$$

The system of equations (4.12) is called a generalized DAD problem [60, page 163]. Note that if $d = 2$, $n_1 = n_2 = n$, $y_1 = y_2$ and $M_1 = M_2^\top = M$, then with the additional constraint that $\lambda = 1$, (4.12) reduces to the classical Sinkhorn equation which aims at finding diagonal matrices $D_1, D_2 \in \mathbb{R}^{n\times n}_+$ such that $D_1 M D_2$ is a doubly stochastic matrix. In this case, if (4.12) has a solution, then $D_1 = \operatorname{diag}(x_1)$ and $D_2 = \operatorname{diag}(x_2)$.

Now, let $\mathcal{F}\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ be given by

$$\mathcal{F}(x_1, \ldots, x_d) = \big(z_1 \circ (M_d x_d)^{-1}, z_2 \circ (M_1 x_1)^{-1}, \ldots, z_d \circ (M_{d-1} x_{d-1})^{-1}\big). \tag{4.13}$$

Then, $\mathcal{F}$ is order-reversing and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}$ with

$$A = \begin{pmatrix} 0 & 0 & \cdots & \cdots & -1 \\ -1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 0 \end{pmatrix}, \qquad \text{i.e.} \qquad A_{i,j} = \begin{cases} -1 & \text{if } i > 1 \text{ and } j = i-1 \\ -1 & \text{if } i = 1 \text{ and } j = d, \\ 0 & \text{otherwise.} \end{cases}$$

We relate the solutions of (4.12) with the eigenvectors of $\mathcal{F}$ in the following:

Proposition 4.2.8. Let $\mathcal{F}\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ be defined as in (4.13). Then every solution to (4.12) is an eigenvector of $\mathcal{F}$. Conversely, if $x \in \operatorname{int}(\mathcal{C})$ is an eigenvector of $\mathcal{F}$, then there exists $\boldsymbol{\lambda} \in \mathbb{R}^d_{++}$ such that $\mathcal{F}(x) = \boldsymbol{\lambda} \otimes x$. Moreover, if $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$ is defined as $\alpha_1 = 1$ and $\alpha_{i+1} = \lambda_{i+1}/\alpha_i$ for $i = 1, \ldots, d-1$, then $(\frac{\alpha_d}{\lambda_1}, \boldsymbol{\alpha} \otimes x)$ is a solution to (4.12).

Proof. Note that $(\lambda, x) \in \mathbb{R}_+ \times \operatorname{int}(\mathcal{C})$ is a solution to (4.12) if and only if

$$z_{i+1}^{-1} \circ M_i x_i = x_{i+1}^{-1} \quad \forall i = 1, \ldots, d-1 \qquad \text{and} \qquad z_1^{-1} \circ M_d x_d = \lambda x_1^{-1}.$$

Hence, $(\lambda, x) \in \mathbb{R}_+ \times \operatorname{int}(\mathcal{C})$ is a solution to (4.12) if and only if $\mathcal{F}(x) = \boldsymbol{\alpha} \otimes x$ with $\boldsymbol{\alpha} = (1, \ldots, 1, \lambda^{-1})$. It follows that every solution to (4.12) is an eigenvector of $\mathcal{F}$. Now, suppose that $x \in \operatorname{int}(\mathcal{C})$ is eigenvector of $\mathcal{F}$ and let $\boldsymbol{\lambda} \in \mathbb{R}^d_+$ be such that $\mathcal{F}(x) = \boldsymbol{\lambda} \otimes x$. Note that $\boldsymbol{\lambda} \in \mathbb{R}^d_{++}$ since $\mathcal{F}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$. Moreover, note that for all $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$ we have $\mathcal{F}(\boldsymbol{\alpha} \otimes x) = \boldsymbol{\alpha}^A \otimes \mathcal{F}(x) = (\boldsymbol{\alpha}^A \circ \boldsymbol{\lambda}) \otimes x$ where A is the multi-homogeneity matrix of $\mathcal{F}$. It follows that for all $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$ it holds

$$z_1 \circ (M_d \alpha_d x_d)^{-1} = \frac{1}{\alpha_d} z_1 \circ (M_d x_d)^{-1} = \Big(\frac{\lambda_1}{\alpha_1 \alpha_d}\Big)\alpha_1 x_1$$

and similarly

$$z_{i+1} \circ (M_i \alpha_i x_i)^{-1} = \Big(\frac{\lambda_{i+1}}{\alpha_i \alpha_{i+1}}\Big)\alpha_{i+1} x_{i+1} \qquad \forall i = 1, \ldots, d-1.$$

So, let $\alpha_1 = 1$ and $\alpha_{i+1} = \lambda_{i+1}/\alpha_i$ for $i = 1, \ldots, d-1$. Then, it holds

$$\frac{\lambda_{i+1}}{\alpha_i \alpha_{i+1}} = 1 \qquad \forall i = 1, \ldots, d-1 \qquad \text{and} \qquad \frac{\lambda_1}{\alpha_1 \alpha_d} = \frac{\lambda_1}{\alpha_d}.$$

It follows that $\mathcal{F}(\boldsymbol{\alpha} \otimes x) = \boldsymbol{\lambda} \otimes (\boldsymbol{\alpha} \otimes x)$ with $\boldsymbol{\lambda} = (1, \ldots, 1, \frac{\lambda_1}{\alpha_d})$ and thus, by the above discussion, $(\frac{\alpha_d}{\lambda_1}, \boldsymbol{\alpha} \otimes x)$ is a solution to (4.12). ☐

4.3 Reduction by substitution

Classically, the generalized DAD problem and the problem of finding the $\ell^{p,q}$-singular vectors of a matrix are solved by substituting the equations defining the eigenvectors of the corresponding multi-homogeneous mapping into each other in order to produce an equivalent problem which consists in finding the eigenvectors of a homogeneous mapping. Such reductions rely on the fact that the homogeneity matrix has a zero diagonal entry. This process is illustrated in the following example.

Example 4.3.1. Let $\mathcal{C} = \mathbb{R}^m_+ \times \mathbb{R}^n_+$, $M \in \mathbb{R}^{m\times n}_+$ and $p, q \in (1, \infty)$. Consider the mapping $\mathcal{F} \colon \mathcal{C} \to \mathcal{C}$ defined as

$$\mathcal{F}(x, y) = ((My)^{1/(p-1)}, (M^\top x)^{1/(q-1)}) \qquad \forall (x, y) \in \mathcal{C}.$$

As discussed in Section 3.2.1, the $\ell^{p,q}$-singular vectors of M are the eigenvectors of $\mathcal{F}$. Let $(u, v) \in \mathcal{C}$ be an eigenvector of $\mathcal{F}$, then there exists $\lambda, \theta \geq 0$ such that

$$\begin{cases} (Mv)^{1/(p-1)} = \lambda u \\ (M^\top u)^{1/(q-1)} = \theta v \end{cases} \tag{4.14}$$

By substituting one equation into the other, we see that if $\lambda, \theta > 0$, then

$$f(u) = \theta^{1/(p-1)} \lambda u \qquad \text{and} \qquad g(v) = \lambda^{1/(q-1)} \theta v,$$

where $f \colon \mathbb{R}^m_+ \to \mathbb{R}^m_+$ is the homogeneous mapping defined as

$$f(x) = (M(M^\top x)^{1/(q-1)})^{1/(p-1)} \qquad \forall x \in \mathbb{R}^m_+,$$

and $g \colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is the homogeneous mapping defined as

$$g(y) = (M^\top (My)^{1/(p-1)})^{1/(q-1)} \qquad \forall y \in \mathbb{R}^n_+.$$

Conversely, note that if $u \in \mathbb{R}^m_+ \setminus \{0\}$ satisfies $f(u) = \lambda u$ for some $\lambda > 0$, then we have $M^\top u \neq 0$ since $\lambda > 0$ and with $v = (M^\top u)^{1/(q-1)} \in \mathbb{R}^n_+ \setminus \{0\}$, we have

$$\mathcal{F}(u, v) = \big((Mv)^{1/(p-1)}, (M^\top u)^{1/(q-1)}\big) = \big(f(u), v\big) = (\lambda u, v),$$

i.e. (u, v) is an eigenvector of $\mathcal{F}$. Similarly, if $v \in \mathbb{R}^n_+ \setminus \{0\}$ satisfies $g(v) = \theta v$ for some $\theta > 0$, then we have $Mv \neq 0$ since $\theta > 0$ and with $u = (Mv)^{1/(p-1)}$, we have

$$\mathcal{F}(u, v) = \big((Mv)^{1/(p-1)}, (M^\top u)^{1/(q-1)}\big) = \big(u, g(v)\big) = (u, \theta v),$$

which, again, shows that (u, v) is an eigenvector of $\mathcal{F}$. Now, consider the sets $E_f \subset \mathbb{R}^m_+, E_g \subset \mathbb{R}^n_+$ and $E_{\mathcal{F}} \subset \mathbb{R}^m_+ \times \mathbb{R}^n_+$ given by

$$\begin{aligned} E_f &= \{u \in \mathbb{R}^m_+ \setminus \{0\} \mid \exists \lambda > 0 \text{ such that } f(u) = \lambda u\} \\ E_g &= \{v \in \mathbb{R}^n_+ \setminus \{0\} \mid \exists \theta > 0 \text{ such that } g(v) = \theta v\} \\ E_{\mathcal{F}} &= \{(u, v) \in \mathbb{R}^m_+ \setminus \{0\} \times \mathbb{R}^n_+ \setminus \{0\} \mid \exists \lambda, \theta > 0 \text{ such that } \mathcal{F}(u, v) = (\lambda u, \theta v)\}, \end{aligned}$$

then the above discussion proves that there exists a bijection $\phi\colon E \to E'$ for all $E, E' \in \{E_f, E_g, E_{\mathcal{F}}\}$. Hence, finding an eigenvector of f with positive eigenvalue is equivalent to finding an eigenvector of g with positive eigenvalue which is equivalent to finding an eigenvector of $\mathcal{F}$ with positive eigenvalue. This equivalence has an interpretation in terms of optimization. We have discussed in Section 3.2, that the nonnegative maximizers of

$$\|M\|_{p,q} = \max_{x \in \mathbb{R}^m \setminus \{0\}, y \in \mathbb{R}^n \setminus \{0\}} \frac{\langle x \,,\, My \rangle}{\|x\|_p \|y\|_q},$$

are eigenvectors of $\mathcal{F}$. Now, let $p' = p/(p-1)$, $q' = q/(q-1)$,

$$\|M\|_{q \to p'} = \max_{y \in \mathbb{R}^n \setminus \{0\}} \frac{\|My\|_{p'}}{\|y\|_q} \qquad \text{and} \qquad \|M^\top\|_{p \to q'} = \max_{x \in \mathbb{R}^m \setminus \{0\}} \frac{\|M^\top x\|_{q'}}{\|x\|_p}.$$

The nonnegative maximizers of $\|M\|_{q \to p'}$ are eigenvectors of g and the nonnegative maximizers of $\|M^\top\|_{p \to q'}$ are eigenvectors of f. Furthermore, as noted in (3.1), it holds $\|M\|_{p,q} = \|M\|_{q \to p'} = \|M^\top\|_{p \to q'}$.

We discuss how such reductions can be done in the case of a general multi-homogeneous mapping defined on the product of cones. Suppose that $d \geq 2$ and let $\mathcal{V} = V_1 \times \ldots \times V_d$. In the next proposition we consider a mapping $\mathcal{F}\colon \mathcal{P} \to \mathcal{P}$ where $\mathcal{P}$ is the part of a cone $\mathcal{C} \subset \mathcal{V}$. We do that in order to use Lemma 3.3.3 which states that if $\mathcal{F}$ is order-preserving or order-reversing and multi-homogeneous of degree $A \in \mathbb{R}^{d \times d}$, then $A_{i,j} = 0$ implies that $y_j \mapsto \mathcal{F}(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_d)_i$ is constant on the part P_j of C_j. We note however that if $\mathcal{F}$ is assumed to be continuous, then the exact same result (and its proof) holds with $\mathcal{P}$ replaced by $\mathcal{C}$. We do not want to assume that $\mathcal{F}$ is continuous on $\mathcal{C}$ because, for instance, the mapping $\mathcal{F}$ of the generalized DAD problem is continuous on $\operatorname{int}(\mathcal{C})$ but can not be extended by continuity on the boundary of $\mathcal{C}$. Indeed, this mapping tends to infinity as x approaches 0.

Proposition 4.3.2. Let $\mathcal{C} \subset \mathcal{V}$ be a cone and let $\mathcal{P}$ be a part of $\mathcal{C}$. Let $\mathcal{F}\colon \mathcal{P} \to \mathcal{P}$ be either order-preserving or order-reversing and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}$. Suppose that $A_{d,d} = 0$ and let $\hat{\mathcal{P}} = P_1 \times \ldots \times P_{d-1}$. For every $x_d \in P_d$, define $\hat{\mathcal{F}}_{x_d}\colon \hat{\mathcal{P}} \to \hat{\mathcal{P}}$ as

$$\hat{\mathcal{F}}_{x_d}(\hat{x})_i = \mathcal{F}\big(\hat{x}_1, \ldots, \hat{x}_{d-1}, \mathcal{F}(\hat{x}_1, \ldots, \hat{x}_{d-1}, x_d)_d\big)_i \qquad \forall \hat{x} \in \hat{\mathcal{P}},\ i = 1, \ldots, d-1.$$

Then, for every $x_d \in P_d$, the following assertions hold:

a) $\hat{\mathcal{F}}_{x_d}$ is multi-homogeneous of degree $\hat{A} \in \mathbb{R}^{(d-1)\times(d-1)}$, with

$$\hat{A}_{i,j} = A_{i,j} + A_{i,d}A_{d,j} \qquad \forall i, j = 1, \ldots, d-1.$$

Furthermore, if $\mathcal{F}$ is order-preserving then $\hat{\mathcal{F}}_{x_d}$ is order-preserving.

b) For every $y_d \in P_d$, it holds $\hat{\mathcal{F}}_{x_d} = \hat{\mathcal{F}}_{y_d}$.

c) Let $\hat{u} \in \hat{\mathcal{P}}$, $u_d \in P_d$ and set $u = (\hat{u}, u_d) \in \mathcal{P}$. Then, the following are equivalent:

   i) There exists $\boldsymbol{\lambda} \in \mathbb{R}^d_{++}$ such that $\mathcal{F}(u) = \boldsymbol{\lambda} \otimes u$.

   ii) There exists $\boldsymbol{\theta} \in \mathbb{R}^{d-1}_{++}$ and $\theta_d > 0$ such that $\hat{\mathcal{F}}_{x_d}(\hat{u}) = \boldsymbol{\theta} \otimes \hat{u}$ and $\mathcal{F}(\hat{u}_1, \ldots, \hat{u}_{d_1}, x_d)_d = \theta_d u_d$.

Proof. Let $\hat{x} \in \mathcal{P}$ and $\hat{\boldsymbol{\alpha}} \in \mathbb{R}^{d-1}_{++}$, then for every $i = 1, \ldots, d-1$, we have

$$\hat{\mathcal{F}}_{x_d}(\hat{\boldsymbol{\alpha}} \otimes \hat{x})_i = \mathcal{F}(\hat{\boldsymbol{\alpha}} \otimes \hat{x}, \mathcal{F}(\hat{\boldsymbol{\alpha}} \otimes \hat{x}, x_d)_d)_i = \Big(\prod_{j=1}^{d-1} \hat{\alpha}_j^{A_{i,j}}\Big)\mathcal{F}(\hat{x}, \mathcal{F}(\hat{\boldsymbol{\alpha}} \otimes \hat{x}, x_d)_d)_i$$

$$= \Big(\prod_{j=1}^{d-1} \hat{\alpha}_j^{A_{i,j}}\Big)\Big(\prod_{j=1}^{d-1} \hat{\alpha}_j^{A_{d,j}}\Big)^{A_{i,d}}\mathcal{F}(\hat{x}, \mathcal{F}(\hat{x}, x_d)_d)_i = \Big(\prod_{j=1}^{d-1} \hat{\alpha}_j^{A_{i,j}+A_{i,d}A_{d,j}}\Big)\hat{\mathcal{F}}_{x_d}(\hat{x})_i,$$

which implies that $\hat{\mathcal{F}}_{x_d}$ is multi-homogeneous of degree $\hat{A}$. Let $\hat{\mathcal{C}} = C_1 \times \ldots \times C_{d-1}$. Suppose that $\mathcal{F}$ is order-preserving and let $\hat{x}, \hat{y} \in \hat{\mathcal{P}}$ be such that $\hat{x} \preceq_{\hat{\mathcal{C}}} \hat{y}$. Set $\tilde{x}_d = \mathcal{F}(\hat{x}, x_d)_d \in P_d$ and $\tilde{y}_d = \mathcal{F}(\hat{y}, x_d)_d \in P_d$. Then, as $\mathcal{F}$ is order-preserving, we have $\tilde{x}_d \preceq_{C_d} \tilde{y}_d$ and thus $(\hat{x}, \tilde{x}_d) \preceq_{\mathcal{C}} (\hat{y}, \tilde{y}_d)$. It follows that $\hat{\mathcal{F}}_{x_d}(\hat{x})_i = \mathcal{F}(\hat{x}, \tilde{x}_d)_i \preceq_{C_i} \mathcal{F}(\hat{y}, \tilde{y}_d)_i = \hat{\mathcal{F}}_{x_d}(\hat{y})_i$ for $i = 1, \ldots, d-1$. This concludes the proof of a). Note that b) follows directly from Lemma 3.3.3. We prove c). Let $\hat{u} \in \hat{\mathcal{P}}$, $u_d \in P_d$ and set $u = (\hat{u}, u_d) \in \mathcal{P}$. Suppose that there exists $\boldsymbol{\lambda} \in \mathbb{R}^d_{++}$ such that $\mathcal{F}(u) = \boldsymbol{\lambda} \otimes u$. Then, in particular, we have $\mathcal{F}(u)_d = \lambda_d u_d$. As $x_d \sim_{C_d} u_d$, Lemma 3.3.3 implies that $\mathcal{F}(u)_d = \mathcal{F}(\hat{u}, x_d)_d$ and thus it holds

$$\lambda_d u_d = \mathcal{F}(u)_d = \mathcal{F}(\hat{u}, x_d)_d.$$

Furthermore, for $i = 1, \ldots, d-1$, we have

$$\mathcal{F}_{x_d}(\hat{u})_i = \mathcal{F}(\hat{u}, \mathcal{F}(\hat{u}, x_d)_d)_i = \mathcal{F}(\hat{u}, \lambda_d u_d)_i = \lambda_d^{A_{i,d}}\mathcal{F}(u)_i = \lambda_d^{A_{i,d}}\lambda_i u_i.$$

Hence, ii) holds with $\boldsymbol{\theta} = (\lambda_d^{A_{1,d}}\lambda_1, \ldots, \lambda_d^{A_{d-1,d}}\lambda_{d-1}) \in \mathbb{R}^n_{++}$ and $\theta_d = \lambda_d$. Now, suppose that ii) holds, i.e. there exists $\boldsymbol{\theta} \in \mathbb{R}^{d-1}_{++}$ such that $\hat{\mathcal{F}}_{x_d}(\hat{u}) = \boldsymbol{\theta} \otimes \hat{u}$. Set $u_d = \mathcal{F}(\hat{u}, x_d)_d \in P_d$ and $u = (\hat{u}, u_d) \in \mathcal{P}$. By Lemma 3.3.3, we have $\mathcal{F}(\hat{u}, x_d)_d = \mathcal{F}(\hat{u}, u_d)_d$ since $A_{d,d} = 0$, and thus $\mathcal{F}(u)_d = u_d$. Furthermore, it follows that for all $i = 1, \ldots, d-1$, it holds

$$\mathcal{F}(u)_i = \mathcal{F}(\hat{u}, u_d)_i = \mathcal{F}(\hat{u}, \mathcal{F}(\hat{u}, x_d)_d)_i = \hat{\mathcal{F}}_{x_d}(\hat{u})_i = \theta_i \hat{u}_i.$$

Therefore, $\mathcal{F}(u) = \boldsymbol{\lambda} \otimes u$ with $\boldsymbol{\lambda} = (\boldsymbol{\theta}, 1) \in \mathbb{R}^d_{++}$ which concludes the proof. □

We note that the transformation $\hat{\mathcal{F}}_{x_d}$ discussed in Proposition 4.3.2 has been used in [36]$^\diamond$ for the study of $\ell^{p_1,\ldots,p_d}$-eigenvectors of nonnegative tensors. Furthermore, it has been used in [67] in order to use classical arguments of the non-linear Perron-Frobenius theory for the study of the eigenvectors of $\mathcal{F}$ defined in (4.11). Finally, we note that the multi-homogeneous mapping $\mathcal{F}\colon \mathbb{R}^{n_1}_{++} \times \ldots \times \mathbb{R}^{n_d}_{++} \to \mathbb{R}^{n_1}_{++} \times \ldots \times \mathbb{R}^{n_d}_{++}$ defined in (4.13) is a particularly nice candidate for Proposition 4.3.2. Indeed, by applying Proposition 4.3.2, d times, it can be shown that the study of the eigenvectors of $\mathcal{F}$ can be reduced to the study of the eigenvectors of a homogeneous mapping $f\colon \mathbb{R}^{n_1}_{++} \to \mathbb{R}^{n_1}_{++}$. This particular reduction for the generalized DAD problem is used in the proof of [60, Theorem 7.1.2].

4.4 Reducible homogeneity matrices

We explain why the study of eigenvectors of multi-homogeneous mappings can be without loss of generality be restricted to the cases where the homogeneity matrix is irreducible. To gain intuition we start with a simple example.

Example 4.4.1. Let $\mathcal{C} = \mathbb{R}^n_+ \times \mathbb{R}^n_+$, $M \in \mathbb{R}^{n\times n}_+$ and $T \in \mathbb{R}^{n\times n\times n}_+$. Define $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ as

$$\mathcal{F}(x, y) = \big(T(\cdot, x, y), My\big) \qquad \forall (x, y) \in \mathcal{C}.$$

Then, $\mathcal{F}$ is order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{2\times 2}$ with

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The eigenvectors of $\mathcal{F}$ are the solutions to the equation

$$\begin{cases} T(\cdot, x, y) = \lambda x, \\ My = \theta y. \end{cases}$$

Hence, in order to find an eigenvector of $\mathcal{F}$, we can first find an eigenvector of M, say $\bar{y} \in \mathbb{R}^n_+ \setminus \{0\}$ and then find an eigenvector of the linear mapping $x \mapsto T(\cdot, x, \bar{y})$.

In the example above, we see that it is sometimes possible to break down the problem of finding the eigenvectors of an order-preserving, resp. order-reversing, multi-homogeneous mappings into smaller subproblems. This can be done when the homogeneity matrix is reducible. We generalize this observation. Let $d \geq 2$ and $\mathcal{V} = V_1 \times \ldots \times V_d$ be the product of finite dimensional real vector spaces.

Proposition 4.4.2. Let $\mathcal{C} \subset \mathcal{V}$ be a cone and let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}$. Suppose that $\mathcal{F}$ is either order-preserving, or order-reversing and there exists $1 \leq k \leq d-1$, $\bar{A} \in \mathbb{R}^{k\times k}$, $\hat{A} \in \mathbb{R}^{(d-k)\times(d-k)}$, $B \in \mathbb{R}^{k\times(d-k)}$ such that

$$A = \begin{pmatrix} \bar{A} & B \\ 0 & \hat{A} \end{pmatrix}.$$

Let $\bar{\mathcal{C}} = C_1 \times \ldots \times C_k$ and $\hat{\mathcal{C}} = C_{k+1} \times \ldots \times C_d$ so that $\mathcal{C} = \bar{\mathcal{C}} \times \hat{\mathcal{C}}$. For $\bar{x} \in \bar{\mathcal{C}}$ and $\hat{x} \in \hat{\mathcal{C}}$ define $\bar{\mathcal{F}}_{\hat{x}}\colon \bar{\mathcal{C}} \to \bar{\mathcal{C}}$ and $\hat{\mathcal{F}}_{\bar{x}}\colon \hat{\mathcal{C}} \to \hat{\mathcal{C}}$ as follows:

$$\begin{aligned} \bar{\mathcal{F}}_{\hat{x}}(\bar{z}) &= \big(\mathcal{F}(\bar{z},\hat{x})_1,\ldots,\mathcal{F}(\bar{z},\hat{x})_k\big), \qquad \forall \bar{z} \in \bar{\mathcal{C}},\\ \hat{\mathcal{F}}_{\bar{x}}(\hat{z}) &= \big(\mathcal{F}(\bar{x},\hat{z})_{k+1},\ldots,\mathcal{F}(\bar{x},\hat{z})_d\big), \qquad \forall \hat{z} \in \hat{\mathcal{C}}. \end{aligned}$$

Then, for every $\bar{x} \in \bar{\mathcal{C}}$ and $\hat{x} \in \hat{\mathcal{C}}$, the following assertions hold:

a) $\bar{\mathcal{F}}_{\hat{x}}$ and $\hat{\mathcal{F}}_{\bar{x}}$ are multi-homogeneous of degree $\bar{A}$ and $\hat{A}$, respectively.

b) If $\mathcal{F}$ is either order-preserving or order-reversing, then so are $\bar{\mathcal{F}}_{\hat{x}}$ and $\hat{\mathcal{F}}_{\bar{x}}$.

c) For every $\bar{y} \in \bar{\mathcal{C}}$ such that $\bar{x} \sim_{\bar{\mathcal{C}}} \bar{y}$ it holds $\hat{\mathcal{F}}_{\bar{x}} = \hat{\mathcal{F}}_{\bar{y}}$.

d) Let $\bar{u} \in \bar{\mathcal{C}}$, $\hat{u} \in \hat{\mathcal{C}}$ and set $u = (\bar{u},\hat{u}) \in \mathcal{C}$. Then, u is an eigenvector of $\mathcal{F}$ if, and only if, $\bar{u}$ is an eigenvector of $\bar{\mathcal{F}}_{\hat{u}}$ and $\hat{u}$ is an eigenvector of $\hat{\mathcal{F}}_{\bar{u}}$.

Proof. a) and b) follow from a direct verification. c) follows from Lemma 3.3.3 and Remark 3.3.4. Finally, d) follows from the fact that for every $\bar{x} \in \bar{\mathcal{C}}$ and $\hat{x} \in \hat{\mathcal{C}}$, if $x = (\bar{x},\hat{x})$, then it holds $\mathcal{F}(x) = (\bar{\mathcal{F}}_{\hat{x}}(\bar{x}), \hat{\mathcal{F}}_{\bar{x}}(\hat{x}))$. □

The key point of Proposition 4.4.2 is c). Indeed, suppose that $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ satisfies the assumptions of Proposition 4.4.2, then one can proceed as follows to find eigenvectors of $\mathcal{F}$: First, find a part $\mathcal{P}$ of $\mathcal{C}$ such that $\mathcal{F}(\mathcal{P}) \subset \mathcal{P}$ and set $\bar{\mathcal{P}} = P_1 \times \ldots \times P_k$ and $\hat{\mathcal{P}} = P_{k+1} \times \ldots \times P_d$. Take any $\bar{x} \in \bar{\mathcal{P}}$. Note that $\hat{\mathcal{F}}_{\bar{x}}(\hat{\mathcal{P}}) \subset \hat{\mathcal{P}}$. Find an eigenvector $\hat{u} \in \hat{\mathcal{P}}$ of $\hat{\mathcal{F}}_{\bar{x}}$. Finally, find an eigenvector $\bar{u} \in \bar{\mathcal{P}}$ of $\bar{\mathcal{F}}_{\hat{u}}$. Then, $u = (\bar{u},\hat{u}) \in \mathcal{P}$ is an eigenvector of $\mathcal{F}$ since $\hat{\mathcal{F}}_{\bar{x}} = \hat{\mathcal{F}}_{\bar{u}}$ by the property c) of Proposition 4.4.2.

The procedure described above implies that if $|A|$ is reducible, then one can find eigenvectors of $\mathcal{F}$ by finding the eigenvectors of multi-homogeneous mappings defined on the product of fewer cones whose homogeneity matrices are all irreducible. This can be done by recursively applying Proposition 4.4.2 to the Frobenius normal form of $|A|$ which exists by the following well-known result:

Theorem 4.4.3. Let $A \in \mathbb{R}^{d\times d}_+$, then there exist nonnegative integers s, t such that $1 \leq s + t \leq d$, positive integers $n_1, \ldots, n_{s+t}$ such that $d_1 + \ldots + d_{s+t} = d$, and a permutation matrix $P \in \mathbb{R}^{d\times d}$ such that $B = PAP^\top$ has the Frobenius normal form:

$$B = \begin{pmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,t} & B_{1,t+1} & B_{1,t+2} & \cdots & B_{1,t+s} \\ 0 & B_{2,2} & \cdots & B_{2,t} & B_{2,t+1} & B_{2,t+2} & \cdots & B_{2,t+s} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & B_{t,t} & B_{t,t+1} & B_{t,t+2} & \cdots & B_{t,t+s} \\ 0 & 0 & \cdots & 0 & B_{t+1,t+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & B_{t+s,t+s} \end{pmatrix},$$

where $B_{i,j} \in \mathbb{R}_+^{d_i\times d_j}$ for $i,j = 1,\dots,t+s$ and $B_{i,i}$ is either irreducible or zero for $i = 1,\dots,t+s$.

Proof. See Theorem 6.4.4 of [30]. □

5 Metrics induced by a cone

A key observation for the nonlinear generalizations of the Perron-Frobenius theorem is the consideration of the Hilbert projective metric induced by a cone. Indeed, it was noted by Birkhoff [14] and Samelson [86] that a linear operator leaving $\mathbb{R}^n_+$ invariant is non-expansive under the Hilbert metric induced by the cone $\mathbb{R}^n_+$. Furthermore, Birkhoff noted that the linear operator associated to a positive matrix is a strict contraction and gave a formula for the contraction ratio. This is the celebrated Birkhoff-Hopf theorem recalled below. Furthermore, Bushell [22] observed that a mapping between cones which is order-preserving and homogeneous of degree $p > 0$, has Lipschitz constant p with respect to the Hilbert metric induced by the cones. In particular, order-preserving 1-homogeneous mappings are non-expansive.

5.1 Hilbert's metric

Let V be a finite dimensional real vector space. Let $C \subset V$ be a cone and for every $x, y \in C \setminus \{0\}$ let

$$M(x/y; C) = \inf\{\beta \geq 0\colon x \preceq_C \beta\, y\}, \quad m(x/y; C) = \sup\{\alpha \geq 0\colon \alpha\, y \preceq_C x\}, \quad (5.1)$$

where we set $M(x/y; C) = \infty$ if there is no $\beta \geq 0$ such that $x \preceq_C \beta y$. In particular, note that if $x \sim_C y$, then $0 < M(x/y; C), m(x/y; C), M(y/x; C), m(y/x; C) < \infty$.

Example 5.1.1. Let $C = \mathbb{R}^n_+$. For $x \in \mathbb{R}^n_+$, let $I(x) = \{i \,|\, x_i > 0\} \subset \{1,\dots,n\}$. Let $x, y \in \mathbb{R}^n_+$, if $I(y) \subset I(x)$, then

$$M(x/y; \mathbb{R}^n_+) = \max_{i\in I(x)} \frac{x_i}{y_i} \qquad \text{and} \qquad m(x/y; \mathbb{R}^n_+) = \min_{i\in I(y)} \frac{x_i}{y_i}.$$

We recall from [22] elementary properties of $M(x/y; C)$ and $m(x/y; C)$.

Lemma 5.1.2. Let $C \subset V$ be a closed cone, $\alpha, \beta > 0$ and let $x, y, u, v \in C \setminus \{0\}$ be such that $x \sim_C y \sim_C u \sim_C v$ and $x \preceq_C u$, $v \preceq_C y$, then

i) $m(x/y;C) \leq M(x/y;C)$,

ii) $m(x/y;C)y \preceq_C x \preceq_C M(x/y;C)y$,

iii) $M(\alpha x + \beta y/y;C) = \alpha M(x/y;C) + \beta$,

iv) $m(\alpha x + \beta y/y;C) = \alpha m(x/y;C) + \beta$,

v) $M(x/y;C)m(y/x;C) = 1$,

vi) $M(x/y;C) \leq M(u/v;C)$ and $m(u/v;C) \leq m(x/y;C)$.

Proof. See [22]. □

We discuss one more useful property of $M(\cdot/\cdot;C)$ and $m(\cdot/\cdot;C)$ in the following lemma:

Lemma 5.1.3. Let $C \subset V$ be a closed cone, let $x, y \in C$ with $x \sim_C y$, and let $\nu\colon C \to \mathbb{R}_+$ be homogeneous of degree 1 and monotonic. If $\nu(x) = \nu(y) = 1$, then

$$m(x/y) \leq 1 \leq M(x/y;C).$$

Proof. As ν is monotonic and homogeneous, it holds

$$1 = \nu(x) \leq \nu(M(x/y;C)y) = M(x/y;C)\nu(y) = M(x/y;C).$$

Similarly, we have

$$1 = \nu(x) \geq \nu(m(x/y;C)y) = m(x/y;C)\nu(y) = m(x/y;C),$$

which concludes the proof. □

Hilbert's metric is defined in terms of $M(\cdot/\cdot;C)$ and $m(\cdot/\cdot;C)$. Let $x, y \in C$ be such that $x \sim_C y$. The Hilbert projective distance $\mu_C(x,y)$ between x and y is defined as follows: Either $x = y = 0$ and we set $\mu_C(0,0) = 0$, or $x, y \neq 0$ and

$$\mu_C(x,y) = \ln\big(M(x/y;C)\, M(y/x;C)\big).$$

If $x \not\sim_C y$, then we set $\mu_C(x,y) = \infty$. We refer to $\mu_C\colon C \times C \to \bar{\mathbb{R}}$, where $\bar{\mathbb{R}} = [-\infty, \infty]$ is the extended real line, as the *Hilbert metric* induced by C. Note that if $x \sim_C y$ and $x, y \neq 0$, then by Lemma 5.1.2, we have

$$\mu_C(x,y) = \ln\Big(\frac{M(x/y;C)}{m(x/y;C)}\Big).$$

We discuss examples of this metric on particular cones.

Example 5.1.4. Let $C = \mathbb{R}^n_+$ and $x, y \in \mathbb{R}^n_{++}$, then it holds

$$\mu_C(x,y) = \ln\Big(\max_{i=1,\dots,n} \frac{x_i}{y_i} \max_{j=1,\dots,n} \frac{y_j}{x_j}\Big) = \max_{i,j=1,\dots,n} \ln\Big(\frac{x_i y_j}{y_i x_j}\Big).$$

If $x, y \in \mathbb{R}^n_+ \setminus \mathbb{R}^n_{++}$ and $x \sim_C y$, then x, y have the same zero pattern, i.e. $x_i > 0$ if and only if $y_i > 0$, and $\mu_C(x,y)$ can be computed as above by restricting the maxima to the nonzero coordinates.

Example 5.1.5. Suppose that $C \subset \mathbb{R}^n$ is a solid polyhedral cone and let $\xi_1, \ldots, \xi_N \in V^*$ be the facet defining functionals of C. Then the proof of Lemma 2.2.2 in [60] shows that for all $x, y \in \operatorname{int}(C)$, it holds

$$M(x/y; C) = \max_{i=1,\ldots,N} \frac{\xi_i(x)}{\xi_i(y)},$$

so that

$$\mu_C(x,y) = \ln\Big(\max_{i,j=1,\ldots,N} \frac{\xi_i(x)\xi_j(y)}{\xi_i(y)\xi_j(x)}\Big) = \max_{1\leq i<j\leq N} \big|\ln\Big(\frac{\xi_i(x)\xi_j(y)}{\xi_i(y)\xi_j(x)}\Big)\big|.$$

Example 5.1.6. Let $C \subset \mathbb{R}^{n\times n}$ be the cone of symmetric positive semi-definite matrices. Then, Proposition 2.4.2 of [60] implies that for every positive definite matrices $A, B \in \operatorname{int}(C)$, it holds

$$\mu_C(A,B) = \ln\big(\rho(A^{-1}B)\rho(B^{-1}A)\big).$$

In facts, μ_C is a pseudometric. We recall that on a set X, $\eta\colon X\times X \to [0,\infty)$ is a *pseudometric* if η satisfies all the conditions of a metric except that $\eta(x,y) = 0$ can happen even if $x \neq y$. The pair (X, η) is called a *pseudometric space*. The Hilbert distance μ_C is a pseudometric since if $x, y \in C$ are linearly dependent, then $\mu_C(x,y) = 0$. More generally, for every $x, y \in C$ with $x \sim_C y$, it holds

$$\mu_C(\alpha\, x, \beta\, y) = \mu_C(x,y) \qquad \forall \alpha, \beta > 0. \tag{5.2}$$

On the other hand, if C is closed, then $x \sim_C y$ and $\mu_C(x,y) = 0$ imply the existence of $\alpha > 0$ such that $x = \alpha y$ (see Lemma 2.7 of [28]). This property allows to formulate eigenvector problems as fixed point problems. Indeed, let $f\colon C \to C$ be a mapping and $x \in C \setminus \{0\}$. Then the above observations imply that x is an eigenvector of f if and only if it is a fixed point of f with respect to the Hilbert metric induced by C. Formally, for every $x \in C \setminus \{0\}$ we have

$$\exists \lambda > 0 \text{ such that } f(x) = \lambda x \quad \iff \quad \mu_C(f(x), x) = 0. \tag{5.3}$$

The equivalence in (5.3) allows then to use results of fixed point theory to study the eigenvectors of mappings leaving a closed cone invariant. The completeness of Hilbert's metric is discussed in the following lemma:

Lemma 5.1.7. Let $C \subset V$ be a closed cone and $P \subset C$ a nonzero part of C. Furthermore, let $\nu\colon C \to [0,\infty)$ be continuous, 1-homogeneous and suppose that $\nu(x) > 0$ for all $x \in P$. Consider the sphere $S_\nu = \{x \in C\colon \nu(x) = 1\}$. Then, $(S_\nu \cap P, \mu_C)$ is a complete metric space. Furthermore, the topology of (S_ν, μ_C) coincides with the norm topology on V.

Proof. See [77, Lemma 2.1] for the completeness and [60, Corollary 2.5.6] for the topology of (S_ν, μ_C). □

As discussed in the next remark, other types of metric were considered on cones.

Remark 5.1.8. The function $F_C(x, y) = \ln(M(x/y; C))$ is known as the Funk metric [78, Chapter 1 and 2]. The Funk metric is not symmetric and is a hemimetric [35]. The Hilbert metric can be seen as a symmetrization of the Funk metric. Indeed, it holds

$$\mu_C(x, y) = F_C(x, y) + F_C(y, x) \qquad \forall x, y \in C \setminus \{0\}, x \sim_C y.$$

The Thompson metric [91] induced by C is a different symmetrization of the Funk metric and is defined as

$$\delta_C(x, y) = \max\{F_C(x, y), F_C(y, x)\} \tag{5.4}$$

for every $x, y \in C \setminus \{0\}$ with $x \sim_C y$. For every nonzero part $P \subset C$, (P, δ_C) is a complete metric space. The Thompson metric is typically used for mappings which are not homogeneous.

The following relationship between μ_C and a monotonic norm $\|\cdot\|$ on V is useful to deduce a convergence rate in terms of a norm on V from a convergence rate in terms of the Hilbert metric.

Lemma 5.1.9. Let $C \subset V$ be a closed cone. Let $\|\cdot\|$ be a norm on V with normality constant γ and for $r > 0$, let $S_r = \{x \in C : \|x\| = r\}$. Then,

$$\|x - y\| \leq r(1 + 2\gamma)(e^{\mu_C(x,y)} - 1), \qquad \forall x, y \in S_r \quad \text{with} \quad x \sim_C y.$$

Proof. By Lemma 2.5.1 of [60], it holds

$$\|x - y\| \leq r(1 + 2\gamma)(e^{\delta_C(x,y)} - 1) \qquad \forall x, y \in S_r \quad \text{with} \quad x \sim_C y, \tag{5.5}$$

where $\delta_C(x, y)$, defined in (5.4), is the Thompson metric induced by C. Let $x, y \in S_r$ with $x \sim_C y$. As $x, y \in S_r$ and $\|\cdot\|$ is monotonic, Lemma 5.1.3 implies that $M(x/y; C), M(y/x; C) \geq 1$ and thus

$$e^{\delta_C(x,y)} = \max\big\{M(x/y; C), M(y/x; C)\big\} \leq M(x/y; C)M(y/x; C) = e^{\mu_C(x,y)}.$$

Combining this observation with (5.5) concludes the proof. ☐

We conclude with a short discussion on the geometry of the Hilbert metric. Suppose that $C \subset V$ is a solid closed cone. Let $w \in \text{int}(C^*)$ and $\Sigma_w = \{x \in C \,|\, w(x) = 1\}$. Then, Lemmas 2.6.1 of [60] imply that for $\epsilon > 0$ and $x \in \text{int}(C)$, the closed ball $B_\epsilon(x) = \{y \in \Sigma_w \,|\, \mu_C(x, y) \leq \epsilon\}$ is a convex subset of V. Finally, let us recall that a metric space (X, μ) is a *geodesic metric space* if for every $x, y \in X$ with $\mu(x, y) > 0$, there exists a geodesic between x and y, i.e. a curve $\gamma\colon [0, 1] \to X$ such that $\gamma(0) = x, \gamma(1) = y$ and $\mu(\gamma(s), \gamma(t)) = |s - t|\mu(x, y)$ for every $s, t \in [0, 1]$. Corollary 2.6.4 implies that (Σ_w, μ_C) is a geodesic space and straight line segments are geodesics.

5.2 Lipschitz constant of homogeneous mappings on cones

We discuss the Lipschitz constant of mappings between cones. We recall that for two pseudometric spaces (X, η), (Y, ζ) and a mapping $f\colon X \to Y$, a scalar $a \geq 0$ is said to be a *Lipschitz constant* of f with respect to η and ζ if

$$\eta(f(y), f(z)) \leq a\,\zeta(y, z) \qquad \forall y, z \in X.$$

When the context makes it clear which pseudometrics are used on the domain and codomain of f, we simply say that a is a Lipschitz constant of f. If f has a bounded Lipschitz constant, then we say that $\bar{a}$ is the *smallest Lipschitz constant* of $f\colon (X, \eta) \to (Y, \zeta)$ if

$$\bar{a} = \inf\{a > 0 \mid \eta(f(x), f(y)) \leq a\zeta(x, y), \quad \forall x, y \in X\}.$$

If 1 is a Lipschitz constant of f, then we say that f is *non-expansive*. If there exists $a < 1$ such that a is a Lipschitz constant of f, then we say that f is a *strict contraction*. Finally, we say that f is *contractive* if

$$\eta(f(y), f(z)) < \zeta(y, z) \qquad \forall y, z \in X \quad \text{with} \quad y \neq z.$$

In the following, we recall results about the Lipschitz constant of mappings between cones. First we discuss the case where the mapping is order-preserving/reversing and homogeneous of degree p. Then we recall the Birkhoff-Hopf theorem which gives a formula for the smallest Lipschitz constant of a linear mapping with respect to the Hilbert metric.

5.2.1 Homogeneous mappings

An order-preserving p-homogeneous mapping $f\colon C \to K$ has p as Lipschitz constant with respect to Hilbert's metric. Indeed, such an f has the property that for all $x, y \in C$ with $x \sim_C y$, it holds

$$f(x) \preceq_K f(M(x/y; C)y) = M(x/y; C)^p f(y),$$

which implies that $M(f(x)/f(y); C) \leq M(x/y; C)^p$. It follows that

$$\mu_C(f(x), f(y)) \leq p\,\mu_C(x, y) \qquad \forall x, y \in C \quad \text{with} \quad x \sim_C y.$$

Similar arguments hold for other classes of mappings.

The next proposition shows that order-preserving and order-reversing p-homogeneous mappings have $|p|$ as Lipschitz constant.

Proposition 5.2.1. Let $C \subset V, K \subset W$ be closed cones and $f\colon C \to K$. If f is either order-preserving and homogeneous of degree $p > 0$ or order-reversing and homogeneous of degree $p < 0$, then

$$\mu_K(f(x), f(y)) \leq |p|\mu_C(x, y) \qquad \forall x, y \in C \quad \text{with} \quad x \sim_C y.$$

Proof. Follows from Corollaries 2.1.4 and 2.1.5 in [60]. □

The scale invariance of the Hilbert metric described in (5.2) implies that any rescaling, constant or not, of a mapping does not change its Lipschitz constant with respect to the Hilbert metric. This is useful as we will study iterates of normalized mappings. Furthermore it can be shown that p-homogeneous strongly order-preserving mappings are contractive. These results are formulated in the following proposition which reduces to Lemma 2.1.6 of [60] in the case $p = 1$ and is proved in a similar way.

Proposition 5.2.2. Let $C \subset V$ be a solid closed cone. Let $\nu\colon C \to [0, \infty)$ be continuous, homogeneous of degree 1 and such that $\nu(x) \neq 0$ for all $x \in C \setminus \{0\}$. Let $f\colon C \to C$ be homogeneous of degree $p \in \mathbb{R} \setminus \{0\}$ and either order-preserving if $p > 0$, or order-reversing if $p < 0$. Suppose that $f(x) \neq 0$ for all $x \in C \setminus \{0\}$. Let $S_\nu = \{x \in C \mid \nu(x) = 1\}$ and let $g\colon S_\nu \to S_\nu$ be given by

$$g(x) = \frac{f(x)}{\nu(f(x))} \qquad \forall x \in S_\nu.$$

Then, for every part P of C such that $P \neq \{0\}$, it holds

$$\mu_C(g(x), g(y)) \leq |p|\, \mu_C(x, y) \qquad \forall x, y \in S_\nu \cap P.$$

Moreover, if f is strongly order-preserving or strongly order-reversing, then

$$\mu_C(g(x), g(y)) < |p|\, \mu_C(x, y) \qquad \forall x, y \in S_\nu \cap P \text{ with } x \neq y.$$

Proof. The first inequality follows from Proposition 5.2.1 and the fact that for all $\alpha, \beta > 0$ and $x, y \in P$ it holds $\mu_C(\alpha x, \beta y) = \mu_C(x, y)$. To prove the second inequality, let $x, y \in S_\nu \cap P$ with $x \neq y$. Then we have $\mu_C(x, y) > 0$ since μ_C is a metric on $S_\nu \cap P$ by Lemma 5.1.7. Hence it holds $x \neq \lambda y$ for all $\lambda > 0$. Set $\alpha = m(y/x; C)$ and $\beta = M(x/y; C)$. It holds $\alpha y \precneqq_C x \precneqq_C \beta y$ since C is closed. Suppose that f is strongly order-preserving and $p > 0$, then we have $\alpha^p f(y) \prec_C f(x) \prec_C \beta^p f(y)$. Hence, there exists $\theta > \alpha$ and $\vartheta < \beta$ such that $\theta^p f(y) \preceq_C f(x) \preceq_C \vartheta^p f(y)$ and thus

$$\mu_C(g(x), g(y)) = \mu_C(f(x), f(y)) \leq \ln(\vartheta^p/\theta^p) = p \ln(\vartheta/\theta) < p \ln(\beta/\alpha) = p\mu_C(x, y).$$

If f is strongly order-reversing and $p < 0$, then we have $\beta^p f(y) \prec_C f(x) \prec_C \alpha^p f(y)$. Therefore, there exists $\theta > \beta$ and $\vartheta < \alpha$ such that $\theta^p f(y) \preceq_C f(x) \preceq_C \vartheta^p f(y)$ and it follows that

$$\mu_C(g(x), g(y)) \leq \ln(\theta^p/\vartheta^p) = |p| \ln(\vartheta/\theta) < |p| \ln(\beta/\alpha) = |p|\mu_C(x, y),$$

which concludes the proof. $\square$

We note that on $\mathbb{R}^n_+$, the element-wise power is a dilatation.

Lemma 5.2.3. Let $C = \mathbb{R}^n_+$, then for all $\alpha \in \mathbb{R}$ it holds

$$\mu_C(x^\alpha, y^\alpha) = |\alpha| \mu_C(x, y) \qquad \forall x, y \in \mathbb{R}^n_+ \quad \text{with} \quad x \sim_C y.$$

Proof. If $\alpha > 0$, then $M(x^\alpha/y^\alpha; C) = M(x/y; C)^\alpha$ and if $\alpha < 0$, then $M(x^\alpha/y^\alpha; C) = M(y/x; C)^{|\alpha|}$. If $\alpha = 0$, then $z^\alpha = (1, \ldots, 1)$ for all $z \in \text{int}(C)$ and thus $M(x^\alpha/y^\alpha; C) = 1$. The claim now follows from the definitions of μ_C. $\square$

Remark 5.2.4. We are not aware of bijective dilatations which are not isometric for cones which are not simplicial. We refer to [61] for a discussion of isometries on polyhedral cones. It is however worth to note that if C is the cone of symmetric positive semi-definite matrices, $A, B \in \text{int}(C)$ and $\alpha \in [-1, 1]$, the discussion in Example 2.2.3 together with Proposition 5.2.1 imply that $\mu_C(A^\alpha, B^\alpha) \leq |\alpha|\mu_C(A, B)$.

Propositions 5.2.1 and 5.2.2 are useful when combined with fixed point theorems. We gather classical fixed point results in the following theorem. For its statement, let us recall that in a metric space (X, η), $u \in X$ is said to be a *locally attracting fixed point* of $f : X \to X$ if it is a fixed point of f and there exists an open neighborhood $U \subset X$ such that $u \in U$ and for all $x \in U$ it holds $\lim_{k\to\infty} f^k(x) = u$.

Theorem 5.2.5. Let (X, η) be a complete metric space and $f : X \to X$ a mapping. Let $c > 0$ be a Lipschitz constant of f with respect to η. Suppose that at least one of the following conditions is satisfied:

a) $c < 1$.

b) $c = 1$, (X, η) is compact and f is contractive.

c) $c = 1$, (X, d) is a geodesic space and $\mathcal{F}$ has a locally attracting fixed point $x \in X$,

Then, f has a unique fixed point $u \in X$, and

$$\lim_{k\to\infty} f^k(x) = u \qquad \forall x \in X.$$

Proof. If $\mathcal{F}$ satisfies a),b) or c), then the claim respectively follows Theorem 3.1 [55], Theorem 3.5 [55], or Proposition 3.2.3 [60]. $\square$

We illustrate a combination of Theorem 5.2.5 with Proposition 5.2.2.

Example 5.2.6. Let $C = \mathbb{R}^n_+$, $M \in \mathbb{R}^{n\times n}_+$ a matrix with at least one nonzero entry per row and $\alpha \in \mathbb{R} \setminus \{0\}$ with $|\alpha| < 1$. Let $f : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ be defined as

$$f(x) = (Mx)^\alpha \qquad \forall x \in \mathbb{R}^n_{++}.$$

Then, f is order-preserving if $\alpha > 0$ and order-reversing if $\alpha < 0$. Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$ and let $S_{++} = \{x \in \mathbb{R}^n_{++} \mid \|x\| = 1\}$. Define $g : S_{++} \to S_{++}$ as

$$g(x) = \frac{f(x)}{\|f(x)\|} \qquad \forall x \in S_{++}.$$

Proposition 5.2.2 implies that $|\alpha|$ is a Lipschitz constant of g with respect to the Hilbert metric μ_C. As (S_{++}, δ_C) is a complete metric space by Lemma 5.1.7, Theorem 5.2.5, a) implies that g has a unique fixed point. It follows that the equation

$$(Mx)^\alpha = \lambda x, \qquad x \in S_{++}, \lambda > 0,$$

has a unique solution (λ, u) and for every $x \in S_{++}$, $\lim_{k\to\infty} g^k(x) = u$.

We conclude by noting that in Corollary 2.1 of [76], a formula is given which characterizes the smallest Lipschitz constant of a locally Lipschitz mapping between cones in terms of its derivatives.

5.2.2 Birkhoff-Hopf theorem

Let V, W be finite dimensional real vector spaces. For a cone $C \subseteq V$ and $S \subseteq C$, the *projective diameter* of S with respect to μ_C is defined as

$$\mathrm{diam}(S; \mu_C) = \sup\{\mu_C(x, y) \colon x, y \in S \text{ and } x \sim_C y\}. \tag{5.6}$$

The projective diameter appears in the Birkhoff-Hopf theorem. This theorem shows that the best Lipschitz constant $\kappa(L)$ of a linear mapping $L \colon V \to W$ between the cones $C \subset V$ and $K \subset W$, i.e. $L(C) \subset K$, can be expressed in terms of the projective diameter $\mathrm{diam}(L(C); \mu_K)$. Different proofs are known, see for instance [14, 23, 28]. The Birkhoff-Hopf theorem is usually stated for a linear mapping $L \colon V \to W$ such that $L(C) \subset K$ where $C \subset V$ and $K \subset W$ are cones. For the discussion, it is convenient to work with mappings defined on C instead of the the whole vector space V. As V is assumed to be finite dimensional, this can be done. We recall from Definition A.6.3 of [60] that $L \colon C \to K$ is *cone linear* if

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y), \qquad \forall x, y \in C, \alpha, \beta \geq 0. \tag{5.7}$$

If $L \colon C \to K$ is cone linear, then the linear mapping $L' \colon (C - C) \to W$ defined as $L'(x) = L'(u) - L(v)$ for all $x = u - v \in C - C$, is a linear extension of L on the subspace $C - C = \{u - v \mid u, v \in C\} \subset V$.

Theorem 5.2.7 (Birkhoff-Hopf)**.** Let $C \subset V$ and $K \subset W$ be cones. Suppose that $L \colon C \to K$ is a cone linear map. Let $\kappa(L)$ be defined as

$$\kappa(L) = \inf\{\lambda \geq 0 \mid \mu_K(Lx, Ly) \leq \mu_C(x, y), \forall x, y \in C, x \sim_C y\},$$

then it holds

$$\kappa(L) = \tanh\left[\tfrac{1}{4}\,\mathrm{diam}(L(C); \mu_K)\right].$$

with the convention $\tanh(\infty) = 1$.

Proof. See Theorem A.6.4 of [60]. □

We refer to the quantity $\kappa(L)$ as the *Birkhoff contraction ratio* of L. Cones can be endowed with other projective metrics, however as discussed in Theorem 3.4 of [22], the Hilbert's metric gives the best contraction ratio for linear mappings among a wide class of projective metrics. If K is the cone of positive definite matrices, then the Hilbert metric is less explicit which makes the computation of $\mathrm{diam}(L(C); \mu_K)$ more difficult. We refer to [66] and [82] for discussions on $\mathrm{diam}(L(C); \mu_K)$ in this case.

When C and K are simplicial cone, the formula of Theorem 5.2.7 has a particularly nice expression. This expression is discussed in Theorem 6.2 of [28]. We recall its derivation in the following. To this end, we first list properties of the projective diameter in the following:

Proposition 5.2.8. Let $C \subset V$ be a cone.

a) If $S \subset C$ and $S' \subset C$ are such that $\{t\,x \mid t \geq 0, x \in S\} = \{t\,y \mid t \geq 0, y \in S'\}$, then $\mathrm{diam}(S; \mu_C) = \mathrm{diam}(S'; \mu_C)$.

b) If $S \subset C$ and all elements of $S \setminus \{0\}$ are comparable, i.e. $x \sim_C y$ for all $x, y \in S$, then $\mathrm{diam}(\mathrm{co}(T); \mu_C) = \mathrm{diam}(T; \mu_C)$, where $\mathrm{co}(T)$ is the convex hull of T.

c) If C is closed and $S \subset C$ is such that all elements of $S \setminus \{0\}$ are comparable, then $\mathrm{diam}(\mathrm{cl}(S); \mu_C) = \mathrm{diam}(S; \mu_C)$.

Proof. See Proposition 2.9 of [28]. □

Now, suppose that $\dim(V) = n, \dim(W) = m$ and let $C \subset V, K \subset W$ be simplicial cones with basis vectors $e_1, \ldots, e_n \subset C$ and $e'_1, \ldots, e'_m \in K$. Let $L\colon C \to K$ be cone linear. We denote by $x_j, y_i, L_{i,j}$ respectively the coordinates of $x \in C$, $y \in K$ and L in the basis $e_1, \ldots, e_n$ and $e'_1, \ldots, e'_n$. First let us assume that $L(C \setminus \{0\}) \subset \mathrm{int}(K)$, i.e. $L_{i,j} > 0$ for all i, j. We want to compute $\mathrm{diam}(L(C); \mu_K)$. Note that

$$L(C) = \Big\{ \sum_{j=1}^{d} \alpha_k L(e_j) \mid \alpha \in \mathbb{R}^d_+ \Big\} = \mathrm{cone}\big(\{L(e_j) \mid j = 1, \ldots, n\}\big).$$

Proposition 5.2.8 (a) and (b) imply that for all $S \subset K$ it holds $\mathrm{diam}(S) = \mathrm{diam}(\mathrm{cone}(S))$. It follows that

$$\mathrm{diam}(L(C); \mu_K) = \mathrm{diam}\big(\{L(e_j) \mid j = 1, \ldots, n\}\big) = \max_{j,j'=1,\ldots,n} \mu_K(L(e_j), L(e_{j'})).$$

Finally, as K is simplicial, we have

$$\mu_K(L(e_j), L(e_{j'})) = \max_{i,i'=1,\ldots,m} \ln\Big(\frac{L(e_j)_i L(e_{j'})_{i'}}{L(e_{j'})_i L(e_j)_{i'}}\Big) = \max_{i,i'=1,\ldots,m} \ln\Big(\frac{L_{i,j} L_{i',j'}}{L_{i,j'} L_{i',j}}\Big).$$

It follows that

$$\kappa(L) = \tanh\big[\tfrac{1}{4}\Delta(\mathrm{diam}(L(C); \mu_K))\big] \tag{5.8}$$

with

$$\mathrm{diam}(L(C); \mu_K) = \max_{\substack{i,i'=1,\ldots,m \\ j,j'=1,\ldots,n}} \ln\Big(\frac{L_{i,j} L_{i',j'}}{L_{i,j'} L_{i',j}}\Big).$$

If $C \subset V$ and $K \subset W$ are solid polyhedral cones then, with a similar argument, the formula in (5.8) can be generalized as follows: Let $v_1, \ldots, v_n \in C$ and $w_1, \ldots, , w_m \in K^*$ be such that $\mathrm{cone}(\{v_1, \ldots, v_n\}) = C$ and $\mathrm{cone}(\{w_1, \ldots, , w_m\}) = K$. Let $L\colon C \to K$ be a cone linear mapping, then it holds

$$\mathrm{diam}(L(C); \mu_K) = \max_{\substack{i,i'=1,\ldots,m \\ j,j'=1,\ldots,n}} \ln\Big(\frac{w_i(Lv_j)\, w_{i'}(Lv_{j'})}{w_i(Lv_{j'})\, w_{i'}(Lv_j)}\Big).$$

The case where L is not strictly positive, i.e. $L_{i,j} = 0$ for some i, j, is discussed in details in Theorem 6.2 of [28]. For later discussion, We recall the following result which states that if a nonnegative matrix has a at least one nonzero entry per column and at least one zero entry, then its contraction rate is 1.

Lemma 5.2.9. Let $M \in \mathbb{R}^{m\times n}_+$, if M has at least one positive entry per column and a zero entry, then the Birkhoff contraction ratio of $M\colon (\mathbb{R}^n_+, \mu_{\mathbb{R}^n_+}) \to (\mathbb{R}^m_+, \mu_{\mathbb{R}^m_+})$ satisfies $\kappa(M) = 1$.

Proof. Follows from Theorem 3.12 of [88]. □

In the next example we show how the result of Example 5.2.6 can be refined with the Birkhoff-Hopf theorem.

Example 5.2.10. Let $C = \mathbb{R}^n_+$, $M \in \mathbb{R}^{n\times n}_+$ a matrix with at least one nonzero entry per row and $\alpha \in \mathbb{R} \setminus \{0\}$. Let $f\colon \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ be defined as

$$f(x) = (Mx)^\alpha \qquad \forall x \in \mathbb{R}^n_{++}.$$

Then, f is order-preserving if $\alpha > 0$ and order-reversing if $\alpha < 0$. By Lemma 5.2.3 we have that

$$\mu_C(f(x), f(y)) = |\alpha|\mu_C(Mx, My) \qquad \forall x, y \in \mathbb{R}^n_{++}.$$

Hence, by Theorem 5.2.7, we deduce that

$$\mu_C(f(x), f(y)) \le |\alpha|\kappa(M)\mu_C(x, y) \qquad \forall x, y \in \mathbb{R}^n_{++}.$$

Now, let $\|\cdot\|$ be any norm on $\mathbb{R}^n$ and let $S_{++} = \{x \in \mathbb{R}^n_{++} \mid \|x\| = 1\}$. Furthermore, define $g\colon S_{++} \to S_{++}$ as

$$g(x) = \frac{f(x)}{\|f(x)\|} \qquad \forall x \in S_{++}.$$

Then, we have

$$\mu_C(g(x), g(y)) = \mu_C(f(x), f(y)) \le |\alpha|\kappa(M)\mu_C(x, y) \qquad \forall x, y \in \mathbb{R}^n_{++}.$$

It follows that if $|\alpha|\kappa(M) < 1$, then g has a unique fixed point by Theorem 5.2.5, a) and thus the equation

$$(Mx)^\alpha = \lambda x \qquad x \in \mathbb{R}^n_{++}, \lambda > 0 \tag{5.9}$$

has a unique solution. If $M \in \mathbb{R}^{n\times n}_{++}$, then $\kappa(M) < 1$ and thus (5.9) has a unique solution for $|\alpha| < 1/\kappa(M)$. In particular, note that if $1 < \alpha < 1/\kappa(M)$ then f is homogeneous of degree strictly greater than one and a strict contraction with respect to the Hilbert metric.

The case $\alpha = -1$ reduces to the particular case of the Sinkhorn-Knopp theorem (see [56, Theorem 2.1]), also known as classical DAD theorem (see [60, Theorem 7.4.4] and [21, Theorem 1]), where the matrix is positive. The latter guarantees the existence of a diagonal matrix D with positive diagonal entries such DMD is a stochastic matrix. Indeed, if $x \in \mathbb{R}^n_{++}$ is such that $(Mx)^{-1} = \lambda x$, then with $u = \lambda^{-1/2}x^{-1}$, we have $Mu = u^{-1}$. It follows that, with $D_u = \operatorname{diag}(u)$, it holds

$$1 = u_i(Mu)_i = \sum_{j=1}^n M_{i,j}u_iu_j = \sum_{j=1}^n (D_uMD_u)_{i,j} \qquad \forall i = 1, \dots, n.$$

i.e. D_uMD_u is a stochastic matrix.

6 Vector valued Hilbert metric

The results of Section 5 motivate the use of the Hilbert metric for the study of eigenvectors of homogeneous mappings leaving a cone invariant. Let $\mathcal{V} = V_1 \times \ldots \times V_d$ be the product of finite dimensional real vector spaces and $\mathcal{C} = C_1 \times \ldots \times C_d \subset \mathcal{V}$ a cone. The Hilbert metric $\mu_{C_1\times\ldots\times C_d}$ induced by the cone $\mathcal{C}$ is not well suited for the study of order-preserving multi-homogeneous mappings on $\mathcal{C}$. This is because the product structure of $\mathcal{C}$ is not preserved. To address this issue, in Section 6.2, we introduce a vector valued version of the Hilbert metric. That is, for $x, y \in \mathcal{C}$, we consider $\boldsymbol{\mu}_{\mathcal{C}}(x, y) = (\mu_{C_1}(x_1, y_1), \ldots, \mu_{C_d}(x_d, y_d))$ where μ_{C_i} is the Hilbert metric induced by C_i for $i = 1, \ldots, d$. Then $\boldsymbol{\mu}_{\mathcal{C}}(x, y) \in \mathbb{R}^d_+$ if $x \sim_{\mathcal{C}} y$. A motivation for using vector valued metrics is that a Lipschitz matrix can be meaningfully defined. In particular we prove in Section 6.2.2 that the homogeneity matrix of an order-preserving mapping is a Lipschitz matrix of the mapping with respect to $\boldsymbol{\mu}_{\mathcal{C}}$. Before proving such statement, we discuss vector valued metrics defined on the product of metric spaces and Lipschitz matrices.

6.1 Vector valued metrics and Lipschitz matrices

Vector valued metric spaces are studied in fixed point theory [1, 2] and consist of a set X and a metric $\boldsymbol{\eta}\colon X \times X \to \mathbb{R}^d_+$ which satisfies the usual properties of a metric where the inequality in the triangle inequality is understood with respect to the partial ordering induced by $\mathbb{R}^d_+$. Furthermore, since $\mathbb{R}^d_+$ is a cone, every vector valued metric is a cone metric. Cone metric spaces generalize usual metric spaces in that the metric takes value in a general cone instead of $[0, \infty)$. This generalization has attracted considerable attention in the fixed point community recently [53, 50, 6]. Nevertheless, we note that, usually, these generalizations do not consider the particular structure we are using, namely finite products of metric spaces. Perhaps a reason is that in a cone metric space (X, β) where $\beta\colon X \times X \to K$ takes values in a cone K of a real vector space, the Lipschitz constant of a mapping $f\colon (X, \beta) \to (X, \beta)$ is defined to be a scalar [6, Section 4], i.e. $\alpha > 0$ is a Lipschitz constant of f if $\beta(f(x), f(y)) \preceq_K \alpha\beta(x, y)$ for all $x, y \in X$. In this case, if $K = \mathbb{R}^d_+$, then for every norm $\|\cdot\|$ on $\mathbb{R}^d$ which is monotonic with respect to $\mathbb{R}^d_+$, $\eta(x, y) = \|\beta(x, y)\|$ is a metric on X and $\eta(f(x), f(y)) \leq \alpha\eta(x, y)$ for all $x, y \in X$. This means that one can use all the results of the classical fixed point theory to study the equation $f(x) = x$. A similar argument can be done in the more general case where K is a normal cone in a real vector space [53]. The above discussion suggests that cone metric spaces are somewhat unnecessary. Nevertheless, it is acknowledged in [12, 54] that cone metric spaces offer a more flexible framework. Such flexibility is extensively exploited here.

6.1.1 d-metric spaces and d-pseudo metric spaces

For the discussion, we consider the following definition of d-(pseudo)metric space.

Definition 6.1.1. For a positive integer d, let $(X_1, \eta_1), \ldots, (X_d, \eta_d)$ be pseudometric spaces. Let $\mathcal{X} = X_1 \times \ldots \times X_d$ and define $\boldsymbol{\eta}\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}^d_+$ as

$$\boldsymbol{\eta}(x, y) = (\eta_1(x_1, y_1), \ldots, \eta_d(x_d, y_d))^\top \qquad \forall x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathcal{X}.$$

Then, we say that $(\mathcal{X}, \boldsymbol{\eta})$ is a *d-pseudometric space* and refer to $\boldsymbol{\eta}$ as a *vector valued pseudometric*. Furthermore, if $(X_1, \eta_1), \ldots, (X_d, \eta_d)$ are metric spaces, then we say that $(\mathcal{X}, \boldsymbol{\eta})$ is a d-metric space and $\boldsymbol{\eta}$ is a *vector valued metric*.

In a d-pseudometric space $(\mathcal{X}, \boldsymbol{\eta})$, we use the same notation as in the product of vector spaces, namely for $x \in \mathcal{X}$ we write x_i to denote the canonical projection of x onto X_i so that $x = (x_1, \ldots, x_d)$ with $x_i \in X_i$ for all i. Similarly, if $\mathcal{S} \subset \mathcal{X}$ then we write S_i to denote the projection of $\mathcal{S}$ onto X_i so that $\mathcal{S} = S_1 \times \ldots \times S_d$.

Up to the product structure on $\mathcal{X}$, the definition of d-metric space coincide with that of d-metric real linear space introduced in [1]. Note that the topology on $\mathcal{X}$ induced by the open balls $B_{\boldsymbol{\epsilon}}(x) = \{y \in \mathcal{X} \,|\, \boldsymbol{\eta}(x, y) < \boldsymbol{\epsilon}\}$, with $\boldsymbol{\epsilon} \in \mathbb{R}^d_{++}$, coincide with the product topology of $(X_1, \eta_1) \times \ldots \times (X_d, \eta_d)$. Therefore we say that $\mathcal{S} \subset \mathcal{X}$ is *open*, *closed*, *complete* or *compact* in $(\mathcal{X}, \boldsymbol{\eta})$ if S_i has the corresponding property in (X_i, η_i) for all $i = 1, \ldots, d$. In particular, we say that $(\mathcal{X}, \boldsymbol{\eta})$ is a *complete d-metric space* if (X_i, η_i) is a complete metric space for every i.

6.1.2 Lipschitz matrices

Lipschitz matrices were considered independently in [1] and [37, 38]$^\diamond$ and they are implicitly used in the proofs of the nonlinear Perron-Frobenius theorems in [40, 36, 39, 74, 92]$^\diamond$.

Definition 6.1.2. Let d, d' be positive integers. Let $(\mathcal{X}, \boldsymbol{\eta})$ be a d-pseudometric space and $(\mathcal{Y}, \boldsymbol{\zeta})$ a d'-pseudometric space. Let $\mathcal{F}\colon (\mathcal{X}, \boldsymbol{\eta}) \to (\mathcal{Y}, \boldsymbol{\zeta})$, then $A \in \mathbb{R}^{d' \times d}_+$ is a *Lipschitz matrix* of $\mathcal{F}$ if

$$\boldsymbol{\zeta}(\mathcal{F}(x), \mathcal{F}(y)) \leq A\, \boldsymbol{\eta}(x, y) \qquad \forall x, y \in \mathcal{X}.$$

If $\mathcal{F}$ has a Lipschitz matrix, then we say that $\mathcal{F}\colon (\mathcal{X}, \boldsymbol{\eta}) \to (\mathcal{Y}, \boldsymbol{\zeta})$ is *Lipschitz continuous*.

We prove in the next section that if $\mathcal{F}\colon (\mathcal{X}, \boldsymbol{\eta}) \to (\mathcal{X}, \boldsymbol{\eta})$ is a mapping on a complete d-metric space, and $\mathcal{F}$ has a Lipschitz matrix $A \in \mathbb{R}^{d \times d}_+$ with spectral radius $\rho(A) < 1$, then one can apply the Banach fixed point theorem to prove that $\mathcal{F}$ has a unique fixed point $u \in \mathcal{X}$ and for every $x \in \mathcal{X}$, the iterates $\mathcal{F}^k(x)$ converge towards u as $k \to \infty$. Furthermore, we will prove that fixed point theorems for non-expansive mappings can be used when $\rho(A) = 1$. First, let us discuss properties of Lipschitz matrices and how to build them. We start with an example

Example 6.1.3. Let $\mathcal{V} = V \times V$, $\mathcal{C} = C \times C$ with $V = \mathbb{R}^n$ and $C = \mathbb{R}^n_+$. Let $M, Q \in \mathbb{R}^{n \times n}_{++}$ and $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ given by $\mathcal{F}(x, y) = (My, Qx)$. Consider the vector valued pseudometric

$$\boldsymbol{\mu}_{\mathcal{C}}((x, y), (x', y')) = (\mu_C(x, y), \mu_C(x', y'))^\top \qquad \forall (x, y), (x', y') \in \mathcal{C}.$$

For every $(x,y),(x',y') \in \mathcal{C}$ with $(x,y) \sim_{\mathcal{C}} (x',y')$, we have

$$\mu_C(My, My') \le \kappa(M)\mu_C(y,y') \qquad \text{and} \qquad \mu_C(Qx, Qx') \le \kappa(Q)\mu_C(x,x')$$

where $\kappa(M)$ and $\kappa(Q)$ are the Birkhoff contraction ratio of M and Q, respectively (see Theorem 5.2.7). It follows that for all $(x,y),(x',y') \in \mathcal{C}$ with $(x,y) \sim_{\mathcal{C}} (x',y')$,

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}(x,y), \mathcal{F}(x',y')) = \begin{pmatrix} \mu_C(My, My') \\ \mu_C(Qx, Qx') \end{pmatrix} \le \underbrace{\begin{pmatrix} 0 & \kappa(M) \\ \kappa(Q) & 0 \end{pmatrix}}_{=A} \begin{pmatrix} \mu_C(x,x') \\ \mu_C(y,y') \end{pmatrix}.$$

Hence, $A \in \mathbb{R}^{2\times 2}_+$ is a Lipschitz matrix of $\mathcal{F}\colon (\mathrm{int}(\mathcal{C}), \boldsymbol{\mu}_{\mathcal{C}}) \to (\mathrm{int}(\mathcal{C}), \boldsymbol{\mu}_{\mathcal{C}})$. Note that by Lemma 5.2.9, for $G \in \mathbb{R}^{2n\times 2n}$ defined as

$$G = \begin{pmatrix} 0 & M \\ Q & 0 \end{pmatrix},$$

it holds $\kappa(G) = 1$. This means that the smallest Lipschitz constant of the mapping $\mathcal{F}\colon (\mathrm{int}(C) \times \mathrm{int}(C), \mu_{C\times C}) \to (\mathrm{int}(C) \times \mathrm{int}(C), \mu_{C\times C})$ equals 1 where $\mu_{C\times C}$ is the Hilbert metric induced by the cone $\mathbb{R}^{2n}_+ \cong C \times C \subset V \times V$. Finally, we note that if the rank of M or Q is at least 2, then $\mathcal{F}\colon (\mathrm{int}(\mathcal{C}), \boldsymbol{\mu}_{\mathcal{C}}) \to (\mathrm{int}(\mathcal{C}), \boldsymbol{\mu}_{\mathcal{C}})$ has no Lipschitz constant in the sense of cone metric space. Indeed, suppose by contradiction that there exists $\lambda > 0$ such that $\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}(x,y), \mathcal{F}(x',y')) \le \lambda \boldsymbol{\mu}_{\mathcal{C}}((x,y),(x',y'))$ for all $(x,y),(x',y') \in \mathcal{C}$ with $(x,y) \sim_{\mathcal{C}} (x',y')$. Then, we have

$$\mu_C(My, My') \le \lambda\mu_C(x,x') \qquad \text{and} \qquad \mu_C(Qx, Qx') \le \lambda\mu_C(y,y') \tag{6.1}$$

for all $(x,y),(x',y') \in \mathcal{C}$. Without loss of generality, suppose that M has rank at least 2, then there exists $\bar{y}, \bar{y}' \in \mathbb{R}^n_{++}$ such that $M\bar{y}$ and $M\bar{y}'$ are linearly independent so that $\mu_C(M\bar{y}, M\bar{y}') > 0$. By setting $x = x'$, $y = \bar{y}$, $y' = \bar{y}'$ in (6.1) we get the contradiction $\mu_C(M\bar{y}, M\bar{y}') = 0$.

Fortunately, as noted in the following remark, Lipschitz matrices enjoy essential properties of the usual Lipschitz constants.

Remark 6.1.4. Let $(\mathcal{X}, \boldsymbol{\eta}), (\mathcal{Y}, \boldsymbol{\zeta}), (\mathcal{Z}, \boldsymbol{\gamma})$ be respectively a d-pseudometric space, a d'-pseudometric space and a d''-pseudometric space. Suppose that $A \in \mathbb{R}^{d'\times d}_+$ is a Lipschitz matrix of $\mathcal{F}\colon (\mathcal{X}, \boldsymbol{\eta}) \to (\mathcal{Y}, \boldsymbol{\zeta})$. Then for every $P \in \mathbb{R}^{d'\times d}_+$ such that $A \preceq_{\mathbb{R}^{d'\times d}_+} P$, P is a Lipschitz matrix of $\mathcal{F}$. Furthermore, if $\mathcal{G}\colon (\mathcal{Y}, \boldsymbol{\zeta}) \to (\mathcal{Z}, \boldsymbol{\gamma})$ has Lipschitz matrix $B \in \mathbb{R}^{d''\times d'}$, then the composition $\mathcal{F} \circ \mathcal{G}\colon (\mathcal{X}, \boldsymbol{\eta}) \to (\mathcal{Z}, \boldsymbol{\gamma})$ has Lipschitz matrix $AB \in \mathbb{R}^{d''\times d}_+$.

In the next lemma we prove that the Lipschitz continuity of Definition 6.1.2 is equivalent to the Lipschitz continuity in the product metric.

Lemma 6.1.5. Let $(\mathcal{X}, \boldsymbol{\eta})$ be a d-pseudometric space and $(\mathcal{Y}, \boldsymbol{\zeta})$ a d'-pseudometric space. Let $\|\cdot\|$ and $\|\cdot\|'$ be norms on $\mathbb{R}^d$ and $\mathbb{R}^{d'}$ respectively. Let $\mathcal{F}\colon (\mathcal{X}, \boldsymbol{\eta}) \to (\mathcal{Y}, \boldsymbol{\zeta})$, then the following statements are equivalent:

i) There exists $a \geq 0$ such that $\|\boldsymbol{\zeta}(\mathcal{F}(x), \mathcal{F}(y))\|' \leq a\|\boldsymbol{\eta}(x, y)\|$ for all $x, y \in \mathcal{X}$.

ii) There exists $A \in \mathbb{R}_+^{d' \times d}$ such that $\boldsymbol{\zeta}(\mathcal{F}(x), \mathcal{F}(y)) \leq A\boldsymbol{\eta}(x, y)$ for all $x, y \in \mathcal{X}$.

Proof. Suppose that i) holds and let us prove that ii) holds. As the norms on a finite dimensional real vector space are equivalent, we may suppose without loss of generality that $\|\cdot\|$ is the ℓ^1-norm $\|\cdot\|_1$ and $\|\cdot\|'$ is the ℓ^∞-norm $\|\cdot\|_\infty$. Let $\mathbf{1} \in \mathbb{R}^{d'}$ be the vector of all ones and $E \in \mathbb{R}^{d' \times d}$ be the matrix of all ones. Note that for every $x, y \in \mathcal{X}$ we have $\mathbf{1}\|\boldsymbol{\eta}(x, y)\|_1 = E\boldsymbol{\eta}(x, y)$. Hence, if $a \geq 0$ satisfies condition i), then for every $x, y \in \mathcal{X}$, we have

$$\boldsymbol{\zeta}(\mathcal{F}(x), \mathcal{F}(y)) \leq \mathbf{1}\|\boldsymbol{\zeta}(\mathcal{F}(x), \mathcal{F}(y))\|_\infty \leq a\mathbf{1}\|\boldsymbol{\eta}(x, y)\|_1 = aE\boldsymbol{\eta}(x, y).$$

It follows that $aE \in \mathbb{R}^{d' \times d}$ is a Lipschitz matrix of $\mathcal{F}$ and thus ii) holds. Now, suppose that $A \in \mathbb{R}_+^{d' \times d}$ is a Lipschitz matrix of $\mathcal{F}$ and let $|||A|||$ be the operator norm of $A\colon (\mathbb{R}^d, \|\cdot\|) \to (\mathbb{R}^{d'}, \|\cdot\|')$, i.e. $|||A||| = \sup\{\|A\mathbf{c}\|' \mid \|\mathbf{c}\| \leq 1\}$. Furthermore, let $\gamma \geq 1$ be the normality constant of $\|\cdot\|'$ with respect to $\mathbb{R}_+^{d'}$. Then, for every $x, y \in \mathcal{X}$, we have

$$\|\boldsymbol{\zeta}(\mathcal{F}(x), \mathcal{F}(y))\|' \leq \gamma\|A\boldsymbol{\eta}(x, y)\| \leq \gamma|||A|||\,\|\boldsymbol{\eta}(x, y)\|,$$

which implies that i) holds with $a = \gamma|||A|||$. This concludes the equivalence between i) and ii). □

In the next lemma, we show that a Lipschitz matrix of $\mathcal{F}\colon (\mathcal{X}, \boldsymbol{\eta}) \to (\mathcal{Y}, \boldsymbol{\zeta})$ can be obtained by analyzing the Lipschitz constant of $\mathcal{F}|_x^{j,i}\colon (X_i, \eta_i) \to (Y_j, \zeta_j)$ for all i, j, where $\mathcal{F}|_x^{j,i}\colon X_i \to Y_j$ is defined for $x \in \mathcal{X}, i = 1, \ldots, d$ and $j = 1, \ldots, d'$ as

$$\mathcal{F}|_x^{j,i}(y_i) = \mathcal{F}(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_d)_j \qquad \forall y_i \in X_i. \tag{6.2}$$

We also discuss the case where $\boldsymbol{\zeta}(\mathcal{F}(x), \mathcal{F}(y)) \lneq A\boldsymbol{\eta}(x, y)$, for every $x, y \in \mathcal{X}$ such that $\boldsymbol{\eta}(x, y) \neq 0$. This condition allows to formulate a vector valued definition of contractive mappings.

Lemma 6.1.6. Let $(\mathcal{X}, \boldsymbol{\eta})$ be a d-pseudometric space and $(\mathcal{Y}, \boldsymbol{\zeta})$ a d'-pseudometric space. Let $\mathcal{F}\colon (\mathcal{X}, \boldsymbol{\eta}) \to (\mathcal{Y}, \boldsymbol{\zeta})$ and $A \in \mathbb{R}_+^{d' \times d}$. If for every $x \in \mathcal{X}$, $i = 1, \ldots, d$, $j = 1, \ldots, d'$ it holds

$$\zeta_j\big(\,\mathcal{F}|_x^{j,i}(y_i)\,,\, \mathcal{F}|_x^{j,i}(z_i)\,\big) \leq A_{j,i}\, \eta_i(y_i, z_i), \qquad \forall y_i, z_i \in X_i, \tag{6.3}$$

then

$$\boldsymbol{\zeta}(\mathcal{F}(x), \mathcal{F}(y)) \leq A\boldsymbol{\eta}(x, y), \quad \forall x, y \in \mathcal{X}.$$

Moreover, if $\mathcal{F}$ satisfies (6.3) and for every $i \in \{1, \ldots, d\}$ there exists $j \in \{1, \ldots, d'\}$ such that $A_{j,i} > 0$ and for all $x \in \mathcal{X}$ it holds

$$\zeta_j\big(\,\mathcal{F}|_x^{j,i}(y_i)\,,\, \mathcal{F}|_x^{j,i}(z_i)\,\big) < A_{j,i}\, \eta_i(y_i, z_i), \qquad \forall y_i, z_i \in X_i \text{ with } \eta_i(x_i, y_i) > 0,$$

then

$$\boldsymbol{\zeta}(\mathcal{F}(x), \mathcal{F}(y)) \lneq A\boldsymbol{\eta}(x, y), \quad \forall x, y \in \mathcal{X} \text{ with } \boldsymbol{\eta}(x, y) \neq 0.$$

Proof. Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be the canonical basis of $\mathbb{R}^d$. Let $x, y \in \mathcal{X}$ and for $k = 1, \dots, d+1$ define $z^k \in \mathcal{X}$ as follows: $z^1 = x$, $z^{d+1} = y$

$$z^k = (y_1, \dots, y_{k-1}, x_k, \dots, x_d), \qquad \forall k = 2, \dots, d.$$

Then, we have $\boldsymbol{\eta}(z^i, z^{i+1}) = \eta_i(x_i, y_i)\mathbf{e}_i$ so that $\boldsymbol{\eta}(x, y) = \sum_{i=1}^d \boldsymbol{\eta}(z^i, z^{i+1})$. It holds $\mathcal{F}(z^i)_j = \mathcal{F}|_{z^i}^{j,i}(x_i)$ and $\mathcal{F}(z^{i+1})_j = \mathcal{F}|_{z^i}^{j,i}(y_i)$ for all $i = 1, \dots, d$. With the triangle inequality, for every $j = 1, \dots, d'$, we have

$$\begin{aligned} \zeta_j(\mathcal{F}(x)_j, \mathcal{F}(y)_j) &\le \sum_{i=1}^d \zeta_j(\mathcal{F}(z^i)_j, \mathcal{F}(z^{i+1})_j) = \sum_{i=1}^d \zeta_j(\mathcal{F}|_{z^i}^{j,i}(x_i), \mathcal{F}|_{z^i}^{j,i}(y_i)) \\ &\le \sum_{i=1}^d A_{j,i}\eta_i(x_i, y_i) = (A\boldsymbol{\eta}(x, y))_j, \end{aligned} \tag{6.4}$$

which shows that A is a Lipschitz matrix of $\mathcal{F}$. Suppose that $\boldsymbol{\eta}(x, y) \neq 0$, then there exists $i \in \{1, \dots, d\}$ such that $\eta_i(x_i, y_i) > 0$. If there exists $j \in \{1, \dots, d'\}$ such that $A_{j,i} > 0$ and for every $u \in \mathcal{X}$, and $v_i, v_i' \in X_i$ with $\eta_i(v_i, v_i') > 0$ it holds $\zeta_j\big(\mathcal{F}|_u^{j,i}(v_i), \mathcal{F}|_u^{j,i}(v_i')\big) < A_{j,i}\eta_i(v_i, v_i')$, then, with $v_i = x_i$, $v_i' = y_i$ and $u = z^i$, we have $\zeta_j\big(\mathcal{F}|_{z^i}^{j,i}(x_i), \mathcal{F}|_{z^i}^{j,i}(y_i)\big) < A_{j,i}\eta_i(x_i, y_i)$. It follows that the inequality in (6.4) becomes strict and therefore $\boldsymbol{\zeta}(\mathcal{F}(x), \mathcal{F}(y)) \lneq A\boldsymbol{\eta}(x, y)$. □

The smallest Lipschitz matrix of $\mathcal{F}\colon (\mathcal{X}, \boldsymbol{\eta}) \to (\mathcal{Y}, \boldsymbol{\zeta})$ is related with the smallest Lipschitz constant of $\mathcal{F}|_x^{j,i}\colon (X_i, \eta_i) \to (Y_j, \zeta_j)$ for all i, j in the following theorem.

Theorem 6.1.7. Let $(\mathcal{X}, \boldsymbol{\eta})$ be a d-pseudometric space, $(\mathcal{Y}, \boldsymbol{\zeta})$ a d'-pseudometric space and $\mathcal{F}\colon (\mathcal{X}, \boldsymbol{\eta}) \to (\mathcal{Y}, \boldsymbol{\zeta})$. For $i = 1, \dots, d$, $j = 1, \dots, d'$ let

$$L_{j,i} \;=\; \sup_{x \in \mathcal{X}} \inf \big\{a \ge 0 \,\big|\, \zeta_j\big(\mathcal{F}|_x^{j,i}(y_i), \mathcal{F}|_x^{j,i}(z_i)\big) \le a\,\eta_i(y_i, z_i),\ \forall y_i, z_i \in X_i\big\}.$$

If $L_{j,i} < \infty$ for all i, j, then $\mathcal{F}$ is Lipschitz continuous and L is the smallest Lipschitz matrix of $\mathcal{F}$ in the following sense:

$$L = \inf \big\{A \in \mathbb{R}_+^{d' \times d} \,\big|\, \boldsymbol{\zeta}(\mathcal{F}(x), \mathcal{F}(y)) \le A\boldsymbol{\eta}(x, y),\ \forall x, y \in \mathcal{X}\big\}, \tag{6.5}$$

where the infimum is taken with respect to the partial ordering induced by $\mathbb{R}_+^{d' \times d}$.

Proof. By Lemma 6.1.6, we know that L is a Lipschitz matrix of $\mathcal{F}$. As $L_{j,i} < \infty$ for all i, j, it follows that $\mathcal{F}$ is Lipschitz continuous. To conclude the proof we show that if $A \in \mathbb{R}_+^{d' \times d}$ is a Lipschitz matrix of $\mathcal{F}$, then $L \preceq_{\mathbb{R}_+^{d \times d}} A$. Fix $i \in \{1, \dots, d\}$ and $j \in \{1, \dots, d'\}$. Let $x \in \mathcal{X}, y_i, z_i \in X_i$ and set $\tilde{y} = (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_d)$, $\tilde{z} = (x_1, \dots, x_{i-1}, z_i, x_{i+1}, \dots, x_d)$. We have

$$\zeta_j\big(\mathcal{F}|_x^{j,i}(y_i), \mathcal{F}|_x^{j,i}(z_i)\big) = \boldsymbol{\zeta}(\mathcal{F}(\tilde{y}), \mathcal{F}(\tilde{z}))_j \le \big(A\boldsymbol{\eta}(\tilde{y}, \tilde{z})\big)_j = A_{j,i}\,\eta_i(y_i, z_i).$$

As the latter is true for every $y_i, z_i \in X_i$, we have that $A_{j,i}$ is a candidate for $a \ge 0$ in the definition of $L_{j,i}$. Hence, $L_{j,i} \le A_{j,i}$ and thus $L \preceq_{\mathbb{R}_+^{d \times d}} A$. It follows that L is a lower bound on the set of Lipschitz matrices of $\mathcal{F}$ with respect to the partial ordering induced by $\mathbb{R}_+^{d' \times d}$. As L is a Lipschitz matrix itself, L is a minimal element. □

As a direct consequence of Theorem 6.1.7, we have that if A and B are both Lipschitz matrices of $\mathcal{F}\colon (\mathcal{X},\boldsymbol{\eta}) \to (\mathcal{Y},\boldsymbol{\zeta})$, then $M \in \mathbb{R}^{d'\times d}$ defined as $M_{i,j} = \min\{A_{i,j}, B_{i,j}\}$, is a Lipschitz matrix of $\mathcal{F}$ as well. Indeed, for L as in Theorem 6.1.7, we have $L \preceq_{\mathbb{R}^{d\times d}_+} A$ and $L \preceq_{\mathbb{R}^{d\times d}_+} B$ which implies that $L \preceq_{\mathbb{R}^{d\times d}_+} M$ and thus M is Lipschitz constant of $\mathcal{F}$.

6.2 Vector valued Hilbert metric and multi-homogeneous mappings

We introduce the vector valued Hilbert metric. Then, we combine the results on Hilbert's metric discussed in Section 5 together with the results on vector valued metrics and Lipschitz matrices discussed in Section 6 to construct Lipschitz matrices for multi-homogeneous mappings on cones.

6.2.1 Vector valued Hilbert metric

Let $\mathcal{V} = V_1 \times \ldots \times V_d$ be the product of finite dimensional real vector spaces and let $\mathcal{C} \subset \mathcal{V}$ be a cone. For every $x, y \in \mathcal{C}$, the *vector valued Hilbert metric* induced by $\mathcal{C}$ is given by

$$\boldsymbol{\mu}_{\mathcal{C}}(x,y) = \big(\mu_{C_1}(x_1,y_1),\ldots,\mu_{C_d}(x_d,y_d)\big)^\top \in \bar{\mathbb{R}}^d_+, \tag{6.6}$$

where $\bar{\mathbb{R}}^d_+ = [0,\infty]^d$ and $\mu_{C_i}\colon C_i \times C_i \to \bar{\mathbb{R}}_+$ is the Hilbert metric induced by C_i for all $i = 1,\ldots,d$.

Example 6.2.1. Let $\mathcal{V} = \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_d}$ and $\mathcal{C} = \mathbb{R}^{n_1}_+ \times \ldots \times \mathbb{R}^{n_d}_+ \subset \mathcal{V}$. Then for every $x, y \in \operatorname{int}(\mathcal{C})$, it holds

$$\boldsymbol{\mu}_{\mathcal{C}}(x,y) = \left[\ln\Big(\max_{1\le i,j\le n_1} \frac{x_{1,i}\,y_{1,j}}{y_{1,i}\,x_{1,j}}\Big),\ldots,\ln\Big(\max_{1\le i,j\le n_d} \frac{x_{d,i}\,y_{d,j}}{y_{d,i}\,x_{d,j}}\Big)\right]^\top.$$

Note that the scale invariance property of the Hilbert metric implies that for every $x, y \in \mathcal{C}$ with $x \sim_C y$, it holds

$$\boldsymbol{\mu}_{\mathcal{C}}(\boldsymbol{\alpha}\otimes x, \boldsymbol{\beta}\otimes y) = \boldsymbol{\mu}_{\mathcal{C}}(x,y) \qquad \forall \boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{R}^d_{++}. \tag{6.7}$$

Furthermore, if $\mathcal{C}$ is closed, then for every $x, y \in \mathcal{C}$ satisfying $x \sim_C y$ and $\boldsymbol{\mu}_{\mathcal{C}}(x,y) = 0$, there exists $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$ such that $\boldsymbol{\alpha}\otimes x = y$. This observation, shows that for every mapping $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ and $x \in \mathcal{C}$ such that $x \sim_{\mathcal{C}} \mathcal{F}(x)$, it holds

$$\boldsymbol{\mu}(\mathcal{F}(x), x) = 0 \quad \iff \quad \exists \boldsymbol{\lambda} \in \mathbb{R}^d_+ \text{ such that } \mathcal{F}(x) = \boldsymbol{\lambda}\otimes x.$$

In other words, likewise eigenvectors of homogeneous mappings are fixed point with respect to the Hilbert metric, eigenvectors of multi-homogeneous mappings are fixed points with respect to $\boldsymbol{\mu}_{\mathcal{C}}$.

The following consequence of Lemma 5.1.7 discusses the completeness of $\boldsymbol{\mu}_{\mathcal{C}}$.

Lemma 6.2.2. Let $\mathcal{C} \subset \mathcal{V}$ be a closed cone. Let $\mathcal{P} \subset \mathcal{C}$ be a part of $\mathcal{C}$ such that $P_i \neq \{0\}$ for $i = 1,\ldots,d$. Then $(\mathcal{P}, \boldsymbol{\mu}_{\mathcal{C}})$ is a d-pseudometric space. Moreover, let $\nu\colon \mathcal{C} \to \mathbb{R}^d_+$ be a multi-normalization of $\mathcal{C}$ and $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C}\colon \boldsymbol{\nu}(x) = \mathbf{1}\}$. Then, $(\mathcal{S}_{\boldsymbol{\nu}} \cap \mathcal{P}, \boldsymbol{\mu}_{\mathcal{C}})$ is a complete d-metric space and its topology coincides with the product norm topology of $\mathcal{V}$.

Proof. $(\mathcal{P}, \boldsymbol{\mu}_\mathcal{C})$ is a d-pseudometric space since (C_i, μ_{C_i}) is a pseudometric space for all i. For $i = 1, \ldots, d$ let $S_i \subset C_i$ be such that $\mathcal{S}_{\boldsymbol{\nu}} = S_1 \times \ldots \times S_d$. Lemma 5.1.7 implies that $(P_i \cap S_i, \mu_{C_i})$ is a complete metric space for $i = 1, \ldots, d$ and therefore $(\mathcal{S}_{\boldsymbol{\nu}} \cap \mathcal{P}, \boldsymbol{\mu}_\mathcal{C})$ is a complete d-metric space. The topology on $(\mathcal{S}_{\boldsymbol{\nu}} \cap \mathcal{P}, \boldsymbol{\mu}_\mathcal{C})$ coincide with the product topology of $(P_1 \cap S_1, \mu_{C_1}) \times \ldots \times (P_d \cap S_d, \mu_{C_d})$. As Lemma 5.1.7 implies that the topology of $(P_i \cap S_i, \mu_{C_i})$ coincide with the norm topology on V_i, this concludes the proof. □

There is an elegant representation of $\boldsymbol{\mu}_\mathcal{C}$ using the partial ordering $\leq$ induced by the simplicial cone $\mathbb{R}^d_+$ discussed in page 10: For $x, y \in \mathcal{C}$ define

$$\mathbf{M}(x/y; \mathcal{C}) = \inf\big\{\boldsymbol{\beta} \in \mathbb{R}^d_+ \,\big|\, x \preceq_\mathcal{C} \boldsymbol{\beta} \otimes y\big\}, \tag{6.8}$$

$$\mathbf{m}(x/y; \mathcal{C}) = \sup\{\boldsymbol{\alpha} \in \mathbb{R}^d_+ \,\big|\, \boldsymbol{\alpha} \otimes y \preceq_\mathcal{C} x\}, \tag{6.9}$$

where the supremum and infimum are defined with respect to the partial ordering induced by $\mathbb{R}^d_+$, and $\mathbf{M}(x/y; \mathcal{C})_i = \infty$ if there is no $\beta_i \geq 0$ such that $x_i \preceq_{C_i} \beta_i y_i$. Note that for all $x, y \in \mathcal{C}$ with $x \sim_\mathcal{C} y$ it holds

$$\mathbf{M}(x/y; \mathcal{C}) = (M(x_1/y_1; C_1), \ldots, M(x_d/y_d; C_d))^\top,$$

$$\mathbf{m}(x/y; \mathcal{C}) = (m(x_1/y_1; C_1), \ldots, M(x_d/y_d; C_d))^\top,$$

where $M(\cdot/\cdot; C_i)$ and $m(\cdot/\cdot; C_i)$ are defined as in Equation (5.1) with $C = C_i$. With this notation, for all $x, y \in \mathcal{C}$ with $x \sim_\mathcal{C} y$ we have

$$\boldsymbol{\mu}_\mathcal{C}(x, y) = \ln\big(\mathbf{M}(x/y; \mathcal{C}) \circ \mathbf{M}(y/x; \mathcal{C})\big) = \ln\big(\mathbf{M}(x/y; \mathcal{C}) \circ \mathbf{m}(x/y; \mathcal{C})^{-I}\big)$$

where the logarithm is applied component-wise.

6.2.2 Lipschitz matrices of multi-homogeneous mappings

We discuss the Lipschitz matrix of multi-homogeneous mappings on the product of cones. To this end, we use Theorem 6.1.7 to prove the multi-homogeneous versions of Propositions 5.2.1, 5.2.2.

Let $\mathcal{V} = V_1 \times \ldots \times V_d$ and $\mathcal{W} = W_1 \times \ldots \times W_{d'}$ be products of finite dimensional real vector spaces. Let $\mathcal{C} \subset \mathcal{V}$ and $\mathcal{K} \subset \mathcal{W}$ be cones. For the reading convenience, let us recall that for a matrix $A \in \mathbb{R}^{d' \times d}$, $|A|$ denotes the component-wise absolute value of A. Furthermore, for a mapping $\mathcal{F}\colon \mathcal{C} \to \mathcal{K}$, we denote by $\mathcal{F}|_x^{j,i}\colon C_i \to K_j$, the mapping defined for $x \in \mathcal{C}$, $i \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, d'\}$ as

$$\mathcal{F}|_x^{j,i}(y_i) = \mathcal{F}(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_d)_j \qquad \forall y_i \in C_i.$$

The following lemma follows from a direct verification and therefore its proof is omitted.

Lemma 6.2.3. Let $\mathcal{C} \subset \mathcal{V}$ and $\mathcal{K} \subset \mathcal{W}$ be cones, $\mathcal{F}\colon \mathcal{C} \to \mathcal{K}$ a mapping, $x \in \mathcal{C}$ and $i \in \{1, \ldots, d\}, j \in \{1, \ldots, d'\}$.

i) If $\mathcal{F}$ is multi-homogeneous of degree $A \in \mathbb{R}^{d' \times d}$, then $\mathcal{F}|_x^{j,i}$ is multi-homogeneous of degree $A_{j,i}$.

ii) If $\mathcal{F}$ is order-preserving, resp. strongly order-preserving, with respect to $\mathcal{C}$ and $\mathcal{K}$, then $\mathcal{F}|_x^{j,i}$ is order-preserving, resp. strongly order-preserving, with respect to C_i and K_j.

iii) If $\mathcal{F}$ is order-reversing, resp. strongly order-reversing, with respect to $\mathcal{C}$ and $\mathcal{K}$, then $\mathcal{F}|_x^{j,i}$ is order-reversing, resp. strongly order-reversing, with respect to C_i and K_j.

Proposition 6.2.4. Let $\mathcal{C} \subset \mathcal{V}$ and $\mathcal{K} \subset \mathcal{W}$ be closed cones and let $\mathcal{F} \colon \mathcal{C} \to \mathcal{K}$ be multi-homogeneous of degree $A \in \mathbb{R}^{d' \times d}$ and either order-preserving or order-reversing. Then for every $x, y \in \mathcal{C}$ with $x \sim_{\mathcal{C}} y$, it holds

$$\boldsymbol{\mu}_{\mathcal{K}}(\mathcal{F}(x), \mathcal{F}(y)) \leq |A| \boldsymbol{\mu}_{\mathcal{C}}(x, y).$$

Proof. Follows from Theorem 6.1.7, Lemma 6.2.3 and Proposition 5.2.1. □

Proposition 6.2.5. Let $\mathcal{C} \subset \mathcal{V}$ be a solid closed cone. Let $\boldsymbol{\nu} \colon \mathcal{C} \to \mathbb{R}^d_+$ be a multi-normalization of $\mathcal{C}$ and set $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$. Let $\mathcal{F} \colon \mathcal{C} \to \mathcal{C}$ be multi-homogeneous of degree $A \in \mathbb{R}^{d \times d}$ and either order-preserving or order-reversing. Suppose that for every $x \in \mathcal{C}$ such that $x_i \neq 0$ for all $i = 1, \ldots, d$ it holds $\mathcal{F}(x)_i \neq 0$ for all $i = 1, \ldots, d$. Define $\mathcal{G} \colon \mathcal{S}_{\boldsymbol{\nu}} \to \mathcal{S}_{\boldsymbol{\nu}}$ as

$$\mathcal{G}(x) = \boldsymbol{\nu}(\mathcal{F}(x))^{-I} \otimes \mathcal{F}(x) \qquad \forall x \in \mathcal{S}_{\boldsymbol{\nu}}.$$

Then, for every part $\mathcal{P}$ of $\mathcal{C}$ such that $P_i \neq \{0\}$ for all $i = 1, \ldots, d$, it holds

$$\boldsymbol{\mu}_{\mathcal{K}}(\mathcal{G}(x), \mathcal{G}(y)) \leq |A| \boldsymbol{\mu}_{\mathcal{C}}(x, y) \qquad \forall x, y \in \mathcal{S}_{\boldsymbol{\nu}} \cap \mathcal{P}.$$

Moreover, if $\mathcal{F}$ is strongly order-preserving or strongly order-reversing and A has at least one nonzero entry per column, then

$$\boldsymbol{\mu}_{\mathcal{K}}(\mathcal{G}(x), \mathcal{G}(y)) \lneq |A| \boldsymbol{\mu}_{\mathcal{C}}(x, y) \qquad \forall x, y \in \mathcal{S}_{\boldsymbol{\nu}} \cap \mathcal{P} \text{ with } x \neq y.$$

Proof. Follows from Theorem 6.1.7, Lemma 6.1.6, Lemma 6.2.3, Proposition 5.2.2 and Equation (6.7). □

7 Fixed point theorems on products of metric spaces

We prove fixed point type results for mappings defined on d-metric spaces. We then discuss the connection between the vector valued Hilbert metric $\boldsymbol{\mu}_{\mathcal{C}}$ and the Hilbert metric $\mu_{C_1 \times \ldots \times C_d}$. Finally, we prove Perron-Frobenius type theorems for eigenvectors of mappings defined on the product of cones.

7.1 Vector valued fixed point theorems

We start by proving a vector valued version of the Banach fixed point theorem. This result was proved independently in [38, Theorem 4.2]$^\diamond$ and [2, Theorem 1].

Theorem 7.1.1. Let $(\mathcal{X}, \boldsymbol{\eta})$ be a complete d-metric space and let $\mathcal{F}\colon (\mathcal{X}, \boldsymbol{\eta}) \to (\mathcal{X}, \boldsymbol{\eta})$ be a mapping with Lipschitz matrix $A \in \mathbb{R}^{d\times d}_+$. If $\rho(A) < 1$, then $\mathcal{F}$ has a unique fixed point $u \in \mathcal{X}$ and $\lim_{k\to\infty} \mathcal{F}^k(x) = u$ for every $x \in \mathcal{X}$. Moreover, for any $x \in \mathcal{X}$ and any positive integer k, it holds

$$\begin{aligned} \boldsymbol{\eta}\big(\mathcal{F}^k(x), u\big) &\leq A^k(I-A)^{-1}\boldsymbol{\eta}\big(\mathcal{F}(x), x\big) \\ \boldsymbol{\eta}\big(\mathcal{F}^k(x), u\big) &\leq A\boldsymbol{\eta}\big(\mathcal{F}^{k-1}(x), u\big) \\ \boldsymbol{\eta}\big(\mathcal{F}^k(x), u\big) &\leq A(I-A)^{-1}\boldsymbol{\eta}\big(\mathcal{F}^k(x), \mathcal{F}^{k-1}(x)\big). \end{aligned} \tag{7.1}$$

Proof. Let $x \in \mathcal{X}$, then we have $\boldsymbol{\eta}\big(\mathcal{F}^2(x), \mathcal{F}(x)\big) \leq A\boldsymbol{\eta}\big(\mathcal{F}(x), x\big)$. Adding $\boldsymbol{\eta}(\mathcal{F}(x), x)$ on both sides of this inequality we get

$$\boldsymbol{\eta}\big(\mathcal{F}^2(x), \mathcal{F}(x)\big) + \boldsymbol{\eta}(\mathcal{F}(x), x) \leq A\boldsymbol{\eta}\big(\mathcal{F}(x), x\big) + \boldsymbol{\eta}(\mathcal{F}(x), x),$$

which can be rearranged into

$$(I-A)\boldsymbol{\eta}(\mathcal{F}(x), x) \preceq \boldsymbol{\eta}(\mathcal{F}(x), x) - \boldsymbol{\eta}\big(\mathcal{F}^2(x), \mathcal{F}(x)\big).$$

It follows from $\rho(A) < 1$ that $I - A$ is invertible and $(I-A)^{-1} = \sum_{j=0}^{\infty} A^j \in \mathbb{R}^{d\times d}_+$. Therefore

$$\boldsymbol{\eta}(\mathcal{F}(x), x) \leq (I-A)^{-1}\Big(\boldsymbol{\eta}(\mathcal{F}(x), x) - \boldsymbol{\eta}\big(\mathcal{F}^2(x), \mathcal{F}(x)\big)\Big).$$

Then, by the triangle inequality, for every $m \geq k \geq 1$ we have

$$\begin{aligned} \boldsymbol{\eta}\big(\mathcal{F}^{m+1}(x), \mathcal{F}^k(x)\big) &\leq \sum_{i=k}^{m} \boldsymbol{\eta}\big(\mathcal{F}^{i+1}(x), \mathcal{F}^i(x)\big) \\ &\leq (I-A)^{-1}\sum_{i=k}^{m}\Big(\boldsymbol{\eta}\big(\mathcal{F}^{i+1}(x), \mathcal{F}^i(x)\big) - \boldsymbol{\eta}\big(\mathcal{F}^{i+2}(x), \mathcal{F}^{i+1}(x)\big)\Big) \\ &= (I-A)^{-1}\big(\boldsymbol{\eta}\big(\mathcal{F}^{k+1}(x), \mathcal{F}^k(x)\big) - \boldsymbol{\eta}\big(\mathcal{F}^{m+2}(x), \mathcal{F}^{m+1}(x)\big)\big) \\ &\leq (I-A)^{-1}\boldsymbol{\eta}\big(\mathcal{F}^{k+1}(x), \mathcal{F}^k(x)\big). \end{aligned} \tag{7.2}$$

In particular, if we set $k = 1$ and let $m \to \infty$, we get

$$\sum_{i=1}^{\infty} \boldsymbol{\eta}\big(\mathcal{F}^i(x), \mathcal{F}^{i+1}(x)\big) \leq (I-A)^{-1}\boldsymbol{\eta}\big(\mathcal{F}^2(x), \mathcal{F}(x)\big).$$

As the above inequality holds component wise, it implies that $(\mathcal{F}^k(x)_i)_{k=1}^{\infty} \subset X_i$ is a Cauchy sequence for every $i = 1, \dots, d$. Since (X_i, η_i) is complete for every i, it follows that $(\mathcal{F}^k(x))_{k=1}^{\infty} \subset \mathcal{X}$ converges towards some $u \in \mathcal{X}$. By Lemma 6.1.5, $\mathcal{F}$ is continuous in the product topology and thus we have

$$u = \lim_{k\to\infty} \mathcal{F}^{k+1}(x) = \mathcal{F}\Big(\lim_{k\to\infty} \mathcal{F}^k(x)\Big) = \mathcal{F}(u), \tag{7.3}$$

i.e. u is a fixed point of $\mathcal{F}$. We prove that u is the unique fixed point of $\mathcal{F}$. Let $y \in \mathcal{X}$ be such that $\mathcal{F}(y) = y$, then

$$0 \le \boldsymbol{\eta}(u,y) = \boldsymbol{\eta}(\mathcal{F}^k(u), \mathcal{F}^k(y)) \le A^k\, \boldsymbol{\eta}(u,y)$$

and by letting $k \to \infty$ we deduce $u = y$ since $\rho(A) < 1$ implies $\lim_{k\to\infty} A^k = 0$. Finally, we prove the convergence rates in (7.1). Note that for all $k \ge 1$ it holds

$$(I-A)^{-1}A^k = \sum_{j=0}^{\infty} A^j A^k = \sum_{j=k}^{\infty} A^j = A^k \sum_{j=0}^{\infty} A^j = A^k (I-A)^{-1}. \tag{7.4}$$

Now, using (7.2) and (7.3) we find that for all $k \ge 1$, it holds

$$\begin{aligned}
\boldsymbol{\eta}\big(u, \mathcal{F}^k(x)\big) &= \lim_{m\to\infty} \boldsymbol{\eta}(\mathcal{F}^{m+1}(x), \mathcal{F}^k(x)) \\
&\le \lim_{m\to\infty} (I-A)^{-1}\Big(\boldsymbol{\eta}\big(\mathcal{F}^{k+1}(x), \mathcal{F}^k(x)\big) - \boldsymbol{\eta}\big(\mathcal{F}^{m+2}(x), \mathcal{F}^{m+1}(x)\big)\Big) \\
&= (I-A)^{-1}\boldsymbol{\eta}\big(\mathcal{F}^{k+1}(x), \mathcal{F}^k(x)\big) \\
&\le (I-A)^{-1}A^k\boldsymbol{\eta}(\mathcal{F}(x), x) = A^k(I-A)^{-1}\boldsymbol{\eta}(\mathcal{F}(x), x).
\end{aligned}$$

Furthermore, it holds

$$\boldsymbol{\eta}(\mathcal{F}^k(x), u) = \boldsymbol{\eta}(\mathcal{F}(\mathcal{F}^{k-1}(x)), \mathcal{F}(u)) \le A\boldsymbol{\eta}(\mathcal{F}^{k-1}(x), u),$$

and, by the triangle inequality,

$$\boldsymbol{\eta}(\mathcal{F}^k(x), u) \le A\boldsymbol{\eta}(\mathcal{F}^{k-1}(x), u) \le A\big(\boldsymbol{\eta}(\mathcal{F}^{k-1}(x), \mathcal{F}^k(x)) + \boldsymbol{\eta}(\mathcal{F}^k(x), u)\big)$$

which can be rearranged into $(I-A)\boldsymbol{\eta}(\mathcal{F}^k(x), u) \le A\boldsymbol{\eta}(\mathcal{F}^{k-1}(x), \mathcal{F}^k(x))$ and, with (7.4), we deduce that

$$\boldsymbol{\eta}(\mathcal{F}^k(x), u) \le (I-A)^{-1}A\boldsymbol{\eta}(\mathcal{F}^{k-1}(x), \mathcal{F}^k(x)) = A(I-A)^{-1}\boldsymbol{\eta}(\mathcal{F}^{k-1}(x), \mathcal{F}^k(x)),$$

which concludes the proof. □

The convergence rates (7.1) of Theorem 7.1.1 are remarkable because they provide information on the convergence of $(\mathcal{F}^k(x))_{k=1}^{\infty}$ in each of the metric spaces (X_i, η_i). The condition $\rho(A) < 1$ of Theorem 7.1.1 has the following interpretation: Given a norm $\|\cdot\|_\alpha$ on $\mathbb{R}^d$, let us denote by $\|\cdot\|_{\alpha\to\alpha}$ the induced operator norm, i.e. $\|M\|_{\alpha\to\alpha} = \sup\{\|M\mathbf{c}\|_\alpha \,|\, \|\mathbf{c}\|_\alpha \le 1\}$ for $M \in \mathbb{R}^{d\times d}$. Now, let $\|\cdot\|_\alpha$ be a norm on $\mathbb{R}^d$ which is monotonic with respect to $\mathbb{R}^d_+$. Suppose that $A \in \mathbb{R}^{d\times d}$ is a Lipschitz matrix of $\mathcal{F}\colon (\mathcal{X}, \boldsymbol{\eta}) \to (\mathcal{X}, \boldsymbol{\eta})$. Then, for every $x, y \in \mathcal{X}$ it hods

$$\|\boldsymbol{\eta}(\mathcal{F}(x), \mathcal{F}(y))\|_\alpha \le \|A\boldsymbol{\eta}(x,y)\|_\alpha \le \|A\|_{\alpha\to\alpha}\, \|\boldsymbol{\eta}(x,y)\|_\alpha. \tag{7.5}$$

This means that $\|A\|_{\alpha\to\alpha}$ is a Lipschitz constant of $\mathcal{F}$ with respect to the metric $\eta_\alpha\colon \mathcal{X}\times\mathcal{X} \to [0,\infty)$ defined as $\eta_\alpha(x,y) = \|\boldsymbol{\eta}(x,y)\|_\alpha$. Hence, in order to apply the classical Banach fixed point theorem to $\mathcal{F}\colon (\mathcal{X}, \eta_\alpha) \to (\mathcal{X}, \eta_\alpha)$, we would like to find the monotonic norm on $\mathbb{R}^d$ which minimizes the quantity $\|A\|_{\alpha\to\alpha}$. The infimum over all induced operator norms of a square matrix equals its spectral radius (see for instance Lemma 5.6.10 [48]), that is

$$\rho(A) = \inf\big\{\|A\|_{\beta\to\beta}\colon \|\cdot\|_\beta \text{ is a norm on } \mathbb{R}^d\big\}. \tag{7.6}$$

It means that the condition $\rho(A) < 1$ of Theorem 7.1.1 could be replaced by the existence of a norm $\|\cdot\|_\alpha$ on $\mathbb{R}^d$ such that $\|A\|_{\alpha\to\alpha} < 1$. We note however that the infimum in (7.6) may not be attained and even if attained at some norm $\|\cdot\|_\alpha$, it could be that $\|\cdot\|_\alpha$ is not monotonic with respect to $\mathbb{R}^d_+$. We prove in Lemma 7.1.2 below that whenever A has a positive left eigenvector then the infimum in (7.6) is attained at a monotonic norm. The construction of such norm is motivated by the following observation: Consider the particular case where $\|\mathbf{c}\|_\alpha = \langle |\mathbf{c}|\,,\,\mathbf{a}\rangle$ for some weight vector $\mathbf{a}\in\mathbb{R}^d_{++}$, then $\|\cdot\|_\alpha$ is a norm on $\mathbb{R}^d$ which is monotonic with respect to $\mathbb{R}^d_+$. In this case, we have $\eta_\alpha(x,y) = \sum_{i=1}^d a_i\eta_i(x_i,y_i)$ for all $x,y\in\mathcal{X}$. Furthermore, note that for every $x,y\in\mathcal{X}$ it holds

$$\begin{aligned}\eta_\alpha(\mathcal{F}(x),\mathcal{F}(y)) &= \sum_{i=1}^d a_i\eta_i(\mathcal{F}(x)_i,\mathcal{F}(y)_i) \le \sum_{i,j=1}^d a_iA_{i,j}\eta_j(x_j,y_j)\\ &= \sum_{j=1}^d (A^\top\mathbf{a})_j\eta_j(x_j,y_j) \le \Big(\max_{j=1,\ldots,d}\frac{(A^\top\mathbf{a})_j}{a_j}\Big)\eta_\alpha(x,y).\end{aligned}$$

Hence, $\max_{j=1,\ldots,d}(A^\top\mathbf{a})_j/a_j$ is a Lipschitz constant of $\mathcal{F}$ with respect to the metric η_α. Again, in order to apply the classical Banach fixed point theorem to $\mathcal{F}\colon(\mathcal{X},\eta_\alpha)\to(\mathcal{X},\eta_\alpha)$, we would like to find the weight vector $\mathbf{a}\in\mathbb{R}^d_{++}$ which minimizes this quantity. This can be done using the Collatz-Wielandt formula (see for instance Theorem 5.6.1 [60]), which states that

$$\rho(A) = \inf_{\mathbf{a}\in\mathbb{R}^d_{++}}\ \max_{j=1,\ldots,d}\frac{(A^\top\mathbf{a})_j}{a_j}, \tag{7.7}$$

and the infimum is attained if $A^\top$ has a positive eigenvector. In particular this means that the condition $\rho(A) < 1$ of Theorem 5.6.1 could be replaced by the existence of $\mathbf{a}\in\mathbb{R}^d_{++}$ such that $A^\top\mathbf{a} < \mathbf{a}$. The interplay between the characterizations (7.6) and (7.7) of $\rho(A)$ is discussed in the following lemma which also summarizes the main conclusions of our observations.

Lemma 7.1.2. Let $\mathcal{F}\colon(\mathcal{X},\boldsymbol{\eta})\to(\mathcal{X},\boldsymbol{\eta})$ be a mapping with Lipschitz matrix $A\in\mathbb{R}^{d\times d}_+$. Suppose that A has a positive left eigenvector $\mathbf{b}\in\mathbb{R}^d_{++}$ and consider the norm $\|\mathbf{c}\|_\alpha = \langle|\mathbf{c}|\,,\,\mathbf{b}\rangle$ for all $\mathbf{c}\in\mathbb{R}^d$. Then, $\|\cdot\|_\alpha$ is a norm on $\mathbb{R}^d$, it is monotonic with respect to $\mathbb{R}^d_+$ and $\|A\|_{\alpha\to\alpha} = \rho(A)$. Moreover,

$$\eta_\alpha(x,y) = \|\boldsymbol{\eta}(x,y)\|_\alpha = \langle\mathbf{b}\,,\,\boldsymbol{\eta}(x,y)\rangle$$

is a metric on $\mathcal{X}$ such that

$$\eta_\alpha(\mathcal{F}(x),\mathcal{F}(y)) \le \rho(A)\eta_\alpha(x,y) \qquad \forall x,y\in\mathcal{X}.$$

Furthermore, if

$$\boldsymbol{\eta}(\mathcal{F}(x),\mathcal{F}(y)) \lneq A\boldsymbol{\eta}(x,y) \qquad \forall x,y\in\mathcal{X} \text{ with } x\neq y,$$

then

$$\eta_\alpha(\mathcal{F}(x),\mathcal{F}(y)) < \rho(A)\eta_\alpha(x,y) \qquad \forall x,y\in\mathcal{X} \text{ with } x\neq y.$$

Proof. It is easily verified that $\|\cdot\|_\alpha$ is a monotonic norm and η_α a metric. Now, as $\mathbf{b}$ is a left eigenvector of A, it holds $A^\top\mathbf{b}=\lambda\mathbf{b}$ for some $\lambda\in\mathbb{R}$. The Collatz-Wielandt formula (7.7) implies that $\lambda=\rho(A)$ since $\mathbf{b}\in\mathbb{R}^d_{++}$. Now, let $\mathbf{c}\in\mathbb{R}^d$, as A is nonnegative we have $|A\mathbf{c}|\leq A|\mathbf{c}|$ and thus

$$\|A\mathbf{c}\|_\alpha = \||A\mathbf{c}|\|_\alpha \leq \|A|\mathbf{c}|\|_\alpha = \langle A|\mathbf{c}|\,,\,\mathbf{b}\rangle = \langle |\mathbf{c}|\,,\,A^\top\mathbf{b}\rangle = \rho(A)\|\mathbf{c}\|_\alpha.$$

Furthermore, if $\mathbf{c}\in\mathbb{R}^d_+$, then $|A\mathbf{c}|=A|\mathbf{c}|$ so that $\|A\mathbf{c}\|_\alpha=\rho(A)\|\mathbf{c}\|_\alpha$ for all $\mathbf{c}\in\mathbb{R}^d_+$. The discussion in Example 2.3.7, and more precisely Equation (2.7), implies that $\|A\|_{\alpha\to\alpha}=\sup\{\|A\mathbf{c}\|_\alpha \mid \mathbf{c}\in\mathbb{R}^d_+, \|\mathbf{c}\|_\alpha\leq 1\}$ since A is nonnegative and $\|\cdot\|_\alpha$ is monotonic with respect to $\mathbb{R}^d_+$. It follows that $\|A\|_{\alpha\to\alpha}=\rho(A)$ and, therefore, the first inequality of the statement is implied by (7.5). To prove the last inequality, note that $\langle\mathbf{b}\,,\,\mathbf{c}\rangle < \langle\mathbf{b}\,,\,\mathbf{c}'\rangle$ for all $\mathbf{c},\mathbf{c}'\in\mathbb{R}^d_+$ with $\mathbf{c}\lneq\mathbf{c}'$ since $\mathbf{b}\in\mathbb{R}^d_{++}$. It follows that if $x\neq y$ and $\boldsymbol{\eta}(\mathcal{F}(x),\mathcal{F}(y))\lneq A\boldsymbol{\eta}(x,y)$, then

$$\begin{aligned}\eta_\alpha(\mathcal{F}(x),\mathcal{F}(y)) = \langle\mathbf{b}\,,\,\boldsymbol{\eta}(\mathcal{F}(x),\mathcal{F}(y))\rangle &< \langle\mathbf{b}\,,\,A\boldsymbol{\eta}(x,y)\rangle\\ &= \rho(A)\langle\mathbf{b}\,,\,\boldsymbol{\eta}(x,y)\rangle = \rho(A)\eta_\alpha(x,y),\end{aligned}$$

which concludes the proof. □

A combination of Lemma 7.1.2 and Theorem 7.1.1 yields the following vector valued Banach fixed point theorem with real valued convergence rates.

Theorem 7.1.3. Let $(\mathcal{X},\boldsymbol{\eta})$ be a complete d-metric space and let $\mathcal{F}\colon(\mathcal{X},\boldsymbol{\eta})\to(\mathcal{X},\boldsymbol{\eta})$ be a mapping with Lipschitz matrix $A\in\mathbb{R}^{d\times d}_+$. Suppose that $A^\top$ has a positive eigenvector $\mathbf{b}\in\mathbb{R}^d_{++}$ and let $\eta_\alpha(x,y)=\langle\mathbf{b}\,,\,\boldsymbol{\eta}(x,y)\rangle$ be the metric of Lemma 7.1.2. If $\rho(A)<1$, then $\mathcal{F}$ has a unique fixed point $u\in\mathcal{X}$ and $\lim_{k\to\infty}\mathcal{F}^k(x)=u$ for every $x\in\mathcal{X}$. Moreover, for any $x\in\mathcal{X}$ and any positive integer k, it holds

$$\begin{aligned}\eta_\alpha\big(\mathcal{F}^k(x),u\big) &\leq \frac{\rho(A)^k}{1-\rho(A)}\,\eta_\alpha\big(\mathcal{F}(x),x\big),\\ \eta_\alpha\big(\mathcal{F}^k(x),u\big) &\leq \rho(A)\,\eta_\alpha\big(\mathcal{F}^{k-1}(x),u\big),\\ \eta_\alpha\big(\mathcal{F}^k(x),u\big) &\leq \frac{\rho(A)}{1-\rho(A)}\,\eta_\alpha\big(\mathcal{F}^k(x),\mathcal{F}^{k-1}(x)\big).\end{aligned} \tag{7.8}$$

Proof. Existence and uniqueness of u as well as $\lim_{k\to\infty}\mathcal{F}^k(x)=u$ follow directly from Theorem 7.1.1. We prove the convergence rates (7.8). First, we recall from (7.4) that for all $k\geq 1$, it holds $A^k(I-A)^{-1}=(I-A)^{-1}A^k$. Hence, for $x\in\mathcal{X}$ and $k\geq 1$, the convergence rates in (7.1) imply

$$\begin{aligned}(I-A)\boldsymbol{\eta}\big(\mathcal{F}^k(x),u\big) &\leq A^k\boldsymbol{\eta}\big(\mathcal{F}(x),x\big),\\ \boldsymbol{\eta}\big(\mathcal{F}^k(x),u\big) &\leq A\boldsymbol{\eta}\big(\mathcal{F}^{k-1}(x),u\big),\\ (I-A)\boldsymbol{\eta}\big(\mathcal{F}^k(x),u\big) &\leq A\boldsymbol{\eta}\big(\mathcal{F}^k(x),\mathcal{F}^{k-1}(x)\big).\end{aligned} \tag{7.9}$$

Now, note that for all $y, z \in \mathcal{X}$ and $k \geq 1$, it holds

$$\begin{aligned}\langle \mathbf{b}\,,\, (I-A)\boldsymbol{\eta}(y,z)\rangle &= \langle (I-A)^\top \mathbf{b}\,,\, \boldsymbol{\eta}(y,z)\rangle = (1-\rho(A))\langle \mathbf{b}\,,\, \boldsymbol{\eta}(y,z)\rangle \\ &= (1-\rho(A))\eta_\alpha(y,z), \\ \langle \mathbf{b}\,,\, A^k\boldsymbol{\eta}(y,z)\rangle &= \langle (A^T)^k\mathbf{b}\,,\, \boldsymbol{\eta}(y,z)\rangle = \rho(A)^k\langle \mathbf{b}\,,\, \boldsymbol{\eta}(y,z)\rangle = \rho(A)^k\eta_\alpha(y,z).\end{aligned}$$

The above identities together with (7.9) imply

$$\begin{aligned}(1-\rho(A))\eta_\alpha\big(\mathcal{F}^k(x),u\big) &\leq \rho(A)^k\,\eta_\alpha\big(\mathcal{F}(x),x\big), \\ \eta_\alpha\big(\mathcal{F}^k(x),u\big) &\leq \rho(A)\,\eta_\alpha\big(\mathcal{F}^{k-1}(x),u\big), \\ (1-\rho(A))\eta_\alpha\big(\mathcal{F}^k(x),u\big) &\leq \rho(A)\,\eta_\alpha\big(\mathcal{F}^k(x),\mathcal{F}^{k-1}(x)\big),\end{aligned}$$

which, after division of the first and last inequations by $(1-\rho(A))$, implies (7.8). □

In facts, Lemma 7.1.2 implies that Theorem 7.1.3 can be directly deduced from the classical Banach fixed point theorem. However, by doing so, the convergence rates become (7.8) which is less informative than (7.1) since η_α blends the distance on all the X_i's. Furthermore, note that Theorem 7.1.1 does not assume that A has a positive left eigenvector. This is not very restrictive because one can always perturb a Lipschitz matrix with a small strictly positive matrix in order to have the existence of a positive eigenvector by the Perron-Frobenius theorem. On the other hand, such a perturbation induces more conservative convergence bounds.

In Section 4.4 we have shown that for the study of eigenvectors of a multi-homogeneous mapping we may assume that the absolute value of the homogeneity matrix is irreducible. A proof similar to that of Proposition 4.4.2 shows that if L is a Lipschitz matrix of $\mathcal{F}\colon (\mathcal{X},\boldsymbol{\eta}) \to (\mathcal{X},\boldsymbol{\eta})$ and L is reducible, then by considering the Frobenius normal form of L (see Theorem 4.4.3), the fixed point equation $\mathcal{F}(x) = x$ can be decomposed into fixed point equations where the corresponding Lipschitz matrices are either irreducible or zero. This approach is particularly relevant in this context since Theorem 6.4.5 of [30] implies that the spectral radius of L equals the maximum of the spectral radii of the diagonal blocks in the Frobenius normal form of L. The latter means that decomposing the fixed point problem into subproblems will neither improve nor damage the contractivity of the whole problem.

Similarly, the substitution technique discussed in Section 4.3 does not allows to transform an expansive problem into a contractive problem. Indeed, if $\mathcal{F}\colon (\mathcal{X},\boldsymbol{\eta}) \to (\mathcal{X},\boldsymbol{\eta})$ has a Lipschitz matrix $L \in \mathbb{R}^{d\times d}_+$ such that $L_{d,d} = 0$, then $\mathcal{F}(x)_d$ does not depend on x_d since $L_{d,d} = 0$ implies that for every $x \in \mathcal{X}$ it holds

$$\eta_d(\mathcal{F}(x_1,\ldots,x_{d-1},y_d)_d, \mathcal{F}(x_1,\ldots,x_{d-1},z_d)_d) = 0 \qquad \forall y_d, z_d \in X_d,$$

so that

$$\mathcal{F}(x_1,\ldots,x_{d-1},y_d)_d = \mathcal{F}(x_1,\ldots,x_{d-1},z_d)_d \qquad \forall y_d, z_d \in X_d.$$

A similar argument as in Proposition 4.3.2 shows that one can transform the fixed point problem $\mathcal{F}(x) = x$ into an equivalent problem of the form $\hat{\mathcal{F}}(\hat{x}) = \hat{x}$ where $\hat{\mathcal{F}}\colon (\hat{\mathcal{X}},\hat{\boldsymbol{\eta}}) \to (\hat{\mathcal{X}},\hat{\boldsymbol{\eta}})$ is defined as

$$\hat{\mathcal{F}}(\hat{x}) = \mathcal{F}(\hat{x}_1,\ldots,\hat{x}_{d-1},\mathcal{F}(\hat{x}_1,\ldots,\hat{x}_{d-1},x_d)_d),$$

with $x_d \in X_d$ fixed (arbitrary) and where $(\hat{\mathcal{X}}, \hat{\boldsymbol{\eta}})$ is the $(d-1)$-metric space with $\hat{\mathcal{X}} = X_1 \times \ldots \times X_{d-1}$ and $\hat{\boldsymbol{\eta}} \colon \hat{\mathcal{X}} \to \mathbb{R}_+^{d-1}$ given by

$$\hat{\boldsymbol{\eta}}(\hat{x}, \hat{y}) = (\eta_1(\hat{x}_1, \hat{y}_1), \ldots, \eta_{d-1}(\hat{x}_{d-1}, \hat{y}_{d-1}))^\top \qquad \forall \hat{x}, \hat{y} \in \hat{\mathcal{X}}.$$

Note that $\hat{\mathcal{F}} \colon (\hat{\mathcal{X}}, \hat{\boldsymbol{\eta}}) \to (\hat{\mathcal{X}}, \hat{\boldsymbol{\eta}})$ has Lipschitz matrix $\hat{L} \in \mathbb{R}^{(d-1)\times(d-1)}$ given by

$$\hat{L}_{i,j} = L_{i,j} + L_{i,d} L_{d,j} \qquad \forall i, j = 1, \ldots, d-1. \tag{7.10}$$

We prove in the next proposition that if L is irreducible, then $\rho(L) = 1$ if and only if $\rho(\hat{L}) = 1$ and $\rho(L) < 1$ if and only if $\rho(\hat{L}) < 1$. The latter implies that $\mathcal{F}$ is non-expansive, respectively a strict contraction, with respect to $\boldsymbol{\eta}$ if and only if $\hat{\mathcal{F}}$ is non-expansive, respectively a strict contraction, with respect to $\hat{\boldsymbol{\eta}}$. We should nevertheless point out that $\rho(L)$ does not equal $\rho(\hat{L})$ in general.

Proposition 7.1.4. Let $L \in \mathbb{R}_+^{d\times d}$ be an irreducible matrix such that $L_{d,d} = 0$ and define $\hat{L} \in \mathbb{R}^{(d-1)\times(d-1)}$ as in (7.10). Then $\hat{L}$ is irreducible and it holds $\rho(L) = 1$, respectively $\rho(L) < 1$, if and only if $\rho(\hat{L}) = 1$, respectively $\rho(\hat{L}) < 1$.

Proof. First we prove that there exists $\hat{\mathbf{b}} \in \mathbb{R}_{++}^{d-1}$ such that $\hat{L}^T\hat{\mathbf{b}} \leq \hat{\mathbf{b}}$, resp. $\hat{L}\hat{\mathbf{b}} < \hat{\mathbf{b}}$, if and only if there exists $\mathbf{b} \in \mathbb{R}_{++}^d$ such that $L\mathbf{b} \leq \mathbf{b}$, resp. $L\mathbf{b} < \mathbf{b}$. Note that for every $\mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{R}_{++}^d$ and $\hat{\mathbf{b}} = (b_1, \ldots, b_{d-1})$, $L_{d,d} = 0$ imply that, for $l = 1, \ldots, d-1$, we have

$$(L\mathbf{b})_l = (\hat{L}\hat{\mathbf{b}})_l + L_{l,d}\big(b_d - (L\mathbf{b})_d\big). \tag{7.11}$$

Now, suppose that $L\mathbf{b} \leq r\mathbf{b}$ for some $r \in (0,1]$. Then, $b_d - (L\mathbf{b})_d \geq 0$ and

$$r\hat{b}_l = rb_l \geq (\hat{L}\hat{\mathbf{b}})_l + L_{l,d}(b_d - (L\mathbf{b})_d) \geq (\hat{L}\hat{\mathbf{b}})_l, \qquad l = 1, \ldots, d-1.$$

Now, let $\hat{\mathbf{b}} \in \mathbb{R}_{++}^{d-1}$ be such that $\hat{L}^T\hat{\mathbf{b}} \leq \hat{\mathbf{b}}$. For $t > 0$, set $\mathbf{b}(t) = (\hat{b}_1, \ldots, \hat{b}_{d-1}, t) \in \mathbb{R}_{++}^d$. For every $k = 1, \ldots, d-1$ and $t > 0$, we have $\delta_d = \big(L\mathbf{b}(t)\big)_d = \big(L\mathbf{b}(0)\big)_d > 0$ and $\big(L\mathbf{b}(t)\big)_k = \big(\hat{L}\hat{\mathbf{b}}\big)_k + L_{k,d}(t - \delta_d)$. It follows that

$$\begin{aligned}\max_{k=1,\ldots,d} \frac{(L\mathbf{b}(t))_k}{(\mathbf{b}(t))_k} &= \max\left\{\frac{\delta_d}{t}\,,\max_{k=1,\ldots,d-1} \frac{\big(\hat{L}\hat{\mathbf{b}}\big)_k + L_{k,d}(t-\delta_d)}{\hat{b}_k}\right\} \\ &\leq \max\left\{\frac{\delta_d}{t}, \max_{k=1,\ldots,d-1} \frac{(\hat{L}\hat{\mathbf{b}})_k}{\hat{b}_k} + (t-\delta_d) \max_{k=1,\ldots,d-1} \frac{L_{k,d}}{\hat{b}_k}\right\}. \end{aligned} \tag{7.12}$$

Hence, with $t = \delta_d$, we have $\mathbf{b}(\delta_d) \in \mathbb{R}_{++}^d$ and

$$\max_{k=1,\ldots,d} \frac{(L\mathbf{b}(\delta_d))_k}{(\mathbf{b}(\delta_d))_k} \leq \max\left\{1, \max_{k=1,\ldots,d-1} \frac{(\hat{L}\hat{\mathbf{b}})_k}{\hat{b}_k}\right\} = 1.$$

Finally, suppose that $\hat{L}^T\hat{\mathbf{b}} < \hat{\mathbf{b}}$, then there exists $\epsilon \in (0,1)$ such that $\epsilon L_{d,k} < \hat{b}_k - \big(\hat{L}\hat{\mathbf{b}}\big)_k$ for $k = 1, \ldots, d-1$. From (7.12), it follows that

$$\max_{k=1,\ldots,d} \frac{(L\mathbf{b}(\delta_d+\epsilon))_k}{(\mathbf{b}(\delta_d+\epsilon))_k} = \max\left\{\frac{\delta_d}{\delta_d+\epsilon}, \max_{k=1,\ldots,d-1} \frac{\big(\hat{L}\hat{\mathbf{b}}\big)_k + \epsilon L_{k,d}}{\hat{b}_k}\right\} < 1.$$

This concludes the proof of the claim that there exists $\hat{\mathbf{b}} \in \mathbb{R}^{d-1}_{++}$ such that $\hat{L}\hat{\mathbf{b}} \leq \hat{\mathbf{b}}$, resp. $\hat{L}\hat{\mathbf{b}} < \hat{\mathbf{b}}$, if and only if there exists $\mathbf{b} \in \mathbb{R}^d_{++}$ such that $L\mathbf{b} \leq \mathbf{b}$, resp. $L\mathbf{b} < \mathbf{b}$. Now, we show that $\hat{L}$ is irreducible. Let $i, j \in \{1, \ldots, d-1\}$. As L is irreducible, there exists a path from i to j in the graph whose adjacency matrix is L, i.e. there exists $k_1, \ldots, k_n \in \{1, \ldots, d\}$ such that $k_1 = i$, $k_n = j$ and $L_{k_l,k_{l+1}} > 0$ for all $l = 1, \ldots, n-1$. Note that if $k_l, k_{l+1} \neq d$, then $\hat{L}_{k_l,k_{l+1}} > 0$ as well. Now, if there exists $l \in \{2, \ldots, n-1\}$ such that $k_l = d$, then $k_{l-1}, k_{l+1} \neq d$ since $L_{d,d} = 0$. Furthermore, we have

$$\hat{L}_{k_{l-1},k_{l+1}} = L_{k_{l-1},k_{l+1}} + L_{k_{l-1},d}L_{d,k_{l+1}} = L_{k_{l-1},k_{l+1}} + L_{k_{l-1},k_l}L_{k_l,k_{l+1}} > 0.$$

This implies that there exists a path from i to j in the graph whose adjacency matrix is $\hat{L}$ and thus, as $i, j \in \{1, \ldots, d-1\}$ are arbitrary, $\hat{L}$ is irreducible as well. As both L and $\hat{L}$ are irreducible, there exists $\mathbf{c} \in \mathbb{R}^d_{++}$ and $\hat{\mathbf{c}} \in \mathbb{R}^d_{++}$ such that $L\mathbf{c} = \rho(L)\mathbf{c}$ and $\hat{L}\hat{\mathbf{c}} = \rho(\hat{L})\hat{\mathbf{c}}$.

We prove that $\rho(L) = 1$ if and only if $\rho(\hat{L}) = 1$. Suppose that $\rho(L) = 1$, then $L\mathbf{c} = \mathbf{c}$ and the above discussion implies that there exists $\hat{\mathbf{b}} \in \mathbb{R}^d_{++}$ such that $\hat{L}\hat{\mathbf{b}} \leq \hat{\mathbf{b}}$. Hence, the Collatz-Wielandt formula 7.7 implies that $\rho(\hat{L}) \leq 1$. Suppose by contradiction that $\rho(\hat{L}) < 1$, then $\hat{L}\hat{\mathbf{c}} < \hat{\mathbf{c}}$ which implies that there exists $\mathbf{b} \in \mathbb{R}^d_{++}$ such that $L\mathbf{b} < \mathbf{b}$ and then the Collatz-Wielandt formula implies $\rho(L) < 1$, a contradiction. It follows that $\rho(\hat{L}) = 1$. By swapping the roles of L and $\hat{L}$ in the above argument, one can deduce that $\rho(\hat{L}) = 1$ implies $\rho(L) = 1$ which proves the first equivalence. Finally, to prove that $\rho(L) < 1$ if and only if $\rho(\hat{L}) < 1$, suppose that $\rho(L) < 1$, resp. $\rho(\hat{L}) < 1$. Then, $L\mathbf{c} < \mathbf{c}$, resp. $\hat{L}\hat{\mathbf{c}} < \hat{\mathbf{c}}$. Hence, there exists $\hat{\mathbf{b}} \in \mathbb{R}^{d-1}_{++}$, resp. $\mathbf{b} \in \mathbb{R}^d_{++}$, such that $\hat{L}\hat{\mathbf{b}} < \hat{\mathbf{b}}$, resp. $L\mathbf{b} < \mathbf{b}$, and thus the Collatz-Wielandt formula implies that $\rho(\hat{L}) < 1$, resp. $\rho(L) < 1$. □

In the next result, we apply classical results of fixed point theory on the metric space induced by Lemma 7.1.2 to prove vector valued fixed point theorems.

Theorem 7.1.5. Let $(\mathcal{X}, \boldsymbol{\eta})$ be a complete d-metric space. Let $\mathcal{F}\colon (\mathcal{X}, \boldsymbol{\eta}) \to (\mathcal{X}, \boldsymbol{\eta})$ and let $L \in \mathbb{R}^{d\times d}_{+}$ be a Lipschitz matrix of $\mathcal{F}$ which has a positive left eigenvector. Suppose that at least one of the following assumptions is satisfied:

a) $\rho(L) < 1$,

b) $\rho(L) = 1$, $(\mathcal{X}, \boldsymbol{\eta})$ is compact and $\boldsymbol{\eta}(\mathcal{F}(x), \mathcal{F}(y)) \lneq L\boldsymbol{\eta}(x, y)$ for every $x, y \in \mathcal{X}$ with $x \neq y$.

c) $\rho(L) = 1$, $(X_1, \eta_1), \ldots, (X_d, \eta_d)$ are geodesic spaces and $\mathcal{F}$ has a locally attracting fixed point $x \in \mathcal{X}$.

Then, $\mathcal{F}$ has a unique fixed point $u \in \mathcal{X}$ and

$$\lim_{k\to\infty} \mathcal{F}^k(x) = u \qquad \forall x \in \mathcal{X}.$$

Proof. If $\mathcal{F}$ satisfies a), then the claim follows from Theorem 7.1.1. Now, let $\mathbf{b} \in \mathbb{R}^d_{++}$ be a positive left eigenvector of L. By Lemma 7.1.2, we have that $(\mathcal{X}, \eta_\alpha)$ is a metric space where η_α is the metric on $\mathcal{X}$ defined as $\eta_\alpha(x, y) = \langle \mathbf{b}\,, \boldsymbol{\eta}(x, y)\rangle$. Note that $(\mathcal{X}, \eta_\alpha)$ is complete since $(\mathcal{X}, \boldsymbol{\eta})$ is a complete d-metric space. Furthermore, we have

$$\eta_\alpha(\mathcal{F}(x), \mathcal{F}(y)) \leq \rho(L)\eta_\alpha(x, y) \qquad \forall x, y \in \mathcal{X}$$

and if $\boldsymbol{\eta}(\mathcal{F}(x), \mathcal{F}(y)) \lneq L\boldsymbol{\eta}(x, y)$ for every $x, y \in \mathcal{X}$ with $x \neq y$, then

$$\eta_\alpha(\mathcal{F}(x), \mathcal{F}(y)) < \rho(L)\eta_\alpha(x, y) \qquad \forall x, y \in \mathcal{X} \quad \text{with} \quad x \neq y.$$

If (X_i, η_i) is compact for all i, then $(\mathcal{X}, \eta_\alpha)$ is a compact metric space. It follows that if $\mathcal{F}$ satisfies b), then Theorem 5.2.5, b) implies that $\mathcal{F}$ has a unique fixed point and the iterates converge to it. It is left to prove the result for $\mathcal{F}$ satisfying c). Let $x, y \in \mathcal{X}$, if $\gamma_i\colon [0, 1] \to X_i$ is a geodesic path between $\gamma_i(0) = x_i$ and $\gamma_i(1) = y_i$, then $g\colon [0, 1] \to \mathcal{X}$ defined as $g(t) = (\gamma_1(t), \ldots, \gamma_d(t))$ is a geodesic path between x and y, with respect to η_α. Hence $(\mathcal{X}, \eta_\alpha)$ is a geodesic space and Theorem 5.2.5, c) concludes the proof. □

7.2 Vector valued and real valued Hilbert metrics

Let $\mathcal{V} = V_1 \times \ldots \times V_d$ be a product of finite dimensional real vector spaces and let $\mathcal{C} = C_1 \times \ldots \times C_d \subset \mathcal{V}$ be a cone. The goal of this section is to explore the connection between the vector valued Hilbert metric $\boldsymbol{\mu}_{\mathcal{C}}$ and the real valued Hilbert metric $\mu_{C_1\times\ldots\times C_d}$. More precisely, let $\mathcal{P} \subset \mathcal{C}$ be a part of $\mathcal{C}$. Let $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d_+$ be a monotonic multi-normalization of $\mathcal{C}$ and set $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{P} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$. For a given mapping $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$, we study the relation between

$$\bar{r}(\mathcal{F}, \boldsymbol{\mu}_{\mathcal{C}}) = \inf_{B\in\mathbb{R}^{d\times d}_+} \big\{\rho(B) \,\big|\, \boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}(x), \mathcal{F}(y)) \leq B\boldsymbol{\mu}_{\mathcal{C}}(x, y),\ \forall x, y \in \mathcal{S}_{\boldsymbol{\nu}}\big\}, \tag{7.13}$$

and

$$\begin{aligned} r(\mathcal{F},&\mu_{C_1\times\ldots\times C_d}) \\ &= \inf_{c\in\mathbb{R}_+} \big\{c \,\big|\, \mu_{C_1\times\ldots\times C_d}(\mathcal{F}(x), \mathcal{F}(y)) \leq c\, \mu_{C_1\times\ldots\times C_d}(x, y), \forall x, y \in \mathcal{S}_{\boldsymbol{\nu}}, \big\}, \end{aligned} \tag{7.14}$$

where any of the above quantities is set to ∞ if the infimum is taken over an empty set. We begin with observations on $\mu_{C_1\times\ldots\times C_d}$. Let $M(x/y; C_1 \times \ldots \times C_d)$ and $m(x/y; C_1 \times \ldots \times C_d)$ be defined as in Equation (5.1) with $C = C_1 \times \ldots \times C_d$. Then, for all $x, y \in \mathcal{C}$, it holds

$$\begin{aligned} M(x/y; C_1 \times \ldots \times C_d) &= \max_{i=1,\ldots,d} M(x_i/y_i; C_i) \\ m(x/y; C_1 \times \ldots \times C_d) &= \min_{i=1,\ldots,d} m(x_i/y_i; C_i). \end{aligned}$$

Hence, the Hilbert metric $\mu_{C_1\times\ldots\times C_d}$ can be expressed for $x, y \in \mathcal{C}$, $x \sim_{\mathcal{C}} y$, as

$$\mu_{C_1\times\ldots\times C_d}(x, y) = \max_{i,j} \ln\big(M(x_i/y_i; C_i)M(y_j/x_j; C_j)\big).$$

The next lemma shows that in general $\mu_{C_1\times\ldots\times C_d}$ cannot be expressed in terms of $\boldsymbol{\mu}_{\mathcal{C}}$ when $d \geq 2$.

Lemma 7.2.1. Suppose that $d \geq 2$. Let $\mathcal{C} \subset \mathcal{V}$ be a closed cone and let $\mathcal{P} \subset \mathcal{C}$ be a part of $\mathcal{C}$ such that $P_1, P_2 \neq \{0\}$. Then, for every function $\phi\colon \mathbb{R}^d_+ \to \mathbb{R}_+$, there exists $x, y \in \mathcal{P}$ such that $\phi(\boldsymbol{\mu}_{\mathcal{C}}(x, y)) \neq \mu_{C_1\times\ldots\times C_d}(x, y)$.

Proof. Suppose by contradiction that there exists $\phi\colon \mathbb{R}^d_+ \to \mathbb{R}_+$ such that for all $x, y \in \mathcal{P}$, it holds $\phi(\boldsymbol{\mu}_{\mathcal{C}}(x, y)) = \mu_{C_1\times\ldots\times C_d}(x, y)$. Let $x \in \mathcal{P}$, then $x_1, x_2 \neq 0$ as $P_1, P_2 \neq \{0\}$. Let $\boldsymbol{\alpha}(s) \in \mathbb{R}^d_{++}$ be defined as $\boldsymbol{\alpha}(s) = (s, s^{-1}, 1, \ldots, 1)$ for all $s > 0$. Then, with (6.7), for every $s > 0$, we have

$$0 = \mu_{C_1\times\ldots\times C_d}(x, x) = \phi(\boldsymbol{\mu}_{\mathcal{C}}(x, x)) = \phi(\boldsymbol{\mu}_{\mathcal{C}}(\boldsymbol{\alpha}(s) \otimes x, x)) = \mu_{C_1\times\ldots\times C_d}(\boldsymbol{\alpha}(s) \otimes x, x)$$

There exists $\delta > 0$ large enough such that for all $s > \delta$, it holds $M(\boldsymbol{\alpha}(s) \otimes x/x; \mathcal{C}) = sM(x_1/x_1; C_1) = s$ and $M(x/\boldsymbol{\alpha}(s) \otimes x; \mathcal{C}) = sM(x_2/x_2; C_2) = s$. It follows that

$$0 = \mu_{C_1\times\ldots\times C_d}(\boldsymbol{\alpha}(s) \otimes x, x) = \ln\big(M(x_1/x_1; C_1)M(x_2/x_2; C_2)\big) + \ln(s^2) = 2\ln(s)$$

for every $s > \delta$, a contradiction. □

Next, we show how to obtain an upper bound on $r(\mathcal{F}, \mu_{C_1\times\ldots\times C_d})$ when $\mathcal{F}$ is order-preserving and multi-homogeneous. To this end, let us recall that for a matrix $A \in \mathbb{R}^{d\times d}$, the $\ell^{\infty,\infty}$-norm of A is defined as the maximal absolute row sum of A, i.e.

$$\|A\|_{\infty,\infty} = \max_{i=1,\ldots,d} \sum_{j=1}^{d} |A_{i,j}|.$$

Lemma 7.2.2. Let $\mathcal{C} \subset \mathcal{V}$ be a closed cone, $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d_+$ a monotonic multi-normalization of $\mathcal{C}$, and set $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$. Let $\mathcal{F} \subset \mathcal{C} \to \mathcal{C}$ be order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}_+$. Then, for every $x, y \in \mathcal{S}_{\boldsymbol{\nu}}$ with $x \sim_{\mathcal{C}} y$, it holds

$$\mu_{C_1\times\ldots\times C_d}(\mathcal{F}(x), \mathcal{F}(y)) \leq \|A\|_{\infty,\infty}\, \mu_{C_1\times\ldots\times C_d}(x, y),$$

Proof. Let $r = \|A\|_{\infty,\infty}$. Note that as $A \in \mathbb{R}^d_+$, it holds $r = \max_{i=1,\ldots,d}(A\mathbf{1})_i$. Let $D = \operatorname{diag}(r\mathbf{1} - A\mathbf{1}) \in \mathbb{R}^{d\times d}_+$ and define $\mathcal{G}\colon \mathcal{C} \to \mathcal{C}$ as $\mathcal{G}(x) = \boldsymbol{\nu}^D(x) \otimes \mathcal{F}(x)$ for all $x \in \mathcal{C}$. Then, $\mathcal{G}$ is order-preserving and multi-homogeneous of degree $D + A$. In particular note that $(D + A)\mathbf{1} = r\mathbf{1}$ and thus, by Lemma 3.3.1, $\mathcal{G}$ is homogeneous of degree r. Hence, Proposition 5.2.1 implies that for every $x, y \in \mathcal{C}$ with $x \sim_{\mathcal{C}} y$ it holds

$$\mu_{C_1\times\ldots\times C_d}(\mathcal{G}(x), \mathcal{G}(y)) \leq r\, \mu_{C_1\times\ldots\times C_d}(x, y).$$

To conclude the proof, note that for every $x \in \mathcal{S}_{\boldsymbol{\nu}}$ it holds $\mathcal{G}(x) = \mathcal{F}(x)$. □

Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}_+$. Lemmas 6.2.4 and 7.2.2 imply that

$$\bar{r}(\mathcal{F}, \boldsymbol{\mu}_{\mathcal{C}}) \leq \rho(A) \qquad \text{and} \qquad r(\mathcal{F}, \mu_{C_1\times\ldots\times C_d}) \leq \|A\|_{\infty,\infty} \tag{7.15}$$

where $\bar{r}(\mathcal{F}, \boldsymbol{\mu}_{\mathcal{C}})$ and $r(\mathcal{F}, \mu_{C_1\times\ldots\times C_d})$ are defined in (7.13) and (7.14), respectively. The construction in the following proposition is inspired from [39, Example 3.3]$^\diamond$ and shows that these bounds are sharp:

Proposition 7.2.3. Let $d \geq 2$, $n \geq 2$ and $A \in \mathbb{R}_+^{d\times d}$. Let $C_i = \mathbb{R}^n_+$ for $i = 1, \ldots, d$ so that $\mathcal{C} = \mathbb{R}^n_+ \times \ldots \times \mathbb{R}^n_+$. Let $\pi\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation. Define $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ as

$$\mathcal{F}(x)_{i,j_i} = \prod_{l=1}^{d} x_{l,\pi(j_i)}^{A_{i,l}} \qquad \forall i = 1, \ldots, d, j_i = 1, \ldots, n.$$

Then, for all $k \geq 1$, it holds

$$\bar{r}(\mathcal{F}^k, \boldsymbol{\mu}_{\mathcal{C}}) = \rho(A^k) \qquad \text{and} \qquad r(\mathcal{F}^k, \mu_{C_1\times\ldots\times C_d}) = \|A^k\|_{\infty,\infty}.$$

Proof. First, we prove the case $k = 1$. Lemma 5.2.3 implies that $\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}(x), \mathcal{F}(y)) = A\boldsymbol{\mu}_{\mathcal{C}}(x, y)$ for all $x, y \in \mathcal{C}$ with $x \sim_{\mathcal{C}} y$. Hence,

$$A = \inf\big\{B \in \mathbb{R}_+^{d\times d} \,\big|\, \boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}(x), \mathcal{F}(y)) \leq B\boldsymbol{\mu}_{\mathcal{C}}(x, y),\, \forall x, y \in \mathcal{S}_{\boldsymbol{\nu}}, x \sim_{\mathcal{C}} y\big\}.$$

By monotonicity of the spectral radius, it follows that $\bar{r}(\mathcal{F}, \boldsymbol{\mu}_{\mathcal{C}}) = \rho(A)$. Now, by Lemma 7.2.2, we have $r(\mathcal{F}, \mu_{C_1\times\ldots\times C_d}) \leq \|A\|_{\infty,\infty}$. To prove the reverse inequality, define $v^{(s,t)} = (s, t, 1, \ldots, 1) \in \mathbb{R}^n_{++}$ and $x^{(s,t)} = (v^{(s,t)}, \ldots, v^{(s,t)}) \in \operatorname{int}(\mathcal{C})$ for all $s, t > 0$. Then, for every $s, t, \tilde{s}, \tilde{t} > 0$, it holds

$$\frac{x^{(\tilde{s},\tilde{t})}_{i,j_i}}{x^{(s,t)}_{i,j_i}} = \begin{cases} \tilde{s}/s & \text{if } j_i = 1,\\ \tilde{t}/t & \text{if } j_i = 2,\\ 1 & \text{otherwise,}\end{cases} \qquad \text{and} \qquad \frac{\mathcal{F}(x^{(\tilde{s},\tilde{t})})_{i,j_i}}{\mathcal{F}(x^{(s,t)})_{i,j_i}} = \begin{cases} \big(\tilde{s}/s\big)^{(\sum_{l=1}^d A_{i,l})} & \text{if } \pi(j_i) = 1,\\ \big(\tilde{t}/t\big)^{(\sum_{l=1}^d A_{i,l})} & \text{if } \pi(j_i) = 2,\\ 1 & \text{otherwise.}\end{cases}$$

Therefore

$$\mu_{\mathcal{C}}(\mathcal{F}(x^{(s,1)}), \mathcal{F}(x^{(1,t)})) = \|A\|_{\infty,\infty}\mu_{\mathcal{C}}(x^{(s,1)}, x^{(1,t)}) \qquad \forall s > t > 1,$$

and thus $\|A\|_{\infty,\infty} \leq r(\mathcal{F}^k, \mu_{C_1\times\ldots\times C_d})$, which concludes the proof for the case $k = 1$.

To generalize the result for $k > 1$, we prove that for every $k \geq 1$, $i \in \{1, \ldots, d\}$ and $j_i \in \{1, \ldots, n\}$, it holds

$$\mathcal{F}^k_{i,j_i}(x) = \prod_{l=1}^{d} x_{l,\pi^k(j_i)}^{(A^k)_{i,l}} \qquad \forall x \in \operatorname{int}(\mathcal{C}). \tag{7.16}$$

To this end, let $B \in \mathbb{R}_+^{d\times d}$ and $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ a permutation. Define $\mathcal{G}\colon \mathcal{C} \to \mathcal{C}$ as

$$\mathcal{G}(x)_{i,j_i} = \prod_{l=1}^{d} x_{l,\sigma(j_i)}^{B_{i,l}} \qquad \forall i = 1, \ldots, d, j_i = 1, \ldots, n.$$

Then, for all $x \in \mathcal{C}$, $i \in \{1, \ldots, d\}$ and $j_i \in \{1, \ldots, n\}$, it holds

$$\mathcal{F}(\mathcal{G}(x))_{i,j_i} = \prod_{l=1}^{d} \mathcal{G}(x)_{l,\pi(j_i)}^{A_{i,l}} = \prod_{l=1}^{d}\Big(\prod_{k=1}^{d} x_{k,\sigma(\pi(j_i))}^{B_{l,k}}\Big)^{A_{i,l}} = \prod_{k=1}^{d} x_{k,\sigma(\pi(j_i))}^{(AB)_{i,k}}.$$

Equation (7.16) is now a direct consequence of the above relation by inductively letting $\mathcal{G} = \mathcal{F}, \mathcal{F}^2, \ldots$ Now, (7.16) implies that the iterates $\mathcal{F}^k$ of $\mathcal{F}$ are all of the same form as $\mathcal{F}$ and thus the first case applies to $\mathcal{F}^k$ for all $k > 1$ which concludes the proof. □

We note that the permutation π in the definition of $\mathcal{F}$ in Proposition 7.2.3 can always be chosen so that the gradient of $\mathcal{F}$ is irreducible at every point in $\operatorname{int}(\mathcal{C})$. This assumption will appear later in the discussion when studying existence and uniqueness of positive eigenvectors of non-expansive mappings. In particular, we note that Proposition 7.2.3 allows to construct examples of mappings which are non-expansive with respect to $\boldsymbol{\mu}_{\mathcal{C}}$ and expansive with respect to $\mu_{C_1\times\ldots\times C_d}$. Such a mapping is discussed in the following example.

Example 7.2.4. Let $d = 3$ and $\mathcal{C} = \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+$. Let $\epsilon \in [0, 1)$ and $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ defined as in Proposition 7.2.3 with

$$A = \frac{1-\epsilon}{4}\begin{pmatrix} 0 & 12 & 0 \\ 1 & 0 & 1 \\ 0 & 4 & 0 \end{pmatrix}.$$

Then, for any integer $m \geq 1$ it holds

$$A^{2m-1} = (1-\epsilon)^{2m-1}A \qquad \text{and} \qquad A^{2m} = \frac{(1-\epsilon)^{2m}}{4}\begin{pmatrix} 3 & 0 & 3 \\ 0 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Furthermore, $\rho(A) = 1-\epsilon$, $\mathbf{b} = (1, 4, 1)^\top \in \mathbb{R}^3_{++}$ satisfies $A^\top \mathbf{b} = \rho(A)\mathbf{b}$, and for all $m \geq 1$ it holds $\|A^{2m-1}\|_\infty = 3(1-\epsilon)^{2m-1}$ and $\|A^{2m}\|_\infty = 3(1-\epsilon)^{2m}/2$. In particular, if $\epsilon = 0$, then we have

$$\bar{r}(\mathcal{F}^k, \boldsymbol{\mu}) = 1 < \frac{3}{2} = r(\mathcal{F}^k, \mu_{C_1\times\ldots\times C_d}) \qquad \forall k \geq 1,$$

i.e. $\mathcal{F}^k$ is non-expansive with respect to the vector valued Hilbert metric whereas every power of $\mathcal{F}$ is expansive with respect to the Hilbert on $\mathbb{R}^{3n}_+$.

Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}_+$. Then $\mathcal{F}^k$ is multi-homogeneous of degree A^k for all $k \geq 1$ by Lemma 3.3.5. Hence (7.15) implies that

$$\bar{r}(\mathcal{F}^k, \boldsymbol{\mu}_{\mathcal{C}}) \leq \rho(A^k) \qquad \text{and} \qquad r(\mathcal{F}^k, \mu_{C_1\times\ldots\times C_d}) \leq \|A^k\|_{\infty,\infty}$$

for all $k \geq 1$. Remarkably the above estimates are linked by the Gelfand formula (see for instance [48, Corollary 5.6.14]). Indeed, the Gelfand formula implies that

$$\rho(A) = \lim_{k\to\infty} \|A^k\|_{\infty,\infty}^{1/k} = \inf_{k\geq 1} \|A^k\|_{\infty,\infty}^{1/k},$$

and as $\rho(A^k)^{1/k} = \rho(A)$ for all $k \geq 1$, it follows that

$$\inf_{k\geq 1} \rho(A^k)^{1/k} = \inf_{k\geq 1} \|A^k\|_{\infty,\infty}^{1/k}.$$

This observation suggests that $\mu_{C_1\times\ldots\times C_d}$ and $\boldsymbol{\mu}_{\mathcal{C}}$ can be connected via the following generalization of the Banach fixed point theorem:

Theorem 7.2.5. Let (X, ζ) be a complete metric space and let $f\colon X \to X$ be a continuous mapping. For $k \geq 1$, let f^k be the k-th composition of f with itself and

$$r(f^k, \zeta) = \inf \{ a > 0 \,|\, \zeta(f^k(x), f^k(y)) \leq a\zeta(x, y),\ \forall x, y \in X \}. \tag{7.17}$$

Suppose that $\inf_{k \geq 1} r(f^k, \zeta)^{1/k} < 1$. Then, f has a unique fixed point $u \in X$, and for each $x \in X$ it holds $\lim_{k\to\infty} f^k(x) = u$.

Proof. See Theorem 3.11 in [55]. □

The condition $\inf_{k\geq 1} r(f^k, \zeta)^{1/k} < 1$ only depends on ζ up to equivalence in the following sense:

Lemma 7.2.6. Let X be a set and let $\zeta\colon X \times X \to [0, \infty)$, $\sigma\colon X \times X \to [0, \infty)$ be metrics on X. Suppose that there exists $c > 0$ such that

$$\tfrac{1}{c}\zeta(x, y) \leq \sigma(x, y) \leq c\,\zeta(x, y) \qquad \forall x, y \in X, \tag{7.18}$$

then for every continuous mapping $f\colon X \to X$ it holds

$$\inf_{k\geq 1} r(f^k, \zeta)^{1/k} = \inf_{k\geq 1} r(f^k, \sigma)^{1/k}. \tag{7.19}$$

Lemma 7.2.6 is a well known result (see for instance [55, page 53]). Nevertheless, we reproduce its proof here as it is insightful for the discussion. The main argument relies on the submultiplicative version of Fekete's lemma recalled in the following:

Lemma 7.2.7 (Fekete)**.** Let $(\alpha_k)_{k=1}^{\infty} \subset \mathbb{R}_+$ be a sequence such that $\alpha_{k+m} \leq \alpha_k\alpha_m$ for all $k, m \geq 1$. Then $\lim_{k\to\infty} \alpha_k^{1/k}$ exists and $\lim_{k\to\infty} \alpha_k^{1/k} = \inf_{k\geq 1} \alpha_k^{1/k}$.

Proof. Note that if there exists $k \geq 1$ such that $\alpha_k = 0$, then it holds $\alpha_{m+k} \leq \alpha_k\alpha_m = 0$ for all $m \geq 1$ which implies that $\alpha_m = 0$ for all $m \geq k$ and the result is obvious. Now, suppose that $\alpha_k > 0$ for all $k \geq 1$. For all $k, m \geq 1$, it holds $\ln(\alpha_{k+m}) \leq \ln(\alpha_k) + \ln(\alpha_m)$ and therefore $(\ln(\alpha_k))_{k=1}^{\infty} \subset \mathbb{R}_+$ is a sub-additive sequence. The conclusion now follows from Theorem 7.6.2 in [46]. □

Proof of Lemma 7.2.6. Since for all $m, k \geq 1$ it holds

$$r(f^{k+m}, \zeta) \leq r(f^k, \zeta)r(f^m, \zeta) \qquad \text{and} \qquad r(f^{k+m}, \sigma) \leq r(f^k, \sigma)r(f^m, \sigma),$$

Lemma 7.2.7 implies that

$$\inf_{k\geq 1} r(f^k, \zeta)^{1/k} = \lim_{k\to\infty} r(f^k, \zeta)^{1/k}, \quad \inf_{k\geq 1} r(f^k, \sigma)^{1/k} = \lim_{k\to\infty} r(f^k, \sigma)^{1/k}. \tag{7.20}$$

Furthermore, (7.18) implies that

$$\tfrac{1}{c^2} r(f^k, \zeta) \leq r(f^k, \sigma) \leq c^2 r(f^k, \zeta) \qquad \forall k \geq 1.$$

Hence, by the sandwich theorem, it holds

$$\lim_{k\to\infty} r(f^k, \zeta)^{1/k} = \lim_{k\to\infty} r(f^k, \sigma)^{1/k}.$$

The above equality together with (7.20), implies (7.19). □

We now generalize these arguments in order to obtain a similar conclusion as in Lemma 7.2.6 which relates $r(\mathcal{F},\boldsymbol{\mu}_{\mathcal{C}})$ with $\bar{r}(\mathcal{F},\mu_{C_1\times\ldots\times C_d})$. Let $(\mathcal{X},\boldsymbol{\eta})$ be a d-metric space and let $\mathcal{F}\colon(\mathcal{X},\boldsymbol{\eta})\to(\mathcal{X},\boldsymbol{\eta})$ be continuous. For every $k\geq 1$, set

$$R(\mathcal{F}^k,\boldsymbol{\eta})=\inf\big\{B\in\mathbb{R}^{d\times d}_+\,\big|\,\boldsymbol{\eta}(\mathcal{F}^k(x),\mathcal{F}^k(y))\leq B\boldsymbol{\eta}(x,y),\ \forall x,y\in\mathcal{X}\big\}. \tag{7.21}$$

Note that by the continuity and monotonicity of the spectral radius, if $\mathcal{F}^k$ has a Lipschitz matrix with respect to $\boldsymbol{\eta}$, then we have

$$\rho(R(\mathcal{F}^k,\boldsymbol{\eta}))=\inf_{B\in\mathbb{R}^{d\times d}_+}\big\{\rho(B)\,\big|\,\boldsymbol{\eta}(\mathcal{F}^k(x),\mathcal{F}^k(y))\leq B\boldsymbol{\eta}(x,y),\ \forall x,y\in\mathcal{X}\big\}.$$

If $\mathcal{F}$ does not have a bounded Lipschitz matrix with respect to $\boldsymbol{\eta}$, then we use the convention that $\rho(R(\mathcal{F}^k,\boldsymbol{\eta}))=\infty$ in the following. The study of the quantity

$$\inf_{k\geq 1}\rho(R(\mathcal{F}^k,\boldsymbol{\eta}))^{1/k}$$

is delicate. Indeed, Fekete's lemma can not be used directly since Lipschitz matrices do not commute and although

$$R(\mathcal{F}^{m+k},\boldsymbol{\eta})\preceq_{\mathbb{R}^{d\times d}_+}R(\mathcal{F}^k,\boldsymbol{\eta})R(\mathcal{F}^m,\boldsymbol{\eta})\qquad\forall k,m\geq 1,$$

the relation $\rho(R(\mathcal{F}^k,\boldsymbol{\eta})R(\mathcal{F}^m,\boldsymbol{\eta}))\leq\rho(R(\mathcal{F}^k,\boldsymbol{\eta}))\rho(R(\mathcal{F}^m,\boldsymbol{\eta}))$ is not true in general. Thus the sequence $\big(\rho(R(\mathcal{F}^k,\boldsymbol{\eta}))\big)_{k=1}^{\infty}$ is not necessarily submultiplicative. We have the following generalization of Lemma 7.2.6:

Theorem 7.2.8. Let $(\mathcal{X},\boldsymbol{\eta})$ be a d-metric space and let $\zeta\colon\mathcal{X}\times\mathcal{X}\to[0,\infty)$ be a metric on $\mathcal{X}$. Suppose that there exists a norm $\|\cdot\|$ on $\mathbb{R}^d$ and $c\geq 1$ such that

$$\tfrac{1}{c}\,\|\boldsymbol{\eta}(x,y)\|\leq\zeta(x,y)\leq c\,\|\boldsymbol{\eta}(x,y)\|\qquad\forall x,y\in\mathcal{X}. \tag{7.22}$$

Let $\mathcal{F}\colon\mathcal{X}\to\mathcal{X}$ and for every $k\geq 1$, let $r(\mathcal{F}^k,\zeta)$ and $R(\mathcal{F}^k,\boldsymbol{\eta})$ be respectively defined as in (7.17) and (7.21), then it holds

$$\inf_{k\geq 1}r(\mathcal{F}^k,\zeta)^{1/k}=\inf_{k\geq 1}\rho\big(R(\mathcal{F}^k,\boldsymbol{\eta})\big)^{1/k}.$$

Proof. Define $r_\star=\inf_{k\geq 1}r(\mathcal{F}^k,\zeta)^{1/k}$ and $R_\star=\inf_{k\geq 1}\rho\big(R(\mathcal{F}^k,\boldsymbol{\eta})\big)^{1/k}$. Note that by Lemma 6.1.5, it holds $r_\star<\infty$ if and only if $R_\star<\infty$. So, let us assume that $r_\star,R_\star<\infty$. First, we prove that $R_\star\leq r_\star$. To this end, let $k\geq 1$ be a positive integer such that $R(\mathcal{F}^k,\boldsymbol{\eta})<\infty$ and let $A\in\mathbb{R}^{d\times d}_+$ be a Lipschitz matrix of $\mathcal{F}^k\colon(\mathcal{X},\boldsymbol{\eta})\to(\mathcal{X},\boldsymbol{\eta})$. Let $\epsilon>0$ and set $A_\epsilon=A+\epsilon\mathbf{1}\mathbf{1}^\top$. As $A\leq A_\epsilon$ we have $\rho(A)\leq\rho(A_\epsilon)$ (see Example 2.3.6) and by continuity of the spectral radius we have $\lim_{\epsilon\to 0}\rho(A_\epsilon)=\rho(A)$. Furthermore, as $A_\epsilon\in\mathbb{R}^{d\times d}_{++}$, the Perron-Frobenius theorem implies the existence of $\mathbf{b}_\epsilon\in\mathbb{R}^d_{++}$ such that $A_\epsilon^\top\mathbf{b}_\epsilon=\rho(A_\epsilon)\mathbf{b}_\epsilon$. Lemma 7.1.2 implies that $\rho(A_\epsilon)$ is a Lipschitz constant of $\mathcal{F}^k\colon(\mathcal{X},\eta_\epsilon)\to(\mathcal{X},\eta_\epsilon)$ where η_ϵ is the metric on $\mathcal{X}$ defined as $\eta_\epsilon(x,y)=\langle\mathbf{b}_\epsilon\,,\,\boldsymbol{\eta}(x,y)\rangle$. It follows that $r(\mathcal{F}^k,\eta_\epsilon)\leq\rho(A_\epsilon)$ and thus for $m\geq 1$ it holds $r(\mathcal{F}^{mk},\eta_\epsilon)\leq\rho(A_\epsilon)^m$. Now, note that

$$\inf_{m\geq 1}r(\mathcal{F}^m,\eta_\epsilon)^{1/m}\leq\inf_{m\geq 1}r(\mathcal{F}^{mk},\eta_\epsilon)^{1/(mk)}\leq\rho(A_\epsilon)^{1/k}. \tag{7.23}$$

As all norms on $\mathbb{R}^d$ are equivalent and $\|\mathbf{c}\|_\epsilon = \langle \mathbf{b}_\epsilon\,,\, |\mathbf{c}|\rangle$ is a norm on $\mathbb{R}^d$, there exists $\gamma > 0$ such that $\frac{1}{\gamma}\|\boldsymbol{\eta}(x,y)\| \leq \eta_\epsilon(x,y) \leq \gamma\|\boldsymbol{\eta}(x,y)\|$ for all $x, y \in \mathcal{X}$. Hence, with (7.22), we have $\frac{1}{c\gamma}\eta_\epsilon(x,y) \leq \zeta(x,y) \leq c\gamma\,\eta_\epsilon(x,y)$ for all $x, y \in \mathcal{X}$. It now follows from (7.19) that $\inf_{m\geq 1} r(\mathcal{F}^m, \eta_\epsilon)^{1/m} = r_\star$ which together with (7.23) implies that $r_\star \leq \rho(A_\epsilon)^{1/k}$. By letting $\epsilon \to 0$, we find that $r_\star \leq \rho(A)^{1/k}$. Taking the infimum over all Lipschitz matrices of $\mathcal{F}^k$ implies that $r_\star \leq R(\mathcal{F}^k, \boldsymbol{\eta})^{1/k}$ and taking the infimum over all $k \geq 1$ shows that $r_\star \leq R_\star$. We prove the reverse direction. Let $\epsilon > 0$ and for $i = 1, \ldots, d$, consider the norm $\|\cdot\|_{i,\epsilon}$ on $\mathbb{R}^d$ defined as $\|\mathbf{c}\|_{i,\epsilon} = |c_i| + \epsilon\sum_{j\neq i}|c_j|$. Note that $\|\cdot\|$ and $\|\cdot\|_{i,\epsilon}$ are equivalent for all $i = 1, \ldots, d$, hence there exists $\gamma_\epsilon \geq 1$ such that

$$\frac{1}{\gamma_\epsilon}\|\mathbf{c}\|_{i,\epsilon} \leq \|\mathbf{c}\| \leq \gamma_\epsilon\|\mathbf{c}\|_{i,\epsilon}, \qquad \forall \mathbf{c} \in \mathbb{R}^d, i = 1, \ldots, d.$$

Now, let $k \geq 1$ be a positive integer such that $r(\mathcal{F}^k, \zeta) < \infty$ and $a \geq 0$, a Lipschitz constant of $\mathcal{F}^k\colon (\mathcal{X}, \zeta) \to (\mathcal{X}, \zeta)$. Then, for every $x, y \in \mathcal{X}$ and $i = 1, \ldots, d$, it holds

$$\begin{aligned}
\eta_i(\mathcal{F}^k(x)_i, \mathcal{F}^k(y)_i) &\leq \|\boldsymbol{\eta}(\mathcal{F}^k(x), \mathcal{F}^k(y))\|_{i,\epsilon} \leq \gamma_\epsilon\|\boldsymbol{\eta}(\mathcal{F}^k(x), \mathcal{F}^k(y))\| \\
&\leq (c\,\gamma_\epsilon)\,\zeta(\mathcal{F}^k(x), \mathcal{F}^k(y)) \leq (c\,\gamma_\epsilon)\,a\,\zeta(x,y) \\
&\leq (c\,\gamma_\epsilon)^2 a\,\|\boldsymbol{\eta}(x,y)\|_{i,\epsilon} = (c\,\gamma_\epsilon)^2\,a\,\Big(\eta_i(x_i, y_i) + \epsilon\sum_{j\neq i}\eta_j(x_j, y_j)\Big).
\end{aligned}$$

Now, let $A_\epsilon \in \mathbb{R}^{d\times d}_{++}$ be defined as $A_\epsilon = a(I + \epsilon(\mathbf{1}\mathbf{1}^\top - I))$. By the above chain of inequalities, for every $x, y \in \mathcal{X}$ we have

$$\boldsymbol{\eta}(\mathcal{F}^k(x), \mathcal{F}^k(y)) \leq (c\gamma_\epsilon)^2 A_\epsilon \boldsymbol{\eta}(x,y).$$

Furthermore, it holds $\rho(A_\epsilon) = a(1+\epsilon(d-1))$, so that $R(\mathcal{F}^k, \boldsymbol{\eta}) \leq a(c\gamma_\epsilon)^2(1+\epsilon(d-1))$. Now, by taking the infimum over all Lipschitz constants a of $\mathcal{F}^k\colon (\mathcal{X}, \zeta) \to (\mathcal{X}, \zeta)$, we deduced that

$$R(\mathcal{F}^k, \boldsymbol{\eta}) \leq (c\gamma_\epsilon)^2(1 + \epsilon(d-1))\, r(\mathcal{F}^k, \zeta).$$

Note that the constant $\alpha_\epsilon = (c\gamma_\epsilon)^2(1 + \epsilon(d-1))$ does not depend on k and thus $R(\mathcal{F}^k, \boldsymbol{\eta})^{1/k} \leq \alpha_\epsilon^{1/k}\, r(\mathcal{F}^k, \zeta)^{1/k}$ for all $k \geq 1$. It follows that, with $\beta_k = \alpha_\epsilon\, r(\mathcal{F}^k, \zeta)$ for all k, we have $R_\star \leq \inf_{k\geq 1}\beta_k$. The sequence $(\beta_k)_{k=1}^\infty \subset [0, \infty)$ is sub-multiplicative. Indeed, for all $k, m \geq 1$, it holds $\beta_{k+m} \leq \alpha_\epsilon\, r(\mathcal{F}^k, \zeta) r(\mathcal{F}^k, \zeta) \leq \beta_k\beta_m$ since $\alpha_\epsilon \geq 1$. Finally, with Lemma 7.2.7, we deduce that

$$R_\star \leq \inf_{k\geq 1}\beta_k^{1/k} = \lim_{k\to\infty}\beta_k^{1/k} = \lim_{k\to\infty} r(\mathcal{F}^k, \zeta)^{1/k} = \inf_{k\geq 1} r(\mathcal{F}^k, \zeta)^{1/k} = r_\star,$$

which concludes the proof. □

As a direct consequence of Theorems 7.2.5 and 7.2.8 we get:

Theorem 7.2.9. Suppose that $(\mathcal{X}, \boldsymbol{\eta})$ is a complete d-metric space and let $\mathcal{F}\colon \mathcal{X} \to \mathcal{X}$ be Lipschitz continuous. For $k \geq 1$, let

$$\bar{r}(\mathcal{F}^k, \boldsymbol{\eta}) = \inf\big\{\rho(B) \,\big|\, B \in \mathbb{R}^{d\times d}_+ \text{ and } \boldsymbol{\eta}(\mathcal{F}^k(x), \mathcal{F}^k(y)) \leq B\boldsymbol{\eta}(x,y),\, \forall x, y \in \mathcal{X}\big\}.$$

If $\inf_{k\geq 1} \bar{r}(\mathcal{F}^k, \boldsymbol{\eta})^{1/k} < 1$, then $\mathcal{F}$ has a unique fixed point $u \in \mathcal{X}$ and

$$\lim_{k\to\infty} \mathcal{F}^k(x) = u \qquad \forall x \in \mathcal{X}.$$

Proof. For $k \geq 1$, let $L_k \in \mathbb{R}^{d\times d}_+$ be defined as

$$L_k = \inf \big\{B \in \mathbb{R}^{d\times d}_+ \,\big|\, \boldsymbol{\eta}(\mathcal{F}^k(x), \mathcal{F}^k(y)) \leq B\boldsymbol{\eta}(x, y),\, \forall x, y \in \mathcal{X}\big\}.$$

Such an L_k exists as the set of Lipschitz matrices of $\mathcal{F}$ is lower bounded by 0 and not empty since $\mathcal{F}$ is Lipschitz continuous. By continuity and monotonicity of the spectral radius with respect to $\mathbb{R}^{d\times d}_+$ (see Example 2.3.6), we have $\rho(L_k) = \bar{r}(\mathcal{F}^k, \boldsymbol{\eta})$. Let $\|\cdot\|$ be a norm $\mathbb{R}^d$ which is monotonic with respect to $\mathbb{R}^d_+$. Then $\zeta(x, y) = \|\boldsymbol{\eta}(x, y)\|$ is a metric on $\mathcal{X}$ and $(\mathcal{X}, \zeta)$ is a complete metric space since $(\mathcal{X}, \boldsymbol{\eta})$ is complete. Theorem 7.2.8 implies that $\inf_{k\geq 1} r(\mathcal{F}^k, \zeta)^{1/k} < 1$ where $r(\mathcal{F}^k, \zeta)$ is defined as in (7.17). The claim now follows from Theorem 7.2.5. □

Finally, we need the following lemma in order to use Theorem 7.2.8 to compare the real valued and vector valued Hilbert metrics. The main idea of the proof is similar to that of [94, Theorem 4].

Lemma 7.2.10. Let $\mathcal{C} \subset \mathcal{V}$ be a closed cone and let $\mathcal{P} \subset \mathcal{C}$ be a part of $\mathcal{C}$ such that $P_i \neq \{0\}$ for all $i = 1, \ldots, d$. Let $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d_+$ be a monotonic multi-normalization of $\mathcal{C}$ and $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{P} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$. For every norm $\|\cdot\|$ on $\mathbb{R}^d$, there exists $r \geq 1$ such that

$$\tfrac{1}{r}\|\boldsymbol{\mu}_{\mathcal{C}}(x, y)\| \leq \mu_{C_1\times\ldots\times C_d}(x, y) \leq r\,\|\boldsymbol{\mu}_{\mathcal{C}}(x, y)\| \qquad \forall x, y \in \mathcal{S}_{\nu}. \tag{7.24}$$

Proof. Let $\|\cdot\|_1$ and $\|\cdot\|_\infty$ be respectively the ℓ^1 and ℓ^∞ norm on $\mathbb{R}^d$. As all the norms on $\mathbb{R}^d$ are equivalent, is suffices to prove that

$$\|\boldsymbol{\mu}_{\mathcal{C}}(x, y)\|_\infty \leq \mu_{C_1\times\ldots\times C_d}(x, y) \leq \|\boldsymbol{\mu}_{\mathcal{C}}(x, y)\|_1 \qquad \forall x, y \in \mathcal{S}_{\nu}.$$

Let $x, y \in \mathcal{S}_\nu$ and for $i = 1, \ldots, d$ let $\alpha_i = M(x_i/y_i; C_i)$ and $\beta_i = M(y_i/x_i; C_i)$. As $\boldsymbol{\nu}$ is monotonic and $x, y \in \mathcal{S}_{\boldsymbol{\nu}}$, Lemma 5.1.3 implies that $\alpha_i, \beta_i \geq 1$ for all $i = 1, \ldots, d$. Furthermore, note that

$$\mu_{C_1\times\ldots\times C_d}(x, y) = \max_{i=1,\ldots,d} \ln(\alpha_i) + \max_{j=1,\ldots,d} \ln(\beta_j),$$

and $\boldsymbol{\mu}_{\mathcal{C}}(x, y) = \big(\ln(\alpha_1) + \ln(\beta_1), \ldots, \ln(\alpha_d) + \ln(\beta_d)\big)$ so that

$$\|\boldsymbol{\mu}_{\mathcal{C}}(x, y)\|_\infty = \max_{i=1,\ldots,d} \ln(\alpha_i) + \ln(\beta_i) \qquad \text{and} \qquad \|\boldsymbol{\mu}_{\mathcal{C}}(x, y)\|_1 = \sum_{i=1}^d \big(\ln(\alpha_i) + \ln(\beta_j)\big).$$

Finally, as $\ln(\alpha_i), \ln(\beta_i) \geq 0$ for all i, we have

$$\max_{i=1,\ldots,d} \ln(\alpha_i) + \ln(\beta_i) \leq \max_{i=1,\ldots,d} \ln(\alpha_i) + \max_{j=1,\ldots,d} \ln(\beta_j) \leq \sum_{i=1}^d \ln(\alpha_i) + \sum_{j=1}^d \ln(\beta_j),$$

which concludes the proof. □

Lemma 7.2.10 together with Theorem 7.2.8 directly imply the following corollary.

Corollary 7.2.11. Let $\mathcal{C} \subset \mathcal{V}$ be a closed cone and let $\mathcal{P} \subset \mathcal{C}$ be a part of $\mathcal{C}$ such that $P_i \neq \{0\}$ for all $i = 1, \ldots, d$. Let $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d_+$ be a monotonic multi-normalization of $\mathcal{C}$ and $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{P} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$, then it holds

$$\inf_{k\geq 1} \bar{r}(\mathcal{F}^k, \boldsymbol{\mu}_{\mathcal{C}})^{1/k} = \inf_{k\geq 1} r(\mathcal{F}^k, \mu_{C_1\times\ldots\times C_d})^{1/k},$$

where $\bar{r}(\mathcal{F}^k, \boldsymbol{\mu}_{\mathcal{C}})$ and $r(\mathcal{F}^k, \mu_{C_1\times\ldots\times C_d})$ are defined as in (7.13) and (7.14), respectively.

It follows that for $\mathcal{S}_{\boldsymbol{\nu}}$ as in Corollary 7.2.11, $(\mathcal{X}, \zeta) = (\mathcal{S}_{\boldsymbol{\nu}}, \mu_{C_1\times\ldots\times C_d})$, $(\mathcal{X}, \boldsymbol{\eta}) = (\mathcal{S}_{\boldsymbol{\nu}}, \boldsymbol{\mu}_{\mathcal{C}})$ and $\mathcal{G}\colon \mathcal{S}_{\boldsymbol{\nu}} \to \mathcal{S}_{\boldsymbol{\nu}}$, the assumptions of Theorem 7.2.5 hold if, and only if, the assumptions of Theorem 7.2.9 hold. Nevertheless, for every $N \geq 1$, there exists order-preserving multi-homogeneous mapping $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ such that

$$r(\mathcal{F}^k, \boldsymbol{\mu}_{\mathcal{C}})^{1/k} < 1 < r(\mathcal{F}^k, \mu_{C_1\times\ldots\times C_d})^{1/k} \qquad \forall k = 1, \ldots, N.$$

Such mappings can be constructed by letting $\epsilon > 0$ small enough in the definition of the mapping $\mathcal{F}$ discussed in Example 7.2.4.

7.3 Fixed point theorems on the product of cones

We prove fixed point theorems for mappings leaving invariant a cone in the product of finite dimensional real vector spaces $\mathcal{V} = V_1 \times \ldots \times V_d$. To this end, we use the properties of the vector valued Hilbert metric together with the vector valued fixed point theorems proved in Section 7.

Our first result shows that mappings which contracts under the vectors valued Hilbert metric have a unique eigenvector.

Theorem 7.3.1. Let $\mathcal{C} \subset \mathcal{V}$ be a closed cone and let $\mathcal{P}$ be a part of $\mathcal{C}$ such that $P_i \neq \{0\}$ for $i = 1, \ldots, d$. Let $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d_+$ be a multi-normalization of $\mathcal{C}$ and set $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$. Furthermore, let $\mathcal{F}\colon \mathcal{P} \cap \mathcal{S}_{\boldsymbol{\nu}} \to \mathcal{P}$ and define $\mathcal{G}\colon \mathcal{P} \cap \mathcal{S}_{\boldsymbol{\nu}} \to \mathcal{P} \cap \mathcal{S}_{\boldsymbol{\nu}}$ as

$$\mathcal{G}(x) = \boldsymbol{\nu}(\mathcal{F}(x))^{-I} \otimes \mathcal{F}(x) \qquad \forall x \in \mathcal{U} \cap \mathcal{S}_{\boldsymbol{\nu}}.$$

If there exists $L \in \mathbb{R}^{d\times d}_+$ and $m \geq 1$ such that $\rho(L) < 1$ and

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}^m(x), \mathcal{F}^m(y)) \leq L\boldsymbol{\mu}_{\mathcal{C}}(x, y) \qquad \forall x, y \in \mathcal{U} \cap \mathcal{S}_{\boldsymbol{\nu}}. \tag{7.25}$$

Then, the equation $\mathcal{F}(x) = \boldsymbol{\lambda} \otimes x$ with $(\boldsymbol{\lambda}, x) \in \mathbb{R}^d_+ \times (\mathcal{P} \cap \mathcal{S}_{\boldsymbol{\nu}})$ has a unique solution $(\boldsymbol{\theta}, u)$. Furthermore, $\boldsymbol{\theta} \in \mathbb{R}^d_{++}$ and for every $x \in \mathcal{P} \cap \mathcal{S}_{\boldsymbol{\nu}}$, it holds

$$\lim_{j\to\infty} \mathcal{G}^j(x) = u \quad \text{and} \quad \boldsymbol{\mu}_{\mathcal{C}}(\mathcal{G}^{km}(x), u) \leq L^k(I - L)^{-1}\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{G}^m(x), u) \tag{7.26}$$

for every $k \geq 1$.

For the proof we need the following lemma which relates the eigenvectors of $\mathcal{F}$ with the fixed points of $\mathcal{G}$.

Lemma 7.3.2. Let $\mathcal{C} \subset \mathcal{V}$ be a cone, $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}_+$ a multi-normalization of $\mathcal{C}$ and set $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$. Let $\mathcal{U} \subset \mathcal{C}$ be such that $\mathcal{U} \cap \mathcal{S}_{\boldsymbol{\nu}} \neq \emptyset$. Furthermore, let $\mathcal{F}\colon \mathcal{U} \cap \mathcal{S}_{\boldsymbol{\nu}} \to \mathcal{C}$ and suppose that $\mathcal{F}(x)_i \neq 0$ for all $i = 1, \ldots, d$ and $x \in \mathcal{U}$. Define $\mathcal{G}\colon \mathcal{U} \cap \mathcal{S}_{\boldsymbol{\nu}} \to \mathcal{C} \cap \mathcal{S}_{\boldsymbol{\nu}}$ as $\mathcal{G}(x) = \boldsymbol{\nu}(\mathcal{F}(x))^{-I} \otimes \mathcal{F}(x)$ for every $x \in \mathcal{U} \cap \mathcal{S}_{\boldsymbol{\nu}}$. Then, the following hold:

i) For every $x, y \in \mathcal{U}$, it holds $\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{G}(x), \mathcal{G}(y)) = \boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}(x), \mathcal{F}(y))$.

ii) If $u \in \mathcal{U} \cap \mathcal{S}_{\boldsymbol{\nu}}$ is such that $\mathcal{F}(u) = \boldsymbol{\lambda} \otimes u$ with $\boldsymbol{\lambda} \in \mathbb{R}^d_+$, then $\boldsymbol{\lambda} \in \mathbb{R}^d_{++}$ and $\boldsymbol{\nu}(u)^{-I} \otimes u \in \mathcal{U} \cap \mathcal{S}_{\boldsymbol{\nu}}$ is a fixed point of $\mathcal{G}$.

iii) If $u \in \mathcal{U} \cap \mathcal{S}_{\boldsymbol{\nu}}$ is a fixed point of $\mathcal{G}$, then it holds $\mathcal{F}(u) = \boldsymbol{\lambda} \otimes u$ with $\boldsymbol{\lambda} = \boldsymbol{\nu}(\mathcal{F}(u))$.

iv) There is a unique solution to the equation $\mathcal{F}(x) = \boldsymbol{\lambda} \otimes x$ with $(\boldsymbol{\lambda}, x) \in \mathbb{R}^d_+ \times (\mathcal{U} \cap \mathcal{S}_{\boldsymbol{\nu}})$ if, and only if, $\mathcal{G}$ has a unique fixed point.

Proof. First of all note that the condition $\mathcal{F}(x)_i \neq 0$ for all $i = 1, \ldots, d$ and $x \in \mathcal{U}$ implies that $\boldsymbol{\nu}(\mathcal{F}(x)) \in \mathbb{R}^d_{++}$ for all $x \in \mathcal{U}$ and thus $\mathcal{G}$ is well defined. Furthermore, note that if $x \in \mathcal{U} \cap \mathcal{S}_{\boldsymbol{\nu}}$ then $x_i \neq 0$ for all $i = 1, \ldots, d$ since $\boldsymbol{\nu}(x)_i > 0$ for all $i = 1, \ldots, d$.

i) Let $x, y \in \mathcal{U}$ and $i \in \{1, \ldots, d\}$. Note that if $\mathcal{F}(x)_i \sim_{C_i} \mathcal{F}(y)_i$ then, as $\boldsymbol{\nu}(\mathcal{F}(x))_i > 0$ and $\boldsymbol{\nu}(\mathcal{F}(y))_i > 0$, it holds $\mathcal{G}(x)_i \sim_{C_i} \mathcal{G}(y)_i$. It follows from (5.2) that

$$\mu_{C_i}(\mathcal{F}(x)_i, \mathcal{F}(y)_i) = \mu_{C_i}\Big(\frac{\mathcal{F}(x)_i}{\boldsymbol{\nu}(\mathcal{F}(x))_i}, \frac{\mathcal{F}(y)_i}{\boldsymbol{\nu}(\mathcal{F}(y))_i}\Big) = \mu_{C_i}(\mathcal{G}(x)_i, \mathcal{G}(y)_i).$$

If $\mathcal{F}(x)_i \not\sim_{C_i} \mathcal{F}(y)_i$, then $\mathcal{G}(x)_i \not\sim_{C_i} \mathcal{G}(y)_i$ and thus

$$\mu_{C_i}(\mathcal{F}(x)_i, \mathcal{F}(y)_i) = \infty = \mu_{C_i}(\mathcal{G}(x)_i, \mathcal{G}(y)_i).$$

In both cases we have $\mu_{C_i}(\mathcal{F}(x)_i, \mathcal{F}(y)_i) = \mu_{C_i}(\mathcal{G}(x)_i, \mathcal{G}(y)_i)$. As the latter is true for all $i \in \{1, \ldots, d\}$, it follows that $\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}(x), \mathcal{F}(y)) = \boldsymbol{\mu}_{\mathcal{C}}(\mathcal{G}(x), \mathcal{G}(y))$.

ii) Suppose that $u \in \mathcal{U} \cap \mathcal{S}_{\boldsymbol{\nu}}$ is such that $\mathcal{F}(u) = \boldsymbol{\lambda} \otimes u$ with $\boldsymbol{\lambda} \in \mathbb{R}^d_+$. Then, we have $\boldsymbol{\nu}(\mathcal{F}(u)) = \boldsymbol{\nu}(\boldsymbol{\lambda} \otimes u) = \boldsymbol{\lambda} \circ \boldsymbol{\nu}(u) = \boldsymbol{\lambda}$ and thus $\boldsymbol{\lambda} = \boldsymbol{\nu}(\mathcal{F}(u)) \in \mathbb{R}^d_{++}$. Furthermore, it holds

$$\mathcal{G}(u) = \boldsymbol{\nu}(\mathcal{F}(u))^{-I} \otimes \mathcal{F}(u) = \boldsymbol{\lambda}^{-I} \otimes (\boldsymbol{\lambda} \otimes u) = u,$$

i.e. u is a fixed point of $\mathcal{G}$.

iii) If $u \in \mathcal{U} \cap \mathcal{S}_{\boldsymbol{\nu}}$ is a fixed point of $\mathcal{G}$, then it holds $u = \mathcal{G}(u) = \boldsymbol{\nu}(\mathcal{F}(u))^{-I} \otimes \mathcal{F}(u)$ and thus $\mathcal{F}(u) = \boldsymbol{\nu}(\mathcal{F}(u)) \otimes u$.

iv) If $y, z \in \mathcal{U} \cap \mathcal{S}_{\boldsymbol{\nu}}$ are such that there exists $\boldsymbol{\theta}, \boldsymbol{\vartheta} \in \mathbb{R}^d_+$ with $\mathcal{F}(y) = \boldsymbol{\theta} \otimes y$ and $\mathcal{F}(z) = \boldsymbol{\vartheta} \otimes z$, then ii) implies that y and z are both fixed points of $\mathcal{G}$. Hence, if $\mathcal{G}$ has a unique fixed point, then $y = z$. Furthermore, it holds

$$\boldsymbol{\theta} = \boldsymbol{\theta} \circ \boldsymbol{\nu}(y) = \boldsymbol{\nu}(\boldsymbol{\theta} \otimes y) = \boldsymbol{\nu}(\mathcal{F}(y)) = \boldsymbol{\nu}(\mathcal{F}(z)) = \boldsymbol{\nu}(\boldsymbol{\vartheta} \otimes z) = \boldsymbol{\vartheta} \circ \boldsymbol{\nu}(z) = \boldsymbol{\vartheta}.$$

It follows that $(\boldsymbol{\theta}, y) = (\boldsymbol{\vartheta}, z)$ and thus the equation $\mathcal{F}(x) = \boldsymbol{\lambda} \otimes x$ with $(\boldsymbol{\lambda}, x) \in \mathbb{R}^d_+ \times (\mathcal{U} \cap \mathcal{S}_{\boldsymbol{\nu}})$ has a unique solution. Conversely, suppose that $y, z \in \mathcal{U} \cap \mathcal{S}_{\boldsymbol{\nu}}$ are both fixed points of $\mathcal{G}$, then iii) implies that $\mathcal{F}(y) = \boldsymbol{\nu}(\mathcal{F}(y)) \otimes y$ and $\mathcal{F}(z) = \boldsymbol{\nu}(\mathcal{F}(z)) \otimes z$. It follows that $(\boldsymbol{\nu}(\mathcal{F}(y)), y)$ and $(\boldsymbol{\nu}(\mathcal{F}(z)), z)$ are both solutions to the equation $\mathcal{F}(x) = \boldsymbol{\lambda} \otimes x$ with $(\boldsymbol{\lambda}, x) \in \mathbb{R}^d_+ \times (\mathcal{U} \cap \mathcal{S}_{\boldsymbol{\nu}})$. In particular, if the latter equation has a unique solution, then $(\boldsymbol{\nu}(\mathcal{F}(y)), y) = (\boldsymbol{\nu}(\mathcal{F}(z)), z)$ and thus $y = z$ which implies that $\mathcal{G}$ has a unique fixed point. □

Proof of Theorem 7.3.1. First note that as $P_i \neq \{0\}$ for all i and $\mathcal{F}(x) \in \mathcal{P}$ for all $x \in \mathcal{S}_{\boldsymbol{\nu}} \cap \mathcal{P}$, we have $\mathcal{F}(x)_i \neq 0$ for all $i = 1, \ldots, d$ and thus $\mathcal{G}$ is well defined. Lemma 6.2.2 implies that $(\mathcal{P} \cap \mathcal{S}_{\boldsymbol{\nu}}, \boldsymbol{\mu}_{\mathcal{C}})$ is a complete d-metric space. Now, for $k \geq 1$, let

$$\bar{r}(\mathcal{G}^k, \boldsymbol{\mu}_{\mathcal{C}}) = \inf\big\{\rho(B) \,\big|\, B \in \mathbb{R}^{d\times d}_+ \text{ and } \boldsymbol{\mu}_{\mathcal{C}}(\mathcal{G}^k(x), \mathcal{G}^k(y)) \leq B\boldsymbol{\mu}_{\mathcal{C}}(x, y),\ \forall x, y \in \mathcal{P} \cap \mathcal{S}_{\boldsymbol{\nu}}\big\}.$$

By Lemma 7.3.2 and (7.25), we have

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{G}^m(x), \mathcal{G}^m(y)) = \boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}^m(x), \mathcal{F}^m(y)) \leq L\boldsymbol{\mu}_{\mathcal{C}}(x, y) \quad \forall x, y \in \mathcal{P} \cap \mathcal{S}_{\boldsymbol{\nu}}. \tag{7.27}$$

It follows that $\bar{r}(\mathcal{G}^m, \boldsymbol{\mu}_{\mathcal{C}}) \leq \rho(L) < 1$, and thus $\inf_{k\geq 1} \bar{r}(\mathcal{G}^k, \boldsymbol{\mu}_{\mathcal{C}})^{1/k} < 1$. Hence, by Theorem 7.2.9, we know that $\mathcal{G}$ has a unique fixed point $u \in \mathcal{P} \cap \mathcal{S}_{\boldsymbol{\nu}}$ and $\lim_{k\to\infty} \mathcal{G}^k(x) = u$ for all $x \in \mathcal{P} \cap \mathcal{S}_{\boldsymbol{\nu}}$. Lemma 7.3.2 implies that the equation $\mathcal{F}(x) = \boldsymbol{\lambda} \otimes x$ with $(\boldsymbol{\lambda}, x) \in \mathbb{R}^d_+ \times (\mathcal{P} \cap \mathcal{S}_{\boldsymbol{\nu}})$ has a unique solution $(\boldsymbol{\theta}, u)$ and $\boldsymbol{\theta} \in \mathbb{R}^d_{++}$. To conclude the proof, note that u is a fixed point of $\mathcal{G}^m$ as well and thus (7.26) follows from (7.27) and Theorem 7.1.1. □

The following result allows to deduce from (7.26), a convergence rate in terms of norms on $V_1, \ldots, V_d$.

Proposition 7.3.3. Let $\mathcal{C} \subset \mathcal{V}$ be a closed cone and let $\mathcal{P}$ be a part of $\mathcal{C}$. For $i = 1, \ldots, d$, let $\|\cdot\|_i$ be a norm on V_i with normality constant 1 with respect to C_i. For $z \in \mathcal{V}$ define $\|z\| = (\|z_1\|_1, \ldots, \|z_d\|_d)^\top \in \mathbb{R}^d_+$ and let $\mathcal{S} = \{x \in \mathcal{C} \,|\, \|x\| = \mathbf{1}\}$. Finally, let $A \in \mathbb{R}^{d\times d}_+$ with $\rho(A) < 1$, $u \in \mathcal{P} \cap \mathcal{S}_{\boldsymbol{\nu}}$, and let $(x_k)_{k=0}^\infty \subset \mathcal{P} \cap \mathcal{S}_{\boldsymbol{\nu}}$ be a sequence such that

$$\boldsymbol{\mu}_{\mathcal{C}}(x_k, u) \leq A^k(I - A)^{-1}\boldsymbol{\mu}_{\mathcal{C}}(x_1, x_0) \qquad \forall k \geq 1. \tag{7.28}$$

Then, with $\mathbf{c} = (I - A)^{-1}\boldsymbol{\mu}_{\mathcal{C}}(x_1, x_0)$, it holds

$$\|x_k - u\| \leq 3\exp(A^k\mathbf{c}) \circ A^k\mathbf{c} \qquad \forall k \geq 1,$$

where the exponential is applied component wise. Furthermore, if A has a positive eigenvector $\mathbf{a} \in \mathbb{R}^d_{++}$ and $\omega = \max_{i=1,\ldots,d} \frac{\boldsymbol{\mu}_{\mathcal{C}}(x_1, x_0)_i}{(1-\rho(A))a_i}$, then

$$\|x_k - u\| \leq 3\rho(A)^k\omega(\exp(\rho(A)^k\omega\mathbf{a}) \circ \mathbf{a}) \qquad \forall k \geq 1.$$

Proof. For $x, y \in \mathcal{P} \cap \mathcal{S}_{\boldsymbol{\nu}}$, as the norms $\|\cdot\|_i$ are monotonic, by Lemma 5.1.9, we have $\|x - y\| \leq 3(\exp(\boldsymbol{\mu}_{\mathcal{C}}(x, y)) - \mathbf{1})$. Now, the inequality $e^t - 1 \leq te^t$ for $t \geq 0$ implies $\|x - y\| \leq \exp(\boldsymbol{\mu}_{\mathcal{C}}(x, y)) \circ \boldsymbol{\mu}_{\mathcal{C}}(x, y)$. Letting $x = x_k, y = u$, with (7.28), we get

$$\|x_k - u\| \leq 3\exp(\boldsymbol{\mu}_{\mathcal{C}}(x_k, u)) \circ \boldsymbol{\mu}_{\mathcal{C}}(x_k, u) \leq 3\exp(A^k\mathbf{c}) \circ A^k\mathbf{c}.$$

This proves the first convergence rate. Now, suppose that A has a positive eigenvector $\mathbf{a} \in \mathbb{R}^d_{++}$ and let $\xi = \max_{i=1,\ldots,d} \boldsymbol{\mu}_{\mathcal{C}}(x_1, x_0)_i / a_i$. Then, $A\mathbf{a} = \rho(A)\mathbf{a}$ and for every $k \geq 0$ it holds $A^k\mathbf{c} \leq A^k(I-A)^{-1}\xi\mathbf{a}$. Furthermore, for all $k \geq 0$ we have

$$A^k(I-A)^{-1}\mathbf{a} = A^k \sum_{j=0}^{\infty} A^j \mathbf{a} = \frac{1}{1-\rho(A)} A^k \mathbf{a} = \frac{\rho(A)^k}{1-\rho(A)}\mathbf{a}.$$

It follows that $A^k\mathbf{c} \leq \rho(A)^k \omega \mathbf{a}$ and thus

$$\|x_k - u\| \leq 3\exp(A^k\mathbf{c}) \circ A^k\mathbf{c} \leq 3\rho(A)^k\omega\exp(\rho(A)^k\omega\mathbf{a}) \circ \mathbf{a},$$

which concludes the proof. □

Theorem 7.3.4. Let $\mathcal{C} \subset \mathcal{V}$ be a solid closed cone. Let $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d_+$ be a multi-normalization of $\mathcal{C}$ and set $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \boldsymbol{\nu}(x) = 1\}$. Furthermore, let $\mathcal{F}\colon \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}} \to \operatorname{int}(\mathcal{C})$ and define $\mathcal{G}\colon \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}} \to \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ as $\mathcal{G}(x) = \boldsymbol{\nu}(\mathcal{F}(x))^{-I} \otimes \mathcal{F}(x)$ for every $x \in \mathcal{U} \cap \mathcal{S}_{\boldsymbol{\nu}}$. Suppose that there exists $u \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ and $\boldsymbol{\theta} \in \mathbb{R}^d_+$ such that $\mathcal{F}(u) = \boldsymbol{\theta} \otimes u$. Suppose that there exists $L \in \mathbb{R}^{d\times d}_+$ satisfying the following conditions:

i) L is irreducible and $\rho(L) = 1$.

ii) It holds

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}(x), \mathcal{F}(y)) \leq L\boldsymbol{\mu}_{\mathcal{C}}(x, y) \qquad \forall x, y \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}.$$

iii) There exists an open neighborhood $\mathcal{U} \subset \operatorname{int}(\mathcal{C})$ such that $u \in \mathcal{U}$ and

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}(x), \mathcal{F}(y)) \lneq L\boldsymbol{\mu}_{\mathcal{C}}(x, y) \qquad \forall x, y \in \mathcal{U} \cap \mathcal{S}_{\boldsymbol{\nu}} \text{ with } x \neq y.$$

Then, $(\boldsymbol{\theta}, u)$ is the unique solution to the equation $\mathcal{F}(x) = \boldsymbol{\lambda} \otimes x$ with $(\boldsymbol{\lambda}, x) \in \mathbb{R}^d_+ \times (\operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}})$. Moreover, $\boldsymbol{\theta} \in \mathbb{R}^d_{++}$ and for every $x \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$, it holds $\lim_{j\to\infty} \mathcal{G}^j(x) = u$.

Proof. Note that as $\mathcal{F}(u) \in \operatorname{int}(\mathcal{C})$ and $u \in \operatorname{int}(\mathcal{C})$, it holds $u_i \neq 0$ and $\mathcal{F}(u)_i \neq 0$ for $i = 1, \ldots, d$. Hence, $\mathcal{F}(u) = \boldsymbol{\theta} \otimes u$ implies $\boldsymbol{\theta} \in \mathbb{R}^d_{++}$. As $\rho(L) = 1$ and L is irreducible, there exists $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d_{++}$ such that $L\mathbf{a} = \mathbf{a}$ and $L^\top\mathbf{b} = \mathbf{b}$. Lemma 6.2.2 implies that $(\operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}, \boldsymbol{\mu}_{\mathcal{C}})$ is a complete d-metric space. Now, let $\mathcal{B}(u) = \{x \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}} \,|\, \boldsymbol{\mu}_{\mathcal{C}}(x, u) \leq \epsilon\mathbf{a}\}$ where $\epsilon > 0$ is small enough so that $\mathcal{B}(u) \subset \mathcal{U}$. Lemma 6.2.2 implies that the topology induced by $\boldsymbol{\mu}_{\mathcal{C}}$ coincides with the product norm topology on $\mathcal{V} = V_1 \times \ldots \times V_d$ and thus $\mathcal{B}(u)$ is a compact subset of $\operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$. Furthermore, by Lemma 7.3.2, we have that u is a fixed point of $\mathcal{G}$ and for all $x \in \mathcal{B}(u)$, it holds

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{G}(x), u) = \boldsymbol{\mu}_{\mathcal{C}}(\mathcal{G}(x), \mathcal{G}(u)) = \boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}(x), \mathcal{F}(u)) \leq L\boldsymbol{\mu}_{\mathcal{C}}(x, u) \leq L\epsilon\mathbf{a} = \epsilon\mathbf{a}.$$

If follows that $\mathcal{G}(\mathcal{B}(u)) \subset \mathcal{B}(u)$. Furthermore, it holds

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{G}(x), \mathcal{G}(y)) \lneq L\boldsymbol{\mu}_{\mathcal{C}}(x, y) \qquad \forall x, y \in \mathcal{B}(u), x \neq y.$$

It follows from Theorem 7.1.5, b) that $\mathcal{G}$ has a unique fixed point in $\mathcal{B}(u)$ and for all $x \in \mathcal{B}(u)$ it holds $\lim_{j\to\infty}\mathcal{G}^j(x) = u$. Hence, u is a locally attracting fixed point of $\mathcal{G}$ in $\operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$. Now, for $i = 1,\ldots,d$ let $\nu_i \colon C_i \to \mathbb{R}$ be such that $\boldsymbol{\nu}(x) = (\nu_1(x_1),\ldots,\nu_d(x_d))^\top$ for all $x \in \mathcal{C}$ and set $S_i = \{x_i \in C_i \mid \nu_i(x_i) = 1\}$. Furthermore, let $w \in \operatorname{int}(\mathcal{C}^*)$ and $\Sigma_i = \{x_i \in C_i \mid w_i(x_i) = 1\}$ for $i = 1,\ldots,d$. Let $\zeta_i \colon (\Sigma_i \cap \operatorname{int}(C_i), \mu_{C_i}) \to (S_i \cap \operatorname{int}(C_i), \mu_{C_i})$ be given by $\zeta_i(x_i) = x_i/\nu_i(x_i)$ for all $x_i \in \Sigma_i \cap \operatorname{int}(C_i)$. Then, ζ_i is a continuous bijective isometry with inverse $\zeta_i^{-1}(x_i) = x_i/w_i(x_i)$. Corollary 2.6.4 in [60] implies that $(\Sigma_i \cap \operatorname{int}(C_i), \mu_{C_i})$ is a geodesic space and thus the existence of ζ_i implies that $(S_i \cap \operatorname{int}(C_i), \mu_{C_i})$ is a geodesic space as well. As the latter is true for all $i = 1,\ldots,d$, Theorem 7.1.5, c) implies that $\mathcal{G}$ has a unique fixed point in $\operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ and $\lim_{j\to\infty}\mathcal{G}^j(x) = u$ for all $x \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$. Finally, the uniqueness of u as a fixed point of $\mathcal{G}$ together with Lemma 7.3.2 imply that $(\boldsymbol{\theta}, u)$ is the unique solution to the equation $\mathcal{F}(x) = \boldsymbol{\lambda} \otimes x$ with $(\boldsymbol{\lambda}, x) \in \mathbb{R}^d_+ \times (\operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}})$. □

8 Multi-linear Birkhoff-Hopf theorem

As shown in Example 5.2.10, the Birkhoff-Hopf theorem can be used to exploit the linear structure in non-linear mappings in order to prove smaller Lipschitz constants. Motivated by this observation and the fact that the eigenvector problems discussed in Section 4.2 all exhibit some kind of linear structure, we generalize the Birkhoff-Hopf theorem 5.2.7 for multi-linear mappings defined on the product of cones. Then, we explain how the result can be used to analyze the Lipschitz matrix of multi-homogeneous polynomial mappings. Finally, we consider the particular case $\mathcal{C} = \mathbb{R}^{n_1}_+ \times \ldots \times \mathbb{R}^{n_d}_+$ and discuss bounds on the Lipschitz matrix of such mappings.

8.1 Birkhoff-Hopf theorem for cone multi-linear mappings

Let $\mathcal{V} = V_1 \times \ldots \times V_d$ and $\mathcal{W} = W_1 \times \ldots \times W_{d'}$ be products of finite dimensional vector spaces. For cones $\mathcal{C} \subset \mathcal{V}$ and $\mathcal{K} \subset \mathcal{W}$ let $\boldsymbol{\mu}_{\mathcal{C}}$ and $\boldsymbol{\mu}_{\mathcal{K}}$ be the vector valued Hilbert metrics induced by these cones (see Equation (6.6)). The next result is a Birkhoff-Hopf theorem for multi-linear mappings on cones which generalizes Theorem 5.2.7. The infinite dimensional case is discussed in Theorem 3.4 of [38]$^\diamond$. We have seen in Section 5.2.2 that for the applications of the Birkhoff-Hopf theorem 5.2.7 it is convenient to generalize the concept of linearity to that of cone linearity. Motivated by this observation, we introduce the following definition where for $\mathcal{F} \colon \mathcal{C} \to \mathcal{K}$, $x \in \mathcal{C}$, $i \in \{1,\ldots,d\}$ and $j \in \{1,\ldots,d'\}$, we denote by $\mathcal{F}|_x^{j,i} \colon C_i \to K_j$ the mapping defined as

$$\mathcal{F}|_x^{i,j}(y_i) = \mathcal{F}(x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_d)_j \qquad \forall y_i \in C_i.$$

Definition 8.1.1. Let $\mathcal{C} \subset \mathcal{V}$ and $\mathcal{K} \subset \mathcal{W}$ be cones and $\mathcal{F} \colon \mathcal{C} \to \mathcal{K}$. We say that $\mathcal{F}$ is *cone multi-linear* if for every $x \in \mathcal{V}$, $i = 1,\ldots,d$ and $j = 1,\ldots,d'$, the mapping $\mathcal{F}|_x^{j,i} \colon C_i \to K_j$ is either cone linear or constant.

The reason for allowing constant $\mathcal{F}|_x^{j,i} \colon C_i \to K_j$ in the above definition is to include mappings $\mathcal{F} \colon \mathcal{C} \to \mathcal{K}$ where $x \mapsto \mathcal{F}(x)_i$ does not depend on one of the x_j. This situation typically arise when studying the $\ell^{p,q,r}$-singular vectors of a nonnegative tensor, where, for $p = q = r = 2$, we want to find the eigenvectors of a multi-homogeneous mapping $\mathcal{F}\colon \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n$ of the form

$$\mathcal{F}(x, y, z) = (T(\cdot, y, z), T(x, \cdot, z), T(x, y, \cdot)),$$

with $T \in \mathbb{R}_+^{l\times m\times n}$ (see Section 4.2.1). In particular, note that, for instance, $T(\cdot, y, z)$ is multi-linear in the classical sense as a mapping from $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^l$ but it is not multi-linear on $\mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n$ since for all α, β and $(x, y, z) \in \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}^n$ and $\tilde{x} \in \mathbb{R}^l$ with $\tilde{x} \neq x$, the mapping $f(x, y, z) = T(\cdot, y, z)$ is such that $f(\alpha x + \beta\tilde{x}, y, z) = f(x, y, z)$ and thus, except in some exceptional cases such as $f(x, y, z) = f(\tilde{x}, y, z) = 0$, we have $f(\alpha x + \beta\tilde{x}, y, z) \neq \alpha f(x, y, z) + \beta f(\tilde{x}, y, z)$.

Note that if $\mathcal{F}\colon \mathcal{C} \to \mathcal{K}$ is cone multi-linear, then $\mathcal{F}$ is order-preserving and multi-homogeneous of degree $A \in \{0, 1\}^{d'\times d}$. We have the following generalization of the Birkhoff-Hopf theorem:

Theorem 8.1.2 (Multi-linear Birkhoff-Hopf)**.** Let $\mathcal{C} \subset \mathcal{V}$ and $\mathcal{K} \subset \mathcal{W}$ be cones and $\mathcal{F}\colon \mathcal{C} \to \mathcal{K}$ a cone multi-linear mapping. Let $L \in \mathbb{R}_+^{d'\times d}$ be defined as

$$L = \inf\big\{A \in \mathbb{R}_+^{d'\times d} \,\big|\, \boldsymbol{\mu}_{\mathcal{K}}(\mathcal{F}(x), \mathcal{F}(y)) \leq A\boldsymbol{\mu}_{\mathcal{C}}(x, y),\ \forall x, y \in \mathcal{C}, x \sim_{\mathcal{C}} y\big\}.$$

Then, for every $i = 1, \ldots, d$ and $j = 1, \ldots, d'$, it holds

$$L_{j,i} = \sup_{x\in\mathcal{C}} \tanh\big[\tfrac{1}{4}\,\mathrm{diam}(\mathcal{F}|_x^{j,i}(C_i); \mu_{K_j})\big].$$

Proof. Let $x \in \mathcal{C}$, $i = 1, \ldots, d$ and $j = 1, \ldots, d'$. If $\mathcal{F}|_x^{j,i}$ is constant then it holds $\mu_{K_j}(\mathcal{F}|_x^{j,i}(y_i), \mathcal{F}|_x^{j,i}(z_i)) = 0$ for all $y_i, z_i \in C_i$ and so $\mathrm{diam}(\mathcal{F}|_x^{j,i}(C_i); \mu_{K_j}) = 0$. If $\mathcal{F}|_x^{j,i}$ is cone linear, then Theorem 5.2.7 implies that the smallest Lipschitz constant of $\mathcal{F}|_x^{j,i}\colon (C_i, \mu_{C_i}) \to (K_j, \mu_{K_j})$, satisfies $\kappa(\mathcal{F}|_x^{j,i}) = \tanh\big[\tfrac{1}{4}\,\mathrm{diam}(\mathcal{F}|_x^{j,i}(C_i); \mu_{K_j})\big]$. Finally, Theorem 6.1.7 implies that $L_{j,i} = \sup_{x\in\mathcal{C}} \kappa(\mathcal{F}|_x^{j,i})$. □

Remark 8.1.3. We make observations which are helpful for the estimation of $L_{j,i}$ in Theorem 8.1.2. So let $\mathcal{C} \subset \mathcal{V}$ and $\mathcal{K} \subset \mathcal{W}$ and suppose that $\mathcal{F}\colon \mathcal{C} \to \mathcal{K}$ is a cone multi-linear mapping.

a) By the monotonicity of the hyperbolic tangent, for every i, j it holds

$$\sup_{x\in\mathcal{C}} \tanh\big[\tfrac{1}{4}\,\mathrm{diam}(\mathcal{F}|_x^{j,i}(C_i); \mu_{K_j})\big] = \tanh\big[\tfrac{1}{4}\sup_{x\in\mathcal{C}} \mathrm{diam}(\mathcal{F}|_x^{j,i}(C_i); \mu_{K_j})\big].$$

b) For every i, j, as $\mathcal{F}|_x^{j,i}$ does not depend on x_i, it holds

$$\begin{aligned}
&\sup_{x\in\mathcal{C}} \tanh\big[\tfrac{1}{4}\,\mathrm{diam}(\mathcal{F}|_x^{j,i}(C_i); \mu_{K_j})\big] \\
&\qquad= \sup_{x\in\mathcal{C}} \tanh\Big[\tfrac{1}{4} \sup_{y_i, z_i\in C_i} \mu_{K_j}(\mathcal{F}|_x^{j,i}(y_i), \mathcal{F}|_x^{j,i}(z_i))\Big] \\
&\qquad= \sup_{x\in\mathcal{C}} \tanh\Big[\tfrac{1}{4} \sup_{z_i\in C_i} \mu_{K_j}(\mathcal{F}|_x^{j,i}(x_i), \mathcal{F}|_x^{j,i}(z_i))\Big] \\
&\qquad= \sup_{x\in\mathcal{C}, z_i\in C_i} \tanh\big[\tfrac{1}{4}\mu_{K_j}(\mathcal{F}(x), \mathcal{F}|_x^{j,i}(z_i))\big].
\end{aligned}$$

c) Let $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d$ be a multi-normalization of $\mathcal{C}$, then by the scaling invariance property of $\boldsymbol{\mu}_{\mathcal{K}}$ (see Equation (6.7)) and the multi-homogeneity of $\mathcal{F}$, it holds

$$\sup_{x\in\mathcal{C}} \tanh\left[\tfrac{1}{4}\operatorname{diam}(\mathcal{F}|_x^{j,i}(C_i);\mu_{K_j})\right] = \sup_{x\in\mathcal{S}} \tanh\left[\tfrac{1}{4}\operatorname{diam}(\mathcal{F}|_x^{j,i}(S_i);\mu_{K_j})\right],$$

where $\mathcal{S} = S_1 \times \ldots \times S_d = \{x \in \mathcal{C} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$.

d) If there exists $i, i' \in \{1,\ldots,d\}$ with $i < i'$ such that $C_i = C_{i'}$ and there exists $j \in \{1,\ldots,d'\}$ such that $x \mapsto \mathcal{F}(x)_j$ is partially symmetric with respect to i and i' in the sense that for all $x \in \mathcal{C}$ it holds

$$\mathcal{F}(x_1,\ldots,x_i,\ldots,x_{i'},\ldots,x_d)_j = \mathcal{F}(x_1,\ldots,x_{i'},\ldots,x_i,\ldots,x_d)_j,$$

then, it holds $L_{j,i} = L_{j,i'}$. Indeed, for $x \in \mathcal{C}$, set

$$x' = (x_1,\ldots,x_{i-1},x_{i'},x_{i+1},\ldots,x_{i'-1},x_i,x_{i'+1},\ldots,x_d).$$

Then, for all $y_i \in C_i$ we have

$$\begin{aligned}\mathcal{F}|_x^{j,i}(y_i) &= \mathcal{F}(x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_{i'-1},x_{i'},x_{i'+1},\ldots,x_d)_j\\ &= \mathcal{F}(x_1,\ldots,x_{i-1},x_{i'},x_{i+1},\ldots,x_{i'-1},y_i,x_{i'+1},\ldots,x_d)_j = \mathcal{F}|_{x'}^{j,i'}(y_i).\end{aligned}$$

It follows that

$$\begin{aligned}\sup_{x\in\mathcal{C}}\operatorname{diam}(\mathcal{F}|_x^{j,i}(C_i);\mu_{K_j}) &= \sup_{x\in\mathcal{C}}\sup_{y_i,\tilde{y}_i\in C_i} \mu_{K_j}(\mathcal{F}|_x^{j,i}(y_i),\mathcal{F}|_x^{j,i}(\tilde{y}_i))\\ &= \sup_{x'\in\mathcal{C}}\sup_{y_i,\tilde{y}_i\in C_i} \mu_{K_j}(\mathcal{F}|_{x'}^{j,i'}(y_i),\mathcal{F}|_{x'}^{j,i'}(\tilde{y}_i))\\ &= \sup_{x'\in\mathcal{C}}\operatorname{diam}(\mathcal{F}|_{x'}^{j,i'}(C_{i'});\mu_{K_j}),\end{aligned}$$

and thus $L_{j,i} = L_{j,i'}$ by a).

We illustrate an application of Theorem 8.1.2 with an example.

Example 8.1.4. Let $\epsilon > 0$ and let $T \in \mathbb{R}^{2\times2\times2}_{++}$ be the symmetric tensor given by

$$T_{i,j,k} = \begin{cases} 1 & \text{if } i=j=k,\\ \epsilon & \text{otherwise}\end{cases} \qquad \forall i,j,k = 1,2.$$

Let $\mathcal{C} = \mathbb{R}^2_+ \times \mathbb{R}^2_+$, $K = \mathbb{R}^2_+$ and $L = (L_1, L_2) \in \mathbb{R}^{1\times 2}_+$ defined as

$$\begin{aligned}L = \inf\Big\{A \in \mathbb{R}^{1\times2}_+ \,\Big|\, &\boldsymbol{\mu}_{\mathcal{K}}(T(\cdot,y,z),T(\cdot,\tilde{y},\tilde{z})) \le A\boldsymbol{\mu}_{\mathcal{C}}((y,z),(\tilde{y},\tilde{z})),\\ &\forall (y,z),(\tilde{y},\tilde{z}) \in \mathcal{C} \text{ with } (y,z) \sim_{\mathcal{C}} (\tilde{y},\tilde{z})\Big\}.\end{aligned}$$

Then, by Theorem 8.1.2, we have

$$L_1 = \sup_{z\in\mathbb{R}^2_+} \tanh\left[\tfrac{1}{4}\operatorname{diam}(T(\cdot,\mathbb{R}^2_+,z);\mu_K)\right],\; L_2 = \sup_{y\in\mathbb{R}^2_+} \tanh\left[\tfrac{1}{4}\operatorname{diam}(T(\cdot,y,\mathbb{R}^2_+);\mu_K)\right].$$

As T is symmetric, we have $T(\cdot, y, z) = T(\cdot, z, y)$ for all $y, z \in \mathbb{R}^2$ and thus $L_1 = L_2$. We compute L_1. Note that for all $y, z \in \mathbb{R}^2$, it holds

$$T(\cdot, y, z) = (1-\epsilon) y \circ z + \epsilon \langle y\,,\,\mathbf{1}\rangle\langle z\,,\,\mathbf{1}\rangle \mathbf{1}, \tag{8.1}$$

where $\mathbf{1} = (1,1)^\top \in \mathbb{R}^2$. By Remark 8.1.3, (a), we have

$$L_1 = \tanh\big[\tfrac{1}{4}\sup_{z\in\mathbb{R}^2_+} \operatorname{diam}(T(\cdot, \mathbb{R}^2_+, z); \mu_K)\big].$$

The cone $\mathbb{R}^2_+$ is simplicial and has the canonical basis e_1, e_2 of $\mathbb{R}^2$ as generators. With Proposition 5.2.8 and (8.1), for every $z \in \mathbb{R}^2_+ \setminus \{0\}$, we have

$$\begin{aligned}
\operatorname{diam}(T(\cdot, \mathbb{R}^2_+, z); \mu_K) = \mu_K(T(\cdot, e_1, z), T(\cdot, e_2, z)) &= \big|\ln\Big(\frac{T(\cdot, e_1, z)_1 T(\cdot, e_2, z)_2}{T(\cdot, e_2, z)_1 T(\cdot, e_1, z)_2}\Big)\big| \\
&= \big|\ln\Big(\frac{\big((1-\epsilon)z_1 + \epsilon(z_1+z_2)\big)\big((1-\epsilon)z_2 + \epsilon(z_1+z_2)\big)}{\epsilon\,(z_1+z_2)\epsilon(z_1+z_2)}\Big)\big| \\
&= \big|\ln\Big(\frac{(z_1+\epsilon z_2)(z_2+\epsilon z_1)}{\epsilon^2\,(z_1+z_2)^2}\Big)\big|
\end{aligned}$$

Now, by Remark 8.1.3 (c), with $\Delta_+ = \{(t, 1-t)\,|\,t\in[0,1]\}$, we have that

$$\begin{aligned}
\sup_{z\in\mathbb{R}^2_+} \operatorname{diam}(T(\cdot, \mathbb{R}^2_+, z); \mu_K) &= \sup_{z\in\Delta_+} \operatorname{diam}(T(\cdot, \mathbb{R}^2_+, z); \mu_K) \\
&= \sup_{t\in[0,1]} \big|\ln\Big(\frac{(t+\epsilon(1-t))((1-t)+\epsilon t)}{\epsilon^2}\Big)\big| = \big|\ln\Big(\frac{(1+\epsilon)^2}{4\epsilon^2}\Big)\big|,
\end{aligned} \tag{8.2}$$

and thus

$$L_1 = L_2 = \tanh\Big|\frac{1}{2}\ln\Big(\frac{1+\epsilon}{2\epsilon}\Big)\Big| = \frac{|1-\epsilon|}{1+3\epsilon}. \tag{8.3}$$

In Example 8.1.4, we have computed

$$L_{j,i} = \sup_{x\in\mathcal{C}} \tanh\big[\tfrac{1}{4}\operatorname{diam}(\mathcal{F}|^{j,i}_x(C_i); \mu_{K_j})\big]$$

exactly, but our computations heavily use the fact that the cones in $\mathcal{C}$ were two dimensional. In general, even when $\mathcal{C}$ is a simplicial cone, the computation of $L_{j,i}$ can be difficult. In the next section, we discuss how to derive upper bounds on $L_{j,i}$ for the particular case where $\mathcal{C} = \mathbb{R}^{n_1}_+ \times \ldots \times \mathbb{R}^{n_d}_+$.

In the next result, we discuss how the multi-linear Birkhoff-Hopf theorem 8.1.2 can be used to derive Lipschitz matrices for polynomial mappings on cones. There are mainly two motivations for such a result. The first is that, for instance, the mapping characterizing the rectangular $\ell^{p,q}$-singular vectors of a nonnegative tensor with $p = q = 2$ is not cone multi-linear. Let us recall from Section 4.2.3 that for a third order tensor $T \in \mathbb{R}^{m\times n\times n}$, the latter mapping is given by

$$\mathcal{F}(x, y) = \big(\,T(\cdot, y, y)\,,\,T(x, \cdot, y)\,\big) \qquad \forall (x, y) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+.$$

Clearly $(x, y) \mapsto T(x, \cdot, y)$ is cone multi-linear, however $y \mapsto T(\cdot, y, y)$ is quadratic. Similarly, the mapping characterizing the Z-eigenvectors of a third order tensor, i.e. the ℓ^2-eigenvectors, suffers the same problem as in this case the mapping is given by $\mathcal{F}(x) = T(\cdot, x, x)$ for some $T \in \mathbb{R}_+^{n\times n\times n}$. The situation becomes worst as the order increase. Indeed, the Z-eigenvectors of a forth order tensor $T \in \mathbb{R}_+^{n\times n\times n\times n}$ are the eigenvectors of $\mathcal{F}(x) = T(\cdot, x, x, x)$ which is cubic in x. The second motivation is that, unlike cone linear mappings, the composition of two cone multi-linear mappings is not cone multi-linear in general. To address these cases, we have the following result:

Theorem 8.1.5. Let $\mathcal{C} \subset \mathcal{V}$ and $\mathcal{K} \subset \mathcal{W}$ be cones and let a be a positive integer. Let $\sigma_1, \ldots, \sigma_m$ be a partition of $\{1, \ldots, d\}$ into non-empty disjoint subsets. Suppose that $l_i \leq l_j$ for all $l_i \in \sigma_i$, $l_j \in \sigma_j$ such that $i \leq j$. Furthermore, for all $i = 1, \ldots, m$, suppose that $C_{l_i} = C_{l_i'}$ for all $l_i, l_i' \in \sigma_i$ and let $s_i = \min\{k_i \,|\, k_i \in \sigma_i\}$. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{K}$ be a cone multi-linear mapping and define $\mathcal{G}\colon \hat{\mathcal{C}} \to \mathcal{K}$ as

$$\mathcal{G}(\hat{x}) = \mathcal{F}(\underbrace{\hat{x}_1, \ldots, \hat{x}_1}_{|\sigma_1| \text{ times}}, \ldots, \underbrace{\hat{x}_m, \ldots, \hat{x}_m}_{|\sigma_m| \text{ times}}) \qquad \forall \hat{x} = (\hat{x}_1, \ldots, \hat{x}_m) \in \hat{\mathcal{C}},$$

where $\hat{\mathcal{C}} = C_{s_1} \times \ldots \times C_{s_m}$ and for $i = 1, \ldots, m$, $|\sigma_i|$ denotes the cardinality of σ_i. Let $\hat{L} \in \mathbb{R}^{d'\times m}$ be defined as

$$\hat{L} = \inf\big\{A \in \mathbb{R}_+^{d'\times m} \,\big|\, \boldsymbol{\mu}_{\mathcal{K}}(\mathcal{G}(\hat{x}), \mathcal{G}(\hat{y})) \leq A\boldsymbol{\mu}_{\hat{\mathcal{C}}}(\hat{x}, \hat{y}),\, \forall \hat{x}, \hat{y} \in \hat{\mathcal{C}}, \hat{x} \sim_{\hat{\mathcal{C}}} \hat{y}\big\}.$$

Then, for all $j = 1, \ldots, d'$ and $i = 1, \ldots, m$ it holds

$$\hat{L}_{j,i} \leq \sum_{l_i \in \sigma_i} \sup_{x \in \mathcal{C}} \tanh\big[\tfrac{1}{4}\operatorname{diam}(\mathcal{F}|_x^{j,l_i}(C_{l_i}); \mu_{K_j})\big]. \tag{8.4}$$

Proof. Let $L \in \mathbb{R}^{d'\times d}$ be defined as

$$L = \inf\big\{A \in \mathbb{R}_+^{d'\times d} \,\big|\, \boldsymbol{\mu}_{\mathcal{K}}(\mathcal{F}(x), \mathcal{F}(y)) \leq A\boldsymbol{\mu}_{\mathcal{C}}(x, y),\, \forall x, y \in \mathcal{C}, x \sim_{\mathcal{C}} y\big\}.$$

Then, by Theorem 8.1.2, for every $i = 1, \ldots, d$ and $j = 1, \ldots, d'$, it holds

$$L_{j,i} = \sup_{x \in \mathcal{C}} \tanh\big[\tfrac{1}{4}\operatorname{diam}(\mathcal{F}|_x^{j,i}(C_i); \mu_{K_j})\big].$$

For all $\hat{x}, \hat{y} \in \hat{C}$ such that $\hat{x} \sim_{\mathcal{C}} \hat{y}$, for $x = (\hat{x}_1, \ldots, \hat{x}_1, \ldots, \hat{x}_m, \ldots, \hat{x}_m)$ and $y = (\hat{y}_1, \ldots, \hat{y}_1, \ldots, \hat{y}_m, \ldots, \hat{y}_m)$, we have $x \sim_{\mathcal{C}} y$ and $\boldsymbol{\mu}_{\mathcal{C}}(x, y) = P\boldsymbol{\mu}_{\hat{\mathcal{C}}}(\hat{x}, \hat{y})$, where $P \in \{0, 1\}^{m\times d}$ is defined as $P_{l,i} = 1$ if $l \in \sigma_i$ and $P_{l,i} = 0$ otherwise. Furthermore, note that $\hat{L} = LP$ and thus

$$\boldsymbol{\mu}_{\mathcal{K}}(\mathcal{G}(\hat{x}), \mathcal{G}(\hat{y})) = \boldsymbol{\mu}_{\mathcal{K}}(\mathcal{F}(x), \mathcal{F}(y)) \leq L\boldsymbol{\mu}_{\mathcal{C}}(x, y) = LP\boldsymbol{\mu}_{\hat{\mathcal{C}}}(x, y) = \hat{L}\boldsymbol{\mu}_{\hat{\mathcal{C}}}(\hat{x}, \hat{y}),$$

which concludes the proof. □

The following example shows that the bound on $\hat{L}$ given by Theorem 8.1.5 is sharp, nevertheless we believe that there are cone multi-linear mappings for which the inequality in (8.4) is strict and leave the improvement of this upper bound to future work.

Example 8.1.6. Let $C = \mathbb{R}^2_+$, $\epsilon > 0$ and let $T \in \mathbb{R}^{2\times 2\times 2}_{++}$ be the tensor of Example 8.1.4. We recall that the smallest Lipschitz matrix $L = (L_1, L_2) \in \mathbb{R}^{1\times 2}$ of $(y, z) \mapsto T(\cdot, y, z)$ is given in (8.3). By Theorem 8.1.5, we have that for all $x, y \in \mathbb{R}^2_+$ with $x \sim_C y$ it holds

$$\mu_C(T(\cdot, x, x), T(\cdot, y, y)) \leq \hat{L}\mu_C(x, y),$$

with

$$\hat{L} \leq \sup_{z\in\mathbb{R}^2_+} \operatorname{diam}(T(\cdot, \mathbb{R}^2_+, z); \mu_K) + \sup_{y\in\mathbb{R}^2_+} \operatorname{diam}(T(\cdot, y, \mathbb{R}^2_+); \mu_K) = 2\frac{|1-\epsilon|}{1+3\epsilon}.$$

Now, we prove that this bound is attained. Let $y(\cdot)\colon (0, \infty) \to \mathbb{R}^2_{++}$ be defined as $y(t) = (t, 1)$. Then, we have $\mu(\mathbf{1}, y(t)) = |\ln(t)|$ for all $t > 0$. Furthermore, by (8.1), it holds $T(\cdot, \mathbf{1}, \mathbf{1}) = (1 + 3\epsilon)\mathbf{1}$ and

$$T(\cdot, y(t), y(t)) = \begin{pmatrix} (1-\epsilon)t^2 + \epsilon(1+t)^2 \\ (1-\epsilon) + \epsilon(1+t)^2 \end{pmatrix} \qquad \forall t > 0.$$

By L'Hospital's rule, we have

$$\begin{aligned}\hat{L} &\geq \lim_{t\to 1} \frac{\mu_C(T(\cdot, y(t), y(t)), T(\cdot, \mathbf{1}, \mathbf{1}))}{\mu(y(t), \mathbf{1})} = \lim_{t\to 1} \left| \frac{\ln\left(\frac{(1-\epsilon)t^2+\epsilon(1+t)^2}{(1-\epsilon)+\epsilon(1+t)^2}\right)}{\ln(t)} \right| \\ &= \lim_{t\to 1} \left| \frac{2(1-\epsilon)\, t\, \left(t^2\epsilon + t\epsilon + t + \epsilon\right)}{(t^2 + 2t\epsilon + \epsilon)\,(t^2\epsilon + 2t\epsilon + 1)} \right| = 2\frac{|1-\epsilon|}{1+3\epsilon}.\end{aligned}$$

which shows that

$$\hat{L} = 2\frac{|1-\epsilon|}{1+3\epsilon}.$$

The mapping $\mathcal{G}$ of Theorem 8.1.5 is multi-homogeneous and order-preserving. Hence, its homogeneity matrix is a natural Lipschitz matrix of the mapping with respect to the vector valued Hilbert metric. We note however in the next result that if $\mathcal{G}$ maps in the interior of a cone, then there always exists a Lipschitz matrix with strictly smaller spectral radius.

Theorem 8.1.7. Let $\mathcal{C} \subset \mathcal{V}$ and $\mathcal{K} \subset \mathcal{W}$ be solid closed cones and let a be a positive integer. Let $\sigma_1, \ldots, \sigma_m$, $\mathcal{F}\colon \mathcal{C} \to \mathcal{K}$, $\hat{\mathcal{C}}$ and $\mathcal{G}\colon \hat{\mathcal{C}} \to \mathcal{K}$ be defined as in Theorem 8.1.5. Suppose that $m = d'$. Then, $\mathcal{G}$ is multi-homogeneous of degree $A \in \mathbb{R}^{d'\times d'}$ with $A_{j,i} = \sum_{l_i\in\sigma_i} B_{j,l_i}$ for all $i, j = 1, \ldots, d'$ where $B \in \mathbb{R}^{d'\times d}$ is the homogeneity matrix of the cone multi-linear mapping $\mathcal{F}$. Let $\hat{L} \in \mathbb{R}^{d'\times m}$ be defined as

$$\hat{L} = \inf\left\{A \in \mathbb{R}^{d'\times m}_+ \;\middle|\; \boldsymbol{\mu}_{\mathcal{K}}(\mathcal{G}(\hat{x}), \mathcal{G}(\hat{y})) \leq A\boldsymbol{\mu}_{\hat{\mathcal{C}}}(\hat{x}, \hat{y}),\ \forall \hat{x}, \hat{y} \in \hat{\mathcal{C}}, \hat{x} \sim_{\hat{\mathcal{C}}} \hat{y}\right\}.$$

If $\mathcal{G}(\operatorname{int}(\hat{\mathcal{C}})) \subset \operatorname{int}(\mathcal{K})$, A is irreducible and there exists $j_\star \in \{1, \ldots, d'\}$ such that $\mathcal{F}(x)_{j_\star} \in \operatorname{int}(C_{j_\star})$ for all $x \in \mathcal{C}$ satisfying $x_1, \ldots, x_d \neq 0$, then $\rho(\hat{L}) < \rho(A)$.

Proof. Note that $\mathcal{G}$ is continuous as $\mathcal{F}$ is a cone multi-linear mapping on the product of finite dimensional vector spaces. $\mathcal{G}$ is order-preserving since $\mathcal{F}$ is order-preserving.

Furthermore, the multi-linearity of $\mathcal{F}$ directly implies the multi-homogeneity of $\mathcal{G}$. Hence, Proposition 6.2.4 implies that A is a Lipschitz matrix of $\mathcal{G}$ and therefore it holds $\hat{L} \preceq_{\mathbb{R}^{d\times d}_+} A$. We recall from Example 2.3.6 that if $M, Q \in \mathbb{R}^{d\times d}_+$ are such that $M \preceq_{\mathbb{R}^{d\times d}_+} Q$, then $\rho(M) \leq \rho(Q)$ and if additionally $M+Q$ is irreducible and $M \neq Q$, then $\rho(M) < \rho(Q)$. Hence, to prove that $\rho(\hat{L}) < \rho(A)$, it is enough to show that $\hat{L} \neq A$ since the irreducibility of A implies that of $\hat{L}+A$. Let $L \in \mathbb{R}^{d'\times d}$ be defined as

$$L = \inf\big\{M \in \mathbb{R}^{d'\times d}_+ \,\big|\, \boldsymbol{\mu}_{\mathcal{K}}(\mathcal{F}(x), \mathcal{F}(y)) \leq M\boldsymbol{\mu}_{\mathcal{C}}(x,y),\ \forall x,y \in \mathcal{C}, x \sim_{\mathcal{C}} y\big\}.$$

Since $\mathcal{F}$ is multi-homogeneous of degree $B \in \mathbb{R}^{d'\times d}$ and order-preserving, B is a Lipschitz matrix of $\mathcal{F}\colon (\mathcal{C}, \boldsymbol{\mu}_{\mathcal{C}}) \to (\mathcal{K}, \boldsymbol{\mu}_{\mathcal{K}})$ by Proposition 6.2.4. It follows from Theorem 8.1.2 that $L \preceq_{\mathbb{R}^{d\times d}_+} B$. Now, we show that the existence of $j_\star \in \{1, \ldots, d'\}$ such that $\mathcal{F}(x)_{j_\star} \in \operatorname{int}(C_{j_\star})$ for all $x \in \mathcal{C}$ satisfying $x_1, \ldots, x_d \neq 0$ implies that $L_{j_\star,l} < B_{j_\star,l}$ for all $l \in \{1, \ldots, d\}$ such that $B_{j_\star,l} > 0$. Let $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d_+$ be a multi-normalization of $\mathcal{C}$ and $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$. For $l = 1, \ldots, d$, with Theorem 8.1.2 and Remark 8.1.3 (c), we have

$$\begin{aligned}
L_{j_\star,l} &= \sup_{x\in\mathcal{C}} \tanh\big[\tfrac{1}{4}\operatorname{diam}(\mathcal{F}|_x^{j_\star,l}(C_l); \mu_{K_{j_\star}})\big] \\
&= \sup_{x\in\mathcal{S}_{\boldsymbol{\nu}}} \tanh\big[\tfrac{1}{4}\operatorname{diam}(\mathcal{F}|_x^{j_\star,l}(C_l); \mu_{K_{j_\star}})\big] \\
&= \sup_{x\in\mathcal{S}_{\boldsymbol{\nu}}}\ \sup_{y,z\in\mathcal{S}_{\boldsymbol{\nu}}} \tanh\big[\tfrac{1}{4}\mu_{K_{j_\star}}(\mathcal{F}|_x^{j_\star,l}(y_l), \mathcal{F}|_x^{j_\star,l}(z_l))\big] \\
&\leq \sup_{x,v\in\mathcal{S}_{\boldsymbol{\nu}}}\ \sup_{y,z\in\mathcal{S}_{\boldsymbol{\nu}}} \tanh\big[\tfrac{1}{4}\mu_{K_{j_\star}}(\mathcal{F}|_x^{j_\star,l}(y_l), \mathcal{F}|_v^{j_\star,l}(z_l))\big] \\
&= \sup_{x,v\in\mathcal{S}_{\boldsymbol{\nu}}} \tanh\big[\tfrac{1}{4}\mu_{K_{j_\star}}(\mathcal{F}(x)_{j_\star}, \mathcal{F}(v)_{j_\star})\big].
\end{aligned}$$

Now, as $\mathcal{S}_{\boldsymbol{\nu}}$ is a compact subset of $\mathcal{C}$, $\mathcal{F}$ is continuous and $\mathcal{F}(x)_{j_\star} \in \operatorname{int}(C_{j_\star})$ for every $x \in \mathcal{C}$ satisfying $x_1, \ldots, x_d \neq 0$, the set $U_{j_\star} = \mathcal{F}(\mathcal{S}_{\boldsymbol{\nu}})_{j_\star}$ is a compact subset of $\operatorname{int}(C_{j_\star})$ and therefore there exists $\gamma_\star < \infty$ such that $\mu_{K_{j_\star}}(\mathcal{F}(x)_{j_\star}, \mathcal{F}(v)_{j_\star}) \leq \gamma_\star$ for all $x, v \in \mathcal{S}_{\boldsymbol{\nu}}$. It follows that for each $l \in \{1, \ldots, d\}$ such that $B_{j_\star,l} > 0$ it holds $L_{j_\star,l} < 1 = B_{j_\star,l}$. Note that such l exists otherwise we would have $A_{j_\star,i} = \sum_{l_i\in\sigma_i} B_{j_\star,l_i} = 0$ for all $i = 1, \ldots, d'$ which contradicts the irreducibility of A. So, let $l_\star \in \{1, \ldots, d\}$ be such that $B_{j_\star,l_\star} > 0$ and let $i_\star \in \{1, \ldots, d'\}$ be such that $l_\star \in \sigma_{i_\star}$. Then, by Theorem 8.1.5, we have

$$\hat{L}_{j_\star,i_\star} \leq \sum_{l\in\sigma_{i_\star}} L_{j_\star,l} < \sum_{l\in\sigma_i} B_{j_\star,l} = A_{j_\star,i_\star},$$

which concludes the proof. □

8.2 Projective diameter of positive tensors

We discuss upper bounds on the quantities $L_{j,i}$ of Theorem 8.1.2 for the particular case of tensors with positive entries in the following theorem and then illustrate applications of this result with examples. We note that similar (but more conservative) upper bounds were derived in Lemma 5.1 of [38]$^\diamond$ for integral operators on cones.

Theorem 8.2.1. Suppose that $d \geq 2$ and let $\mathcal{C} = \mathbb{R}_+^{n_1} \times \ldots \times \mathbb{R}_+^{n_d}$, $K = \mathbb{R}_+^{n_1}$ and $T \in \mathbb{R}_{++}^{n_1 \times \ldots \times n_d}$. Then, it holds

$$\sup_{x \in \mathcal{C}} \operatorname{diam}(T(\boldsymbol{\cdot}, \mathbb{R}_+^{n_2}, x_3, \ldots, x_d); \mu_K) \leq \ln(\Delta), \qquad \text{with} \tag{8.5}$$

$$\Delta = \max_{j_1, j_1' \in [n_1], \ldots, j_d, j_d' \in [n_d]} \frac{\displaystyle\sum_{\sigma_3 \in \mathfrak{S}(\{j_3, j_3'\}), \ldots, \sigma_d \in \mathfrak{S}(\{j_d, j_d'\})} T_{j_1, j_2, \sigma_3(j_3), \ldots, \sigma_d(j_d)} T_{j_1', j_2', \sigma_3(j_3') \ldots, \sigma_d(j_d')}}{\displaystyle\sum_{\sigma_3 \in \mathfrak{S}(\{j_3, j_3'\}), \ldots, \sigma_d \in \mathfrak{S}(\{j_d, j_d'\})} T_{j_1', j_2, \sigma_3(j_3), \ldots, \sigma_d(j_d)} T_{j_1, j_2', \sigma_3(j_3') \ldots, \sigma_d(j_d')}},$$

where for $i = 1, \ldots, d$, $[n_i] = \{1, \ldots, n_i\}$ and

$$\mathfrak{S}(\{j_i, j_i'\}) = \big\{\sigma_i \colon \{j_i, j_i'\} \to \{j_i, j_i'\} \,\big|\, \sigma_i \text{ bijective}\big\}.$$

Proof. If $d = 2$, then the statement follows from (5.8), hence we can suppose that $d \geq 3$. Note that if $x \in \mathcal{C}$ is such that $x_i = 0$ for for some $i \geq 3$, then $T(\boldsymbol{\cdot}, y_2, x_3, \ldots, x_d) = 0$ for all $y_2 \in \mathbb{R}_+^{n_2}$ and thus $\operatorname{diam}(T(\boldsymbol{\cdot}, \mathbb{R}_+^{n_2}, x_3, \ldots, x_d); \mu_K) = 0$. It follows that

$$\sup_{x \in \mathcal{C}} \operatorname{diam}(T(\boldsymbol{\cdot}, \mathbb{R}_+^{n_2}, x_3, \ldots, x_d); \mu_K) = \sup_{x \in \mathcal{C}_0} \operatorname{diam}(T(\boldsymbol{\cdot}, \mathbb{R}_+^{n_2}, x_3, \ldots, x_d); \mu_K),$$

with $\mathcal{C}_0 = \{x \in \mathcal{C} \,|\, x_i \neq 0,\, i = 1, \ldots, d\}$. Note that $T(\boldsymbol{\cdot}, x_2, x_3, \ldots, x_d) \in \mathbb{R}_{++}^{n_1}$ for all $x \in \mathcal{C}_0$ since $T \in \mathbb{R}_{++}^{n_1 \times \ldots \times n_d}$. The cone $\mathbb{R}_+^{n_2}$ is simplicial and generated by the canonical basis $e_1, \ldots, e_{n_2} \in \mathbb{R}^{n_2}$. Proposition 5.2.8 implies that for every $x \in \mathcal{C}_0$,

$$\begin{aligned}
&\operatorname{diam}(T(\boldsymbol{\cdot}, \mathbb{R}_+^{n_2}, x_3, \ldots, x_d); \mu_K) \\
&\quad = \max_{1 \leq j_2 < j_2' \leq n_2} \mu_K(T(\boldsymbol{\cdot}, e_{j_2}, x_3, \ldots, x_d), T(\boldsymbol{\cdot}, e_{j_2'}, x_3, \ldots, x_d)) \\
&\quad = \max_{\substack{1 \leq j_1 < j_1' \leq n_1 \\ 1 \leq j_2 < j_2' \leq n_2}} \ln\Big(\frac{T(\boldsymbol{\cdot}, e_{j_2}, x_3, \ldots, x_d)_{j_1} T(\boldsymbol{\cdot}, e_{j_2'}, x_3, \ldots, x_d)_{j_1'}}{T(\boldsymbol{\cdot}, e_{j_2}, x_3, \ldots, x_d)_{j_1'} T(\boldsymbol{\cdot}, e_{j_2'}, x_3, \ldots, x_d)_{j_1}}\Big)
\end{aligned}$$

As the logarithm and the maximum are continuous, we have

$$\sup_{x \in \mathcal{C}_0} \operatorname{diam}(T(\boldsymbol{\cdot}, \mathbb{R}_+^{n_2}, x_3, \ldots, x_d); \mu_K) = \sup_{x \in \operatorname{int}(\mathcal{C})} \operatorname{diam}(T(\boldsymbol{\cdot}, \mathbb{R}_+^{n_2}, x_3, \ldots, x_d); \mu_K).$$

By combining the above observations, we see that for the proof it is enough to bound the quantity

$$\frac{T(\boldsymbol{\cdot}, e_{j_2}, x_3, \ldots, x_d)_{j_1} T(\boldsymbol{\cdot}, e_{j_2'}, x_3, \ldots, x_d)_{j_1'}}{T(\boldsymbol{\cdot}, e_{j_2}, x_3, \ldots, x_d)_{j_1'} T(\boldsymbol{\cdot}, e_{j_2'}, x_3, \ldots, x_d)_{j_1}} \tag{8.6}$$

for all $j_1, j_1' \in [n_1]$, $j_2, j_2' \in [n_2]$ and $x \in \operatorname{int}(\mathcal{C})$. To this end, we use the following inequality which can be proved by induction over $k \geq 1$ (see also [41, Lemma 4.3]):

$$\frac{\sum_{i=1}^k a_i}{\sum_{i=1}^k b_i} \leq \max_{1 \leq i \leq k} \frac{a_i}{b_i} \qquad \forall a_1, \ldots, a_k, b_1, \ldots, b_k > 0. \tag{8.7}$$

Despite being sharp as we have equality when $a_i = b_i$ for all $i = 1, \ldots, k$, the estimation in (8.7) can be quite conservative. However it can be made more tight by symmetrizing the sums in the nominator and the denominator. For intuition, we first show how it works in the case $d = 3$ and then prove the general case. Suppose that $d = 3$ and let $i, i' \in [n_1]$, $j, j' \in [n_2]$, $y \in \mathbb{R}^{n_3}_{++}$ be fixed. Then we have

$$\begin{aligned}\frac{T(\cdot, e_j, y)_i T(\cdot, e_{j'}, y)_{i'}}{T(\cdot, e_{j'}, y)_i T(\cdot, e_j, y)_{i'}} &= \frac{\big(\sum_{s=1}^{n_3} T_{i,j,s} y_s\big)\big(\sum_{t=1}^{n_3} T_{i',j',t} y_t\big)}{\big(\sum_{s=1}^{n_3} T_{i,j',s} y_s\big)\big(\sum_{t=1}^{n_3} T_{i',j,t} y_t\big)} \\ &= \frac{\sum_{s,t=1}^{n_3} T_{i,j,s} T_{i',j',t} y_s y_t}{\sum_{s,t=1}^{n_3} T_{i,j',s} T_{i',j,t} y_s y_t} \\ &= \frac{\sum_{s,t=1}^{n_3} (T_{i,j,s} T_{i',j',t} + T_{i,j,t} T_{i',j',s}) y_s y_t}{\sum_{s,t=1}^{n_3} (T_{i,j',s} T_{i',j,t} + T_{i,j',t} T_{i',j,s}) y_s y_t} \\ &\leq \max_{k,k' \in [n_3]} \frac{T_{i,j,k} T_{i',j',k'} + T_{i,j,k'} T_{i',j',k}}{T_{i,j',k} T_{i',j,k'} + T_{i,j',k'} T_{i',j,k}},\end{aligned} \tag{8.8}$$

where we have used (8.7) for the last inequality. Now, we prove the general case: Let $j_1, j_1' \in [n_1], j_2, j_2' \in [n_2]$, $x \in \operatorname{int}(\mathcal{C})$ and for all $l_3, l_3' \in [n_3], \ldots, l_d, l_d' \in [n_d]$ set

$$X_{l_3,l_3',\ldots,l_d,l_d'} = \prod_{r=3}^{d} x_{r,l_r} x_{r,l_r'} = (x_{3,l_3}\, x_{3,l_3'}) \cdots (x_{d,l_d}\, x_{d,l_d'}).$$

Then, we have

$$\begin{aligned}&\frac{T(\cdot, e_{j_2}, x_3, \ldots, x_d)_{j_1} T(\cdot, e_{j_2'}, x_3, \ldots, x_d)_{j_1'}}{T(\cdot, e_{j_2'}, x_3, \ldots, x_d)_{j_1} T(\cdot, e_{j_2}, x_3, \ldots, x_d)_{j_1'}} \\ &= \frac{\Big(\sum\limits_{l_3 \in [n_3], \ldots, l_d \in [n_d]} T_{j_1,j_2,l_3\ldots,l_d} x_{3,l_3} \cdots x_{d,l_d}\Big)\Big(\sum\limits_{l_3' \in [n_3], \ldots, j_d' \in [n_d]} T_{j_1',j_2',l_3',\ldots,l_d'} x_{3,l_3'} \cdots x_{d,l_d'}\Big)}{\Big(\sum\limits_{l_3 \in [n_3], \ldots, l_d \in [n_d]} T_{j_1',j_2,l_3\ldots,l_d} x_{3,l_3} \cdots x_{d,l_d}\Big)\Big(\sum\limits_{l_3' \in [n_3], \ldots, l_d' \in [n_d]} T_{j_1,j_2',l_3'\ldots,l_d'} x_{3,l_3'} \cdots x_{d,l_d'}\Big)} \\ &= \frac{\sum\limits_{l_3,l_3' \in [n_3], \ldots, l_d, l_d' \in [n_d]} T_{j_1,j_2,l_3\ldots,l_d} T_{j_1',j_2',l_3',\ldots,l_d'} X_{l_3,l_3',\ldots,l_d,l_d'}}{\sum\limits_{l_3,l_3' \in [n_3], \ldots, l_d, l_d' \in [n_d]} T_{j_1',j_2,l_3\ldots,l_d} T_{j_1,j_2',l_3',\ldots,l_d'} X_{l_3,l_3',\ldots,l_d,l_d'}} \\ &= \frac{\sum\limits_{l_3,l_3' \in [n_3], \ldots, l_d, l_d' \in [n_d]} X_{l_3,l_3',\ldots,l_d,l_d'} \Big(\sum\limits_{\sigma_3 \in \mathfrak{S}(\{l_3,l_3'\}), \ldots, \sigma_d \in \mathfrak{S}(\{l_d,l_d'\})} T_{j_1,j_2,\sigma_3(l_3),\ldots,\sigma_d(l_d)} T_{j_1',j_2',\sigma_3(l_3')\ldots,\sigma_d(l_d')}\Big)}{\sum\limits_{l_3,l_3' \in [n_3], \ldots, l_d, l_d' \in [n_d]} X_{l_3,l_3',\ldots,l_d,l_d'} \Big(\sum\limits_{\sigma_3 \in \mathfrak{S}(\{l_3,l_3'\}), \ldots, \sigma_d \in \mathfrak{S}(\{l_d,l_d'\})} T_{j_1',j_2,\sigma_3(l_3),\ldots,\sigma_d(l_d)} T_{j_1,j_2',\sigma_3(l_3')\ldots,\sigma_d(l_d')}\Big)} \\ &\leq \max_{j_3,j_3' \in [n_3], \ldots, j_d, j_d' \in [n_d]} \frac{\sum\limits_{\sigma_3 \in \mathfrak{S}(\{j_3,j_3'\}), \ldots, \sigma_d \in \mathfrak{S}(\{j_d,j_d'\})} T_{j_1,j_2,\sigma_3(j_3),\ldots,\sigma_d(j_d)} T_{j_1',j_2',\sigma_3(j_3')\ldots,\sigma_d(j_d')}}{\sum\limits_{\sigma_3 \in \mathfrak{S}(\{j_3,j_3'\}), \ldots, \sigma_d \in \mathfrak{S}(\{j_d,j_d'\})} T_{j_1',j_2,\sigma_3(j_3),\ldots,\sigma_d(j_d)} T_{j_1,j_2',\sigma_3(j_3')\ldots,\sigma_d(j_d')}}.\end{aligned}$$

By taking the maximum over $j_1, j_1' \in [n_1]$ and $j_2, j_2' \in [n_2]$, we find that

$$\frac{T(\cdot, e_{j_2}, x_3, \ldots, x_d)_{j_1} T(\cdot, e_{j_2'}, x_3, \ldots, x_d)_{j_1'}}{T(\cdot, e_{j_2}, x_3, \ldots, x_d)_{j_1'} T(\cdot, e_{j_2'}, x_3, \ldots, x_d)_{j_1}} \leq \Delta,$$

which concludes the proof. □

Note that if in Theorem 8.2.1, $d = 2$, i.e. T is a matrix, then by (5.8), we have equality in (8.5). If $d \geq 3$, it can however happen that inequality (8.5) is strict as shown by the following example.

Example 8.2.2. Let $d = 3$, $K = \mathbb{R}^2_+$ and let $T \in \mathbb{R}^{2\times 2\times 2}_{++}$ be the tensor of Example 8.1.4 with $\epsilon > 0$. From (8.2) we know that

$$\sup_{z\in\mathbb{R}^2_+} \operatorname{diam}(T(\cdot, \mathbb{R}^2_+, z); \mu_K) = \Big|\ln\Big(\frac{(1+\epsilon)^2}{4\epsilon^2}\Big)\Big|.$$

Let Δ defined as in Theorem 8.2.1. First, suppose that $\epsilon \in (0, 1]$, then we have

$$\Delta = \max_{i,i',j,j',k,k'=1,2} \frac{T_{i,j,k}T_{i',j',k'} + T_{i,j,k'}T_{i',j',k}}{T_{i,j',k}T_{i',j,k'} + T_{i,j',k'}T_{i',j,k}} = \frac{T_{1,1,1}T_{2,2,2} + T_{1,1,2}T_{2,2,1}}{T_{1,2,1}T_{2,1,2} + T_{1,2,2}T_{2,1,1}} = \frac{1+\epsilon^2}{2\epsilon^2}.$$

This gives the bound

$$\ln\Big(\frac{(1+\epsilon)^2}{4\epsilon^2}\Big) = \sup_{z\in\mathbb{R}^2_+} \operatorname{diam}(T(\cdot, \mathbb{R}^2_+, z); \mu_K) \leq \ln(\Delta) = \ln\Big(\frac{1+\epsilon^2}{2\epsilon^2}\Big).$$

Note that

$$\ln\Big(\frac{1+\epsilon^2}{2\epsilon^2}\Big) = \ln\Big(\frac{(1+\epsilon)^2}{4\epsilon^2} + \frac{(1-\epsilon)^2}{4\epsilon^2}\Big).$$

It follows that, for $\epsilon \in (0, 1]$, the bound given by Theorem 8.2.1 is sharp for $\epsilon = 1$ and gets increasingly more conservative as ϵ tends to 0. Now, suppose that $\epsilon \in [1, \infty)$, then

$$\Delta = \max_{i,i',j,j',k,k'=1,2} \frac{T_{i,j,k}T_{i',j',k'} + T_{i,j,k'}T_{i',j',k}}{T_{i,j',k}T_{i',j,k'} + T_{i,j',k'}T_{i',j,k}} = \frac{T_{1,2,1}T_{2,1,1} + T_{1,2,1}T_{2,1,1}}{T_{1,1,1}T_{2,2,1} + T_{1,1,1}T_{2,2,1}} = \epsilon^2.$$

This gives the bound

$$\ln\Big(\frac{4\epsilon^2}{(1+\epsilon)^2}\Big) = \sup_{z\in\mathbb{R}^2_+} \operatorname{diam}(T(\cdot, \mathbb{R}^2_+, z); \mu_K) \leq \ln(\Delta) = \ln(\epsilon^2).$$

Note that

$$\ln(\epsilon^2) = \ln\Big(\frac{4\epsilon^2}{(1+\epsilon)^2} + (\epsilon-1)\frac{(\epsilon+3)\epsilon^2}{(\epsilon+1)^2}\Big).$$

It follows that, for $\epsilon \in [1, \infty)$, the bound given by Theorem 8.2.1 is sharp for $\epsilon = 1$ and gets increasingly conservative as ϵ tends to ∞.

Remark 8.2.3. The proof of Theorem 8.2.1 shows that the maximum defining Δ is always attained at indexes $j_1, j_1' \in [n_1], \dots, j_d, j_d' \in [n_d]$ such that $j_1 < j_1'$ and $j_2 < j_2'$. However, it can happen that $j_i = j_i'$ for some $i \geq 3$. This is for instance the case in Example 8.2.2 when $\epsilon \geq 1$. Furthermore, numerical experiments suggest that whenever $\epsilon > 1$, the maximum is only attained at indexes $j_1, j_1', j_2, j_2', j_3, j_3'$ such that $j_3 = j_3'$.

Next, we show how to apply Theorem 8.2.1 with an example.

Example 8.2.4. Let $C = \mathbb{R}^n_+$, $\mathcal{C} = C \times C \times C$, $M \in \mathbb{R}^{n\times n}_+, R \in \mathbb{R}^{n\times n\times n}$ and $T \in \mathbb{R}^{n\times n\times n\times n}$. Suppose that T is symmetric, i.e. the entries $T_{i,j,k,l}$ of T are left invariant by any permutation of the indexes i, j, k, l. Define $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ as

$$\mathcal{F}(x, y, z) = \big(Mz, R(x, \cdot, z), T(\cdot, x, y, z)\big) \qquad \forall (x, y, z) \in \mathcal{C}.$$

Let $L \in \mathbb{R}^{3\times 3}_+$ be the smallest Lipschitz matrix of $\mathcal{F}$, i.e.

$$L = \inf\big\{A \in \mathbb{R}^{3\times 3}_+ \,\big|\, \boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}(u), \mathcal{F}(v)) \leq A\boldsymbol{\mu}_{\mathcal{C}}(u, v),\ \forall u, v \in \mathcal{C}, u \sim_{\mathcal{C}} v\big\}.$$

We use Theorem 8.2.1 to derive upper bounds on $L_{i,j}$ for all $i, j = 1, 2, 3$. Let $(\hat{x}, \hat{y}, \hat{z}), (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathcal{C}$ with $(\hat{x}, \hat{y}, \hat{z}) \sim_{\mathcal{C}} (\tilde{x}, \tilde{y}, \tilde{z})$. The mapping $(x, y, z) \mapsto Mz$ is cone multi-linear, hence by Theorem 8.1.2, we have

$$\mu_C(M\hat{z}, M\tilde{z}) \leq L_{1,1}\mu_C(\hat{x}, \tilde{x}) + L_{1,2}\mu_C(\hat{y}, \tilde{y}) + L_{1,3}\mu_C(\hat{z}, \tilde{z}),$$

with

$$L_{1,1} \;=\; L_{1,2} \;=\; 0, \quad L_{1,3} = \max_{1\leq i,i',j,j'\leq n} \tanh\Big[\frac{1}{4}\ln\Big(\frac{M_{i,j}M_{i',j'}}{M_{i,j'}M_{i',j}}\Big)\Big],$$

where the bound $L_{1,3}$ can be derived either with Theorem 8.2.1 or via (5.8). The mapping $(x, y, z) \mapsto R(\cdot, x, z)$ is cone multi-linear, hence by Theorem 8.1.2, we have

$$\mu_C(R(\hat{x}, \cdot, \hat{z}), R(\tilde{x}, \cdot, \tilde{z})) \leq L_{2,1}\mu_C(\hat{x}, \tilde{x}) + L_{2,2}\mu_C(\hat{y}, \tilde{y}) + L_{2,3}\mu_C(\hat{z}, \tilde{z}),$$

with

$$L_{2,1} \leq \tanh(\ln(\Delta_{R,1})/4), \quad L_{2,2} = 0, \quad L_{2,3} \leq \tanh(\ln(\Delta_{R,3})/4),$$

where

$$\Delta_{R,1} = \max_{1\leq i,i',j,j',k,k'\leq n} \frac{R_{i,j,k}R_{i',j',k'} + R_{i,j,k'}R_{i',j',k}}{R_{i,j',k}R_{i',j,k'} + R_{i,j',k'}R_{i',j,k}},$$

and

$$\Delta_{R,3} = \max_{1\leq i,i',j,j',k,k'\leq n} \frac{R_{i,j,k}R_{i',j',k'} + R_{i,j',k}R_{i',j,k'}}{R_{i',j,k}R_{i,j',k'} + R_{i',j',k}R_{i,j,k'}}.$$

The expression for $\Delta_{R,1}$ follows by applying Theorem 8.2.1 to the tensor $\tilde{R} \in \mathbb{R}^{n\times n\times n}_{++}$ defined as $\tilde{R}_{i,j,k} = R_{j,i,k}$ for all $i, j, k = 1, \dots, n$, so that $R(x, \cdot, z) = \tilde{R}(\cdot, x, z)$ for all $x, z \in \mathbb{R}^n_+$ and thus

$$\sup_{z\in\mathbb{R}^n_+} \operatorname{diam}(R(\mathbb{R}^n_+, \cdot, z)) = \sup_{z\in\mathbb{R}^n_+} \operatorname{diam}(\tilde{R}(\cdot, \mathbb{R}^n_+, z)).$$

Similarly, the expression for $\Delta_{R,3}$ follows by applying Theorem 8.2.1 to the tensor $\hat{R} \in \mathbb{R}^{n\times n\times n}_{++}$ defined as $\hat{R}_{i,j,k} = R_{j,k,i}$ for all $i, j, k = 1, \ldots, n$, so that $R(x, \cdot, z) = \hat{R}(\cdot, z, x)$ for all $x, z \in \mathbb{R}^n_+$. The mapping $(x, y, z) \mapsto T(\cdot, x, y, z)$ is cone multi-linear, hence by Theorem 8.1.2 we have

$$\mu_C(T(\cdot, \hat{x}, \hat{y}, \hat{z}), T(\cdot, \tilde{x}, \tilde{y}, \tilde{z})) \leq L_{3,1}\mu_C(\hat{x}, \tilde{x}) + L_{3,2}\mu_C(\hat{y}, \tilde{y}) + L_{3,3}\mu_C(\hat{z}, \tilde{z}),$$

with

$$L_{3,1} = L_{3,2} = L_{3,3} \leq \tanh(\ln(\Delta_T)/4),$$

where

$$\Delta_T = \max_{\substack{1\leq i,i',j,j'\leq n\\ 1\leq k,k',l,l'\leq n}} \frac{T_{i,j,k,l}T_{i',j',k',l'} + T_{i,j,k',l}T_{i',j',k,l'} + T_{i,j,k,l'}T_{i',j',k',l} + T_{i,j,k',l'}T_{i',j',k,l}}{T_{i,j',k,l}T_{i',j,k',l'} + T_{i,j',k',l}T_{i',j,k,l'} + T_{i,j',k,l'}T_{i',j,k',l} + T_{i,j',k',l'}T_{i',j,k,l}}.$$

The equality $L_{3,1} = L_{3,2} = L_{3,3}$ follows from Remark 8.1.3 (d) and the expression for Δ_T follows from Theorem 8.2.1.

The above example shows that in order to obtain bounds for each entry of the Lipschitz matrix of a nonnegative tensor, it suffices to apply Theorem 8.2.1 on the tensors obtained by permuting the indexes. With this procedure, we directly obtain the following corollary of Theorem 8.2.1:

Corollary 8.2.5. Suppose that $d \geq 2$. Let $k, l \in \{1, \ldots, d\}, k \neq l$, $\mathcal{C} = \mathbb{R}^{n_1}_+ \times \ldots \times \mathbb{R}^{n_d}_+$, $K = \mathbb{R}^{n_k}_+$ and $T \in \mathbb{R}^{n_1\times\ldots\times n_d}_{++}$. Then it holds

$$\sup_{x\in\mathcal{C}} \operatorname{diam}(T(x_1, \ldots, x_{k-1}, \cdot, x_{k+1}, \ldots, x_{l-1}, \mathbb{R}^{n_l}_+, x_{l+1}, \ldots, x_d); \mu) \leq \ln(\Delta_{k,l}(T)) \tag{8.9}$$

where

$$\Delta_{k,l}(T) = \max_{j_1,j_1'\in[n_1],\ldots,j_d,j_d'\in[n_d]} \frac{\overline{\mathcal{T}}^{k,l}(j_1, j_1', \ldots, j_d, j_d')}{\underline{\mathcal{T}}^{k,l}(j_1, j_1', \ldots, j_d, j_d')} \tag{8.10}$$

with

$$\overline{\mathcal{T}}^{k,l}(j_1, j_1', \ldots, j_d, j_d') = \sum_{\substack{\sigma_1\in\mathfrak{S}(\{j_1,j_1'\})\\ \vdots\\ \sigma_d\in\mathfrak{S}(\{j_d,j_d'\})}} \begin{pmatrix} T_{\sigma_1(j_1),\ldots,\sigma_{k-1}(j_{k-1}),j_k,\sigma_{k+1}(j_{k+1}),\ldots,\sigma_{l-1}(j_{l-1}),j_l,\sigma_{l+1}(j_{l+1}),\ldots,\sigma_d(j_d)} \\ \cdot T_{\sigma_1(j_1'),\ldots,\sigma_{k-1}(j_{k-1}'),j_k',\sigma_{k+1}(j_{k+1}'),\ldots,\sigma_{l-1}(j_{l-1}'),j_l',\sigma_{l+1}(j_{l+1}'),\ldots,\sigma_d(j_d')} \end{pmatrix},$$

$$\underline{\mathcal{T}}^{k,l}(j_1, j_1', \ldots, j_d, j_d') = \sum_{\substack{\sigma_1\in\mathfrak{S}(\{j_1,j_1'\})\\ \vdots\\ \sigma_d\in\mathfrak{S}(\{j_d,j_d'\})}} \begin{pmatrix} T_{\sigma_1(j_1),\ldots,\sigma_{k-1}(j_{k-1}),j_k',\sigma_{k+1}(j_{k+1}),\ldots,\sigma_{l-1}(j_{l-1}),j_l,\sigma_{l+1}(j_{l+1}),\ldots,\sigma_d(j_d)} \\ \cdot T_{\sigma_1(j_1'),\ldots,\sigma_{k-1}(j_{k-1}'),j_k,\sigma_{k+1}(j_{k+1}'),\ldots,\sigma_{l-1}(j_{l-1}'),j_l',\sigma_{l+1}(j_{l+1}'),\ldots,\sigma_d(j_d')} \end{pmatrix}$$

and, for $i = 1, \ldots, d$, $\mathfrak{S}(\{j_i, j_i'\})$ is defined as in Theorem 8.2.1.

Remark 8.2.6. Note that if $T \in \mathbb{R}^{n_1 \times \ldots \times n_d}_{++}$, then for every $k, l \in \{1, \ldots, d\}, k \neq l$ with $\Delta_{k,l}(T)$ defined as in (8.10), it holds $\Delta_{k,l}(T) = \Delta_{l,k}(T)$. This observation is consistent with the fact that if $d = 2$, then $T = M \in \mathbb{R}^{n_1 \times n_2}_{++}$ is a matrix and $\kappa(M) = \kappa(M^\top)$ (see Theorem 5.2.7 and (5.8)).

In the particular case of a third order tensor $T \in \mathbb{R}^{n \times n \times n}_{++}$, there is a way to compute

$$\sup_{y \in \mathbb{R}^n_+} \operatorname{diam}(T(\cdot, \mathbb{R}^n_+, y); \mu_{\mathbb{R}^n_+}),$$

using the Sinkhorn-Knopp method [56]. Indeed, let $C = \mathbb{R}^n_+$, then by (8.6) and (8.8), we have

$$\sup_{y \in \mathbb{R}^n_+} \operatorname{diam}(T(\cdot, C, y); \mu_K) = \max_{\substack{1 \le i < i' \le n \\ 1 \le k < k' \le n}} \sup_{y \in \mathbb{R}^n_{++}} \frac{\sum_{s,s'=1}^n (T_{i,s,k} T_{i',s',k'} + T_{i,s',k} T_{i',s,k'}) y_s y_{s'}}{\sum_{s,s'=1}^n (T_{i',s,k} T_{i,s',k'} + T_{i',s',k} T_{i,s,k'}) y_s y_{s'}}.$$

The objective in the right hand side of the above optimization problem is the ratio of two symmetric quadratic forms. That is

$$\sup_{y \in \mathbb{R}^n_+} \operatorname{diam}(T(\cdot, C, y); \mu_K) = \max_{\substack{1 \le i < i' \le n \\ 1 \le k < k' \le n}} \sup_{y \in \mathbb{R}^n_{++}} \frac{y^\top B^{(i,k,i',k')} y}{y^\top A^{(i,k,i',k')} y},$$

where $A^{(i,k,i',k')}, B^{(i,k,i',k')} \in \mathbb{R}^{n \times n}_{++}$ are the symmetric matrices defined for every $s, s' = 1, \ldots, n$ as

$$B^{(i,k,i',k')}_{s,s'} = T_{i,s,k} T_{i',s',k'} + T_{i,s',k} T_{i',s,k'} \quad \text{and} \quad A^{(i,k,i',k')}_{s,s'} = T_{i',s,k} T_{i,s',k'} + T_{i',s',k} T_{i,s,k'}.$$

Proposition 8.2.7. Let $A, B \in \mathbb{R}^{n \times n}_{++}$ be symmetric matrices, then there exists a unique $u \in \mathbb{R}^n_{++}$ such that $D_u A D_u$ is a doubly stochastic matrix with $D_u = \operatorname{diag}(u)$. Furthermore, it holds

$$\sup_{y \in \mathbb{R}^n_{++}} \frac{y^\top B y}{y^\top A y} = \max_{1 \le s,s' \le n} u_s B_{s,s'} u_{s'}.$$

Proof. By the discussion in Example 5.2.10, there exists $u \in \mathbb{R}^n_{++}$ such that $D_u A D_u$ is doubly stochastic with $D_u = \operatorname{diag}(u)$. Let $\Sigma_{++} = \{x \in \mathbb{R}^n_+ \,|\, x_1 + \ldots + x_n = 1\}$. Then, for every $y \in \Sigma_{++}$, we have $y^\top D_u A D_u y = 1$. It holds $\mathbb{R}^n_{++} = \{D_u z \,|\, z \in \mathbb{R}^n_{++}\}$ and thus

$$\sup_{y \in \mathbb{R}^n_{++}} \frac{y^\top B y}{y^\top A y} = \sup_{y \in \mathbb{R}^n_{++}} \frac{y^\top D_u B D_u y}{y^\top D_u A D_u y} = \sup_{y \in \Sigma_{++}} y^\top D_u B D_u y.$$

Let $\|\cdot\|_1$ denote the ℓ^1 norm on $\mathbb{R}^n$, then

$$\sup_{y \in \Sigma_{++}} y^\top D_u B D_u y = \sup_{y \in \mathbb{R}^n_{++}} \frac{y^\top D_u B D_u y}{\|y\|_1^2} = \sup_{x,y \in \mathbb{R}^n_{++}} \frac{x^\top D_u B D_u y}{\|x\|_1 \|y\|_1}$$
$$= \max_{1 \le s,s' \le n} (D_u B D_u)_{s,s'} = \max_{1 \le s,s' \le n} u_s B_{s,s'} u_{s'},$$

where the second inequality follows from Theorem 1 in [7] and the before last equality follows from Theorem 1 in [62]. □

We believe that a similar idea can be used for tensors of higher order by solving the generalized Schrödinger equation discussed in Sections 4.2.2 and 11.2.5 and leave such generalization to future work.

9 Existence of positive eigenvectors

Theorem 7.3.1 combined with the results of Section 6.2.2 and/or the multi-linear Birkhoff-Hopf theorem 8.1.2 provide various conditions which ensure that a mapping has an eigenvector in the interior of a solid closed cone. However, all these result rely on the fact that the underlying mapping contracts under the vector valued Hilbert metric induced by the cone.

The purpose of this section is to prove results which guarantee the existence of a positive eigenvector of mappings which are not strict contractions, where by *positive eigenvector* we mean an eigenvector in the interior of the cone. In the context of nonnegative matrices, the classical assumption is that the matrix needs to be irreducible in order to have a positive eigenvector. Irreducibility is not a necessary condition for the existence of a positive eigenvector and we refer to Theorem 6.4.5 of [30] or Theorem 3.10 [11, Chapter 2] for a necessary and sufficient condition. However, irreducibility implies more properties than merely existence of a positive eigenvector such as uniqueness of the positive eigenvector. In facts, it is shown in Lemma 3.17 of [11, Chapter 1] that a linear mapping $L\colon C \to C$, where $C \subset V$ is a closed cone in the finite dimensional real vector space V is irreducible, i.e. it leaves no proper face of C invariant, if and only if L has exactly one (up to scalar multiples) eigenvector in C, and this vector is in $\operatorname{int}(C)$. Generalizing irreducibility to nonlinear mapping is a delicate task and it turns out that generalizing different characterizations of linear irreducible mappings lead to different results. In this section we consider two generalizations of irreducibility. The first one implies that all the eigenvectors of a multi-homogeneous mapping are in the interior of the cone and generalizes the following characterization of irreducible matrix: $M \in \mathbb{R}^{n\times n}_+$ is irreducible if and only if there exists $k \geq 1$ such that $(I+M)^k \in \mathbb{R}^{n\times n}_{++}$. The second one merely implies the existence of an eigenvector in the interior of the cone and generalizes the characterization of irreducible matrix which states that $M \in \mathbb{R}^{n\times n}_+$ is irreducible if and only if the graph whose adjacency matrix is M is strongly connected. The main motivation for these particular generalizations of irreducibility is that they respectively reduce to the definitions of irreducible and weakly irreducible tensors introduced in [31, 24, 26]. We shall point out that a condition characterizing the existence of an eigenvector in the interior of the cone $\mathbb{R}^n_+$ of order-preserving homogeneous mappings is discussed in [58, Theorem 5.1]. In Section 10.4, we discuss a third generalization of irreducibility which guarantees that a positive eigenvector is unique.

9.1 Irreducible mappings on the product of cones

We consider the following definition of irreducible mapping on the cone $\mathcal{C} \subset \mathcal{V}$ where $\mathcal{V} = V_1 \times \ldots \times V_d$ is the product of finite dimensional real vector spaces.

Definition 9.1.1. Let $\mathcal{C} \subset \mathcal{V}$ be a solid closed cone and let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be a mapping. We say that $\mathcal{F}$ is *irreducible*, if for every $x \in \mathcal{C}$ with $x_i \neq 0$ for all $i = 1, \ldots, d$, there exists a positive integer m_x such that $\mathcal{H}^{m_x}(x) \in \operatorname{int}(\mathcal{C})$, where $\mathcal{H}\colon \mathcal{C} \to \mathcal{C}$ is the mapping defined as $\mathcal{H}(x) = x + \mathcal{F}(x)$ for all $x \in \mathcal{C}$.

If $d = 1$, $C = \mathbb{R}^n_+$ and $\mathcal{F}(x) = Mx$ for some $M \in \mathbb{R}^{n\times n}_+$, then $\mathcal{F}$ is irreducible in the sense of Definition 9.1.1 if and only if M is an irreducible matrix and in this case $\mathcal{H}(x) = (I + M)x$ so that $\mathcal{H}^m(x) = (I + M)^m x$ for all $m \geq 1$.

A linear mapping $L\colon V \to V$, where V is a finite dimensional real vector space V, leaving a solid closed close cone $C \subset V$ invariant is irreducible if it does not leave any proper face of C invariant (see Definition 3.14 [11, Chapter 1]). It follows from Theorem 3.20 of [11, Chapter 1] that the latter definition is equivalent to Definition 9.1.1 in the case where $d = 1$ and $\mathcal{F}$ is a linear mapping. A similar equivalence holds for order-preserving and multi-subhomogeneous mapping and is proved in the following result which generalizes [40, Lemma 6.10 (v)]$^\diamond$:

Proposition 9.1.2. Let $\mathcal{C} \subset \mathcal{V}$ be a solid closed cone and let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$. Then, $\mathcal{F}$ is irreducible if, and only if, for every face $\mathcal{Q}$ of $\mathcal{C}$ such that $\mathcal{Q} \neq \mathcal{C}$ and $Q_i \neq \{0\}$ for all $i = 1, \ldots, d$, it holds $\mathcal{F}(\mathcal{Q}) \not\subset \mathcal{Q}$.

Proof. Let $\mathcal{H}\colon \mathcal{C} \to \mathcal{C}$ be defined as $\mathcal{H}(x) = x + \mathcal{F}(x)$ for all $x \in \mathcal{C}$. Let $\mathcal{Q}$ be a face of $\mathcal{C}$ such that $\mathcal{Q} \neq \mathcal{C}$ and $Q_i \neq \{0\}$ for all $i = 1, \ldots, d$. If $\mathcal{F}(\mathcal{Q}) \subset \mathcal{Q}$, then we have $\mathcal{H}(\mathcal{Q}) \subset \mathcal{Q}$ and thus $\mathcal{F}$ is not irreducible since $\mathcal{H}^m(x) \in \mathcal{Q}$ for all $m \geq 1$ and $\mathcal{Q} \neq \mathcal{C}$ implies that $\mathcal{Q} \cap \operatorname{int}(\mathcal{C}) = \emptyset$. We prove the reverse direction. For $z \in \mathcal{C}$, let $\mathcal{Q}(z) = \{y \in \mathcal{C} \,|\, \exists \boldsymbol{\alpha} \in \mathbb{R}^d_{++} \text{ s.t. } \boldsymbol{\alpha} \otimes y \preceq_{\mathcal{C}} z\}$ be the face of $\mathcal{C}$ generated by z. Now, let $x \in \mathcal{C}$ with $x_i \neq 0$ for all $i = 1, \ldots, d$. If $x \in \operatorname{int}(\mathcal{C})$, then $\mathcal{H}(x) \in \operatorname{int}(\mathcal{C})$ and we are done. Suppose that $x \notin \operatorname{int}(\mathcal{C})$. We have $\mathcal{Q}(x) \subset \mathcal{Q}(\mathcal{H}(x))$. As $\mathcal{F}(\mathcal{Q}(x)) \not\subset \mathcal{Q}(x)$ by assumption, we have $\mathcal{Q}(x) \subsetneq \mathcal{Q}(\mathcal{H}(x))$. For all $y \in \mathcal{C}$, it holds $\operatorname{span}(\mathcal{Q}(y)) = \mathcal{Q}(y) - \mathcal{Q}(y)$ since $\mathcal{Q}(y)$ is a cone in $\mathcal{V}$. It follows that $\mathcal{Q}(x) \subsetneq \mathcal{Q}(\mathcal{H}(x))$ implies $\dim(\operatorname{span}(\mathcal{Q}(x))) < \dim(\operatorname{span}(\mathcal{Q}(\mathcal{H}(x))))$. Now, for $k \geq 1$, we have $x \preceq_{\mathcal{C}} \mathcal{H}(x) \preceq_{\mathcal{C}} \mathcal{H}^2(x) \preceq_{\mathcal{C}} \ldots \preceq_{\mathcal{C}} \mathcal{H}^k(x)$ and thus $\mathcal{H}^k(x)_i \neq 0$ for all $i = 1, \ldots, d$. Hence, we can repeat the argument, and find $m \leq \dim(\mathcal{V}) - d$ such that $\operatorname{span}(\mathcal{Q}(\mathcal{H}^m(x))) = \mathcal{V}$. It follows that $\mathcal{H}^m(x) \in \operatorname{int}(\mathcal{C})$. Hence, $\mathcal{F}$ is irreducible. ☐

Remark 9.1.3. The proof of Proposition 9.1.2 shows that if $\mathcal{F}$ is irreducible, then the integer m_x of Definition 9.1.1 satisfies $m_x \leq \dim(\mathcal{V}) - d$.

The condition of Definition 9.1.1 is less restrictive than requiring that no proper face of $\mathcal{C}$ is left invariant by the mapping. Indeed, as shown in Example 9.1.4 below, there are mappings which are irreducible in the sense of Definition 9.1.1 and leave a proper face $\mathcal{Q}$ of $\mathcal{C}$ invariant. This does not contradict Proposition 9.1.2 since there exists i such that $Q_i = \{0\}$. The motivation behind Definition 9.1.1 is to prove that a multi-homogeneous irreducible mapping has all eigenvectors in the interior of the cone. Since an eigenvector x of $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ is required to satisfy $x_i \neq 0$ for all $i = 1, \ldots, d$ (see Definition 4.1.1), it turns out that the condition in Definition 9.1.1 is more appropriate than requiring that for all $y \in \mathcal{C} \setminus \{0\}$ there exists m_y such that $\mathcal{H}^{m_y}(y) \in \operatorname{int}(\mathcal{C})$, where $\mathcal{H}(z) = z + \mathcal{F}(z)$ for $z \in \mathcal{C}$, which is equivalent to the condition that $\mathcal{F}$ does not leave a proper face of $\mathcal{C}$ invariant.

Example 9.1.4. Let $S, T \in \mathbb{R}_{++}^{n\times n\times n}$, $\mathcal{C} = \mathbb{R}_+^n \times \mathbb{R}_+^n$ and $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ defined as

$$\mathcal{F}(x, y) = \big(T(\cdot, x, y), S(\cdot, x, y)\big).$$

Then, for all $(x, y) \in \mathcal{C}$ such that $x, y \neq 0$ we have $\mathcal{F}(x, y) \in \operatorname{int}(\mathcal{C})$ and thus $\mathcal{F}$ is irreducible. However, note that $\mathcal{F}(0, y) = \mathcal{F}(x, 0) = 0$ for all $(x, y) \in \mathcal{C}$. It follows that for every face Q of $\mathbb{R}_+^n$, $\mathcal{F}$ leave the proper faces $\{0\} \times Q$ and $Q \times \{0\}$ of $\mathcal{C}$ invariant.

Next, we prove that the eigenvectors of an irreducible mapping are all in the interior of the cone.

Proposition 9.1.5. Let $\mathcal{C} \subset \mathcal{V}$ be a solid closed cone and let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}$ and irreducible. Every eigenvector $u \in \mathcal{C}$ of $\mathcal{F}$ satisfies $u \in \operatorname{int}(\mathcal{C})$.

Proof. Let $\mathcal{H}\colon \mathcal{C} \to \mathcal{C}$ be defined as $\mathcal{H}(x) = x + \mathcal{F}(x)$ for all $x \in \mathcal{C}$. Furthermore, let $\boldsymbol{\lambda} \in \mathbb{R}_+^d$ be such that $\mathcal{F}(u) = \boldsymbol{\lambda} \otimes u$. We prove by induction over $k \geq 1$ that $\mathcal{H}^k(u) \preceq_{\mathcal{C}} \boldsymbol{\delta}_k \otimes u$ with $\boldsymbol{\delta}_k = \boldsymbol{\delta}_{k-1} + \boldsymbol{\lambda} \circ \boldsymbol{\delta}_{k-1}^A \in \mathbb{R}_{++}^d$ and $\boldsymbol{\delta}_1 = \mathbf{1} + \boldsymbol{\lambda} \in \mathbb{R}_{++}^d$. For $k = 1$, we have $\mathcal{H}(u) = (\mathbf{1} + \boldsymbol{\lambda}) \otimes u = \boldsymbol{\delta}_1 \otimes u$. Let $k \geq 1$ and suppose that $\mathcal{H}^k(u) \preceq_{\mathcal{C}} \boldsymbol{\delta}_k \otimes u$. Then,

$$\mathcal{F}(\mathcal{H}^k(u)) \preceq_{\mathcal{C}} \mathcal{F}(\boldsymbol{\delta}_k \otimes u) = \boldsymbol{\delta}_k^A \otimes \mathcal{F}(u) = (\boldsymbol{\delta}_k^A \circ \boldsymbol{\lambda}) \otimes u,$$

so that

$$\mathcal{H}^{k+1}(u) = \mathcal{H}^k(u) + \mathcal{F}(\mathcal{H}^k(u)) \preceq_{\mathcal{C}} \boldsymbol{\delta}_k \otimes u + (\boldsymbol{\delta}_k^A \circ \boldsymbol{\lambda}) \otimes u = \boldsymbol{\delta}_{k+1} \otimes u,$$

which concludes the induction. Finally, as $\mathcal{F}$ is irreducible, there exists a positive integer m such that $\mathcal{H}^m(u) \in \operatorname{int}(\mathcal{C})$. It follows that $u \in \operatorname{int}(\mathcal{C})$ since $\mathcal{H}^m(u) \preceq_{\mathcal{C}} \boldsymbol{\delta}_m \otimes u \in \operatorname{int}(\mathcal{C})$ and $\boldsymbol{\delta}_m \in \mathbb{R}_{++}^d$. □

The above results together with Theorem 4.1.5 imply that a continuous multi-homogeneous mapping always has a positive eigenvector.

Theorem 9.1.6. Let $\mathcal{C} \subset \mathcal{V}$ be a solid closed cone and let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be continuous, multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}$ and irreducible. There exists $u \in \operatorname{int}(\mathcal{C})$ and $\boldsymbol{\lambda} \in \mathbb{R}_+^d$ such that $\mathcal{F}(u) = \boldsymbol{\lambda} \otimes u$.

Proof. As $\mathcal{F}$ is continuous, Theorem 4.1.5 implies that there exists $u \in \mathcal{C}$ and $\boldsymbol{\lambda} \in \mathbb{R}_+^d$ such that $u_1, \ldots, u_d \neq 0$ and $\mathcal{F}(u) = \boldsymbol{\lambda} \otimes u$. Then, Proposition 9.1.5 implies that $u \in \operatorname{int}(\mathcal{C})$. □

We conclude by noting that when the cone $\mathcal{C}$ is polyhedral and the mapping defined on $\mathcal{C}$ is order-preserving and multi-homogeneous, irreducibility can be checked in a finite number of steps. This is proved in the next proposition which generalizes [97, Theorem 5.2] and [40, Lemma 6.10 (iii)]$^\diamond$. The latter results hold for the case where $\mathcal{F}$ is a polynomial mapping on $\mathcal{C} = \mathbb{R}_+^{n_1} \times \ldots \times \mathbb{R}_+^{n_d}$. Let us recall that for a positive integer N we let $[N] = \{1, \ldots, N\}$.

Proposition 9.1.7. Let $\mathcal{C} \subset \mathcal{V}$ be a solid polyhedral cone and for $i = 1, \ldots, d$, consider $v_{i,1}, \ldots, v_{i,N_i} \in C_i \setminus \{0\}$ such that $C_i = \operatorname{cone}(\{v_{i,1}, \ldots, v_{i,N_i}\})$. Furthermore, let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be order-preserving and multi-homogeneous and define $\mathcal{H}\colon \mathcal{C} \to \mathcal{C}$ as $\mathcal{H}(x) = x + \mathcal{F}(x)$ for all $x \in \mathcal{C}$. Then $\mathcal{F}$ is irreducible if and only if there exists a positive integer $m \le (\dim(\mathcal{V}) - d)$ such that $\mathcal{H}^m(v_{1,j_1}, \ldots, v_{d,j_d}) \in \operatorname{int}(\mathcal{C})$ for all $j_1 \in [N_1], \ldots, j_d \in [N_d]$.

Proof. Suppose that $\mathcal{F}$ is irreducible and let $j_1 \in [N_1], \ldots, j_d \in [N_d]$. Then, there exists an integer $m_{j_1,\ldots,j_d} \ge 1$ such that $\mathcal{H}^{m_{j_1,\ldots,j_d}}(v_{1,j_1}, \ldots, v_{d,j_d}) \in \operatorname{int}(\mathcal{C})$. By Remark 9.1.3, we know that $m_{j_1,\ldots,j_d} \le (\dim(\mathcal{V}) - d)$. Note that for all $x \in \operatorname{int}(\mathcal{C})$, it holds $\mathcal{H}(x) \in \operatorname{int}(\mathcal{C})$ since $x \preceq_{\mathcal{C}} \mathcal{H}(x)$. It follows that $m = \max_{j_1 \in [N_1],\ldots,j_d \in [N_d]} m_{j_1,\ldots,j_d}$ satisfies the desired property. Now, suppose that there exists a positive integer $m \le (\dim(\mathcal{V}) - d)$ such that $\mathcal{H}^m(v_{1,j_1}, \ldots, v_{d,j_d}) \in \operatorname{int}(\mathcal{C})$ for all $j_1 \in [N_1], \ldots, j_d \in [N_d]$. Let $x \in \mathcal{C}$ with $x_i \neq 0$ for all $i = 1, \ldots, d$. For $i \in \{1, \ldots, d\}$, let $\alpha_{i,1}, \ldots, \alpha_{1,N_i} \ge 0$ be such that $x_i = \sum_{l_i=1}^{N_i} \alpha_{i,l_i} v_{i,l_i}$. The coefficients $\alpha_{i,1}, \ldots, \alpha_{1,N_i}$ exist since $C_i = \operatorname{cone}(\{v_{i,1}, \ldots, v_{i,N_i}\})$ by assumption. Furthermore, as $x_i \neq 0$, there exists $j_i \in [N_i]$ such that $\alpha_{i,j_i} > 0$ and it holds $\alpha_{i,j_i} v_{i,j_i} \preceq_{C_i} x_i$. Hence, with $\alpha_* = \sum_{i=1}^d \alpha_{i,j_i}$, $\boldsymbol{\alpha} = (\alpha_{1,j_1}, \ldots, \alpha_{d,j_d})/\alpha_*$ and $v = (v_{1,j_1}, \ldots, v_{d,j_d})$ we have $\boldsymbol{\alpha} \otimes v \preceq_{\mathcal{C}} x$ and $\boldsymbol{\alpha} \in (0,1)^d$. Now, as $\mathcal{F}$ is multi-homogeneous there exists $A \in \mathbb{R}_+^{d \times d}$ such that $\boldsymbol{\beta}^A \otimes \mathcal{F}(y) = \mathcal{F}(\boldsymbol{\beta} \otimes y)$ for all $\boldsymbol{\beta} \in (0,1)^d$ and $y \in \mathcal{C}$. Note that $\mathcal{H}$ is order-preserving since $\mathcal{F}$ is order-preserving. It follows that $\mathcal{H}^m(\boldsymbol{\alpha} \otimes v) \preceq_{\mathcal{C}} \mathcal{H}^m(x)$. To conclude the proof, we show that there exists $\boldsymbol{\alpha}_m \in (0,1)^d$ such that $\boldsymbol{\alpha}_m \otimes \mathcal{H}^m(v) \preceq_{\mathcal{C}} \mathcal{H}^m(\boldsymbol{\alpha} \otimes v)$. As $\mathcal{H}^m(v) \in \operatorname{int}(\mathcal{C})$ this will imply that $\mathcal{H}^m(\boldsymbol{\alpha} \otimes v) \in \operatorname{int}(\mathcal{C})$ and therefore $\mathcal{H}^m(x) \in \operatorname{int}(\mathcal{C})$ since $\mathcal{H}^m$ is order-preserving. To construct $\boldsymbol{\alpha}_m \in (0,1)^d$ we show by induction that for every positive integer $k \ge 1$ there exists $\boldsymbol{\alpha}_k \in (0,1)^d$ such that $\boldsymbol{\alpha}_k \otimes \mathcal{H}^k(v) \preceq_{\mathcal{C}} \mathcal{H}^k(\boldsymbol{\alpha} \otimes v)$. For $k = 1$, let $\boldsymbol{\alpha}_1 = \inf\{\boldsymbol{\alpha}, \boldsymbol{\alpha}^A\}$, where the infimum is taken with respect to the partial ordering induced by $\mathbb{R}_+^d$, i.e. component wise. The multi-homogeneity of $\mathcal{F}$ implies that

$$\begin{aligned}\boldsymbol{\alpha}_1 \otimes \mathcal{H}(v) = \boldsymbol{\alpha}_1 \otimes v + \boldsymbol{\alpha}_1 \otimes \mathcal{F}(v) &\preceq_{\mathcal{C}} \boldsymbol{\alpha} \otimes v + \boldsymbol{\alpha}^A \otimes \mathcal{F}(v)\\ &= \boldsymbol{\alpha} \otimes v + \mathcal{F}(\boldsymbol{\alpha} \otimes v) = \mathcal{H}(\boldsymbol{\alpha} \otimes v).\end{aligned}$$

Suppose that for $k \ge 1$, there exists $\boldsymbol{\alpha}_k \in (0,1)^d$ such that $\boldsymbol{\alpha}_k \otimes \mathcal{H}^k(v) \preceq_{\mathcal{C}} \mathcal{H}^k(\boldsymbol{\alpha} \otimes v)$ and let $\boldsymbol{\alpha}_{k+1} = \inf\{\boldsymbol{\alpha}_k, \boldsymbol{\alpha}_k^A\}$. Then, we have

$$\begin{aligned}\boldsymbol{\alpha}_{k+1} \otimes \mathcal{H}^{k+1}(v) &= \boldsymbol{\alpha}_{k+1} \otimes \mathcal{H}^k(v) + \boldsymbol{\alpha}_{k+1} \otimes \mathcal{F}\big(\mathcal{H}^k(v)\big)\\ &\preceq_{\mathcal{C}} \boldsymbol{\alpha}_k \otimes \mathcal{H}^k(v) + \boldsymbol{\alpha}_k^A \otimes \mathcal{F}\big(\mathcal{H}^k(v)\big)\\ &= \boldsymbol{\alpha}_k \otimes \mathcal{H}^k(v) + \mathcal{F}\big(\boldsymbol{\alpha}_k \otimes \mathcal{H}^k(v)\big)\\ &\preceq_{\mathcal{C}} \mathcal{H}^k(\boldsymbol{\alpha} \otimes v) + \mathcal{F}(\mathcal{H}^k(\boldsymbol{\alpha} \otimes v)) = \mathcal{H}^{k+1}(\boldsymbol{\alpha} \otimes v),\end{aligned}$$

which concludes the inductive proof. As $\mathcal{H}^m(v) \in \operatorname{int}(\mathcal{C})$ and $\boldsymbol{\alpha}_m \otimes \mathcal{H}^m(v) \preceq_{\mathcal{C}} \mathcal{H}^m(\boldsymbol{\alpha} \otimes v) \preceq_{\mathcal{C}} \mathcal{H}^m(x)$ this concludes the proof. □

9.2 Irreducible nonnegative tensors

We relate Definition 9.1.1 with the definitions of irreducible tensors which were introduced in the study of ℓ^p-eigenvectors, rectangular $\ell^{p,q}$-singular vectors and

$\ell^{p_1,\ldots,p_d}$-singular vectors of a nonnegative tensor (see Sections 4.2.4, 4.2.3 and 4.2.1 respectively).

The next lemma implies that the irreducibility of the mappings characterizing the $\ell^{p,q}$-singular values of a nonnegative matrix and the ℓ^p-eigenvectors, the rectangular $\ell^{p,q}$-singular vectors and the $\ell^{p,q,r}$-singular vectors of a nonnegative tensor do not depend on $p,q,r \in (1,\infty)$ and is entirely determined by the zero pattern of the entries of the tensor.

Lemma 9.2.1. Let $\mathcal{C} \subset \mathcal{V}$ be a solid closed cone. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ and $\mathcal{G}\colon \mathcal{C} \to \mathcal{C}$ be such that for all $x \in \mathcal{C}$ it holds $\mathcal{F}(x) \sim_{\mathcal{C}} \mathcal{G}(x)$. Then, $\mathcal{F}$ is irreducible if and only if $\mathcal{G}$ is irreducible.

Proof. Let $\mathcal{H}\colon \mathcal{C} \to \mathcal{C}$ and $\tilde{\mathcal{H}}\colon \mathcal{C} \to \mathcal{C}$ be defined as $\mathcal{H}(x) = x + \mathcal{F}(x)$ and $\tilde{\mathcal{H}}(x) = x + \mathcal{G}(x)$ for all $x \in \mathcal{C}$. We have $\mathcal{H}(x) \sim_{\mathcal{C}} \tilde{\mathcal{H}}(x)$ for all $x \in \mathcal{C}$. It follows that for all $m \geq 1$ and $x \in \mathcal{C}$, $\mathcal{H}^m(x) \sim_{\mathcal{C}} \tilde{\mathcal{H}}^m(x)$. In particular, we have $\mathcal{H}^m(x) \in \operatorname{int}(\mathcal{C})$ if and only if $\tilde{\mathcal{H}}^m(x) \in \operatorname{int}(\mathcal{C})$ which shows that $\mathcal{F}$ is irreducible if and only if $\mathcal{G}$ is irreducible. ☐

Corollary 9.2.2. Let $\mathcal{C} = \mathbb{R}^{n_1}_+ \times \ldots \times \mathbb{R}^{n_d}_+$ and $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$. Let $a_1,\ldots,a_d > 0$ and define $\hat{\mathcal{F}}\colon \mathcal{C} \to \mathcal{C}$ as $\mathcal{F}(x) = \big(\mathcal{F}(x)_1^{a_1},\ldots,\mathcal{F}(x)_d^{a_d}\big)$ for all $x \in \mathcal{C}$. If $\mathcal{F}$ is irreducible, then $\hat{\mathcal{F}}$ is irreducible.

Proof. For $i = 1,\ldots,d$, as $a_i > 0$, we have $x_i \sim_{C_i} x_i^{a_i}$ for all $x_i \in C_i$. It follows that, for all $x \in \mathcal{C}$, we have $x \sim_{\mathcal{C}} (x_1^{a_1},\ldots,x_d^{a_d})$ and thus $\mathcal{F}(x) \sim_{\mathcal{C}} \hat{\mathcal{F}}(x)$. The conclusion follows form Lemma 9.2.1. ☐

The following lemma implies that Definition 9.1.1 is equivalent to the definition of irreducible tensors introduced in [31] when $\mathcal{F}$ is the mapping characterizing the $\ell^{p_1,\ldots,p_d}$-singular values of a nonnegative tensor.

Lemma 9.2.3. Let $\mathcal{C} = \mathbb{R}^{n_1}_+ \times \ldots \times \mathbb{R}^{n_d}_+$, $T \in \mathbb{R}^{n_1\times\ldots\times n_d}_+$ and $p_1,\ldots,p_d \in (1,\infty)$. Define $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ as

$$\mathcal{F}(x) = \big(T(\cdot,x_2,\ldots,x_d)^{1/(p_1-1)},\ldots,T(x_1,\ldots,x_{d-1},\cdot)^{1/(p_d-1)}\big) \qquad \forall x \in \mathcal{C},$$

and let $\mathcal{I} = \cup_{i=1}^d(\{i\} \times [n_i])$. Then, the following statements are equivalent

a) $\mathcal{F}$ is irreducible.

b) For every $\mathcal{J} \subset \mathcal{I}$ such that $\mathcal{J} \neq \emptyset$ and $J_i \neq [n_i]$ for all $i = 1,\ldots,d$ where $J_i = \{l_i \in [n_i] \,|\, (i,l_i) \in \mathcal{J}\}$, the following condition holds: There exists $k \in \{1,\ldots,d\}$ and $j_1 \in [n_1],\ldots,j_d \in [n_d]$ such that $T_{j_1,\ldots,j_d} > 0$, $j_k \in J_k$ and $j_i \in [n_i] \setminus J_i$ for all $i \in \{1,\ldots,d\} \setminus \{k\}$.

Proof. For $x \in \mathcal{C}$, let $\mathcal{Q}(x) = \{y \in \mathcal{C} \,|\, \exists \boldsymbol{\alpha} \in \mathbb{R}^d_{++} \text{ s.t. } \boldsymbol{\alpha} \otimes y \preceq_{\mathcal{C}} x\}$. Suppose that $\mathcal{F}$ is irreducible. By Corollary 9.2.2, we may assume that $p_i = 2$ for $i = 1,\ldots,d$. Let $\mathcal{J} \subset \mathcal{I}$ be such that $\mathcal{J} \neq \emptyset$ and $J_i = \{l_i \in [n_i] \,|\, (i,l_i) \in \mathcal{J}\} \neq [n_i]$ for all $i = 1,\ldots,d$. Define $z \in \mathcal{C}$ as $z_{i,j_i} = 0$ if $(i,j_i) \in \mathcal{J}$ and $z_{i,j_i} = 1$ otherwise. By Proposition 9.1.2, we know that $\mathcal{Q}(\mathcal{F}(z)) \not\subset \mathcal{Q}(z)$. Hence, there exists $(k, l_k) \in \mathcal{I}$ such that $\mathcal{F}(z)_{k,l_k} > 0$ and $z_{k,l_k} = 0$. Now, $\mathcal{F}(z)_{k,l_k} > 0$ implies the existence of $j_1, \ldots, j_d$ such that

$$T_{j_1,\ldots,j_d} z_{1,j_1} \cdots z_{k-1,j_{k-1}} z_{k+1,j_{k+1}} \cdots z_{d,j_d} > 0.$$

It follows that $T_{j_1,\ldots,j_d} > 0$ and $z_{i,j_i} > 0$ for all $i \in \{1, \ldots, d\} \setminus \{k\}$. As $z_{i,j_i} > 0$ implies that $(i, j_i) \notin \mathcal{J}$, this proves b). Now, suppose that b) holds. Again, by Corollary 9.2.2, to prove that $\mathcal{F}$ is irreducible, it is enough to consider the case $p_1 = \ldots = p_d = 2$. Let $\mathcal{Q}$ be a face of $\mathcal{C}$ such that $\mathcal{Q} \neq \operatorname{int}(\mathcal{C})$ and $Q_i \neq \{0\}$ for all $i = 1, \ldots, d$ and set $\mathcal{J} = \{(i, j_i) \,|\, x_{i,j_i} = 0, \forall x \in \mathcal{Q}\}$. Then, $\mathcal{J} \neq \emptyset$ and $J_i \neq [n_i]$ for all $i = 1, \ldots, d$ where $J_i = \{l_i \in [n_i] \,|\, (i, l_i) \in \mathcal{J}\}$. Hence, there exists $k \in \{1, \ldots, d\}$ and $j_1 \in [n_1], \ldots, j_d \in [n_d]$ such that $T_{j_1,\ldots,j_d} > 0$, $j_k \in J_k$ and $j_i \in [n_i] \setminus J_i$ for all $i \in \{1, \ldots, d\} \setminus \{k\}$. Furthermore, as $\mathcal{Q}$ is a face, there exists $z \in \mathcal{Q}$ such that $z_{i,l_i} = 0$ if and only if $(i, l_i) \in \mathcal{J}$ (z is any element of the relative interior of $\mathcal{Q}$). In particular, we have $z_{i,j_i} > 0$ for all $i \neq k$. It follows that

$$\mathcal{F}(z)_{k,j_k} \geq T_{j_1,\ldots,j_d} z_{1,j_1} \cdots z_{k-1,j_{k-1}} z_{k+1,j_{k+1}} \cdots z_{d,j_d} > 0.$$

As $x_{k,j_k} = 0$ for all $x \in \mathcal{Q}$, we have $\mathcal{F}(\mathcal{Q}) \not\subset \mathcal{Q}$. Finally, Proposition 9.1.2 implies that $\mathcal{F}$ is irreducible. □

The following lemma implies that Definition 9.1.1 is equivalent to the definition of irreducible tensors introduced in [24] when $\mathcal{F}$ is the mapping characterizing the ℓ^p-eigenvectors of a nonnegative tensor.

Lemma 9.2.4. Let $\mathcal{C} = \mathbb{R}^n_+$, $T \in \mathbb{R}^{n\times\ldots\times n}_+$ an m-th order tensor, and $p \in (1, \infty)$. Define $\mathcal{F} \colon \mathcal{C} \to \mathcal{C}$ as

$$\mathcal{F}(x) = T(\cdot, x, \ldots, x)^{1/(p-1)} \qquad \forall x \in \mathcal{C},$$

and let $\mathcal{I} = \{1, \ldots, n\}$. Then, the following statements are equivalent

a) $\mathcal{F}$ is irreducible.

b) For every $\mathcal{J} \subset \mathcal{I}$ such that $\mathcal{J} \neq \emptyset$ and $\mathcal{J} \neq \mathcal{I}$, the following condition holds: There exists $j_1, \ldots, j_d \in \mathcal{I}$ such that $T_{j_1,\ldots,j_d} > 0$, $j_1 \notin \mathcal{J}$ and $j_2, \ldots, j_d \in \mathcal{J}$.

Proof. By Corollary 9.2.2, we may assume that $p = 2$. The result now follows from Theorem 6.1 in [98]. □

Next we prove that Definition 9.1.1 is less restrictive than the definition of irreducible tensors introduced in [26] when $\mathcal{F}$ is the mapping characterizing the rectangular $\ell^{p,q}$-singular vectors of a nonnegative tensor.

Lemma 9.2.5. Let $\mathcal{C} = \mathbb{R}^m_+ \times \mathbb{R}^n_+$, $T \in \mathbb{R}^{m\times\ldots\times m\times n\ldots\times n}_+$ a d-th order tensor, and $p, q \in (1, \infty)$. Let $\mathcal{F} \colon \mathcal{C} \to \mathcal{C}$ be defined as

$$\mathcal{F}(x, y) = \big(T(\cdot, \underbrace{x, \ldots, x}_{(a-1)\text{ times}}, \underbrace{y, \ldots, y}_{(d-a)\text{ times}})^{1/(p-1)}, T(\underbrace{x, \ldots, x}_{a\text{ times}}, \cdot, \underbrace{y, \ldots, y}_{(d-a-1)\text{ times}})^{1/(q-1)}\big)$$

for all $(x, y) \in \mathcal{C}$, and let $\mathcal{I}_1 = \{1, \ldots, m\}, \mathcal{I}_2 = \{1, \ldots, n\}$. Consider the following statements:

a) $\mathcal{F}$ is irreducible.

b) For every $i_1 \in \mathcal{I}_1, i_2 \in \mathcal{I}_2$ and $\mathcal{J}_1 \subset \mathcal{I}_1, \mathcal{J}_2 \subset \mathcal{I}_2$ such that $\mathcal{J}_1, \mathcal{J}_2 \neq \emptyset$ and $\mathcal{J}_1 \neq \mathcal{I}_1, \mathcal{J}_2 \neq \mathcal{I}_2$, there exists $j_1, \ldots, j_a \in \mathcal{I}_1$ and $j_{a+1}, \ldots, j_d \in \mathcal{I}_2$ such that $T_{i_1,\ldots,i_1,j_{a+1},\ldots,j_d} > 0$, $T_{j_1,\ldots,j_a,i_2,\ldots,i_2} > 0$, $j_2, \ldots, j_a \in \mathcal{J}_1, j_{a+2}, \ldots, j_d \in \mathcal{J}_2$, $j_1 \notin \mathcal{J}_1$, $j_{a+1} \notin \mathcal{J}_2$.

Then, b) implies a).

Proof. Follows from Theorem 2.4 in [99]. □

The following example shows that, in Lemma 9.2.5, a) does not imply b).

Example 9.2.6. Let $T \in \mathbb{R}_+^{3\times3\times3}$ be defined as

$$T_{1,1,1} = T_{1,1,2} = T_{1,1,3} = T_{1,3,1} = T_{2,1,2} = T_{2,1,3} = T_{2,2,1} = T_{3,1,2} = T_{3,1,3} = 1,$$

and $T_{i,j,k} = 0$ otherwise. Let $\mathcal{C} = \mathbb{R}_+^3 \times \mathbb{R}_+^3$ and define $\mathcal{F} \colon \mathcal{C} \to \mathcal{C}$ as

$$\mathcal{F}(x, y) = \big(T(\cdot, y, y), T(x, \cdot, y)\big) \qquad \forall (x, y) \in \mathcal{C}.$$

Then, for every $x = (x_1, x_2, x_3)^\top, y = (y_1, y_2, y_3)^\top \in \mathbb{R}_+^3$, it holds

$$\mathcal{F}\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = \left(\begin{pmatrix} y_1(y_1 + y_2 + 2y_3) \\ y_1(2y_2 + y_3) \\ y_1(y_2 + y_3) \end{pmatrix}, \begin{pmatrix} x_1(y_1 + y_2 + y_3) + (x_2 + x_3)(y_2 + y_3) \\ x_2 y_1 \\ x_1 y_1 \end{pmatrix}\right)$$

In particular $\mathcal{F}$ is irreducible. However, for $x = (1, 0, 0)^\top$ and $y = (0, 1, 0)^\top$ it holds $\mathcal{F}(x, y) = \big((0, 0, 0)^\top, (1, 0, 0)^\top\big)$ and thus, by Lemma 2 in [26], T does not satisfy b).

We refer to [40, Section 6.4]$^\diamond$ for a detailed discussion on irreducibility of polynomial mappings induced by a nonnegative tensors, i.e. mappings of the form

$$\mathcal{F}(x) = \nabla f_T(\underbrace{x_1, \ldots, x_1}_{a_1 \text{ times}}, \ldots, \underbrace{x_d, \ldots, x_d}_{a_d \text{ times}}),$$

where $T \in \mathbb{R}_+^{\tilde{n}_1 \ldots \times \tilde{n}_m}$ and $m = a_1 + \ldots + a_d$.

9.3 Weakly irreducible mappings

In [34], a definition of graphs associated to order-preserving 1-homogeneous mapping on $\mathbb{R}_+^n$ is introduced. It is proved that when the graph is strongly connected, the mapping has a positive eigenvector. This result was then reused to prove the existence of positive ℓ^p-eigenvectors [31], rectangular $\ell^{p,q}$-singular vectors [65] and $\ell^{p_1,\ldots,p_d}$-singular vectors [36]$^\diamond$, [31] of nonnegative tensors. The nonnegative tensors for which the result can be applied are called weakly irreducible tensors.

Let $\mathcal{V} = \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_d}$ and $\mathcal{C} = \mathbb{R}_+^{n_1} \times \ldots \times \mathbb{R}_+^{n_d}$. We introduce the definition of a graph induced by an order-preserving multi-homogeneous mapping with a similar approach as [34]. For the definition of this graph, consider, for all $i = 1,\dots,d, j_i = 1,\dots,n_i$, the mapping $u^{(i,j_i)}\colon \mathbb{R}_+ \to \mathcal{C}$ defined as

$$\big(u^{(i,j_i)}(t)\big)_{k,l_k} = \begin{cases} t & \text{if } (k,l_k) = (i,j_i) \\ 1 & \text{otherwise,} \end{cases} \qquad \forall (k,l_k) \in \mathcal{I} \; = \bigcup_{\nu=1}^{d} \big(\{\nu\} \times [n_\nu]\big). \quad (9.1)$$

Then, the graph associated to an order-preserving multi-homogeneous mapping is given by the following:

Definition 9.3.1. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be order-preserving and multi-homogeneous. The *directed graph induced by* $\mathcal{F}$, denoted $\mathsf{G}(\mathcal{F}) = (\mathcal{I},\mathcal{E})$, is defined as follows: There is an edge from (k,l_k) to (i,j_i), i.e. $\big((k,l_k),(i,j_i)\big) \in \mathcal{E}$, if

$$\lim_{t\to\infty} \mathcal{F}\big(u^{(i,j_i)}(t)\big)_{k,l_k} = \infty.$$

If $d = 1$, then the graph of Definition 9.3.1 reduces to that introduced in [34]. Moreover, if $\mathcal{F}(x) = Mx$ for some nonnegative matrix $M \in \mathbb{R}^{n\times n}_+$, then $\mathsf{G}(\mathcal{F}) = (\{1\}\times[n],\mathcal{E})$ is the graph with M as adjacency matrix. If $G(x,y) = (My, M^\top x)$, then $\mathsf{G}(G) = ((\{1\}\times[m])\cup(\{2\}\times[n]),\mathcal{E})$ is the graph with $\begin{pmatrix} 0 & M \\ M^\top & 0\end{pmatrix}$ as adjacency matrix. Weakly irreducible mappings are then defined as follows:

Definition 9.3.2. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be order-preserving and multi-homogeneous, and let $\mathsf{G}(\mathcal{F}) = (\mathcal{I},\mathcal{E})$ be its associated graph. $\mathcal{F}$ is said to be weakly irreducible if $\mathcal{F}(\mathrm{int}(\mathcal{C})) \subset \mathrm{int}(\mathcal{C})$ and for all $\nu \in [d]$, $l_\nu \in [n_\nu]$ and $(j_1,\dots,j_d) \in [n_1]\times\ldots\times[n_d]$, there exists $i_\nu \in [d]$ so that there is a path from (i_ν, j_{i_ν}) to (ν, l_ν) in $\mathsf{G}(\mathcal{F})$,

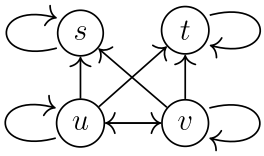
The condition in Definition 9.3.2 is equivalent to requiring that $\mathsf{G}(\mathcal{F})$ is strongly connected when $d = 1$. Generally, if $\mathsf{G}(\mathcal{F})$ is strongly connected, then $\mathsf{G}(\mathcal{F})$ satisfies the assumption of Definition 9.3.2 but the converse implication may fail when $d > 1$, as shown by the mapping of Example 9.3.3 below. The condition $\mathcal{F}(\mathrm{int}(\mathcal{C})) \subset \mathrm{int}(\mathcal{C})$ for weak irreducibility is meaningful as our goal is to prove the existence of a positive eigenvector. Indeed, if $\mathcal{F}$ is order-preserving, multi-homogeneous and has an eigenvector $u \in \mathrm{int}(\mathcal{C})$ then $\mathcal{F}(\mathrm{int}(\mathcal{C})) \subset \mathrm{int}(\mathcal{C})$. $\mathcal{F}(\mathrm{int}(\mathcal{C})) \subset \mathrm{int}(\mathcal{C})$ is implied by the strong connectivity of $\mathsf{G}(\mathcal{F})$ but not by the condition on $\mathsf{G}(\mathcal{F})$ discussed in Definition 9.3.2. Furthermore, this condition is necessary to prove that for the particular case where $\mathcal{F}$ is the mapping characterizing rectangular $\ell^{p,q}$-singular vectors of tensors, Definition 9.3.2 reduces to the corresponding definition of the literature (see Lemma 9.4.5).

Example 9.3.3. Let $n_1 = n_2 = 2$ and $F \in \mathcal{H}^2$ with

$$F((s,t),(u,v)) = \Big(\begin{pmatrix} \min\{su,sv\} \\ \min\{tu,tv\}\end{pmatrix}^{1/4}, \begin{pmatrix} \max\{su,tv\} \\ \max\{sv,tu\}\end{pmatrix}^{1/4} \Big)$$

Then, F is multi-homogeneous of degree $A = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix}$, $F(\mathbf{1},\mathbf{1}) = (\mathbf{1},\mathbf{1})$ and $\mathsf{G}(F)$

is given by



Next we show that if $\mathsf{G}(\mathcal{F}) = (\mathcal{I}, \mathcal{E})$ is undirected, then $\mathsf{G}(\mathcal{F})$ satisfies the connectivity assumption of Definition 9.3.2 if and only if $\mathsf{G}(\mathcal{F})$ is strongly connected.

Lemma 9.3.4. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be order-preserving and multi-homogeneous and let $\mathsf{G}(\mathcal{F}) = (\mathcal{I}, \mathcal{E})$ be its associated graph. If for every $(i, j_i), (\nu, l_\nu) \in \mathcal{I}$, $\big((i, j_i), (\nu, l_\nu)\big) \in \mathcal{E}$ implies $\big((\nu, l_\nu), (i, j_i)\big) \in \mathcal{E}$, then $\mathcal{F}$ is weakly irreducible if and only if $\mathsf{G}(\mathcal{F})$ is strongly connected.

Proof. Let $A \in \mathbb{R}^{d\times d}$ be the homogeneity matrix of $\mathcal{F}$. If $\mathsf{G}(\mathcal{F})$ is strongly connected, then $\mathsf{G}(\mathcal{F})$ satisfies the condition of Definition 9.3.2. Furthermore, as $\mathsf{G}(\mathcal{F})$ is strongly connected, for each $(k, l_k) \in \mathcal{I}$ there exists $(i, j_i) \in \mathcal{I}$ such that $\big((k, l_k), (i, j_i)\big) \in \mathcal{E}$ and therefore there exists $t > 0$ large enough so that $\mathcal{F}(u^{(i,j_i)}(t))_{k,l_k} > 0$. As $u^{(i,j_i)}(t) \in \operatorname{int}(\mathcal{C})$, for every $x \in \operatorname{int}(\mathcal{C})$ there exists $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$ such that $u^{(i,j_i)}(t) \leq \boldsymbol{\alpha} \otimes x$. It follows that $\mathcal{F}(u^{(i,j_i)}(t))_{k,l_k} \preceq_{\mathcal{C}} \mathcal{F}(\boldsymbol{\alpha} \otimes x)_{k,l_k} = \boldsymbol{\alpha}^A \otimes \mathcal{F}(x)_{k,l_k}$ since $\mathcal{F}$ is order-preserving and multi-homogeneous, and thus $\mathcal{F}(x)_{k,l_k} > 0$. As the latter is true for every $(k, l_k) \in \mathcal{I}$, it follows that $\mathcal{F}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$ and thus $\mathcal{F}$ is weakly irreducible. Now, suppose that $\mathcal{F}$ is weakly irreducible and let $(\nu, l_\nu), (\tilde\nu, l_{\tilde\nu}) \in \mathcal{I}$. Furthermore, let $(j_1, \ldots, j_d) \in [n_1] \times \ldots \times [n_d]$. As $\mathcal{F}$ is weakly irreducible, there exists $i_\nu, i_{\tilde\nu} \in [d]$ such that there is a path from (i_ν, j_{i_ν}) to (ν, l_ν) and a path from $(i_{\tilde\nu}, j_{i_{\tilde\nu}})$ to $(\tilde\nu, l_{\tilde\nu})$ in $\mathcal{E}$. If $i_\nu = i_{\tilde\nu}$, then there is a path from (ν, l_ν) to $(\tilde\nu, l_{\tilde\nu})$ since the paths can be walked back by assumption. If $i_\nu \neq i_{\tilde\nu}$, note that, by weak irreducibility of $\mathcal{F}$, for all $k = 1, \ldots, d$, there is a path from one of $(1, j_1), \ldots, (d, j_d)$ to (k, j_k). As every path can be walked in both directions, it follows that for every $i, k \in [d]$ there is a path from (i, j_i) to (k, j_k). In particular, there is a path from (i_ν, j_{i_ν}) to $(i_{\tilde\nu}, j_{i_{\tilde\nu}})$. Hence, there is a path from (ν, l_ν) to $(\tilde\nu, l_{\tilde\nu})$ which shows that $\mathsf{G}(\mathcal{F})$ is strongly connected. □

We prove the following generalization of [34, Theorem 2] which is discussed in [39, Theorem 4.3]$^\diamond$:

Theorem 9.3.5. Let $\mathcal{C} = \mathbb{R}^{n_1}_+ \times \ldots \times \mathbb{R}^{n_d}_+$ and let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be continuous, order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}_+$. Suppose that A is irreducible and $\rho(A) = 1$. If $\mathcal{F}$ is weakly irreducible, then $\mathcal{F}$ has an eigenvector in $\operatorname{int}(\mathcal{C})$.

For the proof of Theorem 9.3.5, we need the following lemma:

Lemma 9.3.6. Let $\mathcal{C} = \mathbb{R}^{n_1}_+ \times \ldots \times \mathbb{R}^{n_d}_+$ and let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be continuous, order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}_+$. Let $w \in \operatorname{int}(\mathcal{C}^*)$, $\boldsymbol{\xi}_w(x) = (w_1(x), \ldots, w_d(x_d)) \in \mathbb{R}^d_+$ for all $x \in \mathcal{C}$ and define $\boldsymbol{\Sigma}_w = \{x \in \mathcal{C} \,|\, \boldsymbol{\xi}_w(x) = \mathbf{1}\}$. For every $\epsilon > 0$, define the mapping $\mathcal{F}^{(\epsilon)}\colon \mathcal{C} \to \mathcal{C}$ as

$$\mathcal{F}^{(\epsilon)}(x) = \mathcal{F}(x) + \epsilon\, \boldsymbol{\xi}_w(x)^A \otimes \mathbf{1} \qquad \forall x \in \mathcal{C}. \tag{9.2}$$

Then, $\mathcal{F}^{(\epsilon)}$ is continuous, order-preserving and multi-homogeneous of degree A. Furthermore, there exist $(\epsilon_k)_{k=1}^{\infty} \subset (0,1)$, $(x^{(\epsilon_k)})_{k=1}^{\infty} \subset \boldsymbol{\Sigma}_w \cap \operatorname{int}(\mathcal{C})$, $(\boldsymbol{\lambda}^{(\epsilon_k)})_{k=1}^{\infty} \subset \mathbb{R}^d_{++}$, $x^* \in \boldsymbol{\Sigma}_w$ and $\boldsymbol{\lambda}^* \in \mathbb{R}^d_+$ such that for all $k \geq 1$ it holds $\mathcal{F}^{(\epsilon_k)}(x^{(\epsilon_k)}) = \boldsymbol{\lambda}^{(\epsilon_k)} \otimes x^{(\epsilon_k)}$,

$$\lim_{k\to\infty} \epsilon_k = 0, \quad \lim_{k\to\infty} x^{(\epsilon_k)} = x^*, \quad \lim_{k\to\infty} \boldsymbol{\lambda}^{(\epsilon_k)} = \boldsymbol{\lambda}^* \quad \text{and} \quad \mathcal{F}(x^*) = \boldsymbol{\lambda}^* \otimes x^*.$$

Proof. Let $\epsilon > 0$, then it is straightforward to verify that $\mathcal{F}^{(\epsilon)}$ is continuous, order-preserving and multi-homogeneous of degree A. Furthermore, note that $\mathcal{F}(\boldsymbol{\Sigma}_w) \subset \operatorname{int}(\mathcal{C})$ and thus the existence of $x^{(\epsilon)}$ and $\boldsymbol{\lambda}^{(\epsilon)}$ follows form Lemma 4.1.6. Finally, the existence of $(\epsilon_k)_{k=1}^{\infty}$, $x^* \in \boldsymbol{\Sigma}_w$ and $\boldsymbol{\lambda}^* \in \mathbb{R}^d$ with the desired properties can be proved in the same way as in the proof of Theorem 4.1.5. □

The proof of Theorem 9.3.5 relies on the following construction which generalizes the technique proposed in Section 3.2 of [34]: Let $\mathcal{F} \colon \mathcal{C} \to \mathcal{C}$ be continuous, order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}_+$. Suppose that A is irreducible and let $\mathsf{G}(\mathcal{F}) = (\mathcal{I}, \mathcal{E})$ be the graph associated to $\mathcal{F}$, and $\mathbf{b} \in \mathbb{R}^d_{++}$ the Perron vector of $A^\top$. For $r > 0$, define

$$\Psi(r) = \sup\left\{ t \geq 0 \;\middle|\; \min_{\substack{((i,j_i),(k,l_k))\in\mathcal{E} \\ (a_1,\ldots,a_d)\in\mathcal{J}}} \mathcal{F}\big(u^{(k,l_k)}(t)\big)_{i,j_i}^{b_i} \prod_{\substack{s=1\\ s\neq i}}^{d} \mathcal{F}\big(u^{(k,l_k)}(t)\big)_{s,a_s}^{b_s} \leq r \right\}.$$

Note that, by definition of $\mathsf{G}(\mathcal{F})$, if $\mathcal{F}$ is weakly irreducible, then $\Psi(r) < \infty$ for any $r > 0$ and Ψ is an increasing function. Moreover, note that Ψ has the following property: Let $(j_1, \ldots, j_d) \in \mathcal{J}$, $i \in [d]$, $(k, l_k) \in \mathcal{I}$ and $t > 0$, if $\big((i, j_i), (k, l_k)\big) \in \mathcal{E}$ then

$$\prod_{s=1}^{d} \mathcal{F}\big(u^{(k,l_k)}(t)\big)_{s,j_s}^{b_s} \leq r \qquad \Longrightarrow \qquad t \leq \Psi(r). \tag{9.3}$$

In the case $d = 1$, the proof of Theorem 6.2.3 [60] uses on the following idea: if $\mathcal{F} \colon \mathcal{C} \to \mathcal{C}$ is order-preserving, homogeneous, $\mathsf{G}(\mathcal{F})$ is strongly connected and the eigenvector $x^* \in \mathbb{R}^{n_1} \setminus \{0\}$ given by Lemma 9.3.6 has a zero entry, then one gets the contradiction $x^* = 0$. We use a similar idea for the case $d > 1$ and prove that if $\mathcal{F} \colon \mathcal{C} \to \mathcal{C}$ is a mapping satisfying the assumptions of Theorem 4.1.5 and x^*, the eigenvector of $\mathcal{F}$ given by Lemma 9.3.6, has a zero entry then $x^*_i = 0$ for some $i \in [d]$, contradicting the fact that x^* is an eigenvector.

Proof of Theorem 9.3.5. Let $w \in \operatorname{int}(\mathcal{C}^*)$, $\boldsymbol{\xi}_w(x) = (w_1(x), \ldots, w_d(x_d)) \in \mathbb{R}^d_+$ for all $x \in \mathcal{C}$ and define $\boldsymbol{\Sigma}_w = \{x \in \mathcal{C} \,|\, \boldsymbol{\xi}_w(x) = \mathbf{1}\}$. Let $(\epsilon_k)_{k=1}^{\infty} \subset (0,1)$, $(x^{(\epsilon_k)})_{k=1}^{\infty} \subset \boldsymbol{\Sigma}_w \cap \operatorname{int}(\mathcal{C})$, $(\boldsymbol{\lambda}^{(\epsilon_k)})_{k=1}^{\infty} \subset \mathbb{R}^d_{++}$, $x^* \in \boldsymbol{\Sigma}_w$ and $\boldsymbol{\lambda}^* \in \mathbb{R}^d_+$ be the sequences given by Lemma 9.3.6 and for all $k \geq 1$, let $\mathcal{F}^{(\epsilon_k)} \colon \mathcal{C} \to \mathcal{C}$ be defined as in (9.2) with $\epsilon = \epsilon_k$. Since $\boldsymbol{\lambda}^{(\epsilon_k)} \to \boldsymbol{\lambda}$, there exists a constant $M_0 > 0$ such that

$$\prod_{s=1}^{d} (\lambda_s^{(\epsilon_k)})^{b_s} \leq M_0 \qquad \forall k \in \mathbb{N}. \tag{9.4}$$

Let $\mathcal{J} = [n_1] \times \ldots \times [n_d]$ and suppose by contradiction that $x^* \in \boldsymbol{\Sigma}_w \setminus \text{int}(\mathcal{C})$. By taking a subsequence if necessary, we may assume that there exists $(j_1, \ldots, j_d) \in \mathcal{J}$ and $\omega \in [d]$ such that $\min_{t_s \in [n_s]} x^{(\epsilon_k)}_{s,t_s} = x^{(\epsilon_k)}_{s,j_s}$, $\forall s \in [d], k \in \mathbb{N}$, and $\lim_{k\to\infty} x^{(\epsilon_k)}_{\omega,j_\omega} = x^*_{\omega,j_\omega} = 0$. By the compactness of $\boldsymbol{\Sigma}_w$, there exists $\tilde{\gamma} > 0$ such that $y_{s,t_s} \leq \tilde{\gamma}$ for all $y \in \boldsymbol{\Sigma}_w$. It follows that

$$0 \leq \lim_{k\to\infty} \prod_{s=1}^{d} (x^{(\epsilon_k)}_{s,j_s})^{b_s} \leq \tilde{\gamma}^{1-b_\omega} \lim_{k\to\infty} (x^{(\epsilon_k)}_{\omega,j_\omega})^{b_\omega} = 0. \tag{9.5}$$

Since $x^* \in \boldsymbol{\Sigma}_w$, there exists $(l_1, \ldots, l_d) \in \mathcal{J}$ with $x^*_{s,l_s} > 0$ for all $s \in [d]$. Thus,

$$\lim_{k\to\infty} \prod_{s=1}^{d} (x^{(\epsilon_k)}_{s,l_s})^{b_s} = \prod_{s=1}^{d} (x^*_{s,l_s})^{b_s} > 0. \tag{9.6}$$

Let $\nu \in [d]$, by assumption on $\mathsf{G}(\mathcal{F})$, there exists $i_\nu \in [d]$ and a path $(i_\nu, j_{i_\nu}) = (m_1, \xi_{m_1}) \to (m_2, \xi_{m_2}) \to \ldots \to (m_{N_\nu}, \xi_{m_{N_\nu}}) = (\nu, l_\nu)$ in $\mathsf{G}(\mathcal{F})$ with $N_\nu \leq n_1 + \ldots + n_d$. Define $\mathbf{i}(1), \mathbf{i}(2), \ldots, \mathbf{i}(N_\nu) \in \mathcal{J}$ as

$$\mathbf{i}_s(a) = \begin{cases} \xi_{m_a} & \text{if } s = m_a, \\ j_s & \text{otherwise.} \end{cases} \qquad \forall s \in [d],\ a \in [N_\nu].$$

Fix $k \in \mathbb{N}$ and let $t = x^{(\epsilon_k)}_{m_2,\xi_{m_2}} / x^{(\epsilon_k)}_{m_2,j_{m_2}}$ and $\boldsymbol{\alpha} = \big((x^{(\epsilon_k)}_{1,j_1})^{-1}, \ldots, (x^{(\epsilon_k)}_{d,j_d})^{-1}\big)$. Note that $\boldsymbol{\alpha}$ is well defined since $x^{(\epsilon_k)} \in \text{int}(\mathcal{C})$ for all k. We have $u^{(m_2,\xi_{m_2})}(t) \preceq_{\mathcal{C}} \boldsymbol{\alpha} \otimes x^{(\epsilon_k)}$ and thus $\mathcal{F}(u^{(m_2,\xi_{m_2})}(t)) \preceq_{\mathcal{C}} \mathcal{F}(\boldsymbol{\alpha} \otimes x^{(\epsilon_k)})$. Furthermore, as $\mathcal{F}(y) \preceq_{\mathcal{C}} \mathcal{F}^{(\epsilon_k)}(y)$ for all $\mathbf{y} \in \mathcal{C}$, it holds $\mathcal{F}(\boldsymbol{\alpha} \otimes x^{(\epsilon_k)}) \preceq_{\mathcal{C}} \mathcal{F}^{(\epsilon_k)}(\boldsymbol{\alpha} \otimes x^{(\epsilon_k)})$. It follows that

$$\begin{aligned}
&(x^{(\epsilon_k)}_{m_1,j_{m_1}})^{b_{m_1}} \prod_{s=1}^{d} \mathcal{F}\big(u^{(m_2,\xi_{m_2})}(t)\big)^{b_s}_{s,\mathbf{i}_s(1)} \leq (x^{(\epsilon_k)}_{m_1,j_{m_1}})^{b_{m_1}} \prod_{s=1}^{d} \mathcal{F}(\boldsymbol{\alpha} \otimes x^{(\epsilon_k)})^{b_s}_{s,\mathbf{i}_s(1)} \\
&\leq (x^{(\epsilon_k)}_{m_1,j_{m_1}})^{b_{m_1}} \prod_{s=1}^{d} \mathcal{F}^{(\epsilon_k)}(\boldsymbol{\alpha} \otimes x^{(\epsilon_k)})^{b_s}_{s,\mathbf{i}_s(1)} = \Bigg(\prod_{\substack{s=1,\\ s\neq m_1}}^{d} (x^{(\epsilon_k)}_{s,j_s})^{b_s}\Bigg)^{-1} \prod_{s=1}^{d} \mathcal{F}^{(\epsilon_k)}(x^{(\epsilon_k)})^{b_s}_{s,\mathbf{i}_s(1)} \\
&= \Bigg(\prod_{\substack{s=1,\\ s\neq m_1}}^{d} (x^{(\epsilon_k)}_{s,j_s})^{b_s}\Bigg)^{-1} \prod_{s=1}^{d} (\lambda^{(\epsilon_k)}_s x^{(\epsilon_k)}_{s,\mathbf{i}_s(1)})^{b_s} = (x^{(\epsilon_k)}_{m_1,j_{m_1}})^{b_{m_1}} \prod_{s=1}^{d} (\lambda^{(\epsilon_k)}_s)^{b_s} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \leq (x^{(\epsilon_k)}_{m_1,j_{m_1}})^{b_{m_1}} M_0,
\end{aligned} \tag{9.7}$$

where $M_0 > 0$ satisfies (9.4). Hence, by (9.3), $t = x^{(\epsilon_k)}_{m_2,\xi_{m_2}} / x^{(\epsilon_k)}_{m_2,j_{m_2}} \leq \Psi(M_0)$ and

$$\prod_{s=1}^{d} (x^{(\epsilon_k)}_{s,\mathbf{i}_s(2)})^{b_s} \leq M_1 \prod_{s=1}^{d} (x^{(\epsilon_k)}_{s,j_s})^{b_s} \qquad \text{with} \qquad M_1 = \Psi(M_0)^{b_{m_2}}.$$

Applying this procedure again to (m_3, ξ_{m_3}), we get the existence of a constant $M_2 > 0$ independent of k, such that

$$\prod_{s=1}^{d} (x_{s,\mathbf{i}_s(3)}^{(\epsilon_k)})^{b_s} \leq M_2 \prod_{s=1}^{d} (x_{s,j_s}^{(\epsilon_k)})^{b_s}.$$

Indeed, let $t = x_{m_3,\xi_{m_3}}^{(\epsilon_k)} / x_{m_3,j_{m_3}}^{(\epsilon_k)}$, then $u^{(m_3,\xi_{m_3})}(t) \preceq_{\mathcal{C}} \boldsymbol{\alpha} \otimes x^{(\epsilon_k)}$ and, similarly to (9.7), we get

$$(x_{m_2,j_{m_2}}^{(\epsilon_k)})^{b_{m_2}} \prod_{s=1}^{d} \mathcal{F}\big(u^{(m_3,\xi_{m_3})}(t)\big)_{s,\mathbf{i}_s(2)}^{b_s} \leq (x_{m_2,j_{m_2}}^{(\epsilon_k)})^{b_{m_2}} \prod_{s=1}^{d} \mathcal{F}(\boldsymbol{\alpha} \otimes x^{(\epsilon_k)})_{s,\mathbf{i}_s(2)}^{b_s}$$
$$\leq (x_{m_2,j_{m_2}}^{(\epsilon_k)})^{b_{m_2}} \prod_{s=1}^{d} \mathcal{F}^{(\epsilon_k)}(\boldsymbol{\alpha} \otimes x^{(\epsilon_k)})_{s,\mathbf{i}_s(2)}^{b_s} = \Bigg(\prod_{\substack{s=1,\\ s\neq m_2}}^{d} (x_{s,j_s}^{(\epsilon_k)})^{b_s} \Bigg)^{-1} \prod_{s=1}^{d} \mathcal{F}^{(\epsilon_k)}(x^{(\epsilon_k)})_{s,\mathbf{i}_s(2)}^{b_s}$$
$$= \Bigg(\prod_{\substack{s=1,\\ s\neq m_2}}^{d} (x_{s,j_s}^{(\epsilon_k)})^{b_s} \Bigg)^{-1} \prod_{s=1}^{d} \big(\lambda_s x_{s,\mathbf{i}_s(2)}^{(\epsilon_k)}\big)^{b_s} = (x_{m_2,\xi_{m_2}}^{\epsilon_k})^{b_{m_2}} \prod_{s=1}^{d} (\lambda_s^{(\epsilon_k)})^{b_s} \leq (x_{m_2,j_{m_2}}^{(\epsilon_k)})^{b_{m_2}} M_1 M_0.$$

Hence, with $M_2 = \Psi(M_0 M_1)^{b_{m_3}}$, we get $(x_{m_3,\xi_{m_3}}^{(\epsilon_k)} / x_{m_3,j_{m_3}}^{(\epsilon_k)})^{b_{m_3}} \leq M_2$ which implies the desired inequality. Repeating this process at most N_ν times, we obtain $\gamma_\nu > 0$ independent of k, such that

$$(x_{\nu,l_\nu}^{(\epsilon_k)})^{b_\nu} \prod_{\substack{s=1\\ s\neq\nu}}^{d} (x_{s,j_s}^{(\epsilon_k)})^{b_s} = \prod_{s=1}^{d} (x_{s,\mathbf{i}_s(N_\nu)}^{(\epsilon_k)})^{b_s} \leq \gamma_\nu \prod_{s=1}^{d} (x_{s,j_s}^{(\epsilon_k)})^{b_s} \qquad \forall k \in \mathbb{N}. \tag{9.8}$$

Taking the product over $\nu \in [d]$ in (9.8) and dividing by $\prod_{s=1}^{d} (x_{s,j_s}^{(\epsilon_k)})^{(d-1)b_s}$ shows

$$\prod_{\nu=1}^{d} (x_{\nu,l_\nu}^{(\epsilon_k)})^{b_\nu} \leq \gamma \prod_{s=1}^{d} (x_{s,j_s}^{(\epsilon_k)})^{b_s} \qquad \forall k \in \mathbb{N},$$

where $\gamma = \prod_{\nu=1}^{d} \gamma_\nu$. Finally, using (9.5) and (9.6) we get a contradiction. $\square$

9.4 Weakly irreducible nonnegative tensors

We relate Definition 9.3.2 with the notions of weak irreducibility introduced in the literature of nonnegative tensors. To facilitate the discussion, we first prove the following simple but useful lemma which implies that the weak irreducibility of the mappings characterizing the ℓ^p-eigenvectors, the rectangular $\ell^{p,q}$-singular vectors and the $\ell^{p_1,\ldots,p_d}$-singular vectors (see Sections 4.2.4, 4.2.3, 4.2.1) does not depend on p, q or $p_1, \ldots, p_d$:

Lemma 9.4.1. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be order-preserving and multi-homogeneous. Let $a_1, \ldots, a_d > 0$ and let $\tilde{\mathcal{F}}\colon \mathcal{C} \to \mathcal{C}$ be given by

$$\tilde{\mathcal{F}}(x) = (\mathcal{F}(x)_1^{a_1}, \ldots, \mathcal{F}(x)_d^{a_d}) \qquad \forall x \in \mathcal{C}.$$

Let $\mathsf{G}(\mathcal{F}) = (\mathcal{I}, \mathcal{E})$ and $\mathsf{G}(\tilde{\mathcal{F}}) = (\mathcal{I}, \tilde{\mathcal{E}})$ be the graphs respectively associated to $\mathcal{F}$ and $\tilde{\mathcal{F}}$, then it holds $\mathcal{E} = \tilde{\mathcal{E}}$.

Proof. Let $(i, j_i), (k, l_k) \in \mathcal{I}$ and $u(t)$ defined as in (9.1). By continuity of $s \to s^{a_i}$, we have

$$\lim_{t\to\infty} \tilde{\mathcal{F}}\big(u^{(i,j_i)}(t)\big)_{k,l_k} = \lim_{t\to\infty} \mathcal{F}\big(u^{(i,j_i)}(t)\big)_{k,l_k}^{a_k} = \Big(\lim_{t\to\infty} \mathcal{F}\big(u^{(i,j_i)}(t)\big)_{k,l_k}\Big)^{a_k}$$

and therefore it holds $\big((i, j_i), (k, l_k)\big) \in \mathcal{E}$ if and only if $\big((i, j_i), (k, l_k)\big) \in \tilde{\mathcal{E}}$. $\square$

Given a nonnegative tensor $T \in \mathbb{R}^{n\times n\times n}$, depending on which eigenvector problem is considered for T, the graph of the induced multi-homogeneous mapping can be quite different. This is illustrated in the following example:

Example 9.4.2. Let $T \in \mathbb{R}^{3\times 3\times 3}$ be defined as

$$T_{2,2,1} = T_{3,2,1} = T_{1,3,1} = T_{2,2,2} = T_{1,1,3} = 1 \quad \text{and} \quad T_{i,j,k} = 0 \quad \text{otherwise.}$$

Let $\mathcal{F}_1\colon \mathbb{R}^3_+ \to \mathbb{R}^3_+$, $\mathcal{F}_2\colon \mathbb{R}^3_+ \times \mathbb{R}^3_+ \to \mathbb{R}^3_+ \times \mathbb{R}^3_+$ and $\mathcal{F}_3\colon \mathbb{R}^3_+ \times \mathbb{R}^3_+ \times \mathbb{R}^3_+ \to \mathbb{R}^3_+ \times \mathbb{R}^3_+ \times \mathbb{R}^3_+$ be defined for all $x, y, z \in \mathbb{R}^3_+$ as

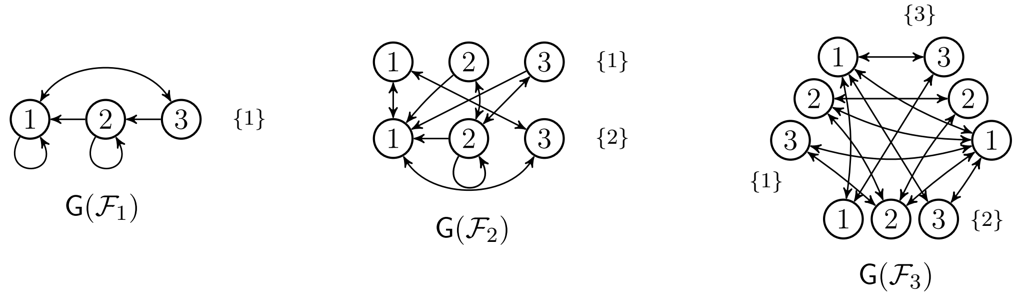
$$\mathcal{F}_1(x) = T(\cdot, x, x), \qquad \mathcal{F}_2(x, y) = \big(T(\cdot, y, y), T(x, \cdot, y)\big),$$

$$\mathcal{F}_3(x, y, z) = \big(T(\cdot, y, z), T(x, \cdot, z), T(x, y, \cdot)\big).$$

Then, with $p = q = r = 2$, the eigenvectors of $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ respectively characterize the ℓ^p-eigenvectors, the rectangular $\ell^{p,q}$-singular vectors and the $\ell^{p,q,r}$-singular vectors of T. The graphs $\mathsf{G}(\mathcal{F}_i) = (\mathcal{I}_i, \mathcal{E}_i), i = 1, 2, 3$ associated to these mappings are such that

$$\mathcal{I}_1 = \{1\}\times[3], \quad \mathcal{I}_2 = (\{1\}\times[3])\cup(\{2\}\times[3]), \quad \mathcal{I}_3 = (\{1\}\times[3])\cup(\{2\}\times[3])\cup(\{3\}\times[3]),$$

with $[3] = \{1, 2, 3\}$ and are given by



Note in particular that $\mathcal{F}_1$ and $\mathcal{F}_3$ are weakly irreducible but not $\mathcal{F}_2$. Indeed, if we let $(j_1, j_2) = (1, 3) \in [3] \times [3]$ and $(\nu, l_\nu) = (1, 3) \in (\{1\} \times [3]) \cup (\{2\} \times [3])$, then the condition of Definition 9.3.2 fails to be satisfied. Furthermore, we note that none of the $\mathcal{F}_i$ are irreducible.

The following lemma implies that Definition 9.3.2 is equivalent to the definition of weakly irreducible tensors introduced in [31] when $\mathcal{F}$ is the mapping characterizing the $\ell^{p_1,\ldots,p_d}$-singular values of a nonnegative tensor.

Lemma 9.4.3. Let $\mathcal{C} = \mathbb{R}^{n_1}_+ \times \ldots \times \mathbb{R}^{n_d}_+$, $T \in \mathbb{R}^{n_1\times\ldots\times n_d}_+$ and $p_1, \ldots, p_d \in (1, \infty)$. Define $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ as

$$\mathcal{F}(x) = \big(T(\cdot, x_2, \ldots, x_d)^{1/(p_1-1)}, \ldots, T(x_1, \ldots, x_{d-1}, \cdot)^{1/(p_d-1)}\big) \qquad \forall x \in \mathcal{C}.$$

Consider the following graph induced by T: Let $\hat{\mathsf{G}}(T) = (\mathcal{I}, \hat{\mathcal{E}})$ be defined so that there is an edge $((k, l_k), (\nu, l_\nu)) \in \hat{\mathcal{E}}$ if there exists $j_1 \in [n_1], \ldots, j_d \in [n_d]$ such that $j_k = l_k$, $j_\nu = l_\nu$ and $T_{j_1,j_2,\ldots,j_d} > 0$. Then, the following statements are equivalent:

a) $\mathcal{F}$ is weakly irreducible.

b) $\hat{\mathsf{G}}(T)$ is strongly connected.

Furthermore, if $\mathcal{F}$ is irreducible, then $\mathcal{F}$ is weakly irreducible.

Proof. By Lemma 9.4.1 we may suppose without loss of generality that $p_1 = \ldots = p_d = 2$. Now, let $\mathsf{G}(\mathcal{F}) = (\mathcal{I}, \mathcal{E})$ be the graph associated to $\mathcal{F}$. Recalling that for all $l = 1, \ldots, d$ and $j_l = 1, \ldots, n_l$ it holds

$$\mathcal{F}(x)_{l,j_l} = \sum_{\substack{j_1\in[n_1],\ldots,j_{l-1}\in[n_{l-1}]\\ j_{l+1}\in[n_{l+1}],\ldots,j_d\in[n_d]}} T_{j_1,\ldots,j_m} x_{1,j_1}\cdots x_{l-1,j_{l-1}} x_{l+1,j_{l+1}}\cdots x_{d,j_d} \qquad \forall x \in \mathcal{C},$$

together with the fact that the function $u^{(i,j_i)}\colon [0, \infty) \to \mathcal{C}$ of (9.1) satisfies $u^{(i,j_i)}(t) \in \operatorname{int}(\mathcal{C}) = \mathbb{R}^{n_1}_{++} \times \ldots \times \mathbb{R}^{n_d}_{++}$ for all $t > 0$, we see that $((k, l_k), (\nu, l_\nu)) \in \hat{\mathcal{E}}$ if and only if $((k, l_k), (\nu, l_\nu)) \in \mathcal{E}$. It follows that $\hat{\mathsf{G}}(T) = \mathsf{G}(\mathcal{F})$. Now, note that $\mathsf{G}(T)$ is undirected and therefore Lemma 9.3.4 implies that $\mathcal{F}$ is weakly irreducible if and only if $\mathsf{G}(\mathcal{F}) = \hat{\mathsf{G}}(T)$ is strongly connected. Finally, if $\mathcal{F}$ is irreducible, then Lemma 9.2.3 and [31, Lemma 3.1] imply that T satisfies b) and thus $\mathcal{F}$ is weakly irreducible. $\square$

The following lemma implies that Definition 9.3.2 is equivalent to the definition of weakly irreducible tensors introduced in [31] when $\mathcal{F}$ is the mapping characterizing the ℓ^p-eigenvectors of a nonnegative tensor recalled in Section 4.2.4.

Lemma 9.4.4. Let $\mathcal{C} = \mathbb{R}^n_+$, $T \in \mathbb{R}^{n\times\ldots\times n}_+$ an m-th order tensor, and $p \in (1, \infty)$. Define $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ as

$$\mathcal{F}(x) = T(\cdot, x, \ldots, x)^{1/(p-1)} \qquad \forall x \in \mathcal{C}.$$

Furthermore, consider the following graph induced by T: Let $\hat{\mathsf{G}}(T) = (\mathcal{I}, \hat{\mathcal{E}})$ be defined so that there is an edge $(l, k) \in \hat{\mathcal{E}}$ if there exists $j_2, \ldots, j_m \in [n]$ such that $k \in \{j_2, \ldots, j_m\}$ and $T_{l,j_2,\ldots,j_m} > 0$. Then, the following statements are equivalent

a) $\mathcal{F}$ is weakly irreducible.

b) $\hat{\mathsf{G}}(T)$ is strongly connected.

Furthermore, if $\mathcal{F}$ is irreducible, then $\mathcal{F}$ is weakly irreducible.

Proof. By Lemma 9.4.1 we may suppose without loss of generality that $p = 2$. Let $\mathsf{G}(\mathcal{F}) = (\mathcal{I}, \mathcal{E})$ be the graph associated to $\mathcal{F}$. Recalling that for all $l = 1, \ldots, n$ it holds

$$\mathcal{F}(x)_l = \sum_{j_2,\ldots,j_m=1}^{n} T_{l,j_2,\ldots,j_m} x_{j_2} \cdots x_{j_m} \qquad \forall x \in \mathcal{C},$$

together with the fact that the function $u^{(1,k)} \colon [0, \infty) \to \mathcal{C}$ of (9.1) satisfies $u^{(1,k)}(t) \in \operatorname{int}(\mathcal{C}) = \mathbb{R}^n_{++}$ for all $t > 0$, we see that $(l, k) \in \hat{\mathcal{E}}$ if and only if $(l, k) \in \mathcal{E}$. It follows that $\mathsf{G}(\mathcal{F}) = \hat{\mathsf{G}}(T)$. Hence, $\mathsf{G}(\mathcal{F})$ is strongly connected if and only if $\hat{\mathsf{G}}(T)$ is strongly connected. Now, if $\hat{\mathsf{G}}(T)$ is strongly connected, then for every $l \in [n]$ there exists $k \in [n]$ such that $(l, k) \in \hat{\mathcal{E}}$. It follows that there exists $j_2, \ldots, j_d \in [n]$ such that $T_{l,j_2,\ldots,j_d} > 0$ and thus $\mathcal{F}(x)_l > 0$ for all $x \in \operatorname{int}(\mathcal{C})$. Hence, $\mathcal{F}$ is weakly irreducible if and only if $\hat{\mathsf{G}}(T)$ is strongly connected. Finally, if $\mathcal{F}$ is irreducible, then Lemma 9.2.4 and [31, Lemma 3.1] imply that T satisfies b) and thus $\mathcal{F}$ is weakly irreducible. $\square$

Finally, we prove that Definition 9.3.2 directly reduces to the definition of weakly irreducible tensors introduced in [26] when $\mathcal{F}$ is the mapping characterizing the rectangular $\ell^{p,q}$-eigenvectors of a nonnegative tensor recalled in Section 4.2.3.

Lemma 9.4.5. Let $\mathcal{C} = \mathbb{R}^m_+ \times \mathbb{R}^n_+$, $T \in \mathbb{R}_+^{m \times \ldots \times m \times n \ldots \times n}$ a d-th order tensor, and $p, q \in (1, \infty)$. Define $\mathcal{F} \colon \mathcal{C} \to \mathcal{C}$ as

$$\mathcal{F}(x, y) = \big(T(\cdot, \underbrace{x, \ldots, x}_{(a-1) \text{ times}}, \underbrace{y, \ldots, y}_{(d-a) \text{ times}})^{1/(p-1)}, T(\underbrace{x, \ldots, x}_{a \text{ times}}, \cdot, \underbrace{y, \ldots, y}_{(d-a-1) \text{ times}})^{1/(q-1)} \big)$$

for all $(x, y) \in \mathcal{C}$. Consider the following graph induced by T: Let $\hat{\mathsf{G}}(T) = (\mathcal{I}, \hat{\mathcal{E}})$ be defined so that there is an edge $((k, l_k), (\nu, s_\nu)) \in \hat{\mathcal{E}}$ if there exists $j_1, \ldots, j_a \in [m], j_{a+1}, \ldots, j_d \in [n]$ such that $T_{j_1,\ldots,j_d} > 0$ and one of the following is satisfied:

i) $k = 1, \nu = 1$, $j_1 = l_k$ and $s_\nu \in \{j_2, \ldots, j_d\}$.

ii) $k = 2, \nu = 2$, $j_{a+1} = l_k$ and $s_\nu \in \{j_{a+2}, \ldots, j_d\}$.

iii) $k = 1$, $\nu = 2$, $j_1 = l_k$ and $s_\nu \in \{j_{a+1}, \ldots, j_d\}$.

iv) $k = 2$, $\nu = 1$, $j_{a+1} = l_k$ and $s_\nu \in \{j_1, \ldots, j_a\}$.

Then the following statements are equivalent:

a) $\mathcal{F}$ is weakly irreducible.

b) $\hat{\mathsf{G}}(T)$ is strongly connected.

Furthermore, if $\mathcal{F}$ is irreducible, then $\mathcal{F}$ is weakly irreducible.

Proof. By Lemma 9.4.1 we may suppose without loss of generality that $p = q = 2$. Let $\mathsf{G}(\mathcal{F}) = (\mathcal{I}, \mathcal{E})$ be the graph associated to $\mathcal{F}$. Recalling that for all $j_1 = 1, \ldots, m$ and $j_2 = 1, \ldots, n$ it holds

$$\mathcal{F}(x,y)_{1,j_1} = \sum_{\substack{j_2,\ldots,j_a \in [m],\\ j_{a+1},\ldots,j_d \in [n]}} T_{j_1,\ldots,j_d} x_{j_2} \cdots x_{j_a} y_{j_{a+1}} y_{j_d} \qquad \forall (x,y) \in \mathcal{C},$$

$$\mathcal{F}(x)_{2,j_2} = \sum_{\substack{j_1,\ldots,j_a \in [m],\\ j_{a+2},\ldots,j_d \in [n]}} T_{j_1,\ldots,j_d} x_{j_1} \cdots x_{j_a} y_{j_{a+2}} \cdots y_{j_d} \qquad \forall (x,y) \in \mathcal{C},$$

together with the fact that the function $u^{(k,l_k)} \colon [0,\infty) \to \mathcal{C}$ of (9.1) satisfies $u^{(k,l_k)}(t) \in \operatorname{int}(\mathcal{C}) = \mathbb{R}^m_{++} \times \mathbb{R}^n_{++}$ for all $t > 0$, we see that $((k,l_k),(\nu,l_\nu)) \in \hat{\mathcal{E}}$ if and only if $((k,l_k),(\nu,l_\nu)) \in \mathcal{E}$, i.e. $\mathcal{E} = \hat{\mathcal{E}}$. Hence, if $\hat{\mathsf{G}}(T)$ is strongly connected then $\mathsf{G}(\mathcal{F})$ is strongly connected as well. Moreover, if $\hat{\mathsf{G}}(T)$ is strongly connected, then for all $(k,l_k) \in \mathcal{I}$ there exists $(\nu,l_\nu) \in \mathcal{I}$ such that $((k,l_k),(\nu,l_\nu)) \in \hat{\mathcal{E}}$ and therefore we have $\mathcal{F}(x)_{k,l_k} > 0$ for all $x \in \operatorname{int}(\mathcal{C})$ which implies that $\mathcal{F}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$. It follows that if $\hat{\mathsf{G}}(T)$ is strongly connected, then $\mathcal{F}$ is weakly irreducible. On the other hand, suppose that $\mathcal{F}$ is weakly irreducible and let $(k,l_k),(\nu,l_\nu) \in \mathcal{I}$. As $\mathcal{F}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$, there exists $j_1,\ldots,j_d$ such that $T_{j_1,\ldots,j_d} > 0$ and $j_k = l_k$. If $k = 1$, let $k' = 2$ and $l' = j_a$ and if $k = 2$, let $k' = 1$ and $l' = j_1$. Then, we have $((k,l_k),(k',l')) \in \mathcal{E}$ and $((k',l'),(k,l_k)) \in \mathcal{E}$. Furthermore as $\mathcal{F}$ is weakly irreducible, there exists either a path form (k',l') to (ν,l_ν) or a path from (k,l_k) to (ν,l_ν). Both cases implies the existence of a path from (k,l_k) to (ν,l_ν) and thus $\mathsf{G}(\mathcal{F})$ is strongly connected which implies that $\hat{\mathsf{G}}(T)$ is strongly connected and thus proves the equivalence between (a) and (b). Finally, we prove that irreducibility implies weak irreducibility. Suppose that $\mathcal{F}$ is not weakly irreducible and let us prove that $\mathcal{F}$ is not irreducible. If $\mathcal{F}$ is not weakly irreducible, then $\mathsf{G}(\mathcal{F})$ is not strongly connected and thus there exists a nonempty proper set $J \subset \mathcal{I}$ such that there exists no edge in $\mathcal{E}$ between J and $\mathcal{I} \setminus J$. Let $J_1 = \{j \in [m] \,|\, (1,j) \in J\}$, $J_2 = \{j \in [n] \,|\, (2,j) \in J\}$ and consider the faces $Q_1 = \{x \in \mathbb{R}^m_+ \,|\, x_{j_1} = 0, \forall j_1 \in J_1\}$ and $Q_2 = \{y \in \mathbb{R}^n_+ \,|\, y_{j_2} = 0, \forall j_2 \in J_2\}$ of $\mathbb{R}^m_+$ and $\mathbb{R}^n_+$ respectively. Then, as there is no edge in $\mathcal{E}$ between J and $\mathcal{I} \setminus J$, it holds $\mathcal{F}(\mathcal{Q}) \subset \mathcal{Q}$. Hence, to show that $\mathcal{F}$ is not irreducible, it is enough, by Proposition 9.1.2, to show that $Q_1 \neq \{0\}, Q_2 \neq \{0\}$ and $\mathcal{Q} \neq \mathcal{C}$. It holds $\mathcal{Q} \neq \mathcal{C}$ since J is a proper subset of $\mathcal{I}$. The end of the proof is about showing that $J_1 \neq \emptyset$ and $J_2 \neq \emptyset$ in order to ensure that $Q_1 \neq \{0\}$ and $Q_2 \neq \{0\}$. Note that if either $J_1 = \emptyset$ and $J_2 = [n]$ or $J_1 = [m]$ and $J_2 = \emptyset$ then, as $\hat{\mathsf{G}}(T) = \mathsf{G}(\mathcal{F})$, we have $T = 0$ which contradicts $\mathcal{F}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$. Furthermore, if $J_1 = \emptyset$, then by the previous argument we have $J_2 \neq [n]$ and furthermore it holds $J_2 \neq \emptyset$ since $J \neq \emptyset$. But then it holds $T_{j_1,\ldots,j_d} = 0$ for all $j_1,\ldots,j_d$ such that $j_{a+1} \in [n] \setminus J_2$ which again contradicts $\mathcal{F}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$. Hence, we must have $J_1 \neq \emptyset$. Similarly, if $J_2 = \emptyset$, then we have shown that $J_1 \neq [m]$ and $J_1 \neq \emptyset$. However, the latter implies that $T_{j_1,\ldots,j_d} = 0$ for all $j_1,\ldots,j_d$ such that $j_1 \in [m] \setminus J_1$ which contradicts $\mathcal{F}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$. It follows that $J_1, J_2 \neq \emptyset$ and thus, by the above discussion, $\mathcal{F}$ is not irreducible. ☐

We refer to [40, Section 6.3]$^\diamond$ for a detailed discussion on weak irreducibility of polynomial mappings induced by a nonnegative tensors, i.e. mappings of the form

$$\mathcal{F}(x) = \nabla f_T(\underbrace{x_1,\ldots,x_1}_{a_1\text{ times}},\ldots,\underbrace{x_d,\ldots,x_d}_{a_d\text{ times}}), \tag{9.9}$$

where $T \in \mathbb{R}_+^{\tilde{n}_1\ldots\times\tilde{n}_m}$ and $m = a_1 + \ldots + a_d$. The following lemma provides a convenient way to construct the graph associated to such mappings and is needed for later discussion.

Lemma 9.4.6. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be as in (9.9) and $\mathsf{G}(\mathcal{F}) = (\mathcal{I}, \mathcal{E})$, its associated graph. Then, $\mathcal{F}$ is differentiable and for every $x \in \mathcal{C}$ it holds $((k, l_k), (i, j_i)) \in \mathcal{E}$ if and only if $\frac{\partial}{\partial x_{i,j_i}}\mathcal{F}(x)_{k,l_k} > 0$.

Proof. Note that if $((k, l_k), (i, j_i)) \in \mathcal{E}$, then x_{i,j_i} appears effectively in the expression of the polynomial $\mathcal{F}(x)_{k,l_k}$ and thus $\frac{\partial}{\partial x_{i,j_i}}\mathcal{F}(x)_{k,l_k} > 0$ since $x \in \operatorname{int}(\mathcal{C})$. Conversely, if $\frac{\partial}{\partial x_{i,j_i}}\mathcal{F}(x)_{k,l_k} > 0$, then x_{i,j_i} appears in the expression of the polynomial $\mathcal{F}(x)_{k,l_k}$ and thus $\mathcal{F}(u^{i,j_i}(t))_{k,l_k} \to \infty$ for $t \to \infty$. □

10 Maximality and uniqueness of positive eigenvectors

Given a nonnegative matrix $M \in \mathbb{R}_+^{n\times n}$ with spectral radius $\rho(M)$, the Collatz-Wielandt formula states that

$$\sup_{x\in\mathbb{R}_+^n\setminus\{0\}} \min_{x_i\neq 0} \frac{(Mx)_i}{x_i} = \rho(M) = \inf_{x\in\mathbb{R}_{++}^n} \max_{i=1,\ldots,n} \frac{(Mx)_i}{x_i}. \tag{10.1}$$

The left-hand side of (10.1) is useful to prove that the maximal eigenvalue of M in magnitude corresponds to an eigenvector with nonnegative entries. Indeed, if $Mv = \theta v$ with $v \in \mathbb{C}^n \setminus \{0\}$ and $\theta \in \mathbb{C}$, then by the triangle inequality we have $|Mv| \leq M|v|$ and thus

$$|\theta| = \min_{v_i\neq 0} \frac{|(Mv)_i|}{|v_i|} \leq \min_{v_i\neq 0} \frac{(M|v|)_i}{|v_i|} \leq \rho(M).$$

A generalization of this argument is used in [31] to prove that the maximal $\ell^{p_1,\ldots,p_d}$-eigenvalue of a nonnegative tensor $T \in \mathbb{R}_+^{n_1\times\ldots\times n_d}$ is attained in $\mathbb{R}_+^{n_1} \times \ldots \times \mathbb{R}_+^{n_d}$. On the other hand, the right hand side of (10.1) is useful to obtain upper bounds on $\rho(M)$. Furthermore, it implies that the eigenvalue corresponding to a positive eigenvector of a nonnegative matrix equals the spectral radius. Indeed, if $v \in \mathbb{R}_{++}^n$ is an eigenvector of M with eigenvalue $\lambda \geq 0$, then

$$\rho(M) \leq \max_{i=1,\ldots,n} \frac{(Mv)_i}{v_i} = \lambda,$$

and thus $\lambda = \rho(M)$. Collatz-Wielandt type formulas are known for homogeneous mappings on cones and are discussed in [60, Section 5.6], [4] and [35]. Furthermore, a Collatz-Wielandt formula for order-preserving multi-homogeneous mappings on $\mathcal{C} = \mathbb{R}_+^{n_1} \times \ldots \times \mathbb{R}_+^{n_d}$ is discussed in [39, Theorem 5.1]$^\diamond$.

Eigenvalues of multi-homogeneous mappings on the product of cones are defined in Definition 4.1.3. For the reading convenience, we recall the definition here: Let $\mathcal{V} = V_1 \times \ldots \times V_d$ be a product of finite dimensional real vector spaces and $\mathcal{C} \subset \mathcal{V}$ a cone. Furthermore, let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be a mapping which is multi-homogeneous of degree $A \in \mathbb{R}_+^{d\times d}$ where A is an irreducible matrix. Let $\mathbf{b} \in \mathbb{R}_{++}^d$ be the Perron vector of $A^\top$, i.e. $A^\top \mathbf{b} = \rho(A)\mathbf{b}$ and $\sum_{i=1}^d b_i = 1$. If $x \in \mathcal{C}$ is an eigenvector of $\mathcal{F}$, i.e. $x_i \neq 0$ for $i = 1, \ldots, d$ and there exists $\boldsymbol{\lambda} \in \mathbb{R}_+^d$ such that $\mathcal{F}(x) = \boldsymbol{\lambda} \otimes x$, then the eigenvalue corresponding to x is the scalar $\theta \in \mathbb{R}_+$ defined as $\theta = \prod_{i=d}^d \lambda_i^{b_i}$. We have proved in Lemma 4.1.4 that if $x, y \in \mathcal{C}$ are eigenvectors of $\mathcal{F}$ with corresponding eigenvalues θ, ϑ such that $\theta \leq \vartheta$, then for every $\boldsymbol{\alpha} \in \mathbb{R}_{++}^d$, the eigenvalues $\tilde{\theta}, \tilde{\vartheta}$ respectively corresponding to $\boldsymbol{\alpha} \otimes x$ and $\boldsymbol{\alpha} \otimes y$ satisfy $\tilde{\theta} = \tilde{\alpha}\theta \leq \tilde{\alpha}\vartheta = \tilde{\vartheta}$ where $\tilde{\alpha} > 0$ is a positive scalar. The scaling dependence of eigenvectors has been addressed in the spectral theory of nonnegative tensors by fixing the eigenvectors to have unit norms. Indeed, in the definition of ℓ^p-eigenvalues, rectangular $\ell^{p,q}$-singular values and $\ell^{p_1,\ldots,p_d}$-singular values of a nonnegative tensors (see Section 4.2), the eigenvectors are scaled on a product of ℓ^p-spheres. We adopt a similar approach in the following.

10.1 Maximality of positive eigenvectors

We prove an analogue to the left hand side of (10.1). In particular, it implies that if a multi-homogeneous mappings on a solid closed cone has a positive eigenvector, then the corresponding eigenvalue is maximal. In the case $d = 1$, the proof reduces to that of Lemma 5.2.1 of [60].

Theorem 10.1.1. Let $\mathcal{C} \subset \mathcal{V}$ be a closed cone and let $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}_+^d$ be a monotonic multi-normalization of $\mathcal{C}$. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}$. Suppose that $\rho(A) \leq 1$, A is irreducible and let $\mathbf{b} \in \mathbb{R}_{++}^d$ be the Perron vector of $A^\top$. Furthermore, let $x, y \in \mathcal{C}$ be such that $\boldsymbol{\nu}(x) = \boldsymbol{\nu}(y) = \mathbf{1}$ and there exists $\boldsymbol{\lambda}, \boldsymbol{\theta} \in \mathbb{R}_+^d$ and $\boldsymbol{\alpha} \in \mathbb{R}_{++}^d$ satisfying

$$\mathcal{F}(x) \preceq_{\mathcal{C}} \boldsymbol{\lambda} \otimes x, \qquad \boldsymbol{\theta} \otimes y \preceq_{\mathcal{C}} \mathcal{F}(y) \qquad \text{and} \qquad \boldsymbol{\alpha} \otimes y \preceq_{\mathcal{C}} x.$$

Then, it holds $\prod_{i=1}^d \theta_i^{b_i} \leq \prod_{i=1}^d \lambda_i^{b_i}$.

Proof. First of all note that, as $\mathcal{F}$ is order-preserving, we have

$$\boldsymbol{\theta} \otimes y \preceq_{\mathcal{C}} \mathcal{F}(y) \preceq_{\mathcal{C}} \mathcal{F}(\boldsymbol{\alpha}^{-I} \otimes x) = \boldsymbol{\alpha}^{-A} \otimes \mathcal{F}(x) \preceq_{\mathcal{C}} (\boldsymbol{\alpha}^{-A} \circ \boldsymbol{\lambda}) \otimes x.$$

Hence, if $\lambda_i = 0$ for some $i \in \{1, \ldots, d\}$, we have $\theta_i = 0$ since $\boldsymbol{\alpha}^{-A} \in \mathbb{R}_{++}^d$ and $x_i, y_i \in C_i \setminus \{0\}$. It follows that $\prod_{i=1}^d \theta_i^{b_i} = 0 \leq 0 = \prod_{i=1}^d \lambda_i^{b_i}$. If $\theta_i = 0$ for some $i \in \{1, \ldots, d\}$, then $\prod_{i=1}^d \theta_i^{b_i} = 0 \leq \prod_{i=1}^d \lambda_i^{b_i}$. Now, suppose that $\boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathbb{R}_{++}^d$. Let $\delta = (1 - \rho(A))$ and define $\tilde{\mathcal{F}}\colon \mathcal{C} \to \mathcal{C}$ as $\tilde{\mathcal{F}}(x) = \boldsymbol{\nu}(x)^{\delta I} \otimes \mathcal{F}(x)$. Then, $\tilde{\mathcal{F}}$ is order-preserving and multi-homogeneous of degree $\tilde{A} = A + (1 - \rho(A))I$. In particular, $\tilde{A}$ is irreducible, $\rho(\tilde{A}) = 1$, and $\tilde{A}^\top \mathbf{b} = \mathbf{b}$. Furthermore, as $\boldsymbol{\nu}(x) = \boldsymbol{\nu}(y) = \mathbf{1}$, we have

$$\tilde{\mathcal{F}}(x) \preceq_{\mathcal{C}} \boldsymbol{\lambda} \otimes x \qquad \text{and} \qquad \boldsymbol{\theta} \otimes y \preceq_{\mathcal{C}} \tilde{\mathcal{F}}(y).$$

Lemma 3.3.8 implies that for every $k \geq 1$, it holds

$$\tilde{\mathcal{F}}^k(x) \preceq_{\mathcal{C}} \boldsymbol{\lambda}^{\sum_{j=0}^{k-1}\tilde{A}^j} \otimes x \qquad \text{and} \qquad \boldsymbol{\theta}^{\sum_{j=0}^{k-1}\tilde{A}^j} \otimes y \preceq_{\mathcal{C}} \tilde{\mathcal{F}}^k(y).$$

As $\tilde{\mathcal{F}}^k$ is order-preserving and multi-homogeneous of degree $\tilde{A}^k$, for $k \geq 1$, we have

$$\big(\boldsymbol{\alpha}^{\tilde{A}^k} \circ \boldsymbol{\theta}^{\sum_{j=0}^{k-1}\tilde{A}^j}\big) \otimes y \preceq_{\mathcal{C}} \boldsymbol{\alpha}^{\tilde{A}^k} \otimes \tilde{\mathcal{F}}^k(y) = \tilde{\mathcal{F}}^k(\boldsymbol{\alpha} \otimes y) \preceq_{\mathcal{C}} \tilde{\mathcal{F}}^k(\mathbf{x}) \preceq_{\mathcal{C}} \boldsymbol{\lambda}^{\sum_{j=0}^{k-1}\tilde{A}^j} \otimes x.$$

Using that $\boldsymbol{\nu}$ is a monotonic normalization and $\boldsymbol{\nu}(x) = \boldsymbol{\nu}(y) = \mathbf{1}$, we get

$$\begin{aligned}
\Big(\prod_{i=1}^d \alpha_i^{b_i}\theta_i^{kb_i}\Big) = \prod_{i=1}^d \alpha_i^{(\tilde{A}^k b)_i}\theta_i^{(\sum_{j=0}^{k-1}\tilde{A}^j b)_i} &= \prod_{i=1}^d \big(\boldsymbol{\alpha}^{\tilde{A}^k} \circ \boldsymbol{\theta}^{\sum_{j=0}^{k-1}\tilde{A}^j}\big)_i^{b_i}\boldsymbol{\nu}(y)_i^{b_i} \\
&= \prod_{i=1}^d \boldsymbol{\nu}\Big(\big(\boldsymbol{\alpha}^{\tilde{A}^k} \circ \boldsymbol{\theta}^{\sum_{j=0}^{k-1}\tilde{A}^j}\big) \otimes y\Big)_i^{b_i} \\
&\leq \prod_{i=1}^d \boldsymbol{\nu}\Big(\boldsymbol{\lambda}^{\sum_{j=0}^{k-1}\tilde{A}^j} \otimes x\Big)_i^{b_i} = \prod_{i=1}^d \Big(\boldsymbol{\lambda}^{\sum_{j=0}^{k-1}\tilde{A}^j}\Big)^{b_i}\boldsymbol{\nu}(x)_i = \prod_{i=1}^d \lambda_i^{kb_i}.
\end{aligned}$$

It follows that

$$0 < \prod_{i=1}^d \alpha_i^{b_i} \leq \left(\frac{\prod_{i=1}^d \lambda_i^{b_i}}{\prod_{i=1}^d \theta_i^{b_i}}\right)^k \qquad \forall k \geq 1.$$

If $\prod_{i=1}^d \lambda_i^{b_i} / \prod_{i=1}^d \theta_i^{b_i} < 1$, then we obtain a contraction by letting $k \to \infty$. Hence, $\prod_{i=1}^d \theta_i^{b_i} \leq \prod_{i=1}^d \lambda_i^{b_i}$ which concludes the proof. □

We get the following corollary:

Corollary 10.1.2. Let $\mathcal{C} \subset \mathcal{V}$ be a solid closed cone and let $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d_+$ be a monotonic multi-normalization of $\mathcal{C}$. Set $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$ and let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}$. Suppose that $\rho(A) \leq 1$, A is irreducible and let $\mathbf{b} \in \mathbb{R}^d_{++}$ be the Perron vector of $A^\top$. If $\mathcal{F}$ has an eigenvector $u \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}$ with corresponding eigenvalue λ, then

$$\sup_{x \in \mathcal{S}_{\boldsymbol{\nu}}} \prod_{i=1}^d \mathrm{m}(\mathcal{F}(x)_i/x_i; C_i)^{b_i} = \lambda.$$

In particular, for every eigenvector $x \in \mathcal{S}_{\boldsymbol{\nu}}$ of $\mathcal{F}$ with corresponding eigenvalue θ, it holds $\theta \leq \lambda$.

Proof. Let $x \in \mathcal{S}_{\boldsymbol{\nu}}$, as $\mathcal{C}$ is closed, we have $\mathbf{m}(\mathcal{F}(x)/x; \mathcal{C}) \otimes x \preceq_{\mathcal{C}} \mathcal{F}(x)$. As $u \in \operatorname{int}(\mathcal{C})$, there exists $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$ such that $x \preceq_{\mathcal{C}} \boldsymbol{\alpha} \otimes u$. Let $\boldsymbol{\delta} \in \mathbb{R}^d_+$ be such that $\mathcal{F}(u) = \boldsymbol{\delta} \otimes u$. Then, by Theorem 10.1.1, we have

$$\prod_{i=1}^d \mathrm{m}(\mathcal{F}(x)_i/x_i; C_i)^{b_i} \leq \prod_{i=1}^d \delta_i^{b_i} = \lambda.$$

This shows that

$$\sup_{x\in\mathcal{S}_{\boldsymbol{\nu}}}\prod_{i=1}^d \mathrm{m}(\mathcal{F}(x)_i/x_i;C_i)^{b_i}\leq \lambda. \tag{10.2}$$

Finally, suppose that $x\in\mathcal{S}_{\boldsymbol{\nu}}$ is an eigenvector of $\mathcal{F}$ with corresponding eigenvalue θ. There exists $\boldsymbol{\vartheta}\in\mathbb{R}^d_+$ such that $\mathcal{F}(x)=\boldsymbol{\vartheta}\otimes x$ and $\theta=\prod_{i=1}^d\vartheta_i^{b_i}$. It follows that

$$\theta=\prod_{i=1}^d\vartheta_i^{b_i}\,\mathrm{m}(x_i/x_i;C_i)^{b_i}=\prod_{i=1}^d\mathrm{m}(\vartheta_ix_i/x_i;C_i)^{b_i}=\prod_{i=1}^d\mathrm{m}(\mathcal{F}(x)_i/x_i;C_i)^{b_i}\leq\lambda.$$

Finally, note that if $x=u$, then $\theta=\lambda$ and thus we have equality in (10.2). □

Theorem 10.1.1 implies that an order-preserving multi-homogeneous mapping cannot have more than one scaled eigenvalue on each part of the cone. In particular, if the cone is polyhedral, then one gets an upper bound on the number of eigenvalues of the mapping. This is stated in the following corollary which is a consequence of [60, Theorem 5.2.3] when $d=1$.

Corollary 10.1.3. Let $\mathcal{C}\subset\mathcal{V}$ be a closed polyhedral cone and let $\boldsymbol{\nu}\colon\mathcal{C}\to\mathbb{R}^d_+$ be a monotonic multi-normalization of $\mathcal{C}$. Let $\mathcal{F}\colon\mathcal{C}\to\mathcal{C}$ be order-preserving and multi-homogeneous of degree $A\in\mathbb{R}^{d\times d}$. Suppose that $\rho(A)\leq 1$, A is irreducible and let $\mathbf{b}\in\mathbb{R}^d_{++}$ be the Perron vector of $A^\top$. Furthermore, let

$$\sigma(\mathcal{F})=\Big\{\prod_{i=1}^d\lambda_i^{b_i}\,\Big|\,\exists x\in\mathcal{C}\text{ s.t. }\boldsymbol{\nu}(x)=\mathbf{1}\text{ and }\mathcal{F}(x)=\boldsymbol{\lambda}\otimes x\Big\}.$$

Then, it holds $|\sigma(\mathcal{F})|\leq\prod_{i=1}^d(N_i-1)$ where N_i is the number of parts of C_i for all $i=1,\ldots,d$.

Proof. Let $\mathcal{P}$ be a part of $\mathcal{C}$. Note that if $\mathcal{P}$ contains an eigenvector of $\mathcal{F}$, then $P_i\neq\{0\}$ for $i=1,\ldots,d$. In particular, there are $\prod_{i=1}^d(N_i-1)$ such parts. Now, suppose that $x,y\in\mathcal{P}$ are eigenvectors of $\mathcal{F}$. Then there exists $\boldsymbol{\lambda},\boldsymbol{\theta}\in\mathbb{R}^d_+$ such that $\mathcal{F}(x)=\boldsymbol{\lambda}\otimes x$ and $\mathcal{F}(y)=\boldsymbol{\theta}\otimes y$. Furthermore, as x,y belong to the same part, we have $x\sim_{\mathcal{C}}y$ and thus there exists $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{R}^d_{++}$ such that $\boldsymbol{\alpha}\otimes x\preceq_{\mathcal{C}}y$ and $\boldsymbol{\beta}\otimes y\preceq_{\mathcal{C}}x$. It follows from Theorem 10.1.1 that $\prod_{i=1}^d\theta_i^{b_i}\leq\prod_{i=1}^d\lambda_i^{b_i}$ and $\prod_{i=1}^d\lambda_i^{b_i}\leq\prod_{i=1}^d\theta_i^{b_i}$. Hence, x and y have the same eigenvalue which implies that $\mathcal{F}$ has at most one eigenvalue on $\{x\in\mathcal{P}\,|\,\boldsymbol{\nu}(x)=\mathbf{1}\}$ and proves the claim. □

We note that with a generalization of the argument of Theorem 5.2.3 in [60], it can be shown that the bound on $|\sigma(\mathcal{F})|$ of Corollary 10.1.3 is tight.

10.2 Collatz-Wielandt formulas

We prove an analogue of the right hand side of the Collatz-Wielandt formula (10.1) for order-preserving multi-homogeneous mappings. The proof of the following result uses techniques inspired from the proofs of [36, Theorem 21]$^\diamond$ and [13, Lemma 3.3] which give Collatz-Wielandt formulas for the $\ell^{p_1,\ldots,p_d}$-singular values of a nonnegative tensor and the $\ell^{p,q}$-singular values of a positive matrix, respectively. Furthermore, the following theorem reduces to the corresponding result of [39, Theorem 5.1]$^\diamond$ when $\mathcal{C} = \mathbb{R}^{n_1}_+ \times \ldots \times \mathbb{R}^{n_d}_+$.

Theorem 10.2.1. Let $\mathcal{C} \subset \mathcal{V}$ be a solid closed cone and let $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d_+$ be a monotonic multi-normalization of $\mathcal{C}$. Set $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$ and let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}$. Suppose that $\rho(A) \leq 1$, A is irreducible and let $\mathbf{b} \in \mathbb{R}^d_{++}$ be the Perron vector of $A^\top$. Suppose that there exists $u \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ and $\boldsymbol{\theta} \in \mathbb{R}^d_{++}$ such that $\mathcal{F}(u) = \boldsymbol{\theta} \otimes u$. Then, it holds

$$\prod_{i=1}^d \theta_i^{b_i} = \inf_{x\in\operatorname{int}(\mathcal{C})\cap\mathcal{S}_{\boldsymbol{\nu}}} \prod_{i=1}^d \mathrm{M}(\mathcal{F}(x)_i/x_i; C_i)^{b_i}.$$

Proof. Let $x \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$. Lemma 5.1.2 implies that $\mathbf{m}(u/x;\mathcal{C}) = \mathbf{M}(x/u;\mathcal{C})^{-I}$ and that for all $y, v \in \operatorname{int}(\mathcal{C})$ and $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$, it holds $\mathbf{M}(\boldsymbol{\alpha} \otimes y/v;\mathcal{C}) = \boldsymbol{\alpha} \otimes \mathbf{M}(y/v;\mathcal{C})$. It follows that

$$\begin{aligned}\boldsymbol{\theta} \circ \mathbf{M}(u/x;\mathcal{C})^{I-A} &= \boldsymbol{\theta} \circ \mathbf{M}(u/x;\mathcal{C})^{-A} \circ \mathbf{M}(u/x;\mathcal{C}) = \mathbf{m}(x/u;\mathcal{C})^{A} \circ \mathbf{M}(\boldsymbol{\theta} \otimes u/x;\mathcal{C})\\ &= \mathbf{M}(\mathbf{m}(x/u;\mathcal{C})^{A} \otimes \mathcal{F}(u)/x;\mathcal{C}) = \mathbf{M}(\mathcal{F}(\mathbf{m}(x/u;\mathcal{C}) \otimes u)/x;\mathcal{C})\\ &\leq \mathbf{M}(\mathcal{F}(x)/x;\mathcal{C}). \qquad (10.3)\end{aligned}$$

By Lemma 5.1.3, we have $\mathbf{1} \leq \mathbf{M}(u/x;\mathcal{C})$ and thus $1 \geq \prod_{i=1}^d \mathrm{M}(u_i/x_i;C_i)^{(1-\rho(A))b_i}$. It now follows from (10.3) that

$$\prod_{i=1}^d \theta_i^{b_i} \leq \prod_{i=1}^d \theta_i^{b_i}\, \mathrm{M}(u_i/x_i;C_i)^{((I-A)\mathbf{b})_i} \leq \prod_{i=1}^d \mathrm{M}(\mathcal{F}(x)_i/x_i;C_i)^{b_i}.$$

Hence, we have

$$\prod_{i=1}^d \theta_i^{b_i} \leq \inf_{x\in\operatorname{int}(\mathcal{C})\cap\mathcal{S}_{\boldsymbol{\nu}}} \prod_{i=1}^d \mathrm{M}(\mathcal{F}(x)_i/x_i; C_i)^{b_i}.$$

To prove that equality holds, note that $\boldsymbol{\theta} = \mathbf{M}(\mathcal{F}(u)/u;\mathcal{C})$ and thus

$$\prod_{i=1}^d \theta_i^{b_i} = \prod_{i=1}^d \mathrm{M}(\mathcal{F}(u)_i/u_i;C_i)^{b_i},$$

which concludes the proof. □

In Section 4.1, and more particularly Lemma 4.1.2, we have seen that if $\rho(A) = 1$, then the eigenvalues are scale invariant. A similar argument shows that when $\rho(A) = 1$, the multi-normalization is not needed in the Collatz-Wielandt formulas of Corollary 10.1.2 and Theorem 10.2.1. Indeed, we have the following result, which for $d = 1$ and $\mathcal{F}$ linear reduces to (10.1) in the case where the matrix is assumed to have a positive eigenvector. The assumption on existence of a positive eigenvector can be removed at the price of a considerably more technical proof. The main difficulty lies in defining the spectral radius of a multi-homogeneous mapping without assuming the existence of an eigenvector. We refer to [39]$^\diamond$ for such a discussion in the case of order-preserving multi-homogeneous mappings defined on $\mathcal{C} = \mathbb{R}^{n_1}_+ \times \ldots \mathbb{R}^{n_d}_+$.

Theorem 10.2.2. Let $\mathcal{C} \subset \mathcal{V}$ be a solid closed cone. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}$. Suppose that $\rho(A) = 1$, A is irreducible and let $\mathbf{b} \in \mathbb{R}^d_{++}$ be the Perron vector of $A^\top$. Suppose that $\mathcal{F}$ has an eigenvector $u \in \operatorname{int}(\mathcal{C})$ with corresponding eigenvalue λ. Then, it holds

$$\sup_{x\in\mathcal{C}_0} \prod_{i=1}^d \mathrm{m}(\mathcal{F}(x)_i/x_i; C_i)^{b_i} = \lambda = \inf_{x\in\operatorname{int}(\mathcal{C})} \prod_{i=1}^d \mathrm{M}(\mathcal{F}(x)_i/x_i; C_i)^{b_i},$$

where $\mathcal{C}_0 = \{x \in \mathcal{C} \,|\, x_1, \ldots, x_d \neq 0\}$.

Proof. Let $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d_+$ be a monotonic multi-normalization of $\mathcal{C}$ and set $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$. By Lemma 4.1.4, (a), we have that the eigenvalue corresponding to $\boldsymbol{\nu}(u)^{-I} \otimes u$ equals λ since $\big(\prod_{i=1}^d \boldsymbol{\nu}(u)_i^{-b_i}\big)^{\rho(A)-1} = \big(\prod_{i=1}^d \boldsymbol{\nu}(u)_i^{-b_i}\big)^0 = 1$. Now, let $x \in \mathcal{C}_0$, and $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$, then we have

$$\begin{aligned}
\prod_{i=1}^d \mathrm{m}(\mathcal{F}(\boldsymbol{\alpha}\otimes x)_i/\alpha_i x_i; C_i)^{b_i} &= \prod_{i=1}^d \mathrm{m}((\boldsymbol{\alpha}^A \otimes \mathcal{F}(x))_i/(\boldsymbol{\alpha}\otimes x)_i; C_i)^{b_i} \\
&= \prod_{i=1}^d \alpha_i^{((A-I)\mathbf{b})_i}\, \mathrm{m}(\mathcal{F}(x)_i/x_i; C_i)^{b_i} \\
&= \prod_{i=1}^d \mathrm{m}(\mathcal{F}(x)_i/x_i; C_i)^{b_i}. \qquad (10.4)
\end{aligned}$$

The same argument as above with $\mathrm{m}(\cdot/\cdot; C_i)$ replaced by $\mathrm{M}(\cdot/\cdot; C_i)$, shows that

$$\prod_{i=1}^d \mathrm{M}(\mathcal{F}(\boldsymbol{\alpha}\otimes x)_i/\alpha_i x_i; C_i)^{b_i} = \prod_{i=1}^d \mathrm{M}(\mathcal{F}(x)_i/x_i; C_i)^{b_i}. \qquad (10.5)$$

With $\boldsymbol{\alpha} = \boldsymbol{\nu}(x)^{-I}$, we have $\boldsymbol{\alpha}\otimes x \in \mathcal{S}_{\boldsymbol{\nu}}$. Hence, (10.4) and Corollary 10.1.2 imply

$$\begin{aligned}
\sup_{x\in\mathcal{C}_0} \prod_{i=1}^d \mathrm{m}(\mathcal{F}(x)_i/x_i; C_i)^{b_i} &= \sup_{x\in\mathcal{C}_0} \prod_{i=1}^d \mathrm{m}(\mathcal{F}(\boldsymbol{\nu}(x)^{-I}\otimes x)_i/(\boldsymbol{\nu}(x)^{-I}\otimes x)_i; C_i)^{b_i} \\
&= \sup_{x\in\mathcal{S}_{\boldsymbol{\nu}}} \prod_{i=1}^d \mathrm{m}(\mathcal{F}(x)_i/x_i; C_i)^{b_i} = \lambda,
\end{aligned}$$

Similarly, (10.5) and Theorem 10.2.1 imply

$$\lambda = \inf_{x\in\operatorname{int}(\mathcal{C})\cap\mathcal{S}_{\boldsymbol{\nu}}} \prod_{i=1}^d \mathrm{M}(\mathcal{F}(x)_i/x_i; C_i)^{b_i} = \inf_{x\in\operatorname{int}(\mathcal{C})} \prod_{i=1}^d \mathrm{M}(\mathcal{F}(x)_i/x_i; C_i)^{b_i},$$

which concludes the proof. □

We conclude this section with a result which shows that the iterates of an order-preserving multi-subhomogeneous mapping induce two monotonic sequences which converge to the maximal eigenvalue of the mapping. Similar results were proved in the literature of nonnegative tensors as for instance in [71, Theorem 2.4], [26, Theorem 7], [36, Proposition 28]$^\diamond$, and for multi-homogeneous mappings on $\mathcal{C} = \mathbb{R}^{n_1}_+ \times \ldots \times \mathbb{R}^{n_d}_+$ in [39, Lemma 6.4]$^\diamond$.

Theorem 10.2.3. Let $\mathcal{C} \subset \mathcal{V}$ be a solid closed cone and let $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d_+$ be a monotonic multi-normalization of $\mathcal{C}$. Set $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$ and let $\mathcal{F}\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ be order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}$. Suppose that $\rho(A) \leq 1$, A is irreducible and let $\mathbf{b} \in \mathbb{R}^d_{++}$ be the Perron vector of $A^\top$. Furthermore, let $\mathcal{G}\colon \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}} \to \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ be defined as

$$\mathcal{G}(x) = \boldsymbol{\nu}(\mathcal{F}(x))^{-I} \otimes \mathcal{F}(x) \qquad \forall x \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}.$$

Then, for every $x \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$, it holds

$$\prod_{i=1}^d \mathrm{m}(\mathcal{F}(\mathcal{G}(x))_i/\mathcal{G}(x)_i; C_i)^{b_i} \geq \prod_{i=1}^d \mathrm{m}(\mathcal{F}(x)_i/x_i; C_i)^{b_i},$$

$$\prod_{i=1}^d \mathrm{M}(\mathcal{F}(\mathcal{G}(x))_i/\mathcal{G}(x)_i; C_i)^{b_i} \leq \prod_{i=1}^d \mathrm{M}(\mathcal{F}(x)_i/x_i; C_i)^{b_i}.$$

Proof. Let $x \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$, then $\mathcal{G}(x) \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ and by Lemma 5.1.3, we have that $\mathbf{m}(\mathcal{G}(x)/x;\mathcal{C}) \leq \mathbf{1} \leq \mathbf{M}(\mathcal{G}(x)/x;\mathcal{C})$. As $\mathcal{F}$ is order-preserving and multi-homogeneous, we have

$$\begin{aligned}\mathbf{m}(\mathcal{G}(x)/x;\mathcal{C})^A \otimes \mathcal{F}(x) = \mathcal{F}(\mathbf{m}(\mathcal{G}(x)/x;\mathcal{C}) \otimes x) \preceq_{\mathcal{C}} \mathcal{F}(\mathcal{G}(x)),\\ \mathcal{F}(\mathcal{G}(x)) \preceq_{\mathcal{C}} \mathcal{F}(\mathbf{M}(\mathcal{G}(x)/x;\mathcal{C}) \otimes x) = \mathbf{M}(\mathcal{G}(x)/x;\mathcal{C})^A \otimes \mathcal{F}(x).\end{aligned}$$

It follows that

$$\begin{aligned}\mathbf{M}(\mathcal{F}(\mathcal{G}(x))/\mathcal{G}(x);\mathcal{C}) &= \boldsymbol{\nu}(\mathcal{F}(x)) \circ \mathbf{M}(\mathcal{F}(\mathcal{G}(x))/\mathcal{F}(x);\mathcal{C})\\ &\leq \boldsymbol{\nu}(\mathcal{F}(x)) \circ \mathbf{M}\left(\mathbf{M}(\mathcal{G}(x)/x;\mathcal{C})^A \otimes \mathcal{F}(x)/\mathcal{F}(x);\mathcal{C}\right)\\ &= \boldsymbol{\nu}(\mathcal{F}(x)) \circ \mathbf{M}(\mathcal{G}(x)/x;\mathcal{C})^A \circ \mathbf{M}\left(\mathcal{F}(x)/\mathcal{F}(x);\mathcal{C}\right)\\ &= \boldsymbol{\nu}(\mathcal{F}(x)) \circ \mathbf{M}(\mathcal{G}(x)/x;\mathcal{C})^A \circ \mathbf{M}(\mathcal{F}(x)/x;\mathcal{C})^{-I} \circ \mathbf{M}(\mathcal{F}(x)/x;\mathcal{C})\\ &= \mathbf{M}(\mathcal{G}(x)/x;\mathcal{C})^A \circ \mathbf{M}(\mathcal{G}(x)/x;\mathcal{C})^{-I} \circ \mathbf{M}(\mathcal{F}(x)/x;\mathcal{C})\\ &= \mathbf{M}(\mathcal{G}(x)/x;\mathcal{C})^{A-I} \circ \mathbf{M}(\mathcal{F}(x)/x;\mathcal{C}), \qquad (10.6)\end{aligned}$$

and

$$\begin{aligned}\mathbf{m}(\mathcal{G}(x)/x;\mathcal{C})^{A-I} \circ \mathbf{m}(\mathcal{F}(x)/x;\mathcal{C}) &= \mathbf{m}(\mathcal{G}(x)/x;\mathcal{C})^A \circ \mathbf{m}(\mathcal{G}(x)/x;\mathcal{C})^{-I} \circ \mathbf{m}(\mathcal{F}(x)/x;\mathcal{C})\\ &= \boldsymbol{\nu}(\mathcal{F}(x)) \circ \mathbf{m}(\mathcal{G}(x)/x;\mathcal{C})^A \circ \mathbf{m}(\mathcal{F}(x)/x;\mathcal{C})^{-I} \circ \mathbf{m}(\mathcal{F}(x)/x;\mathcal{C})\\ &= \boldsymbol{\nu}(\mathcal{F}(x)) \circ \mathbf{m}(\mathcal{G}(x)/x;\mathcal{C})^A \circ \mathbf{m}\left(\mathcal{F}(x)/\mathcal{F}(x);\mathcal{C}\right)\\ &= \boldsymbol{\nu}(\mathcal{F}(x)) \circ \mathbf{m}\left(\mathbf{m}(\mathcal{G}(x)/x;\mathcal{C})^A \otimes \mathcal{F}(x)/\mathcal{F}(x);\mathcal{C}\right)\\ &\leq \boldsymbol{\nu}(\mathcal{F}(x)) \circ \mathbf{m}(\mathcal{F}(\mathcal{G}(x))/\mathcal{F}(x);\mathcal{C}) = \mathbf{m}(\mathcal{F}(\mathcal{G}(x))/\mathcal{G}(x);\mathcal{C}). \qquad (10.7)\end{aligned}$$

Now, as $\mathbf{m}(\mathcal{G}(x)/x;\mathcal{C}) \leq \mathbf{1} \leq \mathbf{M}(\mathcal{G}(x)/x;\mathcal{C})$ and $-(A^\top - I)\mathbf{b} = -(\rho(A)-1)\mathbf{b} \in \mathbb{R}^d_+$, we have

$$\prod_{i=1}^d (\mathbf{M}(\mathcal{G}(x)/x;\mathcal{C})^{A-I})_i^{b_i} = \prod_{i=1}^d \mathrm{M}(\mathcal{G}(x)_i/x_i;C_i)^{(\rho(A)-1)b_i} \leq 1,$$

$$1 \leq \prod_{i=1}^d \mathrm{m}(\mathcal{G}(x)_i/x_i;C_i)^{(\rho(A)-1)b_i} = \prod_{i=1}^d (\mathbf{m}(\mathcal{G}(x)/x;\mathcal{C})^{A-I})_i^{b_i}.$$

Hence, with (10.6), we obtain

$$\begin{aligned}\prod_{i=1}^d \mathrm{M}(\mathcal{F}(\mathcal{G}(x))_i/\mathcal{G}(x)_i;C_i)^{b_i} &= \prod_{i=1}^d \mathbf{M}(\mathcal{F}(\mathcal{G}(x))/\mathcal{G}(x);\mathcal{C})_i^{b_i} \\ &\leq \prod_{i=1}^d \big(\mathbf{M}(\mathcal{G}(x)/x;\mathcal{C})^{A-I} \circ \mathbf{M}(\mathcal{F}(x)/x;\mathcal{C})\big)_i^{b_i} \\ &= \Big(\prod_{i=1}^d (\mathbf{M}(\mathcal{G}(x)/x;\mathcal{C})^{A-I})_i^{b_i}\Big) \prod_{i=1}^d \mathrm{M}(\mathcal{F}(x)_i/x_i;C_i)^{b_i} \\ &\leq \prod_{i=1}^d \mathrm{M}(\mathcal{F}(x)_i/x_i;C_i)^{b_i},\end{aligned}$$

and, with (10.7), we obtain

$$\begin{aligned}\prod_{i=1}^d \mathbf{m}(\mathcal{F}(x)_i/x_i;C_i)^{b_i} &\leq \Big(\prod_{i=1}^d (\mathbf{m}(\mathcal{G}(x)/x;\mathcal{C})^{A-I})_i^{b_i}\Big) \prod_{i=1}^d \mathrm{m}(\mathcal{F}(x)_i/x_i;C_i)^{b_i} \\ &= \prod_{i=1}^d \big(\mathbf{m}(\mathcal{G}(x)/x;\mathcal{C})^{A-I} \circ \mathbf{m}(\mathcal{F}(x)/x;\mathcal{C})\big)_i^{b_i} \\ &\leq \prod_{i=1}^d \mathrm{m}(\mathcal{F}(\mathcal{G}(x))_i/\mathcal{G}(x)_i;C_i)^{b_i},\end{aligned}$$

which concludes the proof. □

10.3 Semi-derivative of mappings on cones

As discussed in [5] and [60], many results of the non-linear Perron-Frobenius theory still hold with a weaker notion of derivative, namely semi-differentiability. We recall the definition of semi-differentiability from [60, page 124]: Let $(V, \|\cdot\|_V)$ be a finite dimensional normed real vector space. For $x \in V$ and $r > 0$ define $B_r(x) = \{v \in V \mid \|v - x\|_V < r\}$. If $U \subset V$ and $x \in U$, then U is locally convex at x if there exists $r > 0$ such that $B_r(x) \cap U$ is convex. Given $U \subset V$ locally convex at $x \in U$, define

$$S_x = \{v \in V \mid \text{ there exist } t_v > 0 \text{ such that } x + tv \in U \text{ for all } t \in [0, t_v]\}.$$

The set S_x is convex and $tS_x = S_x$ for all $t > 0$. Suppose that U is locally convex at x and let $f : U \to V$ be such that the one-sided Gateaux derivative

$$f'_x(v) = \lim_{t \to 0^+} \frac{f(x + tv) - f(x)}{t}$$

exists for all $v \in S_x$. For all $v \in S_x$ with $x + v \in U$, we can write

$$R(v) = f(x + v) - f(x) - f'_x(v).$$

If $f'_x : S_x \to V$ is continuous and

$$\lim_{\|v\|_V \to 0} \frac{\|R(v)\|_V}{\|v\|_V} = 0,$$

we say that $f : U \to V$ is *semi-differentiable* at $x \in U$. For $S \subset U$, if f is semi-differentiable at every $x \in S$, then we say that f is semi-differentiable on S and if is semi-differentiable at every $x \in U$, then we say that f is semi-differentiable. The map $f'_x : S_x \to V$ is called the *semi-derivative* of f at x. In general, f'_x may be nonlinear. However, $f'_x : S_x \to V$ is positively homogeneous of degree one. If $f : U \to V$ is Fréchet differentiable at $x \in U$, then f is semi-differentiable at x and $f'_x(v) = Df(x)v$ for all $v \in S_x$. The main motivation for the consideration of semi-differentiable mappings here is that the maximum and the minimum functions are semi-differentiable [5]. Note that if $C \subset V$ is a solid closed cone and $f : C \to V$ is order-preserving, then for all $x \in \mathrm{int}(C)$ we have $C \subset S_x$ and if f is semi-differentiable at x, then

$$f'_x(v) = \lim_{t \to 0^+} \frac{f(x + tv) - f(x)}{t} \in C.$$

We refer to [5] for a detailed discussion on the properties of semi-differentiability and its relationship with other types of derivatives.

In the following lemma we discuss properties of the semi-derivative of a mapping on $\mathcal{C} \subset \mathcal{V}$ where $\mathcal{C}$ is a solid closed cone in the product of finite dimensional real vectors spaces $\mathcal{V} = V_1 \times \ldots \times V_d$.

Lemma 10.3.1. Let $\mathcal{C} \subset \mathcal{V}$ be a solid closed cone and $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ an order-preserving mapping. Let $u \in \mathrm{int}(\mathcal{C})$ and suppose that $\mathcal{F}$ is semi-differentiable at u. Then, for every $v, w \in \mathcal{C}$ such that $v \sim_{\mathcal{C}} w$, it holds $\mathcal{F}'_u(v) \sim_{\mathcal{C}} \mathcal{F}'_u(w)$. Furthermore, for every $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$, if $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ is given by $\mathcal{G}(x) = \boldsymbol{\alpha} \otimes \mathcal{F}(x)$, then $\mathcal{G}$ is semi-differentiable at u and it holds $(\mathcal{F}'_u)^k(v) \sim_{\mathcal{C}} (\mathcal{G}'_u)^k(v)$ for every $k \geq 1$ and $v \in \mathcal{C}$.

Proof. Let $v, w \in \mathcal{C}$ with $v \sim_{\mathcal{C}} w$, then, there exists $\alpha, \beta > 0$ such that $\alpha v \preceq_{\mathcal{C}} w \preceq_{\mathcal{C}} \beta v$ and thus $t\alpha v \preceq_{\mathcal{C}} tw \preceq_{\mathcal{C}} t\beta v$ for every $t > 0$. Hence, for every t small enough so that $u + t\alpha v, u + tw, u + t\beta v \in \mathcal{C}$ we have $\mathcal{F}(u + t\alpha v) \preceq_{\mathcal{C}} \mathcal{F}(u + tw) \preceq_{\mathcal{C}} \mathcal{F}(u + t\beta w)$, it follows that

$$\mathcal{F}'_u(\alpha v) = \lim_{t \to 0^+} \frac{\mathcal{F}(u + t\alpha v) - \mathcal{F}(u)}{t} \preceq_{\mathcal{C}} \lim_{t \to 0^+} \frac{\mathcal{F}(u + tw) - \mathcal{F}(u)}{t} = \mathcal{F}'_u(w).$$

And by the homogeneity of the semi-derivative it follows that $\alpha\mathcal{F}'_u(v) = \mathcal{F}'_u(\alpha u) \preceq_{\mathcal{C}} \mathcal{F}'_u(w)$. Similarly, it holds

$$\mathcal{F}'_u(w) = \lim_{t\to 0^+} \frac{\mathcal{F}(u+tw)-\mathcal{F}(u)}{t} \preceq_{\mathcal{C}} \lim_{t\to 0^+} \frac{\mathcal{F}(u+t\beta v)-\mathcal{F}(u)}{t} = \mathcal{F}'_u(\beta v) = \beta\mathcal{F}'_u(v).$$

It follows that $\alpha\mathcal{F}'_u(v) \preceq_{\mathcal{C}} \mathcal{F}'_u(w) \preceq_{\mathcal{C}} \beta\mathcal{F}'_u(v)$, i.e. $\mathcal{F}'_u(v) \sim_{\mathcal{C}} \mathcal{F}'_u(w)$. Let $v \in \mathcal{C}$, we show by induction over $k \geq 1$, that $(\mathcal{G}'_u)^k(v) \sim_{\mathcal{C}} (\mathcal{F}'_u)^k(v)$. If $k = 1$, then $\mathcal{G}'_u(v) = \boldsymbol{\alpha}^{-I} \otimes \mathcal{F}'_u(v) \sim_{\mathcal{C}} \mathcal{F}'_u(v)$. Now, let $k \geq 1$ and suppose that $(\mathcal{G}'_u)^k(v) \sim_{\mathcal{C}} (\mathcal{F}'_u)^k(v)$. Then, by the above argument, we have $(\mathcal{G}'_u)((\mathcal{G}'_u)^k(v)) \sim_{\mathcal{C}} (\mathcal{G}'_u)((\mathcal{F}'_u)^k(v))$ and thus

$$\begin{aligned}(\mathcal{G}'_u)^{k+1}(v) = (\mathcal{G}'_u)((\mathcal{G}'_u)^k(v)) &\sim_{\mathcal{C}} (\mathcal{G}'_u)((\mathcal{F}'_u)^k(v)) \\ &= \boldsymbol{\alpha}^{-I} \otimes (\mathcal{F}'_u)((\mathcal{F}'_u)^k(v)) \sim_{\mathcal{C}} (\mathcal{F}'_u)^{k+1}(v),\end{aligned}$$

which concludes the induction proof. □

10.4 Uniqueness of positive eigenvectors

We prove a sufficient condition for the uniqueness of an eigenvector in the interior of a solid cone in the product of finite dimensional real vector spaces $\mathcal{V} = V_1 \times \ldots \times V_d$. The proof resembles that of [39, Theorem 5.3]$^\diamond$ for the case of multi-homogeneous mappings on $\mathcal{C} = \mathbb{R}^{n_1}_+ \times \ldots \times \mathbb{R}^{n_d}_+$ and [60, Theorem 6.1.7] for the case $d = 1$. The later results show that the eigenvalues of an eigenvector on the boundary of the cone must be necessarily strictly smaller than the eigenvalue of an eigenvector in the interior of the cone. This conclusion is also implied by the next theorem which further shows the uniqueness of an eigenvector in the interior of the cone up to scale. We should nevertheless point out that the uniqueness of an eigenvector in the interior of the cone are proved under less restrictive conditions in [39, Theorem 5.5]$^\diamond$ and [60, Theorem 6.4.6] for the particular cases mentioned above.

Theorem 10.4.1. Let $\mathcal{C} \subset \mathcal{V}$ be a solid closed cone, $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d_+$ a monotonic multi-normalization of $\mathcal{C}$ and set $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ be continuous, order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}_+$. Suppose that A is irreducible, $\rho(A) = 1$ and let $\mathbf{b} \in \mathbb{R}^d_{++}$ be the Perron vector of $A^\top$. Furthermore, suppose that $\mathcal{F}$ has an eigenvector $u \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ with corresponding eigenvalue $\theta > 0$. Suppose that $\mathcal{F}$ is semi-differentiable at u and let $\mathcal{F}'_u\colon \mathcal{C} \to \mathcal{C}$ be its semi-derivative. If for every $y \in \mathcal{C} \setminus \{0\}$ there exists $i \in \{1,\ldots,d\}$ and an integer $N \geq 1$ such that

$$-\sum_{k=1}^{N} (\mathcal{F}'_u)^k(-y) \in C_1 \times \ldots \times C_{i-1} \times \operatorname{int}(C_i) \times C_{i+1} \times \ldots \times C_d, \tag{10.8}$$

then for every $x \in \mathcal{S}_{\boldsymbol{\nu}}$ such that $x \neq u$, it holds $\prod_{i=1}^d \mathrm{m}(\mathcal{F}(x)_i/x_i; C_i)^{b_i} < \theta$. Furthermore, u is the unique eigenvector of $\mathcal{F}$ in $\mathcal{S}_{\boldsymbol{\nu}}$ and if $v \in \mathcal{S}_{\boldsymbol{\nu}} \setminus \operatorname{int}(\mathcal{C})$ is an eigenvector of $\mathcal{F}$ with corresponding eigenvalue ϑ, then $\vartheta < \theta$.

Remark 10.4.2. Note that if in Theorem 10.4.1, $\mathcal{F}$ is a polynomial induced by a nonnegative tensor, then the weak irreducibility of $\mathcal{F}$ and condition (10.8) are related by Lemma 9.4.6.

Proof of Theorem 10.4.1. Let $x \in \mathcal{S}_{\boldsymbol{\nu}}$ with $x \neq u$ and set $\boldsymbol{\alpha} = \mathbf{m}(u/x)$. Note that $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$ since $x_i \neq 0$ and $u_i \in \operatorname{int}(C_i)$ for all $i = 1, \dots, d$. Furthermore, $\boldsymbol{\alpha} \otimes x \precneqq_{\mathcal{C}} u$ since $\boldsymbol{\alpha} \otimes x = u$ would contradict $x \neq u$. Indeed, $(\operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}, \boldsymbol{\mu}_{\mathcal{C}})$ is a d-metric space by Lemma 6.2.2 and $\boldsymbol{\mu}_{\mathcal{C}}(\boldsymbol{\alpha} \otimes x, u) = \boldsymbol{\mu}_{\mathcal{C}}(x, u)$.

First, suppose that $\mathcal{F}(u) = u$ so that $\theta = 1$. Then, $\mathcal{F}^k(u) = u$ for all $k \geq 1$ and the chain rule for semi-differentials (see [60, Lemma 6.1.6]) imply that $\mathcal{F}^k$ is semi-differentiable at u for every k and $(\mathcal{F}'_u)^k = (\mathcal{F}^k)'_u$ for every $k \geq 1$. Let $z \in \mathcal{C} \setminus \{0\}$ be defined as $z = u - \boldsymbol{\alpha} \otimes x$. Then, by assumption, there exist $i \in \{1, \dots, d\}$ and $N \geq 1$ satisfying

$$-\sum_{k=1}^{N} (\mathcal{F}'_u)^k(-z) \in \mathcal{C}_i^\circ,$$

with $\mathcal{C}_i^\circ = C_1 \times \dots \times C_{i-1} \times \operatorname{int}(C_i) \times C_{i+1} \times \dots \times C_d$. For $t \in [0,1]$, let

$$u_t = (1-t)\, u + t\, \boldsymbol{\alpha} \otimes x = u - t\, z \in \mathcal{C}.$$

As $\mathcal{F}^k$ is semi-differentiable at u,

$$\mathcal{F}^k(u_t) = \mathcal{F}^k(u) + t\, (\mathcal{F}^k)'_u(-z) + \|t\, z\|_{\mathcal{V}} \tilde{\mathcal{R}}_k(-t\, z)$$

where $\|\cdot\|_{\mathcal{V}}$ is a norm on $\mathcal{V}$ and $\lim_{y \to 0} \mathcal{R}_k(y) = 0$. Hence,

$$\sum_{k=1}^{N} \big(\mathcal{F}^k(u) - \mathcal{F}^k(u_t)\big) = t\Big(- \sum_{k=1}^{N} (\mathcal{F}'_u)^k(-z) - \|z\|_{\mathcal{V}} \mathcal{R}_k(-t\, z)\Big).$$

As, $-\sum_{k=1}^{N} (\mathcal{F}'_u)^k(-z) \in \operatorname{int}(\mathcal{C})$ and $\lim_{y \to 0} \mathcal{R}_k(y) = 0$, there exists $t_1 \in (0,1)$ such that

$$\sum_{k=1}^{N} \big(\mathcal{F}^k(u) - \mathcal{F}^k(u_t)\big) \in \mathcal{C}_i^\circ \qquad \forall t \in (0, t_1).$$

Note that $\boldsymbol{\alpha} \otimes x \leq u_t$ for all $t \in [0,1]$ and thus

$$\sum_{k=1}^{N} \big(\mathcal{F}^k(u_t) - \mathcal{F}^k(\boldsymbol{\alpha} \otimes x)\big) \in \mathcal{C} \qquad \forall t \in (0, t_1).$$

It follows that

$$\sum_{k=1}^{N} \big(\mathcal{F}^k(u) - \mathcal{F}^k(\boldsymbol{\alpha} \otimes x)\big) = \overbrace{\sum_{k=1}^{N} \big(\mathcal{F}^k(u) - \mathcal{F}^k(u_t)\big)}^{\in \mathcal{C}_i^\circ} + \overbrace{\sum_{k=1}^{N} \big(\mathcal{F}^k(u_t) - \mathcal{F}^k(\boldsymbol{\alpha} \otimes x)\big)}^{\in \mathcal{C}} \in \mathcal{C}_i^\circ.$$

With $\mathcal{F}^k(u) = u$ and $\mathcal{F}^k(\boldsymbol{\alpha} \otimes x) = \boldsymbol{\alpha}^{A^k} \otimes \mathcal{F}^k(x)$, it follows that

$$\sum_{k=1}^{N} \big(\mathcal{F}^k(u) - \mathcal{F}^k(\boldsymbol{\alpha} \otimes x)\big) = \sum_{k=1}^{N} \big(u - \boldsymbol{\alpha}^{A^k} \otimes \mathcal{F}^k(x)\big) \in \mathcal{C}_i^\circ.$$

Now, let $\boldsymbol{\beta} = \mathbf{m}(\mathcal{F}(x)/x;\mathcal{C})$, then $\boldsymbol{\beta}\otimes x \preceq_{\mathcal{C}} \mathcal{F}(x)$ and Lemma 3.3.8 implies that

$$\boldsymbol{\beta}^{\sum_{j=0}^{k-1} A^j} \otimes x \preceq_{\mathcal{C}} \mathcal{F}^k(x) \qquad \forall k \geq 1.$$

It follows that

$$\frac{1}{N}\sum_{k=1}^{N}\big(u-\boldsymbol{\alpha}^{A^k}\otimes\mathcal{F}^k(x)\big) \preceq_{\mathcal{C}} u-\Big(\frac{1}{N}\sum_{k=1}^{N}\boldsymbol{\alpha}^{A^k}\circ\boldsymbol{\beta}^{\sum_{j=0}^{k-1}A^j}\Big)\otimes x\in\mathcal{C}_i^{\circ}.$$

By the inequality between arithmetic and geometric means, we have

$$\boldsymbol{\alpha}^{\frac{1}{N}\sum_{k=1}^{N}A^k}\circ\boldsymbol{\beta}^{\frac{1}{N}\sum_{k=1}^{N}\sum_{j=0}^{k-1}A^j} = \Big(\prod_{k=1}^{N}\boldsymbol{\alpha}^{A^k}\circ\boldsymbol{\beta}^{\sum_{j=0}^{k-1}A^j}\Big)^{\frac{1}{N}I} \leq \frac{1}{N}\sum_{k=1}^{N}\boldsymbol{\alpha}^{A^k}\circ\boldsymbol{\beta}^{\sum_{j=0}^{k-1}A^j}$$

and thus

$$u-\Big(\boldsymbol{\alpha}^{\frac{1}{N}\sum_{k=1}^{N}A^k}\circ\boldsymbol{\beta}^{\frac{1}{N}\sum_{k=1}^{N}\sum_{j=0}^{k-1}A^j}\Big)\otimes x\in\mathcal{C}_i^{\circ}.$$

In particular, as $\boldsymbol{\alpha} = \mathbf{m}(u/x;\mathcal{C})$ and $\mathcal{C}$ is closed, we have

$$\Big(\boldsymbol{\alpha}^{\frac{1}{N}\sum_{k=1}^{N}A^k}\circ\boldsymbol{\beta}^{\frac{1}{N}\sum_{k=1}^{N}\sum_{j=0}^{k-1}A^j}\Big)\lneq\boldsymbol{\alpha}.$$

It follows that

$$\Big(\prod_{i=1}^{d}\alpha_i^{b_i}\Big)\Big(\prod_{i=1}^{d}\beta_i^{b_i}\Big)^{\frac{N+1}{2}} = \prod_{i=1}^{d}\Big(\boldsymbol{\alpha}^{\frac{1}{N}\sum_{k=1}^{N}A^k}\circ\boldsymbol{\beta}^{\frac{1}{N}\sum_{k=1}^{N}\sum_{j=0}^{k-1}A^j}\Big)_i^{b_i} < \prod_{i=1}^{d}\alpha_i^{b_i},$$

and thus

$$\prod_{i=1}^{d}\mathrm{m}(\mathcal{F}(x)_i/x_i;C_i)^{b_i} = \prod_{i=1}^{d}\beta_i^{b_i} < 1^{\frac{2}{N+1}} = 1 = \theta.$$

To conclude the proof, suppose that $\mathcal{F}(u) = \boldsymbol{\lambda}\otimes u$ with $\boldsymbol{\lambda}\in\mathbb{R}^d_+$. Note that if $\boldsymbol{\lambda}\notin\mathbb{R}^d_{++}$, then $\theta = \prod_{i=1}^d \lambda_i^{b_i} = 0$ which contradicts our assumption that $\theta > 0$. Define $\mathcal{G}\colon\mathcal{C}\to\mathcal{C}$ as $\mathcal{G}(x) = \boldsymbol{\lambda}^{-I}\otimes\mathcal{F}(x)$ for every $x\in\mathcal{C}$. Then, $\mathcal{G}(u) = u$ and $\mathbf{m}(\mathcal{G}(x)/x;\mathcal{C}) = \boldsymbol{\lambda}^{-I}\circ\mathbf{m}(\mathcal{F}(x)/x;\mathcal{C})$. Furthermore, $\mathcal{G}$ is continuous, order-preserving, and multi-homogeneous of degree A. Note that $\mathcal{G}$ satisfy (10.8). Indeed, if $y\in\mathcal{C}\setminus\{0\}$, then there exists $N\geq 1$ and $i\in\{1,\ldots,d\}$ such that $\sum_{k=1}^N(\mathcal{F}'_u)^k(y)\in\mathcal{C}_i^{\circ}$ and by Lemma 10.3.1, we have that $(\mathcal{F}'_u)^k(y)\sim_{\mathcal{C}}(\mathcal{G}'_u)^k(y)$ for all $k = 1,\ldots,N$. It follows that $\sum_{k=1}^N(\mathcal{G}'_u)^k(y)\in\mathcal{C}_i^{\circ}$ which implies that $\mathcal{G}$ satisfies (10.8). Hence the above discussion implies that for $x\in\mathcal{S}_{\boldsymbol{\nu}}$, we have

$$\prod_{i=1}^{d}\lambda_i^{-b_i}\,\mathrm{m}(\mathcal{F}(x)_i/x_i;C_i)^{b_i} = \prod_{i=1}^{d}\mathrm{m}(\mathcal{G}(x)_i/x_i;C_i)^{b_i} < 1,$$

which implies that $\prod_{i=1}^d\mathrm{m}(\mathcal{F}(x)_i/x_i;C_i)^{b_i} < \prod_{i=1}^d\lambda_i^{b_i} = \theta$.

Finally, if $v\in\mathcal{S}_{\boldsymbol{\nu}}$ is an eigenvector of $\mathcal{F}$ with corresponding eigenvalue ϑ, then there exists $\boldsymbol{\beta}\in\mathbb{R}^d_+$ such that $\mathcal{F}(v) = \boldsymbol{\beta}\otimes v$ so that $\prod_{i=1}^d\mathrm{m}(\mathcal{F}(v)_i/v_i;C_i)^{b_i} = \prod_{i=1}^d\beta_i^{b_i} = \vartheta$. If $v\notin\mathrm{int}(\mathcal{C})$ then $v\neq u$ since $u\in\mathrm{int}(\mathcal{C})$ and thus the above discussion implies $\vartheta < \theta$. If $v\in\mathrm{int}(\mathcal{C})$, then Theorem 10.2.1 implies that $\vartheta = \theta$. Therefore, we must have $v = u$ otherwise the above discussion would imply the contradiction $\vartheta < \theta$. ☐

Next we prove another result which guarantees the uniqueness of an eigenvector in the interior of the cone. In the linear case, it reduces to say that the positive eigenvector of a primitive matrix is unique up to scale. While this is more restrictive than the condition of the previous Theorem 10.4.1, it has the advantage of not assuming semi-differentiability and implies the convergence of the normalized iterates towards the eigenvector in the interior of a cone. For its statement, we introduce the following definition which reduces to Definition 6.5.2 of [60] in the case $d = 1$.

Definition 10.4.3. Let $\mathcal{C} \subset \mathcal{V}$ be a solid closed cone and let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$. We say that $\mathcal{F}$ satisfies condition L_d at $x \in \operatorname{int}(\mathcal{C})$ if for every $y \in \operatorname{int}(\mathcal{C})$ with $y \lneq_{\mathcal{C}} x$, there exists a positive integer $k_y \geq 1$ and $i_y \in \{1, \ldots, d\}$ such that $\mathcal{F}^{k_y}(y)_{i_y} \prec_{\mathcal{C}} \mathcal{F}^{k_y}(x)_{i_y}$. We say that $\mathcal{F}$ satisfies condition U_d at $x \in \operatorname{int}(\mathcal{C})$ if for every $y \in \operatorname{int}(\mathcal{C})$ with $x \lneq_{\mathcal{C}} y$, there exists a positive integer $k_y \geq 1$ and $i_y \in \{1, \ldots, d\}$ such that $\mathcal{F}^{k_y}(x)_{i_y} \prec_{\mathcal{C}} \mathcal{F}^{k_y}(y)_{i_y}$.

Note that every strongly order-preserving mapping satisfies both, condition L_d and condition U_d, at any $x \in \operatorname{int}(\mathcal{C})$. Furthermore, as discussed above, if $M \in \mathbb{R}^{n\times n}_{+}$, then M is strongly order-preserving if $M \in \mathbb{R}^{n\times n}_{++}$ and $x \mapsto Mx$ satisfies condition L_d or U_d at $x \in \mathbb{R}^n_{++}$ if it is a primitive matrix. Indeed, M being primitive implies the existence of $k \geq 1$ such that $M^k \in \mathbb{R}^n_{++}$. The connection between the conditions of Definition 10.4.3 and primitivity is made more clear by the following result which generalizes Lemma 6.5.7 of [60]:

Lemma 10.4.4. Let $\mathcal{C} \subset \mathcal{V}$ be a solid closed cone and $\mathcal{F}\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ be order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}$. Suppose that there exists $u \in \operatorname{int}(\mathcal{C})$ and $\boldsymbol{\lambda} \in \mathbb{R}^d_{+}$ such that $\mathcal{F}(u) = \boldsymbol{\lambda} \otimes u$ and $\mathcal{F}$ is semi-differentiable at u with semi-derivative $\mathcal{F}'_u\colon \mathcal{V} \to \mathcal{V}$ at u. Finally, for $i = 1, \ldots, d$, let

$$\mathcal{C}_i^{\circ} = C_1 \times \ldots \times C_{i-1} \times \operatorname{int}(C_i) \times C_{i+1} \ldots \times C_d.$$

Then:

i) If there exists $k^{\star} \geq 1$ and $i^{\star} \in \{1, \ldots, d\}$ such that $(\mathcal{F}'_u)^{k^{\star}}(v) \in \mathcal{C}^{\circ}_{i^{\star}}$ for every $v \in \mathcal{C} \setminus \{0\}$, then $\mathcal{F}$ satisfies condition U_d at u.

ii) If there exists $k_{\star} \geq 1$ and $i_{\star} \in \{1, \ldots, d\}$ such that $(\mathcal{F}'_u)^{k_{\star}}(-v) \in -\mathcal{C}^{\circ}_{i_{\star}}$ for every $v \in \mathcal{C} \setminus \{0\}$, then $\mathcal{F}$ satisfies condition L_d at u.

Proof. First of all, note that as $\mathcal{F}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$, we have $\boldsymbol{\lambda} \in \mathbb{R}^d_{++}$. So, let $\mathcal{H}\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ be given by $\mathcal{H}(x) = \boldsymbol{\lambda}^{-I} \otimes \mathcal{F}(x)$ for all $x \in \operatorname{int}(\mathcal{C})$. Then, it holds $\mathcal{H}(u) = u$ and, by Lemma 10.3.1, $(\mathcal{H}'_u)^k(v) \sim_{\mathcal{C}} (\mathcal{F}'_u)^k(v)$ for all $v \in \mathcal{C}$ and $k \geq 1$. Furthermore, note that $\mathcal{H}$ satisfies condition U_d, resp. condition L_d, at u if and only if $\mathcal{F}$ does. For $i = 1, \ldots, d$ let $\|\cdot\|_i$ be a norm on V_i and set $\|v\|_{\mathcal{V}} = \max_{i=1,\ldots,d} \|v_i\|_i$ for all $v \in \mathcal{V}$. Furthermore, let $\mathcal{S} = \{x \in \mathcal{C} \mid \|x_1\|_1 = \ldots = \|x_d\|_d = 1\}$. Then $\mathcal{S}$ is a compact set since $\mathcal{C}$ is closed.

i) Let $k^{\star} \geq 1$ and $i^{\star} \in \{1, \ldots, d\}$ be such that $(\mathcal{H}'_u)^{k^{\star}}(v) \in \mathcal{C}^{\circ}_{i^{\star}}$ for every $v \in \mathcal{C}\setminus\{0\}$. Note that $k^{\star}$ and $i^{\star}$ exist since $(\mathcal{H}'_u)^k(v) \sim_{\mathcal{C}} (\mathcal{F}'_u)^k(v)$ for all $v \in \mathcal{C}$ and $k \geq 1$.

By the chain rule for semi-derivatives (see [60, Lemma 6.1.6]), for $v \in \mathcal{C} \setminus \{0\}$, we have

$$\mathcal{H}^{k^\star}(u+v) = \mathcal{H}^{k^\star}(u) + (\mathcal{H}'_u)^{k^\star}(v) + \|v\|_{\mathcal{V}}\mathcal{R}_{k^\star}(v), \tag{10.9}$$

where $\|\cdot\|_{\mathcal{V}}$ is a norm on $\mathcal{V}$ and $\lim_{\|v\|_{\mathcal{V}}\to 0}\mathcal{R}_{k^\star}(v) = 0$. Now, the continuity of $(\mathcal{H}'_u)^{k^\star}$ together with the compactness of $\mathcal{S}$ imply that $(\mathcal{H}'_u)^{k^\star}(\mathcal{S}) \subset \mathcal{C}^\circ_{i^\star}$. Let $(\mathcal{H}'_u)^{k^\star}(\mathcal{S})_i = \{x_i \,|\, x \in (\mathcal{H}'_u)^{k^\star}(\mathcal{S})\} \subset \operatorname{int}(C_i)$. There exists $\tau > 0$ such that the τ-neighborhood $N_{i,\tau}((\mathcal{H}'_u)^{k^\star}(\mathcal{S})_i)$ is contained in $\operatorname{int}(C_i)$. By the positive homogeneity of the semi-derivative, for all $t \in (0,1]$, we have

$$N_{i,t\tau}((\mathcal{H}'_u)^{k^\star}(t\mathcal{S})_i) = tN_{i,\tau}((\mathcal{H}'_u)^{k^\star}(\mathcal{S})_i) \subset \operatorname{int}(C_i).$$

There exists $\delta > 0$ such that $\|\mathcal{R}_{k^\star}(v)\|_{\mathcal{V}} < \tau$ for $\|v\|_{\mathcal{V}} \leq \delta$. As $\|\mathcal{R}_{k^\star}(v)_i\|_i \leq \|\mathcal{R}_{k^\star}(v)\|_{\mathcal{V}}$, it follows that for every $v \in \mathcal{C}$ such that $0 < \|v\|_{\mathcal{V}} \leq \delta$, it holds $(\mathcal{H}'_u)^{k^\star}(v)_i + \|v\|_{\mathcal{V}}\mathcal{R}_{k^\star}(v)_i \in \operatorname{int}(C_i)$ and therefore by (10.9) we have

$$\mathcal{H}^{k^\star}(u)_i \prec_{C_i} \mathcal{H}^{k^\star}(u+v)_i \qquad \forall v \in \mathcal{C}\setminus\{0\}, \|v\|_{\mathcal{V}} \leq \delta.$$

To conclude, note that if $y \in \mathcal{C}\setminus\{0\}$ is such that $\|y\| > \delta$ then, with $t = \delta/\|y\| \in (0,1)$, we have $y = t\,y + (1-t)y$, $\|ty\|_{\mathcal{V}} = \delta$ and $(1-t)y \in \mathcal{C}$. Hence, as $\mathcal{H}$ is order-preserving, by the above argument, we have

$$\mathcal{H}^{k^\star}(u)_i \prec_{C_i} \mathcal{H}^{k^\star}(u+ty)_i \preceq_{C_i} \mathcal{H}^{k^\star}(u+ty+(1-t)y)_i = \mathcal{H}^{k^\star}(u+y)_i,$$

which proves that $\mathcal{H}$ satisfies condition U_d at u.

ii) The proof is essentially the same as for (i). Therefore, we only mention the main changes. Equation (10.9) becomes

$$\mathcal{H}^{k_\star}(u-v) = \mathcal{H}^{k^\star}(u) + (\mathcal{H}'_u)^{k_\star}(-v) + \|v\|_{\mathcal{V}}\mathcal{R}_{k_\star}(v),$$

and τ is chosen so that $N_{i,\tau}((\mathcal{H}'_u)^{k_\star}(-\mathcal{S})_i) \subset -\operatorname{int}(C_i)$. Then, by choosing δ small enough, we have that for every $v \in \mathcal{C}$ with $0 < \|v\|_{\mathcal{V}} \leq \delta$, it holds $(\mathcal{H}'_u)^{k^\star}(v)_i + \|v\|_{\mathcal{V}}\mathcal{R}_{k^\star}(v)_i \in -\operatorname{int}(C_i)$ which implies $\mathcal{H}^{k_\star}(u-v)_i \prec_{C_i} \mathcal{H}^{k_\star}(u)_i$ and therefore $\mathcal{H}$ satisfies condition L_d at u. □

The following result is a special case of [60, Corollary 6.5.8] when $d = 1$ and generalizes [39, Theorem 6.3]$^\diamond$ which holds for differentiable multi-homogeneous mappings on $\mathcal{C} = \mathbb{R}^{n_1}_+ \times \ldots \times \mathbb{R}^{n_d}_+$.

Theorem 10.4.5. Let $\mathcal{C} \subset \mathcal{V}$ be a solid closed cone, $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d_+$ a multi-normalization of $\mathcal{C}$ and $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$. Let $\mathcal{F}\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ be order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}_+$ with A is primitive and $\rho(A) = 1$. Suppose that $\mathcal{F}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$ and let $\mathcal{G}\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ be given by $\mathcal{G}(x) = \boldsymbol{\nu}(\mathcal{F}(x))^{-I} \otimes \mathcal{F}(x)$ for every $x \in \operatorname{int}(\mathcal{C})$. If $\mathcal{F}$ has an eigenvector $u \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$, and $\mathcal{F}$ satisfies condition L_d or U_d at u, then u is the unique eigenvector of $\mathcal{F}$ in $\operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ and for every $x \in \operatorname{int}(\mathcal{C})$ it holds $\lim_{k\to\infty}\mathcal{G}^k(x) = u$.

For the proof, we need the following theorem which formulates results of fixed point theory in our particular setting. For its statement, let us recall that for a solid closed cone $\mathcal{C} \subset \mathcal{V}$, $x \in \operatorname{int}(\mathcal{C})$ and $\mathcal{F}\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$, the *orbit of x under $\mathcal{F}$*, denoted $\mathcal{O}(\mathcal{F}, x)$, is defined as $\mathcal{O}(\mathcal{F}, x) = \big\{\mathcal{F}^k(x) \,|\, k \in \mathbb{N}\big\}$. The *$\omega$-limit set of x under F*, denoted $\omega(F, x)$, is the set of accumulation points of $\mathcal{O}(F, x)$, i.e.

$$\omega(\mathcal{F}, x) = \big\{y \in \operatorname{int}(\mathcal{C}) \,\big|\, \exists (k_j)_{j=1}^{\infty} \subset \mathbb{N} \text{ with } \lim_{j\to\infty} \mathcal{F}^{k_j}(x) = y\big\}.$$

Theorem 10.4.6. Let $\mathcal{C} \subset \mathcal{V}$ be a solid closed cone and let $\mathcal{F}\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ be order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}$. Suppose that A is irreducible and $\rho(A) = 1$. Then, the following hold:

i) If there exists $u \in \operatorname{int}(\mathcal{C})$ such that $(\mathcal{F}^k(u))_{k=1}^{\infty} \subset \operatorname{int}(\mathcal{C})$ has a bounded subsequence in $\operatorname{int}(\mathcal{C})$, then $\mathcal{O}(\mathcal{F}, x)$ is bounded in $\operatorname{int}(\mathcal{C})$ for each $x \in \operatorname{int}(\mathcal{C})$.

ii) If $x \in \operatorname{int}(\mathcal{C})$ is such that $\mathcal{O}(\mathcal{F}, x)$ has a compact closure in $\operatorname{int}(\mathcal{C})$, then $\omega(\mathcal{F}, x)$ is a non-empty compact set and $\mathcal{F}\big(\omega(\mathcal{F}, x)\big) \subset \omega(\mathcal{F}, x)$.

iii) If $x \in \operatorname{int}(\mathcal{C})$ is such that $\mathcal{O}(\mathcal{F}, x)$ has a compact closure in $\operatorname{int}(\mathcal{C})$ and $|\omega(\mathcal{F}, x)| = p$, then there exists $z \in \operatorname{int}(\mathcal{C})$ such that $\lim_{k\to\infty} \mathcal{F}^{pk}(x) = z$ and $\omega(\mathcal{F}, x) = \mathcal{O}(\mathcal{F}, z)$.

iv) For every $x \in \operatorname{int}(\mathcal{C})$ and $y \in \omega(\mathcal{F}, x)$, it holds $\omega(F, y) = \omega(F, x)$.

Proof. In order to apply fixed point results on metric spaces, we consider a weighted Thompson metric on $\mathcal{C}$. We define the metric and discuss its properties related to the assumptions of the results we use. As A is irreducible, let $\mathbf{b} \in \mathbb{R}^d_{++}$ be the Perron vector of $A^\top$. For $i = 1, \ldots, d$, let $\delta_{C_i}\colon \operatorname{int}(C_i)\times\operatorname{int}(C_i) \to \mathbb{R}_+$ be defined as in Equation (5.4) of Remark 5.1.8. δ_{C_i} is the Thompson metric induced by C_i. Consider the vector valued Thompson metric defined as $\boldsymbol{\delta}_{\mathcal{C}}(x, y) = (\delta_{C_1}(x_1, y_1), \ldots, \delta_{C_d}(x_d, y_d))$ for all $x, y \in \operatorname{int}(\mathcal{C})$. Corollary 2.5.6 of [60] implies that $(\operatorname{int}(C_i), \delta_{C_i})$ is a complete metric space and its topology coincide with the norm topology on V_i. It follows that $(\operatorname{int}(\mathcal{C}), \boldsymbol{\delta}_{\mathcal{C}})$ is a complete d-metric space and its topology coincide with the norm topology on $\mathcal{V}$. Now, by [60, Corollary 2.1.4] and Theorem 6.1.7, we know that A is a Lipschitz matrix of $\mathcal{F}$ with respect to $\boldsymbol{\delta}_{\mathcal{C}}$. Lemma 7.1.2 implies that $\mathcal{F}$ is non-expansive with respect to the metric $\delta_\alpha\colon \operatorname{int}(\mathcal{C}) \times \operatorname{int}(\mathcal{C}) \to [0, \infty)$ given by $\delta_\alpha(x, y) = \langle \boldsymbol{\delta}_{\mathcal{C}}(x, y)\,, \mathbf{b}\rangle$ for every $x, y \in \operatorname{int}(\mathcal{C})$. Furthermore, we note that $(\operatorname{int}(\mathcal{C}), \delta_\alpha)$ is a complete metric space. The topology of $(\operatorname{int}(\mathcal{C}), \delta_\alpha)$ coincides with the product topology on $(\operatorname{int}(\mathcal{C}), \boldsymbol{\delta}_{\mathcal{C}})$ and therefore to the norm topology of $\mathcal{V}$. It follows that the closed balls of δ_α are compact sets in $\mathcal{V}$ and $\mathcal{F}\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ is continuous. Now, i), ii), iii) and iv) respectively follow from Theorem 3.1.7 and Lemmas 3.1.2, 3.1.3 and 3.1.6 of [60]. □

Proof of Theorem 10.4.5. Let $\mathbf{b} \in \mathbb{R}^d_{++}$ be the Perron vector of $A^\top$ and let $\mathbf{a} \in \mathbb{R}^d_{++}$. First, suppose that $\mathcal{F}(u) = u$. Let $x \in \operatorname{int}(\mathcal{C})$, we prove that there exists $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$ such that $\omega(\mathcal{F}, x) = \{\boldsymbol{\alpha} \otimes u\}$. As $\mathcal{O}(\mathcal{F}, u) = \{u\}$, Theorem 10.4.6, i) implies that $\mathcal{O}(\mathcal{F}, x)$ is bounded. It follows that $\mathcal{O}(\mathcal{F}, x)$ has compact closure and thus Theorem

10.4.6, ii) implies that $\omega(\mathcal{F},x)$ is a non-empty compact set and $\mathcal{F}(\omega(\mathcal{F},x))\subset\omega(\mathcal{F},x)$. Now, for $k\geq 1$, define

$$\xi_k=\prod_{i=1}^d \mathrm{m}(\mathcal{F}^k(x)_i/u_i;C_i)^{b_i} \qquad \text{and} \qquad \zeta_k=\prod_{i=1}^d \mathrm{M}(\mathcal{F}^k(x)_i/u_i;C_i)^{b_i}.$$

Then, for all $k\geq 1$, it holds

$$\begin{aligned}\xi_{k+1}&=\prod_{i=1}^d \mathrm{m}(\mathcal{F}(\mathcal{F}^k(x))_i/u_i;C_i)^{b_i}\geq \prod_{i=1}^d \mathrm{m}(\mathcal{F}(\mathbf{m}(\mathcal{F}^k(x)/u;\mathcal{C})\otimes u)_i/u_i;C_i)^{b_i}\\&=\prod_{i=1}^d \mathrm{m}(\mathcal{F}^k(x)_i/u_i;C_i)^{b_i}\,\mathrm{m}(\mathcal{F}(u)_i/u_i;C_i)^{b_i}=\xi_k,\end{aligned}$$

$$\begin{aligned}\zeta_{k+1}&=\prod_{i=1}^d \mathrm{M}(\mathcal{F}(\mathcal{F}^k(x))_i/u_i;C_i)^{b_i}\leq \prod_{i=1}^d \mathrm{M}(\mathcal{F}(\mathbf{M}(\mathcal{F}^k(x)/u;\mathcal{C})\otimes u)_i/u_i;C_i)^{b_i}\\&=\prod_{i=1}^d \mathrm{M}(\mathcal{F}^k(x)_i/u_i;C_i)^{b_i}\,\mathrm{M}(\mathcal{F}(u)_i/u_i;C_i)^{b_i}=\zeta_k.\end{aligned}$$

It follows that $\xi_k\leq\xi_{k+1}\leq\zeta_{k+1}\leq\zeta_k$ for all $k\geq 1$ and therefore there exists $\xi,\eta>0$ such that $\lim_{k\to\infty}\xi_k=\xi$ and $\lim_{k\to\infty}\zeta_k=\zeta$. Hence, for every $y\in\omega(\mathcal{F},x)$ it holds $\prod_{i=1}^d\mathrm{m}(y_i/u_i;C_i)^{b_i}=\xi$ and $\prod_{i=1}^d\mathrm{M}(y_i/u_i;C_i)^{b_i}=\zeta$. Furthermore, by Theorem 10.4.6, ii), we have $\mathcal{F}^k(\omega(\mathcal{F},x))\subset\omega(\mathcal{F},x)$ for all $k\geq 1$. It follows that

$$\prod_{i=1}^d \mathrm{m}(\mathcal{F}^k(z)_i/u_i;C_i)^{b_i}=\xi, \qquad \prod_{i=1}^d \mathrm{M}(\mathcal{F}^k(z)_i/u_i;C_i)^{b_i}=\zeta, \tag{10.10}$$

for every $k\geq 1$ and $z\in\omega(\mathcal{F},x)$. Now, let $z\in\omega(\mathcal{F},x)$ and suppose by contradiction that $z\neq\boldsymbol{\alpha}\otimes u$ for all $\boldsymbol{\alpha}\in\mathbb{R}^d_{++}$. Then, it holds $\mathbf{m}(z/u;\mathcal{C})\otimes u\lneq_{\mathcal{C}} z$ and $z\lneq_{\mathcal{C}}\mathbf{M}(z/u;\mathcal{C})\otimes u$. If $\mathcal{F}$ satisfy condition L_d at u, then there exists $k_\star\geq 1$ and $i_\star\in\{1,\ldots,d\}$ such that $\mathcal{F}^{k_\star}(\mathbf{m}(z/u;\mathcal{C})\otimes u)_{i_\star}\prec_{C_{i_\star}}\mathcal{F}^{k_\star}(z)_{i_\star}$. It follows that $\mathbf{m}(\mathcal{F}^{k_\star}(\mathbf{m}(z/u;\mathcal{C})\otimes u)/u;\mathcal{C})\lneq\mathbf{m}(\mathcal{F}^{k_\star}(z)/u;\mathcal{C})$ and therefore, with (10.10), we get the contradiction

$$\begin{aligned}\xi=\prod_{i=1}^d \mathrm{m}\left(\mathcal{F}^{k_\star}(z)/u;C_i\right)^{b_i}&>\prod_{i=1}^d \mathrm{m}\left(\mathcal{F}^{k_\star}(\mathbf{m}(z/u;\mathcal{C})\otimes u)/u;C_i\right)^{b_i}\\&=\prod_{i=1}^d \mathrm{m}(z_i/u_i;C_i)^{(A^{k_\star}\mathbf{b})_i}\,\mathrm{m}\left(\mathcal{F}^{k_\star}(u)/u;C_i\right)^{b_i}\\&=\prod_{i=1}^d \mathrm{m}(z_i/u_i;C_i)^{b_i}=\xi.\end{aligned}$$

Similarly, if $\mathcal{F}$ satisfies condition U_d at u, then there exists $k^\star \geq 1$ and $i^\star \in \{1,\ldots,d\}$ such that $\mathcal{F}^{k^\star}(z)_{i^\star} \prec_{C_{i^\star}} \mathcal{F}^{k^\star}(\mathbf{m}(z/u;\mathcal{C})\otimes u)_{i^\star}$ so that $\mathbf{M}(\mathcal{F}^{k^\star}(\mathbf{m}(z/u;\mathcal{C})\otimes u)/u;\mathcal{C}) \lneq \mathbf{M}(\mathcal{F}^{k^\star}(z)/u;\mathcal{C})$ and, with (10.10), we get the contradiction

$$\begin{aligned}\zeta = \prod_{i=1}^d \mathrm{M}\left(\mathcal{F}^{k^\star}(z)/u;C_i\right)^{b_i} &< \prod_{i=1}^d \mathrm{M}\left(\mathcal{F}^{k^\star}(\mathbf{M}(z/u;\mathcal{C})\otimes u)/u;C_i\right)^{b_i}\\ &= \prod_{i=1}^d \mathrm{M}(z_i/u_i;C_i)^{(A^{k^\star}\mathbf{b})_i}\,\mathrm{M}\left(\mathcal{F}^{k_\star}(u)/u;C_i\right)^{b_i}\\ &= \prod_{i=1}^d \mathrm{M}(z_i/u_i;C_i)^{b_i} = \zeta.\end{aligned}$$

The above argument shows that there exists $\boldsymbol{\alpha}_z \in \mathbb{R}^d_{++}$ such that $z = \boldsymbol{\alpha}_z \otimes u$. As $z \in \omega(\mathcal{F},x)$, Theorem 10.4.6, iv), implies $\omega(\mathcal{F},x) = \omega(\mathcal{F},\boldsymbol{\alpha}_z\otimes u)$. Finally, the primitivity of A implies that $\lim_{k\to\infty} A^k = B$, where $B = \mathbf{a}\mathbf{b}^\top$ and $\mathbf{a}\in\mathbb{R}^d_{++}$ is the unique eigenvector of A such that $A\mathbf{a} = \mathbf{a}$ and $\langle \mathbf{a}\,,\,\mathbf{b}\rangle = 1$. It follows that

$$\lim_{k\to\infty} \mathcal{F}^k(\boldsymbol{\alpha}_z\otimes u) = \lim_{k\to\infty} \boldsymbol{\alpha}_z^{A^k}\otimes\mathcal{F}^k(u) = \lim_{k\to\infty}\boldsymbol{\alpha}_z^{A^k}\otimes u = \boldsymbol{\alpha}_z^B\otimes u,$$

and thus $\omega(\mathcal{F},x) = \omega(\mathcal{F},\boldsymbol{\alpha}_z\otimes u) = \{\boldsymbol{\alpha}_z^B\otimes u\}$. Finally, Theorem 10.4.6, iii) implies that $\lim_{k\to\infty}\mathcal{F}^k(x) = \boldsymbol{\alpha}_z^B\otimes u$. Now, we show by induction over $k\geq 1$, that $\mathcal{G}^k(x) = \boldsymbol{\nu}(\mathcal{F}^k(x))^{-I}\otimes\mathcal{F}^k(x)$. The case $k=1$ holds by definition of $\mathcal{G}$. Suppose it is true for $k\geq 1$, then

$$\mathcal{F}(\mathcal{G}^k(x)) = \mathcal{F}(\boldsymbol{\nu}(\mathcal{F}^k(x))\otimes\mathcal{F}^k(x)) = \boldsymbol{\nu}(\mathcal{F}^k(x))^A\otimes\mathcal{F}^{k+1}(x)$$

so that

$$\begin{aligned}\mathcal{G}^{k+1}(x) &= \boldsymbol{\nu}(\mathcal{F}(\mathcal{G}^k(x)))^{-I}\otimes\mathcal{F}(\mathcal{G}^k(x))\\ &= \big(\boldsymbol{\nu}(\mathcal{F}^k(x))^{-A}\circ\boldsymbol{\nu}(\mathcal{F}^{k+1}(x))^{-I}\big)\otimes\big(\boldsymbol{\nu}(\mathcal{F}^k(x))^A\otimes\mathcal{F}^{k+1}(x)\big)\\ &= \boldsymbol{\nu}(\mathcal{F}^{k+1}(x))^{-I}\otimes\mathcal{F}^{k+1}(x),\end{aligned}$$

which concludes the induction proof. By continuity of $\boldsymbol{\nu}$, it follows that

$$\lim_{k\to\infty}\mathcal{G}^k(x) = \boldsymbol{\nu}(\lim_{k\to\infty}\mathcal{F}^k(x))^{-I}\otimes\big(\lim_{k\to\infty}\mathcal{F}^k(x)\big) = \boldsymbol{\nu}(u)^{-I}\otimes u = u.$$

To prove uniqueness, note that if $v\in\mathrm{int}(\mathcal{C})\cap\mathcal{S}_{\boldsymbol{\nu}}$ is an eigenvector of $\mathcal{F}$, then by Lemma 7.3.2, v is a fixed point of $\mathcal{G}$ and by the above argument we have

$$u = \lim_{k\to\infty}\mathcal{G}^k(v) = \lim_{k\to\infty} v = v,$$

which implies that u is the unique eigenvector of $\mathcal{F}$ in $\mathrm{int}(\mathcal{C})\cap\mathcal{S}_{\boldsymbol{\nu}}$.

Finally, if $\mathcal{F}(u) = \boldsymbol{\lambda}\otimes u$ with $\boldsymbol{\lambda}\in\mathbb{R}^d_+$, then it holds $\boldsymbol{\lambda}\in\mathbb{R}^d_{++}$ since $\mathcal{F}(u)\in\mathrm{int}(\mathcal{C})$. Define $\hat{\mathcal{F}}\colon \mathrm{int}(\mathcal{C})\to\mathrm{int}(\mathcal{C})$ as $\hat{\mathcal{F}}(x) = \boldsymbol{\lambda}^{-I}\otimes\mathcal{F}(x)$ for all $x\in\mathrm{int}(\mathcal{C})$. Then, $\hat{\mathcal{F}}$ satisfies condition U_d, resp. condition L_d, at u if and only if $\mathcal{F}$ does. Hence, by the above discussion, u is the unique eigenvector of $\hat{\mathcal{F}}$ in $\operatorname{int}(\mathcal{C})\cap\mathcal{S}_{\boldsymbol{\nu}}$. As $x\in\operatorname{int}(\mathcal{C})\cap\mathcal{S}_{\boldsymbol{\nu}}$ is an eigenvector of $\mathcal{F}$ if and only if it is an eigenvector of $\hat{\mathcal{F}}$, we conclude that u is the unique eigenvector of $\mathcal{F}$ in $\operatorname{int}(\mathcal{C})\cap\mathcal{S}_{\boldsymbol{\nu}}$. Finally, note that for every $x\in\operatorname{int}(\mathcal{C})\cap\mathcal{S}_{\boldsymbol{\nu}}$ it holds $\mathcal{G}(x)=\boldsymbol{\nu}(\mathcal{F}(x))^{-I}\otimes\mathcal{F}(x)=\boldsymbol{\nu}(\hat{\mathcal{F}}(x))^{-I}\otimes\hat{\mathcal{F}}(x)$, and thus $\lim_{k\to\infty}\mathcal{G}^k(x)=u$ for all $x\in\operatorname{int}(\mathcal{C})\cap\mathcal{S}_{\boldsymbol{\nu}}$. □

11 Perron-Frobenius theorems on the product of cones

We collect here the main results of this thesis. We group the results in two main theorems. The first is a Perron-Frobenius theorem for multi-homogeneous mappings and the second refines some results by taking the multi-linearity into account. We discuss the consequences of such results on the existing literature for the examples discussed in Section 4.2. To facilitate the statements, we make assumptions that are sometimes unnecessarily restrictive for a given result. The finer assumptions can be found by looking at the corresponding results referred in the proof. Furthermore, the convergence rates in the following results are stated in terms of the vector valued Hilbert metric $\boldsymbol{\mu}_{\mathcal{C}}$. We recall however that they can be formulated in terms of norms on the underlying vector space using Proposition 7.3.3.

11.1 Perron-Frobenius theorems

Let $\mathcal{V}=V_1\times\ldots\times V_d$ be the product of finite dimensional real vector spaces. For the reading convenience, we recall that multi-normalizations are introduced in Definition 3.1.1 (p. 14). Multi-homogeneous mappings and their eigenvectors are defined in Section 3.2 (p. 15) and Section 4 (p. 23), respectively. The mappings $\mathbf{M}(\cdot/\cdot;\mathcal{C}),\mathbf{m}(\cdot/\cdot;\mathcal{C})$ and the vector valued Hilbert metric $\boldsymbol{\mu}_{\mathcal{C}}$ are introduced in Section 6.2 (p. 54). Irreducible and weakly irreducible mappings are discussed in Section 9.1 (p. 89) and 9.3 (p. 95), respectively. Semi-differentiability is recalled in Section 10.3 (p. 112).

Our first main result is the following:

Theorem 11.1.1 (Multi-homogeneous Perron-Frobenius theorem)**.** Let $\mathcal{C}\subset\mathcal{V}$ be a solid closed cone, let $\boldsymbol{\nu}\colon\mathcal{C}\to\mathbb{R}^d$ be a monotonic multi-normalization of $\mathcal{C}$ and set $\mathcal{S}_{\boldsymbol{\nu}}=\{x\in\mathcal{C}\,|\,\boldsymbol{\nu}(x)=\mathbf{1}\}$. Let $\mathcal{F}\colon\mathcal{C}\to\mathcal{C}$ be continuous, order-preserving and multi-homogeneous of degree $A\in\mathbb{R}^{d\times d}$. Suppose that A is irreducible and $\mathcal{F}(\operatorname{int}(\mathcal{C}))\subset\operatorname{int}(\mathcal{C})$. Let $\mathbf{b}\in\mathbb{R}^d_{++}$ be the Perron vector of $A^\top$.

I) **Existence:** $\mathcal{F}$ has an eigenvector $u\in\operatorname{int}(\mathcal{C})$ if at least one of the following conditions hold:

a) $\mathcal{F}$ is irreducible.

b) $\rho(A)<1$.

c) $\rho(A)=1$, $\mathcal{C}=\mathbb{R}^{n_1}_+\times\ldots\times\mathbb{R}^{n_d}_+$ and $\mathcal{F}$ is weakly irreducible.

Furthermore, if any of the above is satisfied, then there exists $\boldsymbol{\lambda}\in\mathbb{R}^d_{++}$ such that $\mathcal{F}(u)=\boldsymbol{\lambda}\otimes u$.

II) **Maximality:** If $\rho(A) \leq 1$ and $\mathcal{F}$ has an eigenvector $u \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ with corresponding eigenvalue θ, i.e. $\theta = \prod_{i=1}^d \lambda_i^{b_i}$ where $\boldsymbol{\lambda} \in \mathbb{R}^d_+$ satisfies $\mathcal{F}(u) = \boldsymbol{\lambda} \otimes u$, then θ is a maximal eigenvalue in the following sense

$$\theta = \sup \Big\{ \prod_{i=1}^d \vartheta_i^{b_i} \,\Big|\, \exists x \in \mathcal{S}_{\boldsymbol{\nu}} \text{ such that } \mathcal{F}(x) = \boldsymbol{\vartheta} \otimes x \Big\}.$$

Furthermore, the following Collatz-Wielandt formula holds,

$$\sup_{x \in \mathcal{S}_{\boldsymbol{\nu}}} \prod_{i=1}^d \mathbf{m}(\mathcal{F}(x)/x; \mathcal{C})_i^{b_i} = \theta = \inf_{x \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}} \prod_{i=1}^d \mathbf{M}(\mathcal{F}(x)/x; \mathcal{C})_i^{b_i}.$$

Moreover, if $\rho(A) = 1$, then

$$\sup_{x \in \mathcal{C}_0} \prod_{i=1}^d \mathbf{m}(\mathcal{F}(x)/x; \mathcal{C})_i^{b_i} = \theta = \inf_{x \in \operatorname{int}(\mathcal{C})} \prod_{i=1}^d \mathbf{M}(\mathcal{F}(x)/x; \mathcal{C})_i^{b_i},$$

where $\mathcal{C}_0 = \{x \in \mathcal{C} \,|\, x_1, \ldots, x_d \neq 0\}$.

III) **Uniqueness:** If $\mathcal{F}$ has an eigenvector $u \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$, then u is the unique eigenvector of $\mathcal{F}$ in $\operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ if at least one of the following conditions hold:

a) $\rho(A) < 1$,

b) $\rho(A) = 1$, A is primitive and there exists an integer $k \geq 1$ and an open neighborhood $\mathcal{U} \subset \operatorname{int}(\mathcal{C})$ such that $u \in \mathcal{U}$ and $\mathcal{F}^k$ is strongly order-preserving in $\mathcal{U}$,

c) $\rho(A) = 1$, $\mathcal{F}$ is semi-differentiable at u with semi-derivative $\mathcal{F}'_u \colon \mathcal{C} \to \mathcal{C}$ and for every $y \in \mathcal{C} \setminus \{0\}$, there exists $i \in \{1, \ldots, d\}$ and $N_y \geq 1$ such that

$$-\sum_{k=1}^{N_y} (\mathcal{F}'_u)^k(-y) \in C_1 \times \ldots \times C_{i-1} \times \operatorname{int}(C_i) \times C_{i+1} \times \ldots \times C_d.$$

Furthermore, if any of the above conditions is satisfied and $\mathcal{F}$ is irreducible, then u is the unique eigenvector of $\mathcal{F}$ in $\mathcal{S}_{\boldsymbol{\nu}}$.

IV) **Convergence:** Let $x \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ and consider the sequences $(x_k)_{k=1}^\infty \subset \operatorname{int}(\mathcal{C})$, $(\alpha_k)_{k=1}^\infty, (\beta_k)_{k=1}^\infty \subset [0, \infty)$ defined by $x_0 = x$ and

$$x_{k+1} = \boldsymbol{\nu}(\mathcal{F}(x_k))^{-I} \otimes \mathcal{F}(x_k) \qquad \forall k \geq 1,$$

and

$$\alpha_k = \prod_{i=1}^d \mathbf{m}(\mathcal{F}(x_k)/x_k; \mathcal{C})_i^{b_i}, \quad \beta_k = \prod_{i=1}^d \mathbf{M}(\mathcal{F}(x_k)/x_k; \mathcal{C})_i^{b_i} \qquad \forall k \geq 1.$$

If $\mathcal{F}$ has an eigenvector $u \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ with corresponding eigenvalue θ, then it holds

$$\lim_{k \to \infty} x_k = u, \quad \lim_{k \to \infty} \alpha_k = \lim_{k \to \infty} \beta_k = \theta$$

and

$$\alpha_{k+1} \le \alpha_k \le \theta \le \beta_{k+1} \le \beta_k \qquad \forall k \ge 1, \tag{11.1}$$

if at least one of the following conditions is satisfied:

a) $\rho(A) < 1$.
b) $\rho(A) = 1$, A is primitive and there exists an integer $k \ge 1$ and an open neighborhood $\mathcal{U} \subset \operatorname{int}(\mathcal{C})$ such that $u \in \mathcal{U}$ and $\mathcal{F}^k$ is strongly order-preserving in $\mathcal{U}$.
c) $\rho(A) = 1$, A is primitive, $\mathcal{F}$ is semi-differentiable at u with semi-derivative $\mathcal{F}'_u \colon \mathcal{C} \to \mathcal{C}$ and there exists $i \in \{1, \ldots, d\}$ and $N \ge 1$ such that

$$(\mathcal{F}'_u)^N(y) \in C_1 \times \ldots \times C_{i-1} \times \operatorname{int}(C_i) \times C_{i+1} \times \ldots \times C_d$$

for every $y \in \mathcal{C} \setminus \{0\}$.

Furthermore, if $\rho(A) < 1$, then it holds

$$\boldsymbol{\mu}_{\mathcal{C}}(x_k, u) \le A^k \mathbf{v} \qquad \forall k \ge 1, \tag{11.2}$$

with $\mathbf{v} = (I - A)^{-1} \boldsymbol{\mu}_{\mathcal{C}}(x_1, x_0)$.

Proof of Theorem 11.1.1. For the conclusions which follows from different possible conditions, we refer to the result which implies the statement.

i) Note that if $u \in \operatorname{int}(\mathcal{C})$ is an eigenvector of $\mathcal{F}$, then there exists $\boldsymbol{\lambda} \in \mathbb{R}^d_+$ such that $\mathcal{F}(u) = \boldsymbol{\lambda} \otimes u$. Now, as $\mathcal{F}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$, we necessarily have $\boldsymbol{\lambda} \in \mathbb{R}^d_{++}$. The existence of u is implied by the following results:
   a) Follows from Theorem 9.1.6.
   b) Follows from Proposition 5.2.1 and Theorem 7.3.1.
   c) Follows from Theorem 9.3.5.

ii) Follows from Corollary 10.1.2, Theorem 10.2.1 and Theorem 10.2.2.

iii) a) Follows from Proposition 5.2.1 and Theorem 7.3.1.
   b) Follows from Theorem 10.4.5.
   c) Follows from Theorem 10.4.1.

   Furthermore, Proposition 9.1.5 implies that if $\mathcal{F}$ is irreducible and $x \in \mathcal{S}_{\boldsymbol{\nu}}$ is an eigenvector of $\mathcal{F}$, then $x \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$. Hence, if u is the unique eigenvector of $\mathcal{F}$ in $\operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$, then it holds $x = u$.

iv) Note that if $\lim_{k\to\infty} x_k = u$, then, by continuity of $\mathcal{F}$, it holds $\lim_{k\to\infty} \mathcal{F}(x_k) = \mathcal{F}(u)$. Hence, with $\boldsymbol{\lambda} \in \mathbb{R}^d_+$ such that $\mathcal{F}(u) = \boldsymbol{\lambda} \otimes u$, it holds

$$\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \prod_{i=1}^d \mathbf{m}(\mathcal{F}(x_k)/x_k; \mathcal{C})_i^{b_i}$$
$$= \prod_{i=1}^d \mathbf{m}(\boldsymbol{\lambda} \otimes u/u; \mathcal{C})_i^{b_i} = \prod_{i=1}^d \lambda_i^{b_i} = \theta. \tag{11.3}$$

A similar argument shows that $\lim_{k\to\infty}\beta_k = \theta$. Now, $\lim_{k\to\infty} x_k = u$ is implied by the following results:

a) Follows from Proposition 5.2.1 and Theorem 7.3.1.

b) Follows from Theorem 10.4.5.

c) Follows from Lemma 10.4.4 and Theorem 10.4.5.

Finally, (11.1) and (11.2) respectively follow from Theorem 10.2.3 and Theorem 7.3.1.

□

Except those related to irreducible mappings, the conclusions of Theorem 11.1.1 hold for order-preserving multi-homogeneous mappings with homogeneity matrix $A \in \mathbb{R}^{d\times d}$ such that $\rho(A) \leq 1$. By imposing more structure on the mapping such as (cone) multi-linearity (see Definition 8.1.1, p. 76), some of the results can be further refined using the multi-linear Birkhoff-Hopf theorem 8.1.2. As a motivation for the assumptions in the next theorem, let us recall that all the examples discussed in Section 3.2.5 are eigenvector problems of multi-homogeneous mappings of the form $\mathcal{F}(x) = \Psi(\mathcal{G}(x))$ where Ψ is a non-linear mapping and $\mathcal{G}$ is a polynomial mapping.

Theorem 11.1.2 (Multi-linear Perron-Frobenius theorem)**.** Let $\mathcal{C} \subset \mathcal{V}$ be a solid closed cone, let $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d$ be a multi-normalization of $\mathcal{C}$ and set $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$. Furthermore, let $s_1, \ldots, s_d$ be positive integers, $m = s_1 + \ldots + s_d$ and define $\hat{\mathcal{C}} = \hat{C}_1 \times \ldots \times \hat{C}_m$ as

$$\hat{\mathcal{C}} = \underbrace{C_1 \times \ldots \times C_1}_{s_1 \text{ times}} \times \ldots \times \underbrace{C_d \times \ldots \times C_d}_{s_d \text{ times}}.$$

Let $\mathcal{L}\colon \hat{\mathcal{C}} \to \mathcal{C}$ be a cone multi-linear mapping such that $\mathcal{L}(\operatorname{int}(\hat{\mathcal{C}})) \subset \operatorname{int}(\mathcal{C})$ and let $\hat{L} \in \mathbb{R}^{d\times m}_+$ be such that for every $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, d\}$ it holds

$$L_{j,i} \geq \sup_{\hat{x}\in\hat{\mathcal{C}}, \hat{y}_i \in \hat{C}_i} \tanh\Big[\tfrac{1}{4}\mu_{C_j}\big(\mathcal{L}(\hat{x})_j, \mathcal{L}(\hat{x}_1, \ldots, \hat{x}_{i-1}, \hat{y}_i, \hat{x}_{i+1}, \ldots, \hat{x}_m)_j\big)\Big].$$

Let $s_0 = 0$ and let $B \in \{0,1\}^{d\times m}$, the binary matrix given by $B_{j,l} = 1$ if $s_1 + \ldots + s_{j-1} < l \leq s_1 + \ldots + s_j$ and $B_{j,l} = 0$ otherwise. Moreover, let $\Psi\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ be a multi-homogeneous mapping and let $Q \in \mathbb{R}^{d\times d}_+$ be such that

$$\boldsymbol{\mu}_{\mathcal{C}}(\Psi(x), \Psi(y)) \leq Q\boldsymbol{\mu}_{\mathcal{C}}(x, y) \qquad \forall x, y \in \operatorname{int}(\mathcal{C}).$$

Finally, let $M = QLB \in \mathbb{R}^{d\times d}_+$ and define $\mathcal{F}\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ as

$$\mathcal{F}(x) = \Psi\big(\mathcal{L}(\underbrace{x_1, \ldots, x_1}_{s_1 \text{ times}}, \ldots, \underbrace{x_d, \ldots, x_d}_{s_d \text{ times}})\big) \qquad \forall x \in \operatorname{int}(\mathcal{C}).$$

If $\rho(M) < 1$, then $\mathcal{F}$ has a unique eigenvector $u \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ and for every $x \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ the sequence $(x_k)_{k=1}^{\infty}$ given by

$$x_{k+1} = \boldsymbol{\nu}(\mathcal{F}(x_k))^{-I} \otimes \mathcal{F}(x_k) \qquad \forall k \geq 1,$$

satisfies

$$\lim_{k\to\infty} x_k = u \qquad \text{and} \qquad \boldsymbol{\mu}_{\mathcal{C}}(x_k,u) \le M^k\mathbf{v} \qquad \forall k\ge 1,$$

with $\mathbf{v} = (I-M)^{-1}\boldsymbol{\mu}_{\mathcal{C}}(x_1,x_0)$.

Remark 11.1.3. Useful properties for the estimation of

$$\sup_{\hat{x}\in\hat{\mathcal{C}},\hat{y}_i\in\hat{C}_i} \tanh\big[\tfrac{1}{4}\mu_{C_j}\big(\mathcal{L}(\hat{x})_j, \mathcal{L}(\hat{x}_1,\ldots,\hat{x}_{i-1},\hat{y}_i,\hat{x}_{i+1},\ldots,\hat{x}_m)_j\big)\big],$$

are discussed in Remark 8.1.3 (p. 77). Moreover, if $\mathcal{C} = \mathbb{R}^{n_1}_+ \times \ldots \times \mathbb{R}^{n_d}_+$ and $\mathcal{L}$ is expressed in terms of nonnegative tensors, upper bounds on the above supremum are discussed in Section 8.2 (p. 82).

Proof of Theorem 11.1.2. Let $\mathcal{G}\colon \mathcal{C}\to\mathcal{C}$ be defined as

$$\mathcal{G}(x) = \mathcal{L}(\underbrace{x_1,\ldots,x_1}_{s_1\text{ times}},\ldots,\underbrace{x_d,\ldots,x_d}_{s_d\text{ times}})\big) \qquad \forall x\in\mathcal{C}.$$

Let $i\in\{1,\ldots,m\}$ and $j\in\{1,\ldots,d\}$, then by Remark 8.1.3 (b), we have

$$\begin{aligned}\sup_{\hat{x}\in\hat{\mathcal{C}},\hat{y}_i\in\hat{C}_i} &\tanh\big[\tfrac{1}{4}\mu_{C_j}\big(\mathcal{L}(\hat{x})_j, \mathcal{L}(\hat{x}_1,\ldots,\hat{x}_{i-1},\hat{y}_i,\hat{x}_{i+1},\ldots,\hat{x}_d)_j\big)\big]\\ &= \sup_{\hat{x}\in\hat{\mathcal{C}}} \tanh\big[\tfrac{1}{4}\operatorname{diam}(\mathcal{L}|^{j,i}_{\hat{x}}(\hat{C}_i);\mu_{C_j})\big],\end{aligned}$$

where we recall that for every $\hat{x}\in\hat{\mathcal{C}}$, the mapping $\mathcal{L}|^{j,i}_{\hat{x}}\colon \hat{C}_i\to C_j$ is given by

$$\mathcal{L}|^{j,i}_{\hat{x}}(\hat{y}_i) = \mathcal{L}(\hat{x}_1,\ldots,\hat{x}_{i-1},\hat{y}_i,\hat{x}_{i+1},\ldots,\hat{x}_d)_j \qquad \forall \hat{y}_i\in\hat{C}_i.$$

By Theorem 8.1.5, we have

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{G}(x),\mathcal{G}(y)) \le LB\boldsymbol{\mu}_{\mathcal{C}}(x,y) \qquad \forall x,y\in\operatorname{int}(\mathcal{C}).$$

Now, as $\mathcal{F}(x) = \Psi(\mathcal{G}(x))$ for all $x\in\operatorname{int}(\mathcal{C})$, we have

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}(x),\mathcal{F}(y)) \le Q\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{G}(x),\mathcal{G}(y)) \le M\boldsymbol{\mu}_{\mathcal{C}}(x,y) \qquad \forall x,y\in\operatorname{int}(\mathcal{C}).$$

Hence, if $\rho(M)<1$, the existence and uniqueness of u as well as the convergence of $(x_k)_{k=1}^\infty$ follow from Theorem 7.3.1. □

Remark 11.1.4. Note that if, in Theorem 11.1.2, Ψ is order-preserving, then $\mathcal{F}$ is order-preserving as well. Hence, if the homogeneity matrix $A\in\mathbb{R}^{d\times d}$ of $\mathcal{F}$ satisfies $\rho(A)\le 1$, Theorem 11.1.1 can be used to obtain further results on $\mathcal{F}$.

11.2 Applications

We discuss the consequences of Theorems 11.1.1 and 11.1.2 on each of the examples discussed in Section 4.2. Again we stress that the convergence rates in the following results are given in terms of the vector valued Hilbert metric $\boldsymbol{\mu}_{\mathcal{C}}$. They can be formulated in terms of norms on the underlying vector space using Proposition 7.3.3.

11.2.1 $\ell^{p,q}$-singular values of a nonnegative matrix

Let $M \in \mathbb{R}^{m\times n}_+$, $p, q \in (1,\infty)$ and let $\|\cdot\|_p$, $\|\cdot\|_q$ be the ℓ^p-norm on $\mathbb{R}^m$ and the ℓ^q-norm on $\mathbb{R}^n$, respectively. Let $\tilde{\mathcal{C}} = \mathbb{R}^m_+ \times \mathbb{R}^n_+$, consider the monotonic multi-normalization $\boldsymbol{\nu}\colon \tilde{\mathcal{C}} \to \mathbb{R}^2_+$ defined as $\boldsymbol{\nu}(x,y) = (\|x\|_p, \|y\|_q)$ and set $\tilde{\mathcal{S}}_{\boldsymbol{\nu}} = \{(x,y) \in \tilde{\mathcal{C}} \mid \boldsymbol{\nu}(x) = \mathbf{1}\}$. We have seen in Section 4.2.1 that the $\ell^{p,q}$-singular vectors and values of M are characterized by the solutions to the following system of equations:

$$\begin{cases} My = \lambda x^{p-1} \\ M^\top x = \lambda y^{q-1} \end{cases} \qquad \text{with} \qquad (\lambda, (x,y)) \in \mathbb{R}_+ \times \tilde{\mathcal{S}}_{\boldsymbol{\nu}}. \tag{11.4}$$

We recall from (3.2) and (4.3) that the above system characterizes the nonnegative critical points of the objective function defining the $\ell^{p,q}$-norm of M, defined as

$$\|M\|_{p,q} = \max_{x\neq 0, y\neq 0} \frac{\langle x\,,\, My\rangle}{\|x\|_p\|y\|_q}, \tag{11.5}$$

where $\langle\cdot\,,\,\cdot\rangle$ is the Euclidean inner product on $\mathbb{R}^m$. In particular, it holds

$$\|M\|_{p,q} = \sup\{\tilde{\lambda} \mid \tilde{\lambda} \text{ is an } \ell^{p,q}\text{-singular value of } M\}.$$

Moreover, we have proved in Proposition 4.2.1 that the $\ell^{p,q}$-singular vectors of M are characterized by the eigenvectors of the multi-homogeneous mapping $\tilde{\mathcal{F}}\colon \tilde{\mathcal{C}} \to \tilde{\mathcal{C}}$ given by

$$\tilde{\mathcal{F}}(x,y) = \big((My)^{1/(p-1)}, (M^\top x)^{1/(q-1)}\big) \qquad \forall (x,y) \in \tilde{\mathcal{C}}, \tag{11.6}$$

with homogeneity matrix

$$\tilde{A} = \begin{pmatrix} 0 & 1/(p-1) \\ 1/(q-1) & 0 \end{pmatrix}.$$

Now, note that $\tilde{A}_{2,2} = 0$ and therefore, by the discussion in Example 4.3.1, more generally by Proposition 4.3.2, we know that $(x,y) \in \tilde{\mathcal{S}}_{\boldsymbol{\nu}}$ is an eigenvector of $\tilde{\mathcal{F}}$ if and only if $x \in \mathcal{S}_{\boldsymbol{\nu}}$ is an eigenvector of $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ where $\mathcal{C} = \mathbb{R}^m_+$, $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \mid \|x\|_p = 1\}$ and

$$\mathcal{F}(x) = (M(M^\top x)^{1/(q-1)})^{1/(p-1)} \qquad \forall x \in \mathbb{R}^n_+. \tag{11.7}$$

Note that $\mathcal{F}$ is order-preserving and homogeneous of degree r with $r^{-1} = (p-1)(q-1)$.

In Example 4.3.1 we have shown that the eigenvectors of $\mathcal{F}$ and $\tilde{\mathcal{F}}$ corresponding to positive eigenvalues are in bijection. The connection between the eigenvalues of $\mathcal{F}$ and $\tilde{\mathcal{F}}$ is discussed in the following lemma.

Lemma 11.2.1. In the above setting, suppose that $(x,y) \in \tilde{\mathcal{S}}_{\boldsymbol{\nu}}$ satisfies one of the following conditions:

i) (x,y) is an eigenvector of $\tilde{\mathcal{F}}$,

ii) x is an eigenvector of $\mathcal{F}$ and $y = v/\|v\|_q$ with $v = (M^\top x)^{1/(q-1)}$.

If $\xi = \langle x\,,\,My\rangle \neq 0$, then it holds

$$\tilde{\mathcal{F}}(x,y) = (\xi^{1/(p-1)}x, \xi^{1/(q-1)}y) \qquad \text{and} \qquad \mathcal{F}(x) = \xi^{q/((p-1)(q-1))}x.$$

Proof. The proof of Proposition 4.2.1 implies that if $(x,y) \in \tilde{\mathcal{S}}_{\boldsymbol{\nu}}$ is an eigenvector of $\tilde{\mathcal{F}}$ then it holds $\tilde{\mathcal{F}}(x,y) = \tilde{\boldsymbol{\lambda}} \otimes (x,y)$ with $\tilde{\boldsymbol{\lambda}} = (\xi^{1/(p-1)}, \xi^{1/(q-1)})$ and $\xi = \langle x\,,\,My\rangle$. The discussion in Example 4.3.1 further implies that $\mathcal{F}(x) = \theta x$ with $\theta = \tilde{\lambda}_2^{1/(p-1)}\tilde{\lambda}_1 = \xi^{q/((p-1)(q-1))}$. Conversely, the discussion in Example 4.3.1 implies that if $x \in \mathcal{S}_{\boldsymbol{\nu}}$ satisfies $\mathcal{F}(x) = \lambda x$ and $M^\top x \neq 0$, then $(x,y) \in \tilde{\mathcal{S}}_{\boldsymbol{\nu}}$ is an eigenvector of $\tilde{\mathcal{F}}$ with $y = v/\|v\|_q$ and $v = (M^\top x)^{1/(q-1)}$. Furthermore, it holds $\tilde{F}(x,y) = \tilde{\boldsymbol{\theta}} \otimes (x,y)$ with $\tilde{\boldsymbol{\theta}} = (\lambda\|v\|_q^{-1/(p-1)}, \|v\|_q)$. Proposition 4.2.1 then implies that $\lambda^{p-1}\|v\|_q^{-1} = \xi = \|v\|_q^{q-1}$ so that $\|v\|_q = \xi^{1/(q-1)}$ and $\lambda = \xi^{q/((p-1)(q-1))}$. To conclude, note that with the above expressions we have $\lambda\|v\|_q^{-1/(p-1)} = \xi^{1/(p-1)}$. □

Next, we discuss properties of $\tilde{\mathcal{F}}$. First, note that $\tilde{\mathcal{F}}$ can be written in the form $\tilde{\mathcal{F}}(x,y) = \Psi(\mathcal{L}(x,y))$ where $\mathcal{L}\colon \tilde{\mathcal{C}} \to \tilde{\mathcal{C}}$ is the cone multi-linear mapping defined as

$$\mathcal{L}(x,y) = (My, M^\top x) \qquad \forall (x,y) \in \tilde{\mathcal{C}},$$

and $\Psi\colon \tilde{\mathcal{C}} \to \tilde{\mathcal{C}}$ is the nonlinear mapping defined as

$$\Psi(x,y) = (x^{1/(p-1)}, y^{1/(q-1)}) \qquad \forall (x,y) \in \tilde{\mathcal{C}}.$$

Let $\kappa(\cdot)$ denote the Birkhoff contraction ratio discussed in Theorem 5.2.7. By Theorem 8.1.2 we have that for all $(x,y), (\tilde{x},\tilde{y}) \in \tilde{\mathcal{C}}$ with $(x,y) \sim_{\tilde{\mathcal{C}}} (\tilde{x},\tilde{y})$, it holds

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{L}(x,y), \mathcal{L}(\tilde{x},\tilde{y})) \leq L\boldsymbol{\mu}_{\mathcal{C}}((x,y),(\tilde{x},\tilde{y}))$$

where $L \in \mathbb{R}_+^{2\times 2}$ is given by

$$L = \begin{pmatrix} 0 & \kappa(M) \\ \kappa(M^\top) & 0 \end{pmatrix}.$$

As $p > 1$ and $q > 1$, Ψ is order-preserving and multi-homogeneous of degree $Q \in \mathbb{R}^{2\times 2}$ with

$$Q = \begin{pmatrix} 1/(p-1) & 0 \\ 0 & 1/(q-1) \end{pmatrix} = \operatorname{diag}\Big(\frac{1}{p-1}, \frac{1}{q-1}\Big).$$

By Lemma 5.2.3, for all $(x,y), (\tilde{x},\tilde{y}) \in \tilde{\mathcal{C}}$ with $(x,y) \sim_{\tilde{\mathcal{C}}} (\tilde{x},\tilde{y})$, we have

$$\boldsymbol{\mu}_{\mathcal{C}}(\Psi(x,y), \Psi(\tilde{x},\tilde{y})) = Q\boldsymbol{\mu}_{\mathcal{C}}((x,y),(\tilde{x},\tilde{y})).$$

It follows that

$$\boldsymbol{\mu}_{\mathcal{C}}(\tilde{\mathcal{F}}(x,y), \tilde{\mathcal{F}}(\tilde{x},\tilde{y})) \leq \tilde{L}\boldsymbol{\mu}_{\mathcal{C}}((x,y),(\tilde{x},\tilde{y})),$$

with

$$\tilde{L} = QL = \begin{pmatrix} 0 & \kappa(M)/(p-1) \\ \kappa(M^\top)/(q-1) & 0 \end{pmatrix}.$$

Note that the spectral radius of $\tilde{L}$ is given by

$$\rho(\tilde{L}) = \sqrt{\frac{\kappa(M)\kappa(M^\top)}{(p-1)(q-1)}}.$$

As $\Psi(x,y) \sim_{\tilde{\mathcal{C}}} (x,y)$ for all $(x,y) \in \tilde{\mathcal{C}}$, Lemmas 9.2.1 and 9.2.2 imply that $\tilde{\mathcal{F}}$ is irreducible, $\tilde{\mathcal{F}}$ is weakly irreducible, respectively $\tilde{\mathcal{F}}(\operatorname{int}(\tilde{\mathcal{C}})) \subset \operatorname{int}(\tilde{\mathcal{C}})$ if and only if $\mathcal{L}$ has the corresponding property. We further note that Lemma 3.1 of [31] implies that $\mathcal{L}$ is irreducible if and only if $\mathcal{L}$ is weakly irreducible. Note that $\mathcal{L}(\operatorname{int}(\tilde{\mathcal{C}})) \subset \operatorname{int}(\tilde{\mathcal{C}})$ if and only if M has at least one positive entry per row and per column. Furthermore, it can be shown that $\mathcal{L}$ is irreducible if and only if $MM^\top \in \mathbb{R}^{m\times m}$ and $M^\top M \in \mathbb{R}^{n\times n}$ are irreducible matrices.

Next, we discuss the properties of $\mathcal{F}$. By the Birkhoff-Hopf theorem 5.2.7 and Lemma 5.2.3, for every $x, \tilde{x} \in \mathcal{C}$ with $x \sim_{\mathcal{C}} \tilde{x}$ we have

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}(x), \mathcal{F}(\tilde{x})) \leq c\boldsymbol{\mu}_{\mathcal{C}}(x, \tilde{x}) \quad \text{with} \quad c = \frac{\kappa(M)\kappa(M^\top)}{(p-1)(q-1)}. \tag{11.8}$$

Furthermore, as $\mathcal{F}(x) \sim_{\mathcal{C}} M^\top M x$ for all $x \in \mathcal{C}$ we have that $\mathcal{F}$ is irreducible if and only if $MM^\top$ is an irreducible matrix. Similarly, it can be verified that $\mathcal{F}$ is weakly irreducible, resp. $\mathcal{F}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$ if and only if $MM^\top$ is irreducible, resp. $MM^\top$ has at least one positive entry per row. From the above observations, we deduce that if $\tilde{\mathcal{F}}$ is irreducible, resp. $\tilde{\mathcal{F}}(\operatorname{int}(\tilde{\mathcal{C}})) \subset \operatorname{int}(\tilde{\mathcal{C}})$, then $\mathcal{F}$ is irreducible, resp. $\mathcal{F}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$. The converse is however not true in general. Indeed, if $M = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, then $M^\top M \in \mathbb{R}^{2\times 2}_{++}$ and thus $\mathcal{F}(\mathcal{C} \setminus \{0\}) \subset \operatorname{int}(\mathcal{C})$, however $\tilde{\mathcal{F}}(\operatorname{int}(\tilde{\mathcal{C}})) \not\subset \operatorname{int}(\tilde{\mathcal{C}})$ as $M^\top$ does not have a positive entry on the second row. This example shows that the substitution technique of Section 4.3 is helpful to gain irreducibility in the problem.

By combining the above observations with Theorem 11.1.1 we obtain the following Perron-Frobenius theorem for $\ell^{p,q}$-singular values of nonnegative matrices.

Theorem 11.2.2. Let $M \in \mathbb{R}^{m\times n}_+$, $p, q \in (1,\infty)$ and let $\|\cdot\|_p$, $\|\cdot\|_q$ be the ℓ^p-norm on $\mathbb{R}^m$ and the ℓ^q-norm on $\mathbb{R}^n$ respectively. Let $\mathcal{S} = \{x \in \mathbb{R}^m_+ \mid \|x\|_p = 1\}$ and $c > 0$ defined as in (11.8). Then (11.4) has solution and the following assertions hold:

i) If either $MM^\top$ has at least one positive entry per row and $c < 1$ or $(p-1)(q-1) = 1$ and $MM^\top$ is irreducible, then (11.4) has a unique solution $(\lambda, (u,v))$ such that $u \in \mathbb{R}^m_{++}$. Furthermore, for every $x_0 \in \mathbb{R}^m_{++} \cap \mathcal{S}$, the sequence $(x_k)_{k=1}^\infty \subset \mathbb{R}^m_{++} \cap \mathcal{S}$ defined as

$$x_{k+1} = \frac{\mathcal{F}(x_k)}{\|\mathcal{F}(x_k)\|_p} \qquad k = 0, 1, \ldots, \tag{11.9}$$

where $\mathcal{F} \colon \mathbb{R}^m_+ \to \mathbb{R}^m_+$ is defined in (11.7), satisfies $\lim_{k\to\infty} x_k = u$. Moreover, it holds $v = \frac{M^\top u}{\|M^\top u\|_q}$ and if additionally $MM^\top$ is irreducible, then $(\lambda, (u,v))$ it the unique solution of (11.4). Finally, if $c < 1$, then

$$\mu_{\mathbb{R}^m_+}(x_k, u) \leq c^k \left(\frac{\mu_{\mathbb{R}^m_+}(x_1, x_0)}{(1-c)}\right) \quad \forall k \geq 1.$$

ii) If either $(p-1)(q-1) > 1$ and $MM^\top$ has at least one positive entry per row or $(p-1)(q-1) = 1$ and $MM^\top$ is irreducible, then the unique solution $(\lambda, (u, v))$ of (11.10) such that $u \in \mathbb{R}^m_{++}$ is maximal in the following sense:

$$\lambda = \sup\{\tilde{\lambda} \mid \tilde{\lambda} \text{ is an } \ell^{p,q}\text{-singular value of } M\}.$$

Furthermore, the following Collatz-Wielandt formula holds:

$$\sup_{x\in\mathcal{S}} \min_{x_j\neq 0} \frac{(M(M^\top x)^{1/(q-1)})_j}{x_j^{p-1}} = \lambda^{\frac{q}{q-1}} = \inf_{x\in\mathbb{R}^m_{++}\cap\mathcal{S}} \min_{j\in[m]} \frac{(M(M^\top x)^{1/(q-1)})_j}{x_j^{p-1}}.$$

Moreover, the sequences $(\alpha_k)_{k=1}^\infty, (\beta_k)_{k=1}^\infty$ defined in terms of the sequence $(x_k)_{k=1}^\infty$ of (11.9), by

$$\alpha_k = \min_{j\in[m]} \frac{\mathcal{F}(x_k)_j}{(x_k)_j} \qquad \text{and} \qquad \beta_k = \max_{j\in[m]} \frac{\mathcal{F}(x_k)_j}{(x_k)_j}$$

satisfy

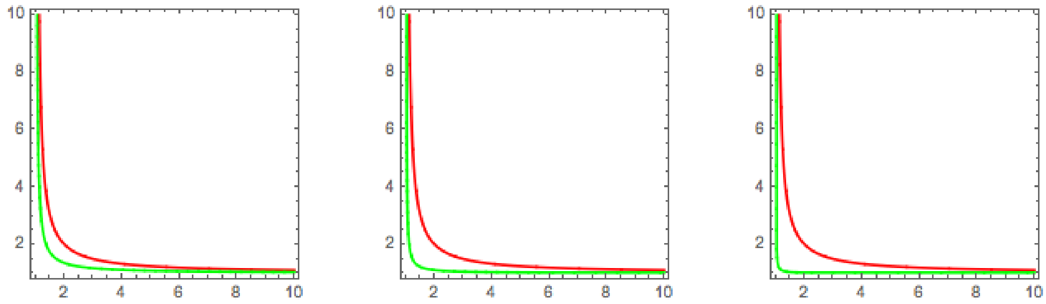
$$\alpha_{k+1} \leq \alpha_k \leq \lambda^{\frac{q}{(q-1)(p-1)}} \leq \beta_{k+1} \leq \beta_k,$$

for every $k \geq 1$, and $\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \beta_k = \lambda^{\frac{q}{(q-1)(p-1)}}$.

Proof. Lemma 11.2.1 implies that for all $x \in \mathbb{R}^m_+$ such that $M^\top x \neq 0$, it holds $\mathcal{F}(x) = \lambda x$ if and only if $(\lambda^{\frac{q}{(p-1)(q-1)}}, (x, y))$ is a solution of (11.4) with $y = \frac{(M^\top x)^{1/(q-1)}}{\|(M^\top x)^{1/(q-1)}\|_q}$. Note that if $MM^\top$ has at least one nonzero entry per row, and $x \in \mathbb{R}^m_{++}$ satisfies $\mathcal{F}(x) = \lambda x$, then $M^\top x \neq 0$. Furthermore, let us show that if $MM^\top$ is irreducible, then for every $x \in \mathbb{R}^m_{++}$, $D\mathcal{F}(x)$ is primitive. First, note that, as $MM^\top$ is irreducible, $MM^\top$ has at least one positive entry per row and thus $\mathcal{F}(x) \in \mathbb{R}^m_{++}$ for all $x \in \mathbb{R}^m_{++}$. Furthermore, note that for all $i, j \in [m]$ it holds

$$D\mathcal{F}(x)_{i,j} = \frac{\mathcal{F}(x)^{2-p}}{(p-1)(q-1)} \sum_{l=1}^m M_{i,l}(M^\top x)_l^{\frac{2-q}{q-1}} M^\top_{l,j}.$$

It follows that $(MM^\top)_{i,j} > 0$ implies $D\mathcal{F}(x)_{i,j} > 0$. Now, as $MM^\top$ is irreducible and positive semi-definite, $\rho(MM^\top)$ is a simple eigenvalue of $MM^\top$ and thus $MM^\top$ is primitive by Theorem 1 of [93]. Hence, $MM^\top$ is primitive and thus $D\mathcal{F}(x)$ is primitive as well. The proof follows now from Theorem 7.3.1 and Theorem 11.1.1 (Ic), (IIIc), (II), (IVa) and (IVc) applied to $\mathcal{F}$. □



$\epsilon = \frac{1}{4}$ $\qquad$ $\epsilon = \frac{1}{2}$ $\qquad$ $\epsilon = \frac{3}{4}$

Figure 1: Plot of the lines $(p-1)(q-1) = 1$ (in red) and $\kappa(M_\epsilon)\kappa(M_\epsilon^T) = (p-1)(q-1)$ (in green) with p on the x-axis and q on the y-axis.

The existence and uniqueness of a solution to (11.4) as well as the convergence of $(x_k)_{k=1}^\infty$, the monotonicity and convergence of $(\alpha_k)_{k=1}^\infty, (\beta_k)_{k=1}^\infty$ and the Collatz-Wielandt formula are proved in Theorems 1 and 2 of [36]$^\diamond$ under the assumption that $(p-1)(q-1) \geq 1$ and $MM^\top$ is irreducible. These results improve previous work in [13, 17, 31]. Indeed, in [17] uniqueness of the positive solution is not proved, in [13] the matrix is assumed to be irreducible and in [31] it is assumed that $p, q \geq 2$ and $MM^\top$, $M^\top M$ are irreducible matrices. The condition $c < 1$ for the existence of a solution $(\lambda, (u, v))$ to (11.4) with $u \in \mathbb{R}^n_{++}$, its uniqueness and the convergence of $(x_k)_{k=1}^\infty$ is new. It is the first known condition which ensure the computability of $\|M\|_{p,q}$ when $(p-1)(q-1) < 1$. We illustrate the improvement offered by this new condition in Figure 11.2.1 where we plot the lines $(p-1)(q-1) = 1$ and $\kappa(M)\kappa(M^T) = (p-1)(q-1)$, i.e. $c = 1$, for the matrix $M_\epsilon \in \mathbb{R}^{2\times 2}_{++}$ defined in terms of $\epsilon \in (0, 1)$ as

$$M_\epsilon = \begin{pmatrix} \epsilon & 1 \\ 1 & \epsilon \end{pmatrix} \qquad \text{so that} \qquad \kappa(M_\epsilon) = \kappa(M_\epsilon^\top) = \frac{1-\epsilon}{1+\epsilon}.$$

11.2.2 $\ell^{p_1,\ldots,p_d}$-singular vectors of a nonnegative tensor

Let $d \geq 2$, $T \in \mathbb{R}_+^{n_1\times\ldots\times n_d}$, $p_1, \ldots, p_d \in (1, \infty)$ and let $\|\cdot\|_{p_i}$ be the ℓ^{p_i}-norm on $\mathbb{R}^{n_i}$. Let $\mathcal{C} = \mathbb{R}^{n_1}_+ \times \ldots \times \mathbb{R}^{n_d}_+$, consider the monotonic multi-normalization $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d_+$ defined as $\boldsymbol{\nu}(x) = (\|x_1\|_{p_1}, \ldots, \|x_d\|_{p_d})$ and set $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$. We have seen in Section 4.2.1 that the $\ell^{p_1,\ldots,p_d}$-singular values/vectors, of T are the solutions to the following system of equations:

$$\begin{cases} T(\cdot, x_2, \ldots, x_d) & = & \lambda\, x_1^{p_1-1} \\ T(x_1, \cdot, x_3, \ldots, x_d) & = & \lambda\, x_2^{p_2-1} \\ \vdots & \vdots & \vdots \\ T(x_1, \ldots, x_{d-1}, \cdot) & = & \lambda\, x_d^{p_d-1} \end{cases} \quad \text{with} \quad (\lambda, x) \in \mathbb{R}_+ \times \mathcal{S}_{\boldsymbol{\nu}}. \tag{11.10}$$

We recall from Equation (4.3) that the above system characterizes the nonnegative critical points of the objective function defining the $\ell^{p_1,\dots,p_d}$-norm of T, defined as

$$\|T\|_{p_1,\dots,p_d} = \max_{x_1\neq 0,\dots,x_d\neq 0} \frac{f_T(x_1,\dots,x_d)}{\|x_1\|_{p_1}\cdots\|x_d\|_{p_d}}.$$

In particular, it holds

$$\|T\|_{p_1,\dots,p_d} = \sup\{\tilde\lambda \,|\, \tilde\lambda \text{ is an } \ell^{p_1,\dots,p_d}\text{-singular value of } T\}.$$

Moreover, we have proved in Proposition 4.2.1 that the $\ell^{p_1,\dots,p_d}$-singular vectors of T are characterized by the eigenvectors of the order-preserving multi-homogeneous mapping $\mathcal{F}\colon \mathcal{C}\to\mathcal{C}$ given by

$$\mathcal{F}(x) = \Psi(\mathcal{L}(x)) \qquad \forall x\in\mathcal{C}, \tag{11.11}$$

where $\mathcal{L}\colon\mathcal{C}\to\mathcal{C}$ is the cone multi-linear mapping defined as

$$\mathcal{L}(x) = \big(T(\cdot,x_2,\dots,x_d),\dots,T(x_1,\dots,x_{d-1},\cdot)\big) \qquad \forall x\in\mathcal{C}, \tag{11.12}$$

and $\Psi\colon\ \mathcal{C}\to\mathcal{C}$ is the nonlinear multi-homogeneous mapping defined as

$$\Psi(x) = (x_1^{1/(p_1-1)},\dots,x_d^{1/(p_d-1)}) \qquad \forall x\in\mathcal{C}.$$

We discuss properties of $\mathcal{L}$. First, note that from Lemmas 9.2.3 and 9.4.3, we have that $\mathcal{L}$ is irreducible, resp. weakly irreducible, if and only if T is an irreducible, resp. weakly irreducible, tensor in the sense of [31]. Next, note that if $T\in\mathbb{R}^{n_1\times\ldots\times n_d}_{++}$, then $\mathcal{L}(\mathcal{C}_0)\in\operatorname{int}(\mathcal{C})$, where $\mathcal{C}_0=\{x\in\mathcal{C}\,|\,x_1,\dots,x_d\neq 0\}$ and in this case $\mathcal{L}$ is irreducible. Furthermore, it holds $\mathcal{L}(\operatorname{int}(\mathcal{C}))\subset\operatorname{int}(\mathcal{C})$ if and only if T is such that for all $i=1,\dots,d$ and $l_i\in[n_i]$, there exists $j_1\in[n_1],\dots,j_d\in[n_d]$ such that $j_i=l_i$ and $T_{j_1,\dots,j_d}>0$. Note that the latter assumption is not very restrictive, as for instance, if $n_1=\ldots=n_d$ and T is the identity tensor, then $\mathcal{L}(\operatorname{int}(\mathcal{C}))\subset\operatorname{int}(\mathcal{C})$. Moreover, by definition of weak irreducibility, we have that if $\mathcal{L}$ is weakly irreducible, then $\mathcal{L}(\operatorname{int}(\mathcal{C}))\subset\operatorname{int}(\mathcal{C})$. To summarize, we have the following chain of implications:

$$\mathcal{L}(\mathcal{C}_0)\subset\operatorname{int}(\mathcal{C}) \Rightarrow \mathcal{L}\text{ irreducible } \Rightarrow \mathcal{L}\text{ weakly irreducible } \Rightarrow \mathcal{L}(\operatorname{int}(\mathcal{C}))\subset\operatorname{int}(\mathcal{C}).$$

Now, let $L\in\mathbb{R}^{d\times d}_+$ be defined as $L_{i,i}=0$ for $i=1,\dots,d$ and

$$L_{k,l} = \begin{cases}\tanh\big[\tfrac14\ln(\Delta_{k,l}(T))\big] & \text{if } T\in\mathbb{R}^{n_1\times\ldots\times n_d}_{++},\\ 1 & \text{otherwise,}\end{cases} \qquad \forall k,l\in\{1,\dots,d\}, k\neq l,$$

where $\Delta_{k,l}(T)$ is defined in (8.10). Then, by Corollary 8.2.5, for all $i,j\in\{1,\dots,d\}$, we have

$$L_{j,i} \geq \sup_{x\in\mathcal{C},y_i\in C_i} \tanh\big[\tfrac14\mu_{C_j}\big(\mathcal{L}(x)_j,\mathcal{L}(x_1,\dots,x_{i-1},y_i,x_{i+1},\dots,x_m)_j\big)\big].$$

Moreover, note that unless all the entries of T are equal, in which case $L = 0$, then all the off diagonal entries of L are positive and thus L is irreducible. By Theorem 8.1.2, we have

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{L}(x), \mathcal{L}(y)) \leq L\boldsymbol{\mu}_{\mathcal{C}}(x, y) \qquad \forall x, y \in \operatorname{int}(\mathcal{C}).$$

We discuss properties of Ψ. Let $Q \in \mathbb{R}_+^{d\times d}$ be defined as

$$Q = \operatorname{diag}\left(\tfrac{1}{p_1-1}, \ldots, \tfrac{1}{p_d-1}\right), \tag{11.13}$$

then Q is the homogeneity matrix of Ψ. Moreover, as $p_i - 1 > 0$ for all $i = 1, \ldots, d$, Ψ is order-preserving and Lemma 5.2.3 implies that

$$\boldsymbol{\mu}_{\mathcal{C}}(\Psi(x), \Psi(y)) = Q\boldsymbol{\mu}_{\mathcal{C}}(x, y) \qquad \forall x, y \in \mathcal{C} \quad \text{with} \quad x \sim_{\mathcal{C}} y.$$

Note also that $\Psi(x) \sim_{\mathcal{C}} x$ for every $x \in \mathcal{C}$ and thus $\mathcal{F}(x) \sim_{\mathcal{C}} \mathcal{L}(x)$ for all $x \in \mathcal{C}$. Hence, by Lemmas 9.2.1 and 9.2.2, we have that $\mathcal{F}(\mathcal{C}_0) \subset \operatorname{int}(\mathcal{C})$, $\mathcal{F}$ is irreducible, $\mathcal{F}$ is weakly irreducible, respectively $\mathcal{F}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$ if and only if $\mathcal{L}$ has the corresponding property.

Finally, note that $\mathcal{F}$ is order-preserving as Ψ and $\mathcal{L}$ are order-preserving. Now, let $A, M \in \mathbb{R}^{d\times d}$ be respectively defined as

$$A = Q(\mathbf{1}\mathbf{1}^\top - I) \qquad \text{and} \qquad M = QL, \tag{11.14}$$

then A is the homogeneity matrix of $\mathcal{F}$ and it holds

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}(x), \mathcal{F}(y)) \leq M\boldsymbol{\mu}_{\mathcal{C}}(x, y) \qquad \forall x, y \in \operatorname{int}(\mathcal{C}).$$

Note in particular that, as $L \preceq_{\mathbb{R}_+^{d\times d}} (\mathbf{1}\mathbf{1}^\top - I)$, we have $M \preceq_{\mathbb{R}_+^{d\times d}} A$ and therefore by monotonicity of the spectral radius (see Example 2.3.6) and the linear Collatz-Wielandt formula (10.1), it holds

$$\rho(M) \leq \rho(A) \leq \max_{i=1,\ldots,d} (A\mathbf{1})_i = \max_{i=1,\ldots,d} \frac{d-1}{p_i - 1}. \tag{11.15}$$

Furthermore, note that if $T \in \mathbb{R}_+^{n_1\times\ldots\times n_d} \setminus \mathbb{R}_{++}^{n_1\times\ldots\times n_d}$, i.e. T has at least one zero entry, then $L = (\mathbf{1}\mathbf{1}^\top - I)$ and thus $M = A$. We discuss the behavior of M and A with respect to $p_1, \ldots, p_d$. To this end, consider the mappings $f_M\colon (1,\infty)^d \to \mathbb{R}_+^{d\times d}$ and $f_A\colon (1,\infty)^d \to \mathbb{R}_+^{d\times d}$ given by

$$f_M(q_1, \ldots, q_d) = \operatorname{diag}\left(\tfrac{1}{q_1-1}, \ldots, \tfrac{1}{q_d-1}\right)L, \quad \text{and}$$
$$f_A(q_1, \ldots, q_d) = \operatorname{diag}\left(\tfrac{1}{q_1-1}, \ldots, \tfrac{1}{q_d-1}\right)(\mathbf{1}\mathbf{1}^\top - I).$$

Then, $M = f_M(p_1, \ldots, p_d)$ and $A = f_A(p_1, \ldots, p_d)$. Moreover, if T does not have all entries equal, then L and $(\mathbf{1}\mathbf{1}^\top - I)$ are irreducible matrices. Hence, by the discussion in Example 2.3.6, we have that for all $q_1, \ldots, q_d, \tilde{q}_1, \ldots, \tilde{q}_d \in (1, \infty)$ such that $(q_1, \ldots, q_d) \lneq (\tilde{q}_1, \ldots, \tilde{q}_d)$ it holds

$$\rho\big(f_M(q_1, \ldots, q_d)\big) > \rho\big(f_M(\tilde{q}_1, \ldots, \tilde{q}_d)\big) \quad \text{and} \quad \rho\big(f_A(q_1, \ldots, q_d)\big) > \rho\big(f_A(\tilde{q}_1, \ldots, \tilde{q}_d)\big).$$

Furthermore, it holds

$$\lim_{t\to\infty} \rho\big(f_M(t\,p_1,\ldots,t\,p_d)\big) = 0 \quad \text{and} \quad \lim_{t\to\infty} \rho\big(f_A(t\,p_1,\ldots,t\,p_d)\big) = 0.$$

It follows that by choosing $p_1,\ldots,p_d$ large enough, it is always possible to have $\rho(M) < 1$ or $\rho(A) < 1$.

Note that A is irreducible, so let $\mathbf{b} \in \mathbb{R}^d_{++}$ be the Perron vector of $A^\top$. The eigenvectors and eigenvalues of $\mathcal{F}$ are related to the solutions of (11.10) in Proposition 4.2.1 where we have shown that $x \in \mathcal{S}_{\boldsymbol{\nu}}$ is an $\ell^{p_1,\ldots,p_d}$-singular vector of T if and only if it is an eigenvector of $\mathcal{F}$. Moreover, let λ be the $\ell^{p_1,\ldots,p_d}$-singular value of T corresponding to x and let θ be the eigenvalue of $\mathcal{F}$ corresponding to x, i.e. $\mathcal{F}(x) = \boldsymbol{\vartheta} \otimes x$ with $\boldsymbol{\vartheta} \in \mathbb{R}^d_+$ and $\theta = \prod_{i=1}^d \vartheta_i^{b_i}$, then there exists $\gamma > 0$, independent of x, λ and $\boldsymbol{\vartheta}$, such that $\theta = \lambda^\gamma$ which implies that the eigenvalues of $\mathcal{F}$ on $\mathcal{S}_{\boldsymbol{\nu}}$ and the $\ell^{p_1,\ldots,p_d}$-singular value of T have the same ordering.

We combine the above observations with Theorems 11.1.1 and 11.1.2 to obtain to the following Perron-Frobenius theorem for $\ell^{p_1,\ldots,p_d}$-singular vectors of nonnegative tensors.

Theorem 11.2.3. Let $d \geq 2$, $T \in \mathbb{R}_+^{n_1\times\ldots\times n_d}$, $p_1,\ldots,p_d \in (1,\infty)$, let $\|\cdot\|_{p_i}$ be the ℓ^{p_i}-norm on $\mathbb{R}^{n_i}$. Let $\mathcal{C} = \mathbb{R}^{n_1}_+ \times \ldots \times \mathbb{R}^{n_d}_+$ and $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \mid \|x_1\|_{p_1} = \ldots = \|x_d\|_{p_d} = 1\}$. Let $\mathcal{L}\colon \mathcal{C} \to \mathcal{C}$ be defined as in (11.12), let $A, M \in \mathbb{R}^{d\times d}_+$ be defined as in (11.14) and let $\mathbf{b} \in \mathbb{R}^d_{++}$ be the Perron vector of $A^\top$. Then (11.10) has a solution and the following assertions hold:

i) If $\mathcal{L}(\mathrm{int}(\mathcal{C})) \subset \mathrm{int}(\mathcal{C})$ and $\rho(M) < 1$, then (11.10) has a unique solution (λ, u) such that $u \in \mathrm{int}(\mathcal{C})$. Furthermore, for every $x_0 \in \mathrm{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$, the sequence $(x_k)_{k=1}^\infty \subset \mathrm{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ defined as

$$x_{k+1} = \Big(\frac{\mathcal{F}(x_k)_1}{\|\mathcal{F}(x_k)_1\|_{p_1}}, \ldots, \frac{\mathcal{F}(x_k)_d}{\|\mathcal{F}(x_k)_d\|_{p_d}}\Big) \qquad k = 0,1,\ldots, \tag{11.16}$$

where $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ is defined in (11.11), satisfies

$$\lim_{k\to\infty} x_k = u \qquad \text{and} \qquad \boldsymbol{\mu}_{\mathcal{C}}(x_k, u) \leq M^k \mathbf{v} \qquad \forall k \geq 1,$$

with $\mathbf{v} = (I - M)^{-1}\boldsymbol{\mu}_{\mathcal{C}}(x_1, x_0)$. If additionally, $\mathcal{L}$ is irreducible, then (λ, u) is the unique solution of (11.10).

ii) If $\rho(A) = 1$ and $\mathcal{L}$ is weakly irreducible, then (11.10) has a unique solution (λ, u) such that $u \in \mathrm{int}(\mathcal{C})$. Furthermore, for every $y_0 \in \mathrm{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$, the sequence $(y_k)_{k=1}^\infty \subset \mathrm{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ defined as

$$y_{k+1} = \Big(\frac{\mathcal{H}(y_k)_1}{\|\mathcal{H}(y_k)_1\|_{p_1}}, \ldots, \frac{\mathcal{H}(y_k)_d}{\|\mathcal{H}(y_k)_d\|_{p_d}}\Big) \qquad k = 0,1,\ldots, \tag{11.17}$$

where $\mathcal{H}\colon \mathcal{C} \to \mathcal{C}$ is defined as $\mathcal{H}(x) = \big(x \circ \mathcal{F}(x)\big)^{1/2}$ and $\mathcal{F}$ is defined in (11.11), satisfies

$$\lim_{k\to\infty} y_k = u.$$

If additionally, $\mathcal{L}$ is irreducible, then (λ, u) is the unique solution of (11.10).

iii) If either $\rho(A) < 1$ and $\mathcal{L}(\mathrm{int}(\mathcal{C})) \subset \mathrm{int}(\mathcal{C})$, or $\rho(A) = 1$ and $\mathcal{L}$ is weakly irreducible, then the unique solution (λ, u) of (11.10) such that $u \in \mathrm{int}(\mathcal{C})$ is maximal in the following sense:

$$\lambda = \sup\{\tilde{\lambda} \mid \tilde{\lambda} \text{ is an } \ell^{p_1,\ldots,p_d}\text{-singular value of } T\}$$

Furthermore, with $\gamma = \sum_{i=1}^d \frac{b_i}{p_i-1}$, the following Collatz-Wielandt formula holds,

$$\sup_{x\in\mathcal{S}_{\boldsymbol{\nu}}} \prod_{i=1}^{d}\Big(\max_{j_i\in[n_i]} \frac{\mathcal{L}(x)_{i,j_i}}{x_{i,j_i}^{p_i-1}}\Big)^{\frac{b_i}{p_i-1}} = \lambda^{\gamma} = \inf_{x\in\mathrm{int}(\mathcal{C})\cap\mathcal{S}_{\boldsymbol{\nu}}} \prod_{i=1}^{d}\Big(\min_{j_i\in[n_i]} \frac{\mathcal{L}(x)_{i,j_i}}{x_{i,j_i}^{p_i-1}}\Big)^{\frac{b_i}{p_i-1}}.$$

Moreover, the sequences $(\alpha_k)_{k=1}^\infty, (\beta_k)_{k=1}^\infty, (\zeta_k)_{k=1}^\infty, (\xi_k)_{k=1}^\infty \subset (0,\infty)$ defined in terms of the sequences $(x_k)_{k=1}^\infty, (y_k)_{k=1}^\infty \subset \mathrm{int}(\mathcal{C})$ of (11.16) and (11.17), by

$$\alpha_k = \prod_{i=1}^{d}\Big(\min_{j_i\in[n_i]} \frac{\mathcal{F}(x_k)_{i,j_i}}{(x_k)_{i,j_i}}\Big)^{b_i}, \quad \beta_k = \prod_{i=1}^{d}\Big(\max_{j_i\in[n_i]} \frac{\mathcal{F}(x_k)_{i,j_i}}{(x_k)_{i,j_i}}\Big)^{b_i}$$
$$\zeta_k = \prod_{i=1}^{d}\Big(\min_{j_i\in[n_i]} \frac{\mathcal{H}(y_k)_{i,j_i}}{(y_k)_{i,j_i}}\Big)^{2b_i}, \quad \xi_k = \prod_{i=1}^{d}\Big(\max_{j_i\in[n_i]} \frac{\mathcal{H}(x_k)_{i,j_i}}{(y_k)_{i,j_i}}\Big)^{2b_i},$$

satisfy

$$\alpha_{k+1} \le \alpha_k \le \lambda^\gamma \le \beta_{k+1} \le \beta_k \qquad \text{and} \qquad \zeta_{k+1} \le \zeta_k \le \lambda^\gamma \le \xi_{k+1} \le \xi_k,$$

for every $k \ge 1$, and if $\rho(A) < 1$, respectively $\rho(A) = 1$ and $\mathcal{L}$ is weakly irreducible, then

$$\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \beta_k = \lambda^\gamma, \quad \text{respectively} \quad \lim_{k\to\infty} \zeta_k = \lim_{k\to\infty} \xi_k = \lambda^\gamma.$$

Proof. We begin with general observations. We recall from Proposition 4.2.1 that for every $u \in \mathcal{S}_{\boldsymbol{\nu}}$, (λ, u) is a solution of (11.10) if and only if $\mathcal{F}(u) = \boldsymbol{\vartheta} \otimes u$ with $\boldsymbol{\vartheta} = (\lambda^{1/(p_1-1)}, \ldots, \lambda^{1/(p_d-1)})$ where $\mathcal{F}$ is defined in (11.11). As $\mathcal{F}$ is continuous, the existence of a solution to (11.10) follows form Theorem 4.1.5. Note that $\mathcal{H}\colon \mathcal{C} \to \mathcal{C}$ is order-preserving and multi-homogeneous of degree $B = (A+I)/2$. Then $\rho(B) = \rho(A)$ and, as A is irreducible, B is primitive. $B^\top$ has the same Perron vector as $A^\top$, namely $\mathbf{b} \in \mathbb{R}^d_{++}$. Furthermore, note that if $x \in \mathrm{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ satisfies $\mathcal{F}(x) = \boldsymbol{\vartheta} \otimes x$ with $\boldsymbol{\vartheta} \in \mathbb{R}^d_+$, then, it holds $\mathcal{H}(x) = (x \circ \mathcal{F}(x))^{1/2} = \big(x \circ (\boldsymbol{\vartheta} \otimes x)\big)^{1/2} = \boldsymbol{\vartheta}^{1/2} \otimes x$. It follows that if θ is the eigenvalue of $\mathcal{F}$ corresponding to x, then the eigenvalue of $\mathcal{H}$ corresponding to x equals $\sqrt{\theta}$. Conversely, if it holds $\mathcal{H}(x) = \boldsymbol{\beta} \otimes x$ with $\boldsymbol{\beta} \in \mathbb{R}^d_+$, then $\mathcal{F}(x) = x^{-1} \circ \mathcal{H}(x)^2 = \boldsymbol{\beta}^2 \otimes x$ and thus if α is the eigenvalue of $\mathcal{H}$ corresponding to x, then the eigenvalue of $\mathcal{F}$ corresponding to x equals α^2.

i) Follows from Theorem 11.1.2.

ii) $\mathcal{F}$ is weakly irreducible since $\mathcal{L}$ is weakly irreducible. Thus, the existence of (λ, u) follows from Theorem 11.1.1 (Ic). As $\mathcal{L}$ is weakly irreducible, Lemma 9.4.6 implies that its Jacobian matrix $D\mathcal{L}(u) \in \mathbb{R}_+^{(n_1+\ldots+n_d)\times(n_1+\ldots+n_d)}$ is irreducible. The Jacobian matrix of $\mathcal{F}$ at u is given by

$$D\mathcal{F}(u) = \operatorname{diag}((p_1-1)\mathcal{L}(u)_1^{\frac{2-p_1}{p_1-1}}, \ldots, (p_d-1)\mathcal{L}(u)_d^{\frac{2-p_d}{p_d-1}})D\mathcal{L}(u).$$

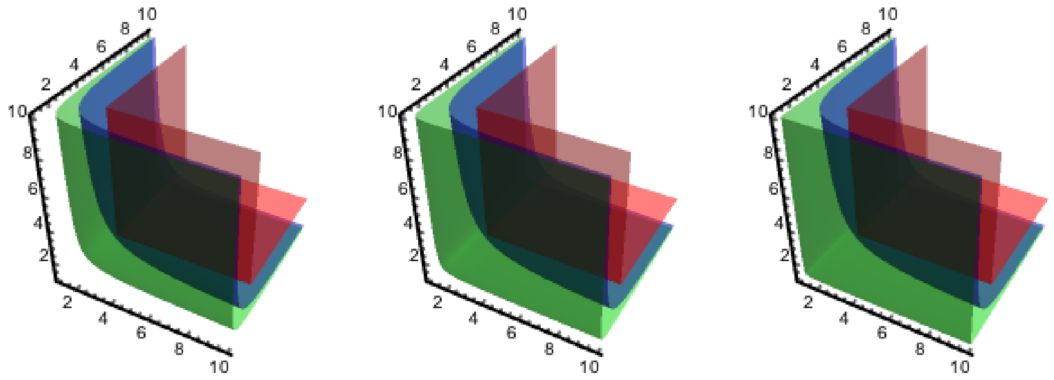
It follows that $D\mathcal{F}(u)$ is irreducible as well and thus the uniqueness of (λ, u) follows from Theorem 11.1.1 (IIIc). The Jacobian matrix of $\mathcal{H}$ at u is given by

$$D\mathcal{H}(u) = \frac{1}{2}\operatorname{diag}(\mathcal{H}(u))^{-1/2}\big(\operatorname{diag}(\mathcal{F}(u)) + \operatorname{diag}(x)DF(u)\big).$$

Hence, $D\mathcal{H}(u)$ is a primitive matrix since it is the sum of a positive diagonal matrix and an irreducible matrix. The convergence of $(y_k)_{k=1}^\infty$ now follows from Theorem 11.1.1 (IVc).

iii) The maximality of λ as well as the Collatz-Wielandt formula follow from Theorem 11.1.1 (II). Finally, the monotonicity and the convergence of the sequences $(\alpha_k)_{k=1}^\infty$,$(\beta_k)_{k=1}^\infty$,$(\zeta_k)_{k=1}^\infty$,$(\xi_k)_{k=1}^\infty$ follow from Theorem 11.1.1, (IVa) and (IVc). □

The results of Theorem 11.2.3 assuming $\rho(A) \leq 1$ are implied by Theorems 3.2 and 3.3 in [40]$^\diamond$ and improve previous results of the literature: The existence and uniqueness of solutions to (11.10) and a Collatz-Wielandt formula were proved in Theorem 1 and Corollary 4.3 of [31] under the assumption that $\min\{p_1, \ldots, p_d\} \geq d$ and $\mathcal{L}$ is weakly irreducible. Furthermore, the convergence of the sequence $(x_k)_{k=1}^\infty$ towards a unique $\ell^{p_1,\ldots,p_d}$-singular vector $u \in \operatorname{int}(\mathcal{C})$ is proved in Corollary 5.2 [31] under the assumption that $\min\{p_1, \ldots, p_d\} \geq d$ and $D\mathcal{L}(u)$ is primitive. These assumptions are more restrictive than those of the corresponding result in the above theorem. Indeed, we see from (11.15) that $\min\{p_1, \ldots, p_d\} \geq d$ implies $\rho(A) \leq 1$ but the converse is not true in general. Furthermore, as $\mathcal{L}$ is weakly irreducible if and only if $D\mathcal{L}(u)$ is an irreducible matrix (by Lemmas 9.4.3 and 9.4.6), the assumption that $D\mathcal{L}(u)$ needs to be primitive is more restrictive than requiring $\mathcal{L}$ to be weakly irreducible. The convergence of $(x_k)_{k=1}^\infty$ for $\rho(A) \leq 1$, $d \geq 3$ and $D\mathcal{L}(u)$ primitive is implied by Theorem 11.1.1, (IVc). Furthermore, the convergence of $(x_k)_{k=1}^\infty$ under the assumption that $\mathcal{L}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$ and $\rho(A) < 1$ implied by Theorem 11.2.3, (i) is strikingly less restrictive than asking for $D\mathcal{L}(u)$ to be primitive. By using the substitution strategy discussed in Section 4.3, a Perron-Frobenius theorem for $\ell^{p_1,\ldots,p_d}$-singular vectors of nonnegative tensors is proved in [36]$^\diamond$ under a slightly less restrictive conditions on $p_1, \ldots, p_d$, and similar irreducibility conditions, than that of [31]. We note however that the condition on $p_1, \ldots, p_d$ discussed in [36]$^\diamond$ implies $\rho(A) \leq 1$ (see Proposition 7.1.4). We stress that, when $T \in \mathbb{R}_{++}^{n_1\times\ldots\times n_d}$, the condition $\rho(M) < 1$ of Theorem 11.2.3, (i) is the first known condition which implies existence and uniqueness of a positive solution to (11.10) as well as the convergence of $(x_k)_{k=1}^\infty$ for choices of $p_1, \ldots, p_d$ satisfying $p_1, \ldots, p_d < m$. We illustrate the improvement



$\epsilon = \frac{1}{4}$ $\qquad$ $\epsilon = \frac{1}{2}$ $\qquad$ $\epsilon = \frac{3}{4}$

Figure 2: Plot of the surfaces $\rho(A(p_1, p_2, p_3)) = 1$ (in blue), $\rho(M(p_1, p_2, p_3, \epsilon)) = 1$ (in green) and $\min\{p_1, p_2, p_3\} = 1$ (in red) with p_1 on the x-axis, p_2 on the y-axis and p_3 on the z-axis.

offered by Theorem 11.2.3 in Figure 2 where we plot the conditions discussed above on p_1, p_2, p_3 for the tensor $T_\epsilon \in \mathbb{R}^{2\times 2\times 2}_{++}$ defined for $\epsilon \in (0, 1)$ as

$$(T_\epsilon)_{i,j,k} = \begin{cases} 1 & \text{if } i = j = k, \\ \epsilon & \text{otherwise} \end{cases} \qquad \forall i, j, k = 1, 2. \tag{11.18}$$

From Example 8.2.2, we know that $\Delta_{i,j}(T_\epsilon) = \frac{1+\epsilon^2}{2\epsilon^2}$ for all $i \neq j$, so that

$$f(\epsilon) = \tanh\left[\tfrac{1}{4}\ln(\Delta_{i,j}(T_\epsilon))\right] = \frac{\sqrt{1+\epsilon^2} - \sqrt{2}\epsilon}{\sqrt{1+\epsilon^2} + \sqrt{2}\epsilon}. \tag{11.19}$$

In particular, we have

$$f\Big(\frac{1}{4}\Big) \leq \frac{47}{96}, \qquad f\Big(\frac{1}{2}\Big) \leq \frac{22}{96}, \qquad f\Big(\frac{3}{4}\Big) \leq \frac{5}{61}. \tag{11.20}$$

For all $p_1, p_2, p_3 \in (1, \infty)$, let

$$A(p_1, p_2, p_3) = \begin{pmatrix} 0 & \frac{1}{p_1 - 1} & \frac{1}{p_1 - 1} \\ \frac{1}{p_2 - 1} & 0 & \frac{1}{p_2 - 1} \\ \frac{1}{p_3 - 1} & \frac{1}{p_3 - 1} & 0 \end{pmatrix}, \qquad M(p_1, p_2, p_3, \epsilon) = f(\epsilon) A(p_1, p_2, p_3).$$

Note that $\rho(A(p, p, p)) \leq 1$ if and only if $p \geq 3$ while $\rho(M(2, 2, 2, \epsilon)) < 1$ if $\epsilon \in [1/2, 1)$. Thus, the best rank one approximation [45, 63] (an NP-hard problem in general) of T_ϵ can be computed in polynomial time when $\epsilon \in [1/2, 1)$. As can be seen in Figure 2, the condition $\rho(M(p_1, p_2, p_3, \epsilon)) < 1$ which holds for all triples (p_1, p_2, p_3) above the green surface, improves the condition $\min\{p_1, p_2, p_3\} \geq 3$ which is satisfied for all pair (p_1, p_2, p_3) on and above above the red surface, especially when ϵ is closed to 1. The red and the blue surfaces meet at $(p_1, p_2, p_3) = (3, 3, 3)$, i.e. $p_1 = p_2 = p_3 = d$.

Finally, we note that in facts the condition $\rho(M) < 1$ of Theorem 11.2.3, (i) can be further improved by using the smallest L satisfying the assumptions of Theorem 11.1.2. As discussed in Example 8.2.2, $\Delta_{i,j}(T)$ only provides an upper bound on the coefficients of the smallest L in Theorem 11.1.2. Indeed, by using the exact bound L (computed for $T = T_\epsilon$ in Example 8.1.4), it can for instance be shown that for $p_1 = p_2 = p_3 = 2$, the results of Theorem 11.2.3 (i) still holds for every $\epsilon \in (2/5, 1)$ (rather than $\epsilon \in [1/2, 1)$).

11.2.3 Rectangular $\ell^{p,q}$-singular vectors of a nonnegative tensor

Let $d \geq 2$, $T \in \mathbb{R}_+^{n_1\times\ldots\times n_d}$ with $n_1, \ldots, n_d$ so that there are integers m, n such that $m = n_1 = \ldots = n_a$ and $n = n_{a+1} = \ldots = n_d$. For $p, q \in (1, \infty)$, let $\|\cdot\|_p$ be the ℓ^p-norm on $\mathbb{R}^m$ and $\|\cdot\|_q$ the ℓ^q-norm on $\mathbb{R}^n$. Define $\mathcal{C} = \mathbb{R}_+^m \times \mathbb{R}_+^n$, and consider the monotonic multi-normalization $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}_+^2$ of $\mathcal{C}$ defined as $\boldsymbol{\nu}(x, y) = (\|x\|_p, \|y\|_q)$ and set $\mathcal{S}_{\boldsymbol{\nu}} = \{(x, y) \in \mathcal{C} \,|\, \boldsymbol{\nu}(x, y) = \mathbf{1}\}$. We have seen in Section 4.2.3 that the rectangular $\ell^{p,q}$-singular values/vectors of T are the solutions to the following system of equations:

$$\begin{cases} T(\cdot, x, \ldots, x, y, \ldots, y) &= \lambda x^{p-1} \\ T(x, \ldots, x, \cdot, y, \ldots, y) &= \lambda y^{q-1} \end{cases} \quad \text{and} \quad (\lambda, (x, y)) \in \mathbb{R}_+ \times \mathcal{S}_{\boldsymbol{\nu}}. \tag{11.21}$$

Moreover, we have discussed that the rectangular $\ell^{p,q}$-singular vectors of T are characterized by the eigenvectors of the order-preserving multi-homogeneous mapping $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ given by

$$\mathcal{F}(x, y) = \Psi(\mathcal{L}(x, y)) \qquad \forall (x, y) \in \mathcal{C}, \tag{11.22}$$

where $\mathcal{L}\colon \mathcal{C} \to \mathcal{C}$ is the mapping defined for every $(x, y) \in \mathcal{C}$ as

$$\mathcal{L}(x, y) = \big(T(\cdot, \underbrace{x, \ldots, x}_{(a-1)\text{ times}}, \underbrace{y, \ldots, y}_{(d-a)\text{ times}}), T(\underbrace{x, \ldots, x}_{a\text{ times}}, \cdot, \underbrace{y, \ldots, y}_{(d-a-1)\text{ times}})\big) \tag{11.23}$$

and $\Psi\colon\ \mathcal{C} \to \mathcal{C}$ is the nonlinear multi-homogeneous mapping defined as

$$\Psi(x, y) = (x^{1/(p-1)}, y^{1/(q-1)}) \qquad \forall (x, y) \in \mathcal{C}.$$

We discuss properties of $\mathcal{L}$. Note that if $T \in \mathbb{R}_{++}^{n_1\times\ldots\times n_d}$, then $\mathcal{L}(\mathcal{C}_0) \in \operatorname{int}(\mathcal{C})$, where $\mathcal{C}_0 = \{(x, y) \in \mathcal{C} \,|\, x, y \neq 0\}$ and in this case $\mathcal{L}$ is irreducible. Furthermore, it holds $\mathcal{L}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$ if and only if T is such that for all $i \in [m]$ and all $i' \in [n]$, there exists $j_1, j_1' \in [n_1], \ldots, j_d, j_d' \in [n_d]$ such that $j_1 = i$, $j_{a+1}' = i'$, $T_{j_1,\ldots,j_d} > 0$ and $T_{j_1',\ldots,j_d'} > 0$. Moreover, if $\mathcal{L}$ is weakly irreducible then $\mathcal{L}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$. Finally, note that by Lemma 9.4.5 we know that if $\mathcal{L}$ is irreducible, then $\mathcal{L}$ is weakly irreducible. To summarize, we have the following chain of implications:

$$\mathcal{L}(\mathcal{C}_0) \subset \operatorname{int}(\mathcal{C}) \Rightarrow \mathcal{L} \text{ irreducible } \Rightarrow \mathcal{L} \text{ weakly irreducible } \Rightarrow \mathcal{L}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C}).$$

Now, let $P \in \mathbb{R}_+^{2\times 2}$ be defined as

$$P = \begin{pmatrix} a-1 & d-a \\ a & d-a-1 \end{pmatrix}$$

and let $L \in \mathbb{R}_+^{2\times 2}$ be defined as $L = P$ if $T \in \mathbb{R}_+^{n_1\times\ldots\times n_d} \setminus \mathbb{R}_{++}^{n_1\times\ldots\times n_d}$ and

$$L = \begin{pmatrix} \sum_{l=2}^{a} \tanh\left[\frac{1}{4}\ln(\Delta_{1,l}(T))\right] & \sum_{l=a+1}^{d} \tanh[\frac{1}{4}\ln(\Delta_{1,l}(T))] \\ \sum_{l=1}^{a} \tanh[\frac{1}{4}\ln(\Delta_{a+1,l}(T))] & \sum_{l=a+2}^{d} \tanh[\frac{1}{4}\ln(\Delta_{a+1,l}(T))] \end{pmatrix}$$

if $T \in \mathbb{R}_{++}^{n_1\times\ldots\times n_d}$, where $\Delta_{k,l}(T)$ is defined in (8.10). Then, by Theorem 8.1.5, for every $(x, y), (\tilde{x}, \tilde{y}) \in \mathcal{C}$ with $(x, y) \sim_{\mathcal{C}} (\tilde{x}, \tilde{y})$, we have

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{L}(x, y), \mathcal{L}(\tilde{x}, \tilde{y})) \leq L\boldsymbol{\mu}_{\mathcal{C}}((x, y), (\tilde{x}, \tilde{y})).$$

We discuss properties of Ψ. Let $Q \in \mathbb{R}_+^{2\times 2}$ be defined as

$$Q = \begin{pmatrix} 1/(p-1) & 0 \\ 0 & 1/(q-1) \end{pmatrix}, \tag{11.24}$$

then Q is the homogeneity matrix of Ψ. Moreover, as $p - 1 > 0$ and $q - 1 > 0$, Ψ is order-preserving and Lemma 5.2.3 implies that for every $(x, y), (\tilde{x}, \tilde{y}) \in \mathcal{C}$ with $(x, y) \sim_{\mathcal{C}} (\tilde{x}, \tilde{y})$, we have

$$\boldsymbol{\mu}_{\mathcal{C}}(\Psi(x, y), \Psi(\tilde{x}, \tilde{y})) = Q\boldsymbol{\mu}_{\mathcal{C}}((x, y), (\tilde{x}, \tilde{y})).$$

Note also that $\Psi(x, y) \sim_{\mathcal{C}} (x, y)$ for every $(x, y) \in \mathcal{C}$ and thus $\mathcal{F}(x, y) \sim_{\mathcal{C}} \mathcal{L}(x, y)$ for all $(x, y) \in \mathcal{C}$. Hence, by Lemmas 9.2.1 and 9.2.2, we have that $\mathcal{F}(\mathcal{C}_0) \subset \operatorname{int}(\mathcal{C})$, $\mathcal{F}$ is irreducible, $\mathcal{F}$ is weakly irreducible, respectively $\mathcal{F}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$ if and only if $\mathcal{L}$ has the corresponding property.

Finally, note that $\mathcal{F}$ is order-preserving as Ψ and $\mathcal{L}$ are order-preserving. Now, let $A, M \in \mathbb{R}^{d\times d}$ be respectively defined as

$$A = QP \qquad \text{and} \qquad M = QL, \tag{11.25}$$

then A is the homogeneity matrix of $\mathcal{F}$ and it holds

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}(x), \mathcal{F}(y)) \leq M\boldsymbol{\mu}_{\mathcal{C}}(x, y) \qquad \forall x, y \in \operatorname{int}(\mathcal{C}).$$

Note in particular that, as $L \preceq_{\mathbb{R}_+^{d\times d}} P$, we have $M \preceq_{\mathbb{R}_+^{d\times d}} A$ and therefore by monotonicity of the spectral radius and the linear Collatz-Wielandt formula (10.1), it holds

$$\rho(M) \leq \rho(A) \leq \max_{i=1,2}(A\mathbf{1})_i = \max\Big\{\frac{d-1}{p-1}, \frac{d-1}{q-1}\Big\}. \tag{11.26}$$

Furthermore, note that if $T \in \mathbb{R}_+^{n_1\times\ldots\times n_d} \setminus \mathbb{R}_{++}^{n_1\times\ldots\times n_d}$, i.e. T has at least one zero entry, then $L = P$ and thus $M = A$. We comment on the behavior of M and A with respect to p, q. To this end, consider the mappings $f_M \colon (1,\infty)^2 \to \mathbb{R}_+^{2\times 2}$ and $f_A \colon (1,\infty)^2 \to \mathbb{R}_+^{2\times 2}$ given by

$$f_M(r, s) = \begin{pmatrix} 1/(r-1) & 0 \\ 0 & 1/(s-1) \end{pmatrix} L \quad \text{and} \quad f_A(r, s) = \begin{pmatrix} 1/(r-1) & 0 \\ 0 & 1/(s-1) \end{pmatrix} P.$$

Then, $M = f_M(p,q)$ and $A = f_A(p,q)$. Moreover, if T does not have all entries equal, then L and P are irreducible matrices and by the discussion in Example 2.3.6, we have that for all $r, s, \tilde{r}, \tilde{s} \in (1,\infty)$ such that $(r,s) \lneq (\tilde{r},\tilde{s})$ it holds

$$\rho\big(f_M(r,s)\big) > \rho\big(f_M(\tilde{r},\tilde{s})\big) \quad \text{and} \quad \rho\big(f_A(r,s)\big) > \rho\big(f_A(\tilde{r},\tilde{s})\big).$$

Furthermore, it holds

$$\lim_{t\to\infty} \rho\big(f_M(t\,p,t\,q)\big) = 0 \quad \text{and} \quad \lim_{t\to\infty} \rho\big(f_A(t\,p,t\,q)\big) = 0.$$

It follows that by choosing p, q large enough, it is always possible to have $\rho(M) < 1$ or $\rho(A) < 1$.

Note that A is irreducible, so let $\mathbf{b} \in \mathbb{R}^2_{++}$ be the Perron vector of $A^\top$. The eigenvectors and eigenvalues of $\mathcal{F}$ are related to the solutions of (11.21) in Proposition 4.2.5 where we have shown that $x \in \mathcal{S}_{\boldsymbol{\nu}}$ is a rectangular $\ell^{p,q}$-singular vector of T if and only if it is an eigenvector of $\mathcal{F}$. Moreover, let λ be the $\ell^{p,q}$-singular value of T corresponding to x and θ the eigenvalue of $\mathcal{F}$ corresponding to x, i.e. $\mathcal{F}(x) = \boldsymbol{\vartheta} \otimes x$ with $\boldsymbol{\vartheta} \in \mathbb{R}^d_+$ and $\theta = \vartheta_1^{b_1}\vartheta_2^{b_2}$, then there exists $\gamma > 0$, independent of x, λ and $\boldsymbol{\vartheta}$, such that $\theta = \lambda^\gamma$ which implies that the eigenvalues of $\mathcal{F}$ on $\mathcal{S}_{\boldsymbol{\nu}}$ and the rectangular $\ell^{p,q}$-singular values of T have the same ordering.

We combine the above observations with Theorems 11.1.1 and 11.1.2 to obtain the following Perron-Frobenius theorem for rectangular $\ell^{p,q}$-singular vectors of nonnegative tensors.

Theorem 11.2.4. Let $d \geq 2$, $T \in \mathbb{R}_+^{n_1\times\ldots\times n_d}$ with $n_1,\ldots,n_d$ so that there are integers m, n such that $m = n_1 = \ldots = n_a$ and $n = n_{a+1} = \ldots = n_d$. For $p, q \in (1,\infty)$, let $\|\cdot\|_p$ be the ℓ^p-norm on $\mathbb{R}^m$ and $\|\cdot\|_q$ the ℓ^q-norm on $\mathbb{R}^n$. Define $\mathcal{C} = \mathbb{R}^m_+ \times \mathbb{R}^n_+$ and $\mathcal{S}_{\boldsymbol{\nu}} = \{(x,y) \in \mathcal{C} \mid \|x\|_p = \|y\|_q = 1\}$. Let $\mathcal{L}\colon \mathcal{C} \to \mathcal{C}$ be defined as in (11.23), let $A, M \in \mathbb{R}^{2\times 2}_+$ be defined as in (11.25) and let $\mathbf{b} \in \mathbb{R}^2_{++}$ be the Perron vector of $A^\top$. Then (11.21) has a solution and the following assertions hold:

i) If $\mathcal{L}(\mathrm{int}(\mathcal{C})) \subset \mathrm{int}(\mathcal{C})$ and $\rho(M) < 1$, then (11.21) has a unique solution $(\lambda,(u,v))$ such that $(u,v) \in \mathrm{int}(\mathcal{C})$. Furthermore, for every $(x_0,y_0) \in \mathrm{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$, the sequence $(x_k,y_k)_{k=1}^\infty \subset \mathrm{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ defined as

$$(x_{k+1},y_{k+1}) = \Big(\frac{\mathcal{F}(x_k,y_k)_1}{\|\mathcal{F}(x_k,y_k)_1\|_p}, \frac{\mathcal{F}(x_k,y_k)_2}{\|\mathcal{F}(x_k,y_k)_d\|_q}\Big) \qquad k = 0,1,\ldots, \quad (11.27)$$

where $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ is defined in (11.22), satisfies

$$\lim_{k\to\infty} (x_k,y_k) = (u,v) \qquad \text{and} \qquad \boldsymbol{\mu}_{\mathcal{C}}\big((x_k,y_k),(u,v)\big) \leq M^k\mathbf{c} \qquad \forall k \geq 1,$$

with $\mathbf{c} = (I - M)^{-1}\boldsymbol{\mu}_{\mathcal{C}}\big((x_1,y_1),(x_0,y_0)\big)$. If additionally, $\mathcal{L}$ is irreducible, then $(\lambda,(u,v))$ is the unique solution of (11.21).

ii) If $\rho(A) = 1$ and $\mathcal{L}$ is weakly irreducible, then (11.21) has a unique solution $(\lambda,(u,v))$ such that $(u,v) \in \mathrm{int}(\mathcal{C})$. Furthermore, for every $(w_0,z_0) \in \mathrm{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$, the sequence $(w_k,z_k)_{k=1}^\infty \subset \mathrm{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ defined as

$$(w_{k+1},z_{k+1}) = \Big(\frac{\mathcal{H}(w_k,z_k)_1}{\|\mathcal{H}(w_k,z_k)_1\|_p}, \frac{\mathcal{H}(w_k,z_k)_2}{\|\mathcal{H}(w_k,z_k)_d\|_q}\Big) \qquad k = 0,1,\ldots, \quad (11.28)$$

where $\mathcal{H}\colon \mathcal{C} \to \mathcal{C}$ is defined as $\mathcal{H}(x,y) = \big(x \circ \mathcal{F}(x,y)_1, y \circ \mathcal{F}(x,y)_2\big)^{1/2}$ and $\mathcal{F}$ is defined in (11.22), satisfies

$$\lim_{k\to\infty}(w_k, z_k) = (u, v).$$

If additionally, $\mathcal{L}$ is irreducible, then $(\lambda, (u, v))$ is the unique solution of (11.21).

iii) If either $\rho(A) < 1$ and $\mathcal{L}(\mathrm{int}(\mathcal{C})) \subset \mathrm{int}(\mathcal{C})$ or $\rho(A) = 1$ and $\mathcal{L}$ is weakly irreducible, then the unique solution $(\lambda, (u, v))$ of (11.21) such that $(u, v) \in \mathrm{int}(\mathcal{C})$, is maximal in the following sense:

$$\lambda = \sup\{\tilde{\lambda} \,|\, \tilde{\lambda} \text{ is a rectangular } \ell^{p,q}\text{-singular value of } T\}$$

Furthermore, with $\gamma = \frac{b_1}{p-1} + \frac{b_2}{q-1}$, the following Collatz-Wielandt formula holds,

$$\begin{aligned}
\lambda^\gamma &= \sup_{(x,y)\in\mathcal{S}_\nu} \max_{i\in[m],j\in[n]} \left(\frac{\mathcal{L}(x,y)_{1,i}}{x_i^{p-1}}\right)^{\frac{b_1}{p-1}} \left(\frac{\mathcal{L}(x,y)_{2,j}}{y_j^{q-1}}\right)^{\frac{b_2}{q-1}} \\
&= \inf_{(x,y)\in\mathrm{int}(\mathcal{C})\cap\mathcal{S}_\nu} \min_{i\in[m],j\in[n]} \left(\frac{\mathcal{L}(x,y)_{1,i}}{x_i^{p-1}}\right)^{\frac{b_1}{p-1}} \left(\frac{\mathcal{L}(x,y)_{2,j}}{y_j^{q-1}}\right)^{\frac{b_2}{q-1}}.
\end{aligned}$$

Moreover, the sequences $(\alpha_k)_{k=1}^\infty, (\beta_k)_{k=1}^\infty, (\zeta_k)_{k=1}^\infty, (\xi_k)_{k=1}^\infty \subset (0,\infty)$ defined in terms of the sequences $(x_k, y_k)_{k=1}^\infty, (w_k, z_k)_{k=1}^\infty \subset \mathrm{int}(\mathcal{C})$ of (11.27) and (11.28), by

$$\begin{aligned}
\alpha_k &= \min_{i\in[m],j\in[n]} \left(\frac{\mathcal{F}(x_k,y_k)_{1,i}}{(x_k)_i}\right)^{b_1} \left(\frac{\mathcal{F}(x_k,y_k)_{2,j}}{(y_k)_j}\right)^{b_2}, \\
\beta_k &= \max_{i\in[m],j\in[n]} \left(\frac{\mathcal{F}(x_k,y_k)_{1,i}}{(x_k)_i}\right)^{b_1} \left(\frac{\mathcal{F}(x_k,y_k)_{2,j}}{(y_k)_j}\right)^{b_2}, \\
\zeta_k &= \min_{i\in[m],j\in[n]} \left(\frac{\mathcal{H}(w_k,z_k)_{1,i}}{(w_k)_i}\right)^{2b_1} \left(\frac{\mathcal{H}(w_k,z_k)_{2,j}}{(z_k)_j}\right)^{2b_2}, \\
\xi_k &= \max_{i\in[m],j\in[n]} \left(\frac{\mathcal{H}(w_k,z_k)_{1,i}}{(w_k)_i}\right)^{2b_1} \left(\frac{\mathcal{H}(w_k,z_k)_{2,j}}{(z_k)_j}\right)^{2b_2},
\end{aligned}$$

satisfy

$$\alpha_{k+1} \le \alpha_k \le \lambda^\gamma \le \beta_{k+1} \le \beta_k \qquad \text{and} \qquad \zeta_{k+1} \le \zeta_k \le \lambda^\gamma \le \xi_{k+1} \le \xi_k,$$

for every $k \ge 1$, and if $\rho(A) < 1$, respectively $\rho(A) = 1$ and $\mathcal{L}$ is weakly irreducible, then

$$\lim_{k\to\infty}\alpha_k = \lim_{k\to\infty}\beta_k = \lambda^\gamma, \quad \text{respectively} \quad \lim_{k\to\infty}\zeta_k = \lim_{k\to\infty}\xi_k = \lambda^\gamma.$$

Proof. We recall from Proposition 4.2.5 that for every $(u,v) \in \mathcal{S}_{\boldsymbol{\nu}}$, $(\lambda,(u,v))$ is a solution of (11.21) if and only if $\mathcal{F}(u,v) = \boldsymbol{\vartheta} \otimes (u,v)$ with $\boldsymbol{\vartheta} = (\lambda^{1/(p-1)}, \lambda^{1/(q-1)})$ where $\mathcal{F}$ is defined in (11.11). As $\mathcal{F}$ is continuous, the existence of a solution to (11.21) follows form Theorem 4.1.5. Note that $\mathcal{H}\colon \mathcal{C} \to \mathcal{C}$ is order-preserving and multi-homogeneous of degree $B = (A+I)/2$. Note that $\rho(B) = \rho(A)$ and B is primitive since A is irreducible. $B^\top$ has the same Perron vector as $A^\top$, namely $\mathbf{b} \in \mathbb{R}^2_{++}$. Furthermore, note that if $(x,y) \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ satisfies $\mathcal{F}(x,y) = \boldsymbol{\vartheta} \otimes x$ with $\boldsymbol{\vartheta} \in \mathbb{R}^2_+$, then, it holds

$$\mathcal{H}(x,y) = ((x,y) \circ \mathcal{F}(x,y))^{1/2} = \big((x,y) \circ (\boldsymbol{\vartheta} \otimes (x,y))\big)^{1/2} = \boldsymbol{\vartheta}^{1/2} \otimes (x,y).$$

It follows that if θ is the eigenvalue of $\mathcal{F}$ corresponding to (x,y), then the eigenvalue of $\mathcal{H}$ corresponding to (x,y) equals $\sqrt{\theta}$. Conversely, if it holds $\mathcal{H}(x,y) = \boldsymbol{\beta} \otimes (x,y)$ with $\boldsymbol{\beta} \in \mathbb{R}^2_+$, then $\mathcal{F}(x) = (x,y)^{-1} \circ \mathcal{H}(x,y)^2 = \boldsymbol{\beta}^2 \otimes (x,y)$ and thus if α is the eigenvalue of $\mathcal{H}$ corresponding to (x,y), then the eigenvalue of $\mathcal{F}$ corresponding to (x,y) equals α^2.

i) Follows from Theorem 11.1.2.

ii) $\mathcal{F}$ is weakly irreducible since $\mathcal{L}$ is weakly irreducible. Thus, the existence of $(\lambda,(u,v))$ follows from Theorem 11.1.1 (Ic). As $\mathcal{L}$ is weakly irreducible, Lemma 9.4.6 implies that its Jacobian matrix $D\mathcal{L}(u,v) \in \mathbb{R}^{(m+n)\times(m+n)}_+$ is irreducible. The Jacobian matrix of $\mathcal{F}$ at u is given by

$$D\mathcal{F}(u,v) = \operatorname{diag}((p-1)\mathcal{L}(u,v)_1^{\frac{2-p}{p-1}}, (q-1)\mathcal{L}(u,v)_2^{\frac{2-q}{q-1}})D\mathcal{L}(u,v).$$

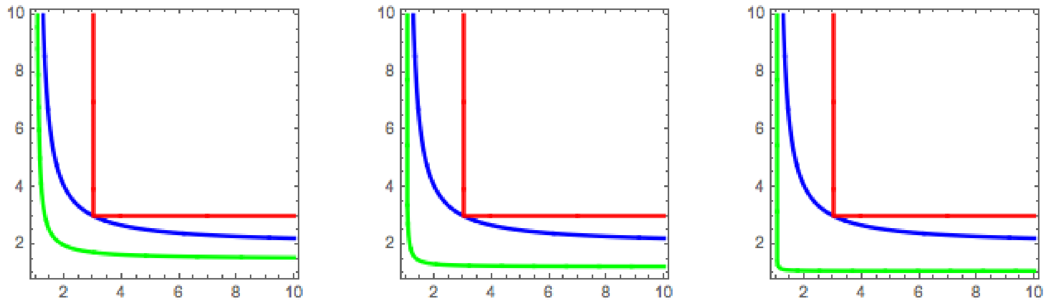
It follows that $D\mathcal{F}(u,v)$ is irreducible as well and thus the uniqueness of $(\lambda,(u,v))$ follows from Theorem 11.1.1 (IIIc). The Jacobian matrix of $\mathcal{H}$ at (u,v) is given by

$$D\mathcal{H}(u,v) = \frac{1}{2}\operatorname{diag}(\mathcal{H}(u,v))^{-1/2}\big(\operatorname{diag}(\mathcal{F}(u,v)) + \operatorname{diag}(u,v)DF(u,v)\big).$$

Hence, $D\mathcal{H}(u,v)$ is a primitive matrix since it is the sum of a positive diagonal matrix and an irreducible matrix. The convergence of $(w_k,z_k)_{k=1}^\infty$ now follows from Theorem 11.1.1 (IVc).

iii) The maximality of λ as well as the Collatz-Wielandt formula follow from Theorem 11.1.1 (II). Finally, the monotonicity and the convergence of the sequences $(\alpha_k)_{k=1}^\infty$,$(\beta_k)_{k=1}^\infty$,$(\zeta_k)_{k=1}^\infty$,$(\xi_k)_{k=1}^\infty$ follow from Theorem 11.1.1, (IVa) and (IVc). □

The results of Theorem 11.2.4 assuming $\rho(A) \leq 1$ are implied by Theorems 3.2 and 3.3 in [40]$^\diamond$ and improve previous results of the literature: The existence and uniqueness of solutions to (11.10) were proved in Theorem 4.3 of [65] under the assumption that $\min\{p,q\} \geq m$ and $\mathcal{L}$ is weakly irreducible. Furthermore, the convergence of the sequence $(x_k,y_k)_{k=1}^\infty$ towards a unique rectangular $\ell^{p,q}$-singular vector $(u,v) \in \operatorname{int}(\mathcal{C})$ is proved in Theorem 7 of [26] under the assumption that $p =$



$\epsilon = \frac{1}{4}$ $\epsilon = \frac{1}{2}$ $\epsilon = \frac{3}{4}$

Figure 3: Plot of the lines $\rho(A(p,q)) = 1$ (in blue), $\rho(M(p,q,\epsilon)) = 1$ (in green) and $\min\{p,q\} = 3$ (in red) with p on the x-axis and q on the y-axis.

$q = m$ and T is irreducible in the sense of [26]. The latter definition of irreducibility is a more restrictive condition than requiring $\mathcal{L}$ to be irreducible in the sense of Definition 9.1.1 (see Lemma 9.2.5 and Example 9.2.6). A Collatz-Wielandt formula is proved in Theorem 4.6 of [97] under the assumption that $\mathcal{L}$ is weakly irreducible and $p = q = m$. These assumptions are more restrictive than those of the corresponding result in the above theorem. Indeed, we see from (11.26) that $p, q \geq m$ implies $\rho(A) \leq 1$ but the converse is not true in general. Furthermore, note that if $\mathcal{L}$ is irreducible then $\mathcal{L}$ is weakly irreducible by Lemma 9.4.5. Furthermore, we note that the convergence of $(x_k, y_k)_{k=1}^{\infty}$ under the assumption that $\mathcal{L}(\text{int}(\mathcal{C})) \subset \text{int}(\mathcal{C})$ and $\rho(A) < 1$ implied by Theorem 11.2.3, (i) is less restrictive than asking for $\mathcal{L}$ to be weakly irreducible. We stress that, when $T \in \mathbb{R}_{++}^{n_1 \times \ldots \times n_d}$, the condition $\rho(M) < 1$ of Theorem 11.2.3, (i) is the first known condition which implies existence and uniqueness of a positive solution to (11.21) as well as the convergence of $(x_k, y_k)_{k=1}^{\infty}$ for choices of p, q satisfying $p, q < m$. We illustrate the improvement offered by Theorem 11.2.4 in Figure 3 where the conditions discussed above on p, q are plotted for the tensor $T_\epsilon \in \mathbb{R}_{++}^{2\times 2\times 2}$ defined in (11.18). For all $p, q \in (1, \infty)$, let

$$A(p,q) = \begin{pmatrix} 0 & \frac{2}{p-1} \\ \frac{1}{q-1} & \frac{1}{q-1} \end{pmatrix}, \quad M(p,q,\epsilon) = f(\epsilon) A(p,q),$$

where $f(\epsilon)$ is defined in (11.19). Figure 3 shows that the condition $\rho(M(p,q,\epsilon)) < 1$, which holds for all pairs (p,q) above the green line, improve the condition $\min\{p,q\} \geq 3$ which is satisfied for all pair (p,q) above the red line, especially when ϵ is closed to 1. Finally, we note that the red and the blue lines meet at $(p,q) = (3,3)$, i.e. $p = q = m$.

11.2.4 ℓ^p-eigenvectors of a nonnegative tensor

Let $T \in \mathbb{R}_{+}^{n\times \ldots \times n}$ be an m-th order tensor, $p \in (1, \infty)$ and let $\|\cdot\|_p$ be the ℓ^p-norm on $\mathbb{R}^n$. Define $\mathcal{C} = \mathbb{R}^n_+$, and consider the monotonic multi-normalization $\boldsymbol{\nu} \colon \mathcal{C} \to \mathbb{R}_+$ of $\mathcal{C}$ defined as $\boldsymbol{\nu}(x) = \|x\|_p$ and set $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \boldsymbol{\nu}(x) = 1\}$. We have seen in Section

4.2.4 that the rectangular ℓ^p-eigenvalues/eigenvectors, of T are the solutions to the following system of equations:

$$T(\cdot, x, \ldots, x) = \lambda\, x^p \quad \text{and} \quad (\lambda, x) \in \mathbb{R}_+ \times \mathcal{S}_{\boldsymbol{\nu}}. \tag{11.29}$$

Moreover, we have discussed that the ℓ^p-eigenvectors of T are characterized by the eigenvectors of the order-preserving homogeneous mapping $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ given by

$$\mathcal{F}(x) = \Psi(\mathcal{L}(x)) \qquad \forall x \in \mathcal{C}, \tag{11.30}$$

where $\mathcal{L}\colon \mathcal{C} \to \mathcal{C}$ is the mapping defined as

$$\mathcal{L}(x) = T(\cdot, x, \ldots, x) \qquad \forall x \in \mathcal{C}, \tag{11.31}$$

and $\Psi\colon\ \mathcal{C} \to \mathcal{C}$ is the nonlinear homogeneous mapping defined as

$$\Psi(x) = x^{1/(p-1)} \qquad \forall (x, y) \in \mathcal{C}.$$

We discuss properties of $\mathcal{L}$. Note that if $T \in \mathbb{R}^{n\times\ldots\times n}_{++}$, then $\mathcal{L}(\mathcal{C}_0) \in \operatorname{int}(\mathcal{C})$, where $\mathcal{C}_0 = \{x \in \mathcal{C} \,|\, x \neq 0\}$ and in this case $\mathcal{L}$ is irreducible. Furthermore, it holds $\mathcal{L}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$ if and only if T is such that for all $j_1 \in [n]$, there exists $j_2, \ldots, j_d \in [n_d]$ such that $T_{j_1,\ldots,j_d} > 0$. Moreover, if $\mathcal{L}$ is weakly irreducible then $\mathcal{L}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$. Finally, note that, by Lemma 9.4.4, we know that if $\mathcal{L}$ is irreducible, then $\mathcal{L}$ is weakly irreducible. To summarize, we have the following chain of implications:

$$\mathcal{L}(\mathcal{C}_0) \subset \operatorname{int}(\mathcal{C}) \Rightarrow \mathcal{L} \text{ irreducible } \Rightarrow \mathcal{L} \text{ weakly irreducible } \Rightarrow \mathcal{L}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C}).$$

Now, let $L \in \mathbb{R}^{1\times 1}_+$ be defined as $L = m-1$ if $T \in \mathbb{R}^{n\times\ldots\times n}_+ \setminus \mathbb{R}^{n\times\ldots\times n}_{++}$ and

$$L = \sum_{l=2}^{m} \tanh[\tfrac{1}{4}\ln(\Delta_{1,l}(T))] \qquad \text{if } T \in \mathbb{R}^{n\times\ldots\times n}_{++},$$

where $\Delta_{k,l}(T)$ is defined in (8.10). Then, by Theorem 8.1.5, for every $x, y \in \mathcal{C}$ with $x \sim_{\mathcal{C}} y$, we have

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{L}(x), \mathcal{L}(y)) \leq L \boldsymbol{\mu}_{\mathcal{C}}(x, y).$$

We discuss properties of Ψ. Note that Ψ is homogeneous of degree $1/(p-1)$. Moreover, as $p-1 > 0$, Ψ is order-preserving and Lemma 5.2.3 implies that for every $x, y \in \mathcal{C}$ with $x \sim_{\mathcal{C}} y$, we have

$$\boldsymbol{\mu}_{\mathcal{C}}(\Psi(x), \Psi(y)) = \frac{1}{p-1} \boldsymbol{\mu}_{\mathcal{C}}(x, y).$$

Note also that $\Psi(x) \sim_{\mathcal{C}} x$ for every $x \in \mathcal{C}$ and thus $\mathcal{F}(x) \sim_{\mathcal{C}} \mathcal{L}(x)$ for all $x \in \mathcal{C}$. Hence, by Lemmas 9.2.1 and 9.2.2, we have that $\mathcal{F}(\mathcal{C}_0) \subset \operatorname{int}(\mathcal{C})$, $\mathcal{F}$ is irreducible, $\mathcal{F}$ is weakly irreducible, respectively $\mathcal{F}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$ if and only if $\mathcal{L}$ has the corresponding property.

Finally, note that $\mathcal{F}$ is order-preserving as Ψ and $\mathcal{L}$ are order-preserving. Now, let $A, M \in \mathbb{R}^{d\times d}$ be respectively defined as

$$A = \frac{m-1}{p-1} \qquad \text{and} \qquad M = \frac{L}{p-1}, \tag{11.32}$$

then A is the homogeneity matrix of $\mathcal{F}$ and it holds

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}(x),\mathcal{F}(y)) \leq M\boldsymbol{\mu}_{\mathcal{C}}(x,y) \qquad \forall x,y\in\operatorname{int}(\mathcal{C}).$$

Note in particular that, as $L \leq m-1$, we have $M \leq A$. Furthermore, note that if T has a zero entry, then $L = m-1$ and thus $M = A$. Furthermore, it holds

$$\lim_{p\to\infty}\frac{m-1}{p-1} = 0 \quad \text{and} \quad \lim_{p\to\infty}\frac{L}{p-1} = 0.$$

It follows that by choosing p large enough, it is always possible to have $M < 1$ or $A < 1$.

The eigenvectors and eigenvalues of $\mathcal{F}$ are clearly related to the solutions of (11.21). Indeed, $x \in \mathcal{S}_{\boldsymbol{\nu}}$ is an ℓ^p-eigenvector of T if and only if it is an eigenvector of $\mathcal{F}$. Moreover, let λ be the ℓ^p-eigenvalue of T corresponding to x and θ the eigenvalue of $\mathcal{F}$ corresponding to x, then $\lambda = \theta^{p-1}$ which implies that the eigenvalues of $\mathcal{F}$ on $\mathcal{S}_{\boldsymbol{\nu}}$ and the ℓ^p-eigenvalues of T have the same ordering.

We combine the above observations with Theorems 11.1.1 and 11.1.2 to obtain the following Perron-Frobenius theorem for ℓ^p-eigenvectors of nonnegative tensors.

Theorem 11.2.5. Let $T \in \mathbb{R}_+^{n\times\ldots\times n}$ be a tensor of order m, $p \in (1,\infty)$ and let $\|\cdot\|_p$ be the ℓ^p-norm on $\mathbb{R}^n$. Define $\mathcal{C} = \mathbb{R}^n_+$ and $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \|x\|_p = 1\}$. Let $\mathcal{L}\colon \mathcal{C}\to\mathcal{C}$ be defined as in (11.31) and let $A, M \in \mathbb{R}_+$ be defined as in (11.32). Then (11.29) has a solution and the following assertions hold:

i) If $\mathcal{L}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$ and $M < 1$, then (11.29) has a unique solution (λ, u) such that $u \in \operatorname{int}(\mathcal{C})$. Furthermore, for every $x_0 \in \operatorname{int}(\mathcal{C})\cap\mathcal{S}_{\boldsymbol{\nu}}$, the sequence $(x_k)_{k=1}^{\infty} \subset \operatorname{int}(\mathcal{C})\cap\mathcal{S}_{\boldsymbol{\nu}}$ defined as

$$x_{k+1} = \frac{\mathcal{F}(x_k)}{\|\mathcal{F}(x_k)\|_p} \qquad k = 0,1,\ldots, \tag{11.33}$$

where $\mathcal{F}\colon\mathcal{C}\to\mathcal{C}$ is defined in (11.30), satisfies

$$\lim_{k\to\infty} x_k = u \qquad \text{and} \qquad \boldsymbol{\mu}_{\mathcal{C}}(x_k, u) \leq M^k c \qquad \forall k \geq 1,$$

with $c = (1-M)^{-1}\boldsymbol{\mu}_{\mathcal{C}}(x_1, x_0)$. If additionally, $\mathcal{L}$ is irreducible, then (λ, u) is the unique solution of (11.21).

ii) If $\rho(A) = 1$ and $\mathcal{L}$ is weakly irreducible, then (11.21) has a unique solution (λ, u) such that $u \in \operatorname{int}(\mathcal{C})$. Furthermore, for every $y_0 \in \operatorname{int}(\mathcal{C})\cap\mathcal{S}_{\boldsymbol{\nu}}$, the sequence $(y_k)_{k=1}^{\infty} \subset \operatorname{int}(\mathcal{C})\cap\mathcal{S}_{\boldsymbol{\nu}}$ defined as

$$y_{k+1} = \frac{\mathcal{H}(y_k)}{\|\mathcal{H}(y_k)\|_p} \qquad k = 0,1,\ldots, \tag{11.34}$$

where $\mathcal{H}\colon\mathcal{C}\to\mathcal{C}$ is defined as $\mathcal{H}(x) = \big(x\circ\mathcal{F}(x)\big)^{1/2}$ and $\mathcal{F}$ is defined in (11.30), satisfies

$$\lim_{k\to\infty} y_k = u.$$

If additionally, $\mathcal{L}$ is irreducible, then (λ, u) is the unique solution of (11.21).

iii) If either $\rho(A) < 1$ and $\mathcal{L}(\mathrm{int}(\mathcal{C})) \subset \mathrm{int}(\mathcal{C})$ or $\rho(A) = 1$ and $\mathcal{L}$ is weakly irreducible, then the unique solution (λ, u) of (11.21) such that $u \in \mathrm{int}(\mathcal{C})$, is maximal in the following sense:

$$\lambda = \sup\{\tilde{\lambda} \mid \tilde{\lambda} \text{ is an } \ell^p\text{-eigenvalue of } T\}.$$

Furthermore, the following Collatz-Wielandt formula holds,

$$\sup_{x \in \mathcal{S}_\nu} \max_{j \in [n]} \frac{\mathcal{L}(x)_j}{x_j^{p-1}} = \lambda = \inf_{x \in \mathrm{int}(\mathcal{C}) \cap \mathcal{S}_\nu} \min_{j \in [n]} \frac{\mathcal{L}(x)_j}{x_j^{p-1}}.$$

Moreover, the sequences $(\alpha_k)_{k=1}^\infty, (\beta_k)_{k=1}^\infty, (\zeta_k)_{k=1}^\infty, (\xi_k)_{k=1}^\infty \subset (0, \infty)$ defined in terms of the sequences $(x_k)_{k=1}^\infty, (y_k)_{k=1}^\infty \subset \mathrm{int}(\mathcal{C})$ of (11.33) and (11.34), by

$$\alpha_k = \min_{j \in [n]} \frac{\mathcal{L}(x_k)_j}{(x_k)_j^{p-1}}, \quad \zeta_k = \min_{j \in [n]} \Big(\frac{\mathcal{L}(y_k)_j}{(y_k)_j^{p-1}}\Big)^2,$$
$$\beta_k = \max_{j \in [n]} \frac{\mathcal{L}(x_k)_j}{(x_k)_j^{p-1}}, \quad \xi_k = \max_{j \in [n]} \Big(\frac{\mathcal{L}(y_k)_j}{(y_k)_j^{p-1}}\Big)^2,$$

satisfy

$$\alpha_{k+1} \le \alpha_k \le \lambda \le \beta_{k+1} \le \beta_k \qquad \text{and} \qquad \zeta_{k+1} \le \zeta_k \le \lambda \le \xi_{k+1} \le \xi_k,$$

for every $k \ge 1$, and if $\rho(A) < 1$, respectively $\rho(A) = 1$ and $\mathcal{L}$ is weakly irreducible, then

$$\lim_{k \to \infty} \alpha_k = \lim_{k \to \infty} \beta_k = \lambda, \qquad \text{respectively} \qquad \lim_{k \to \infty} \zeta_k = \lim_{k \to \infty} \xi_k = \lambda.$$

Proof. As $\mathcal{F}$ is continuous, the existence of a solution to (11.29) follows form Theorem 4.1.5. Note that $\mathcal{H}\colon \mathcal{C} \to \mathcal{C}$ is order-preserving and homogeneous of degree $B = (A+1)/2$. Furthermore, note that if $x \in \mathrm{int}(\mathcal{C}) \cap \mathcal{S}_\nu$ satisfies $\mathcal{F}(x) = \theta x$ with $\theta \ge 0$, then, it holds $\mathcal{H}(x) = (x \circ \mathcal{F}(x))^{1/2} = \big(x \circ (\theta x)\big)^{1/2} = \theta^{1/2} x$. It follows that if θ is the eigenvalue of $\mathcal{F}$ corresponding to x, then the eigenvalue of $\mathcal{H}$ corresponding to x equals $\sqrt{\theta}$. Conversely, if it holds $\mathcal{H}(x) = \beta x$ with $\beta \ge 0$, then $\mathcal{F}(x) = x^{-1} \circ \mathcal{H}(x)^2 = \beta^2 x$ and thus if β is the eigenvalue of $\mathcal{H}$ corresponding to x, then the eigenvalue of $\mathcal{F}$ corresponding to x equals β^2.

1. Follows from Theorem 11.1.2.

2. $\mathcal{F}$ is weakly irreducible since $\mathcal{L}$ is weakly irreducible. Thus, the existence of (λ, u) follows from Theorem 11.1.1 (Ic). As $\mathcal{L}$ is weakly irreducible, Lemma 9.4.6 implies that its Jacobian matrix $D\mathcal{L}(u) \in \mathbb{R}_+^{n \times n}$ is irreducible. The Jacobian matrix of $\mathcal{F}$ at u is given by

$$D\mathcal{F}(u) = Q\, \mathrm{diag}(\mathcal{L}(u)^{1/(p-1)-1}) D\mathcal{L}(u).$$

It follows that $D\mathcal{F}(u)$ is irreducible as well and thus the uniqueness of (λ, u) follows from Theorem 11.1.1 (IIIc). The Jacobian matrix of $\mathcal{H}$ at u is given by

$$D\mathcal{H}(u) = \frac{1}{2}\operatorname{diag}(\mathcal{H}(u))^{-1}\big(\operatorname{diag}(\mathcal{F}(u)) + \operatorname{diag}(u)DF(u)\big).$$

Hence, $D\mathcal{H}(u)$ is a primitive matrix since it is the sum of a positive diagonal matrix and an irreducible matrix. The convergence of $(y_k)_{k=1}^{\infty}$ now follows from Theorem 11.1.1 (IVc).

3. The maximality of λ as well as the Collatz-Wielandt formula follow from Theorem 11.1.1 (II). Finally, the monotonicity and the convergence of the sequences $(\alpha_k)_{k=1}^{\infty}$,$(\beta_k)_{k=1}^{\infty}$,$(\zeta_k)_{k=1}^{\infty}$,$(\xi_k)_{k=1}^{\infty}$ follow from Theorem 11.1.1, (IVa) and (IVc). □

The results of Theorem 11.2.3 assuming $\rho(A) \leq 1$ are implied by Theorems 3.2 and 3.3 in [40]$^\diamond$ and improve previous results of the literature: The existence and uniqueness of solutions to (11.29) and a Collatz-Wielandt formula were proved in Theorem 4.1 and Corollary 4.3 of [31] under the assumption that $p \geq m$ and $\mathcal{L}$ is weakly irreducible. Furthermore, the convergence of the sequence $(x_k)_{k=1}^{\infty}$ towards a unique ℓ^p-eigenvector $u \in \operatorname{int}(\mathcal{C})$ is proved in Corollary 5.2 of [31] under the assumption that $p \geq m$ and $D\mathcal{L}(u)$ is primitive. These assumptions are more restrictive than those of the corresponding result in the above theorem. Indeed, as $\mathcal{L}$ is weakly irreducible if and only if $D\mathcal{L}(u)$ is an irreducible matrix (by Lemmas 9.4.4 and 9.4.6), the assumption that $D\mathcal{L}(u)$ needs to be primitive is more restrictive than requiring $\mathcal{L}$ to be weakly irreducible. Furthermore, the convergence of $(x_k)_{k=1}^{\infty}$ for $p \geq m$ and $D\mathcal{L}(u)$ primitive is implied by Theorem 11.1.1, (IVc). Nevertheless, for $p = m$, we note that the convergence result of Theorem 11.2.5, (ii) is equivalent to Theorem 5.4 in [49] in terms of assumptions. However, note that the converging sequence in [49] uses an additive shift while we have a multiplicative shift. The convergence of $(x_k)_{k=1}^{\infty}$ under the assumption that $\mathcal{L}(\operatorname{int}(\mathcal{C})) \subset \operatorname{int}(\mathcal{C})$ and $p > m$ implied by Theorem 11.2.5, (i) is a strong improvement on previous results in terms of irreducibility. In Theorems 7 and 8 of [20], it is shown that the conclusions of Theorem 11.2.5, (i), holds for the case $p = 2$ and T is an irreducible stochastic tensor. Nevertheless for general positive tensors (not necessarily stochastic), the condition $M < 1$ of Theorem 11.2.5, (i) is the first known condition which implies existence and uniqueness of a positive solution to (11.10) as well as the convergence of $(x_k)_{k=1}^{\infty}$ for choices of p satisfying $p < m$. To appreciate the improvement offered by Theorem 11.2.3 we note that for the tensor $T_\epsilon \in \mathbb{R}_{++}^{2\times 2\times 2}$ defined in (11.18), with $M(p,\epsilon) = \frac{2f(\epsilon)}{p-1}$ it holds $M(p, 1/4) < 1$ for all $p > 95/48 \approx 1.9792$, $M(p, 1/2) < 1$ for all $p > 35/24 \approx 1.4583$ and $M(p, 3/4) < 1$ for all $p > 71/61 \approx 1.1639$, where we have used the bounds on $f(\epsilon)$ discussed in (11.20).

11.2.5 Discrete generalized Schrödinger equation

Let $T \in \mathbb{R}_{++}^{n\times\ldots\times n}$ be a tensor of order $d \geq 3$. Let $\mathcal{C} = \mathbb{R}_+^n \times \ldots \times \mathbb{R}_+^n$ and $\mathcal{C}_0 = \{x \in \mathcal{C} \,|\, x_1, \ldots, x_d \neq 0\}$. We recall from Section 4.2.2 that the following system of equations is a particular case of the generalized discrete Shrödinger equation:

$$\begin{cases} T(\cdot, x_2, \ldots, x_d) & = & x_1^{-1} \\ T(x_1, \cdot, x_3, \ldots, x_d) & = & x_2^{-1} \\ \vdots & \vdots & \vdots \\ T(x_1, \ldots, x_{d-1}, \cdot) & = & x_d^{-1} \end{cases} \quad \text{and} \quad x \in \mathcal{C}_0. \tag{11.35}$$

Now, let $\mathcal{F}\colon \mathcal{C}_0 \to \mathcal{C}_0$ be defined as

$$\mathcal{F}(x) = \Psi(\mathcal{L}(x)) \qquad \forall x \in \mathcal{C}_0, \tag{11.36}$$

where $\mathcal{L}\colon \mathcal{C} \to \mathcal{C}$ is defined as

$$\mathcal{L}(x) = \big(T(\cdot, x_2, \ldots, x_d)^{-1}, \ldots, T(x_1, \ldots, x_{d-1}, \cdot)^{-1}\big) \qquad \forall x \in \mathcal{C} \tag{11.37}$$

and $\Psi\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ is defined as

$$\Psi(x) = (x_1^{-1}, \ldots, x_d^{-1}) \qquad \forall x \in \operatorname{int}(\mathcal{C}).$$

We discuss properties of $\mathcal{L}$. Note that $\mathcal{L}(\mathcal{C}_0) \subset \operatorname{int}(\mathcal{C})$ as $T \in \mathbb{R}_{++}^{n\times \ldots \times n}$. Let $L \in \mathbb{R}_+^{d\times d}$ be defined as

$$L_{k,l} = \begin{cases} \tanh\big[\frac{1}{4}\ln(\Delta_{k,l}(T))\big] & \text{if } k \neq l, \\ 0 & \text{if } k = l \end{cases} \quad \forall k, l \in \{1, \ldots, d\}, \tag{11.38}$$

where $\Delta_{k,l}(T)$ is defined in (8.10). Then, by Corollary 8.2.5, for all $i, j \in \{1, \ldots, d\}$, we have

$$L_{j,i} \geq \sup_{x\in\mathcal{C}, y_i \in C_i} \tanh\big[\tfrac{1}{4}\mu_{C_j}\big(\mathcal{L}(x)_j, \mathcal{L}(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_m)_j\big)\big].$$

Moreover, by Theorem 8.1.2 we have

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{L}(x), \mathcal{L}(y)) \leq L\boldsymbol{\mu}_{\mathcal{C}}(x, y) \qquad \forall x, y \in \operatorname{int}(\mathcal{C}).$$

We discuss properties of Ψ. Note that Ψ is multi-homogeneous of degree $-I$, Ψ is order-reversing and Lemma 5.2.3 implies that

$$\boldsymbol{\mu}_{\mathcal{C}}(\Psi(x), \Psi(y)) = \boldsymbol{\mu}_{\mathcal{C}}(x, y) \qquad \forall x, y \in \operatorname{int}(\mathcal{C}).$$

Finally, note that $\mathcal{F}$ is order-reversing as Ψ is order-reversing and $\mathcal{L}$ is order-preserving. Now, let $A, M \in \mathbb{R}^{d\times d}$ be respectively defined as

$$A = (I - \mathbf{1}\mathbf{1}^\top) \qquad \text{and} \qquad M = L, \tag{11.39}$$

then A is the homogeneity matrix of $\mathcal{F}$ and it holds

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}(x), \mathcal{F}(y)) \leq L\boldsymbol{\mu}_{\mathcal{C}}(x, y) \qquad \forall x, y \in \operatorname{int}(\mathcal{C}).$$

Furthermore, we have proved in Proposition 4.2.4 that if $x \in \mathcal{C}_0$ is an eigenvector of $\mathcal{F}$, then $x \in \operatorname{int}(\mathcal{C})$ and there exists $t > 0$ such that tx is a solution of (11.35).

With the above observations we obtain the following:

Theorem 11.2.6. Let $T \in \mathbb{R}^{n\times\ldots\times n}_{++}$ be a tensor of order $d \geq 2$ and let $\mathcal{C} = \mathbb{R}^n_+ \times \ldots \times \mathbb{R}^n_+$. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ which is monotonic with respect to $\mathbb{R}^n_+$. Furthermore, let $L \in \mathbb{R}^{d\times d}_+$ and $\mathcal{F}\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ be defined as in (11.36) and (11.38) respectively. If $\rho(L) < 1$, then the following hold:

i) Equation (11.35) has a solution $u \in \operatorname{int}(\mathcal{C})$ and $\bar{u} \in \operatorname{int}(\mathcal{C})$ is a solution of (11.35) if and only if there exists $\alpha_1, \ldots, \alpha_d > 0$ such that $\prod_{i=1}^d \alpha_i = 1$ and $\bar{u} = (\alpha_1 u_1, \ldots, \alpha_d u_d)$.

ii) For every $x_0 \in \operatorname{int}(\mathcal{C})$, the sequence $(x_k)_{k=1}^\infty \subset \operatorname{int}(\mathcal{C})$ defined as

$$x_{k+1} = \Big(\frac{\mathcal{F}(x_k)_1}{\|\mathcal{F}(x_k)_1\|}, \ldots, \frac{\mathcal{F}(x_k)_d}{\|\mathcal{F}(x_k)_d\|}\Big) \qquad \forall k \geq 1,$$

satisfies

$$\lim_{k\to\infty} x_k = \hat{u} = \Big(\frac{u_1}{\|u_1\|}, \ldots, \frac{u_d}{\|u_d\|}\Big).$$

Furthermore, $\big(\frac{n}{f_T(\hat{u})}\big)^{1/d} \hat{u}$ is a solution of (11.35) and

$$\boldsymbol{\mu}_{\mathcal{C}}(x_k, u) \leq L^k \mathbf{v} \quad \forall k \geq 1,$$

with $\mathbf{v} = (I - L)^{-1} \boldsymbol{\mu}_{\mathcal{C}}(x_1, x_0)$.

Proof. Let $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d_+$ be defined as $\boldsymbol{\nu}(x) = (\|x_1\|, \ldots, \|x_d\|)$ and let $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$. Then, as $\rho(L) < 1$, Theorem 7.3.1 implies that $\mathcal{F}$ has a unique eigenvector $\hat{u} \in \mathcal{S}_{\boldsymbol{\nu}}$. Furthermore, Theorem 7.3.1 implies the convergence of $(x_k)_{k=1}^\infty \subset \operatorname{int}(\mathcal{C})$ towards $\hat{u}$ and the convergence rate in (ii). By Proposition 4.2.4, as $\hat{u}$ is an eigenvector of $\mathcal{F}$, we have that $u = \big(\frac{n}{f_T(\hat{u})}\big)^{1/d}\hat{u}$ is a solution of (11.35) and thus $\mathcal{F}(u) = u$. Moreover, by the scaling invariance of $\boldsymbol{\mu}_{\mathcal{C}}$ (see Equation (6.7)), we have $\boldsymbol{\mu}_{\mathcal{C}}(x_k, u) = \boldsymbol{\mu}_{\mathcal{C}}(x_k, \hat{u})$ for all $k \geq 0$ which proves the convergence rate. We show that the set of solutions to (11.35) is given by $\{\boldsymbol{\alpha} \otimes u \,|\, \boldsymbol{\alpha} \in \mathbb{R}^d_{++} \text{ and } \prod_{i=1}^d \alpha_i = 1\}$. Let $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$ be such that $\prod_{i=1}^d \alpha_i = 1$. As $\mathcal{F}$ is multi-homogeneous of degree $A = (I - \mathbf{1}\mathbf{1}^\top)$, we have

$$\mathcal{F}(\boldsymbol{\alpha} \otimes u) = \boldsymbol{\alpha}^A \otimes \mathcal{F}(u) = \boldsymbol{\alpha}^A \otimes u = (\boldsymbol{\alpha} \circ \boldsymbol{\alpha}^{-\mathbf{1}\mathbf{1}^\top}) \otimes u = \boldsymbol{\alpha} \otimes u.$$

It follows that $\boldsymbol{\alpha} \otimes u$ is a solution of (11.35). Now, let $\bar{u}$ be a solution of (11.35). Then, $\bar{u}$ is an eigenvector of $\mathcal{F}$ and, as $\mathcal{F}$ has a unique eigenvector in $\mathcal{S}_{\boldsymbol{\nu}}$, we have

$$\Big(\frac{\bar{u}_1}{\|\bar{u}_1\|}, \ldots, \frac{\bar{u}_d}{\|\bar{u}_d\|}\Big) = \hat{u} = \Big(\frac{u_1}{\|u_1\|}, \ldots, \frac{u_d}{\|u_d\|}\Big).$$

It follows that $\bar{u} = \boldsymbol{\alpha} \otimes u$ for some $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$. We prove that $\prod_{i=1}^d \alpha_i = 1$. As $\bar{u}$ and u are solutions of (11.35), it follows that $\boldsymbol{\alpha} \otimes u$ and u are solutions of (11.35). Hence, it holds $\mathcal{F}(u) = u$ and $\mathcal{F}(\boldsymbol{\alpha} \otimes u) = \boldsymbol{\alpha} \otimes u$. Now, as $\mathcal{F}$ is multi-homogeneous of degree $A = (I - \mathbf{1}\mathbf{1}^\top)$, we have

$$\boldsymbol{\alpha}^A \otimes u = \boldsymbol{\alpha}^A \otimes \mathcal{F}(u) = \mathcal{F}(\boldsymbol{\alpha} \otimes u) = \boldsymbol{\alpha} \otimes u.$$

It follows that $\boldsymbol{\alpha}^{A-I} = \mathbf{1}$ and thus $\prod_{j=1}^d \alpha_j = 1$ which concludes the proof. □

We point out that the convergence of the sequence $(x_k)_{k=1}^\infty$ in Theorem 11.2.6 is new.

11.2.6 Quantum copulas

Let V_1 be the space of symmetric matrices in $\mathbb{R}^{m\times m}$ and V_2 the space of symmetric matrices in $\mathbb{R}^{n\times n}$. Let $C_1 \subset V_1$ and $C_2 \subset V_2$ be the cones of positive semi-definite matrices, $\mathcal{V} = V_1 \times V_2$ and $\mathcal{C} = C_1 \times C_2$. Let $\Phi\colon V_1 \to V_2$ be a linear mapping and denote by Φ^* its adjoint with respect to $\langle \cdot\,,\,\cdot\rangle$, the Hilbert-Schmidt inner products on V_1 and V_2. Suppose that $\Phi(C_1\setminus\{0\}) \subset \operatorname{int}(C_2)$, then by Lemma 2 of [67], it holds $\Phi^*(C_2\setminus\{0\}) \subset \operatorname{int}(C_1)$. Let $\mathcal{C} = C_1 \times C_2$ and $\mathcal{S} = \{(X,Y) \in \mathcal{C} \mid \operatorname{Tr}(X) + \operatorname{Tr}(Y) = 1\}$ where $\operatorname{Tr}(\cdot)$ denotes the trace operator. The following equation is discussed in the context of quantum copulas in [67]:

$$\begin{cases}\Phi(Y^{-1}) = \frac{1}{m}X^{-1}\\ \Phi^*(X) = \frac{1}{n}Y\end{cases} \qquad \text{with} \qquad (X,Y) \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}. \tag{11.40}$$

In particular, it is shown in Theorem 5 of [67] that Equation 11.40 has a unique solution. The main idea of the proof in [67] is to use the substitution technique of Section 4.3 to characterize the solutions of (11.40) as the eigenvectors of an order-preserving homogeneous mapping on $\operatorname{int}(C_1)$. We explain how to derive the same result by keeping the problem in its multi-homogeneous form. Define $\mathcal{F}\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ as

$$\mathcal{F}(X,Y) = \big(\Phi(Y^{-1})^{-1},\, \Phi^*(X)\big) \qquad \forall (X,Y) \in \operatorname{int}(\mathcal{C}). \tag{11.41}$$

Then, $\mathcal{F}$ is order-preserving and multi-homogeneous of degree $A \in \mathbb{R}^{2\times 2}$ with

$$A = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

By Proposition 4.2.7, we know that the solutions of (11.40) are eigenvectors of $\mathcal{F}$ and that the eigenvectors of $\mathcal{F}$ can be uniquely rescaled to be a solution of (11.40). Let $\kappa(\Phi)$ and $\kappa(\Phi^*)$ be the Birkhoff contraction ratio given by the Birkhoff-Hopf theorem 5.2.7. Furthermore, note that by the discussion in Example 2.3.5 we know that the map $X \mapsto X^{-1}$ is order-reversing. Hence, by Proposition 5.2.1, we have have $\mu_{C_1}(X^{-1}, \tilde X^{-1}) \leq \mu_{C_1}(X, \tilde X)$ and $\mu_{C_2}(Y^{-1}, \tilde Y^{-1}) \leq \mu_{C_1}(Y, \tilde Y)$ for all $(X,Y), (\tilde X, \tilde Y) \in \operatorname{int}(\mathcal{C})$. It follows that for every $(X,Y), (\tilde X, \tilde Y) \in \operatorname{int}(\mathcal{C})$, it holds

$$\begin{aligned}\mu_{C_1}(\mathcal{F}(X,Y)_1, \mathcal{F}(\tilde X, \tilde Y)_1) \leq \mu_{C_1}(\Phi(Y^{-1}), \Phi(\tilde Y^{-1})) &\leq \kappa(\Phi)\mu_{C_2}(Y^{-1}, \tilde Y^{-1})\\ &\leq \kappa(\Phi)\mu_{C_2}(Y, \tilde Y).\end{aligned}$$

It now follows from Theorem 6.1.7 that for all $(X,Y), (\tilde X, \tilde Y) \in \operatorname{int}(\mathcal{C})$ it holds

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}(X,Y), \mathcal{F}(\tilde X, \tilde Y)) \leq L\boldsymbol{\mu}_{\mathcal{C}}((X,Y), (\tilde X, \tilde Y)), \tag{11.42}$$

with

$$L = \begin{pmatrix} 0 & \kappa(\Phi)\\ \kappa(\Phi^*) & 0\end{pmatrix}. \tag{11.43}$$

With these observations one can prove the following:

Theorem 11.2.7. Equation (11.40) has a unique solution (X, Y). Moreover, for every $(Y_0, X_0) \in \operatorname{int}(\mathcal{C})$, the sequence $(Y_k, X_k)_{k=1}^{\infty} \in \operatorname{int}(\mathcal{C})$ defined for all $k \geq 0$ as

$$(Y_{k+1}, X_{k+1}) = \Big(\frac{\Phi(Y_k^{-1})^{-1}}{\operatorname{Tr}(\Phi(Y_k^{-1})^{-1})}, \frac{\Phi^*(X_k)}{\operatorname{Tr}(\Phi^*(X_k))}\Big) \tag{11.44}$$

satisfies

$$\lim_{k\to\infty}(Y_k, X_k) = (\hat{X}, \hat{Y}) \qquad \text{with} \qquad (\hat{X}, \hat{Y}) = \Big(\frac{X}{\operatorname{Tr}(X)}, \frac{Y}{\operatorname{Tr}(Y)}\Big).$$

Furthermore, it holds

$$(X, Y) = \frac{\big(\hat{X}\,,\, n\operatorname{Tr}(\Phi^*(\hat{X}))\,\hat{Y}\big)}{\operatorname{Tr}(\hat{X}) + n\operatorname{Tr}(\Phi^*(\hat{X}))\operatorname{Tr}(\hat{Y})},$$

the matrix $L \in \mathbb{R}_+^{2\times 2}$ of (11.43) has spectral radius $\rho(L) = \sqrt{\kappa(\Phi)\kappa(\Phi^*)} < 1$ and

$$\boldsymbol{\mu}_{\mathcal{C}}((X_k, Y_k), (X, Y)) \leq L^k\mathbf{v} \qquad \forall k \geq 1,$$

with $\mathbf{v} = (I - L)^{-1}\boldsymbol{\mu}_{\mathcal{C}}((X_1, Y_1), (X_0, Y_0))$.

Proof. It holds $\rho(L) = \sqrt{\kappa(\Phi)\kappa(\Phi^*)}$ as L is irreducible and has $(\sqrt{\kappa(\Phi)}, \sqrt{\kappa(\Phi^*)})^\top$ as positive eigenvector. As $\Phi(C_1\setminus\{0\}) \subset \operatorname{int}(C_2)$ and $\Phi^*(C_2\setminus\{0\}) \subset \operatorname{int}(C_1)$, we have $\kappa(\Phi) < 1$ and $\kappa(\Phi^*) < 1$ by Theorem 8.1.7. It follows that $\rho(L) < 1$. Let $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}_+^2$ be the monotonic multi-normalization given by $\boldsymbol{\nu}(X, Y) = (\operatorname{Tr}(X), \operatorname{Tr}(Y))$ and let $\mathcal{S}_{\boldsymbol{\nu}} = \{(X, Y) \in \mathcal{C} \,|\, \boldsymbol{\nu}(X, Y) = \mathbf{1}\}$. Then (11.42) and Theorem 7.3.1 imply that $\mathcal{F}$ defined in (11.41) has a unique eigenvector $(\hat{X}, \hat{Y}) \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ and for all $(X_0, Y_0) \in \operatorname{int}(\mathcal{C})$, the sequence $(\hat{X}_k, \hat{Y}_k)_{k=1}^{\infty}$ defined as

$$(\hat{X}_{k+1}, \hat{Y}_{k+1}) = \boldsymbol{\nu}(\mathcal{F}(\hat{X}_k, \hat{Y}_k))^{-I} \otimes \mathcal{F}(\hat{X}_k, \hat{Y}_k),$$

satisfies

$$\lim_{k\to\infty}(\hat{X}_{k+1}, \hat{Y}_{k+1}) = (\hat{X}, \hat{Y}) \quad \text{and} \quad \boldsymbol{\mu}_{\mathcal{C}}((\hat{X}_k, \hat{Y}_k), (\hat{X}, \hat{Y})) \leq L^k\mathbf{v} \qquad \forall k \geq 1,$$

with $\mathbf{v} = (I - L)^{-1}\boldsymbol{\mu}_{\mathcal{C}}((\hat{X}_1, \hat{Y}_1), (\hat{X}_0, \hat{Y}_0))$. As $(\hat{X}, \hat{Y})$ is the limit of the sequence $(\hat{X}_k, \hat{Y}_k)_{k=1}^{\infty}$, we have $\mathcal{F}(\hat{X}, \hat{Y}) = (\lambda\hat{X}, \theta\hat{Y})$ with

$$(\lambda, \theta) = \boldsymbol{\nu}(\mathcal{F}(\hat{X}, \hat{Y})) = (\operatorname{Tr}(\Phi(\hat{Y}^{-1})^{-1}), \operatorname{Tr}(\Phi^*(\hat{X}))).$$

Proposition 4.2.7 implies that $\operatorname{Tr}(\Phi(\hat{Y}^{-1})^{-1}) = \frac{m}{n}\operatorname{Tr}(\Phi^*(\hat{X}))^{-1}$ and the pair $(X, Y) = t\,(\hat{X}, n\operatorname{Tr}(\Phi^*(\hat{X}))\hat{Y})$ is a solution of (11.41) with $t = (\operatorname{Tr}(\hat{X}) + n\operatorname{Tr}(\Phi^*(\hat{X}))\operatorname{Tr}(\hat{Y}))^{-1}$. The scaling invariance of $\boldsymbol{\mu}_{\mathcal{C}}$ (see Equation (6.7)) implies that $\boldsymbol{\mu}_{\mathcal{C}}((\hat{X}_k, \hat{Y}_k), (\hat{X}, \hat{Y})) = \boldsymbol{\mu}_{\mathcal{C}}((\hat{X}_k, \hat{Y}_k), (X, Y))$ for all $k \geq 1$ which proves the claimed convergence rate. Next, we prove the uniqueness of the solution (X, Y). Suppose that $(\bar{X}, \bar{Y})$ is a solution of (11.40). Then, the pair $(\bar{X}/\operatorname{Tr}(\bar{X}), \bar{Y}/\operatorname{Tr}(\bar{Y})) \in \operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$ is an eigenvector of $\mathcal{F}$. As $\mathcal{F}$ has a unique eigenvector in $\operatorname{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$, we have

$$\Big(\frac{\bar{X}}{\operatorname{Tr}(\bar{X})}, \frac{\bar{Y}}{\operatorname{Tr}(\bar{Y})}\Big) = (\hat{X}, \hat{Y}) = \Big(\frac{X}{\operatorname{Tr}(X)}, \frac{Y}{\operatorname{Tr}(Y)}\Big).$$

It follows that $\bar{X} = (\operatorname{Tr}(\bar{X})/\operatorname{Tr}(X))X$ and $\bar{Y} = (\operatorname{Tr}(\bar{Y})/\operatorname{Tr}(Y))Y$. To conclude, we prove that for $\alpha, \beta > 0$, $(\alpha X, \beta Y)$ is a solution of (11.44) if and only if $\alpha = \beta = 1$. Suppose that $(\alpha X, \beta Y)$ is a solution of (11.40). Then, it holds $\Phi^*(\alpha X) = \frac{1}{n}(\beta Y)$ and since $\Phi^*(X) = \frac{1}{n}Y$ it follows that $\alpha = \beta$. Now, as $(X, Y), (\alpha X, \beta Y) \in \mathcal{S}_{\boldsymbol{\nu}}$ we necessarily have $\alpha = \beta = 1$ which implies that $(\bar{X}, \bar{Y}) = (X, Y)$ and therefore (X, Y) is the unique solution to (11.40). □

11.2.7 Generalized DAD problem

Let $d \geq 2$. For $i = 1, \ldots, d$, let $V_i = \mathbb{R}^{n_i}$, $C_i = \mathbb{R}^{n_i}_+$, $z_i \in \mathbb{R}^{n_i}_{++}$ and $M_i \in \mathbb{R}^{n_{i+1}\times n_i}_+$ with $n_{d+1} = n_1$. Suppose that M_i has at least one positive entry per row. Let $\mathcal{C} = C_1 \times \ldots \times C_d$. We have discussed in Section 4.2.6 that the generalized DAD problem is formulated as

$$\begin{cases} x_2 \circ M_1 x_1 & = & z_2 \\ x_3 \circ M_2 x_2 & = & z_3 \\ \vdots & \vdots & \vdots \\ x_d \circ M_{d-1} x_{d-1} & = & z_d \\ x_1 \circ M_d x_d & = & \lambda z_1 \end{cases} \qquad \text{with} \qquad (\lambda, x) \in \mathbb{R}_+ \times \operatorname{int}(\mathcal{C}). \tag{11.45}$$

It is proved in Theorem 7.1.4 of [60] that the above equation has a solution if there exists $i \in \{1, \ldots, d\}$ such that $\kappa(M_i) < 1$, where $\kappa(\cdot)$ is the Birkhoff contraction rate discussed in Theorem 5.2.7. The main idea of the proof of [60, Theorem 7.1.4] is to use the substitution strategy discussed in Section 4.3 to formulate an equivalent problem of the form $f(x_1) = \lambda x_1$ where $f\colon \mathbb{R}^{n_1}_{++} \to \mathbb{R}^{n_1}_{++}$ is a mapping which is homogeneous of degree 1 or -1 and is order-preserving or order-reversing, depending on the parity of d. We prove here a similar result by treating the problem directly in its multi-homogeneous form. Let $\mathcal{F}\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ be given by

$$\mathcal{F}(x_1, \ldots, x_d) = \Psi(\mathcal{L}(x)) \qquad \forall x \in \operatorname{int}(\mathcal{C}), \tag{11.46}$$

where $\mathcal{L}\colon \mathcal{C} \to \mathcal{C}$ is defined as

$$\mathcal{L}(x) = (M_d x_d, M_1 x_1, \ldots, M_{d-1} x_{d-1}) \qquad \forall x \in \mathcal{C},$$

and $\Psi\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ is defined as

$$\Psi(x) = \big(z_1 \circ x_1^{-1}, \ldots, z_d \circ x_d^{-1}\big) \qquad \forall x \in \operatorname{int}(\mathcal{C}).$$

We discuss properties of $\mathcal{L}$. Note that $\mathcal{L}$ is cone multi-linear and thus by Theorem 8.1.2, we have

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{L}(x), \mathcal{L}(y)) \leq L\boldsymbol{\mu}_{\mathcal{C}}(x, y) \qquad \forall x, y \in \operatorname{int}(\mathcal{C}),$$

with $L \in \mathbb{R}^{d\times d}_+$ defined as

$$L_{i,j} = \begin{cases} \kappa(M_{i-1}) & \text{if } i > 1 \text{ and } j = i-1 \\ \kappa(M_d) & \text{if } i = 1 \text{ and } j = d, \\ 0 & \text{otherwise.} \end{cases} \tag{11.47}$$

Furthermore, we note that $\rho(L) = \big(\prod_{i=1}^{d} \kappa(M_i)\big)^{1/d}$. Indeed, L is irreducible and thus it has a positive left eigenvector $\mathbf{a} \in \mathbb{R}^d_{++}$ such that $L^\top \mathbf{a} = \rho(L)\mathbf{a}$. Now, note that $L^\top \mathbf{a} = (\kappa(M_d)a_d, \kappa(M_1)a_1, \ldots, \kappa(M_{d-1})a_{d-1})$ and thus $L^\top \mathbf{a} = \rho(L)\mathbf{a}$ implies that $\kappa(M_i)a_i = \rho(L)a_{i+1}$ for $i = 1, \ldots, d-1$ and $\kappa(M_d)a_d = \rho(L)a_1$. Hence, we have

$$\rho(L)^d\Big(\prod_{i=1}^{d} a_i\Big) = \Big(\prod_{i=1}^{d} \rho(L)a_i\Big) = \Big(\prod_{i=1}^{d} \kappa(M_i)\Big)\Big(\prod_{i=1}^{d} a_i\Big)$$

which proves the claim.

We discuss properties of Ψ. Note that Ψ is order-reversing and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}$ with

$$A_{i,j} = \begin{cases} -1 & \text{if } i > 1 \text{ and } j = i-1 \\ -1 & \text{if } i = 1 \text{ and } j = d, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore note that for all $i = 1, \ldots, d$ and $x, y \in \operatorname{int}(\mathcal{C})$, it holds

$$\begin{aligned}\mu_{\mathcal{C}}(\Psi(x), \Psi(y))_i &= \ln\Big(\max_{j_i, j_i' \in [n_i]} \frac{\Psi(x)_{i,j_i}\Psi(y)_{i,j_i'}}{\Psi(y)_{i,j_i}\Psi(x)_{i,j_i'}}\Big) = \ln\Big(\max_{j_i, j_i' \in [n_i]} \frac{z_{i,j_i}x_{i,j_i}^{-1} z_{i,j_i'} y_{i,j_i'}^{-1}}{z_{i,j_i}y_{i,j_i}^{-1} z_{i,j_i'} x_{i,j_i'}^{-1}}\Big) \\ &= \ln\Big(\max_{j_i, j_i' \in [n_i]} \frac{y_{i,j_i}x_{i,j_i'}}{x_{i,j_i}y_{i,j_i'}}\Big) = \boldsymbol{\mu}_{\mathcal{C}}(x,y)_i,\end{aligned}$$

and thus

$$\mu_{\mathcal{C}}(\Psi(x), \Psi(y)) = \boldsymbol{\mu}_{\mathcal{C}}(x,y) \qquad \forall x, y \in \operatorname{int}(\mathcal{C}).$$

Finally, we note that $\mathcal{F}$ is order-reversing and multi-homogeneous of degree $A \in \mathbb{R}^{d\times d}$ and

$$\boldsymbol{\mu}_{\mathcal{C}}(\mathcal{F}(x), \mathcal{F}(y)) \leq L\boldsymbol{\mu}_{\mathcal{C}}(x,y) \qquad \forall x, y \in \operatorname{int}(\mathcal{C}).$$

Furthermore, we have proved in Proposition 4.2.8 that if (x, λ) solves (11.45), then x is an eigenvector of $\mathcal{F}$ and if x is an eigenvector of $\mathcal{F}$, then there exists $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$ and $\lambda > 0$ such that $(\lambda, \boldsymbol{\alpha} \otimes x)$ is a solution of (11.45).

We combine the above observations with Theorem 11.1.2 to obtain the following:

Theorem 11.2.8. Let $\mathcal{F}\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ and $L \in \mathbb{R}^{d\times d}_+$ be defined as in (11.46) and (11.47). For $i = 1, \ldots, d$, let $\|\cdot\|_i$ be a norm on $\mathbb{R}^{n_i}$. If there exists $j \in \{1, \ldots, d\}$ such that $\kappa(M_j) < 1$, then (11.45) has a solution (λ, u). Moreover, for every $x_0 \in \operatorname{int}(\mathcal{C})$, the sequence $(x_k)_{k=1}^{\infty} \subset \operatorname{int}(\mathcal{C})$ defined as

$$x_{k+1} = \Big(\frac{\mathcal{F}(x_k)_1}{\|\mathcal{F}(x_k)_1\|_1}, \ldots, \frac{\mathcal{F}(x_k)_d}{\|\mathcal{F}(x_k)_d\|_d}\Big) \qquad \forall k \geq 1,$$

where $\mathcal{F}\colon \operatorname{int}(\mathcal{C}) \to \operatorname{int}(\mathcal{C})$ is defined in (11.46), satisfies

$$\lim_{k\to\infty} x_k = \hat{u} = \Big(\frac{u_1}{\|u_1\|_1}, \ldots, \frac{u_d}{\|u_d\|_d}\Big).$$

Furthermore, let $\boldsymbol{\alpha} \in \mathbb{R}^d_{++}$ and $\lambda > 0$ be defined as

$$\alpha_1 = 1, \qquad \alpha_{i+1} = \frac{\|z_{i+1} \circ (M_{i+1}\hat{u}_{i+1})^{-1}\|_{i+1}}{\alpha_i} \quad \text{for} \quad i = 1, \ldots, d-1,$$

and

$$\lambda = \frac{\alpha_d}{\|z_1 \circ (M_1\hat{u}_1)^{-1}\|_1}.$$

Then $(\lambda, \boldsymbol{\alpha} \otimes \hat{u})$ is a solution of (11.45) and, with $\mathbf{v} = (I - L)^{-1}\boldsymbol{\mu}_{\mathcal{C}}(x_1, x_0)$, it holds

$$\boldsymbol{\mu}_{\mathcal{C}}(x_k, \boldsymbol{\alpha} \otimes \hat{u}) \leq L^k\mathbf{v} \quad \forall k \geq 1.$$

Proof. Let $\boldsymbol{\nu}\colon \mathcal{C} \to \mathbb{R}^d_+$ be defined as $\boldsymbol{\nu}(x) = (\|x_1\|_1, \ldots, \|x_d\|_d)$ and let $\mathcal{S}_{\boldsymbol{\nu}} = \{x \in \mathcal{C} \,|\, \boldsymbol{\nu}(x) = \mathbf{1}\}$. By the above discussion we know that $\rho(L) = \big(\prod_{i=1}^d \kappa(M_i)\big)^{1/d}$ and thus $\kappa(M_j) < 1$ implies $\rho(L) < 1$. Hence, by Theorem 11.1.2, we know that $\mathcal{F}$ has a unique eigenvector $\hat{u} \in \mathrm{int}(\mathcal{C}) \cap \mathcal{S}_{\boldsymbol{\nu}}$, the sequence $(x_k)_{k=1}^{\infty}$ converges towards $\hat{u}$, and

$$\boldsymbol{\mu}_{\mathcal{C}}(x_k, \hat{u}) \leq L^k\mathbf{v} \quad \forall k \geq 1.$$

Now, note that as $\hat{u}$ is the limit of $(x_k)_{k=1}^{\infty}$, we have $\mathcal{F}(\hat{u}) = \boldsymbol{\nu}(\mathcal{F}(\hat{u})) \otimes \hat{u}$. Hence, Proposition 4.2.8 implies that $(\lambda, \boldsymbol{\alpha} \otimes \hat{u})$ is a solution of (11.45). Finally, by the scaling invariance of $\boldsymbol{\mu}_{\mathcal{C}}$ (see Equation (6.7)), we have $\boldsymbol{\mu}_{\mathcal{C}}(x_k, \boldsymbol{\alpha} \otimes \hat{u}) = \boldsymbol{\mu}_{\mathcal{C}}(x_k, \hat{u})$ for all $k \geq 1$ which proves the convergence rate. □

12 Conclusion

The goal of this thesis was to study, from a Perron-Frobenius perspective, eigenvector equations of the following type:

$$\begin{cases} \mathcal{F}(x_1, \ldots, x_d)_1 &= \lambda_1 x_1 \\ \mathcal{F}(x_1, \ldots, x_d)_2 &= \lambda_2 x_2 \\ &\vdots \\ \mathcal{F}(x_1, \ldots, x_d)_d &= \lambda_d x_d \end{cases} \qquad (\boldsymbol{\lambda}, x) \in \mathbb{R}^d_+ \times \mathcal{C}_0, \tag{12.1}$$

where $\mathcal{C}_0 = (C_1 \setminus \{0\}) \times \ldots \times (C_d \setminus \{0\})$, C_i is a cone in a finite dimensional real vector space for $i = 1, \ldots, d$, and $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}$ is a mapping. To achieve this goal, in Section 3, we have introduced multi-homogeneous mappings. The relevance of such mappings is motivated in Section 4 where we have shown that equations of the form (12.1) involving multi-homogeneous mappings appear in various places of the literature and in particular that of nonnegative tensors. For the analysis of such problems, in Section 6.2, we have considered the vector valued Hilbert metric induced by a product of cones. We have proved that desirable properties of the usual Hilbert metric in the study of homogeneous mappings on cones, such as non-expansiveness and contractivity, can be generalized to multi-homogeneous mappings when using a vector valued metric. To prove such generalizations, we have used results of Section 6.1 where Lipschitz matrices are discussed. Fixed point theorems for vector valued metrics are discussed in Section 7.1. In Section 7.2, we give strong arguments in favor of the vector valued Hilbert metric over the real valued Hilbert for the study of multi-homogeneous mappings. Then, in Section 7.3, we have discussed how to use the vector valued Hilbert metric to study the solutions of (12.1) when $\mathcal{F}$ is a strict contraction. In order to better identify strict contractions on the product of cones, in Section 8, we have extended the Birkhoff-Hopf theorem to multi-linear mappings. Then, in Sections 9 and 10, we address the case of non-expansive mappings and prove multi-homogeneous versions of other classical results of the Perron-Frobenius theory such as the existence, the maximality and the uniqueness of an eigenvector in the interior of the cone, the Collatz-Wielandt formula and the convergence of the normalized iterates towards eigenvectors in the interior of the cone. Our main results are summarized in Section 11.1. In Section 11.2, we have shown how they can be applied to improve, complete and recover results of the literature on ℓ^p-eigenvectors, rectangular $\ell^{p,q}$-singular vectors and $\ell^{p_1,\ldots,p_d}$-singular vectors of nonnegative tensors. Furthermore, we have seen that they allow to deduce new results for the computation of the solution to the discrete generalized Schrödinger equation and offer a new point of view on the generalized DAD problem as well as in the study of quantum copulas.

All in all, multi-homogeneous mappings are useful to unify the study of various problems and allow a natural generalization of the results from the nonlinear Perron-Frobenius theory of homogeneous mappings. The nonlinear Perron-Frobenius theory has numerous further results which were not mentioned in this thesis and could be potentially extended for multi-homogeneous mappings. For instance, extending the definition of spectral radius in [68] to a multi-homogeneous mappings defined on the product of general cones would allow to refine results on the existence of eigenvectors and the Collatz-Wielandt formula as the latter definition does not assume the existence of an eigenvector. Also, we have not discussed Denjoy-Wolff type results [59] which describe the behavior of the normalized iterates of a mapping which does not have an eigenvector in the interior of the cone. Furthermore, we are convinced that introducing the concept of multi-subhomogeneous mappings would broadly extend the range of applications of vector valued metrics on cones. We refer to [3] and [60, Chapter 8] for a discussion on subhomogeneous mappings. We believe however that multi-subhomogeneous mappings should generalize a more refined concept, namely p-subhomogeneous mappings, i.e. mappings $f\colon C \to C$ for which $\alpha^p f(x) \leq f(\alpha x)$ for all $x \in C$ and $\alpha \in (0,1]$. Such approach allows to fully exploit the vector valued version of the Thompson metric (which is typically used for the study of subhomogeneous mappings). Finally, we note that the estimation of the projective diameter in the multi-linear Birkhoff-Hopf theorem, mainly discussed in Section 8.2, can be improved in two ways: First, by working on parts of the cone. This would allow zero entries in the tensor while still having a bounded projective diameter. Second, by sharpening the upper bound $\Delta_{i,j}(T)$ discussed in Theorem 8.2.1. Such result would enlarge the range in which p, q and $p_1, \ldots, p_d$ can be chosen in order to guarantee existence and uniqueness of ℓ^p-eigenvectors, rectangular $\ell^{p,q}$-singular vectors and $\ell^{p_1,\ldots,p_d}$-singular vectors of non-negative tensors.

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