A stochastic approach to path–dependent nonlinear Kolmogorov equations via BSDEs with time–delayed generators and applications to finance

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Abstract

We prove the existence of a viscosity solution of the following path dependent nonlinear Kolmogorov equation:

$$\begin{cases} \partial_t u(t,\phi) + \mathcal{L}u(t,\phi) + f(t,\phi,u(t,\phi),\partial_x u(t,\phi)\sigma(t,\phi),(u(\cdot,\phi))_t) = 0, \ t \in [0,T), \ \phi \in \mathbb{A}, \\ u(T,\phi) = h(\phi), \ \phi \in \mathbb{A}, \end{cases}$$

where $\mathbb{A} = \mathcal{C}([0,T];\mathbb{R}^d)$, $(u(\cdot,\phi))_t := (u(t+\theta,\phi))_{\theta \in [-\delta,0]}$ and

$$\mathcal{L}u(t,\phi) := \langle b(t,\phi), \partial_x u(t,\phi) \rangle + \frac{1}{2}\mathrm{Tr}\big[\sigma(t,\phi)\sigma^*(t,\phi)\partial_{xx}^2 u(t,\phi)\big].$$

The result is obtained by a stochastic approach. In particular we prove a new type of nonlinear Feynman–Kac representation formula associated to a backward stochastic differential equation with time–delayed generator which is of non–Markovian type. Applications to the large investor problem and risk measures via g–expectations are also provided.

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1 Introduction

We aim at providing a probabilistic representation of a viscosity solution to the following path–dependent nonlinear Kolmogorov equation (PDKE)

$$\begin{cases} -\partial_t u(t,\phi) - \mathcal{L}u(t,\phi) - f(t,\phi,u(t,\phi),\partial_x u\,(t,\phi)\,\sigma(t,\phi),(u(\cdot,\phi))_t) = 0, \\ u(T,\phi) = h(\phi), \end{cases} \quad (1)$$

for $T < \infty$ a fixed time horizon, $t \in [0,T)$, $\phi \in \mathbb{A} := \mathcal{C}([0,T];\mathbb{R}^d)$, the space of continuous $\mathbb{R}^d$–valued functions defined on the interval $[0,T]$. For a fixed delay $\delta > 0$, we have set

$$(u(\cdot,\phi))_t := (u(t+\theta,\phi))_{\theta \in [-\delta,0]}.$$

In equation (1) we have denoted by $\mathcal{L}$ the second order differential operator given by

$$\mathcal{L}u(t,\phi) := \frac{1}{2}\mathrm{Tr}\big[\sigma(t,\phi)\sigma^*(t,\phi)\partial^2_{xx}u(t,\phi)\big] + \langle b(t,\phi),\partial_x u(t,\phi)\rangle,$$

with $b : [0,T] \times \mathbb{A} \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{A} \to \mathbb{R}^{d \times d'}$ being two non–anticipative functionals to be better introduced in subsequent section.

We will prove that, under appropriate assumptions on the coefficients, the deterministic non–anticipative functional $u : [0,T] \times \mathbb{A} \to \mathbb{R}$ given by the representation formula

$$u(t,\phi) := Y^{t,\phi}(t)$$

is a viscosity solution, in the sense given in [16], to equation (1), where

$$(X^{t,\phi}(s),Y^{t,\phi}(s),Z^{t,\phi}(s))_{s \in [t,T]}$$

is the unique solution to the decoupled forward–backward stochastic differential system on $[t,T]$

$$\begin{cases} X^{t,\phi}(s) = \phi(t) + \int_t^s b(r,X^{t,\phi})dr + \int_t^s \sigma(r,X^{t,\phi})dW(r), \\ Y^{t,\phi}(s) = h(X^{t,\phi}) + \int_s^T f(r,X^{t,\phi},Y^{t,\phi}(r),Z^{t,\phi}(r),Y_r^{t,\phi})dr - \int_s^T Z^{t,\phi}(r)\,dW(r), \end{cases} \quad (2)$$

with $(t,\phi) \in [0,T] \times \mathbb{A}$, where W is a standard Brownian motion.

Here, the notation $Y_r^{t,\phi}$ appearing in the generator f of the backward component in system (2) stands for the path of the process $Y^{t,\phi}$ restricted to $[r-\delta,r]$, namely

$$Y_r^{t,\phi} := (Y^{t,\phi}(r+\theta))_{\theta \in [-\delta,0]}.$$

In system (2) the forward equation is a functional stochastic differential equation, while the backward equation has time–delayed generator, that is the generator f can depend, unlike the classical backward stochastic differential equations (BSDEs), on the past values of $Y^{t,\phi}$.

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Let us stress that if we do not consider delay neither in the forward nor in the backward component, we retrieve standard results of Markovian forward–backward system, hence in this case $u(t, \phi) = u(t, \phi(t))$ and equation (1) becomes

$$\begin{cases} -\partial_t u(t,x) - \mathcal{L}u(t,x) - f(t,x,u(t,x),\partial_x u(t,x)\,\sigma(t,x)) = 0, \\ u(T,x) = h(x), \quad t \in [0,T), \; x \in \mathbb{R}^d, \end{cases}$$

with

$$\mathcal{L}u(t,x) := \frac{1}{2}\mathrm{Tr}\big[\sigma(t,x)\sigma^*(t,x)\partial^2_{xx}u(t,x)\big] + \langle b(t,x), \partial_x u(t,x) \rangle.$$

Let us recall that BSDEs were first introduced by Bismut [4], in the linear case, whereas the nonlinear case was considered by Pardoux and Peng in [30]. Later, in [31, 37], the connection between BSDEs and semilinear parabolic partial differential equations (PDEs) was established, proving the nonlinear Feynman–Kac formula for Markovian equations. Also, similar deterministic representations associated with suitable PDEs can be proved by taking into account different types of BSDEs, such as BSDEs with random terminal time, see, e.g. [9], reflected BSDEs, see, e.g. [19], or also backward stochastic variational inequalities, see, e.g. [26, 27].

When considering the non-Markovian case, the associated PDE becomes path-dependent; this is shown for the first time in [34]. Subsequently, in [38] the authors proved, in the case of smooth coefficients, the existence and uniqueness of a classical solution for the path–dependent Kolmogorov equation

$$\begin{cases} -\partial_t u(t,\phi) - \dfrac{1}{2}\partial^2_{xx}u(t,\phi) - f(t,\phi,u(t,\phi),\partial_x u(t,\phi)) = 0, \\ u(T,\phi) = h(\phi), \quad t \in [0,T), \; \phi \in \mathbb{A}. \end{cases} \tag{3}$$

In particular, the authors appealed to a representation formula using the non–Markovian BSDE:

$$Y^{t,\phi}(s) = h(W^{t,\phi}) + \int_s^T f(W^{t,\phi}, Y^{t,\phi}(r), Z^{t,\phi}(r))dr - \int_s^T Z^{t,\phi}(r)\,dW(r), \quad s \in [t,T], \tag{4}$$

with the generator and the final condition depending on the Brownian paths

$$W^{t,\phi}(s) = \phi(t) + W(s) - W(t), \quad \text{if } s \in [t,T], \quad \text{and}$$
$$W^{t,\phi}(s) = \phi(s), \quad \text{if } s \in [0,t).$$

After that, in order to deal with such PDEs, in [36] a new type of viscosity solution is introduced.

The definition of viscosity solution we will adopt in the present work is that introduced in [16, 17, 18] for semilinear and fully non linear path-dependent PDE, by using the framework of functional Itô calculus first set by Dupire [15] and Cont & Fournié [7].

In the present work we will generalize the results in [16] along two directions. First we will consider a BSDE whose generator depends not only on past values assumed by a standard Brownian motion W, but the BSDE may depend on a general diffusion process X. Secondly, and perhaps the most important result, we will prove the connection

between path–dependent PDEs and BSDEs with time–delayed generators. We recall that time–delayed BSDEs were first introduced in [13] and [14] where the authors obtained the existence and uniqueness of the solution of the time–delayed BSDE

$$Y(t) = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z(s)dW(s), \quad 0 \le t \le T, \tag{5}$$

where

$$Y_s := (Y(r))_{r \in [0,s]} \quad \text{and} \quad Z_s := (Z(r))_{r \in [0,s]}.$$

In particular, the aforementioned existence and uniqueness result holds true if the time horizon T or the Lipschitz constant for the generator f are sufficiently small. A generalization of the existence and uniqueness theorem for these kind of BSDEs was made in [6] by considering the equation

$$Y(t) = \xi + \int_t^T f(s, Y_s, M_s)ds + M(T) - M(t), \ 0 \le t \le T,$$

where the solution is the pair (Y, M) such that Y is an adapted process and M is a martingale.

To our best knowledge, the link between time–delayed BSDEs and path–dependent PDEs has never been addressed in literature.

We also emphasize that our framework, since the BSDE under study is time–delayed, requires that the backward equation contains a supplementary initial condition to be satisfied, namely

$$Y^{t,\phi}(s) = Y^{s,\phi}(s), \quad \text{if } s \in [0, t).$$

Let us further stress that the Feynman–Kac formula would fail with standard prolongation $Y^{t,\phi}(s) = Y^{t,\phi}(t)$, for $s \in [0, t)$. Although the existence results for equation (5) have already been treated in [13, 14], our new initial condition imposes a more elaborated proof.

The last part of the paper presents two financial models based on our theoretical results. In recent years delay equations have been of growing interest, mainly motivated by many concrete applications where the effect of delay cannot be neglected, see, e.g. [28, 40]. On the contrary, BSDEs with time delayed generator have been first introduced as a pure mathematical tool, with no application of interest. Only later in [10, 11] the author proposed some financial applications to pricing, hedging and investment portfolio management, where backward equations with delayed generator provide a fundamental tool.

Based on the recently introduced path–dependent calculus, together with the mild assumptions of differentiability required, the probabilistic representation for a viscosity solution of a non-linear parabolic equations proved in the present paper finds perhaps its best application in finance. In fact, a wide variety of financial derivatives can be formally treated under the theory developed in what follows, from the more standard European options, to the more exotic path–dependent options, such as Asian options or look-back options.

We propose here two possible applications of forward–backward stochastic differential system (2), where the delay in the backward component arises from two different motivations. The first example we will deal with is a generalization of a well-known model in finance, where we will consider the case of a non standard investor acting on a financial market. We will assume, following [8, 21], that a so called *large investor* wishes to invest on

a given market, buying or selling a stock. This investor has the peculiarity that his actions on the market can affect the stock price. In particular, we will assume that the stock price S and the bond B are a function of the large investor's portfolio (X, π), X being the value of the portfolio and π the amount invested in the risky asset S.

This case has already been treated in the financial literature, see, e.g. [21]. We further generalize the aforementioned results assuming a second market imperfection, that is we assume that it might be a small time delay between the action of the large investor and the reaction of the market, so that we are led to consider the financial system with the presence of the past of X in the coefficients r, μ, σ :

$$\begin{cases} \dfrac{dB(t)}{B(t)} = r(t, X(t), \pi(t), X_t)dt, \quad B(0) = 1, \\[2mm] \dfrac{dS(t)}{S(t)} = \mu(t, X(t), \pi(t), X_t)\, dt + \sigma(t, X(t), X_t)dW(t), \quad S(0) = s_0 > 0, \end{cases}$$

where the notation X_t stands for the path $(X(t+\theta))_{\theta \in [-\delta, 0]}$, δ being a small enough delay.

The second example we deal with arises from a different situation. Recent literature in financial mathematics has been focused in how to measure the riskiness of a given financial investment. To this extent *dynamic risk measures* have been introduced in [2]. In particular, BSDEs have been shown to be perhaps the best mathematical tool for modelling dynamic risk measures, via the so called *g-expectations*. In [11] the author proposed a risk measure that takes also into account the past values assumed by the investment, that is we will suppose that, in making his future choices, the investor will consider not only the present value of the investment, but also the values assumed in a sufficiently small past interval. This has been motivated by empirical studies that show how the memory effect has a fundamental importance in an investor's choices, see, e.g. [11] and references therein for financial studies on the memory effect in financial investment. We therefore consider an investor that tries to quantify the riskiness of a given investment, with Y being his investment, and will assume that the investor looks at the average value of his investment in a sufficiently recent past, that is we consider a generator of the form $\frac{1}{\delta} g_1 \left(\int_{-\delta}^0 Y(t+\theta)d\theta \right) g_2 \left(Z(t) \right)$, with $\delta > 0$ being a sufficiently small delay.

The paper is organized as follows: in Section 2 we introduce notions which are needed for the functional Itô's calculus and also the notion of viscosity solution for path–dependent PDEs. In Section 3 we prove the existence and uniqueness of a solution for the time-delayed BSDE, whereas Section 4 is devoted to the main results of the present work, that is the proof of the continuity of the function $u(t, \phi) := Y^{t,\phi}(t)$ as well as the nonlinear Feynman–Kac formula, that is Theorem 13, the core of the present work. In Section 5 we eventually present the financial applications.

2 Preliminaries

2.1 Pathwise derivatives and functional Itô's formula

Let us first introduce the framework on which we shall construct the solutions of PDKE (1). For a deep treatment of functional Itô calculus we refer the reader to Dupire [15] and Cont & Fournié [7].

Let $\hat{\mathbb{\Lambda}} := \mathbb{D}\left([0,T];\mathbb{R}^d\right)$ be the set of càdlàg (i.e., right continuous, with finite left–hand limits) $\mathbb{R}^d$–valued functions, $\hat{B}$ the canonical process on $\hat{\mathbb{\Lambda}}$, i.e. $\hat{B}(t,\hat{\phi}) := \hat{\phi}(t)$ and $\hat{\mathbb{F}} := (\hat{\mathcal{F}}_s)_{s\in[0,T]}$ the filtration generated by $\hat{B}$. On $\hat{\mathbb{\Lambda}}$ and $[0,T]\times\hat{\mathbb{\Lambda}}$, we introduce the following norm and respectively pseudometric, with respect to whom it becomes a Banach space, respectively a complete pseudometric space. In this regard, we define, for any $(t,\hat{\phi}),(t',\hat{\phi}') \in [0,T]\times\hat{\mathbb{\Lambda}}$,

$$||\hat{\phi}||_T := \sup_{r\in[0,T]}|\hat{\phi}(r)|,$$

$$d((t,\hat{\phi}),(t',\hat{\phi}')) := |t-t'| + \sup_{r\in[0,T]}|\hat{\phi}(r\wedge t) - \hat{\phi}'(r\wedge t')|.$$

Let $\hat{u} : [0,T]\times\hat{\mathbb{\Lambda}} \to \mathbb{R}$ be an $\hat{\mathbb{F}}$–progressively measurable non-anticipative process, that is $\hat{u}(t,\hat{\phi})$ depends only on the restriction of $\hat{\phi}$ on $[0,t]$, i.e. $\hat{u}(t,\hat{\phi}) = \hat{u}(t,\hat{\phi}(\cdot\wedge t))$, for any $(t,\hat{\phi}) \in [0,T]\times\hat{\mathbb{\Lambda}}$. We say that $\hat{u}$ is vertically differentiable at $(t,\hat{\phi}) \in [0,T]\times\hat{\mathbb{\Lambda}}$ if there exists

$$\partial_{x_i}\hat{u}(t,\hat{\phi}) := \lim_{h\to 0}\frac{\hat{u}(t,\hat{\phi}+h\mathbb{1}_{[t,T]}e_i) - \hat{u}(t,\hat{\phi})}{h}$$

for any $i = \overline{1,d}$, where we have denoted by $\{e_i\}_{i=\overline{1,d}}$ the canonical basis of $\mathbb{R}^d$. The second order derivatives, when they exist, are denoted by $\partial^2_{x_ix_j}\hat{u}(t,\hat{\phi}) := \partial_{x_i}(\partial_{x_j}\hat{u})$, for any $i,j = \overline{1,d}$. Let us further denote by $\partial_x\hat{u}(t,\hat{\phi})$ the gradient vector, that is we have $\partial_x\hat{u}(t,\hat{\phi}) = \left(\partial_{x_1}\hat{u}(t,\hat{\phi}),\ldots,\partial_{x_d}\hat{u}(t,\hat{\phi})\right)$, and by $\partial^2_{xx}\hat{u}(t,\hat{\phi})$ the $d\times d$–Hessian matrix, that is $\partial^2_{xx}\hat{u}(t,\hat{\phi}) = \left(\partial^2_{x_ix_j}\hat{u}(t,\hat{\phi})\right)_{i,j=\overline{1,d}}$.

For $t \in [0,T]$ and a path $\phi \in \hat{\mathbb{\Lambda}}$, we denote

$$\phi_{(t)} := \phi(\cdot\wedge t) \in \hat{\mathbb{\Lambda}}. \tag{6}$$

We say that $\hat{u}$ is horizontally differentiable at $(t,\hat{\phi}) \in [0,T]\times\hat{\mathbb{\Lambda}}$ if there exists

$$\partial_t\hat{u}(t,\hat{\phi}) := \lim_{h\to 0_+}\frac{\hat{u}(t+h,\hat{\phi}_{(t)}) - \hat{u}(t,\hat{\phi})}{h},$$

for $t \in [0,T)$ and $\partial_t\hat{u}(T,\hat{\phi}) := \lim_{t\to T_-}\partial_t\hat{u}(t,\hat{\phi})$.

If $\hat{u} : [0,T]\times\hat{\mathbb{\Lambda}} \to \mathbb{R}$ is non-anticipative, we write $\hat{u} \in \mathcal{C}([0,T]\times\hat{\mathbb{\Lambda}})$ if $\hat{u}$ is continuous on $[0,T]\times\hat{\mathbb{\Lambda}}$ under the pseudometric d; we write that $\hat{u} \in \mathcal{C}_b([0,T]\times\hat{\mathbb{\Lambda}})$ if $\hat{u} \in \mathcal{C}([0,T]\times\hat{\mathbb{\Lambda}})$ and $\hat{u}$ is bounded on $[0,T]\times\hat{\mathbb{\Lambda}}$. Eventually we write $\hat{u} \in \mathcal{C}^{1,2}_b([0,T]\times\hat{\mathbb{\Lambda}})$ if $\hat{u} \in \mathcal{C}([0,T]\times\hat{\mathbb{\Lambda}})$ and the derivatives $\partial_x\hat{u}$, $\partial^2_{xx}\hat{u}$, $\partial_t\hat{u}$ exist and they are continuous and bounded.

Having introduced the above notations by following [15], we will now work with processes $u : [0,T]\times\mathbb{\Lambda} \to \mathbb{R}$, with $\mathbb{\Lambda}$ being the space of continuous paths, i.e.

$$\mathbb{\Lambda} = \mathcal{C}([0,T];\mathbb{R}^d).$$

Let B be the canonical process on Λ, i.e. $B(t,\phi) := \phi(t)$ and $\mathbb{F} := (\mathcal{F}_s)_{s\in[0,T]}$ the filtration generated by B.

From the fact that $\mathbb{\Lambda}$ is a closed subspace of $\hat{\mathbb{\Lambda}}$, we have $(\mathbb{\Lambda},||\cdot||_T)$ is also a Banach space; similarly, we claim that $([0,T]\times\mathbb{\Lambda},d)$ is a complete pseudometric space. As above, if $u : [0,T]\times\mathbb{\Lambda} \to \mathbb{R}$ is a non-anticipative process, we write that $u \in \mathcal{C}([0,T]\times\mathbb{\Lambda})$ if u is continuous

on $[0, T] \times \mathbb{A}$ under the pseudometric d; if, moreover, u is continuous and bounded on $[0, T] \times \mathbb{A}$, we write $u \in \mathcal{C}_b([0, T] \times \mathbb{A})$. Eventually, following [16], we write that $u \in \mathcal{C}_b^{1,2}([0, T] \times \mathbb{A})$ if there exists $\hat{u} \in \mathcal{C}_b^{1,2}([0, T] \times \hat{\mathbb{A}})$ such that $\hat{u}\big|_{[0,T] \times \mathbb{A}} = u$ and by definition we take $\partial_{t_i} u := \partial_t \hat{u}$, $\partial_x u := \partial_x \hat{u}$, $\partial_{xx}^2 u := \partial_{xx}^2 \hat{u}$; notice that definitions are independent of the choice of $\hat{u}$.

We introduce now the shifted spaces of càdlàg and continuous paths. If $t \in [0, T]$, $\hat{B}^t$ is the shifted canonical process on $\hat{\mathbb{A}}^t := \mathbb{D}\left([t, T]; \mathbb{R}^d\right)$, $\hat{\mathbb{F}}^t := (\hat{\mathcal{F}}_s^t)_{s \in [t, T]}$ is the shifted filtration generated by $\hat{B}^t$,

$$||\hat{\phi}||_T^t := \sup_{r \in [t,T]} |\hat{\phi}(r)|,$$

$$d^t((s, \hat{\phi}), (s', \hat{\phi}')) := |s - s'| + \sup_{r \in [t,T]} |\hat{\phi}(r \wedge s) - \hat{\phi}'(r \wedge s')|,$$

for any $(s, \hat{\phi}), (s', \hat{\phi}') \in [t, T] \times \hat{\mathbb{A}}^t$. Analogously we define the spaces $\mathcal{C}([t, T] \times \hat{\mathbb{A}}^t)$, $\mathcal{C}_b([t, T] \times \hat{\mathbb{A}}^t)$ and $\mathcal{C}_b^{1,2}([t, T] \times \hat{\mathbb{A}}^t)$. Similarly, we denote $\mathbb{A}^t := \mathcal{C}([t, T]; \mathbb{R}^d)$, B^t the shifted canonical process on $\mathbb{A}^t$, $\mathbb{F}^t := (\mathcal{F}_s^t)_{s \in [t,T]}$ the shifted filtration generated by B^t and we introduce the spaces $\mathcal{C}([t, T] \times \mathbb{A}^t)$, $\mathcal{C}_b([t, T] \times \mathbb{A}^t)$ and $\mathcal{C}_b^{1,2}([t, T] \times \mathbb{A}^t)$.

Let us denote by $\mathcal{T}$ the set of all $\mathbb{F}$–stopping times τ such that for all $t \in [0, T)$, the set $\{\phi \in \mathbb{A} : \tau(\phi) > t\}$ is an open subset of $(\mathbb{A}, || \cdot ||_T)$ and $\mathcal{T}^t$ the be the set of all $\mathbb{F}$–stopping times τ such that for all $s \in [t, T)$, the set $\{\phi \in \mathbb{A}^t : \tau(\phi) > s\}$ is an open subset of $(\mathbb{A}^t, || \cdot ||_T^t)$.

For a càdlàg function $\phi \in \mathbb{D}\left([-\delta, T]; \mathbb{R}^d\right)$, we denote

$$\phi_t := (\phi(t + \theta))_{\theta \in [-\delta, 0]}. \tag{7}$$

We conclude this subsection by recalling the functional version of Itô's formula (see Cont & Fournié [7, Theorem 4.1]).

Theorem 1 (Functional Itô's formula) *Let A be a d–dimensional Itô process, i.e. $A : [0, T] \times \mathbb{A} \to \mathbb{R}^d$ is a continuous $\mathbb{R}^d$–valued semimartingale defined on the probability space $(\mathbb{A}, \mathbb{F}, \mathbb{P})$ which admits the representation*

$$A(t) = A(0) + \int_0^t b(r)\, dr + \int_0^t \sigma(r) dB(r), \quad \text{for all } t \in [0, T], \quad \mathbb{P}\text{–a.s.},$$

where b, σ are progressively measurable stochastic processes such that $\int_0^T \left(|b(r)| + |\sigma(r)|^2\right) dr < \infty$, $\mathbb{P}$–a.s.

If $F \in \mathcal{C}_b^{1,2}([0, T] \times \hat{\mathbb{A}})$ then, for any $t \in [0, T)$, the following change of variable formula holds true:

$$F\left(t, A_{(t)}\right) = F\left(0, A_{(0)}\right) + \int_0^t \partial_t F\left(r, A_{(r)}\right) dr + \int_0^t \left\langle \partial_x F\left(r, A_{(r)}\right), b(r) \right\rangle dr$$

$$+ \frac{1}{2} \int_0^t \text{Tr}\left[\sigma(r) \sigma^*(r)\, \partial_{xx}^2 F(r, A_{(r)})\right] dr + \int_0^t \left\langle \partial_x F\left(r, A_{(r)}\right), \sigma(r)\, dB(r) \right\rangle, \quad \mathbb{P}\text{–a.s..}$$

2.2 Path–dependent PDEs

We give now the notion of viscosity solution to equation (1) as it was first introduced in [16] (see also [17, 18]).

Let $(t, \phi) \in [0, T] \times \mathbb{A}$ be fixed and $(W(t))_{t \geq 0}$ be a d'–dimensional standard Brownian motion defined on some complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$. We denote by $\mathbb{G}^t = \left(\mathcal{G}_s^t\right)_{s \in [0, T]}$ the natural filtration generated by $\left((W(s) - W(t))\mathbb{1}_{\{s \geq t\}}\right)_{s \in [0, T]}$, augmented by $\mathcal{N}$, the set of $\mathbb{P}$–null events of $\mathcal{G}$.

Let us take $L \geq 0$ and $t < T$. We denote by $\mathcal{U}_t^L$ the space of $\mathbb{G}^t$–progressively measurable $\mathbb{R}^d$–valued processes λ such that $|\lambda| \leq L$. We define a new probability measure $\mathbb{P}^{t, \lambda}$ by $d\mathbb{P}^{t, \lambda} := M^{t, \lambda}(T)\, d\mathbb{P}$, where

$$M^{t, \lambda}(s) := \exp\left(\int_t^s \lambda(r)\, dW(r) - \frac{1}{2} \int_t^s |\lambda(r)|^2 dr \right), \quad \mathbb{P}\text{–a.s..}$$

Under some suitable assumptions on the coefficients, to be specified later on (see Theorem 4), there exists and is unique a continuous and adapted stochastic process $\left(X^{t, \phi}(s)\right)_{s \in [0, T]}$ such that, for given $(t, \phi) \in [0, T] \times \mathbb{A}$,

$$
\begin{cases}
X^{t, \phi}(s) = \phi(t) + \displaystyle\int_t^s b(r, X^{t, \phi})dr + \int_t^s \sigma(r, X^{t, \phi})dW(r), \quad s \in [t, T], \\[2mm]
X^{t, \phi}(s) = \phi(s), \quad s \in [0, t).
\end{cases}
$$

We are now ready to define the space of the test functions,

$$\underline{\mathcal{A}}^L u(t, \phi) := \Big\{ \varphi \in \mathcal{C}_b^{1,2}([0, T] \times \mathbb{A}) : \exists \tau_0 \in \mathcal{T}_+^t,$$

$$\varphi(t, \phi) - u(t, \phi) = \min_{\tau \in \mathcal{T}^t} \underline{\mathcal{E}}_t^L\left[(\varphi - u)\left(\tau \wedge \tau_0, X^{t, \phi}\right)\right] \Big\}$$

and

$$\overline{\mathcal{A}}^L u(t, \phi) := \Big\{ \varphi \in \mathcal{C}_b^{1,2}([0, T] \times \mathbb{A}) : \exists \tau_0 \in \mathcal{T}_+^t,$$

$$\varphi(t, \phi) - u(t, \phi) = \max_{\tau \in \mathcal{T}^t} \overline{\mathcal{E}}_t^L\left[(\varphi - u)\left(\tau \wedge \tau_0, X^{t, \phi}\right)\right] \Big\},$$

where $\mathcal{T}_+^t := \{\tau \in \mathcal{T}^t : \tau > t\}$, if $t < T$ and $\mathcal{T}_+^T := \{T\}$. Also, for any $\xi \in L^2\left(\mathcal{F}_T^t; \mathbb{P}\right)$,

$$\underline{\mathcal{E}}_t^L(\xi) := \inf_{\lambda \in \mathcal{U}_t^L} \mathbb{E}^{\mathbb{P}^{t, \lambda}}(\xi) \quad \text{and} \quad \overline{\mathcal{E}}_t^L(\xi) := \sup_{\lambda \in \mathcal{U}_t^L} \mathbb{E}^{\mathbb{P}^{t, \lambda}}(\xi)$$

are nonlinear expectations.

We are now able to give the definition of a viscosity solution of the functional PDE (1), see, e.g. [16, Definition 3.3].

Definition 2 *Let $u \in \mathcal{C}_b([0, T] \times \mathbb{A})$ be such that $u(T, \phi) = h(\phi)$, for all $\phi \in \mathbb{A}$.*
(a) For any $L \geq 0$, we say that u is a viscosity L–subsolution of (1) if at any point $(t, \phi) \in [0, T] \times \mathbb{A}$, for any $\varphi \in \underline{\mathcal{A}}^L u(t, \phi)$, it holds

$$-\partial_t \varphi(t, \phi) - \mathcal{L}\varphi(t, \phi) - f(t, \phi, u(t, \phi), \partial_x \varphi(t, \phi)\, \sigma(t, \phi), (u(\cdot, \phi))_t) \leq 0.$$

(b) For any $L \geq 0$, we say that u is a viscosity L–supersolution of (1) if at any point $(t, \phi) \in [0, T] \times \mathbb{A}$, for any $\varphi \in \overline{\mathcal{A}}^L u(t, \phi)$, we have

$$-\partial_t \varphi(t, \phi) - \mathcal{L}\varphi(t, \phi) - f(t, \phi, u(t, \phi), \partial_x \varphi(t, \phi)\, \sigma(t, \phi), (u(\cdot, \phi))_t) \geq 0.$$

(*c*) *We say that* u *is a viscosity subsolution (respectively, supersolution) of* (1) *if* u *is a viscosity L–subsolution (respectively, L–supersolution) of* (1) *for some* $L \geq 0$.

(*d*) *We say that* u *is a viscosity solution of* (1) *if* u *is a viscosity subsolution and supersolution of* (1).

Remark 3 *It is easy to see that this definition is equivalent to the classical one in the Markovian framework, see, e.g. [16].*

Let us stress that if u is a function from $\mathcal{C}_b^{1,2}([0,T] \times \mathbb{A})$, then it is easily seen that u is a viscosity solution of (1) if and only if u is a classical solution for (1). Indeed, if u is a viscosity solution then $u \in \underline{\mathcal{A}}^L u(t,\phi) \cap \overline{\mathcal{A}}^L u(t,\phi)$ and therefore u satisfies (1). For the reverse statement one can use the nonlinear Feynman–Kac formula proved in this new framework (see Theorem 21 below), together with functional Itô's formula in order to compute $u\left(s, X^{t,\phi}\right)$ and the existence and uniqueness result for the corresponding stochastic system (2).

Let us also mention that, in accordance with the standard theory of viscosity solutions, the viscosity property introduced above is a local property, i.e. to check that u is a viscosity solution in (t,ϕ) it is sufficient to know the value of u on the interval $[t, \tau_\epsilon]$, where $\epsilon > 0$ is arbitrarily fixed and $\tau_\epsilon \in \mathcal{T}_+^t$ is given by $\tau_\epsilon := \inf\{s > t : |\phi(s)| \geq \epsilon\} \wedge (t + \epsilon)$.

Eventually, let us notice that since b and σ are Lipschitz, we have uniqueness in law for $X^{t,\phi}$; also, since the filtration on $(\Omega, \mathcal{G}, \mathbb{P})$ is generated by W, every progressively measurable processes λ is a functional of W. Therefore, the spaces of test functions and the above definition are independent on the choice of $(\Omega, \mathcal{G}, \mathbb{P})$ and W.

3 The forward–backward delayed system

We are now able to state the existence and uniqueness results for a delayed forward-backward system, where both the forward and the backward component exhibit a delayed behaviour, that is we will assume that the generator of the backward equation may depend on past values assumed by its solution (Y, Z). In complete generality, since we will need these results in next sections, we will allow the solution to depend on a general initial time and initial values. Also we remark that in order to ensure the existence and uniqueness result, we need to equip the backward equation with a suitable condition in the time interval $[0, t)$, t being the initial time, fact that implies a different proof than the one provided in [13].

The main goal is to find a family $\left(X^{t,\phi}, Y^{t,\phi}, Z^{t,\phi}\right)_{(t,\phi)\in[0,T]\times\mathbb{A}}$ of stochastic processes such that the following decoupled forward–backward system holds $\mathbb{P}$–a.s.

$$
\begin{cases}
X^{t,\phi}(s) = \phi(t) + \displaystyle\int_t^s b(r, X^{t,\phi})dr + \int_t^s \sigma(r, X^{t,\phi})dW(r), \ s \in [t, T], \\[2mm]
X^{t,\phi}(s) = \phi(s), \quad s \in [0, t), \\[2mm]
Y^{t,\phi}(s) = h(X^{t,\phi}) + \displaystyle\int_s^T F(r, X^{t,\phi}, Y^{t,\phi}(r), Z^{t,\phi}(r), Y_r^{t,\phi}, Z_r^{t,\phi})dr \\[3mm]
\hphantom{Y^{t,\phi}(s) =} - \displaystyle\int_s^T Z^{t,\phi}(r)\,dW(r), \quad s \in [t, T], \\[2mm]
Y^{t,\phi}(s) = Y^{s,\phi}(s), \quad Z^{t,\phi}(s) = 0, \quad s \in [0, t).
\end{cases}
\tag{8}
$$

Let us stress once more that in both the forward and backward equation, the values of $X^{t,\phi}$ and $\left(Y^{t,\phi}, Z^{t,\phi}\right)$ need to be known in the time interval $[0,t]$ and respectively $[t-\delta, t]$; this is one reason for which we have to impose such initial conditions. The above initial condition for Y is absolutely necessary in view of the Feynman–Kac formula, which will be proven later. We also prolong, by convention, $Y^{t,\phi}$ by $Y^{t,\phi}(0)$ on the negative real axis (this is needed in the case that $t < \delta$). For the sake of simplicity, we will take $Z^{t,\phi}(s) := 0$ and $F(s, \cdot, \cdot, \cdot, \cdot, \cdot) := 0$ whenever s becomes negative.

3.1 The forward path-dependent SDE

Let us first focus on the forward component X appearing in system (8); the next theorem states the existence and the uniqueness, as well as estimates, for the process $\left(X^{t,\phi}(r)\right)_{r \in [0,T]}$.

The existence result is a classical one (see, e.g. [28] or [29]) and the estimates can be obtained by applying Itô's formula together with assumptions (A_1)–(A_2), see, e.g. [41], and for these reasons we will not state the proof.

In what follows we will assume the following to hold.

Let us consider two non-anticipative functionals $b : [0,T] \times \mathbb{A} \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{A} \to \mathbb{R}^{d \times d'}$ such that

(A_1) b and σ are continuous;

(A_2) there exists $\ell > 0$ such that for any $t \in [0,T]$, $\phi, \phi' \in \mathbb{A}$,

$$|b(t,\phi) - b(t,\phi')| + |\sigma(t,\phi) - \sigma(t,\phi')| \le \ell ||\phi - \phi'||_T.$$

Theorem 4 *Let b, σ satisfying assumptions (A_1)–(A_2). Let $(t,\phi), (t',\phi') \in [0,T] \times \mathbb{A}$ be given. Then there exists a unique continuous and adapted stochastic process $\left(X^{t,\phi}(s)\right)_{s \in [0,T]}$ such that*

$$\begin{cases} X^{t,\phi}(s) = \phi(t) + \int_t^s b(r, X^{t,\phi})dr + \int_t^s \sigma(r, X^{t,\phi})dW(r), \ s \in [t,T], \\ X^{t,\phi}(s) = \phi(s), \ s \in [0,t). \end{cases} \tag{9}$$

Moreover, for any $q \ge 1$, there exists $C = C(q, T, \ell) > 0$ such that

$$\mathbb{E}\left(||X^{t,\phi}||_T^{2q}\right) \le C(1 + ||\phi||_T^{2q}),$$

$$\mathbb{E}\left(||X^{t,\phi} - X^{t',\phi'}||_T^{2q}\right) \le C\left(||\phi - \phi'||_T^{2q} + (1 + ||\phi||_T^{2q} + ||\phi'||_T^{2q}) \cdot |t - t'|^q \right.$$

$$\left. + \sup_{r \in [t \wedge t', t \vee t']} |\phi(t) - \phi(r)|^{2q}\right),$$

$$\mathbb{E}\left(\sup_{\substack{s,r \in [t,T] \\ |s-r| \le \epsilon}} |X^{t,\phi}(s) - X^{t,\phi}(r)|^{2q}\right) \le C(1 + ||\phi||_T^{2q})\epsilon^{q-1}, \quad \textit{for all } \epsilon > 0.$$

3.2 The backward delayed SDE

Let us now consider the delayed backward SDE appearing in (8). In what follows, d and d' are previously fixed constants, whereas $m \in \mathbb{N}^*$ is a new fixed constant. Let us then introduce the main reference spaces we will consider.

10

Definition 5

(i) *let $\mathcal{H}_t^{2,m\times d'}$ denote the space of (equivalence classes of) $\mathbb{G}^t$–progressively measurable processes*
$$Z : \Omega \times [0,T] \to \mathbb{R}^{m\times d'} \text{ such that } \mathbb{E}\left[\int_0^T |Z(s)|^2 ds\right] < \infty\,;$$

(ii) *let $\mathcal{S}_t^{2,m}$ the space of (equivalence classes of) continuous $\mathbb{G}^t$–progressively measurable processes*
$$Y : \Omega \times [0,T] \to \mathbb{R}^m \text{ such that } \mathbb{E}\left[\sup_{0\leq s\leq T} |Y(s)|^2\right] < \infty\,.$$

Also we will equip the spaces $\mathcal{H}_t^{2,m\times d'}$ and $\mathcal{S}_t^{2,m}$ with the following norms
$$\|Z\|_{\mathcal{H}_t^{2,m\times d'}}^2 = \mathbb{E}\left[\int_0^T e^{\beta s}|Z(s)|^2 ds\right]\,, \qquad \|Y\|_{\mathcal{S}_t^{2,m}}^2 = \mathbb{E}\left[\sup_{0\leq s\leq T} e^{\beta s}|Y(s)|^2\right]\,,$$

for a given constant $\beta > 0$.

Concerning the delayed backward SDE in (8), we will assume the following to hold.

Let $F : [0,T]\times\mathbb{A}\times\mathbb{R}^m\times\mathbb{R}^{m\times d'}\times L^2\left([-\delta,0];\mathbb{R}^m\right)\times L^2([-\delta,0];\mathbb{R}^{m\times d'}) \to \mathbb{R}^m$ and $h : \mathbb{A} \to \mathbb{R}^m$ such that the following holds:

(A_3) There exist $L, K, M > 0$, $p \geq 1$ and a probability measure α on $([-\delta,0],\mathcal{B}\left([-\delta,0]\right))$ such that, for any $t \in [0,T]$, $\phi \in \mathbb{A}$, $(y,z), (y',z') \in \mathbb{R}^m \times \mathbb{R}^{m\times d'}$, $\hat{y},\hat{y}' \in L^2\left([-\delta,0];\mathbb{R}^m\right)$ and $\hat{z},\hat{z}' \in L^2([-\delta,0];\mathbb{R}^{m\times d'})$, we have

(i) $\quad \phi \mapsto F(t,\phi,y,z,\hat{y},\hat{z})$ is continuous,

(ii) $\quad |F(t,\phi,y,z,\hat{y},\hat{z}) - F(t,\phi,y',z',\hat{y},\hat{z})| \leq L(|y-y'| + |z-z'|)$,

(iii) $\quad |F(t,\phi,y,z,\hat{y},\hat{z}) - F(t,\phi,y,z,\hat{y}',\hat{z}')|^2$
$$\leq K\int_{-\delta}^0 \left(\left|\hat{y}(\theta) - \hat{y}'(\theta)\right|^2 + \left|\hat{z}(\theta) - \hat{z}'(\theta)\right|^2\right)\alpha(d\theta),$$

(iv) $\quad |F\left(t,\phi,0,0,0,0\right)| < M(1 + \|\phi\|_T^p)$.

(A_4) The function $F\left(\cdot,\cdot,y,z,\hat{y},\hat{z}\right)$ is $\mathbb{F}$–progressively measurable, for any $(y,z,\hat{y},\hat{z}) \in \mathbb{R}^m \times \mathbb{R}^{m\times d'} \times L^2\left([-\delta,0];\mathbb{R}^m\right) \times L^2([-\delta,0];\mathbb{R}^{m\times d'})$.

(A_5) The function h is continuous and $|h(\phi)| \leq M(1 + \|\phi\|_T^p)$, for all $\phi \in \mathbb{A}$.

Remark 6 *In order to show the existence and uniqueness of a solution to the backward part of system (8) and to obtain the continuity of $Y^{t,\phi}$ with respect to ϕ we are forced to impose that K or δ are small enough.*

More precisely, we will assume that there exists a constant $\gamma \in (0,1)$ such that

$$K\,\frac{\gamma e^{\left(\gamma+\frac{6L^2}{\gamma}\right)\delta}}{(1-\gamma)L^2}\,\max\{1,T\} < \frac{1}{290}\,. \tag{10}$$

We are now ready to state the first result of the this section.

Theorem 7 *Let us assume that assumptions* (A_3)–(A_5) *hold true. If condition (10) is satisfied, then there exists a unique solution* $(Y^{t,\phi}, Z^{t,\phi})_{(t,\phi)\in[0,T]\times\mathbb{A}}$ *for the backward stochastic differential system from (8) such that* $(Y^{t,\phi}, Z^{t,\phi}) \in \mathcal{S}_t^{2,m} \times \mathcal{H}_t^{2,m\times d'}$, *for all* $t \in [0,T]$ *and the application* $t \mapsto (Y^{t,\phi}, Z^{t,\phi})$ *is continuous from* $[0,T]$ *into* $\mathcal{S}_0^{2,m} \times \mathcal{H}_0^{2,m\times d'}$.

Remark 8 *The previous theorem provides only the regularity of* $t \mapsto (Y^{t,\phi}, Z^{t,\phi})$. *In what concerns the continuity of* $(Y^{t,\phi}, Z^{t,\phi})$ *with respect to* ϕ, *see the proof of Theorem 16.*

Remark 9 *Theorem 7 states that if* K *or* δ *are small enough such that condition (10) is satisfied, then there exists a unique solution.*

More precisely, if δ *is fixed, then we can find* K *sufficiently small such that restriction (10) is satisfied.*

On the other hand, if K *is fixed it seems, at first sight, that restriction (10) cannot be made true by letting* δ *to 0; but in this case it is sufficient to take* $\tilde{L} \geq L$ *large enough for which there exists a small enough* $\delta > 0$ *such that condition (10) with* L *replaced by* $\tilde{L}$ *is still satisfied. Of course, assumption* $(A_3 - ii)$ *is still verified with* $\tilde{L}$ *instead* L.

Remark 10 *The proof is mainly based on Banach fixed point theorem; we provide it in the Appendix section.*

The main difference between our result and Theorem 2.1 from [13] is due to the supplementary structure condition $Y^{t,\phi}(s) = Y^{s,\phi}(s)$, *for* $s \in [0,t)$ *which should be satisfied by the unknown process* $Y^{t,\phi}$.

We also allow T *to be arbitrary and we consider that the time horizon is different from the delay* $\delta \in [0,T]$; *moreover, we make the difference between the Lipschitz constant* L *with respect to* (y,z) *and the Lipschitz constant* K *with respect to* $\hat{y}$ *and* $\hat{z}$, *and hence in restriction (10) we can play also with the constant* K.

Remark 11 *Using Itô's formula and proceeding as in the proof of Theorem 7, we can easily show that the solution* $(Y^{t,\phi}, Z^{t,\phi})$ *to equation (8) satisfies the following inequality. For any* $q \geq 1$, *there exists* $C > 0$ *such that*

$$
\mathbb{E}\Big(\sup_{r\in[0,T]} |Y^{t,\phi}(r)|^{2q} \Big) + \mathbb{E}\Big(\int_0^T |Z^{t,\phi}(r)|^2 dr \Big)^q
$$
$$
\leq C\Big[\mathbb{E}|h(X^{t,\phi})|^{2q} + \mathbb{E}\Big(\int_0^T |F(r, X^{t,\phi}, 0, 0, 0, 0)| dr \Big)^{2q} \Big] \leq C(1 + ||\phi||_T^{2q}).
$$

(11)

4 Path–dependent PDE – proof of the existence theorem

The current section is devoted to the study of viscosity solution to the path-dependent equation (1). In particular, in order to obtain existence of a viscosity solution, we will impose some additional assumptions on the generator f and on the terminal condition h in equation (8), in particular we will assume that the generator f depends only on past values assumed by Y and not by past values of Z. In what follows we will assume the following to hold.

Let $f : [0,T] \times \mathbb{A} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{C}([-\delta,0]; \mathbb{R}) \to \mathbb{R}$ and $h : \mathbb{A} \to \mathbb{R}$ such that the following hold.

(A_6) the functions f and h are continuous; also $f(\cdot, \cdot, y, z, \hat{y})$ is non-anticipative;

(A_7) there exist $L, K, M > 0$ and $p \geq 1$ such that for any $(t, \phi) \in [0, T] \times \mathbb{A}$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$ and $\hat{y}, \hat{y}' \in \mathcal{C}([-\delta, 0]; \mathbb{R})$:

$$(i) \quad |f(t, \phi, y, z, \hat{y}) - f(t, \phi, y', z', \hat{y})| \leq L(|y - y'| + |z - z'|),$$

$$(ii) \quad |f(t, \phi, y, z, \hat{y}) - f(t, \phi, y, z, \hat{y}')|^2 \leq K \int_{-\delta}^0 |\hat{y}(\theta) - \hat{y}'(\theta)|^2 \alpha(d\theta),$$

$$(iii) \quad |f(t, \phi, 0, 0, 0)| \leq M(1 + \|\phi\|_T^p),$$

$$(iv) \quad |h(\phi)| \leq M(1 + \|\phi\|_T^p),$$

with α a probability measure on $([-\delta, 0], \mathcal{B}([-\delta, 0]))$.

Remark 12 *For example, the following generators satisfy assumptions (A_6), (A_7):*

$$f_1(t, \phi, y, z, \hat{y}) := K \int_{-\delta}^0 \hat{y}(s)\, ds, \qquad f_2(t, \phi, y, z, \hat{y}) := K\, \hat{y}(t - \delta).$$

In general, if $g : [0, T] \to \mathbb{R}$ is a measurable, bounded function with $g(t) = 0$ for $t < 0$, then the following linear time–delayed generator

$$f(t, \phi, y, z, \hat{y}) = \int_{-\delta}^0 g(t + \theta)\, \hat{y}(\theta)\, \alpha(d\theta),$$

satisfies assumptions (A_6), (A_7).

We state now the main result of the present paper.

Theorem 13 (Existence) *Let us assume that assumptions (A_1), (A_2), (A_6), (A_7) hold. If the delay δ or the Lipschitz constant K are sufficiently small, i.e. condition (10) is verified, then the path–dependent PDE (1) admits at least one viscosity solution.*

Remark 14 *The new and the essential part of the proof of Theorem 13 is the nonlinear representation Feynman–Kac type formula, which links the functional SDE (9) to a suitable BSDE with time–delayed generators. We prove Feynman–Kac type formula first in the case of BSDE independent of the past of Y, see Theorem 18, and after that in the general case of BSDE (15), see Theorem 21.*

We also mention that this theorem is a generalization of the Theorem 4.3 from [16].

Remark 15 (Uniqueness) *The problem of uniqueness is solved* by reducing it to the problem of uniqueness of the viscosity solution from the case of f being independent of the term $(u(\cdot, \phi))_t$. This particular case is presented in Theorem 4.6 from [16].*

Let us take two viscosity solutions u^1 and u^2 of the path–dependent PDE (1). We define

$$f^i(t, \phi, y, z) := f(t, \phi, y, z, (u^i(\cdot, \phi))_t), \quad i = \overline{1, 2}.$$

---

*We are thankful to the Reviewer for his/her solution in what concerns the uniqueness problem.

Using these drivers we associate the following BSDEs:

$$Y^{t,\phi}(s) = h(X^{t,\phi}) + \int_s^T f^i(r, X^{t,\phi}, Y^{t,\phi}(r), Z^{t,\phi}(r))dr - \int_s^T Z^{t,\phi}(r)\, dW(r), \quad i = \overline{1,2}, \quad (12)$$

for which there exist unique solutions $\left(Y^{i,t,\phi}, Z^{i,t,\phi}\right) \in \mathcal{S}_t^{2,1} \times \mathcal{H}_t^{2,1 \times d'}, \quad i = \overline{1,2}$.
    Using Theorem 18 we see that

$$Y^{i,t,\phi}(s) = v^i(s, X^{t,\phi}), \quad \text{for all } s \in [0,T], \quad a.s.,$$

for any $(t,\phi) \in [0,T] \times \mathbb{A}$, where $v^i : [0,T] \times \mathbb{A} \to \mathbb{R}$ are defined by (16), i.e.

$$v^i(t,\phi) := Y^{i,t,\phi}(t), \quad (t,\phi) \in [0,T] \times \mathbb{A}.$$

Hence, using Theorem 13, we obtain that the functions v^i are solutions of the PDE of type (1), but without the delayed terms $\left(v^i(\cdot,\phi)\right)_t$:

$$\begin{cases} -\partial_t v^i(t,\phi) - \mathcal{L}v^i(t,\phi) - f^i(t,\phi,v^i(t,\phi),\partial_x v^i(t,\phi)\,\sigma(t,\phi)) = 0, \\ v^i(T,\phi) = h(\phi), \quad i = \overline{1,2}. \end{cases} \quad (13)$$

Since u^i is also solution to equation (13), by using Theorem 4.6 from [16] we obtain

$$u^i(t,\phi) = v^i(t,\phi), \quad (t,\phi) \in [0,T] \times \mathbb{A}, \quad i = \overline{1,2}.$$

We mention that Theorem 4.6 from [16] is proved in the particular case $b = 0, \sigma = I$, but we can adapt the proof to our case without difficulty. However, some stronger assumptions on f and h (boundedness and uniform continuity w.r.t. ϕ) are required and we will have to adopt them in order to use this result.
    In this case, since

$$Y^{i,t,\phi}(s) = v^i(s, X^{t,\phi}) = u^i(s, X^{t,\phi}), \ i = \overline{1,2},$$

BSDEs (12) become a single equation,

$$Y^{i,t,\phi}(s) = h(X^{t,\phi}) + \int_s^T f(r, X^{t,\phi}, Y^{i,t,\phi}(r), Z^{i,t,\phi}(r), Y_r^{i,t,\phi})dr - \int_s^T Z^{i,t,\phi}(r)\, dW(r), \quad (14)$$

with $i = \overline{1,2}$, for which we have uniqueness (see Theorem 7).
    Hence $Y^{1,t,\phi} = Y^{2,t,\phi}$ and therefore

$$u^1(t,\phi) = Y^{1,t,\phi}(t) = Y^{2,t,\phi}(t) = u^2(t,\phi).$$

Under assumptions $(A_1),(A_2),(A_6),(A_7)$, it follows from Theorem 7 (in the case $m = 1$) that for each $(t,\phi) \in [0,T] \times \mathbb{A}$ there exists a unique triple $\left(X^{t,\phi}, Y^{t,\phi}, Z^{t,\phi}\right)$ of $\mathbb{G}^t$–progressively measurable processes such that $X^{t,\phi}$ satisfies equation (9) and $\left(Y^{t,\phi}, Z^{t,\phi}\right) \in \mathcal{S}_t^{2,1} \times \mathcal{H}_t^{2,1 \times d'}$, with $Y^{t,\phi}(s) = Y^{s,\phi}(s)$, for any $s \in [0,t)$, is a solution of the following BSDE on $[t,T]$:

$$Y^{t,\phi}(s) = h(X^{t,\phi}) + \int_s^T f(r, X^{t,\phi}, Y^{t,\phi}(r), Z^{t,\phi}(r), Y_r^{t,\phi})dr - \int_s^T Z^{t,\phi}(r)\, dW(r). \quad (15)$$

Let us further observe that the generator f depends on ω only via the the process $X^{t,x}$.

Before proving Theorem 13, we need to show some results. For that, let us first define the function $u : [0,T] \times \mathbb{A} \to \mathbb{R}$ by

$$u(t,\phi) := Y^{t,\phi}(t), \; (t,\phi) \in [0,T] \times \mathbb{A}; \tag{16}$$

notice that $u(t,\phi)$ is a deterministic function since $Y^{t,\phi}(t)$ is $\mathcal{G}_t^t \equiv \mathcal{N}$–measurable.

Theorem 16 *Under the assumptions of Theorem 13, the function u is continuous.*

Remark 17 *As in the case of Theorem 7, the proof can be found in the Appendix section.*

In order to prove the generalized Feynman–Kac formula suitable to our framework we first consider the particular case when the generator f is independent of the past values of Y and Z, namely $(Y^{t,\phi}, Z^{t,\phi})$ is the solution of the standard BSDE with Lipschitz coefficients

$$Y^{t,\phi}(s) = h(X^{t,\phi}) + \int_s^T f(r, X^{t,\phi}, Y^{t,\phi}(r), Z^{t,\phi}(r))dr - \int_s^T Z^{t,\phi}(r)\, dW(r), \; s \in [t,T]. \tag{17}$$

Theorem 18 *Let us assume that $(\mathrm{A}_1),(\mathrm{A}_2),(\mathrm{A}_6),(\mathrm{A}_7)$ hold. Then*

$$Y^{t,\phi}(s) = u(s, X^{t,\phi}), \quad \text{for all } s \in [0,T], \quad a.s.,$$

for any $(t,\phi) \in [0,T] \times \mathbb{A}$, where $Y^{t,\phi}$ is the solution of BSDE (17) and $u : [0,T] \times \mathbb{A} \to \mathbb{R}$ is defined by (16).

Remark 19 *We mention that this kind of result was previously proven only in a particular case by Peng, Wang [38] who considered the case of smooth coefficients h and f (they also took $b = 0, \sigma = I$).*

Remark 20 *We also mention that Theorem 4.3 from [16] uses the essential Feynman–Kac formula*

$$Y_s^0 = u\left(s, B^t\right)$$

but without proving it; a similar situation can be found in [17]. Our result fills this gap.

Proof. Again, for the sake of readability, we split the proof into several steps.
Step I.
Let $0 = t_0 < t_1 < \cdots < t_n = T$ and suppose that

$$b(t,\phi) = b_1(t,\phi(t))\mathbb{1}_{[0,t_1)}(t) + b_2(t,\phi(t_1),\phi(t)-\phi(t_1))\mathbb{1}_{[t_1,t_2)}(t) + \dots$$
$$+ b_n(t,\phi(t_1),\phi(t_2)-\phi(t_1),\dots,\phi(t)-\phi(t_{n-1}))\mathbb{1}_{[t_{n-1},T]}(t),$$

$$\sigma(t,\phi) = \sigma_1(t,\phi(t))\mathbb{1}_{[0,t_1)}(t) + \sigma_2(t,\phi(t_1),\phi(t)-\phi(t_1))\mathbb{1}_{[t_1,t_2)}(t) + \dots$$
$$+ \sigma_n(t,\phi(t_1),\phi(t_2)-\phi(t_1),\dots,\phi(t)-\phi(t_{n-1}))\mathbb{1}_{[t_{n-1},T]}(t)$$

and

$$f(t,\phi,y,z) = f_1(t,\phi(t),y,z)\mathbb{1}_{[0,t_1)}(t) + f_2(t,\phi(t_1),\phi(t)-\phi(t_1),y,z)\mathbb{1}_{[t_1,t_2)}(t) + \dots$$
$$+ f_n(t,\phi(t_1),\phi(t_2)-\phi(t_1),\dots,\phi(t)-\phi(t_{n-1}))\mathbb{1}_{[t_{n-1},T]}(t),$$

$$h(\phi) = \varphi(\phi(t_1),\phi(t_2)-\phi(t_1),\dots,\phi(T)-\phi(t_{n-1})),$$

for every $\phi \in \mathbb{A}$.

Let us first show that the terms $(X^{t,\phi}(t_1), \ldots, X^{t,\phi}(r) - X^{t,\phi}(t_k))$, $0 \leq k \leq n-1$ are related to the solution of a SDE of Itô type in $\mathbb{R}^{n \times d}$. Let

$$\tilde{b}(t, x_1, \ldots, x_n) := \begin{pmatrix} b_1(t, x_1) \mathbb{1}_{[0, t_1)}(t) \\ b_2(t, x_1, x_2) \mathbb{1}_{[t_1, t_2)}(t) \\ \vdots \\ b_n(t, x_1, \ldots, x_n) \mathbb{1}_{[t_{n-1}, t_n)}(t) \end{pmatrix},$$

and

$$\tilde{\sigma}(t, x_1, \ldots, x_n) := \begin{pmatrix} \sigma_1(t, x_1) \mathbb{1}_{[0, t_1)}(t) \\ \sigma_2(t, x_1, x_2) \mathbb{1}_{[t_1, t_2)}(t) \\ \vdots \\ \sigma_n(t, x_1, \ldots, x_n) \mathbb{1}_{[t_{n-1}, t_n)}(t) \end{pmatrix}.$$

Let then $\tilde{X}^{t,\boldsymbol{x}}$, with $t \in [0, T]$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^{d \times n}$ be the unique solution of the following stochastic differential equation:

$$\tilde{X}^{t,\boldsymbol{x}}(s) = \boldsymbol{x} + \int_t^s \tilde{b}(r, \tilde{X}^{t,\boldsymbol{x}}) dr + \int_t^s \tilde{\sigma}(r, \tilde{X}^{t,\boldsymbol{x}}) dW(r), \ s \in [t, T].$$

We assert that, for $t \in [t_{k_0}, t_{k_0+1})$, $s \in [t_k, t_{k+1}]$, $s \geq t$, with $0 \leq k_0 \leq k \leq n-1$, we have

$$(X^{t,\phi}(t_1), \ldots, X^{t,\phi}(s) - X^{t,\phi}(t_k)) = \left(\tilde{X}^{i,t,(\phi(t_1), \ldots, \phi(t) - \phi(t_{k_0}), 0, \ldots, 0)}(s) \right)_{i=\overline{1, k+1}}.$$

Let us stress that for $k = 0$ this reads $X^{t,\phi}(s) = \tilde{X}^{1,t,\phi(t)}(s)$; so that for $k > k_0 = 0$, it is interpreted as $(X^{t,\phi}(t_1), \ldots, X^{t,\phi}(s) - X^{t,\phi}(t_k)) = \left(\tilde{X}^{i,t,(\phi(t), 0, \ldots, 0)}(s) \right)_{i=\overline{1, k+1}}$.

We will prove this statement by induction on k. If $k = 0$, then $k_0 = 0$ and we obviously have

$$X^{t,\phi}(s) = \tilde{X}^{1,t,\phi(t)}(s).$$

Let us suppose that the statement holds true for $k - 1$; for the sake of brevity, in what follows we will denote $\boldsymbol{x} := (\phi(t_1), \ldots, \phi(t) - \phi(t_{k_0}), 0, \ldots, 0)$.

If $k_0 \leq k - 1$ then, from the induction hypothesis, we have

$$(X^{t,\phi}(t_1), \ldots, X^{t,\phi}(r) - X^{t,\phi}(t_{k-1})) = \left(\tilde{X}^{i,t,(\phi(t_1), \ldots, \phi(t) - \phi(t_{k_0}), 0, \ldots, 0)}(r) \right)_{i=\overline{1, k}}$$

for every $r \in [t_{k-1}, t_k]$, so that, for $s \in [t_k, t_{k+1}]$ we have,

$$\tilde{X}^{j,t,\boldsymbol{x}}(s) = \tilde{X}^{j,t,\boldsymbol{x}}(t_k) = X^{t,\phi}(t_j) - X^{t,\phi}(t_{j-1}), \ 1 \leq j \leq k,$$

with the convention $X^{t,\phi}(t_0) = 0$.

In the case $k_0 = k$, for $s \in [t, t_{k+1}]$ we also have:

$$\tilde{X}^{j,t,\boldsymbol{x}}(s) = \boldsymbol{x}^j = \phi(t_j) - \phi(t_{j-1}) = X^{t,\phi}(t_j) - X^{t,\phi}(t_{j-1}), \ 1 \leq j \leq k,$$

again with the convention $\phi(t_0) = 0$.

Consequently, on $[t \vee t_k, t_{k+1}]$ it holds

$$\tilde{X}^{k+1,t,\boldsymbol{x}}(s) = \boldsymbol{x}^{k+1} + \int_{t \vee t_k}^{s} b_{k+1}(r, \tilde{X}^{1,t,\boldsymbol{x}}(r), \ldots, \tilde{X}^{k+1,t,\boldsymbol{x}}(r))dr$$

$$+ \int_{t \vee t_k}^{s} \sigma_{k+1}(r, \tilde{X}^{1,t,\boldsymbol{x}}(r), \ldots, \tilde{X}^{k+1,t,\boldsymbol{x}}(r))dW(r)$$

$$= \boldsymbol{x}^{k+1} + \int_{t \vee t_k}^{s} b_{k+1}(r, X^{t,\phi}(t_1), \ldots, X^{t,\phi}(t_k) - X^{t,\phi}(t_{k-1}), \tilde{X}^{k+1,t,\boldsymbol{x}}(r))dr$$

$$+ \int_{t \vee t_k}^{s} \sigma_{k+1}(r, X^{t,\phi}(t_1), \ldots, X^{t,\phi}(t_k) - X^{t,\phi}(t_{k-1}), \tilde{X}^{k+1,t,\boldsymbol{x}}(r))dW(r).$$

If $k_0 \leq k - 1$ then $\boldsymbol{x}^{k+1} = 0$; if $k_0 = k$, then $\boldsymbol{x}^{k+1} = \phi(t) - \phi(t_{k_0}) = X^{t,\phi}(t) - X^{t,\phi}(t_k)$. By uniqueness, since $X^{t,\phi}$ satisfies

$$X^{t,\phi}(s) = X^{t,\phi}(t \vee t_k) + \int_{t \vee t_k}^{s} b_{k+1}(r, X^{t,\phi}(t_1), \ldots, X^{t,\phi}(r) - X^{t,\phi}(t_k))dr$$

$$+ \int_{t \vee t_k}^{s} \sigma_{k+1}(r, X^{t,\phi}(t_1), \ldots, X^{t,\phi}(r) - X^{t,\phi}(t_k))dW(r), \ s \in [t \vee t_k, t_{k+1}].$$

we obtain $\tilde{X}^{k+1,t,\boldsymbol{x}}(s) = X^{t,\phi}(s) - X^{t,\phi}(t_k)$, for all $s \in [t \vee t_k, t_{k+1}]$. We thus have proved that the statement holds true for k.

The next step is to derive a Feynman–Kac type formula linking $X^{t,\phi}$ and $Y^{t,\phi}$. For that, let us consider the following BSDE:

$$\tilde{Y}^{t,\boldsymbol{x}}(s) = \varphi(\tilde{X}^{t,\boldsymbol{x}}(T)) + \int_t^T \tilde{f}(r, \tilde{X}^{t,\boldsymbol{x}}(r), \tilde{Y}^{t,\boldsymbol{x}}(r), \tilde{Z}^{t,\boldsymbol{x}}(r))dr + \int_t^s \tilde{Z}^{t,\boldsymbol{x}}dW(r), \ s \in [t, T],$$

where $\tilde{f}$ is defined by

$$\tilde{f}(t, x_1, \ldots, x_n, y, z)$$
$$= f_1(t, x_1, y, z)\mathbb{1}_{[0,t_1)}(t) + f_2(t, x_1, x_2, y, z)\mathbb{1}_{[t_1,t_2)}(t) + \cdots + f_n(t, x_1, x_2, \ldots, x_n)\mathbb{1}_{[t_{n-1},T]}(t).$$

From [20], there exist some measurable functions $\tilde{u}, \tilde{d}$ such that for all $(t, \boldsymbol{x}) \in [0, T] \times \mathbb{R}^{d \times n}$ it holds

$$\tilde{Y}^{t,\boldsymbol{x}}(s) = \tilde{u}(s, \tilde{X}^{t,\boldsymbol{x}}(s)), \quad \text{for all } s \in [t, T];$$
$$\tilde{Z}^{t,\boldsymbol{x}}(s) = \tilde{d}(s, \tilde{X}^{t,\boldsymbol{x}}(s))\tilde{\sigma}(s, \tilde{X}^{t,\boldsymbol{x}}), \ ds\text{-a.e. on } [t, T].$$

On the other hand, let $t \in [t_{k_0}, t_{k_0+1})$, with $0 \leq k_0 \leq n - 1$ and denote, for simplicity, $\boldsymbol{x} = (\phi(t_1), \ldots, \phi(t) - \phi(t_{k_0}), 0, \ldots, 0)$.

If $s \in [t_k, t_{k+1}]$, $s \geq t$, and therefore $k \geq k_0$, we have

$$(X^{t,\phi}(t_1), \ldots, X^{t,\phi}(s) - X^{t,\phi}(t_k)) = \left(\tilde{X}^{i,t,\boldsymbol{x}}(s)\right)_{i=\overline{1,k+1}}$$

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Thus, on $[t \vee t_k, t_{k+1}]$

$$\tilde{f}(s, \tilde{X}^{t,\boldsymbol{x}}(s), \tilde{Y}^{t,\boldsymbol{x}}(s), \tilde{Z}^{t,\boldsymbol{x}}(s)) = f_{k+1}(s, X^{t,\phi}(t_1), \dots, X^{t,\phi}(s) - X^{t,\phi}(t_k), \tilde{Y}^{t,\boldsymbol{x}}(s), \tilde{Z}^{t,\boldsymbol{x}}(s))$$
$$= f(s, X^{t,\phi}, \tilde{Y}^{t,\boldsymbol{x}}(s), \tilde{Z}^{t,\boldsymbol{x}}(s)).$$

Allowing k to vary, we obtain the equality

$$\tilde{f}(s, \tilde{X}^{t,\boldsymbol{x}}(s), \tilde{Y}^{t,\boldsymbol{x}}(s), \tilde{Z}^{t,\boldsymbol{x}}(s)) = f(s, X^{t,\phi}, \tilde{Y}^{t,\boldsymbol{x}}(s), \tilde{Z}^{t,\boldsymbol{x}}(s)), \quad \text{for all } s \in [t, T].$$

Also,

$$\varphi(\tilde{X}^{t,\boldsymbol{x}}(T)) = \varphi(X^{t,\phi}(t_1), \dots, X^{t,\phi}(T) - X^{t,\phi}(t_{n-1})) = h(X^{t,\phi}),$$

and by the uniqueness of the solution of the BSDE, we get that

$$(\tilde{Y}^{t,\boldsymbol{x}}, \tilde{Z}^{t,\boldsymbol{x}}) = (Y^{t,\phi}, Z^{t,\phi})$$

and, consequently,

$$Y^{t,\phi}(s) = \tilde{u}(s, X^{t,\phi}(t_1), \dots, X^{t,\phi}(s) - X^{t,\phi}(t_k), 0, \dots, 0), \quad \text{for all } s \in [t \vee t_k, t_{k+1}],$$
$$Z^{t,\phi}(s) = \tilde{d}_{k+1}(s, X^{t,\phi}(t_1), \dots, X^{t,\phi}(s) - X^{t,\phi}(t_k), 0, \dots, 0) \cdot$$
$$\cdot \sigma_{k+1}(s, X^{t,\phi}(t_1), \dots, X^{t,\phi}(s) - X^{t,\phi}(t_k)), \ ds\text{-a.e. on } [t \vee t_k, t_{k+1}].$$

By setting, for $0 \leq k \leq n-1$, $t \in [t_k, t_{k+1})$ and $\phi \in \mathbb{A}$,

$$u(t, \phi) := \tilde{u}(t, \phi(t_1), \dots, \phi(t) - \phi(t_k), 0, \dots, 0);$$
$$d(t, \phi) := \tilde{d}_{k+1}(t, \phi(t_1), \dots, \phi(t) - \phi(t_k), 0, \dots, 0),$$

we get that, for every $t \in [0, T]$ and $\phi \in \mathbb{A}$,

$$Y^{t,\phi}(s) = u(s, X^{t,\phi}), \quad \text{for all } s \in [t, T];$$
$$Z^{t,\phi}(s) = d(s, X^{t,\phi})\sigma(t, X^{t,\phi}), \ ds\text{-a.e. on } [t, T].$$

Step II.
   Let us notice that the same conclusion holds for b, σ and f of the form

$$b(t, \phi) = b_1(t, \phi(t))\mathbf{1}_{[0,t_1)}(t) + b_2(t, \phi(t_1), \phi(t))\mathbf{1}_{[t_1,t_2)}(t) + \dots$$
$$+ b_n(t, \phi(t_1), \dots, \phi(t_{n-1}), \phi(t))\mathbf{1}_{[t_{n-1},T]}(t),$$

$$\sigma(t, \phi) = \sigma_1(t, \phi(t))\mathbf{1}_{[0,t_1)}(t) + \sigma_2(t, \phi(t_1), \phi(t))\mathbf{1}_{[t_1,t_2)}(t) + \dots$$
$$+ \sigma_n(t, \phi(t_1), \dots, \phi(t_{n-1}), \phi(t))\mathbf{1}_{[t_{n-1},T]}(t)$$

and

$$f(t, \phi, y, z) = f_1(t, \phi(t), y, z)\mathbf{1}_{[0,t_1)}(t) + f_2(t, \phi(t_1), \phi(t), y, z)\mathbf{1}_{[t_1,t_2)}(t) + \dots$$
$$+ f_n(t, \phi(t_1), \dots, \phi(t_{n-1}), \phi(t))\mathbf{1}_{[t_{n-1},T]}(t),$$

$$h(\phi) = \varphi(\phi(t_1), \dots, \phi(t_{n-1}), \phi(T)),$$

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for every $\phi \in \mathbb{A}$.

Step III.

For $0 = t_0 \leq t_1 < \cdots < t_k \leq T$ and $x_1, \ldots, x_k \in \mathbb{R}^d$, let $\Phi^{x_1,\ldots,x_k}_{t_1,\ldots,t_k} : [0, T] \to \mathbb{R}^d$ be such that

$$\Phi^{x_1,\ldots,x_k}_{t_1,\ldots,t_k}(t_i) = x_i, \ i = \overline{1, k}; \quad \Phi^{x_1,\ldots,x_k}_{t_1,\ldots,t_k}(T) = x_k, \quad \Phi^{x_1,\ldots,x_k}_{t_1,\ldots,t_k}(0) = x_1$$

and is prolonged to $[0, T]$ by linear interpolation.

Let us consider partitions of $[0, T]$, $0 = t_0^n < t_1^n < \cdots < t_n^n = T$, $t_k^n := \frac{kT}{n}$. For $k \in \{1, \ldots, n\}$, $t \in [0, T]$ and $x_1, \ldots, x_k \in \mathbb{R}^d$, we define

$$b_k^n(t, x_1, \ldots, x_k) := b\big(t, \Phi^{x_1,\ldots,x_k}_{t_1^n,\ldots,t_k^n}\big).$$

Notice that $b_k^n(t, \cdot)$ are continuous functions.

Finally, for $t \in [0, T]$ and $x_1, \ldots, x_n \in \mathbb{R}^d$ we set

$$\bar{b}_n(t, x_1, \ldots, x_n) := b_1^n(t, x_1)\mathbb{1}_{[0, t_1^n)} + \cdots + b_n^n(t, x_1, \ldots, x_n)\mathbb{1}_{[t_{n-1}^n, t_n^n]}$$

and, for $(t, \phi) \in [0, T] \times \mathbb{A}$,

$$b_n(t, \phi) := \bar{b}_n(t, \phi(t \wedge t_1^n), \ldots, \phi(t \wedge t_n^n)).$$

If $t \in [t_{k-1}^n, t_k^n)$ with $k \in \{1, \ldots, n\}$, or if $t = T$ and $k = n$, then

$$b_n(t, \phi) = b_k^n(t, \phi(t_1^n), \ldots, \phi(t_{k-1}^n), \phi(t)),$$

so b_n has the form described in ***Step II***. In this case it also holds

$$
\begin{aligned}
|b_n(t, \phi) - b(t, \phi)| &= \big|b\big(t, \Phi^{\phi(t_1^n),\ldots,\phi(t_{k-1}^n),\phi(t)}_{t_1^n,\ldots,t_{k-1}^n,t_k^n}\big) - b(t, \phi)\big| \\
&\leq \sup_{\|\psi - \phi\|_T \leq \omega(\phi, T/n)} |b(t, \psi) - b(t, \phi)| \leq \ell\omega(\phi, T/n),
\end{aligned}
$$

where $\omega(\phi, \epsilon) := \sup_{|s - r| \leq \epsilon} |\phi(s) - \phi(r)|$.

In a similar way, one introduces σ_n, f_n, h_n and we have, for every $(t, \phi, y, z) \in [0, T] \times \mathbb{A} \times \mathbb{R} \times \mathbb{R}^{d'}$,

$$
\begin{aligned}
|\sigma_n(t, \phi) - \sigma(t, \phi)| &\leq \ell\omega(\phi, T/n); \\
|f_n(t, \phi, y, z) - f(t, \phi, y, z)| &\leq \sup_{\|\psi - \phi\|_T \leq \omega(\phi, T/n)} |f(t, \psi, y, z) - f(t, \phi, y, z)|; \\
|h_n(\phi) - h(\phi)| &\leq \sup_{\|\psi - \phi\|_T \leq \omega(\phi, T/n)} |h(\psi) - h(\phi)|.
\end{aligned}
$$

We also have that b_n, σ_n, f_n and h_n satisfy assumptions $(A_1),(A_2),(A_6),(A_7)$ with the same constants. Corresponding to these coefficients, we define the solution of the associated FB-SDE,

$$\left(X^{n,t,\phi}, Y^{n,t,\phi}, Z^{n,t,\phi}\right).$$

By the Feynman–Kac formula, already proven in this case, we have the existence of some non-anticipative functions u_n and d_n such that, for every $(t, \phi) \in [0, T] \times \Lambda$ we have:

$$Y^{n,t,\phi}(s) = u_n(s, X^{n,t,\phi}), \quad \text{for all } s \in [t, T];$$
$$Z^{n,t,\phi}(s) = d_n(s, X^{n,t,\phi})\sigma(t, X^{n,t,\phi}), \quad ds\text{–a.e. on } [t, T].$$

In order to pass to the limit in the first formula above, we need to show that $X^{n,t,\phi} \to X^{t,\phi}$ in probability in Λ and that u_n converges to u on compact subsets of Λ.

Let $t \in [0, T]$ and $\phi, \phi' \in \Lambda$. By Itô's formula we have

$$\left|X^{n,t,\phi'}(s) - X^{t,\phi}(s)\right|^2 = |\phi'(t) - \phi(t)|^2$$

$$+2\int_t^s \langle b_n(r, X^{n,t,\phi'}) - b(r, X^{t,\phi}), X^{n,t,\phi'}(r) - X^{t,\phi}(r)\rangle dr + \int_t^s \left|\sigma_n(r, X^{n,t,\phi'}) - \sigma(r, X^{t,\phi})\right|^2 dr$$

$$+2\int_t^s \langle \sigma_n(r, X^{n,t,\phi'}) - \sigma(r, X^{t,\phi}), \left(X^{n,t,\phi'}(r) - X^{t,\phi}(r)\right)dW(r)\rangle.$$

By standard computations using Gronwall's lemma, Burkholder–Davis–Gundy inequalities and Theorem 4, we get, for some arbitrary $p > 1$,

$$\begin{aligned}
\mathbb{E} \sup_{s\in[0,T]} \left|X^{n,t,\phi'}(s) - X^{t,\phi}(s)\right|^{2p} &\leq C\left(\|\phi' - \phi\|_T^{2p} + \mathbb{E}\omega(X^{t,\phi}, T/n)^{2p}\right) \\
&\leq C\left(\|\phi' - \phi\|_T^{2p} + \frac{1 + \|\phi\|_T^{2p}}{n^{p-1}} + \omega(\phi, T/n)^{2p}\right),
\end{aligned} \tag{18}$$

where the positive constant C is depending only on p and the parameters of our FBSDE; also, as above, the constant C is allowed to change value from line to line during this proof.

Applying now Itô's formula to $e^{\beta s}\left|Y^{n,t,\phi'}(s) - Y^{t,\phi}(s)\right|^2$, we obtain, for $\beta > 0$,

$$e^{\beta s}\left|\Delta_Y^{n,t,\phi,\phi'}(s)\right|^2 + \beta\int_s^T e^{\beta r}\left|\Delta_Z^{n,t,\phi,\phi'}(r)\right|^2 dr + \int_s^T e^{\beta r}\left|\Delta_Z^{n,t,\phi,\phi'}(r)\right|^2 dr$$

$$= e^{\beta T}\left|\Delta_h^{n,t,\phi,\phi'}\right|^2 + 2\int_s^T \langle \Delta_f^{n,t,\phi,\phi'}(r), e^{\beta r}\Delta_Y^{n,t,\phi,\phi'}(r)\rangle dr$$

$$+2\int_s^T \langle f_n(s, X^{n,t,\phi'}, Y^{n,t,\phi'}(s), Z^{n,t,\phi'}(s)) - f_n(s, X^{n,t,\phi'}, Y^{t,\phi}(s), Z^{t,\phi}(s)), e^{\beta r}\Delta_Y^{n,t,\phi,\phi'}(r)\rangle dr$$

$$-2\int_s^T \langle e^{\beta r}\Delta_Y^{n,t,\phi,\phi'}(r), \Delta_Z^{n,t,\phi,\phi'}(r)dW(r)\rangle,$$

where, for the sake of simplicity, we have denoted

$$\Delta_Y^{n,t,\phi,\phi'}(s) := Y^{n,t,\phi'}(s) - Y^{t,\phi}(s);$$
$$\Delta_Z^{n,t,\phi,\phi'}(s) := Z^{n,t,\phi'}(s) - Z^{t,\phi}(s);$$
$$\Delta_h^{n,t,\phi,\phi'} := h_n(X^{n,t,\phi'}) - h(X^{t,\phi});$$
$$\Delta_f^{n,t,\phi,\phi'}(s) := f_n(s, X^{n,t,\phi'}, Y^{t,\phi}(s), Z^{t,\phi}(s)) - f(s, X^{t,\phi}, Y^{t,\phi}(s), Z^{t,\phi}(s)).$$

Again, for β sufficiently large, exploiting Burkholder–Davis–Gundy inequalities and the Lipschitz property in $(y, z) \in \mathbb{R} \times \mathbb{R}^{d'}$ of f_n, we infer that

$$\mathbb{E} \sup_{s \in [t,T]} \left| \Delta_Y^{n,t,\phi,\phi'}(s) \right|^2 + \mathbb{E} \int_t^T \left| \Delta_Z^{n,t,\phi,\phi'}(r) \right|^2 dr \leq C\mathbb{E} \left[\left| \Delta_h^{n,t,\phi,\phi'} \right|^2 + \int_t^T \left| \Delta_f^{n,t,\phi,\phi'}(r) \right|^2 dr \right]. \tag{19}$$

We now have

$$\left| \Delta_h^{n,t,\phi,\phi'} \right| \leq \left| h_n(X^{n,t,\phi'}) - h(X^{n,t,\phi'}) \right| + \left| h(X^{n,t,\phi'}) - h(X^{t,\phi}) \right|$$

$$\leq \sup_{||\psi - X^{n,t,\phi'}||_T \leq \omega(X^{n,t,\phi'}, T/n)} \left| h(\psi) - h(X^{n,t,\phi'}) \right| + \left| h(X^{n,t,\phi'}) - h(X^{t,\phi}) \right|$$

$$\leq \sup_{||\psi - X^{t,\phi}||_T \leq \omega(X^{t,\phi}, T/n) + 3||X^{n,t,\phi'} - X^{t,\phi}||_T} \left| h(\psi) - h(X^{t,\phi}) \right| + 2\left| h(X^{n,t,\phi'}) - h(X^{t,\phi}) \right| \tag{20}$$

$$\leq 3 \sup_{||\psi - X^{t,\phi}||_T \leq \omega(X^{t,\phi}, T/n) + 3||X^{n,t,\phi'} - X^{t,\phi}||_T} \left| h(\psi) - h(X^{t,\phi}) \right|,$$

since

$$\omega(X^{n,t,\phi'}, T/n) \leq \omega(X^{t,\phi}, T/n) + 2||X^{n,t,\phi'} - X^{t,\phi}||_T.$$

Similarly, we obtain

$$\left| \Delta_f^{n,t,\phi,\phi'}(s) \right| \leq 3 \sup_{||\psi - X^{t,\phi}||_T \leq \omega(X^{t,\phi}, T/n) + 3||X^{n,t,\phi'} - X^{t,\phi}||_T} \left| f\left(s, \psi, Y^{t,\phi}(s), Z^{t,\phi}(s)\right) \right. \tag{21}$$
$$\left. - f(s, X^{t,\phi}, Y^{t,\phi}(s), Z^{t,\phi}(s)) \right|.$$

Let now $(t, \phi) \in [0, T] \times \mathbb{A}$ and let (ϕ_n) be a sequence converging to ϕ in $\mathbb{A}$. It is clear from relation (18) that $\left(X^{n,t,\phi_n}\right)$ converges in $L^{2p}(\Omega; \mathbb{A})$ to $X^{t,\phi}$, therefore there exists a subsequence converging a.s. in $\mathbb{A}$ to $X^{t,\phi}$. Without restricting the generality, we will still denote $\left(X^{n,t,\phi_n}\right)$ this subsequence. Since $\omega(X^{t,\phi}, T/n) + 3||X^{n,t,\phi'} - X^{t,\phi}||_T$ converges to 0 a.s., it is clear by relations (20) and (18) that $\Delta_h^{n,t,\phi,\phi'}$ and $\Delta_f^{n,t,\phi,\phi'}(s)$ converge to 0, a.s., respectively $ds\mathbb{P}$-a.e.

Then, by the dominated convergence theorem, we obtain

$$\mathbb{E} \left[\left| \Delta_h^{n,t,\phi,\phi_n} \right|^2 + \int_t^T \left| \Delta_f^{n,t,\phi,\phi_n}(r) \right|^2 dr \right] \to 0,$$

which, combined with estimate (19), gives that $\mathbb{E} \sup_{s \in [t,T]} |Y^{n,t,\phi_n}(s) - Y^{t,\phi}(s)|^2 \to 0$. On the one hand, assuming $\phi_n \equiv \phi$, this implies

$$\mathbb{E} \sup_{s \in [t,T]} |Y^{n,t,\phi}(s) - Y^{t,\phi}(s)|^2 \to 0 \, ;$$

on the other hand, letting $s = t$, we obtain

$$u_n(t, \phi_n) \to u(t, \phi).$$

Therefore we can pass to the limit in the relation

$$Y^{n,t,\phi}(s) = u_n(s, X^{n,t,\phi}), \quad \text{for all } s \in [t, T], \text{ a.s.},$$

and find the conclusion of the theorem. ∎

We are able now to prove the Feynman–Kac formula in the case where the generator f depends of the past values of Y.

Theorem 21 *Let us assume that* $(A_1),(A_2),(A_6),(A_7)$ *hold and condition ([10]) is verified. Then*

$$Y^{t,\phi}(s) = u(s, X^{t,\phi}), \quad \text{for all } s \in [0,T], \quad a.s.,$$

for any $(t,\phi) \in [0,T] \times \mathbb{A}$, *where* $Y^{t,\phi}$ *is the solution of BSDE ([15]) and* $u : [0,T] \times \mathbb{A} \to \mathbb{R}$ *is defined by ([16]).*

Proof. The continuity of u was already asserted in Theorem [16].

Let now consider BSDE with delayed generator

$$Y^{t,\phi}(s) = h(X^{t,\phi}) + \int_s^T f(r, X^{t,\phi}, Y^{t,\phi}(r), Z^{t,\phi}(r), Y_r^{t,\phi})dr - \int_s^T Z^{t,\phi}(r)\,dW(r), \quad (22)$$

and the corresponding iterative process, given by the equations

$$Y^{n+1,t,\phi}(s) = h(X^{t,\phi}) + \int_s^T f(r, X^{t,\phi}, Y^{n+1,t,\phi}(r), Z^{n+1,t,\phi}(r), Y_r^{n,t,\phi})dr$$

$$- \int_s^T Z^{n+1,t,\phi}(r)\,dW(r), \quad s \in [t,T], \quad (23)$$

with $Y^{0,t,\phi} \equiv 0$ and $Z^{0,t,\phi} \equiv 0$.

Let us suppose that there exists a $\mathbb{F}$–progressively measurable functional $u_n : [0,T] \times \mathbb{A} \to \mathbb{R}$ such that u_n is continuous and $Y^{n,t,\phi}(s) = u_n(s, X^{t,\phi})$, for every $t, s \in [0,T]$ and $\phi \in \mathbb{A}$.

Let us now consider the term

$$Y_r^{n,t,\phi} = \left(Y^{n,t,\phi}(r+\theta)\right)_{\theta \in [-\delta,0]};$$

since $Y^{n,t,\phi}(r+\theta) = u_n(r+\theta, X^{t,\phi})$ if $r+\theta \geq 0$ and $Y^{n,t,\phi}(r+\theta) = Y^{n,t,\phi}(0) = u_n(0, X^{t,\phi})$ if $r+\theta < 0$, by defining

$$\tilde{u}_n(t,\phi) := \left(u_n(\mathbb{1}_{[0,T]}(t+\theta), \phi)\right)_{\theta \in [-\delta,0]},$$

we have

$$Y_r^{n,t,\phi} = \tilde{u}_n(r, X^{t,\phi}).$$

We can then apply Theorem [18] in order to infer that

$$Y^{n+1,t,\phi}(s) = u_{n+1}(s, X^{t,\phi}),$$

for a continuous non-anticipative functional $u_{n+1} : [0,T] \times \mathbb{A} \to \mathbb{R}$.

Notice that $\left(Y^{n,t,\phi}, Z^{n,t,\phi}\right)$ is the Picard iteration sequence used for constructing the solution $\left(Y^{t,\phi}, Z^{t,\phi}\right)$:

$$\left(Y^{n+1,\cdot,\phi}, Z^{n+1,\cdot,\phi}\right) = \Gamma(Y^{n,\cdot,\phi}, Z^{n,\cdot,\phi}),$$

where Γ is the contraction defined in the proof of Theorem [7]. We then have:

$$\lim_{n\to\infty} \mathbb{E}\left(\sup_{s\in[0,T]} \left|Y^{n,t,\phi}(s) - Y^{t,\phi}(s)\right|^2\right) = 0.$$

Of course, $u_n(t,\phi)$ converges to $u(t,\phi) := \mathbb{E}Y^{t,\phi}(t)$, for every $t \in [0,T]$ and $\phi \in \mathbb{A}$. This implies that the nonlinear Feynman–Kac formula $Y^{t,\phi}(s) = u(s, X^{t,\phi})$ holds. ∎

Remark 22 *From the above proof it can be seen why we have renounced to the dependence of the driver on the past of z. More precisely, even if we had a representation of the form*

$$Z^{n,t,\phi}(s) = d^n(s, X^{t,\phi})$$

in Theorem 18, we couldn't use it because of the lack of continuity of d^n .

We are now able to prove Theorem 13, which shows the existence of a viscosity solution to equation (1).

Proof of Theorem 13.

We will prove that function u defined by (16) is a viscosity solution to (1). In particular we will only show that u is a viscosity subsolution, the supersolution case being similar.

Suppose by contrary that u is not a viscosity subsolution. Then, for any $L_0 \geq 0$, there exists $L \geq L_0$ such that u is not a viscosity L–subsolution in the sense of Definition 2. Therefore, there exist $(t, \phi) \in [0, T] \times \mathbb{A}$ and $\varphi \in \underline{\mathcal{A}}^L u(t, \phi)$ such that, for some $c > 0$,

$$\partial_t \varphi(t, \phi) + \mathcal{L}\varphi(t, \phi) + f(t, \phi, u(t, \phi), \partial_x \varphi(t, \phi)\,\sigma(t, \phi), (u(\cdot, \phi))_t) \leq -c < 0$$

Using the definition of $\underline{\mathcal{A}}^L u(t, \phi)$ we see that there exists $\tau_0 \in \mathcal{T}_+^t$ such that

$$\varphi(t, \phi) - u(t, \phi) = \min_{\tau \in \mathcal{T}^t} \underline{\mathcal{E}}_t^L \left[(\varphi - u)\left(\tau \wedge \tau_0, X^{t,\phi}\right) \right] .$$

Let us now take

$$\tilde{\tau} := T \wedge \tau_0 \wedge \inf\{s > t : \partial_t \varphi(s, X^{t,\phi}) + \mathcal{L}\varphi(s, X^{t,\phi})$$
$$+ f(s, X^{t,\phi}, u(s, X^{t,\phi}), \partial_x \varphi(s, X^{t,\phi})\sigma(s, X^{t,\phi}), (u(\cdot, X^{t,\phi}))_s) > -c/2\}.$$

By the continuity of the coefficients, we deduce that $\tilde{\tau} \in \mathcal{T}_+^t$.

Let us denote

$$(Y^1(s), Z^1(s)) := (\varphi(s, X^{t,\phi}), \partial_x \varphi(s, X^{t,\phi})\sigma(s, X^{t,\phi})), \quad \text{for } s \in [t, T],$$
$$(Y^2(s), Z^2(s)) := (Y^{t,\phi}(s), Z^{t,\phi}(s)), \quad \text{for } s \in [0, T]$$

and

$$\Delta Y(s) := Y^1(s) - Y^2(s), \quad \Delta Z(s) := Z^1(s) - Z^2(s), \text{ for } s \in [t, T].$$

Using Itô's formula we deduce, for any $s \in [t, T]$,

$$\varphi(s, X^{t,\phi}) = \varphi(t, \phi) + \int_t^s \left(\partial_t \varphi(r, X^{t,\phi}) + \mathcal{L}\varphi(r, X^{t,\phi})\right) dr + \int_t^s \langle \partial_x \varphi(r, X^{t,\phi}), \sigma(r, X^{t,\phi}) dW(r) \rangle,$$

so that we obtain

$$\Delta Y(\tilde{\tau}) - \Delta Y(t)$$
$$= \int_t^{\tilde{\tau}} \left(\partial_t \varphi(r, X^{t,\phi}) + \mathcal{L}\varphi(r, X^{t,\phi}) + f(r, X^{t,\phi}, Y^2(r), Z^2(r), Y_r^2)\right) dr + \int_t^{\tilde{\tau}} \Delta Z(r)\, dW(r) .$$

Since for any $r \in [t, \tilde{\tau}]$ we have

$$\partial_t \varphi(r, X^{t,\phi}) + \mathcal{L}\varphi(r, X^{t,\phi}) + f(r, X^{t,\phi}, Y^2(r), Z^2(r), Y_r^2)$$
$$\leq -\frac{c}{2} + f(r, X^{t,\phi}, Y^2(r), Z^2(r), Y_r^2) - f(r, X^{t,\phi}, u(r, X^{t,\phi}), \partial_x \varphi(r, X^{t,\phi})\sigma(r, X^{t,\phi}), (u(\cdot, X^{t,\phi}))_r),$$

we deduce, using the Feynman–Kac formula $Y^2(r) = u\left(r, X^{t,\phi}\right)$, that

$$\Delta Y(\tilde{\tau}) - \Delta Y(t) \leq -\frac{c}{2}(\tilde{\tau} - t) + \int_t^{\tilde{\tau}} \Delta Z(r) \, dW(r)$$
$$+ \int_t^{\tilde{\tau}} \left(f(r, X^{t,\phi}, u(r, X^{t,\phi}), Z^2(r), (u(\cdot, X^{t,\phi}))_r)) - f(r, X^{t,\phi}, u(r, X^{t,\phi}), Z^1(r), (u(\cdot, X^{t,\phi}))_r) \right) dr.$$

We have that there exists $\lambda \in \mathcal{U}_T^L$ such that

$$f(r, X^{t,\phi}, u(r, X^{t,\phi}), Z^1(r), (u(\cdot, X^{t,\phi}))_r)) - f(r, X^{t,\phi}, u(r, X^{t,\phi}), Z^2(r), (u(\cdot, X^{t,\phi}))_r)$$
$$= \langle \lambda(r), \Delta Z(r) \rangle$$

and therefore

$$\Delta Y(\tilde{\tau}) - \Delta Y(t) \leq -\frac{c}{2}(\tilde{\tau} - t) + \int_t^{\tilde{\tau}} \Delta Z(r) \left(dW(r) - \lambda(r) \, dr \right).$$

Noticing now that $W(s) - \int_t^s \lambda(r) \, dr$ is a $\mathbb{P}^{t,\lambda}$–martingale, we obtain

$$\Delta Y(t) = \mathbb{E}^{\mathbb{P}^{t,\lambda}}(\Delta Y(t))$$
$$\geq \mathbb{E}^{\mathbb{P}^{t,\lambda}}(\Delta Y(\tilde{\tau})) + \frac{c}{2}\mathbb{E}^{\mathbb{P}^{t,\lambda}}(\tilde{\tau} - t) + \mathbb{E}^{\mathbb{P}^{t,\lambda}} \int_t^{\tilde{\tau}} \Delta Z(r) \left(dW(r) - \lambda(r) \, dr \right)$$
$$> \mathbb{E}^{\mathbb{P}^{t,\lambda}}(\Delta Y(\tilde{\tau})) = \mathbb{E}^{\mathbb{P}^{t,\lambda}}(\varphi(\tilde{\tau}, X^{t,\phi}) - Y^{t,\phi}(\tilde{\tau})) = \mathbb{E}^{\mathbb{P}^{t,\lambda}} \left[(\varphi - u)(\tilde{\tau}, X^{t,\phi}) \right]$$
$$\geq \underline{\mathcal{E}}_t^L \left[(\varphi - u)(\tilde{\tau}, X^{t,\phi}) \right] = \underline{\mathcal{E}}_t^L \left[(\varphi - u)(\tau_0 \wedge \tilde{\tau}, X^{t,\phi}) \right]$$
$$\geq \min_{\tau \in \mathcal{T}^t} \underline{\mathcal{E}}_t^L \left[(\varphi - u)(\tau_0 \wedge \tilde{\tau}, X^{t,\phi}) \right] = \varphi(t, X^{t,\phi}) - Y^{t,\phi}(t) = \Delta Y(t),$$

which is a contradiction.

The proof is complete now. ∎

5 Financial applications

In what follows we shall apply theoretical results developed in previous sections in order to analyze some particular models of great interest in modern finance.

Financial literature that shows how delay naturally arises when dealing with asset price evolution or in general with certain financial instruments, is nowadays wide and developed, see, for instance, [1, 5, 24, 25] and references therein. On the other hand, not much is done

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when the delay enters the backward component. We aim here to give some financial applications where also the backward equation exhibits a delayed behaviour. We remark that since the goal of the present work is purely theoretical, the examples provided will not be stated in complete generality. Actually in this section we will show how the study of BSDEs with delayed generators, together with the associated path–dependent Kolmogorov equation, may lead to the study of a completely new class of financial problems that have not been studied before. Nevertheless, we intend to address in a future work the study of these problems in complete generality.

BSDEs with delay have been introduced in [13] as a pure mathematical tool with no financial application of interest. Later on, some works showing that the delay in the backward component arises naturally in several applications have appeared, see, e.g. [10, 11].

In what follows we will provide two extensions of forward–backward models that have been proposed in past literature where the backward components can exhibit a short–time delay.

5.1 The large investor problem

Following the model studied in [8], see also, e.g. [21], we will consider in the current example a non–standard investor acting on a financial market. We assume that this investor, usually referred to as *the large investor* in the literature, has superior information about the stock prices and/or he is willing to invest a large amount of money in the stock. This fact implies that the large investor may influence the behaviour of the stock price with his actions. It is further natural to assume that there is a short time delay between the investor's actions and the reaction of the market to the large investor's actions. In particular we assume that the drift coefficient of the underlying S at time t depends on how the large investor acts on the market in the interval $(t - \delta, t)$.

Let us then consider a risky asset S and a riskless bond B evolving according to

$$
\begin{cases}
\dfrac{dB(t)}{B(t)} = r(t, X(t), \pi(t), X_t)dt , \quad B(0) = 1 , \\[2mm]
\dfrac{dS(t)}{S(t)} = \mu(t, X(t), \pi(t), X_t) \, dt + \sigma(t, X(t), X_t)dW(t), \quad S(0) = s_0 > 0.
\end{cases}
\tag{24}
$$

Here we have denoted by X the portfolio of the large investor. Also we used the notations introduced by (6) and (7). We suppose that the coefficients r, μ and σ satisfy some suitable regularity assumptions.

We have that the portfolio X, composed at any time $t \in [0, T]$ by $\pi(t)$, the amount invested in the risky asset S and by $X(t) - \pi(t)$, the amount invested in the riskless bond B, evolves according to

$$
dX(t) = \frac{\pi(t)}{S(t)} dS(t) + \frac{X(t) - \pi(t)}{B(t)} dB(t)
$$

$$
= \pi(t) \cdot [\mu(t, X(t), \pi(t), X_t) \, dt + \sigma(t, X(t), X_t)dW(t)] + [X(t) - \pi(t)] \cdot r(t, X(t), \pi(t), X_t)dt ,
$$

with the final condition $X(T) = h(S)$.

Hence, for $t \in [0, T]$,

$$X(t) = h(S) + \int_t^T F(s, X(s), \pi(s), X_s, \pi_s) \, ds - \int_t^T \pi(s) \sigma(s, X(s), X_s) dW(s), \qquad (25)$$

where we have denoted for short

$$F(s, X(s), \pi(s), X_s, \pi_s) := -[X(s) - \pi(s)] \cdot r(s, X(s), \pi(s), X_s) - \pi(s) \cdot \mu(s, X(s), \pi(s), X_s). \tag{26}$$

Since the forward equations in (24) can be explicitly solved by

$$S(t) = s_0 \exp \left[\int_0^t \left(\mu(s, X(s), \pi(s), X_s) - \frac{1}{2} \sigma^2(s, X(s), X_s) \right) ds + \int_0^t \sigma(s, X(s), X_s) \, dW(s) \right],$$

we deduce that S is a functional of X, π and W, i.e. there exists $\tilde{h}$ such that the final condition becomes $X(T) = \tilde{h}(W, X, \pi)$.

A first remark is that we can impose some suitable assumptions on the functions r, μ and σ such that the function $\bar{F} : [0, T] \times \mathbb{R} \times \mathbb{R} \times L^2([-\delta, 0]; \mathbb{R}) \to \mathbb{R}$, defined by

$$\bar{F}(s, y, z, \hat{y}) := -\left(y - z\sigma^{-1}(s, y, \hat{y}) \right) \cdot r(s, y, z\sigma^{-1}(s, y, \hat{y}), \hat{y}) - z \cdot \mu\left(s, y, z\sigma^{-1}(s, y, \hat{y}), \hat{y} \right),$$

satisfies assumptions (A_3)–(A_5).

The second remark concerns the fact that Theorem 7 is still true, with a slight adjustment of the proof, if we consider in the backward equation (8) the final condition

$$\bar{h}(W, X, Z) := \tilde{h}(W, X, \sigma^{-1}(\cdot, X, X_\cdot) Z),$$

instead of a functional of W only (which represents the forward part within the theoretical framework from Section 3), with $\bar{h}$ satisfying a Lipschitz condition:

$$\left| \bar{h}(x, y, z) - \bar{h}(x, y', z') \right|^2 \leq L \left[\int_0^T |y(s) - y'(s)|^2 \, ds + \int_0^T |z(s) - z'(s)|^2 \, ds \right].$$

Of course, $\bar{h}$ will satisfy (A_5) and the above condition if we impose some suitable assumptions on μ and σ (for instance, if we consider μ bounded and σ constant).

Therefore we can rewrite (25) as

$$X(t) = \bar{h}(W, X, Z) + \int_t^T \bar{F}(s, X(s), Z(s), X_s) \, ds - \int_t^T Z(s) \, dW(s), \quad t \in [0, T] \qquad (27)$$

and we deduce from Theorem 7 that, under proper assumptions on the coefficients, there exists a unique solution (X, Z) to equation (27).

Hence equation (25) admits a unique solution (X, π), where

$$\pi(s) := Z(s) \sigma^{-1}(s, X(s), X_s).$$

In order to obtain the connection with the PDE we consider first the decoupled forward–backward stochastic system:

$$
\begin{cases}
\bar{W}^{t,\phi}(s) = \phi(t) + \int_t^s dW(r), \quad s \in [t,T], \\[2mm]
\bar{W}^{t,\phi}(s) = \phi(s), \quad s \in [0,t), \\[2mm]
X^{t,\phi}(s) = \bar{h}(\bar{W}^{t,\phi}, X^{t,\phi}, Z^{t,\phi}) + \int_s^T \bar{F}(r, X^{t,\phi}(r), Z^{t,\phi}(r), X_r^{t,\phi}) dr \\[3mm]
\hspace{3cm} - \int_s^T Z^{t,\phi}(r) dW(r), \quad s \in [t,T], \\[2mm]
X^{t,\phi}(s) = X^{s,\phi}(s), \quad \pi^{t,\phi}(s) = 0, \quad s \in [0,t).
\end{cases}
$$

Using Theorem 7 we see that, under suitable assumptions on the coefficients, there exists a unique solution $\left(X^{t,\phi}, \pi^{t,\phi}\right)_{(t,\phi)\in[0,T]\times\mathbb{A}}$ of the above system.

From the results of the previous sections, in particular from theorem 21, we have the following representation for the solution (X, Z) of the backward component in the above system. In particular we have, for every $(t, \phi) \in [0, T] \times \mathbb{A}$,

$$
X^{t,\phi}(s) = u(s, \bar{W}^{t,\phi}), \quad \text{for all } s \in [t, T],
$$

where $u(t, \phi) = X^{t,\phi}(t)$ is a viscosity solution of the following path–dependent PDE:

$$
\begin{cases}
\partial_t u(t, \phi) + \dfrac{1}{2} \partial_{xx}^2 u(t, \phi) + \bar{F}(t, u(t, \phi), \partial_x u(t, \phi), (u(\cdot, \phi))_t) = 0, \quad \phi \in \mathbb{A}, \ t \in [0, T), \\[2mm]
u(T, \phi) = \bar{h}(\phi, (u(\cdot, \phi))), \quad \phi \in \mathbb{A}.
\end{cases}
$$

An example of this type has been developed first in [8] and then treated by many authors, see, e.g. [21], where they considered a case where the drift and the diffusion component of the price equation depend both on the wealth process X and the underlying process S. This would lead to a fully coupled forward–backward system which does not fit in our setting. On the other hand in our model we have assumed that the drift μ and the interest rate r may be influenced also by past values of the wealth process X, whereas in cited papers no delay in the backward component is assumed.

5.2 Risk measures via g-expectations

A key problem in financial mathematics is the risk management of an investment. Such a problem has been widely studied in finance since the introductory paper [2] where the notion of *risk measure* has been first introduced. Since then, several empirical studies concerning the key task of risk-management have been conducted, showing in particular that the best way to quantify the risk of a given financial position should be a *dynamic risk measure*, rather than a classic static one. Starting from this fact, the notion of *g-expectation* has been first introduced in [33], as a fundamental mathematical tool when dealing with *dynamic risk measures*; we refer also to [35, 39] for a comprehensive and exhaustive introduction to *dynamic risk measures*.

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The main purpose of a risk measure is to quantify in a single number the riskiness of a given financial position. The next one is the mathematical formulation of the notion of *risk measure*, see, e.g. [12, Definition 13.1.1].

Definition 23 *A family* $(\rho_t)_{t \in [0,T]}$ *of mappings* $\rho_t : L^2(\Omega, \mathcal{F}_T) \to L^2(\Omega, \mathcal{F}_t)$, *such that* $\rho_T(\xi) = -\xi$ *is called a dynamic risk measure.*

From a practical point of view, if we denote by ξ the terminal value of a given financial position, $\rho_t(\xi)$ quantifies the risk the investor takes in the position ξ at terminal time T. Clearly, in order to have a concrete financial use, a risk measure has to satisfy a set of properties, usually referred to as *axioms of risk measures*; we refer to [12] to a complete list of the aforementioned axioms.

From a mathematical point of view, it has been shown that BSDEs and the related forward–backward system are a perfect tool to tackle the problem of risk management. In particular, one possible way to define a *dynamic risk measure* is to specify the generator g of the driving BSDE, from here the name g−expectation, where the generator g determines the properties of the *dynamic risk measure*. A direct approach to g-expectation is therefore to introduce a BSDE of the form

$$Y(t) = \xi + \int_t^T g\left(s, Y(s), Z(s)\right) ds - \int_t^T Z(s) dW(s) \,, \tag{28}$$

where the generator g is called the generator of a g−expectation. In this sense we have

$$\rho_t(\xi) = \mathbb{E}^g\left[\xi \,\middle|\, \mathcal{F}_t\right] = Y(t) \,,$$

where we have used the subscript g to emphasize the role played by the generator g. Heuristically speaking we have the relation

$$\mathbb{E}^g\left[dY(t) \,\middle|\, \mathcal{F}_t\right] = -g\left(t, Y(t), Z(t)\right) dt \,, \quad 0 \le t \le T \,,$$

so that intuitively the coefficient g reflects the agent's belief on the expected change of risk.

Once we have chosen a risk measure g, such that it is financially reasonable, we then solve the BSDE (28) endowed with a suitable final condition which represents the investor's wealth at terminal time T, we refer to [3, 39] for a detailed introduction to the usage of BSDEs as *dynamic risk measures*.

In the literature it has always been considered a generator g that depends only on the present value at time t of the risk measure $Y(t)$ and its variability $Z(t)$, but, as pointed out in [11], if we want to model an investor preference we cannot leave aside the memory effect, that is it is reasonable to assume that an investor makes his choices based also on what happened on the past. In [11] the author proposed to consider a g−expectation which incorporates a disappointment effect through a BSDE of moving average. In fact in the just mentioned work the author suggests that when dealing with the investor's preferences, to consider Markovian systems is restrictive since it is natural that an investor takes into account the past history of a given investment when he is to make some choice. We refer to [11] for a short but exhaustive review of different economical studies of how memory effect cannot be neglected when dealing with an investor's choice. Regarding the case considered

in [11], we make the assumption that the investor has a *short memory*, that is, in making his choices he considers only what has happened in the recent past.

Let us take two bounded and Lipschitz functions g_1, g_2 such that $g_2(0) = 0$. We consider a g−expectation of the form $g(s, y, z) = \beta\, g_1(\bar{y})\, g_2(z)$, where $\bar{y}$ is the time–average in a sufficiently small time interval and $\beta \in \mathbb{R}$ a given financial parameter (and we remark that, in this case, the assumption $g(s, y, 0) = 0$ is satisfied). Hence we will deal with a BSDE with delayed generator of the form

$$Y(t) = \xi + \frac{\beta}{\delta} \int_t^T g_1\Big(\int_{-\delta}^0 Y(s+r)dr \Big) g_2(Z(s))\, ds - \int_t^T Z(s)dW(s), \qquad (29)$$

with ξ the terminal payoff of the investment to be introduced soon and δ small enough such that inequality (10) is well-posed.

Let us now assume that the financial market is composed by one risky asset S and one riskless bond B. The generalization to d risky assets can be easily derived from the present case. We assume, in a complete generality, that both the bond and the asset may exhibit delay. For the case of delay in the forward component a more established theory exists, with existence and uniqueness, as well as regularity results, see, e.g. [1, 22, 23].

We consider in what follows the delayed market model introduced in [5]. In what concerns the stock price we assume that S evolves according to the following stochastic delay differential equation:

$$\frac{dS(t)}{S(t)} = \mu(t, S)dt + \sigma(t, S)dW(t), \qquad (30)$$

with $S_0 = s_0 \in \mathbb{R}$, where $\mu, \sigma : [0, T] \times \mathbb{A} \to \mathbb{R}$ are some given functions, where the notation is introduced in Section 1.

Let us assume that μ and σ satisfy assumptions of type (A_1)–(A_2), so that there exists a unique solution of equation (30) (this is a consequence of Theorem 4).

Also we assume the investor subscribes a claim with terminal payoff $h : \mathbb{A} \to \mathbb{R}$ so that the BSDE (29) becomes

$$Y(t) = h(S_T) + \frac{\beta}{\delta} \int_t^T g_1\Big(\int_{-\delta}^0 Y(s+r)dr \Big) g_2(Z(s))\, ds - \int_t^T Z(s)dW(s), \qquad (31)$$

where we assume h to satisfy (A_7)−(iv); also, let us stress that the generator $g : L^2([-\delta, 0]; \mathbb{R}) \to \mathbb{R}$ defined above satisfies assumptions $(A_6), (A_7)$, see, e.g. Remark 12. We are naturally led to consider the following forward-backward system with delay

$$\begin{cases} S^{t,\phi}(s) = \phi(t) + \int_t^s S^{t,\phi}(r)\mu(t, S^{t,\phi})dr + \int_t^s S^{t,\phi}(r)\sigma(t, S^{t,\phi})dW(r), \quad s \in [t, T], \\[2mm] S^{t,\phi}(s) = \phi(s), \quad s \in [0, t), \\[2mm] Y(s) = h(S^{t,\phi}) + \dfrac{\beta}{\delta} \int_s^T g_1\Big(\int_{-\delta}^0 Y^{t,\phi}(r+\theta)d\theta \Big) g_2(Z^{t,\phi}(r))\, dr \\[4mm] \hspace{4cm} - \int_s^T Z^{t,\phi}(r)dW(r), \quad s \in [t, T], \\[2mm] Y^{t,\phi}(s) = Y^{s,\phi}(s), \quad s \in [0, t), \end{cases} \qquad (32)$$

and by theorems 4–7, the forward-backward system (32) admits a unique solution.

Let us also stress that the great majority of possible claims that can be considered in finance satisfies the above assumptions on the terminal payoff h; also we allow the option to possibly be *path–dependent*, that is its terminal value at time T depends explicitly on past values assumed by the asset S.

From the results of the previous sections, we have the following characterization for the FBSDE (32). In particular we obtain theorem 21 holds, so that, for every $(t, \phi) \in [0, T] \times \mathbb{A}$,

$$Y^{t,\phi}(s) = u(s, S^{t,\phi}), \quad \text{for all } s \in [t, T],$$

where $u(t, \phi) = Y^{t,\phi}(t)$ is a viscosity solution of the following path–dependent PDE:

$$
\begin{cases}
\partial_t u(t, \phi) + \dfrac{1}{2} \phi^2 \sigma^2(t, \phi) \partial^2_{xx} u(t, \phi) + \phi \mu(t, \phi) \partial_x u(t, \phi) \\[2mm]
\quad + \dfrac{\beta}{\delta} g_1 \Big(\displaystyle\int_{-\delta}^{0} u(t+r, \phi) dr \Big) g_2 \big(\phi \sigma(t, \phi) \partial_x u(t, \phi) \big) = 0, \quad \phi \in \mathbb{A}, \; t \in [0, T), \\[2mm]
u(T, \phi) = h(\phi), \quad \phi \in \mathbb{A}.
\end{cases}
$$

6   Appendix: Proofs of Theorem 7 and Theorem 16

Proof of Theorem 7.   The existence and uniqueness will be obtained by the Banach fixed point theorem. Let $\phi \in \mathbb{A}$ be arbitrarily fixed and let us consider the map Γ defined on $\mathcal{A} \times \mathcal{B}$, with $\mathcal{A} := \mathcal{C}\big([0, T]; \mathcal{S}_0^{2,m}\big)$ and $\mathcal{B} := \mathcal{C}\big([0, T]; \mathcal{H}_0^{2,m \times d'}\big)$, in the following way: for $(U, V) \in \mathcal{A} \times \mathcal{B}$, $\Gamma(U, V) = (Y, Z)$, where for $t \in [0, T]$, the couple of adapted processes $\big(Y^t(s), Z^t(s)\big)_{s \in [t, T]}$ is the unique solution of the BSDE

$$Y^t(s) = h(X^{t,\phi}) + \int_s^T F(r, X^{t,\phi}, Y^t(r), Z^t(r), U_r^t, V_r^t) dr - \int_s^T Z^t(r) \, dW(r), \quad s \in [t, T]. \tag{33}$$

Now we prolong the solution by taking $Y^t(s) := Y^s(s)$ and $Z^t(s) := 0$, for $s \in [0, t)$.

Step I.

Let us first show that Γ takes values in the Banach space $\mathcal{A} \times \mathcal{B}$. For that, let us take $(U, V) \in \mathcal{A} \times \mathcal{B}$; we will prove that $(Y, Z) := \Gamma(U, V) \in \mathcal{A} \times \mathcal{B}$, i.e. for every $t \in [0, T]$ we have

$$Y^t \in \mathcal{S}_t^{2,m} \subseteq \mathcal{S}_0^{2,m}, \quad Z^t \in \mathcal{H}_t^{2,m \times d'} \subseteq \mathcal{H}_0^{2,m \times d'} \tag{34}$$

and the applications

$$
\begin{aligned}
&[0, T] \ni t \mapsto Y^t \in \mathcal{S}_0^{2,m}, \\
&[0, T] \ni t \mapsto Z^t \in \mathcal{H}_0^{2,m \times d'}
\end{aligned}
\tag{35}
$$

are continuous.

Let $t \in [0, T]$ be fixed and $t' \in [0, T]$; with no loss of generality, we will suppose that $t < t'$ and $t' - t < \delta$.

We have, using (33),

$$\mathbb{E}\big(\sup_{s\in[0,T]}|Y^t(s)-Y^{t'}(s)|^2\big)$$

$$\leq \mathbb{E}\big(\sup_{s\in[0,t']}|Y^t(s)-Y^{t'}(s)|^2\big) + \mathbb{E}\big(\sup_{s\in[t',T]}|Y^t(s)-Y^{t'}(s)|^2\big)$$

$$\leq 2\mathbb{E}\big(\sup_{s\in[t,t']}|Y^t(s)-Y^t(t)|^2\big) + 2\mathbb{E}\big(\sup_{s\in[t,t']}|Y^t(t)-Y^s(s)|^2\big)$$

$$+\mathbb{E}\big(\sup_{s\in[t',T]}|Y^t(s)-Y^{t'}(s)|^2\big).$$

From the continuity of the solution of equation (33) with respect to time, we have

$$\mathbb{E}\big(\sup_{s\in[t,t']}|Y^t(s)-Y^t(t)|^2\big)\to 0,$$

as $t'\to t$.

Concerning the term $\mathbb{E}\big(\sup_{s\in[t',T]}|Y^t(s)-Y^{t'}(s)|^2\big)$ let us denote for short, only throughout this step,

$$\Delta Y(r):=Y^t(r)-Y^{t'}(r),\quad \Delta Z(r):=Z^t(r)-Z^{t'}(r)$$

$$\Delta U(r):=U^t(r)-U^{t'}(r),\quad \Delta V(r):=V^t(r)-V^{t'}(r)$$

and

$$\Delta h := h(X^{t,\phi}) - h(X^{t',\phi}),$$

$$\Delta F(r) := F(r,X^{t,\phi},Y^t(r),Z^t(r),U^t_r,V^t_r) - F(r,X^{t',\phi},Y^t(r),Z^t(r),U^t_r,V^t_r).$$

Exploiting Itô's formula we have, for any $\beta>0$ and any $s\in[t',T]$,

$$e^{\beta s}|\Delta Y(s)|^2 + \beta\int_s^T e^{\beta r}|\Delta Y(r)|^2 dr + \int_s^T e^{\beta r}|\Delta Z(r)|^2 dr$$

$$= e^{\beta T}|\Delta Y(T)|^2 - 2\int_s^T e^{\beta r}\langle\Delta Y(r),\Delta Z(r)\rangle dW(r)$$

$$+2\int_s^T e^{\beta r}\langle\Delta Y(r),F(r,X^{t,\phi},Y^t(r),Z^t(r),U^t_r,V^t_r) - F(r,X^{t',\phi},Y^{t'}(r),Z^{t'}(r),U^{t'}_r,V^{t'}_r)\rangle dr.$$

From assumptions (A_3)–(A_5), and noting that it holds

$$\int_s^T e^{\beta r}\big(\int_{-\delta}^0 \big(|\Delta U(r+\theta)|^2 + |\Delta V(r+\theta)|^2\big)\alpha(d\theta)\big)dr$$

$$= \int_{-\delta}^0\Big[\int_s^T e^{\beta r}\big(|\Delta U(r+\theta)|^2 + |\Delta V(r+\theta)|^2\big)dr\Big]\alpha(d\theta)$$

$$= \int_{-\delta}^0\big(\int_{s+\theta}^{T+\theta} e^{\beta(r-\theta)}\big(|\Delta U(r)|^2 + |\Delta V(r)|^2\big)dr\big)\alpha(d\theta)$$

$$\leq e^{\beta\delta}\cdot\int_{-\delta}^0\alpha(d\theta)\cdot\int_0^T e^{\beta r}\big(|\Delta U(r)|^2 + |\Delta V(r)|^2\big)dr$$

$$\leq Te^{\beta\delta}\sup_{r\in[0,T]}\big(e^{\beta r}|\Delta U(r)|^2\big) + e^{\beta\delta}\int_0^T e^{\beta r}|\Delta V(r)|^2 dr,$$

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we have for any $a > 0$,

$$2 \int_s^T e^{\beta r} \langle \Delta Y(r), F(r, X^{t,\phi}, Y^t(r), Z^t(r), U_r^t, V_r^t) - F(r, X^{t',\phi}, Y^{t'}(r), Z^{t'}(r), U_r^{t'}, V_r^{t'}) \rangle dr$$

$$\leq a \int_s^T e^{\beta r} |\Delta Y(r)|^2 dr + \frac{3}{a} \int_s^T e^{\beta r} |\Delta F(r)|^2 dr + \frac{6L^2}{a} \int_s^T e^{\beta r} \left(|\Delta Y(r)|^2 + |\Delta Z(r)|^2 \right) dr$$

$$+ \frac{3TKe^{\beta \delta}}{a} \sup_{r \in [0,T]} \left(e^{\beta r} |\Delta U(r)|^2 \right) + \frac{3Ke^{\beta \delta}}{a} \int_0^T e^{\beta r} |\Delta V(r)|^2 dr.$$

Therefore we have

$$e^{\beta s} |\Delta Y(s)|^2 + \left(\beta - a - \frac{6L^2}{a} \right) \int_s^T e^{\beta r} |\Delta Y(r)|^2 dr + \left(1 - \frac{6L^2}{a} \right) \int_s^T e^{\beta r} |\Delta Z(r)|^2 dr$$

$$\leq e^{\beta T} |\Delta Y(T)|^2 + \frac{3}{a} \int_s^T e^{\beta r} |\Delta F(r)|^2 dr - 2 \int_s^T e^{\beta r} \langle \Delta Y(r), \Delta Z(r) \rangle dW(r)$$

$$+ \frac{3TKe^{\beta \delta}}{a} \sup_{r \in [0,T]} e^{\beta r} |\Delta U(r)|^2 + \frac{3Ke^{\beta \delta}}{a} \int_0^T e^{\beta r} |\Delta V(r)|^2 dr.$$

We now choose $\beta, a > 0$ such that

$$a + \frac{6L^2}{a} < \beta \quad \text{and} \quad \frac{6L^2}{a} < 1, \tag{36}$$

hence we obtain

$$\left(1 - \frac{6L^2}{a} \right) \mathbb{E} \int_s^T e^{\beta r} |\Delta Z(r)|^2 dr \leq \mathbb{E} \left(e^{\beta T} |\Delta h|^2 \right) + \frac{3}{a} \mathbb{E} \int_s^T e^{\beta r} |\Delta F(r)|^2 dr$$

$$+ \frac{3TKe^{\beta \delta}}{a} \mathbb{E} \left(\sup_{r \in [0,T]} e^{\beta r} |\Delta U(r)|^2 \right) + \frac{3Ke^{\beta \delta}}{a} \mathbb{E} \int_0^T e^{\beta r} |\Delta V(r)|^2 dr. \tag{37}$$

and, exploiting Burkholder–Davis–Gundy's inequality, we have

$$2 \mathbb{E} \left(\sup_{s \in [t',T]} \left| \int_s^T e^{\beta r} \langle \Delta Y(r), \Delta Z(r) \rangle dW(r) \right| \right)$$

$$\leq \frac{1}{4} \mathbb{E} \left(\sup_{s \in [t',T]} e^{\beta s} |\Delta Y(s)|^2 \right) + 144 \mathbb{E} \int_{t'}^T e^{\beta r} |\Delta Z(r)|^2 dr.$$

which immediately implies

$$\frac{3}{4} \mathbb{E} \left(\sup_{s \in [t',T]} e^{\beta s} |\Delta Y(s)|^2 \right) \leq \mathbb{E} \left(e^{\beta T} |\Delta h|^2 \right) + \frac{3}{a} \mathbb{E} \int_{t'}^T e^{\beta r} |\Delta F(r)|^2 dr$$

$$+ \frac{3TKe^{\beta \delta}}{a} \mathbb{E} \left(\sup_{r \in [0,T]} e^{\beta r} |\Delta U(r)|^2 \right) + \frac{3Ke^{\beta \delta}}{a} \mathbb{E} \int_0^T e^{\beta r} |\Delta V(r)|^2 dr + 144 \mathbb{E} \int_{t'}^T |\Delta Z(r)|^2 dr.$$

Hence, we have

$$\frac{3}{4}\mathbb{E}\big(\sup_{s\in[t',T]}e^{\beta s}|\Delta Y(s)|^2\big) \leq \mathbb{E}\big(e^{\beta T}|\Delta h|^2\big) + \frac{3}{a}C_1\mathbb{E}\int_{t'}^{T}e^{\beta r}|\Delta F(r)|^2 dr$$

$$+\frac{3TKe^{\beta\delta}C_1}{a}\mathbb{E}\big(\sup_{r\in[0,T]}e^{\beta r}|\Delta U(r)|^2\big) \tag{38}$$

$$+\frac{3Ke^{\beta\delta}C_1}{a}\mathbb{E}\int_{0}^{T}e^{\beta r}|\Delta V(r)|^2 dr,$$

where

$$C_1 := 1 + \frac{144}{1-6L^2/a}.$$

Exploiting thus assumptions (A_3) and (A_5) together with the fact that $X^{\cdot,\phi}$ is continuous and bounded, we have

$$C_1\mathbb{E}\big(e^{\beta T}|\Delta h|^2\big) + \frac{3}{a}C_1\mathbb{E}\int_{t'}^{T}e^{\beta r}|\Delta F(r)|^2 dr \to 0 \quad \text{as } t' \to t.$$

Since $(U,V) \in \mathcal{A} \times \mathcal{B}$, and therefore we have

$$\mathbb{E}\big(\sup_{r\in[0,T]}e^{\beta r}|\Delta U(r)|^2\big) \to 0 \quad \text{and} \quad \mathbb{E}\int_{0}^{T}e^{\beta r}|\Delta V(r)|^2 dr \to 0,$$

as $t' \to t$, we have

$$\mathbb{E}\big(\sup_{s\in[t',T]}e^{\beta s}|\Delta Y(s)|^2\big) \to 0 \quad \text{and} \quad \mathbb{E}\int_{t'}^{T}e^{\beta r}|\Delta Z(r)|^2 dr \to 0, \quad \text{as } t' \to t. \tag{39}$$

We are left to show that the term $\mathbb{E}\big(\sup_{s\in[t,t']}|Y^t(t) - Y^s(s)|^2\big)$ is also converging to 0 as $t' \to t$.

Since the map $t \mapsto Y^t(t)$ is deterministic, we have from equation (33),

$$Y^t(t) - Y^s(s) = \mathbb{E}\big[Y^t(t) - Y^s(s)\big]$$

$$= \mathbb{E}\big[h(X^{t,\phi}) - h(X^{s,\phi})\big] + \mathbb{E}\int_{t}^{T}F(r,X^{t,\phi},Y^t(r),Z^t(r),U_r^t,V_r^t)dr$$

$$-\mathbb{E}\int_{s}^{T}F(r,X^{s,\phi},Y^s(r),Z^s(r),U_r^s,V_r^s)dr$$

$$= \mathbb{E}\big[h(X^{t,\phi}) - h(X^{s,\phi})\big] + \mathbb{E}\int_{t}^{s}F(r,X^{t,\phi},Y^t(r),Z^t(r),U_r^t,V_r^t)dr$$

$$+\mathbb{E}\int_{s}^{T}\big[F(r,X^{t,\phi},Y^t(r),Z^t(r),U_r^t,V_r^t) - F(r,X^{s,\phi},Y^s(r),Z^s(r),U_r^s,V_r^s)\big]dr.$$

Using then assumption (A$_3$) we have

$$|Y^t(t) - Y^s(s)| \le \mathbb{E}|h(X^{t,\phi}) - h(X^{s,\phi})| + \mathbb{E}\int_t^s L\big(|Y^t(r)| + |Z^t(r)|\big)dr$$

$$+ \sqrt{K \int_t^s \mathbb{E}\Big(\int_{-\delta}^0 (|U^t(r+\theta)|^2 + |V^t(r+\theta)|^2)\,\alpha(d\theta)\Big)dr} \cdot \sqrt{s-t}$$

$$+ \mathbb{E}\int_t^s \big|F(r, X^{t,\phi}, 0, 0, 0, 0)\big|dr$$

$$+ \mathbb{E}\int_s^T \big|F(r, X^{t,\phi}, Y^t(r), Z^t(r), U^t_r, V^t_r) - F(r, X^{s,\phi}, Y^t(r), Z^t(r), U^t_r, V^t_r)\big|dr$$

$$+ \mathbb{E}\int_s^T L\big(|Y^t(r) - Y^s(r)| + |Z^t(r) - Z^s(r)|\big)dr$$

$$+ \sqrt{K(T-s)\int_s^T \mathbb{E}\Big(\int_{-\delta}^0 (|U^t(r+\theta) - U^s(r+\theta)|^2 + |V^t(r+\theta) - V^s(r+\theta)|^2)\,\alpha(d\theta)\Big)dr}$$

and therefore we obtain

$$|Y^t(t) - Y^s(s)|$$

$$\le \mathbb{E}|h(X^{t,\phi}) - h(X^{s,\phi})| + L\sqrt{s-t}\sqrt{T\mathbb{E}\sup_{r\in[0,T]}|Y^t(r)|^2 + \mathbb{E}\int_0^T |Z^t(r)|^2 dr}$$

$$+ \sqrt{K}\sqrt{s-t}\sqrt{T\mathbb{E}\sup_{r\in[0,T]}|U^t(r)|^2 + \mathbb{E}\int_0^T |V^t(r)|^2 dr} + (s-t)M(1 + \mathbb{E}\|X^{t,\phi}\|_T^p)$$

$$+ \mathbb{E}\int_s^T \big|F(r, X^{t,\phi}, Y^t(r), Z^t(r), U^t_r, V^t_r) - F(r, X^{s,\phi}, Y^t(r), Z^t(r), U^t_r, V^t_r)\big|dr$$

$$+ L\sqrt{T-s}\sqrt{T\mathbb{E}\sup_{r\in[s,T]}|Y^t(r) - Y^s(r)|^2 + \mathbb{E}\int_s^T |Z^t(r) - Z^s(r)|^2 dr}$$

$$+ \sqrt{K}\sqrt{T-s}\sqrt{T\mathbb{E}\sup_{r\in[0,T]}|U^t(r) - U^s(r)|^2 + \mathbb{E}\int_0^T |V^t(r) - V^s(r)|^2 dr}\,.$$

Taking again into account the fact that $(U, V) \in \mathcal{A} \times \mathcal{B}$, previous step and assumptions (A$_3$) and (A$_5$), we infer that

$$\mathbb{E}\big(\sup_{s\in[t,t']} |Y^t(t) - Y^s(s)|\big) \to 0, \quad \text{as } t' \to t. \tag{40}$$

Concerning the term $\mathbb{E}\int_0^T |Z^t(r) - Z^{t'}(r)|^2 dr$, we see that

$$\mathbb{E}\int_0^T |Z^t(r) - Z^{t'}(r)|^2 dr = \mathbb{E}\int_0^{t'} |Z^t(r) - Z^{t'}(r)|^2 dr + \mathbb{E}\int_{t'}^T |Z^t(r) - Z^{t'}(r)|^2 dr$$

$$= \mathbb{E}\int_t^{t'} |Z^t(r)|^2 dr + \mathbb{E}\int_{t'}^T |Z^t(r) - Z^{t'}(r)|^2 dr\,,$$

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hence, by (37),

$$\mathbb{E}\int_0^T |Z^t(r) - Z^{t'}(r)|^2 dr \to 0, \quad \text{as } t' \to t. \tag{41}$$

Step II.

We are now to prove that Γ is a contraction on the space $\mathcal{A} \times \mathcal{B}$ with respect to the norms

$$||| (Y, Z) |||_{\mathcal{A} \times \mathcal{B}} := \left(|||Y|||_1^2 + |||Z|||_2^2 \right)^{1/2},$$

where

$$|||Y|||_1^2 := \sup_{t \in [0,T]} \mathbb{E}\left(\sup_{r \in [0,T]} e^{\beta r} |Y^t(r)|^2 \right),$$

$$|||Z|||_2^2 := \sup_{t \in [0,T]} \mathbb{E}\int_0^T e^{\beta r} |Z^t(r)|^2 dr.$$

Let us recall that $\Gamma : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \times \mathcal{B}$ is defined by $\Gamma(U, V) = (Y, Z)$, where (Y, Z) is the solution of the BSDE (33).

Let us consider $(U^1, V^1), (U^2, V^2) \in \mathcal{A} \times \mathcal{B}$ and $(Y^1, Z^1) := \Gamma(U^1, V^1), (Y^2, Z^2) := \Gamma(U^2, V^2)$. For the sake of brevity, we will denote in what follows

$$\Delta F^t(r) := F(r, X^{t,\phi}, Y^{1,t}(r), Z^{1,t}(r), U_r^{1,t}, V_r^{1,t})$$
$$- F(r, X^{t,\phi}, Y^{2,t}(r), Z^{2,t}(r), U_r^{2,t}, V_r^{2,t}),$$
$$\Delta U^t(r) := U^{1,t}(r) - U^{2,t}(r), \quad \Delta V^t(r) := V^{1,t}(r) - V^{2,t}(r),$$
$$\Delta Y^t(r) := Y^{1,t}(r) - Y^{2,t}(r), \quad \Delta Z^t(r) := Z^{1,t}(r) - Z^{2,t}(r).$$

Proceeding as in ***Step I***, we have from Itô's formula, for any $s \in [t, T]$ and $\beta > 0$,

$$e^{\beta s}|\Delta Y^t(s)|^2 + \beta \int_s^T e^{\beta r}|\Delta Y^t(r)|^2 dr + \int_s^T e^{\beta r}|\Delta Z^t(r)|^2 dr$$

$$= 2\int_s^T e^{\beta r}\langle \Delta Y^t(r), \Delta F^t(r)\rangle dr - 2\int_s^T e^{\beta r}\langle \Delta Y^t(r), \Delta Z^t(r)\rangle dW(r). \tag{42}$$

Noticing that it holds

$$\frac{2K}{a}\int_s^T e^{\beta r}\left(\int_{-\delta}^0 \left(|\Delta U^t(r+\theta)|^2 + |\Delta V^t(r+\theta)|^2 \right)\alpha(d\theta) \right) dr$$

$$\leq \frac{2K}{a}\int_{-\delta}^0 \left(\int_s^T e^{\beta r}\left(|\Delta U^t(r+\theta)|^2 + |\Delta V^t(r+\theta)|^2 \right) dr \right)\alpha(d\theta)$$

$$\leq \frac{2K}{a}\int_{-\delta}^0 \left(\int_{s+r}^{T+r} e^{\beta(r'-\theta)}\left(|\Delta U^t(r')|^2 + |\Delta V^t(r')|^2 \right) dr' \right)\alpha(d\theta)$$

$$\leq \frac{2K}{a}\int_{-\delta}^0 e^{-\beta\theta}\alpha(d\theta) \cdot \int_{s-\delta}^T e^{\beta r}\left(|\Delta U^t(r)|^2 + |\Delta V^t(r)|^2 \right) dr$$

$$\leq \frac{2Ke^{\beta\delta}}{a}\int_{s-\delta}^T e^{\beta r}\left(|\Delta U^t(r)|^2 + |\Delta V^t(r)|^2 \right) dr.$$

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we immediately have, from assumptions (A_3)–(A_5), that for any $a > 0$,

$$2\left| \int_s^T e^{\beta r} \langle \Delta Y^t(r), \Delta F^t(r) \rangle dr \right| \leq 2 \int_s^T e^{\beta r} |\langle \Delta Y^t(r), \Delta F^t(r) \rangle| dr$$

$$\leq a \int_s^T e^{\beta r} |\Delta Y^t(r)|^2 + \frac{1}{a} \int_s^T e^{\beta r} |\Delta F^t(r)|^2 dr$$

$$\leq a \int_s^T e^{\beta r} |\Delta Y^t(r)|^2 + \frac{2}{a} \int_s^T e^{\beta r} L^2 \left(|\Delta Y^t(r)| + |\Delta Z^t(r)| \right)^2 dr$$

$$+ \frac{2}{a} \int_s^T e^{\beta r} \left(K \int_{-\delta}^0 \left(|\Delta U^t(r+\theta)|^2 + |\Delta V^t(r+\theta)|^2 \right) \alpha(d\theta) \right) dr \tag{43}$$

$$\leq a \int_s^T e^{\beta r} |\Delta Y^t(r)|^2 + \frac{4L^2}{a} \int_s^T e^{\beta r} \left(|\Delta Y^t(r)|^2 + |\Delta Z^t(r)|^2 \right) dr$$

$$+ \frac{2Ke^{\beta\delta}}{a} \int_{s-\delta}^T e^{\beta r} \left(|\Delta U^t(r)|^2 + |\Delta V^t(r)|^2 \right) dr.$$

Therefore equation ([42](#)) yields

$$e^{\beta s} |\Delta Y^t(s)|^2 + \left(\beta - a - \frac{4L^2}{a} \right) \int_s^T e^{\beta r} |\Delta Y^t(r)|^2 dr + \left(1 - \frac{4L^2}{a} \right) \int_s^T e^{\beta r} |\Delta Z^t(r)|^2 dr$$

$$\leq \frac{2Ke^{\beta\delta}}{a} T \sup_{r \in [0,T]} e^{\beta r} |\Delta U^t(r)|^2 + \frac{2Ke^{\beta\delta}}{a} \int_0^T e^{\beta r} |\Delta V^t(r)|^2 dr$$

$$- 2 \int_s^T e^{\beta r} \langle \Delta Y^t(r), \Delta Z^t(r) \rangle dW(r). \tag{44}$$

Let now $\beta, a > 0$ satisfying

$$\beta > a + \frac{4L^2}{a} \quad \text{and} \quad 1 > \frac{4L^2}{a}, \tag{45}$$

we have

$$\left(1 - \frac{4L^2}{a} \right) \mathbb{E} \int_s^T e^{\beta r} |\Delta Z^t(r)|^2 dr$$

$$\leq \frac{2TKe^{\beta\delta}}{a} \mathbb{E} \left(\sup_{r \in [0,T]} e^{\beta r} |\Delta U^t(r)|^2 \right) + \frac{2Ke^{\beta\delta}}{a} \mathbb{E} \int_0^T e^{\beta r} |\Delta V^t(r)|^2 dr. \tag{46}$$

Exploiting now Burkholder–Davis–Gundy's inequality, we have

$$2\mathbb{E} \left[\sup_{s \in [t,T]} \left| \int_s^T e^{\beta r} \langle \Delta Y^t(r), \Delta Z^t(r) \rangle dW(r) \right| \right]$$

$$\leq 4\mathbb{E} \left[\sup_{s \in [t,T]} \left| \int_t^s e^{\beta r} \langle \Delta Y^t(r), \Delta Z^t(r) \rangle dW(r) \right| \right]$$

$$\leq \frac{1}{2} \mathbb{E} \left(\sup_{s \in [t,T]} e^{\beta s} |\Delta Y^t(s)|^2 \right) + 72 \mathbb{E} \int_t^T e^{\beta r} |\Delta Z^t(r)|^2 dr,$$

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which implies

$$\mathbb{E}\big(\sup_{s\in[t,T]} e^{\beta s}|\Delta Y^t(s)|^2\big) \le \frac{2Ke^{\beta\delta}}{a}T\mathbb{E}\big(\sup_{s\in[0,T]} e^{\beta s}|\Delta U^t(s)|^2\big) + \frac{2Ke^{\beta\delta}}{a}\mathbb{E}\int_0^T e^{\beta r}|\Delta V^t(r)|^2 dr$$

$$+2\mathbb{E}\Big[\sup_{s\in[t,T]}\Big|\int_s^T e^{\beta r}\langle \Delta Y^t(r), \Delta Z^t(r)\rangle dW(r)\Big|\Big]$$

$$\le \frac{2Ke^{\beta\delta}}{a}T\mathbb{E}\big(\sup_{s\in[0,T]} e^{\beta s}|\Delta U^t(s)|^2\big) + \frac{2Ke^{\beta\delta}}{a}\mathbb{E}\int_0^T e^{\beta r}|\Delta V^t(r)|^2 dr$$

$$+\frac{1}{2}\mathbb{E}\big(\sup_{s\in[t,T]} e^{\beta s}|\Delta Y^t(s)|^2\big) + 72\mathbb{E}\int_t^T e^{\beta r}|\Delta Z^t(r)|^2 dr.$$

Hence, we have

$$\mathbb{E}\big(\sup_{s\in[t,T]} e^{\beta s}|\Delta Y^t(s)|^2\big)$$

$$\le \frac{4TKe^{\beta\delta}}{a}C_1\mathbb{E}\big(\sup_{s\in[0,T]} e^{\beta s}|\Delta U^t(s)|^2\big) + \frac{4Ke^{\beta\delta}}{a}C_1\mathbb{E}\int_0^T e^{\beta r}|\Delta V^t(r)|^2 dr, \tag{47}$$

where we have denoted by $C_1 := 1 + \frac{72}{1-4L^2/a}$.

Let us now consider the term $\mathbb{E}\big(\sup_{s\in[0,t]} e^{\beta s}|\Delta Y(s)|^2\big)$. From equation (33), we see that,

$$\mathbb{E}\big(\sup_{s\in[0,t]} e^{\beta s}|\Delta Y^t(s)|^2\big) = \mathbb{E}\big(\sup_{s\in[0,t]} e^{\beta s}|Y^{1,t}(s) - Y^{2,t}(s)|^2\big)$$

$$= \mathbb{E}\big(\sup_{s\in[0,t]} e^{\beta s}|Y^{1,s}(s) - Y^{2,s}(s)|^2\big) = \sup_{s\in[0,t]} e^{\beta s}|\Delta Y^s(s)|^2 = \sup_{s\in[0,t]} \mathbb{E}\big(e^{\beta s}|\Delta Y^s(s)|^2\big) \tag{48}$$

so that, exploiting Itô's formula and proceeding as above, we obtain

$$\mathbb{E}\big(e^{\beta s}|\Delta Y^s(s)|^2\big) \le \frac{2TKe^{\beta\delta}}{a}\mathbb{E}\big(\sup_{r\in[0,T]} e^{\beta r}|\Delta U^s(r)|^2\big) + \frac{2Ke^{\beta\delta}}{a}\mathbb{E}\int_0^T e^{\beta r}|\Delta V^s(r)|^2 dr. \tag{49}$$

Thus from inequalities (46–49) we obtain

$$\mathbb{E}\big(\sup_{s\in[0,T]} e^{\beta r}|\Delta Y^t(s)|^2\big) + \mathbb{E}\int_0^T e^{\beta r}|\Delta Z^t(r)|^2 dr$$

$$\le \frac{4TKe^{\beta\delta}}{a}C_1\mathbb{E}\big(\sup_{s\in[0,T]} e^{\beta s}|\Delta U^t(s)|^2\big) + \frac{4Ke^{\beta\delta}}{a}C_1\mathbb{E}\int_0^T e^{\beta r}|\Delta V^t(r)|^2 dr$$

$$+\frac{2TKe^{\beta\delta}}{a(1-4L^2/a)}\mathbb{E}\big(\sup_{r\in[0,T]} e^{\beta r}|\Delta U^t(r)|^2\big) + \frac{2Ke^{\beta\delta}}{a(1-4L^2/a)}\mathbb{E}\int_0^T e^{\beta r}|\Delta V^t(r)|^2 dr$$

$$+\frac{2TKe^{\beta\delta}}{a}\cdot \sup_{s\in[0,t]}\mathbb{E}\big(\sup_{r\in[0,T]} e^{\beta r}|\Delta U^s(r)|^2\big) + \frac{2Ke^{\beta\delta}}{a}\cdot \sup_{s\in[0,t]}\mathbb{E}\int_0^T e^{\beta r}|\Delta V^s(r)|^2 dr.$$

Passing then to the supremum for $t \in [0, T]$ we get

$$|||Y^1 - Y^2|||_1^2 + |||Z^1 - Z^2|||_2^2 \leq \frac{2Ke^{\beta\delta}}{a}\left(3 + \frac{145}{1 - 4L^2/a}\right)\max\{1, T\}\left[|||U^1 - U^2|||_1^2 + |||V^1 - V^2|||_2^2\right].$$

By choosing now $a := \frac{4L^2}{\gamma}$ and β slightly bigger than $\gamma + \frac{4L^2}{\gamma}$, condition (45) is satisfied and, by restriction (C) we have

$$\frac{2Ke^{\beta\delta}}{a}\left(3 + \frac{145}{1 - 4L^2/a}\right)\max\{1, T\} < 1. \tag{50}$$

Eventually, since U and V were chosen arbitrarily, it follows that the application Γ is a contraction on the space $\mathcal{A} \times \mathcal{B}$. Therefore there exists a unique fixed point $\Gamma(Y, Z) = (Y, Z) \in \mathcal{A} \times \mathcal{B}$ and this finishes the proof of the existence and a uniqueness of a solution to our BSDE with delay. ∎

Proof of Theorem 16. Let us first prove the continuity of $\mathbb{A} \ni \phi \mapsto u(t, \phi)$, uniformly with respect to $t \in [0, T]$.

Let us thus take $t \in [0, T]$, $\phi, \phi' \in \mathbb{A}$, and let us denote

$$\Delta Y(r) := Y^{t,\phi}(r) - Y^{t,\phi'}(r), \quad \Delta Z(r) := Z^{t,\phi}(r) - Z^{t,\phi'}(r)$$

and

$$\Delta f(r) := f(r, X^{t,\phi}, Y^{t,\phi}(r), Z^{t,\phi}(r), Y_r^{t,\phi}) - f(r, X^{t,\phi'}, Y^{t,\phi}(r), Z^{t,\phi}(r), Y_r^{t,\phi}).$$

Using similar calculus as in the proof of Theorem 7, for $\beta, a > 0$ such that

$$\beta > a + \frac{6L^2}{a} \quad \text{and} \quad 1 > \frac{6L^2}{a}, \tag{51}$$

we obtain

$$\left(1 - \frac{6L^2}{a}\right)\mathbb{E}\int_t^T e^{\beta r}|\Delta Z(r)|^2 dr$$
$$\leq \mathbb{E}\left(e^{\beta T}|\Delta Y(T)|^2\right) + \frac{3}{a}\mathbb{E}\int_t^T e^{\beta r}|\Delta f(r)|^2 dr + \frac{3Ke^{\beta\delta}}{a}\mathbb{E}\int_{t-\delta}^T e^{\beta r}|\Delta Y(r)|^2 dr. \tag{52}$$

Denoting

$$C_1 := 1 + \frac{144}{1 - 6L^2/a} \tag{53}$$

and choosing $a := \frac{6L^2}{\gamma}$ and β bigger than $\gamma + \frac{6L^2}{\gamma}$, condition (51) is satisfied and

$$T\frac{3Ke^{\beta\delta}}{a}C_1 < \frac{1}{4}, \tag{54}$$

by restriction (C). We then have

$$\frac{1}{2}\mathbb{E}\Big(\sup_{s\in[t,T]} e^{\beta s}|\Delta Y(s)|^2\Big)$$

$$\leq C_1\mathbb{E}\big(e^{\beta T}|\Delta Y(T)|^2\big) + \frac{3C_1}{a}\mathbb{E}\int_t^T e^{\beta r}|\Delta f(r)|^2 dr + \frac{3K\delta e^{\beta\delta}}{a}C_1\mathbb{E}\Big(\sup_{s\in[t-\delta,t]} e^{\beta s}|\Delta Y(s)|^2\Big).$$

(55)

Exploiting the initial conditions satisfied by the $Y^{t,\phi}$, we can rewrite equation (55) as

$$\mathbb{E}\Big(\sup_{s\in[t,T]} e^{\beta s}|\Delta Y(s)|^2\Big) \leq 2C_1\mathbb{E}\big(e^{\beta T}|\Delta Y(T)|^2\big) + \frac{6C_1}{a}\mathbb{E}\int_t^T e^{\beta r}|\Delta f(r)|^2 dr$$

$$+ \frac{6K\delta e^{\beta\delta}}{a}C_1 \sup_{s\in[t-\delta,t]} \mathbb{E}\Big(\sup_{r\in[s,T]} e^{\beta(s-r)}e^{\beta r}|Y^{s,\phi}(r) - Y^{s,\phi'}(r)|^2\Big).$$

Passing to the supremum for $t\in[0,T]$ we have

$$\sup_{t\in[0,T]} \mathbb{E}\Big(\sup_{s\in[t,T]} e^{\beta s}|Y^{t,\phi}(s) - Y^{t,\phi'}(s)|^2\Big)$$

$$\leq 2C_1 \sup_{t\in[0,T]} \mathbb{E}\big(e^{\beta T}|h(X^{t,\phi}) - h(X^{t,\phi'})|^2\big) + \frac{6C_1}{a} \sup_{t\in[0,T]} \mathbb{E}\int_t^T e^{\beta r}|\Delta f(r)|^2 dr \qquad (56)$$

$$+ \frac{6K\delta e^{\beta\delta}}{a}C_1 \sup_{s\in[0,T]} \mathbb{E}\Big(\sup_{r\in[s,T]} e^{\beta r}|Y^{s,\phi}(r) - Y^{s,\phi'}(r)|^2\Big).$$

We can now see that

$$\mathbb{E}\Big(\sup_{s\in[0,t]} e^{\beta s}|Y^{t,\phi}(s) - Y^{t,\phi'}(s)|^2\Big) = \sup_{s\in[0,t]} e^{\beta s}|Y^{s,\phi}(s) - Y^{s,\phi'}(s)|^2$$

$$\leq \sup_{s\in[0,T]} \mathbb{E}\Big(\sup_{r\in[s,T]} e^{\beta r}|Y^{s,\phi}(r) - Y^{s,\phi'}(r)|^2\Big), \qquad (57)$$

so that we can apply again inequality (56).

From inequalities (56) and (57) we can conclude that

$$\sup_{t\in[0,T]} \mathbb{E}\Big(\sup_{s\in[0,T]} e^{\beta s}|Y^{t,\phi}(s) - Y^{t,\phi'}(s)|^2\Big)$$

$$\leq 4C_1 \sup_{t\in[0,T]} \mathbb{E}\big(e^{\beta T}|h(X^{t,\phi}) - h(X^{t,\phi'})|^2\big) + \frac{12C_1}{a} \sup_{t\in[0,T]} \mathbb{E}\int_t^T e^{\beta r}|\Delta f(r)|^2 dr$$

$$+ \frac{12K\delta e^{\beta\delta}}{a}C_1 \sup_{s\in[0,T]} \mathbb{E}\Big(\sup_{r\in[s,T]} e^{\beta r}|Y^{s,\phi}(r) - Y^{s,\phi'}(r)|^2\Big).$$

Since $\delta \leq T$, by (54) we also have $\frac{12K\delta e^{\beta\delta}}{a}C_1 < 1$ and so

$$\left(1 - \frac{12K\delta e^{\beta\delta}}{a}C_1\right)\sup_{t\in[0,T]}\mathbb{E}\left(\sup_{s\in[0,T]}e^{\beta s}|Y^{t,\phi}(s) - Y^{t,\phi'}(s)|^2\right)$$

$$\leq 4C_1\sup_{t\in[0,T]}\mathbb{E}\left(e^{\beta T}|h(X^{t,\phi}) - h(X^{t,\phi'})|^2\right)$$

$$+\frac{12C_1}{a}\sup_{t\in[0,T]}\mathbb{E}\int_t^T e^{\beta r}|f(r, X^{t,\phi}, Y^{t,\phi}(r), Z^{t,\phi}(r), Y_r^{t,\phi})+$$

$$-f(r, X^{t,\phi'}, Y^{t,\phi}(r), Z^{t,\phi}(r), Y_r^{t,\phi})|^2 dr.$$

Let us now fix $\phi \in \mathbb{\Lambda}$. In order to prove that u is continuous in ϕ, uniformly with respect to $t \in [0,T]$, it is enough to show that

$$\mathbb{E}\left(|h(X^{t,\phi}) - h(X^{t,\phi'})|^2\right)$$
$$+\mathbb{E}\int_0^T |f(r, X^{t,\phi}, Y^{t,\phi}(r), Z^{t,\phi}(r), Y_r^{t,\phi}) - f(r, X^{t,\phi'}, Y^{t,\phi}(r), Z^{t,\phi}(r), Y_r^{t,\phi})|^2 dr$$

converge to 0 as $\phi' \to \phi$, uniformly in $t \in [0,T]$.

Since we have no guarantee that the family $\{|Z^{t,\phi}|^2\}_{t\in[0,T]}$ is uniformly integrable, we will use the Lipschitz property of f in the argument (y, z, u) in order to replace $[0,T]$ with a finite subset. By Theorem 7, the mapping $t \mapsto (Y^{t,\phi}, Z^{t,\phi})$ is continuous from $[0,T]$ into $\mathcal{S}_0^{2,1} \times \mathcal{H}_0^{2,d'}$ and therefore uniformly continuous. Consequently, as $n \to \infty$, we have

$$\sup_{|t-t'|\leq\frac{1}{n}}\mathbb{E}\left[\sup_{s\in[0,T]}\left(Y^{t,\phi}(s) - Y^{t',\phi}(s)\right)^2 + \int_0^T \left(Z^{t,\phi}(s) - Z^{t',\phi}(s)\right)^2 ds\right] \to 0.$$

Let, for $n \in \mathbb{N}^*$, $\pi_n := \{0, \frac{T}{n}, \ldots, \frac{(n-1)T}{n}, T\}$, then, by (A_6), we see that

$$\sup_{t\in[0,T]}\sup_{t'\in\pi_n}\mathbb{E}\int_0^T |f(r, X^{t,\phi}, Y^{t',\phi}(r), Z^{t',\phi}(r), Y_r^{t',\phi}) - f(r, X^{t,\phi'}, Y^{t',\phi}(r), Z^{t',\phi}(r), Y_r^{t',\phi})|^2 dr,$$

converges to

$$\sup_{t\in[0,T]}\mathbb{E}\int_0^T |f(r, X^{t,\phi}, Y^{t,\phi}(r), Z^{t,\phi}(r), Y_r^{t,\phi}) - f(r, X^{t,\phi'}, Y^{t,\phi}(r), Z^{t,\phi}(r), Y_r^{t,\phi})|^2 dr,$$

uniformly in ϕ'. We are thus left to prove that

$$\mathbb{E}\left(|h(X^{t,\phi}) - h(X^{t,\phi'})|^2\right)$$
$$+\mathbb{E}\int_0^T |f(r, X^{t,\phi}, Y^{t',\phi}(r), Z^{t',\phi}(r), Y_r^{t',\phi}) - f(r, X^{t,\phi'}, Y^{t',\phi}(r), Z^{t',\phi}(r), Y_r^{t',\phi})|^2 dr$$

converge to 0 as $\phi' \to \phi$, uniformly in $t \in [0,T]$, for fixed $n \in \mathbb{N}^*$ and $t' \in \pi_n$.

Let us thus introduce the modulus of continuity of the functions h and f:

$$m_{h,f}(\epsilon, \mathcal{K}, \mathcal{U}, \kappa) := \sup_{\substack{\phi', \phi'' \in \mathcal{K}, \, t \in [0,T], \, (y,u) \in \mathcal{U} \\ |z| \leq \kappa, \, ||\phi - \phi'||_T \leq \epsilon}} \left(|h(\phi') - h(\phi'')| + |f(t, \phi', y, z, u) - f(t, \phi'', y, z, u)| \right),$$

where $\epsilon > 0$, $\mathcal{K}$ is a compact in $\mathbb{A}$, $\mathcal{U}$ is a compact in $\mathbb{R} \times L^2([-\delta, 0]; \mathbb{R})$ and $\kappa \in \mathbb{R}_+$.

Let $\epsilon > 0$ be fixed, but arbitrary; with no loss generality, we can suppose that the function ϕ' lies in a compact $\mathcal{K} \subseteq \mathbb{A}$.

Using the estimates satisfied by the solution, we deduce that the family $\left\{ \left(X^{t,\phi'}, Y^{t',\phi} \right) \right\}_{(t,\phi') \in [0,T] \times \mathcal{K}}$ is tight with respect to the product topology on $\mathbb{A} \times \mathcal{C}([0,T])$, and therefore, for every $\epsilon > 0$, there exist compact subsets $\mathcal{K}_\epsilon \subseteq \mathbb{A}$ and $\mathcal{K}'_\epsilon \subseteq \mathcal{C}([0,T])$ such that

$$\mathbb{P}\left(X^{t,\phi'} \in \mathcal{K}_\epsilon, \, Y^{t',\phi} \in \mathcal{K}'_\epsilon \right) \geq 1 - \epsilon, \quad \text{for all } (t, \phi') \in [0,T] \times \mathcal{K}.$$

For ease of the notation, let us define $\Phi : \mathbb{A} \times \mathbb{A} \times \mathcal{C}([0,T]) \times \mathbb{R}^d \to \mathbb{R}$ by

$$\Phi(r, \phi', \phi'', y, z) := \frac{1}{T} \left| h(\phi') - h(\phi'') \right|^2 + |f(r, \phi', y(r), z, y_r) - f(r, \phi'', y(r), z, y_r)|^2.$$

We can see by (A_6), that it holds

$$\Phi(\phi', \phi'', y, z) \leq C \left(1 + ||\phi'||_T^{2p} + ||\phi''||_T^{2p} + ||y||_T^2 + |z|^2 \right),$$

where in what follows we will denote by C several possibly different constants depending only on K, L, M and T. Then, for all $t \in [0,T]$, $\phi', \phi'' \in \mathbb{A}$, we have, from the a priori estimates on the processes $Y^{t,\phi}$ and $Z^{t,\phi}$,

$$\mathbb{E}\left[\int_0^T \Phi\left(r, X^{t,\phi'}, X^{t,\phi''}, Y^{t',\phi}, Z^{t',\phi}(r) \right) dr \right]^{p'} \leq C \left(1 + ||\phi'||_T^{pp'} + ||\phi''||_T^{pp'} \right).$$

Let now $\mathcal{U}_\epsilon$ be the image of $[0,T] \times \mathcal{K}'_\epsilon$ through the continuous application $(r, y) \mapsto (y(r), y_r)$, and we also have that $\mathcal{U}$ is compact in $\mathbb{R} \times L^2([-\delta, 0]; \mathbb{R})$.

For arbitrary $\epsilon', \kappa > 0$, we see that,

$$\mathbb{E} \int_0^T \Phi\left(r, X^{t,\phi}, X^{t,\phi'}, Y^{t',\phi}, Z^{t',\phi}(r) \right) dr$$

$$\leq \mathbb{E} \int_0^T \Phi\left(X^{t,\phi}, X^{t,\phi'}, Y^{t',\phi}, Z^{t',\phi}(r) \right)$$

$$\cdot \mathbb{1}_{\left\{ (X^{t,\phi}, Y^{t',\phi}), (X^{t,\phi'}, Y^{t',\phi}) \in \mathcal{K}_\epsilon \times \mathcal{K}'_\epsilon, |Z^{t',\phi}(r)| \leq \kappa, ||X^{t,\phi} - X^{t,\phi'}||_T \leq \epsilon' \right\}} dr$$

$$+ \mathbb{E} \int_0^T \Phi\left(X^{t,\phi}, X^{t,\phi'}, Y^{t',\phi}, Z^{t',\phi}(r) \right) \mathbb{1}_{\left\{ (X^{t,\phi}, Y^{t',\phi}) \notin \mathcal{K}_\epsilon \times \mathcal{K}'_\epsilon \right\}} dr$$

$$+ \mathbb{E} \int_0^T \Phi\left(X^{t,\phi}, X^{t,\phi'}, Y^{t',\phi}, Z^{t',\phi}(r) \right) \mathbb{1}_{\left\{ (X^{t,\phi}, Y^{t',\phi}) \notin \mathcal{K}_\epsilon \times \mathcal{K}'_\epsilon \right\}} dr$$

$$+ \mathbb{E} \int_0^T \Phi\left(X^{t,\phi}, X^{t,\phi'}, Y^{t',\phi}, Z^{t',\phi}(r) \right) \mathbb{1}_{\left\{ |Z^{t',\phi}(r)| > \kappa \right\}} dr$$

$$+ \mathbb{E} \int_0^T \Phi\left(X^{t,\phi}, X^{t,\phi'}, Y^{t',\phi}, Z^{t',\phi}(r) \right) \mathbb{1}_{\left\{ ||X^{t,\phi} - X^{t,\phi'}||_T > \epsilon' \right\}} dr$$

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and therefore

$$\mathbb{E} \int_0^T \Phi\left(r, X^{t,\phi}, X^{t,\phi'}, Y^{t',\phi}, Z^{t',\phi}(r)\right) dr$$

$$\leq T m_{h,f}\left(\epsilon', \mathcal{K}_\epsilon, \mathcal{U}_\epsilon, \kappa\right) + 2 \left\{ \mathbb{E}\left[\int_0^T \Phi\left(r, X^{t,\phi}, X^{t,\phi'}, Y^{t',\phi}, Z^{t',\phi}(r)\right) dr\right]^{p'} \right\}^{1/p'} \epsilon^{1-\frac{1}{p'}}$$

$$+ C \mathbb{E}\left[\left(1 + ||X^{t,\phi}||_T^{2p} + ||X^{t,\phi'}||_T^{2p} + ||Y^{t',\phi}||_T^2 \right) \right] \int_0^T \mathbb{1}_{\left\{|Z^{t',\phi}(r)|>\kappa\right\}} dr$$

$$+ C \mathbb{E} \int_0^T \left| Z^{t',\phi}(r) \right|^2 \mathbb{1}_{\left\{|Z^{t',\phi}(r)|>\kappa\right\}} dr$$

$$+ \left\{ \mathbb{E}\left[\int_0^T \Phi\left(r, X^{t,\phi'}, X^{t,\phi''}, Y^{t',\phi}, Z^{t',\phi}(r)\right) dr\right]^{p'} \right\}^{1/p'} \left[\mathbb{P}\left(||X^{t,\phi} - X^{t,\phi'}||_T > \epsilon' \right)\right]^{1-\frac{1}{p'}}$$

$$\leq T m_{h,f}\left(\epsilon', \mathcal{K}_\epsilon, \mathcal{U}, \kappa\right) + C\, \mathbb{E} \int_0^T |Z^{t',\phi}(r)|^2 \mathbb{1}_{\left\{|Z^{t',\phi}(r)|>\kappa\right\}} dr$$

$$+ C\left(1 + ||\phi||_T^p + ||\phi'||_T^p\right) \left[\epsilon^{1-\frac{1}{p'}} + \frac{\mathbb{E}||X^{t,\phi} - X^{t,\phi'}||_T^{p'-1}}{(\epsilon')^{p'-1}} + \frac{\left(\mathbb{E} \int_0^T |Z^{t',\phi}(r)|^2 dr\right)^{1/2}}{\kappa^2} \right].$$

Therefore we have

$$\sup_{t\in[0,T]} \mathbb{E} \int_0^T \Phi\left(r, X^{t,\phi}, X^{t,\phi'}, Y^{t',\phi}, Z^{t',\phi}(r)\right) dr$$

$$\leq T m_{h,f}\left(\epsilon', \mathcal{K}_\epsilon, \mathcal{U}_\epsilon, \kappa\right) + C\mathbb{E} \int_0^T |Z^{t',\phi}(r)|^2 \mathbb{1}_{\left\{|Z^{t',\phi}(r)|>\kappa\right\}} dr$$

$$+ C\left(1 + ||\phi||_T^p + ||\phi'||_T^p\right) \left[\epsilon^{1-\frac{1}{p'}} + \frac{\left(1 + ||\phi||_T^{p-1} + ||\phi'||_T^{p-1}\right) \mathbb{E}||\phi - \phi'||_T^{p'-1}}{(\epsilon')^{p'-1}} + \frac{1}{\kappa^2} \right].$$

Passing now to the limit as $\phi' \to \phi$, $\epsilon' \to 0$, $(\epsilon, \kappa) \to (0, +\infty)$, we obtain the claim.

Concerning the continuity of $[0,T] \ni t \to u(t,\phi)$, this is an immediate consequence of the continuity of the stochastic process $Y^{t,\phi}$, together with the continuity of the mapping $t \mapsto Y^{t,\phi}$ from $[0,T]$ into $\mathcal{S}_0^{2,1}$. ∎

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