Games and Puzzles

# THE MAXIMUM QUEENS PROBLEM WITH PAWNS

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**Abstract:** *The classic n-queens problem asks for placements of just n mutually non-attacking queens on an $n \times n$ board. By adding enough pawns, we can arrange to fill roughly one-quarter of the board with mutually non-attacking queens. How many pawns do we need? We discuss that question for square boards as well as rectangular $m \times n$ boards.*

**Keywords:** chess, $n$-queens problem, combinatorics.

## The n-queens problem

Take a standard chessboard and as many queens as possible. How many queens can you put on the chessboard, with at most one queen in each square, so that no queen "attacks" (i.e. is a vertical, horizontal, or diagonal move away from) any other queen? Figure 1 shows a placement of eight mutually non-attacking queens. Finding such arrangements is the classic "8-queens problem", first proposed by M. Bezzel in 1848. In 1869 this problem was generalized to the "n-queens problem" of placing $n$ mutually non-attacking queens on an $n \times n$ board. The n-queens problem has solutions for $N = 1$ and $N \geqslant 4$. Hundreds of papers have been written on this problem and its variations, including versions on other board shapes (including cylinders, toruses, and three-dimensional boards) and other piece types (including bishops, rooks, and fairy pieces) . We refer interested readers to the book *Across the Board: The Mathematics of Chessboard Problems* by J. J. Watkins [5], the n-queens survey article by J. Bell and B. Stevens [1], and the online n-queens bibliography maintained by W. Kosters [4].

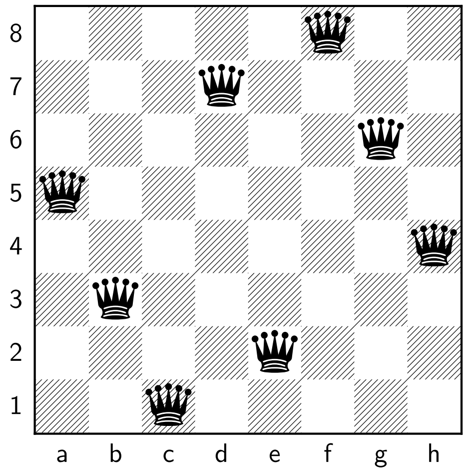


Figure 1: Eight mutually non-attacking queens on a standard chessboard.

## Peppering the problem with pawns

Can we do better? Can we put more than eight queens on a standard chessboard or more than $n$ queens on an $n \times n$ chessboard? If we follow the rules of the classic problem, the answer is clearly "no": each queen attacks every other square on its row (or column), so there is at most one queen per row (or column) and thus at most $n$ queens on an $n \times n$ board.

We now change the rules in order to allow more queens on the board. Recall that in chess, queens do not move through other pieces. If we put a pawn in a square between two queens that are in the same row, column, or diagonal, those queens no longer attack each other. If we allow the placement of some pawns, how many mutually non-attacking queens can we place on the board? This is the "maximum queens problem", posed in 1998 by K. Zhao in her disseration [6].

For $n \geqslant 6$, we can put $n + 1$ queens on the board with 1 pawn [3, Theorem 1]. Figure 2 shows an example arrangement. Zhao proved that we need 3 pawns to put 6 queens on a $5 \times 5$ board [6]. For each $k > 0$, we can place $n + k$ mutually non-attacking queens on an $n \times n$ board with $k$ pawns, if $n$ is sufficiently large [2, Theorem 11].


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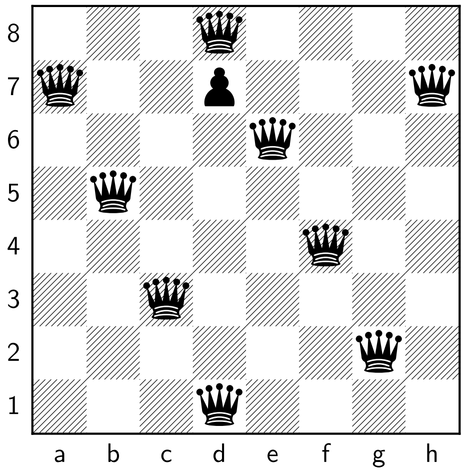


Figure 2: Nine mutually non-attacking queens with one pawn on a standard chessboard.

How much further can we go? If we don't care how many pawns we place on the board, we can place $\frac{n^2}{4}$ queens if $n$ is even and $\frac{(n+1)^2}{4}$ queens if $n$ is odd. To see this, take an $n \times n$ board and divide it into two-column strips (with a one-column strip at the end if $n$ is odd) and divide each strip into two-row block (with one-row blocks at the bottom if $n$ is odd) as shown in Figure 3. Now consider the blocks. If we put two queens in any of these blocks, they will be in adjacent squares and will attack each other, regardless of how many pawns are on the board. So, the maximm number of queens we can put on the board equals the number of blocks, which is $\frac{n^2}{4}$ queens if $n$ is even and $\frac{(n+1)^2}{4}$ queens if $n$ is odd. To see that we can place that many queens, on the first, third, etc. rows place a queen in the first, third, etc. squares and then place pawns in every empty square, as indicated in Figure 3.

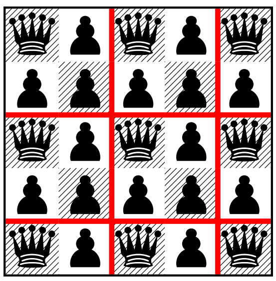

Figure 3: A $5 \times 5$ chessboard divided into blocks, each of which can hold at most one queen if no queens can attack other queens.


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Can we place the maximum number of queens with fewer pawns? If $n$ is odd, the answer is "no". Consider Figure 3 again. When $n$ is odd, the partition produces a $1 \times 1$ block in the last row and column. To get the maximum number of queens, each block must have a queen, so the $1 \times 1$ corner must have a queen. This forces the placement of all the other queens. There is only one way to place the queens, and the squares without queens are each between two queens and therefore require pawns. We now consider the case where $n$ is even.

**Proposition 1.** *For $n = 4k + 2$ with $k \geqslant 1$, it is possible to place $\frac{n^2}{4}$ queens and $\frac{n^2}{4} - 3$ pawns on an $n \times n$ board so that no queens attack each other.*

**P**roof Sketch: We present a pattern with $\frac{n^2}{4}$ queens and $\frac{n^2}{4}$ pawns. Label the rows and columns $0, 1 \ldots n - 1$ as shown in Figure 4. In rows with labels of the form $4k + 1$ (i.e. rows 1, 5, 9,etc.) in the squares whose row number exceeds their column number, place queens in even-numbered columns and pawns in odd-numbered columns. In rows with labels of the form $4k + 3$ (i.e. rows 3, 7, etc.) , in the squares whose row number exceeds their column number, place pawns in even-numbered columns and queens in odd numbered columns. To obtain the rest of the pattern, reflect across the main diagonal and change the piece type, so for each position $(x, y)$, the piece in $(y, x)$ is a queen if and only if the piece in $(x, y)$ is a pawn.

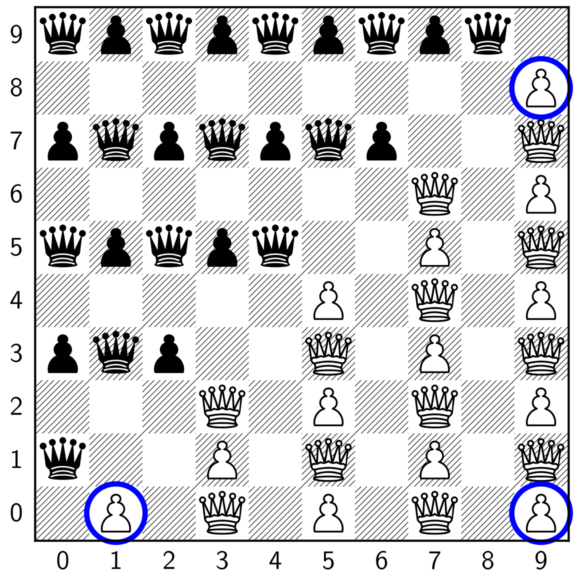


Figure 4: "Corner" pattern.


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We can show that the board has $\frac{n^2}{4}$ queens and $\frac{n^2}{4}$ pawns and that none of the queens attack each other. Finally, we note that we can remove the pawns in positions $(0,1),(0,n-1)$, and $(n-2,n-1)$ and the queens will still not attack each other. ■

When $n$ is a multiple of 4, we can do slightly better.

**Proposition 2.** *For $n = 4k$ with $k \geqslant 1$, it is possible to place $\frac{n^2}{4}$ queens and $\frac{n^2}{4} - 4$ pawns on an $n \times n$ board so that no queens attack each other.*

**P**roof Sketch: Again we present a pattern (as illustrated in Figure 5) that produces the desired results. Given a $4k \times 4k$ board, label the rows and columns $-2k \ldots 2k-1$. In the upper-right quadrant (i.e. the positions with both coordinates nonnegative), place queens and pawns in the "corner" pattern of the previous Proposition, except do not remove any pawns like we did previously. Rotate the board a quarter-turn, relabel rows and columns, and place queens and pawns in the new upper-right quadrant in the corner pattern. Repeat the previous sentence two more times.

We now have a "dartboard" pattern, symmetric with respect to quarter-turn rotation, with $\frac{n^2}{4}$ queens and $\frac{n^2}{4}$ pawns. We can check that none of the queens attack any of the other queens. We conclude by noting that four pawns in the outer ring formed by the first and last rows and the first and last columns (circled in Figure 5) can be removed. ■

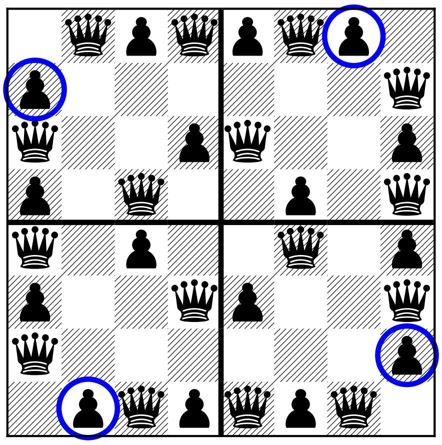

Figure 5: "Dartboard" pattern

Can we do better? Are there patterns on even-order boards with the maximum number of queens but fewer pawns? Computer experiments for small board sizes indicate that the answer is "no", but we don't have a general proof.


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## Stretching to Rectangles

We can extend the results to rectangular boards of size $m \times n$ where $m$ is not necessarily equal to $n$. Dividing the board into blocks as we did for square boards, we get that the maximum number of non-attacking queens we can place on such boards is $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$, where $\lceil x \rceil$ is the smallest integer greater than or equal to $x$. For example, on a $4 \times 7$ board we can place at most $\lceil \frac{4}{2} \rceil \lceil \frac{7}{2} \rceil = \lceil 2 \rceil \lceil 3.5 \rceil = (2)(4) = 8$ queens, as illustrated in Figure 6. If both $m$ and $n$ are odd, there is only one way to place the queens and every other square requires a pawn.

So we make at least one of the dimensions even.

**Proposition 3.** *If $m$ or $n$ is even, we can place $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ queens and $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil - 2$ pawns on an $m \times n$ board so that no two queens attack each other.*

**P**roof sketch: Suppose without loss of generality that $m$ is even. Label the rows $0, 1, \ldots, m-1$ and the columns $0, 1, \ldots, n-1$. In columns with labels of the form $4k$, place queens in the even-numbered rows and pawns in the odd-numbered rows. In columns with labels of the form$4k+2$, place pawns in the even-numbered rows and queens in the odd-numbered rows. We get a striped pattern as illustrated in Figure 6. We can check that none of the queens attack each other and that there are the right number of queens. We conclude by eliminating a pawn in the first and last non-empty columns, as indicated in Figure 6.

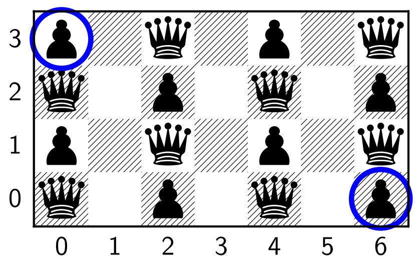


Figure 6: "Stripe" pattern

If $m$ or $n$ is 2, that is the best possible result: A 2-row board without pawns can hold at most 2 queens, one per row. Each pawn increases the number of available "rows" by at most 1. So the total number of queens on such a board is at most 2 more than the number of pawns. The number of pawns needed to get $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ queens on a 2-row (or 2-column) board is $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil - 2$.

If both $m$ and $n$ are even and larger than 2, we can do better, using portions of the "corner" pattern.


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**Proposition 4.** *If $m > 2$ and $n > 2$ are even, then we can place $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$ queens and $\lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil - 3$ pawns on an $m \times n$ board so that no two queens attack each other.*

**P**roof Sketch: Suppose without loss of generality that $m < n$. Take an $n \times n$ board and place queens and pawns according to the procedure for the corner pattern, without the final pawn removal. If $m$ is a multiple of 4, remove rows $m, \ldots, n-1$, and note that the pawns at positions $(0,1), (m-1,0)$ and either $(0, n-1)$ or $(m-1, n-1)$ can be removed. If $m$ is not a multiple of 4, remove columns $m, \ldots, n-1$, note that pawns at $(0,1), (0, m-1)$, and either $(n-1, m-1)$ or $(n-1, 0)$ can be removed, and then transpose rows and columns. ■

Can we do better? The results of computer experimentation with low values of $m$ and $n$ indicate that the answer is "no", but, again, we have no general proof.

## Other Directions

This paper leaves many questions open for future research. Here are a few:

1. A common thing to do with the classic $n$-queens problem is to count the number of solutions for particular values of $n$. Except when $m$ and $n$ are both odd, we haven't done that with the maximum queens problem. Given the maximum number of queens and the minimum necessary number of pawns, how many ways can those pieces be arranged on an $m \times n$ board (with $m$ or $n$ even) so that none of the queens attack each other?

2. Clearly, with enough pawns, we can place $q$ mutually non-attacking queens for $0 \leqslant q \leqslant \lceil \frac{m}{2} \rceil \lceil \frac{n}{2} \rceil$. How many pawns are needed?

3. What results can we get on other types of board, such as cylindrical boards, Mobius strips, and toruses? Observe that the maximum number of queens is the same as the maximum number of kings you can put on the board so that no kings are on adjacent squares: If no two pieces are on adjacent squares, we can separate them with pawns. The maximum number of nonadjacent kings is referred to as the "kings independence number" and it is known for many board types [5]. For example, the kings independence number on an $m \times n$ torus is $\min\{\lfloor \frac{m \lfloor \frac{n}{2} \rfloor}{2} \rfloor, \lfloor \frac{n \lfloor \frac{m}{2} \rfloor}{2} \rfloor\}$ [5, Theorem 11.1] (where $\lfloor x \rfloor$ is the largest integer less than or equal to $x$) and that number is also the maximum number of non-attacking queens we can put on that board with sufficiently many pawns. So, the interesting question is "How many pawns do we need?"

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