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Embeddings of Harary Graphs in Orientable Surfaces

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EMBEDDINGS OF HARARY GRAPHS IN ORIENTABLE SURFACES

By

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Thesis Advisor: Dr. Robert Franzosa

An Abstract of the Thesis Presented
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The purpose of this thesis is to study embeddings of Harary graphs in orientable surfaces. In particular, our goal is to provide a complete description of one method of constructing a maximal embedding in an orientable surface for any Harary graph. Rotation systems, which describe the ordering of edges around the vertices of a graph, can be used to represent graph embeddings in orientable surfaces. Together with the Boundary Walk Algorithm, this representation provides a method of constructing a corresponding graph embedding. By switching adjacent edges in a rotation system, we can control the genus of the constructed embedding surface. We will explore how certain series of adjacent edge switches may be used to take standard rotation systems (which will be defined) to rotation systems corresponding to maximal embeddings of Harary graphs.

DEDICATION

Dedicated to my original teachers, Leon and Amy Smith.

ACKNOWLEDGEMENTS

Firstly, I would like to recognize Dr. Robert Franzosa for his vital contributions to the structure of both the thesis at large and the logic of its arguments, and for inspiring me to look beyond the boundaries of what is explored in the thesis. I would also like to recognize Dr. Eisso Atzema and Dr. Andrew Knightly for their feedback and support throughout this process, and for their role as outstanding educators in my undergraduate and graduate career. I must also recognize the department of Mathematics and Statistics and the Graduate School for providing me with opportunity and resources. Lastly, thank you to my two greatest pillars of strength, my parents. This thesis could not have been written without their continued support.

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CHAPTER 1

INTRODUCTION

1.1 Subject and Scope

A Harary graph, $H_{k,n}$, is a particular kind of k-connected graph that has the smallest number of edges connecting n vertices. We describe them in more detail in the next section. It will be our goal in the following work to provide a method of constructing maximal embeddings of Harary graphs. In doing so, we will also establish the upper embeddability of Harary graphs and provide a method of calculating the genus of that maximal embedding.

It is noteworthy that the collection of Harary graphs contains the collection of complete graphs on n vertices (K_n). In fact, complete graphs are Harary graphs of the form $H_{n-1,n}$. In the second chapter, we will determine a method of constructing maximal embeddings for complete graphs (K_n) via rotation systems (which encode graph embeddings as cyclic orderings of the edges of the graph around each vertex in the embedding). We begin with complete graphs because they are highly symmetric, and because they have historically been a subject of great interest to graph theorists. Then, modeling our approach to constructing maximal embeddings for Harary graphs on the method that we will have established for complete graphs, we will consider $H_{k,n}$ with k even (another highly symmetric sub-collection). In the final chapters of the thesis, we will extend the latter method separately to each of n even and n odd for $H_{k,n}$ with k odd. In each case, we identify a rotation system corresponding to a maximal embedding. It should be noted that such an identification also precisely describes the structure of that maximal embedding when paired with a process called the Boundary Walk Algorithm. The details of this construction process are covered in the following section, after establishing our most basic definitions.

Remark Our result on the upper embeddability of Harary graphs is not new. In fact, it is known that all k-connected graphs, with $k \geq 4$, are upper embeddable [Mohar and Thomassen, 2001]. While that powerful result asserts the upper embeddability of all Harary graphs with $k \geq 4$, it does not provide a direct means for obtaining an embedding. Our results do so, by providing specific rotation systems that yield a maximal embedding via the Boundary Walk Algorithm.

1.2 Preliminary Definitions

For the purposes of this thesis, the word **graph** will always refer to a simple topological graph. A **simple** graph has at most one edge between each pair of vertices and no edges which begin and end at the same vertex. A **topological graph** is a quotient space constructed by gluing the endpoints of a finite set of closed bounded intervals in $\mathbb{R}$ (edges) to elements in a finite set of points (vertices). This definition allows us to consider objects from the field of graph theory as topological spaces.

A graph is **connected** if there is at least one path of edges between any two pairs of vertices in the graph, and **disconnected** otherwise. A graph is **k-connected** if there does not exist a set of $k-1$ vertices whose removal (along with all edges incident to the vertices) disconnects the graph. Therefore, a k-connected graph has *at least* k edges incident to every vertex.

The Harary graphs $H_{k,n}$ ($n \geq 3$, $k \leq n-1$) were introduced by Frank Harary in [Harary, 1962] and can be constructed by first placing n vertices equally spaced around a circle and labeling them counter-clockwise 1 to n, then

- If k is even, connect each vertex to the closest $\frac{k}{2}$ vertices on each side of it.
- If k is odd and n is even, connect each vertex to the closest $\frac{k-1}{2}$ vertices on each

side of it as well as the diametrically opposed vertex (by adding edges between vertex j and vertex $(j + \frac{n}{2}) \mod n$ for $j = 1, 2, 3 \ldots \frac{n}{2}$.

• If k and n are both odd, connect each vertex to the closest $\frac{k-1}{2}$ vertices on each side of it. Then, add edges between vertex j and vertex $(j + \frac{n-1}{2}) \mod n$ for $j = n, 1, 2, 3 \ldots \frac{n-1}{2}$.

Each Harary graph, $H_{k,n}$, is k-connected and has the smallest number of edges connecting n vertices. Figures 1.1 through 1.6 (shown on the following page) are examples of Harary graphs which illustrate how the structures vary with the number of vertices and degree of connectivity.

Harary graphs represent a particular example of the minimal case of k-connectivity for a graph on n vertices. Furthermore, they generalize the collection of complete graphs (K_n) into a larger collection of highly symmetric k-connected graphs. Complete graphs are $(n-1)$-connected graphs having the smallest number of edges on n vertices. That is, every vertex of the complete graph on n vertices has $n - 1$ edges incident to it, since each vertex is adjacent to every other vertex. The collection of complete graphs is precisely the collection of all Harary graphs of the form $H_{n-1,n}$.

In the following chapters we will always label the vertices of the Harary graphs in a simple and consistent manner. Notice that, due to the construction process, there is always an underlying regular n-gon structure for $H_{k,n}$ with $n > 2$, to which additional edges between vertices may be added to satisfy the degree of connectivity specified by k. Then we can label adjacent vertices with integers from 1 to n counter-clockwise along this regular n-gon. Moving forward then, we will always use arithmetic modulo n when referring to vertex labels, and therefore vertex n may also be referred to as vertex 0 in certain contexts.

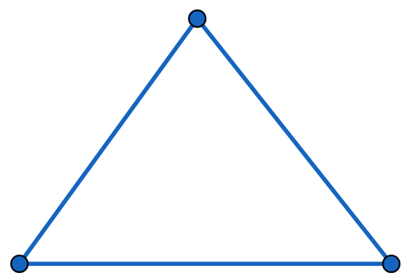

Figure 1.1. $H_{2,3}$

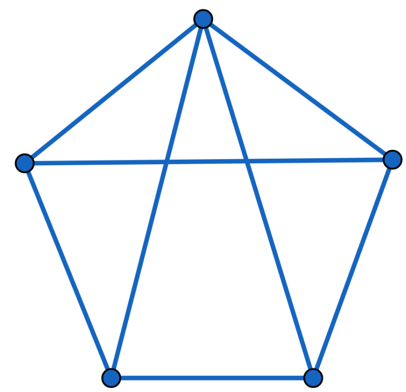

Figure 1.2. $H_{3,5}$

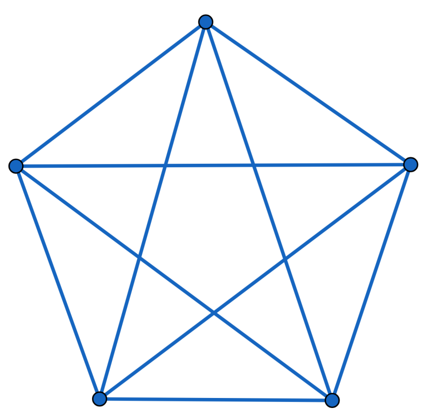

Figure 1.3. $H_{4,5}$

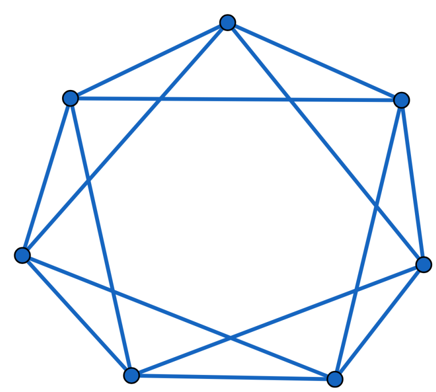

Figure 1.4. $H_{4,7}$

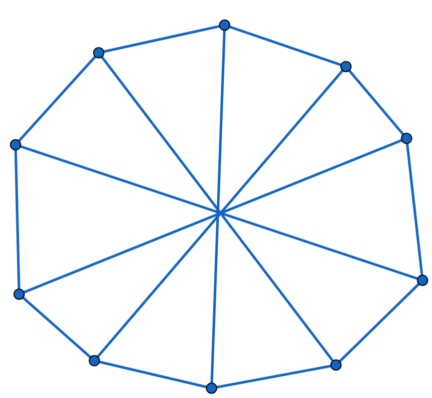

Figure 1.5. $H_{3,10}$

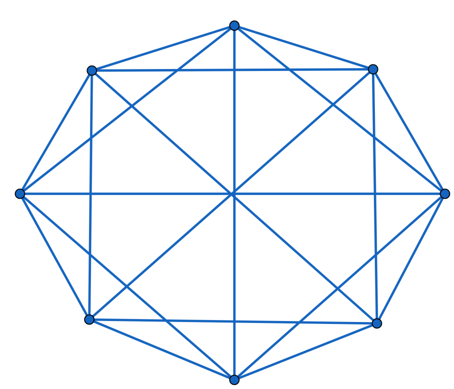

Figure 1.6. $H_{5,8}$

We will begin a discussion of graph embeddings by laying out some preliminary results from elementary topology to which we will refer throughout the work. For further information on topology, see [Adams and Franzosa, 2008]. Technically, a surface S is **orientable** if there does not exist a subspace of S that is homeomorphic to a Möbius band, and is **compact** if for every open cover of S there exists a finite subcover. We will pass over the details and take for granted the result that any compact orientable surface is homeomorphic to one of either the sphere or an n-hole torus (for some $n \in \mathbb{N}$).

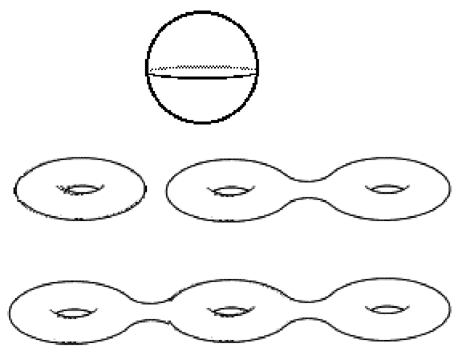

Figure 1.7. Several Orientable Surfaces

We will also use the fact that the **genus** of a compact orientable surface is, put simply, the number of holes in the surface. That is, a sphere has genus 0 and an n-hole torus has genus n. Notably, a compact orientable surface of genus $n \geq 1$ can be represented as a $4n$-gon with edges glued in pairs. For example, a torus can be obtained by gluing the edges of a square as seen in Figure 1.8.

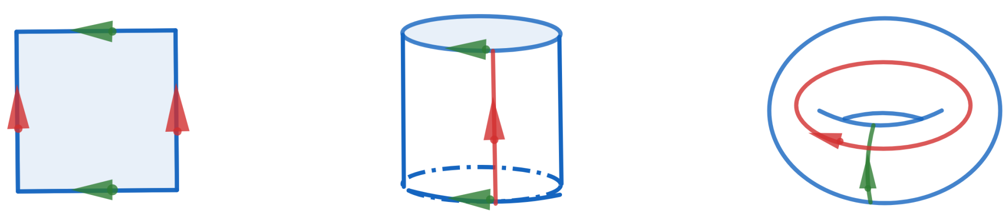

Figure 1.8. A Glued Torus

Furthermore, any collection of polygons with edges glued in pairs results in a compact surface. To ensure that such a pairwise gluing results in a compact **orientable** surface (as opposed to a compact non-orientable surface), we require that each pair of glued edges be oriented in opposite directions with respect to the clockwise orientation of the boundary of the corresponding polygon. A gluing that satisfies this condition is referred to as **orientation preserving**. For example, the polygon collection with vertices (and consequently, edges) labeled for gluing in Figure 1.9, results in the 2-hole torus in Figure 1.10.

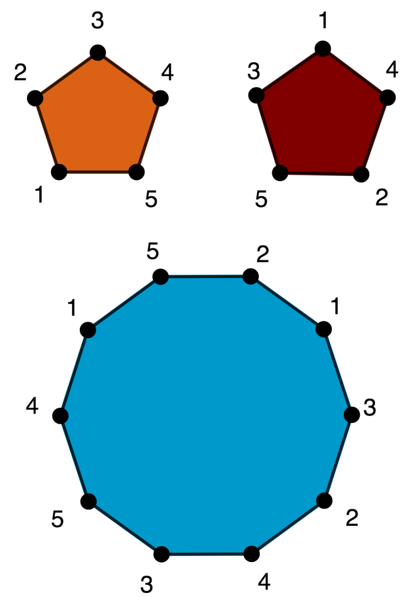

Figure 1.9. A Polygon Collection

Eventually we will introduce an algorithm to identify an orientation preserving gluing of polygons corresponding to an embedding of a given graph, and this requires that we first define graph embeddings and explore their structure.

An **embedding** of a graph G into a surface S is a function $f : G \to S$ that maps G homeomorphically onto the subspace $f(G)$ in S. Thus, an embedding essentially places a copy of G into S, that is, a copy of the graph is drawn on the surface in the colloquial sense. Clearly, such an embedding partitions the surface into components of the complement of the embedded graph. We say that f is a **2-cell embedding** if

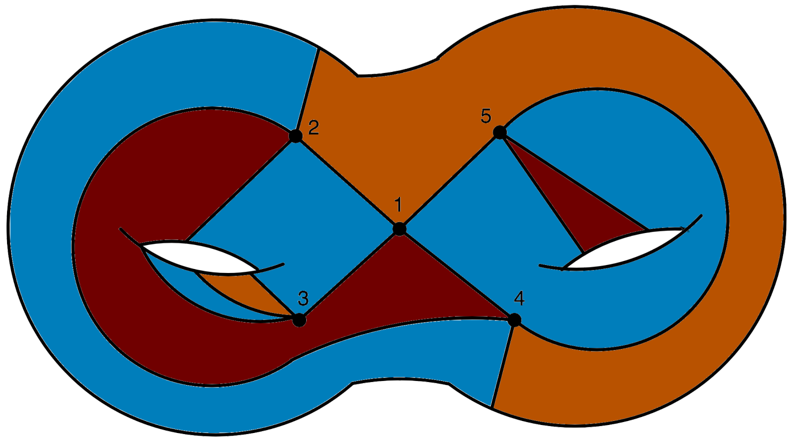

Figure 1.10. A Glued 2-hole Torus

these components are each homeomorphic to the open 2-disk. Note that the interior of a polygon is homeomorphic to the open 2-disk, as depicted by the pentagon under the homeomorphism f shown in Figure 1.11.

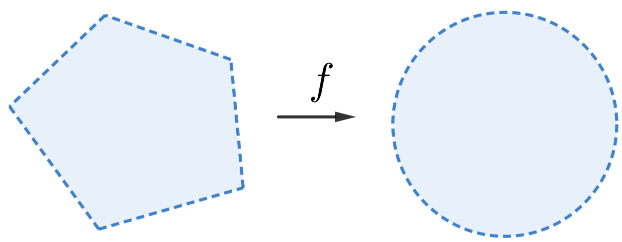

Figure 1.11. A Homeomorphism f of 2-d Interiors

We limit our study of Harary graph embeddings to those which are 2-cell embeddings. This is because 2-cell embeddings decompose an orientable surface into what should be considered the most topologically simple components.

Now, any embedding of a graph in an orientable surface naturally defines a cyclic ordering of the edges around each vertex:

1. Choose an orientation for the surface (clockwise or counterclockwise)

2. At each vertex, take the order of the edges around the vertex in the orientation direction

Figure 1.12 shows a 2-hole torus with an embedding of the complete graph on 5 vertices arising from the polygon gluing shown in Figure 1.9. Choose a counter-clockwise orientation for the surface. Then, for example, we see that a natural cyclic ordering of the edges around vertex 1 would be 12, 13, 14, 15. Around vertex 2 it is 21, 23, 24, 25; around vertex 3 it is 31, 32, 34, 35; around vertex 4 it is 41, 42, 43, 45; and around vertex 5 it is 51, 52, 53, 54.

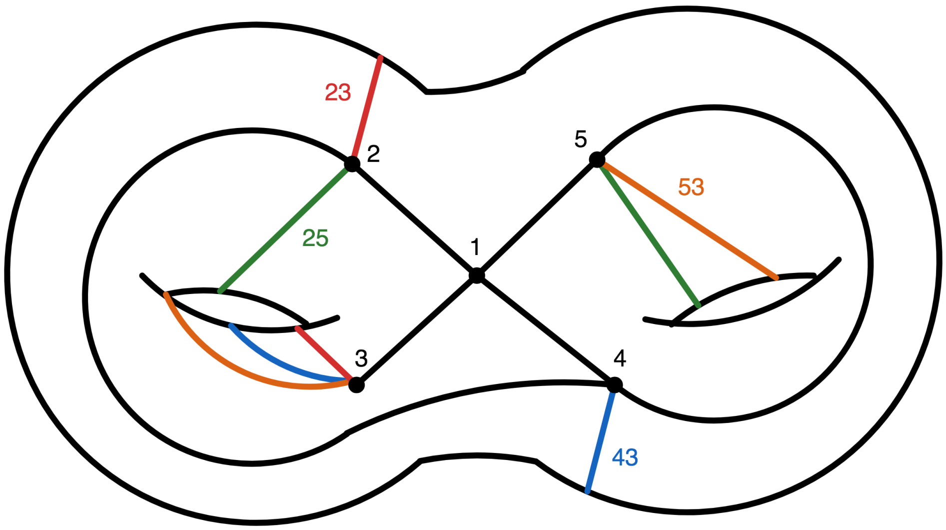

Figure 1.12. A 2-cell Embedding of K_5 in a 2-hole Torus

It is natural to ask if the cyclic ordering of the edges around each vertex necessarily provides enough information to identify the embedding. That is, can we construct a graph embedding using only the knowledge of the ordering of the edges in a given graph embedding such that the constructed graph embedding is topologically equivalent to the original? In the case of 2-cell embeddings, it turns out that the answer is yes, via a construction process involving what are known as rotation systems and boundary walks. We describe that construction process in what follows. For more on rotation systems, see [Mohar and Thomassen, 2001].

A **rotation system** for a graph is a cyclic ordering of the edges around each vertex of the graph. For example, the complete graph on four vertices, K_4, has a rotation system written as follows.

1: 12, 13, 14

2: 21, 23, 24

3: 31, 32, 34

4: 41, 42, 43

We simplify this notation by just listing the vertices at the terminal end of the edge directed away from the central vertex. We would write this rotation system more concisely as

1: 234, 2: 134, 3: 124, 4: 123

Note that the ordering of the edges is cyclic, so that

1: 342, 2: 413, 3: 124, 4: 312

is the same rotation system. It will be helpful to visualize a rotation system as the cyclic ordering which exists in the corresponding embedding as seen in Figure 1.13.

Note that we label the vertices of a graph with integer values, and define an edge between two vertices, say 1 and 2, by either the concatenation 12 or 21. Now, by considering **directed edges**, that is, 12 as the edge directed from 1 to 2 and vice versa, we can remove this ambiguity. A **path** in a graph is any sequence of directed edges $p = (12, 23, 34...(n-1)n)$ transversing vertices and edges of the graph such that each directed edge after the first directed edge begins at the vertex at which

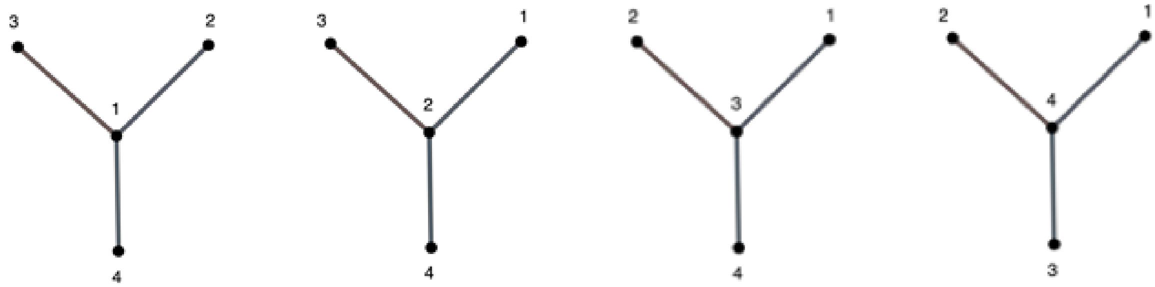

Figure 1.13. A Rotation Sytem for K_4

the previous directed edge ended. We will denote paths as concatenated strings of vertex labels which describe the order of the directed edges in the path. For example, the path p above is written as the string $1234...(n-1)n$. A **closed path** is a path that ends at the same vertex where it begins. For example, the path $p_0 = 1234...(n-1)n1$ is a closed path beginning and ending at vertex 1.

Any 2-cell embedding of a graph in an orientable surface can be seen as a collection of polygons with edges glued in pairs. Given such a 2-cell embedding of a graph, we can always do the following:

1. Label the vertices of the graph in the embedding and identify the glued polygons corresponding to the embedding.

2. Identify the boundary of each polygon as a closed path by listing the directed edges of its boundary in the clockwise direction.

3. List the order in which the edges directed away from each vertex appear in the counter-clockwise direction around each vertex.

The choice of clockwise in step 2 and counter-clockwise in step 3 is arbitrary, in the sense that we could choose different orientations. We only make this particular choice to be consistent with the construction process to come.

We define the closed paths in step 2 as **boundary walks**. We will show below that a given rotation system, as in step 3, corresponds to a boundary walk collection, as in step 2.

Definition Given a rotation system ordering around a vertex v (as in Figure 1.14) and a directed edge $v'v$, the **succeeding edge** to $v'v$ is the directed edge vv'' immediately following the undirected edge vv' in the rotation system ordering at v. Similarly, the **preceding edge** to a directed edge vv''' is the directed edge $v''v$ where vv'' is the undirected edge immediately preceding vv''' in the rotation system ordering at v.

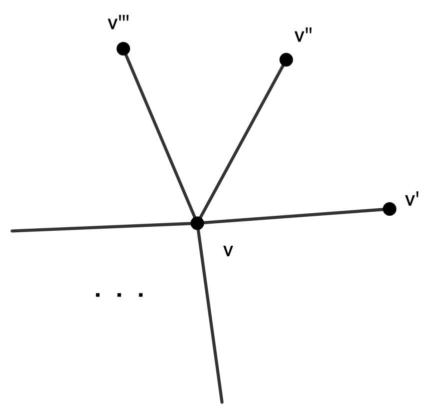

Figure 1.14. A Rotation Sytem at Vertex V

In particular, we can show that by making a path $uv'vv''w...$ of succeeding edges in the rotation system of an embedded graph, we actually trace out a cyclic path along the boundary of a polygon (see Figure 1.15) that bounds a component of the complement of the embedded graph.

For further illustration, refer back to Figure 1.10, which depicts an embedding of K_5 in an orientable surface of genus 2. The rotation system 1: 2345, 2: 1345, 3: 1245, 4: 1235, 5: 1234 describes the ordering of the edges around each vertex in the embedding. Notice that the boundaries of the polygons may be read clockwise as

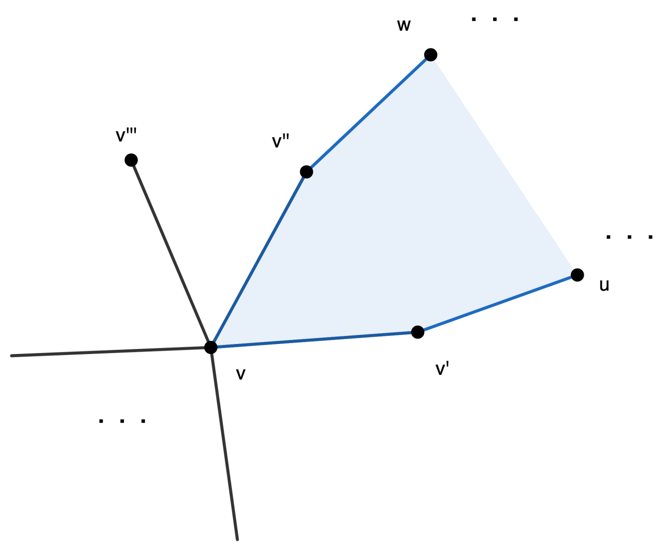

Figure 1.15. A Polygonal Boundary Formed with Succeeding Edges

the closed paths 123451, in the case of the orange pentagon, 531425 for the maroon pentagon, and 54152132435 for the blue decagon. Moreover, the succeeding edge to 12 at vertex 2 is 23, the succeeding edge to 23 at vertex 3 is 34, and so on. Notice that the list of succeeding edges we are forming, $p = (12, 23, 34, 45, 51)$, is precisely the boundary walk corresponding to the orange pentagon. The following algorithm lays out the general method of determining the boundary walk collection associated to a rotation system for a graph, so that the boundary walk collection corresponds to the polygons to be glued to obtain an embedding of the graph in an orientable surface. It will follow that the resulting 2-cell embedding of the graph has the edges ordered around the vertices according to the rotation system. This latter fact is straightforward to prove, but we omit the proof.

Definition The **Boundary Walk Algorithm** takes a rotation system for a graph and generates a corresponding boundary walk collection, and it proceeds as follows:

1. Begin a walk with any directed edge of the graph, say vv', and append the succeeding edge to vv' in the rotation system of v', say $v'v''$, to obtain a path $vv'v''$.

2. Consider the last directed edge designated by the walk, in this case, $v'v''$ and continue appending the succeeding edge in the rotation system of the last vertex, in this case v''. The process terminates at the directed edge immediately preceding the first directed edge to repeat (each boundary walk will end at the preceding edge to the directed edge from which it departed, as in Theorem 1.1 below).

3. If every directed edge of the graph is contained in some boundary walk in the collection, then the algorithm ends. Otherwise, the previous steps are repeated beginning with any directed edge which is not contained in some boundary walk already in the collection.

Notice that in addition to each boundary walk being finite, the boundary walk collection will also be finite, since there is a finite number of directed edges in any graph. Thus, given any rotation system, the Boundary Walk Algorithm eventually terminates.

Theorem 1.1 (The Boundary Walk Theorem) *Given a rotation system, every boundary walk ends at the preceding edge to the directed edge from which it departed, and every directed edge in the walk is distinct.*

Proof Begin a walk from a directed edge g_0g_1. There is a finite number of directed edges in any graph. Therefore, after a finite number of concatenations, some directed edge g must be the first to appear for a second time. Since the rotation system is fixed, the preceding edge to any directed edge is also fixed. Therefore, the first directed edge to repeat is that which has no preceding edge listed in the path. Therefore, $g = g_0g_1$. The Boundary Walk Algorithm terminates the walk at the preceding edge to g_0g_1, and thus every directed edge included in the walk is distinct.

∎

Example Consider the complete graph on five vertices, K_5. Take a rotation system at K_5 given by following orderings:

1: 2345, 2: 1345, 3: 1245, 4: 1235, and 5: 1234.

Apply the Boundary Walk Algorithm, beginning a walk with directed edge 12. We obtain the first boundary walk in the list below. Then begin a walk with a directed edge not contained in the first boundary walk, directed edge 54. We obtain the second boundary walk in the list. Finally, begin a walk with another directed edge not already contained in the first two boundary walks, directed edge 53. We obtain the third boundary walk.

123451

54152132435

531425

We can see that this boundary walk collection contains 20 distinct directed edges, and we know that K_5 has 10 edges. Therefore, each directed edge of K_5 is contained in a boundary walk listed in the collection, and the Boundary Walk Algorithm was terminated. Notice that these boundary walks correspond to the boundary of the polygons in Figure 1.9, which glue to give the embedding of K_5 in a 2-hole torus shown in Figure 1.10.

Now, since the Boundary Walk Algorithm acts together with a rotation system to generate the polygons associated to an embedding, and the rotation system ordering at any vertex is a cyclic ordering of the edges appearing in the embedding that result from gluing these polygons, it will be natural to consider how the corners of the polygons appear relative to vertices in the embeddings.

Definition A **corner** of a boundary walk (polygon) is a path consisting of exactly two directed edges contained in the boundary walk.

Take an unspecified graph and consider a rotation system ordering around a vertex, say 1, given by 1 : 234. The boundary walk algorithm will generate boundary walks, one of which will necessarily include the corner 213. The same is true of 314 and 412. For the time being, assume each corner belongs to a distinct boundary walk. Then we may color the interior of each corresponding polygon a different color to represent each boundary walk. Say we color that of 213 red, 314 green, and 412 blue. Then we can represent each of these three corners in a diagram of the rotation system ordering at vertex 1 in the corresponding embedding shown in Figure 1.16.

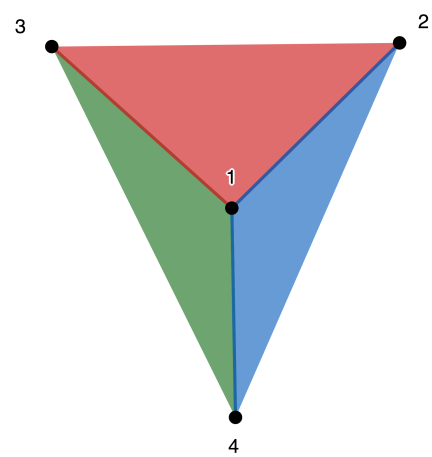

Figure 1.16. A Rotation System at Vertex 1 with Coloring

Notice that this precisely represents the corners of the polygonal faces of the boundary walks as as they actually meet in the embedding in the orientable surface. It could also be the case that two or more corners in Figure 1.16 are in the same boundary walk, such as in Figure 1.17. Such diagrams will be of great use as a visual aid to the thesis.

To recap, we may assign a cyclic ordering of the edges adjacent to each vertex in a graph as a rotation system. This assignment describes an embedding of the graph

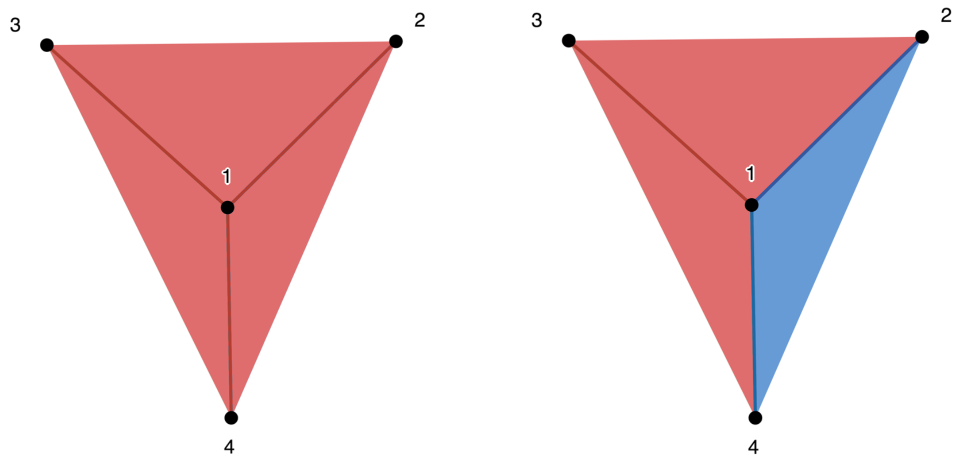

Figure 1.17. Rotation Systems at Vertex 1 with Colorings

in an orientable surface in which the cyclic ordering of the edges around the embedded vertices is identical to the cyclic ordering of the rotation system. Furthermore, the embedding corresponding to the rotation system is found by gluing the edges of the collection of polygons generated via the Boundary Walk Algorithm acting on the rotation system. Each boundary walk generated by the algorithm is a closed path, which, given a fixed orientation to preserve the relationship of interior to the exterior, transverses the boundary of a polygon. Therefore, the embedding of the graph corresponding to a particular rotation system can be seen as the seams of the glued polygons. In turn, each polygon can be seen as a geometric face of the embedding surface. Together, the embedding and its complement decompose a surface into various polygons, or two-dimensional surfaces. Hence, faces, polygons, and boundary walks are considered to be synonymous terms in further discussions.

Many readers will be familiar with Euler's Formula for convex polyhedra. The formula asserts that for any convex polyhedron, the sum of the vertices and faces is exactly two more than the number of edges ($V - E + F = 2$). The octahedron (Figure 1.18, for example, has 6 vertices, 12 edges, and 8 faces, giving $6 - 12 + 8 = 2$. Notice that the octahedron, or any polyhedron for that matter, is homeomorphic to the sphere.

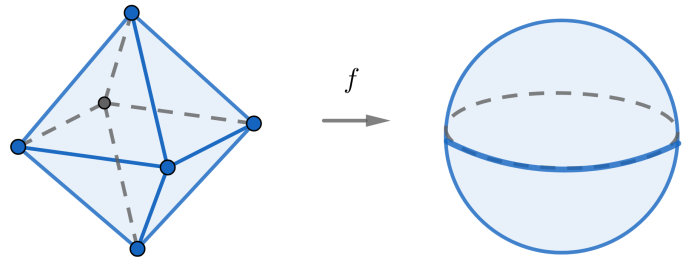

Figure 1.18. A Homeomorphism f of the Octahedron to the Sphere

Euler's Formula applies more generally to topological surfaces of genus other than 0. For orientable surfaces, a more general form of Euler's Formula is given by

$$V - E + F = 2 - 2g \tag{1.1}$$

where g is the genus of the surface, and the vertices, edges, and faces are those corresponding to a 2-cell embedding of a graph in the surface. This is important for us because in Formula 1.1, V and E are fixed by the graph, while F may vary across embeddings (that is, we may glue together various numbers of different polygons to embed the same graph in different surfaces).

Example We will pick up from the previous example with the same rotation system for K_5. Notice that Euler's Formula for the embedding surface formed by gluing the polygons in Figure 1.9 gives the following:

$$\begin{aligned} V - E + F &= 2 - 2g \\ 5 - 10 + 3 &= 2 - 2g \\ g &= 2 \end{aligned} \tag{1.2}$$

Thus g=2, confirming that the gluing of the polygons does in fact result in a 2-hole torus.

A **maximal embedding** is a 2-cell embedding of a graph into a surface of the largest possible genus. Therefore, to identify a maximal embedding, we must minimize F. Given a graph, by Formula 1.1, F is either always odd or always even across all 2-cell embeddings. Therefore, the maximal embedding will theoretically have either 1 or 2 faces. A graph is called **upper embeddable** if this maximum can be realized.

Example Now, consider K_5 with the rotation system given by the orderings:

1: 2345, 2: 1345, 3: 1245, 4: 1235, and 5: 2134.

Apply the Boundary Walk Algorithm beginning with directed edge 12 to obtain the boundary walk:

123452132435415314251

This boundary walk contains every directed edge of K_5. By gluing the edges of the polygon we obtain a maximal embedding of K_5 in an orientable surface, since we have minimized the number of faces F in Formula 1.1. We may now compute the genus of the maximal embedding surface:

$$\begin{aligned} V - E + F &= 2 - 2g \\ 5 - 10 + 1 &= 2 - 2g \\ g &= 3 \end{aligned} \tag{1.3}$$

It is with the tools described in this section that we will establish methods of constructing a maximal embedding for any Harary graph. That is, in the chapters that follow, our goal will be to determine a rotation system corresponding to a maximal embedding for various sub-collections of Harary graphs. We will begin with the case $H_{n-1,n}$ (complete graphs K_n) to model the approach for $H_{k,n}$ with k even.

CHAPTER 2

A STANDARD ROTATION SYSTEM FOR K_n

In order to analyze embeddings of the family of complete graphs (K_n) in orientable surfaces, we will define a standard rotation system for K_n. We will assume that $n > 2$, and assign integer labels 1 through n to the vertices of K_n as specified in Chapter 1. For each vertex, we will choose a rotation system which orders the edges counter-clockwise around the vertex in numerical order. It is important to remember that all arithmetic is done modulo n. In what follows then, equivalence is to be understood as modular equivalence.

Definition The standard rotation system for the complete graph K_n is

$1 : 234 \dots n$

$2 : 134 \dots n$

$3 : 124 \dots n$

$\vdots$

$n : 123 ... n - 1$

The standard rotation system gives an associated boundary walk collection generated by the Boundary Walk Algorithm. By gluing the corresponding collection of polygons, we obtain an associated embedding of K_n into an orientable surface. We will use Lemma 2.1 to determine the structure of these boundary walks in Theorem 2.3.

Lemma 2.1 (The First Boundary Walk Lemma) *Assume the standard rotation system for K_n. Then in any boundary walk in the corresponding collection, the succeeding edge to a directed edge pq is*

1. $q(p + 1)$ *if* $q \neq p + 1$

2. $q(p + 2)$ *if* $q = p + 1$.

Proof

1. Assume that $q = p + 1$. Then the standard rotation system orders $q(p+1)$ after edge qp at vertex q. Therefore, the succeeding edge to pq in this case is $q(p+1)$.

2. The standard rotation system orders $p(p+2)$ after $p(p+1)$ at vertex p. Therefore, $(p+1)(p+2)$ is the succeeding edge to $p(p+1)$. ∎

Corollary 2.2 *(The First Boundary Walk Corollary) For the standard rotation system, a boundary walk with initial directed edge $p(p+1)$ has the form*

$$p(p+1)(q+1)(p+2)(q+2)...(p-1)(q-1)p,$$

*and every directed edge xy of K_n such that $y - x = q - p$ **or** $y - x = p + 1 - q$ is contained in the boundary walk.*

Proof Assume that initial directed edge pq is such that $q \neq p+1$. Then it follows from the lemma that pq is followed by $q(p+1)$ in the boundary walk beginning pq. Note that $q \neq p$ since no edge of the graph is a loop. Then $p+1 \neq q+1$, and $q(p+1)$ is thus a directed edge of the form pq above. Let $k \in \mathbb{Z}^+$. By repeating the same argument, $p+k \neq q+k$. Also, $q+k \neq p+1+k$ since $q \neq p+1$. So, any directed edge $(q+k)(p+k+1)$ or $(p+k)(q+k)$ is a directed edge of the form pq above. Thus, beginning a boundary walk with directed edge pq results in the form

$$pq(p+1)(q+1)(p+2)(q+2)...(p-1)(q-1)p.$$

Now, consider the boundary walk

$$pq(p+1)(q+1)(p+2)(q+2)\ldots(p-1)(q-1)p.$$

Notice that

$$\{p, p+1, p+2\ldots p-1\} = \{1, 2, 3\ldots n\} = \{q, q+1, q+2\ldots q-1\},$$

so that every directed edge $(q+k)(p+k+1)$ or $(p+k)(q+k)$ with $k \in \mathbb{Z}^+$ appears in the walk. Any directed edge xy such that $y-x=q-p$ or $y-x=p+1-q$ is of the form $(q+k)(p+k+1)$ or $(p+k)(q+k)$ with $k \in \mathbb{Z}^+$. Therefore, every directed edge xy such that $y-x=q-p$ or $y-x=p+1-q$ appears in the boundary walk. ∎

Theorem 2.3 (The Standard Collection for K_n) *The boundary walk collection associated to the standard rotation system for K_n is:*

$BW_0: \;\; 123\ldots n1$

$BW_1: \;\; 1n2132\ldots(n-2)n(n-1)1$

$BW_2: \;\; 2n3142\ldots(n-3)n(n-2)1(n-1)2$

$BW_3: \;\; 3n4152\ldots(n-4)n(n-3)\ldots(n-1)3$

$\vdots$

$BW_{\lceil\frac{n}{2}-1\rceil}: \;\; (n-\lceil\frac{n}{2}\rceil)n(n-\lceil\frac{n}{2}-1\rceil)1(n-\lceil\frac{n}{2}\rceil)2\ldots(n-1)(n-\lceil\frac{n}{2}\rceil)$

The length of the final boundary walk, $BW_{\lceil\frac{n}{2}-1\rceil}$, is n if n is odd, and $2n$ if n is even.

Proof Begin the Boundary Walk Algorithm with directed edge 12. By Lemma 2.1, the next directed edges are 23, 34, etc. The walk ends when the algorithm encounters the preceding edge to 12. Directed edge $n1$ is the preceding edge to directed edge 12, again by Lemma 2.1. Then the first boundary walk is of the form

$$BW_0: \;\; 123\ldots(n-1)n1.$$

Every directed edge xy such that $y-x=1$ (modulo n) is contained in BW_0 by construction. Therefore, directed edge $1n$ is not included in BW_0, since $n-1 \neq 1$, and the Boundary Walk Algorithm may begin a new walk with directed edge $1n$. By Lemma 2.1, the boundary walk beginning with directed edge $1n$ is of the form

$$BW_1: \ \ 1n2132\ldots(n-2)n(n-1)1.$$

BW_1 contains every directed edge xy of K_n such that $y-x=n-1$ or $y-x=2-n=2$ by Corollary 2.2. More generally, the Boundary Walk Algorithm may begin a new walk with directed edge jn (so long as jn is not included in a boundary walk already listed) giving

$$BW_j: \ \ jn(j+1)1(j+2)2\ldots(n-j-1)n(n-j)\ldots(n-1)j.$$

BW_j contains every directed edge xy of K_n such that $y-x=n-j$ or $y-x=j+1-n=j+1$ by Corollary 2.2.

Assume that $j \geq \lceil \frac{n}{2} \rceil$. Then $n-j \leq j$. It follows that for each $j \geq \lceil \frac{n}{2} \rceil$, one of BW_0 through $BW_{\lceil \frac{n}{2}-1 \rceil}$ already contains every directed edge xy such that $y-x=j$. Thus, every directed edge of the graph is already contained in some boundary walk in the collection, and the Boundary Walk Algorithm terminates after listing $BW_{\lceil \frac{n}{2}-1 \rceil}$.

It should be clear from the construction that BW_0 contains n directed edges, while BW_1 through $BW_{\lceil \frac{n}{2}-2 \rceil}$ each contain $2n$ directed edges. However, notice that $n-j=j+2$ when $j=\frac{n}{2}-1=\lceil \frac{n}{2}-1 \rceil$ and n is even, but that $n-j=j+1$ when $j=\frac{n-1}{2}=\lceil \frac{n}{2}-1 \rceil$ and n is odd. By extension, $n-j-1=j$ in the odd case as well. Then, the boundary walk terminates at the preceding edge to directed edge $(n-j-1)n$, since it is the same as the initial directed edge of the boundary walk, jn. Then, when n is odd, $BW_{\lceil \frac{n}{2}-1 \rceil}$ is of half the length of each of BW_1 through $BW_{\lceil \frac{n}{2}-2 \rceil}$. ∎

Observe that BW_0 contains corner $(n-1)n1$, and that for $1 \leq j \leq \lceil \frac{n}{2}-1 \rceil$, it is the case that BW_j contains corners $jn(j+1)$ and $(n-j-1)n(n-j)$, except in the particular case $j=\frac{n}{2}-1$ and n is odd, where the two corners are equivalent (as shown above). This nicely illustrates the relationship between the form of the boundary walks and how the corresponding polygons are glued together around vertex n (see Figures 2.1-2.5 below). We see that BW_0 glues to BW_1 in edge $n1$, with BW_0 to the right of directed edge $n1$ and BW_1 to the left (relative to the edge direction). Likewise, BW_1 glues to BW_2 in edge $n2$, with BW_1 to the right of $n2$ and BW_2 to the left, and so on... until $BW_{\lceil \frac{n}{2}-2 \rceil}$ glues to $BW_{\lceil \frac{n}{2}-1 \rceil}$ in edge $n\lceil \frac{n}{2} - 1 \rceil$ with $BW_{\lceil \frac{n}{2}-2 \rceil}$ to the right of $n\lceil \frac{n}{2} - 1 \rceil$ and $BW_{\lceil \frac{n}{2}-1 \rceil}$ to the left.

Furthermore, BW_0 glues to to BW_1 in edge $n(n-1)$ with the former to the left and the latter to the right. Likewise, BW_1 glues to to BW_2 in edge $n(n-2)$ with the former to the left and the latter to the right, and so on... until $BW_{\lceil \frac{n}{2}-2 \rceil}$ glues to to $BW_{\lceil \frac{n}{2}-1 \rceil}$ in edge $n\lceil \frac{n}{2} \rceil$ if n is odd, and in edge $n(\frac{n}{2}+1)$ if n is even, with the former BW to the left and the latter to the right of their respective gluing edges. This completes the gluing process if n is odd. If n is even, then we also see that $BW_{\lceil \frac{n}{2}-1 \rceil}$ glues to itself in edge $n\frac{n}{2}$. As a result of these observations we have the following:

Theorem 2.4 *For the standard rotation system on K_n, when the corresponding BW polygons are glued together, the edges around vertex n appear in the rotation system ordering:*

$$n1, n2, n3 \ldots n(n-1)$$

and the boundary walk polygons appear in the ordering:
BW_0 (between edges $n(n-1)$ and $n1$), BW_1, BW_2, ... $BW_{\lceil \frac{n}{2}-2 \rceil}$, $BW_{\lceil \frac{n}{2}-1 \rceil}$, $BW_{\lceil \frac{n}{2}-1 \rceil}$ (if n is even), $BW_{\lceil \frac{n}{2}-2 \rceil}$, ... BW_2, BW_1.

This theorem strongly informs our general method of determining a rotation system associated to a maximal embedding for K_n to be discussed in the following chapter.

Remark The choice of vertex n here is not significant. In fact, by making the same observations with regards to the gluing around any other vertex, we would see that the ordering of the edges around that vertex in the gluing also appear in the rotation system ordering, and the corners of the boundary walks appear in the same arrangement around that vertex as around vertex n in Theorem 2.4.

The standard collection for K_n gives an associated 2-cell embedding into an orientable surface upon gluing. It is a small feat of calculation now to determine the genus of this surface using Equation 1.1, since the number of vertices and edges are fixed by the graph, and the number of faces in the standard collection is $\lceil \frac{n}{2} \rceil$:

$$g = \frac{V - E - F - 2}{-2} = \frac{n - \frac{n(n-1)}{2} + \lceil \frac{n}{2} \rceil - 2}{-2}$$

Given any complete graph K_n, we now immediately know the genus of one embedding surface, as well as the details contained in the Theorems of this chapter regarding the structure of that embedding surface. Note that the genus of this surface gives us a lower bound on the genus of the surface of a maximal embedding. However, we can in fact precisely find the maximal genus by determining a rotation system which gives a maximal embedding. This will be done in the next chapter.

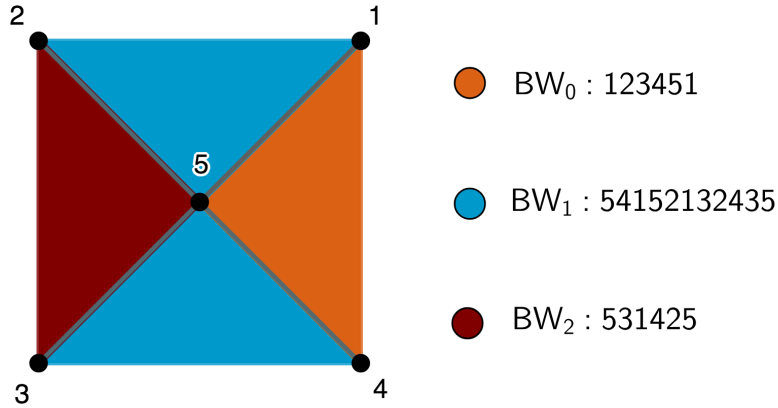

Figure 2.1. Vertex 5, K_5

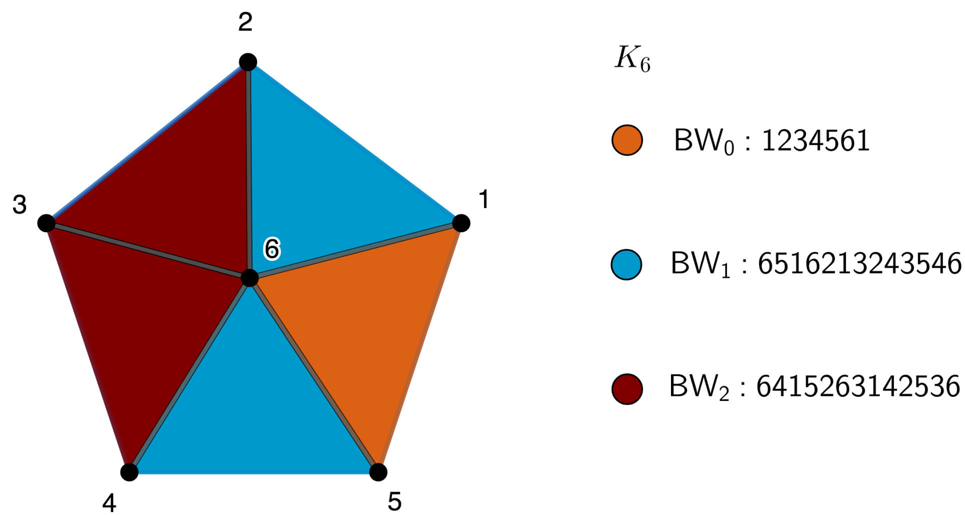

Figure 2.2. Vertex 6, K_6

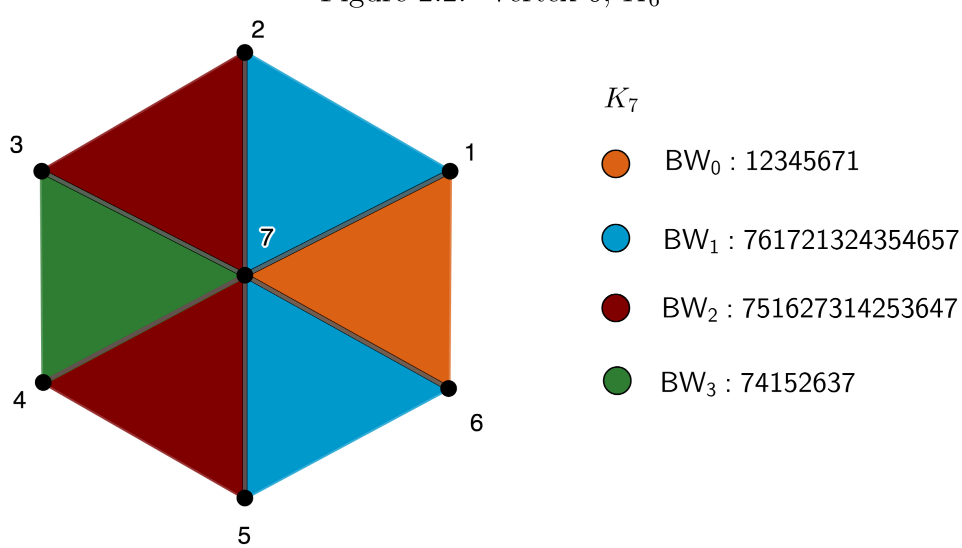

Figure 2.3. Vertex 7, K_7

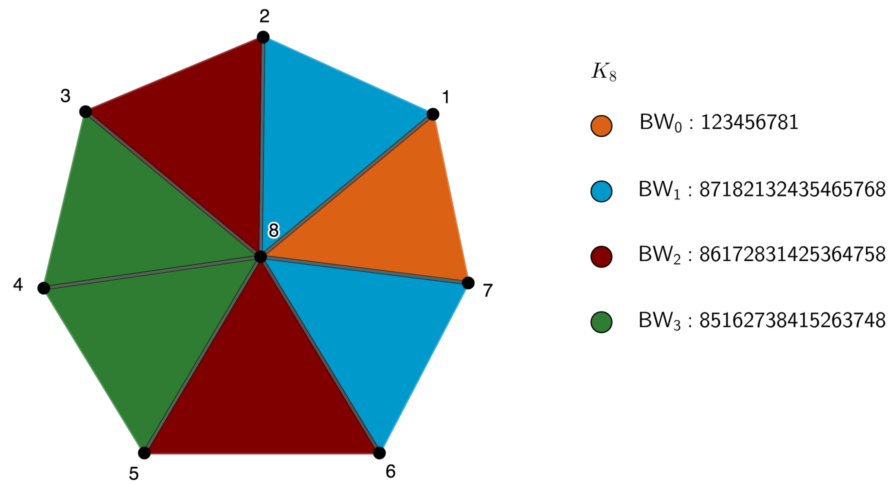

Figure 2.4. Vertex 8, K_8

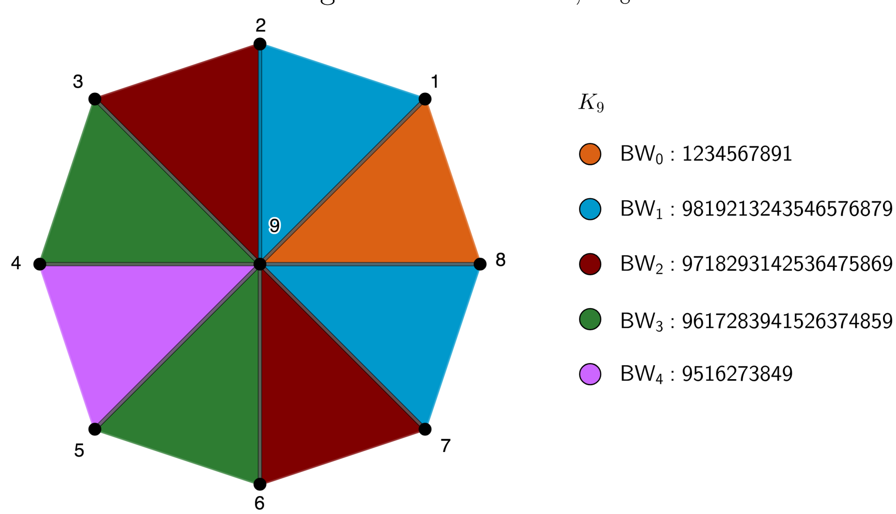

Figure 2.5. Vertex 9, K_9

CHAPTER 3

A MAXIMAL EMBEDDING OF K_n

In order to construct maximal embeddings, we will control the number of faces (i.e. boundary walks) in embeddings associated to rotation systems for K_n by making switches in the ordering of edges at vertices in the standard rotation system. The result of switching adjacent edges in a rotation system ordering at a vertex depends on which of the boundary walks in the collection the switched edges belong to. One such result is described in the following theorem. Note that we will use the notation $\omega_{ab\to cd}$ to denote a path in a boundary walk which is the list of directed edges beginning with the succeeding edge to directed edge ab and ending at the preceding edge to directed edge cd. For example, $ab\omega_{ab\to cd}cd\omega_{cd\to ab}$ describes a complete boundary walk starting at ab, including cd, and ending at the preceding edge to ab.

Theorem 3.1 (The First Edge Switching Theorem) *Assume a rotation system includes the ordering*

$$j : \ldots x12y \ldots$$

at vertex j and generates at least three distinct boundary walks

$$2jy\omega_{jy\to 2j},\ 1j2\omega_{j2\to 1j},\ \textit{and}\ xj1\omega_{j1\to xj}.$$

Then, swapping edge $j1$ with $j2$ at j {see Figure 3.1} unites the three boundary walks into a single boundary walk containing all of their distinct directed edges and expressed as

$$j1\omega_{j1\to xj}xj2\omega_{j2\to 1j}1jy\omega_{jy\to 2j}2j.$$

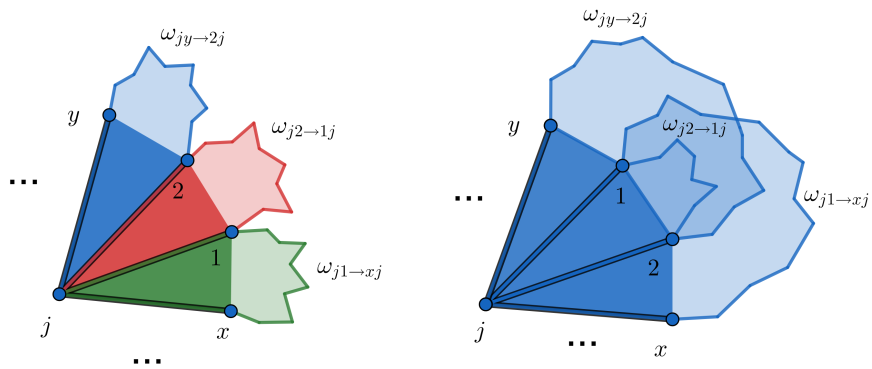

Figure 3.1. An Adjacent Edge Switch with 3 Distinct Boundary Walks

Proof The only boundary walks affected are those containing the directed edges $j1$, $1j$, $j2$, or $2j$. Swapping the vertices 1 and 2 at j, the rotation system ordering at vertex j becomes

$$j : \ldots x21y \ldots$$

Let the Boundary Walk Algorithm act on the initial directed edge $j1$. Then we follow from the succeeding edge to $j1$ through xj as before (since that part of the boundary walk has not been impacted by the edge switch). Now, after edge xj, according to the new ordering at vertex j, we follow $j2$. After $j2$ we follow from the succeeding edge to $j2$ through $1j$ as before. In this way, the Boundary Walk Algorithm determines that the boundary walk with initial directed edge $j1$ is

$$j1\omega_{j1\to xj}xj2\omega_{j2\to 1j}1jy\omega_{jy\to 2j}2j,$$

where the paths

$$\omega_{j1\to xj}, \omega_{j2\to 1j}, \text{and } \omega_{jy\to 2j}$$

between affected directed edges are unchanged by the swap. ∎

We may use Theorem 3.1 as a guide in reducing the number of boundary walks by modifying rotation systems for a graph. This is equivalent to reducing F in Equation 1.1, and thus making progress toward constructing a maximal embedding for a graph. Consider Figures 2.1-2.5 given at the end of the previous chapter. Each figure details the arrangement of the corners of boundary walks in the standard collection for K_n around the nth vertex for K_5-K_9, respectively. By Theorem 3.1, we know that in each of these diagrams, switching edge $n1$ with $n2$ will unite BW_0, BW_1, and BW_2. This switch therefore reduces by two the number of boundary walks in the associated collection for the standard rotation system of K_5-K_9. We also see that in Figure 2.5 we may follow this swap with a second swap, $n3$ with $n4$, in order to unite BW_3 and BW_4 with the single new boundary walk formed as a result of the previous swap, and again reduce by two the number of boundary walks in the collection for the rotation system of K_9. In fact, the pattern that one may see emerging here is sufficient information to write down a rotation system which corresponds to a maximal embedding for K_n. In the following theorem we do just that, essentially by swapping sufficiently many pairs of edges in the standard rotation system ordering at vertex n.

Theorem 3.2 *A maximal embedding for K_n is given by the rotation system:*

$1 : 234...n$

$2 : 134...n$

$3 : 124...n$

$\vdots$

$n-1 : 123...n$

$n : 214365...(j+1)j(j+2)(j+3)...(n-1)$

where $j = 2\lceil \frac{n}{4} \rceil - 3$. The number of boundary walks, F, for the resulting rotation system is given by:

1. $F = 2$ *if* $4|n$ *or* $4|(n+1)$.

2. $F = 1$ *if* $4|(n+2)$ *or* $4|(n+3)$.

The embedding surface has genus $g = \lfloor \frac{(n-1)(n-2)}{4} \rfloor$.

Proof Note that this rotation system is obtained from the standard rotation system for K_n by swapping pairs of edges at vertex n as follows:

$n1$ with $n2$, $n3$ with $n4$, $n5$ with $n6$... and nj with $n(j+1)$.

The number of boundary walks in the standard collection for K_n is $\lceil \frac{n}{2} \rceil$, and the boundary walks including the corners

$(n-1)n1$, $1n2$, $2n3$, $3n4$, $\ldots$ $(n - \lceil \frac{n}{2} \rceil)n(n - \lceil \frac{n}{2} - 1 \rceil)$

are all distinct by Theorem 2.3. Therefore, by Theorem 3.1, each swap indicated above reduces the number of boundary walks in the collection by 2, and there are $\frac{j+1}{2}$ swaps made. Then the total number of boundary walks after the swaps is:

$$F = \left\lceil \frac{n}{2} \right\rceil - 2\Big(\frac{j+1}{2}\Big) = \left\lceil \frac{n}{2} \right\rceil - 1 - j = \left\lceil \frac{n}{2} \right\rceil - 1 - 2\left\lceil \frac{n}{4} \right\rceil + 3 = \left\lceil \frac{n}{2} \right\rceil - 2\left\lceil \frac{n}{4} \right\rceil + 2$$

Therefore,

Case 1: If $4|n$, then $\lceil \frac{n}{2} \rceil = \frac{n}{2}$, and $\lceil \frac{n}{4} \rceil = \frac{n}{4}$, so $F = \frac{n}{2} - 2\frac{n}{4} + 2 = 2$.

Case 2: If $4|(n+1)$, then $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$, and $\lceil \frac{n}{4} \rceil = \frac{n+1}{4}$, so $F = \frac{n+1}{2} - 2(\frac{n+1}{4}) + 2 = 2$.

Case 3: If $4|(n+2)$, then $2|n$, so $\lceil \frac{n}{2} \rceil = \frac{n}{2}$, and $\lceil \frac{n}{4} \rceil = \frac{n+2}{4}$, so

$F = \frac{n}{2} - 2(\frac{n+2}{4}) + 2 = 1$.

Case 4: If $4|(n+3)$, then $2|n+3$, so $2|n+1$. Then $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$, and $\lceil \frac{n}{4} \rceil = \frac{n+3}{4}$, so we have $F = \frac{n+1}{2} - 2\frac{n+3}{4} + 2 = 1$.

Then there is one boundary walk after the specified $\frac{j+1}{2}$ switches are made if $4|(n+3)$ or $4|(n+2)$, and two if $4|(n+1)$ or $4|n$. Since the number of boundary walks cannot be reduced further in either case, we must have a maximal embedding. We may now confirm that the genus of the embedding surface is $\lfloor\frac{(n-2)(n-1)}{4}\rfloor$. Recall that $V = n$ and $E = \frac{n(n-1)}{2}$.

If $4|(n+1)$ or $4|n$, then

$$V - E + F = n - \frac{n(n-1)}{2} + 2 = \frac{3n - n^2}{2} + 2.$$

Then the genus of the embedding surface is

$$g = \frac{V - E + F - 2}{-2} = \frac{3n - n^2}{-4} = \frac{n^2 - 3n}{4}.$$

Also, $\frac{n^2-3n}{4}$ is an integer, so

$$g = \frac{n^2 - 3n}{4} = \left\lfloor\frac{n^2 - 3n}{4}\right\rfloor = \left\lfloor\frac{n^2 - 3n}{4} + \frac{1}{2}\right\rfloor = \left\lfloor\frac{n^2 - 3n + 2}{4}\right\rfloor = \left\lfloor\frac{(n-2)(n-1)}{4}\right\rfloor$$

On the other hand, if $4|(n+3)$ or $4|(n+2)$, then

$$V - E + F = n - \frac{n(n-1)}{2} + 1 = \frac{3n - n^2}{2} + 1.$$

Also, Then the genus of the embedding surface is

$$g = \frac{V - E + F - 2}{-2} = \frac{3n - n^2 - 2}{-4} = \frac{n^2 - 3n + 2}{4}.$$

But also, $\frac{n^2-3n+2}{4}$ is an integer, so

$$g = \frac{n^2 - 3n + 2}{4} = \left\lfloor\frac{n^2 - 3n + 2}{4}\right\rfloor = \left\lfloor\frac{(n-2)(n-1)}{4}\right\rfloor. \quad \blacksquare$$

CHAPTER 4

A STANDARD ROTATION SYSTEM FOR $H_{k,n}$ WITH k EVEN

We will begin by assigning integer labels 1 through n to the vertices of the Harary graph $H_{k,n}$ as specified in Chapter 1. Recall that for k even, each vertex is connected to the $\frac{k}{2}$ vertices on each side of it (as in Figure 1.4). Then, considering vertex labelings modulo n, we can define the standard rotation system as ordering the edges around each vertex j as follows:

Definition The standard rotation system for $H_{k,n}$ with k even orders the edges around any vertex j as

$$j : (j - \frac{k}{2})(j - \frac{k}{2} + 1) \dots (j - 1)(j + 1) \dots (j + \frac{k}{2} - 1)(j + \frac{k}{2}).$$

This standard rotation system gives a standard boundary walk collection and standard embedding, just as in the case of complete graphs. However, the general structure of some of the boundary walks in the collection is slightly different. There are boundary walks of the same form as BW_0, $BW_1, \dots BW_{\frac{k}{2}-1}$ from the standard collection for K_n. However, boundary walks including a directed edge $p(p - \frac{k}{2})$ have a new form. Therefore, we require a second boundary walk lemma to describe the structure of this additional boundary walk form. Before that, we will give an example to clarify these ideas.

Example Consider the Harary graph $H_{8,12}$. The standard rotation system for $H_{8,12}$ is as follows:

1: 9(10)(11)(12)2345

2: (10)(11)(12)13456

3: (11)(12)124567

4: (12)1235678

5: 12346789

6: 2345789(10)

7: 345689(10)(11)

8: 45679(10)(11)(12)

9: 5678(10)(11)(12)1

10: 6789(11)(12)12

11: 789(10)(12)123

12: 89(10)(11)1234

Then the boundary walk collection generated via the Boundary Walk Algorithm, and labeled in the same manner as the standard collection for K_n, is:

BW_0 : 123456789(10)(11)(12)1

BW_1 : 1(12)2132435465768798(10)9(11)(10)(12)(11)1

BW_2 : 2(12)31425364758697(10)8(11)9(12)(10)1(11)2

BW_3 : 3(12)415263748596(10)7(11)8(12)91(10)2(11)3

BW_4 : 4(12)84

BW_5 : 3(11)73

BW_6 : 2(10)62

BW_7 : 1951

Notice that BW_0 through BW_3 have the same form as those in the standard collection for K_n, as claimed. Also notice that $\frac{k}{2} = 4$, and that BW_4 through BW_7 all take a different, shorter, form than any seen in the standard collection for K_n. In fact, consider the cyclic group $\mathbb{Z}/12\mathbb{Z}$. In particular, consider the subgroup generated by the element 4, with corresponding set $\{4, 8, 12\}$. Notice that BW_4 is simply a concatenation of the elements of this set in descending order. Likewise, BW_5 can be seen as a concatenation of the elements of the set $\{4-1, 8-1, 12-1\} = \{3, 7, 11\}$ in descending order, BW_6 as that of $\{4-2, 8-2, 12-2\} = \{2, 6, 10\}$, and BW_7 as that of $\{4-3, 8-3, 12-3\} = \{1, 5, 9\}$. Each of these sets corresponding to BW_5 through BW_7 are cosets of the set $\{4, 8, 12\}$ corresponding to BW_4. The relevance of the subgroup structure of $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/12\mathbb{Z}$ arises due to to the fact that the rule (governed by the standard rotation system) which determines the succeeding edge to a directed edge $p(p-\frac{k}{2})$ is essentially given by subtraction of $\frac{k}{2}$ from the terminal vertex, as we will see in the following Lemma. It is again important to remember that all arithmetic is done modulo n. In what follows then, equivalence is to be understood as modular equivalence.

Lemma 4.1 (The Second Boundary Walk Lemma) *Assume the standard rotation system for $H_{k,n}$. Then in any boundary walk, the succeeding edge to edge pq is*

1. $q(p+1)$ *if* $q \neq p+1$ *and* $q \neq p-\frac{k}{2}$

2. $q(p+2)$ *if* $q = p+1$

3. $q(p-k)$ *if* $q = p-\frac{k}{2}n$

Proof Both 1 and 2 follow the same argument as Lemma 2.1.

For 3, assume that $q = p-\frac{k}{2}$. The standard rotation system orders edge $(p-\frac{k}{2})(p-k)$ after edge $p(p-\frac{k}{2})$ at vertex $p-\frac{k}{2}$, since $p = p-\frac{k}{2}+\frac{k}{2}$ and $p-k = p-\frac{k}{2}-\frac{k}{2}$.

Corollary 4.2 *(The Second Boundary Walk Corollary)*

Assume the standard rotation system for $H_{k,n}$. Then

1. A boundary walk with initial directed edge pq such that $q \neq (p+1)$ *and* $q \neq p - \frac{k}{2}$ *has the form*

$$pq(p+1)(q+1)(p+2)(q+2)...(p-1)(q-1)p,$$

and every directed edge xy of K_n *such that* $y - x = q - p$ ***or*** $y - x = p + 1 - q$ *is contained in the boundary walk.*

2. A boundary walk with initial directed edge pq such that $q = p - \frac{k}{2}$ *has the form*

$$p(p-\frac{k}{2})(p-k)(p-\frac{3k}{2})\dots(p+k)(p+\frac{k}{2})p$$

Proof Part 1 follows the same argument as Corollary 2.2. For part 2, notice that any directed edge of the form $(p-\frac{mk}{2})(p-\frac{(m+1)k}{2})$ with $m \in \mathbb{Z}$ has a succeeding edge of precisely the same form, $(p-\frac{(m+1)k}{2})(p-\frac{(m+2)k}{2})$, by Lemma 4.1. Likewise, then, the preceding edge to $(p-\frac{mk}{2})(p-\frac{(m+1)k}{2})$ is $(p-\frac{(m-1)k}{2})(p-\frac{mk}{2})$. The form of the boundary walk follows. ∎

Definition A k**-gap boundary walk** is any boundary walk of the form

$$p(p-\frac{k}{2})(p-k)(p-\frac{3k}{2})\dots(p+k)(p+\frac{k}{2})p$$

This definition captures the new form of the boundary walks mentioned earlier (those that arise due to the subgroup structure of $\mathbb{Z}/n\mathbb{Z}$). The term "k-gap boundary walks" is used because each corner in the boundary walk includes edges of the form $p(p-\frac{k}{2})$, which arise due to the ordering of $q(q-\frac{k}{2})$ after $q(q+\frac{k}{2})$ in the rotation system ordering at vertex q. Note that the difference between the terminal vertex labels is k. This is a unique feature of the ordering, since otherwise the difference between any two successive terminal vertex labels in the ordering is always 1. The structure of these k-gap boundary walks will be described algebraically in the proof of the following theorem. That description will rely on the use of the elementary result from group theory given in the following lemma, see [Gallian, 2013] for its proof.

Lemma 4.3 *Let a be an element of order n in a group and b be a positive integer. Then $|a^b| = \frac{n}{gcd(n,b)}$.*

Theorem 4.4 (The Standard Collection for $H_{k,n}$, k Even) *The standard boundary walk collection associated to the standard rotation system for $H_{k,n}$ with k even is given as follows (where $d = \gcd(n, \frac{k}{2})$):*

$BW_0 :\ 123 \ldots n1$

$BW_1 :\ 1n2132 \ldots (n-2)n(n-1)1$

$\vdots$

$BW_j :\ jn(j+1)1(j+2)2 \ldots (n-j-1)n(n-j) \ldots (n-1)j$

$\vdots$

$BW_{\frac{k}{2}-1} :\ (\frac{k}{2}-1)n(\frac{k}{2})1(\frac{k}{2}+1)2 \ldots (n-\frac{k}{2})n(n-\frac{k}{2}+1) \ldots (n-1)(\frac{k}{2}-1)$

and

$BW_{\frac{k}{2}} :\ (\frac{k}{2})n(n-\frac{k}{2})(n-k) \ldots k(\frac{k}{2})$

$BW_{\frac{k}{2}+1} :\ (\frac{k}{2}-1)(n-1)(n-\frac{k}{2}-1) \ldots (k-1)(\frac{k}{2}-1)$

$BW_{\frac{k}{2}+2} :\ (\frac{k}{2}-2)(n-2)(n-\frac{k}{2}-2) \ldots (k-2)(\frac{k}{2}-2)$

$\vdots$

$BW_{\frac{k}{2}+d-1} :\ (\frac{k}{2}-d+1)(n-d+1)(n-\frac{k}{2}-d+1) \ldots (k-d+1)(\frac{k}{2}-d+1)$

The associated embedding is in the surface of genus $g = \frac{2(1-d)+(1-n)(2-k)}{4}$.

Proof Begin the Boundary Walk Algorithm with directed edge 12. By The Second Boundary Walk Lemma, the next directed edges are 23, 34, etc. The walk ends when the algorithm encounters the preceding edge to 12. Directed edge $n1$ is the preceding edge to directed edge 12, again by Lemma 4.1. Then the first boundary walk is of the form

$$BW_0: \;\; 123\ldots(n-1)n1.$$

Every directed edge xy such that $y-x=1$ is contained in BW_0 by construction. Therefore, directed edge $1n$ is not included in BW_0, since $n-1 \neq 1$, and the Boundary Walk Algorithm may begin a new walk with directed edge $1n$. By Lemma 4.1, the boundary walk beginning with directed edge $1n$ is of the form

$$BW_1: \;\; 1n2132\ldots(n-2)n(n-1)1$$

and contains every directed edge xy of $H_{k,n}$ such that $y-x=n-1$ or $y-x=2-n=2$ by Corollary 4.2. More generally, the Boundary Walk Algorithm may begin a new walk with directed edge jn for $j \leq \frac{k}{2}-1$ since $n-j > \frac{k}{2} \geq j+1$ for all $j \leq \frac{k}{2}-1$ (because $k<n$ by definition). This boundary walk has the form

$$BW_j: \;\; jn(j+1)1(j+2)2\ldots(n-j-1)n(n-j)\ldots(n-1)j$$

and contains every directed edge xy of $H_{k,n}$ such that $y-x=n-j$ or $y-x=j+1-n=j+1$ by Corollary 4.2. The collection now contains every directed edge xy of $H_{k,n}$ such that $1-\frac{k}{2} < y-x < \frac{k}{2}$, and the only directed edges xy not yet included in these boundary walks are those such that $y-x=n-\frac{k}{2}$. All of these directed edges belong to the k-gap boundary walks.

The Boundary Walk Algorithm may begin a new walk with directed edge $n(n-\frac{k}{2})$. By Corollary 4.2 the next directed edge in the walk is $(n-\frac{k}{2})(n-k)$, then $(n-k)(n-\frac{3k}{2})$, and so on, until the boundary ends at the preceding edge to the initial directed edge $n(n-\frac{k}{2})$, which would be directed edge $(\frac{k}{2})n$.

Now, consider the set of vertices $\{1,2,3,...n\}$ as the cyclic group $\mathbb{Z}/n\mathbb{Z}$, since we generate subsequent vertices in our boundary walk via addition modulo n. Then this boundary walk is simply an ordered concatenation of the elements in the subgroup $\langle\frac{k}{2}\rangle$ of $\mathbb{Z}/n\mathbb{Z}$, that is, the subgroup generated by $\frac{k}{2}$. It follows that we can easily determine the number of edges in the boundary walk (which is equal to the number of the vertices) by applying Lemma 4.3.

Since 1 is an element of order n in $\mathbb{Z}/n\mathbb{Z}$ and $\frac{k}{2}$ is a positive integer, we have $|1^{\frac{k}{2}}| = |\frac{k}{2}| = |\langle\frac{k}{2}\rangle| = \frac{n}{\gcd(n,\frac{k}{2})}$. That is, the order of the subgroup generated by $\frac{k}{2}$ is $\frac{n}{\gcd(n,\frac{k}{2})}$, so there are $\frac{n}{\gcd(n,\frac{k}{2})}$ directed edges in the boundary walk $n(n-\frac{k}{2})\ldots(\frac{k}{2})n$. The value $d = \gcd(n, \frac{k}{2})$ will be heavily referenced in the remainder of the thesis, and any further use of the character d should be taken as $\gcd(n, \frac{k}{2})$. Note that if n and $\frac{k}{2}$ are relatively prime, then every directed edge is included in a boundary walk after listing this one. Otherwise, there are boundary walks that remain to be identified.

Notice that the boundary walk $(n-1)(n-\frac{k}{2}-1)\ldots(\frac{k}{2}-1)(n-1)$ has the same number of directed edges as $n(n-\frac{k}{2})\ldots(\frac{k}{2})n$, with each vertex labeling in the concatenation reduced by one. Clearly, the same is true for
$(n-j)(n-\frac{k}{2}-j)\ldots(\frac{k}{2}-j)$ with a vertex labeling reduced by j. Each of these boundary walks may be seen as a concatenation of the vertices in the coset $\langle\frac{k}{2}\rangle + j$ of the subgroup $\langle\frac{k}{2}\rangle$. We see that we may form d distinct boundary walks, since $\{1, 2, 3 \ldots n\}$ is partitioned into d cosets of size $\frac{n}{\gcd(n,\frac{k}{2})}$. Furthermore, the cosets $\langle\frac{k}{2}\rangle + j$ are distinct for $1 \leq j \leq d-1$. Then we can be certain that every directed edge of the form $v(v-\frac{k}{2})$ for every vertex v is contained in one of the distinct boundary walks associated to each of the distinct cosets. It follows that we have accounted for every directed edge of $H_{k,n}$ after sequentially generating the boundary walks $BW_{\frac{k}{2}}$ through $BW_{\frac{k}{2}+d-1}$ (the boundary walks corresponding to the cosets). Then the Boundary Walk Algorithm terminates after listing $BW_{\frac{k}{2}+d-1}$.

Notice that we have $\frac{k}{2}+d$ boundary walks in total. Then we have that

$$V - E + F = n - \frac{nk}{2} + \frac{k}{2} + d$$

since for $H_{k,n}$ with k even there are clearly $\frac{nk}{2}$ edges. Consequently, by applying

Equation 1.1 we find that we have a standard embedding of $H_{k,n}$ for k even into an orientable surface of genus

$$g = \frac{(n - \frac{nk+k}{2} + d - 2)}{-2}$$

or equivalently,

$$g = \frac{2(1-d) + (1-n)(2-k)}{4} \qquad \blacksquare$$

Notice that one consequence of this construction, that follows directly from Corollary 4.2, is that we can identify the boundary walk BW_j in the standard collection to which any directed edge xy belongs.

Corollary 4.5 *Given a directed edge xy of $H_{k,n}$, the boundary walk in the standard collection to which xy belongs can be determined by the following rules:*
1. If $y - x = 1$, then xy is a directed edge of BW_0.
2. For $1 \leq j \leq \frac{k}{2} - 1$, if $y - x = j + 1$ or $y - x = n - j$, then xy is a directed edge of BW_j.
3. If $y - x = n - \frac{k}{2}$, then xy is a directed edge of one of $BW_{\frac{k}{2}}$-$BW_{\frac{k}{2}+d-1}$, corresponding to the coset of the subgroup generated by $\frac{k}{2}$ in $\mathbb{Z}/n\mathbb{Z}$ to which x and y belong.

CHAPTER 5

A MAXIMAL EMBEDDING OF $H_{k,n}$ WITH k EVEN

Now, as we did for K_n in Chapter 3, we are going to obtain a rotation system for a maximal embedding of $H_{k,n}$, k even, by making adjacent edge switches in the standard rotation system. The standard collection in the previous chapter includes more complicated boundary walk structures than the standard collection for K_n, and we will require an additional edge switching theorem in order to describe a series of edge switches which takes the standard rotation system for $H_{k,n}$, k even, to a rotation system for its maximal embedding. For the remainder of the chapter, we will assume that k is even, and $H_{k,n}$ will refer exclusively to the k even case.

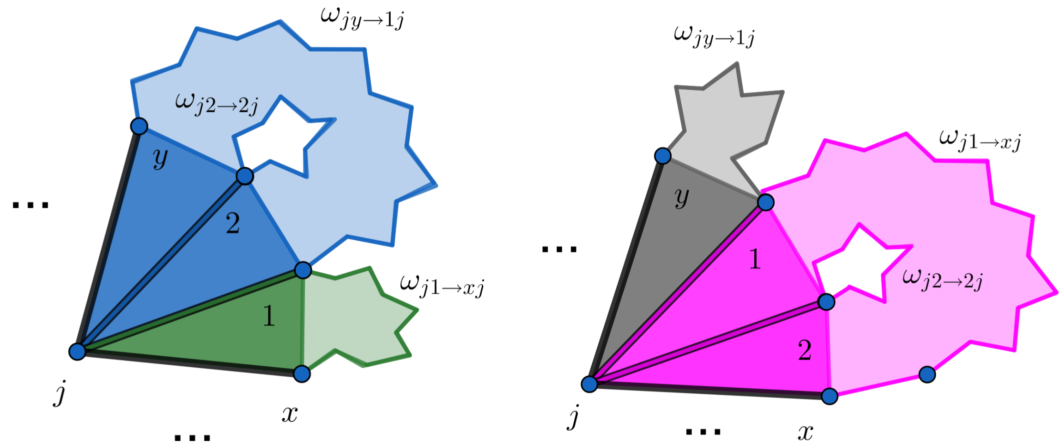

Figure 5.1. An Adjacent Edge Switch with 2 Distinct Boundary Walks

Suppose, as in Figure 5.1, that there are 3 consecutive corners in the corner diagram at vertex j such that, reading counterclockwise, one belongs to one boundary walk (green) while the other two both belong to another (blue). Then switching the edges $j1$ and $j2$ in the rotation system around j changes the arrangement of the corners so that the first two corners are in the same boundary walk (pink), and the third is in a separate one (gray). We can see further details of the effect of the swap on the corners and the boundary walks by carefully examining Figure 5.1. The total effect will be captured in the presentation of the following theorem.

Theorem 5.1 (The Second Edge Switching Theorem) *Assume a rotation system includes*

$$j : \dots x12y \dots$$

and generates at least two boundary walks

$$1j2\omega_{j2\to 2j}2jy\omega_{jy\to 1j} \text{ and } xj1\omega_{j1\to xj}.$$

Then swapping edge $j1$ with $j2$ at vertex j replaces the above two boundary walks with boundary walks

$$xj2\omega_{j2\to 2j}2j1\omega_{j1\to xj} \text{ and } 1jy\omega_{jy\to 1j},$$

leaving the remainder of the boundary walks unchanged. Consequently, this switch causes no change in the total number of boundary walks.

Proof (The Second Edge Switching Theorem)

The only boundary walks affected are those containing the directed edges $j1, 1j, j2$, or $2j$. Swapping the vertices 1 and 2 in the rotation system ordering around j, we have

$$j : \dots x21y \dots$$

Then, in the same manner as in the First Edge Switching Theorem, the Boundary Walk Algorithm determines that the boundary walk with initial directed edge xj is $xj2\omega_{j2\to 2j}2j1\omega_{j1\to xj}$. This walk includes directed edges $2j$, $j1$, and $j2$. The only remaining affected directed edge is $1j$, and the Boundary Walk Algorithm also determines that the boundary walk with initial edge $1j$ is $1jy\omega_{jy\to 1j}$.

∎

While this theorem doesn't directly help us reach a maximal embedding by reducing the number of boundary walks via a rotation system edge switch, as we will see in the next example (and theorem) it helps us by rearranging the boundary walks in such a way that the First Edge Switching Theorem can be used to obtain a maximal embedding. In order to use the First and Second Edge Switching Theorems as guides, it will be helpful to visualize the entire rotation system as an array of vertices with the edge orderings corresponding to the rotation system as well as the corners around the vertex colored according to the corresponding boundary walks. We will refer to the diagram representing the edge orderings and corner colorings at any particular vertex v as the *corner diagram* at v. First, let's extend Example 4, regarding $H_{8,12}$ from the previous chapter.

Example We may pick various colors to represent each of the 8 boundary walks in the standard collection. Let's pick orange for BW_0, a light blue for BW_1, dark red for BW_2, dark green for BW_3. In the future, we will pick these same colors for consistency, choosing additional colors as needed for any further boundary walks BW_i with $i < \frac{k}{2}$. Let's pick pink for BW_4, blue for BW_5, red for BW_6, and dark purple for BW_7. Again, in the future we will pick these same colors for the k-gap boundary walks, $BW_{\frac{k}{2}}$, $BW_{\frac{k}{2}+1}$, $BW_{\frac{k}{2}+2}$, $BW_{\frac{k}{2}+3}$, choosing additional colors as needed for any further k-gap boundary walks. Then the corner diagram at vertex 1 will appear as shown in the first row and column in the array in Figure 5.2. The corner diagram at vertex 2 appears in the first row and second column, with most of the terminal vertices of its edges unlabeled (in order to reduce visual clutter). However, the edges which are labeled are enough to establish the arrangement of the remaining edges as described by the rotation system ordering at vertex 2. After a pattern has been established by the first 4 corner diagrams, all terminal vertex labels are left off to reduce visual clutter without losing information about the ordering of the edges.

- BW_0 : 123456789(10)(11)(12)1
- BW_1 : 1(12)2132435465768798(10)9(11)(10)(12)(11)1
- BW_2 : 2(12)31425364758697(10)8(11)9(12)(10)1(11)2
- BW_3 : 3(12)415263748596(10)7(11)8(12)91(10)2(11)3
- BW_4 : 4(12)84
- BW_5 : 3(11)73
- BW_6 : 2(10)62
- BW_7 : 1951

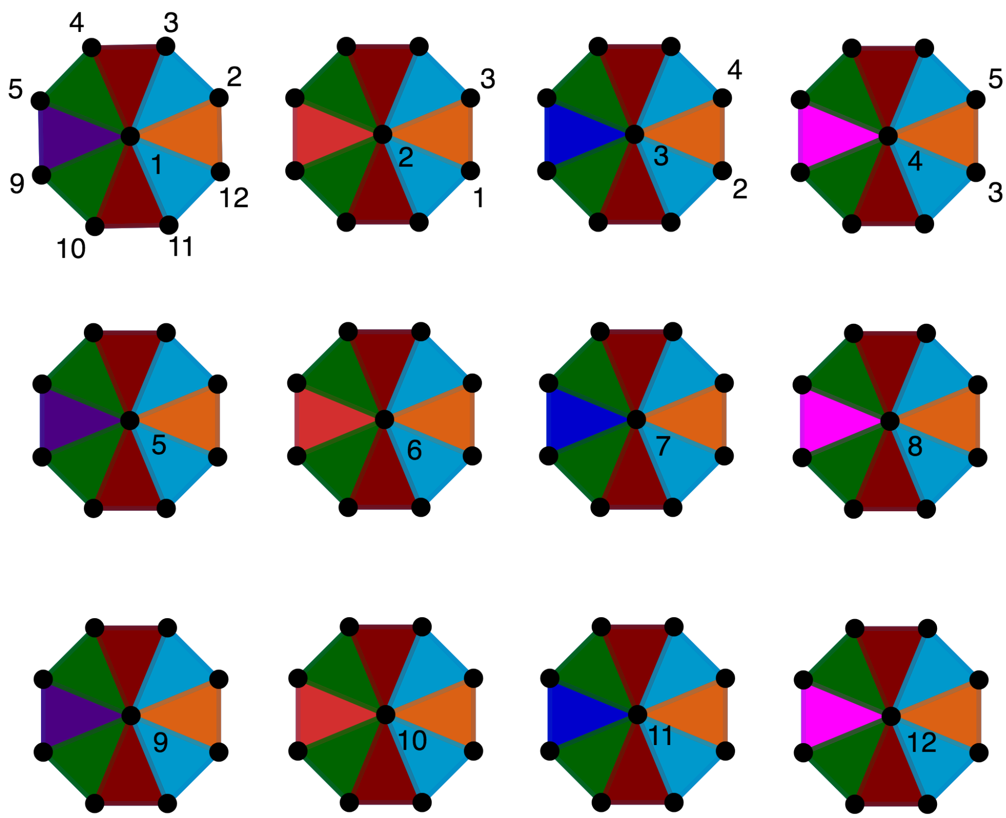

Figure 5.2. A Corner Diagram Array for $H_{8,12}$

To motivate the general edge-switching process that we follow to obtain a maximal embedding for $H_{k,n}$ (k even), we will work through an example with $H_{8,12}$. The primary challenge is to obtain a collection of switches that unites the k-gap boundary walks.

Example Consider $H_{8,12}$ and the first row of the corner array diagram corresponding to the standard collection, as in Example 5. The first row is shown below in Figure 5.3, along with the standard boundary walk collection with boundary walk colors identified.

- BW_0 : 123456789(10)(11)(12)1
- BW_1 : 1(12)2132435465768798(10)9(11)(10)(12)(11)1
- BW_2 : 2(12)31425364758697(10)8(11)9(12)(10)1(11)2
- BW_3 : 3(12)415263748596(10)7(11)8(12)91(10)2(11)3
- BW_4 : 4(12)84
- BW_5 : 3(11)73
- BW_6 : 2(10)62
- BW_7 : 1951

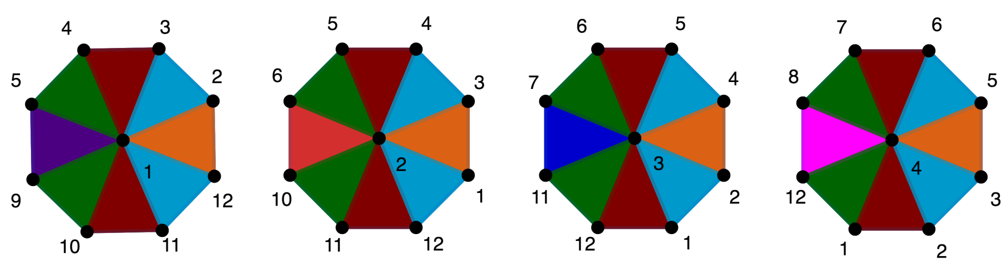

Figure 5.3. First Row Corner Diagram Array for $H_{8,12}$

First, consider the effect of swapping edges 19 and 1(10) in the rotation system ordering at vertex 1, so that the ordering is 1 : (10)9(11)(12)2345. By the First Edge Switching Theorem, BW_7 is united with BW_2 and BW_3 to form a new boundary walk, which we will call boundary walk θ. The resulting boundary walk collection and associated corner diagrams are as shown in Figure 5.4.

- BW_0 : 123456789(10)(11)(12)1
- BW_1 : 1(12)2132435465768798(10)9(11)(10)(12)(11)1
- θ : 2(12)31425364758697(10)8(11)9(12)(10)1951(10)2 . . .
  . . . (11)3(12)415263748596(10)7(11)8(12)91(11)2
- BW_4 : 4(12)84
- BW_5 : 3(11)73
- BW_6 : 2(10)62

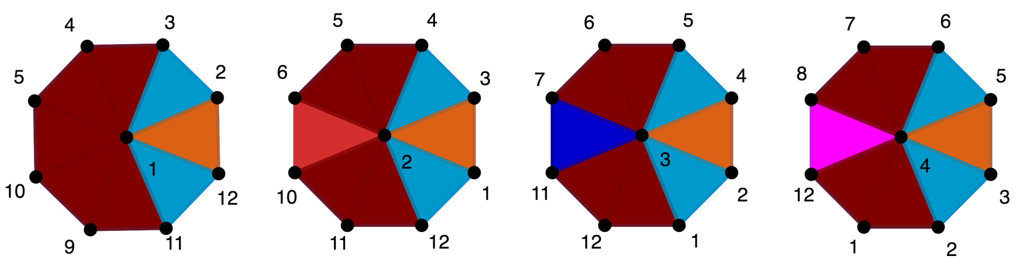

Figure 5.4. First Row Corner Diagram Array for $H_{8,12}$ After 1 Adjacent Edge Swap

Then swap edges 2(10) and 2(11) in the rotation system ordering at vertex 2, so that the ordering is 2 : (11)(10)(12)13456. By the Second Edge Switching Theorem, BW_6 and θ are modified, resulting in two new boundary walks consisting of their directed edges, θ_1 and θ_2. The resulting boundary walk collection and associated corner diagrams are as shown in Figure 5.5.

We will continue this process of swapping edges at successive vertices, alternately applying the First Edge Switching Theorem and the Second Edge Switching Theorem. Notice that the total effect of each pair of swaps is to first reduce the total number of boundary walks in the collection by 2 by uniting k-gap boundary walks, and then to prepare the next corner diagram for another application of the First Edge Switching Theorem. Now, to continue, swap edges 3(11) and 3(12) in the rotation system ordering at vertex 3, so that the ordering is

- BW_0 : 123456789(10)(11)(12)1
- BW_1 : 1(12)2132435465768798(10)9(11)(10)(12)(11)1
- θ_1 : 2(12)31425364758697(10)8(11)9(12)(10)1951(10)2
- θ_2 : (11)3(12)415263748596(10)7(11)8(12)91(11)2(10)62(11)
- BW_4 : 4(12)84
- BW_5 : 3(11)73

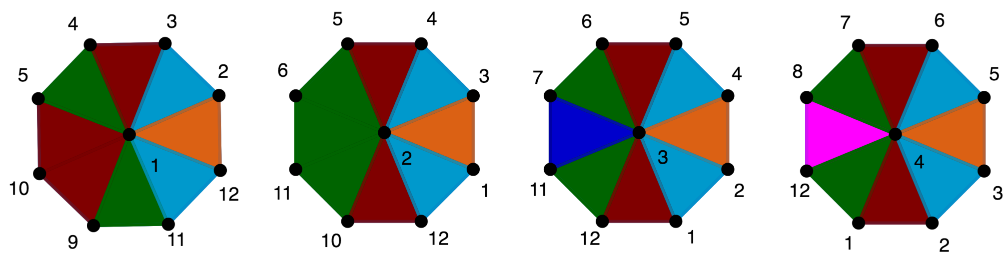

Figure 5.5. First Row Corner Diagram Array for $H_{8,12}$ After 2 Adjacent Edge Swaps

3 : (12)(11)124567. The resulting boundary walk collection and associated corner diagrams are as shown in Figure 5.6.

- BW_0 : 123456789(10)(11)(12)1
- BW_1 : 1(12)2132435465768798(10)9(11)(10)(12)(11)1
- θ_3 : 2(12)3(11)73(12)(12)415263748596(10)7(11)8(12)91(11)2(10)62...
  ...(11)31425364758697(10)8(11)9(12)(10)1951(10)2
- BW_4 : 4(12)84

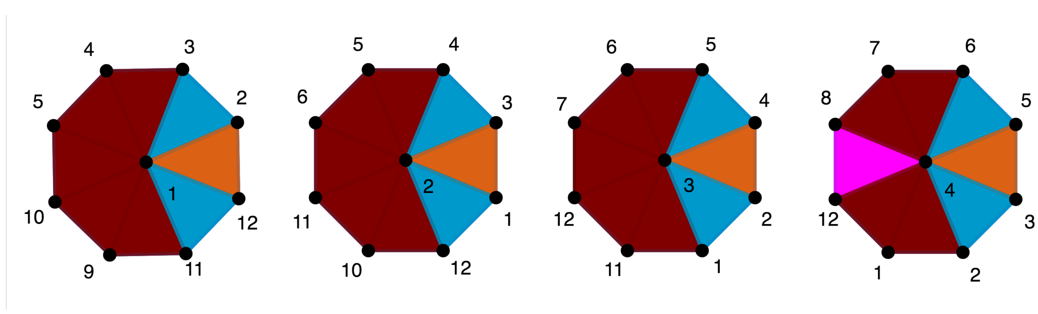

Figure 5.6. First Row Corner Diagram Array for $H_{8,12}$ After 3 Adjacent Edge Swaps

Now swap edges 4(12) and 41 in the rotation system ordering at vertex 4, so that the ordering is $4 : 1(12)235678$. The resulting boundary walk collection and associated corner diagrams are as shown in Figure 5.7.

- BW_0 : 123456789(10)(11)(12)1
- BW_1 : 1(12)2132435465768798(10)9(11)(10)(12)(11)1
- θ_4 : (12)425364758697(10)8(11)9(12)(10)1951(10)2(12)3(11)73(12)
- θ_5 : 8415263748596(10)7(11)8(12)91(11)2(10)62(11)314(12)8

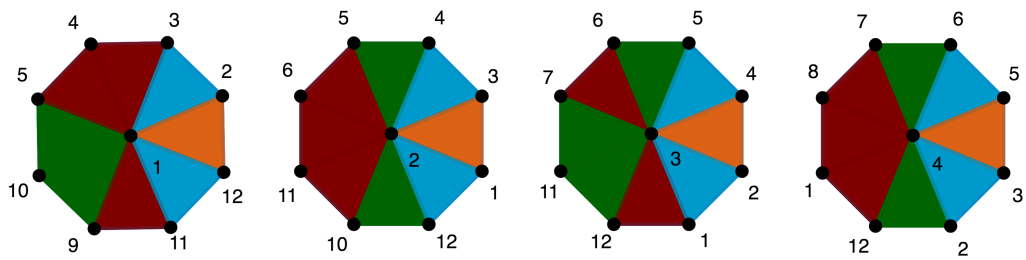

Figure 5.7. First Row Corner Diagram Array for $H_{8,12}$ After 4 Adjacent Edge Swaps

At this point, all boundary walks in the collection are present in the corner diagrams at all vertices. Then at one vertex we can make a sequence of swaps all of which satisfy the conditions of the First Edge Switching Theorem, as in the K_n case. We will choose to make those swaps (only one in this case) at vertex $d = \gcd(n, \frac{k}{2}) = 4$, where we left off with our last swap. Swap edges 43 and 42 in the rotation system ordering at vertex 4, so that the ordering is $4 : 1(12)325678$. By the First Edge Switching Theorem, BW_0, BW_1, and θ_4 unite to form a single boundary walk. The resulting boundary walk collection is as shown in Figure 5.8, where we also see the effect on the arrangement of the remaining boundary walks in the corner diagram at vertex 4.

Notice that there are two boundary walks remaining in the collection, so we have reached a maximal embedding of $H_{8,12}$ via the specified swaps. Also note that the genus of the embedding surface is

- θ_5 : 8415263748596(10)7(11)8(12)91(11)2(10)62(11)314(12)8
- θ_6 : 123456789(10)(11)(12)(10)1951(10)2(12)3(11)73(12)42536475869 . . .
  . . . 7(10)8(11)9(12)(11)1(12)2132435465768798(10)9(11)(10)(12)1

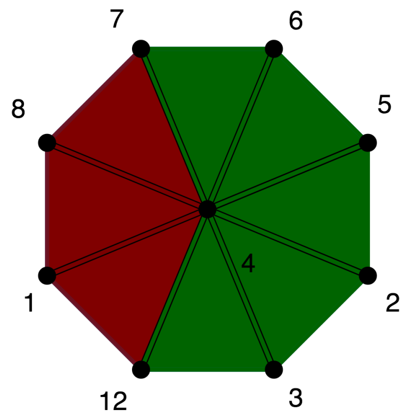

Figure 5.8. First Row Corner Diagram Array for $H_{8,12}$ After 6 Adjacent Edge Swaps

$$g = \frac{V - E + F - 2}{-2} = (12 - \frac{12(8)}{2} + 2 - 2)/-2 = 18.$$

Thus, it follows that we have an embedding of $H_{8,12}$ in an 18-hole torus via the rotation system:

1 : (10)9(11)(12)2345

2 : (11)(10)(12)13456

3 : (12)(11)124567

4 : 1(12)325678

5 through 12 : The Standard Rotation System Ordering for $H_{k,n}$ with k even.

Next we will see that the edge switching strategy employed here can be used generally to obtain a maximal embedding for $H_{k,n}$.

We will move forward with a visualization of the boundary walks of $H_{k,n}$, for general values of k and n via general corner diagrams. We will demonstrate generally how we can obtain a rotation system for a maximal embedding by making a set of edge switches on the standard rotation system. Fix the standard rotation system for $H_{k,n}$. Consider the corner diagrams at the first and second vertices with arbitrary n, k and $d = \gcd(n, \frac{k}{2})$, shown below in Figure 5.9. For simplicity and consistency of representation, we will assume that $k \geq 8$ and $d \geq 4$.

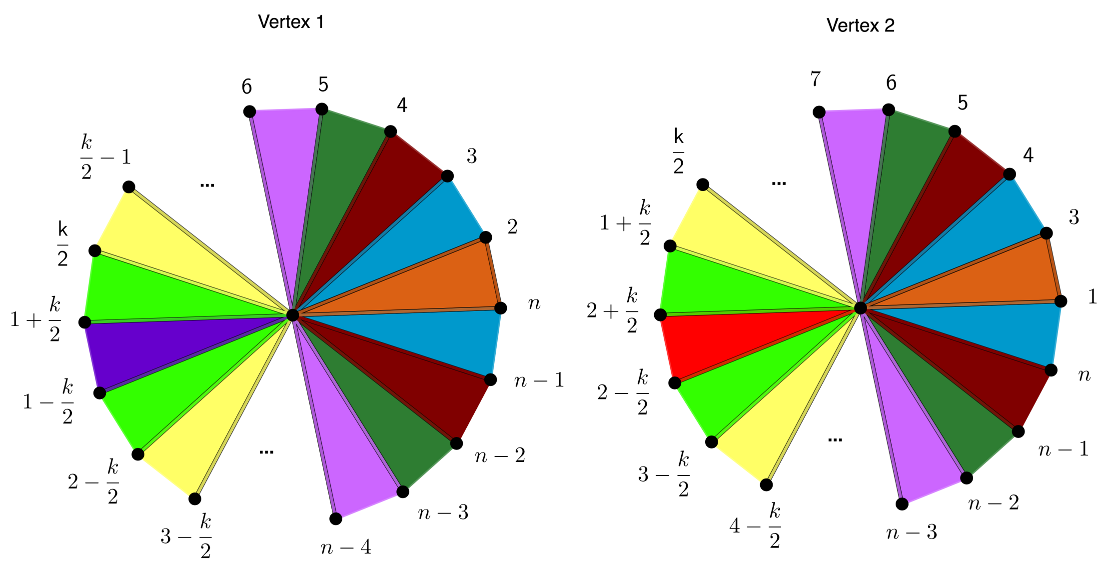

Figure 5.9. Corner Diagrams for $H_{k,n}$, k Even, Vertices 1 and 2

Notice the similarities in the arrangement of the corners belonging to the same boundary walks around each vertex. In fact, the only difference with regard to this arrangement is that the corner $(1+\frac{k}{2})1(1-\frac{k}{2})$ in the rotation system of vertex 1 belongs to a different k-gap boundary walk than the corner $(2+\frac{k}{2})2(2-\frac{k}{2})$ in the same position in the corner diagram at vertex 2, as long as n and $\frac{k}{2}$ are not relatively prime. Recall that there are $d = \gcd(n, \frac{k}{2})$ of these k-gap boundary walks in the standard collection for $H_{k,n}$. We also know that the k-gap boundary walk with corner $(1+\frac{k}{2})1(1-\frac{k}{2})$ also contains the corner $(1+d+\frac{k}{2})(1+d)(1+d-\frac{k}{2})$, the k-gap boundary walk with corner $(2+\frac{k}{2})2(2-\frac{k}{2})$ also contains the corner

$(2+d+\frac{k}{2})(2+d)(2+d-\frac{k}{2})$, and so on. Furthermore, note that for $i \leq d$, the corners $(i+\frac{k}{2})i(i-\frac{k}{2})$ each belong to distinct k-gap boundary walks.

Now we may consider an entire generalized array of corner diagrams representing the standard rotation system for $H_{k,n}$ with k, n, and d arbitrary (Figure 5.10). We will leave off the terminal vertex labels of the edges on each corner diagram to reduce visual clutter.

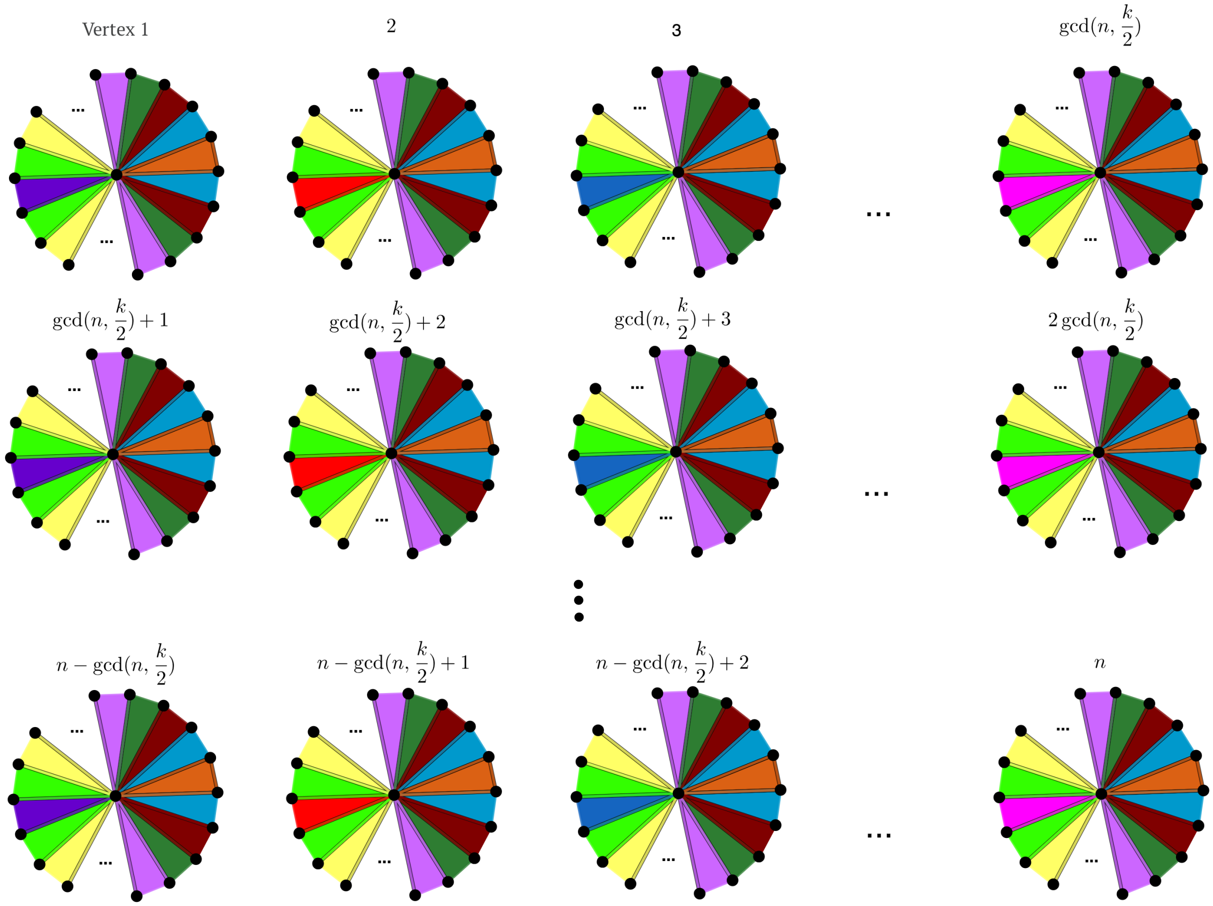

Figure 5.10. Corner Diagram Array for $H_{k,n}$, k even

Notice that, apart from the vertex labels, each row of the array has corner diagrams identical to those of the first row. Therefore, we can focus our attention at any row without losing information about the form of the entire array. Figure 5.11 shows the first row, on which we focus in the proof of Theorem 5.2.

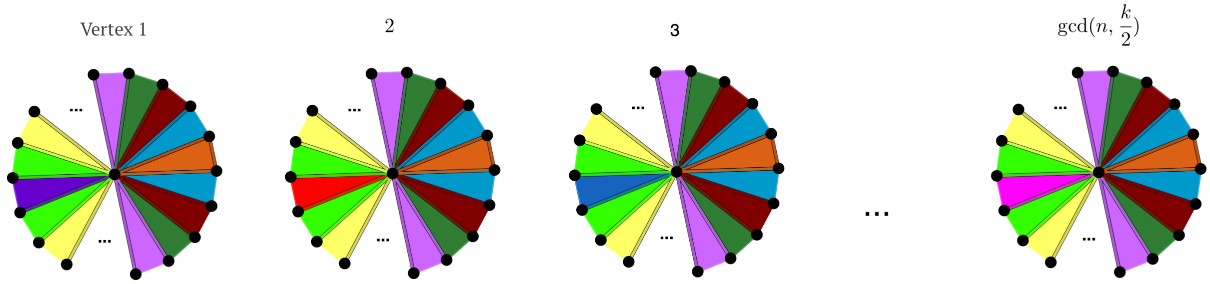

Figure 5.11. Rotation System Array, First Row

This visualization will inform our choices for a series of swaps which takes the standard rotation system to a rotation system corresponding to a maximal embedding of $H_{k,n}$. The reader may find it useful to review Example 5 to put the generalizations in the proof of the following theorem into context. There, one can see the details of each swap and its effect on the first row of the corner diagram array for $H_{8,12}$. The following theorem will assume that $k > 2$. This is because if $k = 2$, then $\frac{k}{2} = 1$, and $d = 1$. Then the total number of boundary walks is $\frac{k}{2} + d = 2$, and the standard rotation system is already a maximal embedding. Another way to see this is to note that if $k = 2$, then the graph is topologically a circle, and therefore clearly has a maximal embedding in the sphere.

Theorem 5.2 *The Rotation System of a Maximal Embedding of $H_{k,n}$ (k Even)* *A maximal embedding of $H_{k,n}$, $k > 2$ even, is given by the rotation system obtained from the standard rotation system with swaps of adjacent edges:*

- $i(i - \frac{k}{2})$ *with* $i(i - \frac{k}{2} + 1)$ *at vertex* i *for all* $i \in \mathbb{Z}^+$ *such that* $i \leq d$
- *If* $k \geq 8$, *then also swap* $d(d-1)$ *with* $d(d-2)$, $d(d-3)$ *with* $d(d-4)$, $\ldots$

$d(d-j+1)$ *with* $d(d-j)$ *at vertex* $d = \gcd(n, \frac{k}{2})$, *with*

$$j = \begin{cases} \frac{k}{2} - 3, & \text{if } \frac{k}{2} \text{ is odd.} \\ \frac{k}{2} - 2, & \text{if } \frac{k}{2} \text{ is even.} \end{cases}$$

The number of boundary walks, F, for the resulting rotation system is given by

1. *F=2, if d and $\frac{k}{2}$ share the same parity.*
2. *F=1, if d is odd and $\frac{k}{2}$ is even.*

We refer to the edge switches listed after the first bullet in the theorem statement as Type 1 switches. Making all of the specified Type 1 switches results in one or two new boundary walks that contain all of the directed edges in the k-gap boundary walks. As a result of these edge switches, all boundary walks are present at (i.e. pass through) each vertex.

We refer to the edge switches listed after the second bullet as Type 2 switches. They are akin to the edge switches in Theorem 3.2 and reduce the number of boundary walks to one or two, resulting in a maximal embedding. A set of edge switches of this type could be done at any of the vertices to obtain a maximal embedding.

Proof Theorem (5.2)

Assume we have the standard rotation system for $H_{k,n}$, k even. By Corollary 6.2, we know that corners $(1+\frac{k}{2})1(1-\frac{k}{2})$, $(1-\frac{k}{2})1(2-\frac{k}{2})$, and $(2-\frac{k}{2})1(3-\frac{k}{2})$ are contained in distinct boundary walks (see Figure 5.12).

In fact, $(1-\frac{k}{2})1(2-\frac{k}{2})$ is contained in $BW_{\frac{k}{2}-1}$ and $(2-\frac{k}{2})1(3-\frac{k}{2})$ is contained in $BW_{\frac{k}{2}-2}$. Swap edge $1(1-\frac{k}{2})$ with edge $1(2-\frac{k}{2})$ in the rotation system ordering at vertex 1. By Theorem 3.1 (The First Edge Switching Theorem), the three boundary walks unite to form one boundary walk,

$$\theta : 1(1-\frac{k}{2})\omega_{1(1-\frac{k}{2})\to(1+\frac{k}{2})1}(1+\frac{k}{2})1(2-\frac{k}{2})\omega_{1(2-\frac{k}{2})\to(1-\frac{k}{2})1}(1-\frac{k}{2})1(3-\frac{k}{2})\omega_{1(3-\frac{k}{2})\to(2-\frac{k}{2})1}(2-\frac{k}{2})1$$

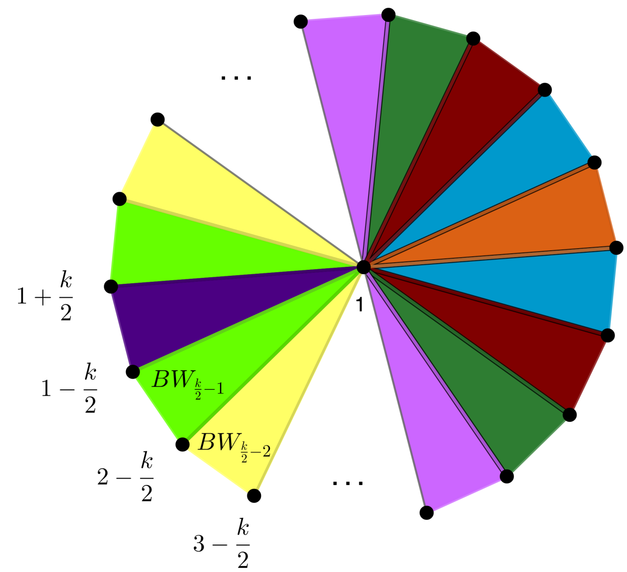

Figure 5.12. Corner Diagram at Vertex 1 of $H_{k,n}$, k Even

containing all of their edges, with the ordering of edges in each $\omega_{ab\to cd}$ unchanged. Figure 5.13 details the affect on the corner diagram at vertex 1.

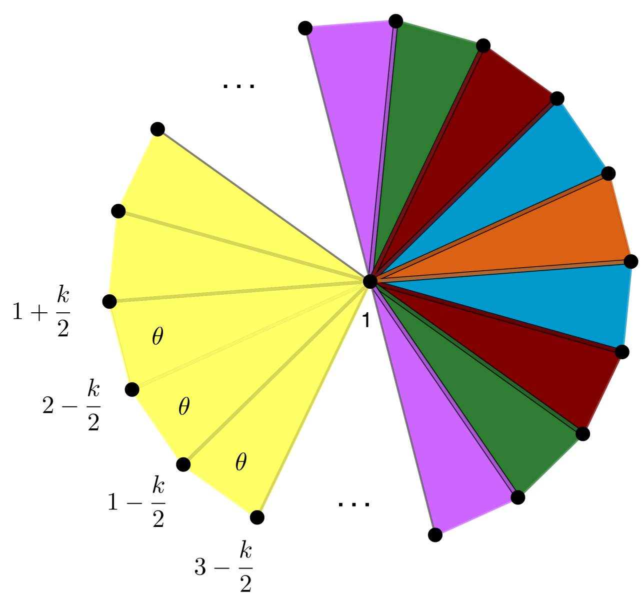

Figure 5.13. Corner Diagram at Vertex 1 of $H_{k,n}$, k Even, After Swap At Vertex 1

This means that boundary walk θ contains the corners $(2-\frac{k}{2})2(3-\frac{k}{2})$ and $(3-\frac{k}{2})2(4-\frac{k}{2})$, originally contained in $BW_{\frac{k}{2}-1}$ and $BW_{\frac{k}{2}-2}$. If $d=1$, then this is the only Type 1 swap performed. On the other hand, if $d \geq 2$, then we note that the corner $(2+\frac{k}{2})2(2-\frac{k}{2})$ is contained in $BW_{\frac{k}{2}+1}$ and thus not contained in θ. Therefore, as we can see in Figure 5.14 detailing the corner diagram at vertex 2 after the swap at vertex 1, the corners $(2+\frac{k}{2})2(2-\frac{k}{2})$, $(2-\frac{k}{2})2(3-\frac{k}{2})$, and $(3-\frac{k}{2})2(4-\frac{k}{2})$ satisfy the necessary conditions to apply Theorem 5.1 (The Second Edge Switching Theorem).

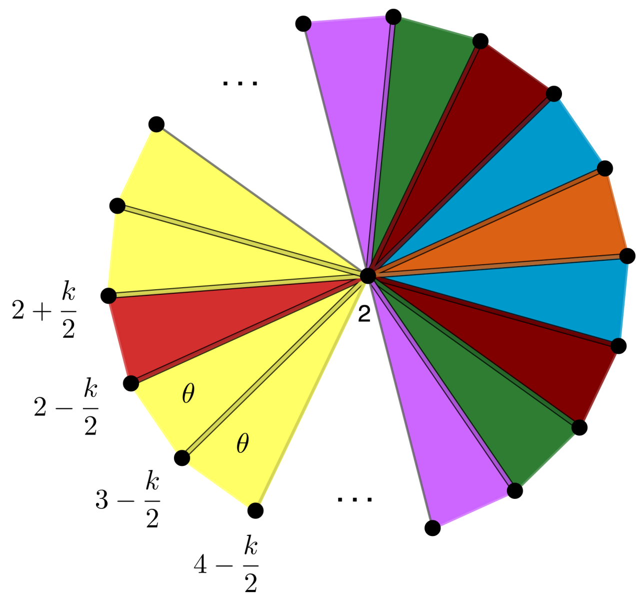

Figure 5.14. Corner Diagram at Vertex 2 of $H_{k,n}$, k Even, After Swap At Vertex 1

Make a second swap, edge $2(2-\frac{k}{2})$ with $2(3-\frac{k}{2})$ in the rotation system ordering around vertex 2. By the Second Edge Switching Theorem, the swap results in the boundary walks

$$\theta_1 : (2-\frac{k}{2})2(4-\frac{k}{2})\omega_{2(4-\frac{k}{2})\to(2-\frac{k}{2})2}$$

and

$$\theta_2 : (2+\frac{k}{2})2(3-\frac{k}{2})\omega_{2(3-\frac{k}{2})\to(3-\frac{k}{2})2}(3-\frac{k}{2})2(2-\frac{k}{2})\omega_{2(2-\frac{k}{2})\to(2+\frac{k}{2})2}$$

The corner diagram at vertex 2 is arranged as shown in Figure 5.15 after the swap at vertex 2.

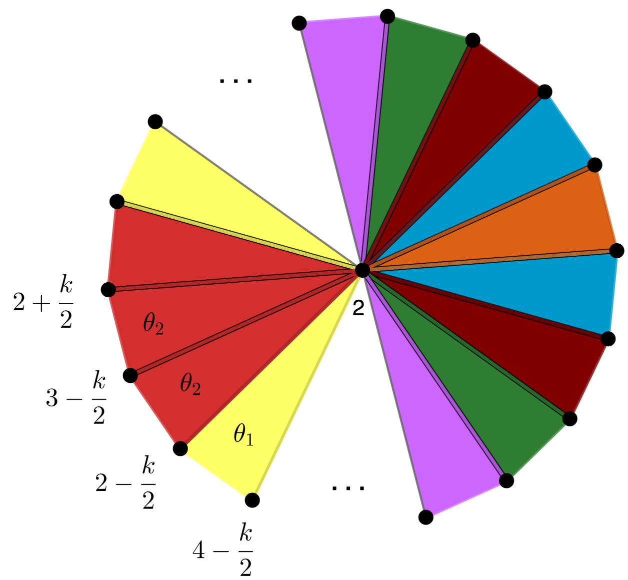

Figure 5.15. Corner Diagram at Vertex 2 of $H_{k,n}$, k Even, After Swap At Vertex 2

If $d = 2$ then we perform only these two Type 1 swaps. If $d \geq 3$, note that boundary walks θ_1 and θ_2 contain only corners in the same order as they appear in the boundary walks from the standard collection, or contain the corners created by a previous swap (one of the corners $(1+\frac{k}{2})1(2-\frac{k}{2})$, $(1-\frac{k}{2})1(3-\frac{k}{2})$, and $(2-\frac{k}{2})1(1-\frac{k}{2})$ from the first swap). Then θ_1 contains corner $(3-\frac{k}{2})3(4-\frac{k}{2})$ and θ_2 contains corner $(4-\frac{k}{2})3(5-\frac{k}{2})$, which are each distinct from the boundary walk containing corner $(3+\frac{k}{2})3(3-\frac{k}{2})$, which belongs to one of $BW_{\frac{k}{2}}$ through $BW_{\frac{k}{2}+d-1}$ from the standard collection. The corner diagram at vertex 3 in Figure 5.16 shows the new corner arrangement at vertex 3 after both swaps.

Notice that the First Edge Switching Theorem may be applied to the swap of edges $3(3-\frac{k}{2})$ and $3(4-\frac{k}{2})$. Note that the overall effect of the pair of swaps $1(1-\frac{k}{2})$ with $1(2-\frac{k}{2})$ and $2(2-\frac{k}{2})$ with $2(3-\frac{k}{2})$ was to reduce the total number of boundary walks by 2 by incorporating the directed edges of a k-gap boundary walk into a boundary walk also containing the directed edges of $BW_{\frac{k}{2}-1}$ and $BW_{\frac{k}{2}-2}$ via

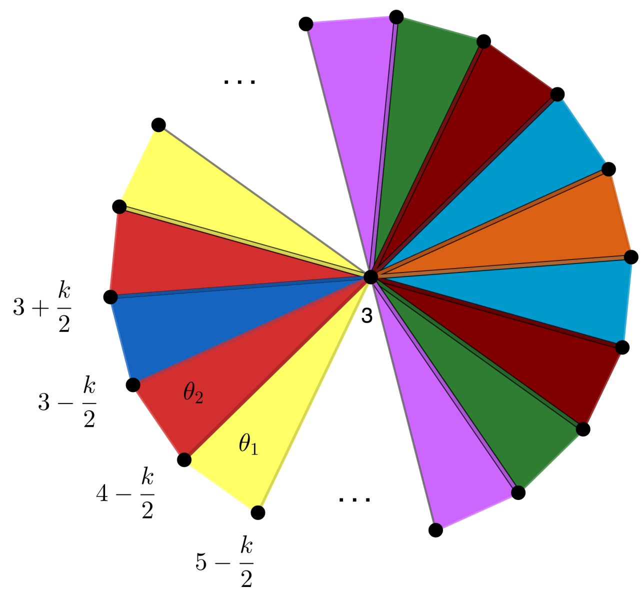

Figure 5.16. Corner Diagram at Vertex 3 of $H_{k,n}$, k Even, After Swap At Vertex 2

the former, and to rearrange the corners at vertex 3 in order to apply the First Edge Switching Theorem again via the latter.

The pair of consecutive swaps (at vertex 1 and 2) comprises a base case to prove by induction that the First and Second Edge Switching Theorems may be applied alternately as above. That is, we may continue this process by successively swapping edges $i(i-\frac{k}{2})$ and $i(i+1-\frac{k}{2})$ for all $i \leq d$, incorporating the directed edges of the k-gap boundary walks into boundary walks also containing the directed edges of $BW_{\frac{k}{2}-1}$ and $BW_{\frac{k}{2}-2}$. We claim that after j such steps, if j is odd, then all of the directed edges of $BW_{\frac{k}{2}-1}$, $BW_{\frac{k}{2}-2}$, and of the first j k-gap boundary walks are now contained in a single boundary walk, while the remaining directed edges (and thus, corners) are unaffected. On the other hand, if j is even, then all of the directed edges of $BW_{\frac{k}{2}-1}$, $BW_{\frac{k}{2}-2}$, and of the first j k-gap boundary walks are now contained in two boundary walks, while the remaining directed edges (and thus, corners) are unaffected. In each case, the arrangement of corners in the corner diagram at vertex $j+1$ is such that the First or Second Edge Switching Theorem will apply again as before. That is, if j is odd, then the two directed edges

$(j+1)(j+2-\frac{k}{2})$ and $(j+2-\frac{k}{2})(j+1)$ belong to the same boundary walk. If j is even, then the two directed edges $(j+1)(j+2-\frac{k}{2})$ and $(j+2-\frac{k}{2})(j+1)$ belong to two distinct boundary walks.

To continue the argument by induction, we will first assume we have made an even number, $s-1<d$, of consecutive swaps (as we had after completing the base case). This means that we had continued with a third swap $3(3-\frac{k}{2})$ with $3(4-\frac{k}{2})$ in the rotation system ordering at vertex 3 and a fourth swap $4(4-\frac{k}{2})$ with $4(5-\frac{k}{2})$ in the rotation system ordering at vertex 4, and so on, until swapping $(s-1)(s-1-\frac{k}{2})$ with $(s-1)(s-\frac{k}{2})$ in the rotation system of vertex $s-1$. We assume that the corners $(s+\frac{k}{2})s(s-\frac{k}{2})$, $(s-\frac{k}{2})s(s+1-\frac{k}{2})$, and $(s+1-\frac{k}{2})s(s+2-\frac{k}{2})$ are contained in distinct boundary walks. That is, the corner diagram at vertex $s \leq d$ (the next smallest vertex at which no swap has yet been made) appears as in Figure 5.17.

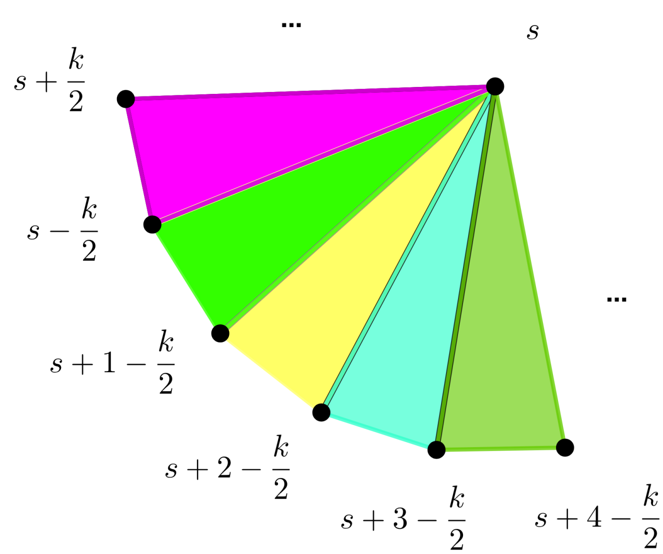

Figure 5.17. Corner Diagram at Vertex s of $H_{k,n}$, k even

Then we may swap edge $s(s-\frac{k}{2})$ with $s(s+1-\frac{k}{2})$ in the rotation system ordering around vertex $s \leq d$ and apply the First Edge Switching Theorem. The three distinct boundary walks are united into a single boundary walk containing all of their directed edges (and no other boundary walks are affected):

$$\theta_a : s(s-\frac{k}{2})\omega_{s(s-\frac{k}{2})\to(s+\frac{k}{2})s}(s+\frac{k}{2})s(s+1-\frac{k}{2})\omega_{s(s+1-\frac{k}{2})\to(s-\frac{k}{2})s}\cdots$$
$$(s-\frac{k}{2})s(s+2-\frac{k}{2})\omega_{s(s+2-\frac{k}{2})\to(s+1-\frac{k}{2})s}(s+1-\frac{k}{2})s$$

Observe the corresponding change in the corner diagram at vertex s from Figure 5.17 to Figure 5.18:

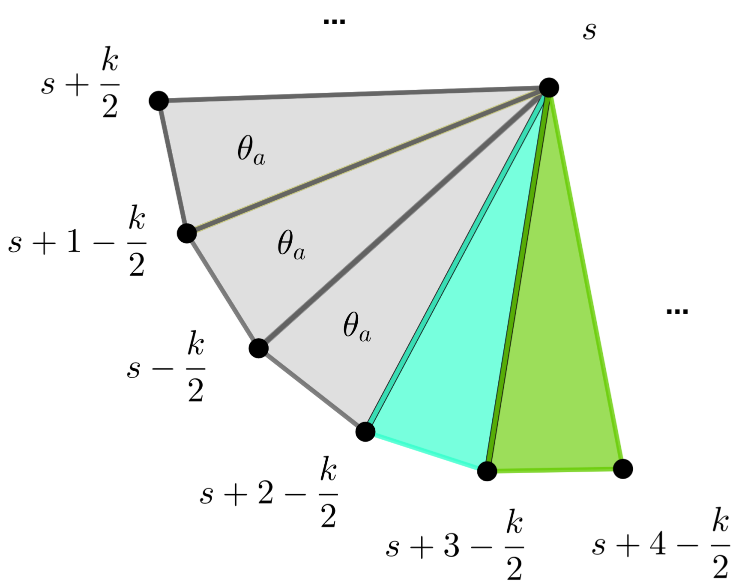

Figure 5.18. Corner Diagram at Vertex s of $H_{k,n}$, k even

If $s = d$, then all boundary walks in the collection are present at vertex s, and we are done. If $s < d$, then we note that the boundary walk θ_a contains only corners in the same order as they appear in the boundary walks from the standard collection other than the previously swapped corners, one of $(i+\frac{k}{2})i(i+1-\frac{k}{2})$, $(i+1-\frac{k}{2})i(i-\frac{k}{2})$, or $(i-\frac{k}{2})i(i+2-\frac{k}{2})$, for $1 \le i \le s$. Then θ_a contains corner $(s+1-\frac{k}{2})(s+1)(s+2-\frac{k}{2})$ in the path $\omega_{s(s+1-\frac{k}{2})\to(s-\frac{k}{2})s}$ and corner $(s+2-\frac{k}{2})(s+1)(s+3-\frac{k}{2})$ in the path $\omega_{s(s+2-\frac{k}{2})\to(s+1-\frac{k}{2})s}$. Also, corner $(s+1+\frac{k}{2})(s+1)(s+1-\frac{k}{2})$ in the corner diagram at vertex $s+1 \le d$ shown in Figure 5.19 is contained in one of $BW_{\frac{k}{2}}$ through $BW_{\frac{k}{2}+d-1}$ from the standard collection, which is distinct from boundary walk θ_a above.

Then we may apply the Second Edge Switching Theorem by swapping edges $(s+1)(s+1-\frac{k}{2})$ and $(s+1)(s+2-\frac{k}{2})$ in the rotation system ordering around vertex $s+1 \leq d$. After the swap we have two boundary walks (and no other boundary walks are affected):

$$\theta_b : (s+1-\frac{k}{2})(s+1)(s+3-\frac{k}{2})\omega_{(s+1)(s+3-\frac{k}{2})\to(s+1-\frac{k}{2})(s+1)}$$

and

$$\theta_c : (s+1+\frac{k}{2})(s+1)(s+2-\frac{k}{2})\omega_{(s+1)(s+2-\frac{k}{2})\to(s+2-\frac{k}{2})(s+1)}\cdots$$
$$(s+2-\frac{k}{2})(s+1)(s+1-\frac{k}{2})\omega_{(s+1)(s+1-\frac{k}{2})\to(s+1+\frac{k}{2})(s+1)}$$

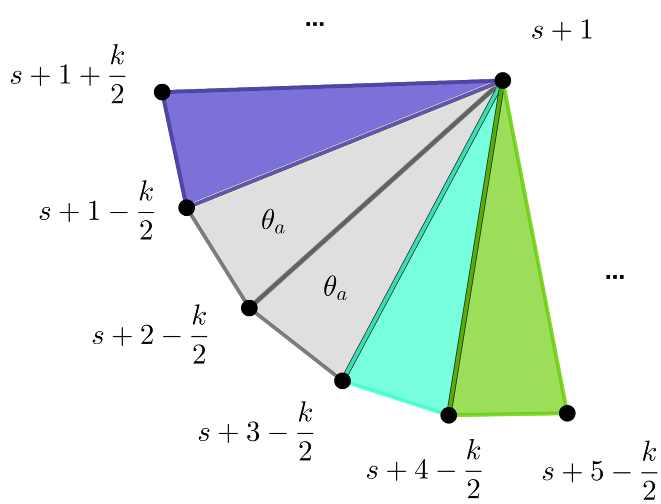

Figure 5.19. Corner Diagram at Vertex $s+1$ of $H_{k,n}$, k Even Before Swap

Observe the corresponding change in the corner diagram at vertex $s+1 \leq d$ from Figure 5.19 to Figure 5.20. If $s+1 = d$, then all boundary walks are present at vertex $s+1$ and are prepared for Type 2 swaps, so we stop. On the other hand, if $s+1 < d$, we note that θ_b and θ_c contain only corners in the same order as they appear in the boundary walks from the standard collection, other than the

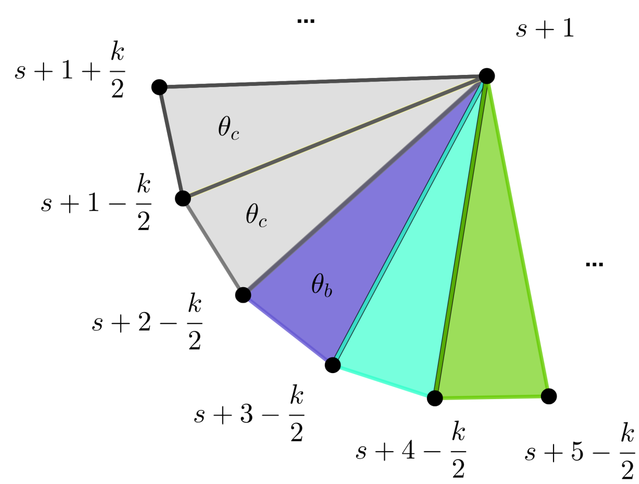

Figure 5.20. Corner Diagram at Vertex $s+1$ of $H_{k,n}$, k Even After Swap

previously swapped corners $(i+\frac{k}{2})i(i+1-\frac{k}{2})$, $(i+1-\frac{k}{2})i(i-\frac{k}{2})$, or $(i-\frac{k}{2})i(i+2-\frac{k}{2})$, for $1 \leq i \leq s+1$. Then θ_c contains corner $(s+2-\frac{k}{2})(s+2)(s+3-\frac{k}{2})$ in the path $\omega_{(s+1)(s+2-\frac{k}{2})\to(s+2-\frac{k}{2})(s+1)}$ and θ_b contains corner $(s+3-\frac{k}{2})(s+2)(s+4-\frac{k}{2})$ in the path $\omega_{(s+1)(s+3-\frac{k}{2})\to(s+1-\frac{k}{2})(s+1)}$. We have assumed that we have made fewer than d swaps, so we also know that corner $(s+2+\frac{k}{2})(s+2)(s+2-\frac{k}{2})$ is still contained in one of $BW_{\frac{k}{2}}$-$BW_{\frac{k}{2}+d-1}$ from the standard collection, which is distinct from boundary walks θ_b and θ_c above. Then we have made $s+1$ total swaps, where $s+1$ is an even number, and have the three distinct boundary walks containing the specified corners above. This completes the argument by induction.

By making d such consecutive swaps $i(i-\frac{k}{2})$ and $i(i-\frac{k}{2}+1)$ at vertex i for $1 \leq i \leq d$, we have reduced the number of boundary walks in the collection by d if d is even, or $d+1$ if d is odd. Notice that the only boundary walks affected by the above swaps are $BW_{\frac{k}{2}}$-$BW_{\frac{k}{2}+d-1}$ as well as $BW_{\frac{k}{2}-1}$ and $BW_{\frac{k}{2}-2}$.

We claim now that if $k=4$ or $k=6$ then the Type 1 swaps result in a maximal embedding and no Type 2 swaps are needed. Recall that if a rotation system yields either exactly one boundary walk or exactly two, then it results in a maximal embedding. Suppose that $k = 4$. Then $\frac{k}{2} = 2$, and $d = 1$ or $d = 2$. If $d = 1$, then the total number of boundary walks, F, is $\frac{k}{2} + d = 3$, and is reduced by 2 by the above swaps, leaving exactly one boundary walk. If $d = 2$, then $F = 4$, and F is reduced by 2 by the above swaps, leaving exactly two boundary walks. Now suppose that $k = 6$. Then $\frac{k}{2} = 3$, and $d = 1$ or $d = 3$. If $d = 1$, then $F = 4$, and F is reduced by 2 by the above swaps, leaving exactly two boundary walks. If $d = 3$, then $F = 6$ and F is reduced by 4 by the above swaps, leaving exactly two boundary walks once more. In all of these cases, a maximal embedding has been reached by the above swaps.

To continue in full generality, we assume that $k \geq 8$ and consider the following cases where the Type 2 swaps are also carried out.

Case 1: d and $\frac{k}{2}$ are both even.

Since every odd numbered Type 1 swap follows the First Edge Switching Theorem and every even numbered Type 1 swap follows the Second Edge Switching Theorem, it follows that we employ each theorem $\frac{d}{2}$ times to make d total swaps, and with the Type 1 swaps we have reduced the number of boundary walks in the standard collection by $2\left(\frac{d}{2}\right)$, leaving

$$\frac{k}{2} + d - 2\left(\frac{d}{2}\right) = \frac{k}{2}$$

total boundary walks in the collection. The remaining boundary walks are arranged around vertex d according to Figure 5.21.

Now consider the Type 2 swaps. After swapping $d(d-1)$ with $d(d-2)$, $d(d-3)$ with $d(d-4)$, $\ldots$ $d(d-(\frac{k}{2}-3))$ with $d(d-(\frac{k}{2}-2))$, we have made $(\frac{k}{2}-2)/2$ Type 2 swaps and thus further reduced the number of boundary walks by $\frac{k}{2}-2$, leaving

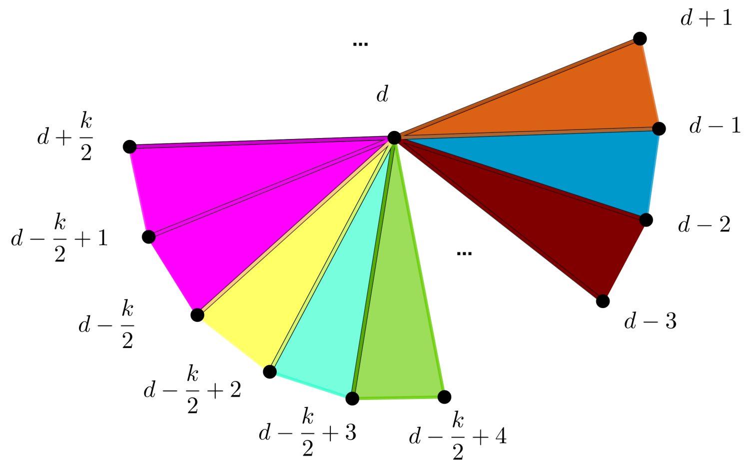

Figure 5.21. Corner Diagram at Vertex $d = \gcd(n, \frac{k}{2})$ of $H_{k,n}$, k even

exactly 2 boundary walks in the collection.

Case 2: d and $\frac{k}{2}$ are both odd.

We make $\frac{d+1}{2}$ Type 1 swaps at odd numbered vertices since d is odd, and thus reduce the total number of boundary walks in the standard collection by $d+1$, leaving $\frac{k}{2} - 1$ boundary walks, which are arranged around vertex d according to Figure 5.22.

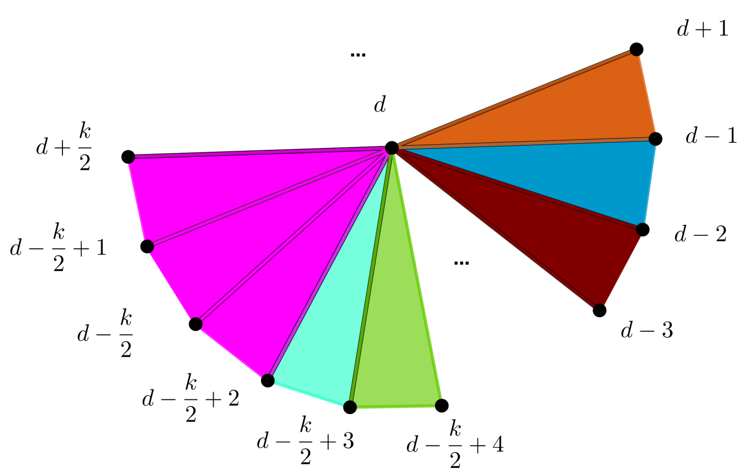

Figure 5.22. Corner Diagram at Vertex $d = \gcd(n, \frac{k}{2})$ of $H_{k,n}$, k even

Now consider the Type 2 swaps. After swapping $d(d-1)$ with $d(d-2)$, $d(d-3)$ with $d(d-4)$, ... $d(d-(\frac{k}{2}-4))$ with $d(d-(\frac{k}{2}-3))$, we have made $(\frac{k}{2}-3)/2$ Type 2 swaps and thus further reduced the number of boundary walks by $\frac{k}{2}-3$, leaving exactly 2 boundary walks in the collection.

Case 3: d is odd and $\frac{k}{2}$ is even.

Again, we make $\frac{d+1}{2}$ Type 1 swaps at odd numbered vertices since d is odd, and thus reduce the total number of boundary walks in the standard collection by $d+1$, leaving $\frac{k}{2}-1$ boundary walks, which are arranged around vertex d according to Figure 5.21 again. After Type 2 swaps $d(d-1)$ with $d(d-2)$, $d(d-3)$ with $d(d-4)$, ... $d(d-(\frac{k}{2}-3))$ with $d(d-(\frac{k}{2}-2))$, we have made $(\frac{k}{2}-2)/2$ Type 2 swaps and thus further reduced the number of boundary walks by $\frac{k}{2}-2$, leaving exactly 1 boundary walk in the collection. ∎

Corollary 5.3 *The maximal genus of $H_{k,n}$ with k even is:*

$$g = \begin{cases} (n - \frac{nk}{2} - 1)/-2, & \textit{if } d \textit{ and } \frac{k}{2} \textit{ do not share the same parity} \\ (n - \frac{nk}{2})/-2, & \textit{if } d \textit{ and } \frac{k}{2} \textit{ share the same parity} \end{cases}$$

Proof The genus of the embedding surface corresponding to the rotation system specified in the Theorem can be computed via Formula 1.1. We have that

$$V - E + F = n - \frac{nk}{2} + 1$$

or

$$V - E + F = n - \frac{nk}{2} + 2$$

if there are 1 or 2 boundary walks in the corresponding boundary walk collection, respectively. The result of the corollary follows from the fact that

$$V - E + F = 2 - 2g. \quad \blacksquare$$

CHAPTER 6

EXTENDING RESULTS TO $H_{k,n}$ WITH k ODD, n EVEN

We will extend the results of the previous chapter, with minor alterations, to cover the remaining Harary graphs, $H_{k,n}$ with k odd. In this chapter, we consider the case of $H_{k,n}$ with k odd and n even. Recall that we construct $H_{k,n}$ with k odd and n even by taking $H_{k-1,n}$ and adding an edge between each pair of diametrically opposed vertices. That is, we add $\frac{n}{2}$ edges of the form $j(j+\frac{n}{2})$ with $j = 1, 2, 3 \ldots \frac{n}{2}$. In Figure 6.1, the 4 additional edges added to $H_{4,8}$ to construct $H_{5,8}$ are shown in red.

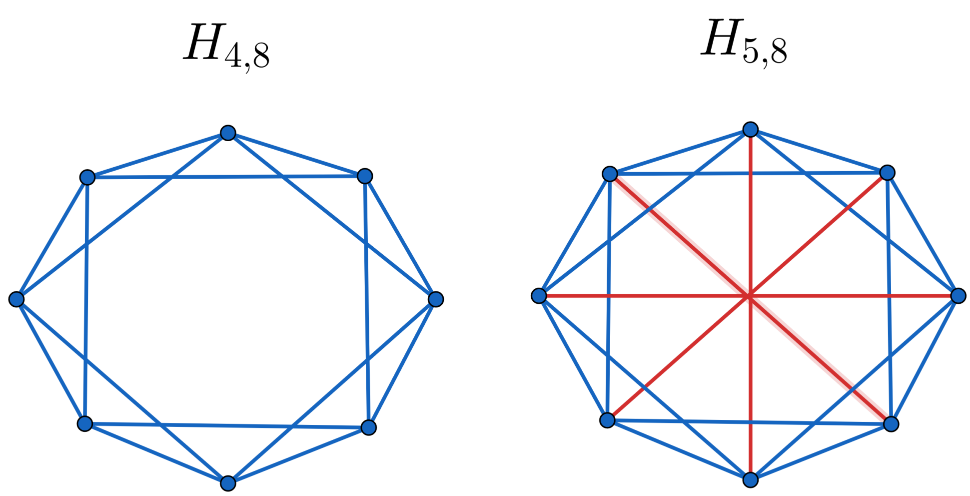

Figure 6.1. Drawing $H_{5,8}$ by adding edges to $H_{4,8}$

Notice that each vertex is then connected to the $\frac{k-1}{2}$ vertices to the left and to the right due to the structure inherited from $H_{4,8}$ as well as the diametrically opposed vertex. Note that diametrically opposed vertex labels will differ by $\frac{n}{2}$. Then there is a straightforward choice of definition for a standard rotation system for $H_{k,n}$

with k odd and n even. This choice will preserve most of the boundary walk structure associated to $H_{k-1,n}$ that has been detailed in the previous chapter.

Definition The standard rotation system for $H_{k,n}$ with k odd and n even orders the edges around any vertex j as

$$j : (j - \frac{k-1}{2})(j - \frac{k-1}{2} + 1) \ldots (j-1)(j + \frac{n}{2})(j+1) \ldots (j + \frac{k-1}{2} - 1)(j + \frac{k-1}{2}).$$

It follows from the definition that the new succeeding edge to $(j-1)j$ is $j(j+\frac{n}{2})$, and the new preceding edge to $j(j+1)$ is $(j+\frac{n}{2})j$. We will take this fact as the Third Boundary Walk Lemma.

Lemma 6.1 (The Third Boundary Walk Lemma) *Assume the standard rotation system for $H_{k,n}$ with k odd and n even. Then,*

1. The succeeding edge to directed edge $(j-1)j$ is $j(j+\frac{n}{2})$.

2. The succeeding edge to $(j+\frac{n}{2})j$ is $j(j+1)$.

Corollary 6.2 *(The Third Boundary Walk Corollary)*

For the standard rotation system, the boundary walk with initial edge $(j-1)j$ has the form

$$(j-1)j(j+\frac{n}{2})(j+1+\frac{n}{2})(j+1)(j+2)(j+3+\frac{n}{2})(j+4+\frac{n}{2}) \ldots (j-2+\frac{n}{2})(j-1+\frac{n}{2})$$

and contains every directed edge xy of $H_{k,n}$ with k odd and n even such that $y - x = 1$ and x has the same parity as $j-1$, as well as every directed edge xy such that $y - x = \frac{n}{2}$ with x of the same parity as j is included in the boundary walk.

Proof The succeeding edge to $(j-1)j$ is $j(j+\frac{n}{2})$ by the Third Boundary Walk Lemma (1). The succeeding edge to $j(j+\frac{n}{2})$ is $(j+\frac{n}{2})(j+1+\frac{n}{2})$ by the Third Boundary Walk Lemma (2). The succeeding edge to $(j+\frac{n}{2})(j+1+\frac{n}{2})$ is $(j+1+\frac{n}{2})(j+1)$ again by the Third Boundary Walk Lemma (1). The succeeding edge to $(j+1+\frac{n}{2})(j+1)$ is $(j+1)(j+2)$, again by the Third Boundary Walk Lemma (2).

Notice that $(j+1)(j+2)$ is another edge of the form $(j-1)j$, and that $(j+1)-(j-1)=2$. By repeatedly iterating the above argument, we see that the boundary walk consists of concatenations of paths of the form $(j-1)j(j+\frac{n}{2})(j+1+\frac{n}{2})$, with j increased by 2 at each iteration. The boundary walk ends with the preceding edge to $(j-1)j$, which is $(j-1+\frac{n}{2})(j-1)$.

Suppose that a directed edge xy is such that $y-x=1$, and x shares the same parity as $j-1$. Then $x=j-1+2m$ for some smallest integer m and thus appears for the first time in the boundary walk after m iterations of the argument above. The same argument also ensures that a directed edge xy such that $y-x=\frac{n}{2}$ with x of the same parity as j is included in the boundary walk, since $(j+\frac{n}{2})-j=\frac{n}{2}$. ∎

Theorem 6.3 (The Standard Collection for $H_{k,n}$, k odd, n even) *The standard boundary walk collection associated to the standard rotation system for $H_{k,n}$ with k even and n odd is identical to the standard boundary walk collection associated to the standard rotation system of the related graph $H_{k-1,n}$, except for BW_0, which has the following form if $\frac{n}{2}$ is even:*

BW_0 :

$$12(2+\tfrac{n}{2})(3+\tfrac{n}{2})34(4+\tfrac{n}{2})(5+\tfrac{n}{2})\ldots n1(1+\tfrac{n}{2})(2+\tfrac{n}{2})23(3+\tfrac{n}{2})(4+\tfrac{n}{2})\ldots(n-1)n(\tfrac{n}{2})(1+\tfrac{n}{2})1,$$

and is split into two distinct boundary walks with the following forms if $\frac{n}{2}$ is odd:

BW_{01} : $12(2+\frac{n}{2})(3+\frac{n}{2})34(4+\frac{n}{2})(5+\frac{n}{2})\ldots(n-1)n(\frac{n}{2})(1+\frac{n}{2})1$

BW_{02} : $23(3+\frac{n}{2})(4+\frac{n}{2})45(5+\frac{n}{2})(6+\frac{n}{2})\ldots n1(1+\frac{n}{2})(2+\frac{n}{2})2.$

Proof Parts 1 and 3 of the Second Boundary Walk Lemma applied to $H_{k-1,n}$ give us the same boundary walks BW_1 through $BW_{\frac{k-1}{2}+\gcd(n,\frac{k-1}{2})-1}$ for $H_{k,n}$ with k odd and n even, as the argument for the boundary walk structures use the same rotation system ordering. However, part 2 of the Second Boundary Walk Lemma no longer applies since $j(j+1)$ no longer succeeds $(j-1)j$. Instead, the Third Boundary Walk Lemma describes the structure of boundary walks including edges of this form. The only such boundary walk in the standard collection containing (the affected) directed edges of the form $j(j+1)$ and $(j-1)j$ in $H_{k-1,n}$, $k-1$ even, is BW_0.

Assume the Boundary Walk Algorithm applied to the standard rotation system for $H_{k,n}$ with k odd and n even has generated BW_1 through $BW_{\frac{k-1}{2}+\gcd(n,\frac{k-1}{2})-1}$. The collection contains every directed edge of the graph, except those of the form xy such that $y-x=1$ or $y-x=\frac{n}{2}$. The algorithm may begin a new walk with edge 12, the initial directed edge of BW_0 in the standard collection for $H_{k-1,n}$, $k-1$ even.

Case 1: If $\frac{n}{2}$ is even, then the boundary walk beginning with directed edge 12 begins

$$BW_0 : 12(2+\frac{n}{2})(3+\frac{n}{2})34(4+\frac{n}{2})(5+\frac{n}{2})\ldots$$

By the Third Boundary Walk Corollary, we see that every directed edge xy such that $y-x=1$ with x odd is included in the walk. Then the boundary walk includes directed edge $(\frac{n}{2}-1)(\frac{n}{2})$, and continues

$$\ldots(\frac{n}{2}-1)(\frac{n}{2})n1(1+\frac{n}{2})(2+\frac{n}{2})23\ldots$$

and by the same argument as above, the boundary walk must include every directed edge xy such that $y-x=1$ with x even, and boundary walk continues:

$$\ldots(\frac{n}{2}-2)(\frac{n}{2}-1)(n-1)n(\frac{n}{2})(1+\frac{n}{2})1$$

The boundary walk ends at directed edge $(1+\frac{n}{2})1$ as above, since it is the preceding edge to the initial directed edge 12. By the Third Boundary Walk Corollary, BW_0 contains every directed edge xy such that $y-x=1$ or $y-x=\frac{n}{2}$.

Case 2: If $\frac{n}{2}$ is odd, then the boundary walk beginning with directed edge 12 has the form

$$BW_{01} : 12(2+\frac{n}{2})(3+\frac{n}{2})34(4+\frac{n}{2})(5+\frac{n}{2})\ldots(\frac{n}{2}-2))(\frac{n}{2}-1)(n-1)n(\frac{n}{2})(1+\frac{n}{2})1.$$

By the same argument as Case 1, this boundary walk includes every directed edge xy such that $y-x=1$ and x is odd, or $y-x=\frac{n}{2}$ and x is even, but does not contain the directed edge 23. Then the Boundary Walk Algorithm may begin a new boundary walk with directed edge 23:

$$BW_{02} : 23(3+\frac{n}{2})(4+\frac{n}{2})45\ldots(\frac{n}{2}-1)(\frac{n}{2})n1(\frac{n}{2}+1)(\frac{n}{2}+2)2.$$

Again, by the same argument as Case 1, this boundary walk includes every directed edge xy such that $y-x=1$ and x is even, or $y-x=\frac{n}{2}$ and x is odd. Then together, BW_{01} and BW_{02} contain every directed edge xy such that $y-x=1$ or $y-x=\frac{n}{2}$. ∎

It follows from the above that the number of boundary walks in the standard collection is $\gcd(n,\frac{k-1}{2})+\frac{k-1}{2}$ if $\frac{n}{2}$ is even, and $\gcd(n,\frac{k-1}{2})+\frac{k-1}{2}+1$ if $\frac{n}{2}$ is odd. It also follows that the arrangement of the corners of the boundary walks around vertices are the same as the $H_{k-1,n}$ case, except for at BW_0. Now, if $\frac{n}{2}$ is even, then we have the arrangement at any vertex i for the standard boundary walk collection shown in the corner diagram at i in Figure 6.2. Notice the addition of the diametrically opposed edge $i(i+\frac{n}{2})$, ordered between $i(i-1)$ and $i(i+1)$, and whose directed edges are included in the new BW_0, shown in the color orange.

On the other hand if $\frac{n}{2}$ is odd, then we have the arrangement at any vertex i for the standard boundary walk collection shown in the corner diagram at i in Figure 6.3. Notice the addition of the diametrically opposed edge $i(i+\frac{n}{2})$, ordered between $i(i-1)$ and $i(i+1)$. Further, the directed edge $(i+\frac{n}{2})i$ is included in BW_{01} (shown in the usual color orange), and directed edge $i(i+\frac{n}{2})$ is included in BW_{02} (shown in a lighter colored orange).

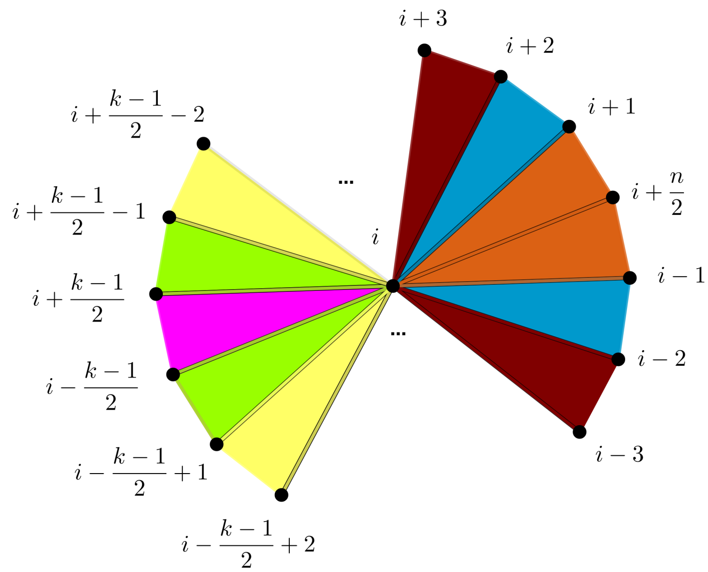

Figure 6.2. Corner Diagram at Vertex i of $H_{k,n}$, k odd, n even

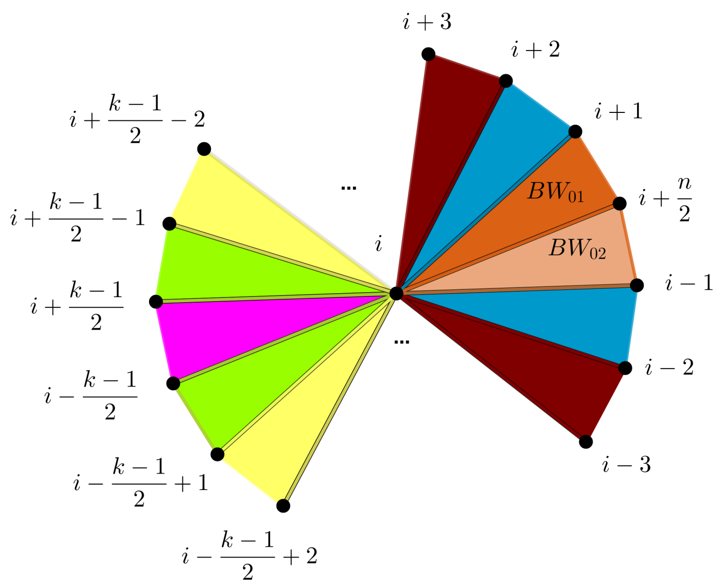

Figure 6.3. Corner Diagram at Vertex i of $H_{k,n}$, k odd, n even

Theorem 6.4 *The Rotation System of a Maximal Embedding of $H_{k,n}$ (k odd, n even)*

A maximal embedding of $H_{k,n}$, k odd, n even, is given by the rotation system obtained from the standard rotation system with the following swaps:

1. The same swaps made for $H_{k-1,n}$ if $\frac{n}{2}$ is even, with the same outcomes.

2. If $\frac{n}{2}$ is odd, then the same Type 1 swaps to unite the k-gap boundary walks as for $H_{k-1,n}$, but with a new set of Type 2 swaps, that is,

- *$i(i-\frac{k-1}{2})$ and $i(i-\frac{k-1}{2}+1)$ at vertex i for all $i \in \mathbb{Z}^+$ such that $i \leq d$*
- *$d(d+\frac{n}{2})$ with $d(d-1)$, $d(d-2)$ with $d(d-3)$, $\ldots$ $d(d-j+1)$ with $d(d-j)$ at vertex $d = \gcd(n, \frac{k-1}{2})$ if $k-1 \geq 8$, with*

$$j = \begin{cases} \frac{k-1}{2} - 3, & \textit{if } \frac{k-1}{2} \textit{ even,} \\ \frac{k-1}{2} - 2, & \textit{if } \frac{k-1}{2} \textit{ odd,} \end{cases}$$

as well as the single additional swap $d(d-\frac{k-1}{2}+2)$ with $d(d-\frac{k-1}{2})$ at vertex d, if $\frac{k-1}{2}$ is even and $\gcd(n, \frac{k-1}{2})$ is also even.

The number of boundary walks, F, for the resulting rotation system is given by

1. $F = 2$, if $\frac{k-1}{2}$ even, $\gcd(n, \frac{k-1}{2})$ odd.

2. $F = 1$, if $\frac{k-1}{2}$ and $\gcd(n, \frac{k-1}{2})$ share the same parity.

Proof For 1, we may use the same argument as Theorem 5.2. For 2, we may use the argument from Theorem 5.2 for the first set of swaps uniting the k-gap boundary walks. We continue the argument with the new arrangement of boundary walks around vertex $d = \gcd(n, \frac{k-1}{2})$ in the cases that follow.

Case 1: $\frac{k-1}{2}$ is even and $\gcd(n, \frac{k-1}{2})$ is odd.

We have made $\frac{\gcd(n, \frac{k-1}{2})+1}{2}$ Type 1 swaps at odd numbered vertices since $\gcd(n, \frac{k-1}{2})$ is odd, and thus reduce the total number of boundary walks in the standard collection by $\gcd(n, \frac{k-1}{2}) + 1$, leaving $\frac{k-1}{2}$ boundary walks, which are arranged around vertex $d = \gcd(n, \frac{k-1}{2})$ according to Figure 6.4.

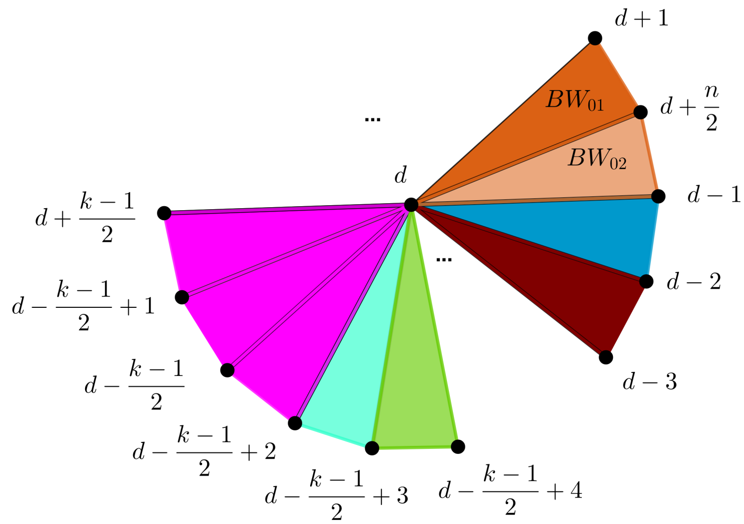

Figure 6.4. Corner Diagram at Vertex $d = \gcd(n, \frac{k-1}{2})$ for Cases 1 and 2

After new Type 2 swaps $d(d+\frac{n}{2})$ with $d(d-1)$, $d(d-2)$ with $d(d-3)$, $\ldots$ $d(d-\frac{k-1}{2}+4)$ with $d(d-\frac{k-1}{2}+3)$ at vertex $d = \gcd(n, \frac{k-1}{2})$, we have made $(\frac{k-1}{2}-2)/2$ total swaps and thus reduced the number of boundary walks by $\frac{k-1}{2}-2$, leaving exactly 2 boundary walks in the collection.

Case 2: $\frac{k-1}{2}$ is odd.

We have made $\frac{\gcd(n,\frac{k-1}{2})+1}{2}$ Type 1 swaps at odd numbered vertices since $\gcd(n, \frac{k-1}{2})$ is odd, and thus reduce the total number of boundary walks in the standard collection by $\gcd(n, \frac{k-1}{2})+1$, leaving $\frac{k-1}{2}$ boundary walks, which are arranged around vertex $d = \gcd(n, \frac{k-1}{2})$ according to Figure 6.4 again. After new Type 2 swaps $d(d+\frac{n}{2})$ with $d(d-1)$, $d(d-2)$ with $d(d-3)$, $\ldots$ $d(d-\frac{k-1}{2}+3)$ with $d(d-\frac{k-1}{2}+2)$ at vertex $d = \gcd(n, \frac{k-1}{2})$, we have made $(\frac{k-1}{2}-1)/2$ total swaps and thus reduced the number of boundary walks by $\frac{k-1}{2}-1$, leaving exactly 1 boundary walk in the collection.

Case 3: $\frac{k-1}{2}$ is even and $\gcd(n, \frac{k-1}{2})$ is even.

We have made $\frac{\gcd(n,\frac{k-1}{2})}{2}$ Type 1 swaps at odd numbered vertices since $\gcd(n, \frac{k-1}{2})$ is even, and thus reduce the total number of boundary walks in the standard collection by $\gcd(n, \frac{k-1}{2})$, leaving $\frac{k-1}{2}+1$ boundary walks, which are arranged around vertex $d = \gcd(n, \frac{k-1}{2})$ according to Figure 6.5.

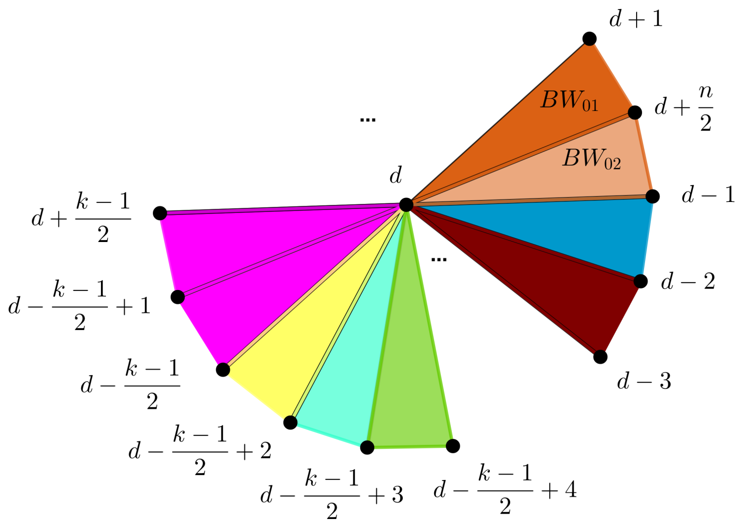

Figure 6.5. Corner Diagram at Vertex $d = \gcd(n, \frac{k-1}{2})$ for Case 3

After new Type 2 swaps $d(d+\frac{n}{2})$ with $d(d-1)$, $d(d-2)$ with $d(d-3)$, … $d(d-\frac{k-1}{2}+4)$ with $d(d-\frac{k-1}{2}+3)$ as well as the single additional swap $d(d-\frac{k-1}{2}+2)$ with $d(d-\frac{k-1}{2})$ at vertex $d = \gcd(n, \frac{k-1}{2})$, we have made $(\frac{k}{2})/2$ total swaps and thus reduced the number of boundary walks by $\frac{k-1}{2}$, leaving exactly 1 boundary walk in the collection. ∎

CHAPTER 7

EXTENDING RESULTS TO $H_{k,n}$ WITH k AND n BOTH ODD

Finally, we will consider the case of $H_{k,n}$ with k odd and n odd. Recall that we construct $H_{k,n}$ with k and n both odd by taking $H_{k-1,n}$ and adding $\frac{n+1}{2}$ edges of the form $j(j + \frac{n-1}{2})$ with $j = n, 1, 2, 3 \ldots \frac{n-1}{2}$. In Figure 7.1, the 4 additional edges added to $H_{4,7}$ to construct $H_{5,7}$ are shown in red.

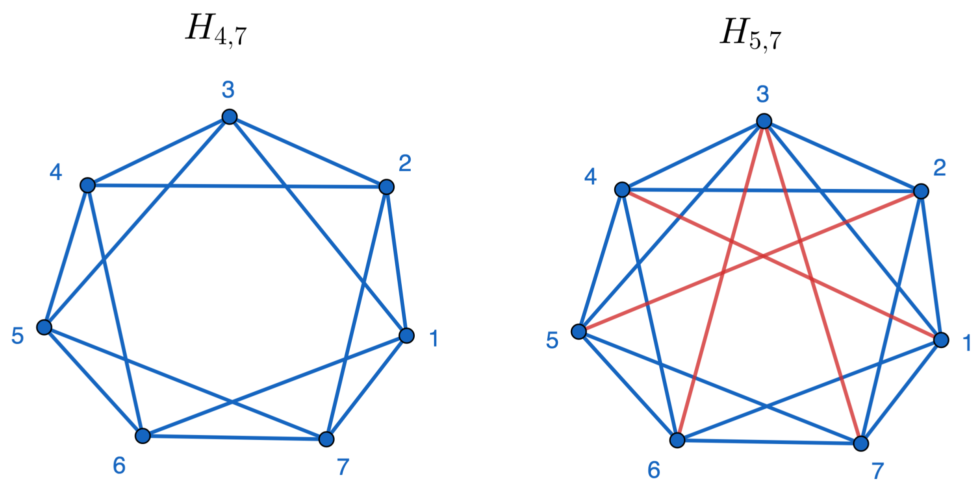

Figure 7.1. Drawing $H_{5,7}$ by adding edges to $H_{4,7}$

Notice that both edges $n(\frac{n-1}{2})$ and $\frac{n-1}{2}(n-1)$ have vertex $\frac{n-1}{2}$ as an endpoint, while no other pair of added edges share common endpoints. Then any rotation system must have one additional edge in the ordering around $\frac{n-1}{2}$, edge $(\frac{n-1}{2})n$, while the remaining vertices are free to order the $\frac{k-1}{2} + 1$ edges as in the same manner as the previous chapter. Then there is again a straightforward choice of definition for a standard rotation system for $H_{k,n}$ with k and n both odd. This choice will again preserve most of the boundary walk structure associated to $H_{k-1,n}$.

Definition The standard rotation system for $H_{k,n}$ with k and n both odd orders the edges around any vertex $j \neq \frac{n-1}{2}$ as

$$j : (j-\frac{k-1}{2})(j-\frac{k-1}{2}+1)\dots(j-1)(j+\frac{n-1}{2})(j+1)\dots(j+\frac{k-1}{2}-1)(j+\frac{k-1}{2}),$$

and around vertex $\frac{n-1}{2}$ as

$$\frac{n-1}{2} : (\frac{n-1}{2}-\frac{k-1}{2})(\frac{n-1}{2}-\frac{k-1}{2}+1)\dots(\frac{n-1}{2}-1)(n-1)n(\frac{n-1}{2}+1)\dots$$
$$\dots(\frac{n-1}{2}+\frac{k-1}{2}-1)(\frac{n-1}{2}+\frac{k-1}{2}).$$

It follows from the definition that the new succeeding edge to $(j-1)j$ is $j(j+\frac{n-1}{2})$, and the new preceding edge to $j(j+1)$ is $(j+\frac{n-1}{2})j$. We will take this fact as the Fourth Boundary Walk Lemma.

Lemma 7.1 (The Fourth Boundary Walk Lemma) *Assume the standard rotation system for $H_{k,n}$ with k and n both odd. Then,*

1. The succeeding edge to directed edge $(j-1)j$ is $j(j+\frac{n-1}{2})$.

2. The succeeding edge to $(j+\frac{n-1}{2})j$ is $j(j+1)$.

Corollary 7.2 *(The Fourth Boundary Walk Corollary)*

For the standard rotation system, the boundary walk with initial edge $(j-1)j$ has the form

$$(j-1)j(j+\frac{n-1}{2})(j+1+\frac{n-1}{2})(j+1)(j+2)(j+3+\frac{n-1}{2})(j+4+\frac{n-1}{2})\dots$$
$$\dots(j-2+\frac{n-1}{2})(j-1+\frac{n-1}{2})$$

and contains every directed edge xy of $H_{k,n}$ with k odd and n even such that $y-x=1$ and x has the same parity as $j-1$, as well as every directed edge xy such that $y-x=\frac{n-1}{2}$ with x of the same parity as j is included in the boundary walk.

The proof of the corollary follows the same argument as the Third Boundary Walk Corollary, so it is omitted here.

Theorem 7.3 (The Standard Collection for $H_{k,n}$, k and n both odd) *The standard boundary walk collection associated to the standard rotation system for $H_{k,n}$ with k even and n odd is identical to the standard boundary walk collection associated to the standard rotation system of the related graph $H_{k-1,n}$, except for BW_0, which has the following form if $\frac{n-1}{2}$ is odd:*

$BW_0:$ $12(2+\frac{n-1}{2})(3+\frac{n-1}{2})34(4+\frac{n-1}{2})(5+\frac{n-1}{2})\ldots(n-2)(n-1)(\frac{n-1}{2})n1(1+\frac{n-1}{2})(2+\frac{n-1}{2})23\ldots(n-1)n(\frac{n-1}{2})(1+\frac{n-1}{2})1,$

and is split into two distinct boundary walks with the following forms if $\frac{n-1}{2}$ is even:

$BW_{01}:$

$12(2+\frac{n-1}{2})(3+\frac{n-1}{2})34(4+\frac{n-1}{2})(5+\frac{n-1}{2})\ldots(\frac{n-1}{2}-1)(\frac{n-1}{2})(n-1)n(\frac{n-1}{2})(1+\frac{n-1}{2})1$

$BW_{02}:$

$23(3+\frac{n-1}{2})(4+\frac{n-1}{2})45(5+\frac{n-1}{2})(6+\frac{n}{2})\ldots(n-2)(n-1)(\frac{n-1}{2})n1(1+\frac{n-1}{2})(2+\frac{n-1}{2})2.$

Proof The argument is the same as for Theorem 6.3, except where the details are altered here. Assume the Boundary Walk Algorithm applied to the standard rotation system for $H_{k,n}$ with k and n both odd has generated BW_1 through $BW_{\frac{k-1}{2}+\gcd(n,\frac{k-1}{2})-1}$. The collection contains every directed edge of the graph, except those of the form xy such that $y-x=1$ or $y-x=\frac{n-1}{2}$. The algorithm may begin a new walk with edge 12, the initial directed edge of BW_0 in the standard collection for $H_{k-1,n}$, $k-1$ even.

Case 1: If $\frac{n-1}{2}$ is odd, then the boundary walk beginning with directed edge 12 begins

$$BW_0: 12(2+\frac{n-1}{2})(3+\frac{n-1}{2})34(4+\frac{n-1}{2})(5+\frac{n-1}{2})\ldots$$

By the Fourth Boundary Walk Corollary, we see that every directed edge xy such that $y-x=1$ with x odd is included in the walk. Then the boundary walk includes directed edge $(\frac{n-1}{2}-1)(\frac{n-1}{2})$, and continues

$$\dots(n-2)(n-1)(\frac{n-1}{2})n1(1+\frac{n-1}{2})(2+\frac{n-1}{2})23\dots$$

and by the same argument as above, the boundary walk must include every directed edge xy such that $y-x=1$ with x even, and boundary walk continues:

$$\dots(n-1)n(\frac{n-1}{2})(1+\frac{n-1}{2})1$$

The boundary walk ends at directed edge $(1+\frac{n-1}{2})1$ as above, since it is the preceding edge to the initial directed edge 12. By the Fourth Boundary Walk Corollary, BW_0 contains every directed edge xy such that $y-x=1$ or $y-x=\frac{n-1}{2}$.

Case 2: If $\frac{n}{2}$ is even, then the boundary walk beginning with directed edge 12 has the form

$$BW_{01}: 12(2+\frac{n-1}{2})(3+\frac{n-1}{2})34(4+\frac{n-1}{2})(5+\frac{n-1}{2})\dots$$
$$\dots(\frac{n-1}{2}-1)(\frac{n-1}{2})(n-1)n(\frac{n-1}{2})(1+\frac{n-1}{2})1.$$

By the same argument as Case 1, this boundary walk includes every directed edge xy such that $y-x=1$ and x is odd, or $y-x=\frac{n-1}{2}$ and x is even, but does not contain the directed edge 23. Then the Boundary Walk Algorithm may begin a new boundary walk with directed edge 23:

$$BW_{02}: 23(3+\frac{n-1}{2})(4+\frac{n-1}{2})45(5+\frac{n-1}{2})(6+\frac{n}{2})\dots$$
$$\dots(n-2)(n-1)(\frac{n-1}{2})n1(1+\frac{n-1}{2})(2+\frac{n-1}{2})2.$$

Again, by the same argument as Case 1, this boundary walk includes every directed edge xy such that $y-x=1$ and x is even, or $y-x=\frac{n}{2}$ and x is odd. Then together, BW_{01} and BW_{02} contain every directed edge xy such that $y-x=1$ or $y-x=\frac{n-1}{2}$. ∎

It follows from the above that the number of boundary walks in the standard collection is $\gcd(n, \frac{k-1}{2}) + \frac{k-1}{2}$ if $\frac{n-1}{2}$ is odd, and $\gcd(n, \frac{k-1}{2}) + \frac{k-1}{2} + 1$ if $\frac{n-1}{2}$ is even. It also follows that the arrangement of the corners of the boundary walks around vertices are the same as the $H_{k-1,n}$ case, except for at BW_0. Now, if $\frac{n-1}{2}$ is odd, then we have the arrangement at any vertex i and the special vertex $\frac{n-1}{2}$ for the standard boundary walk collection shown in the corner diagrams in Figure 7.2 on the following page. Notice the addition of the edge $i(i + \frac{n-1}{2})$, ordered between $i(i-1)$ and $i(i+1)$, and whose directed edges are included in the new BW_0, shown in the color orange in the general case. Also notice the addition of the extra edge $(\frac{n-1}{2})n$ in the special case, whose directed edges are also included in the new BW_0.

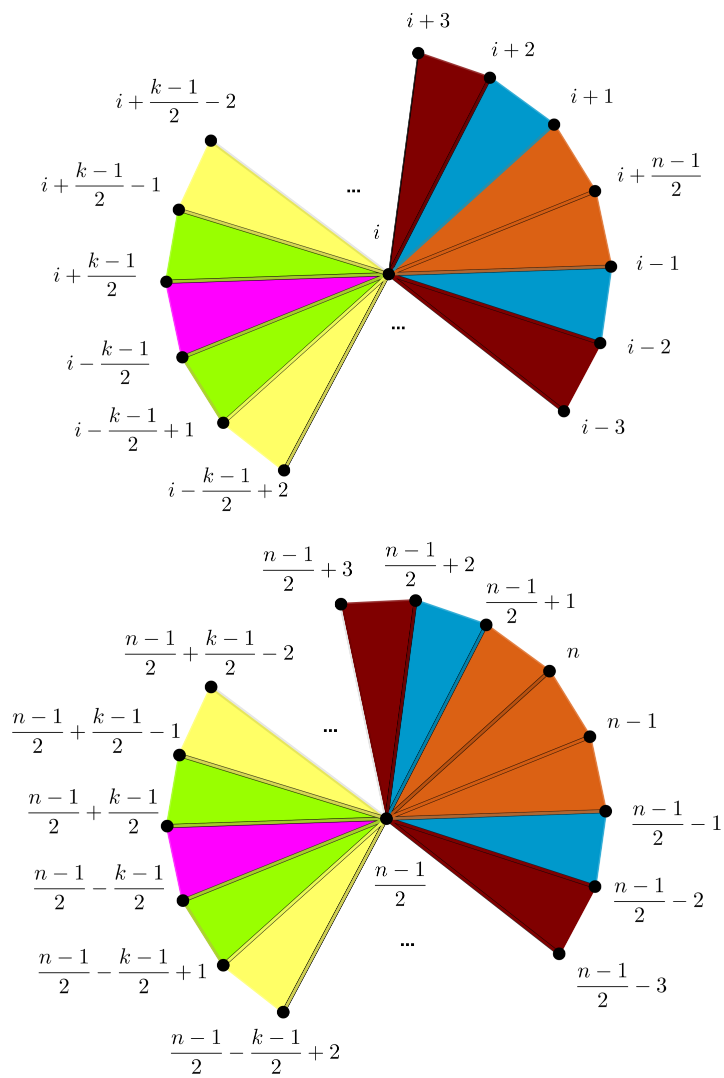

Figure 7.2. Corner Diagrams at Vertex i and $\frac{n-1}{2}$ of $H_{k,n}$, $\frac{n-1}{2}$

On the other hand if $\frac{n-1}{2}$ is even, then we have the arrangement at any vertex i for the standard boundary walk collection shown in the corner diagram at i in Figure 7.3 on the following page. Notice that the directed edge $(i+\frac{n-1}{2})i$ is included in BW_{02} (shown in the lighter colored orange), and directed edge $i(i+\frac{n}{2})$ is included in BW_{01} (shown in the usual color orange). Also notice that directed edges $n(\frac{n-1}{2})$ and $\frac{n-1}{2}(n-1)$ are included in $BW_0 1$, while the directed edges $(n-1)\frac{n-1}{2}$ and $(\frac{n-1}{2})n$ are included in BW_{02}.

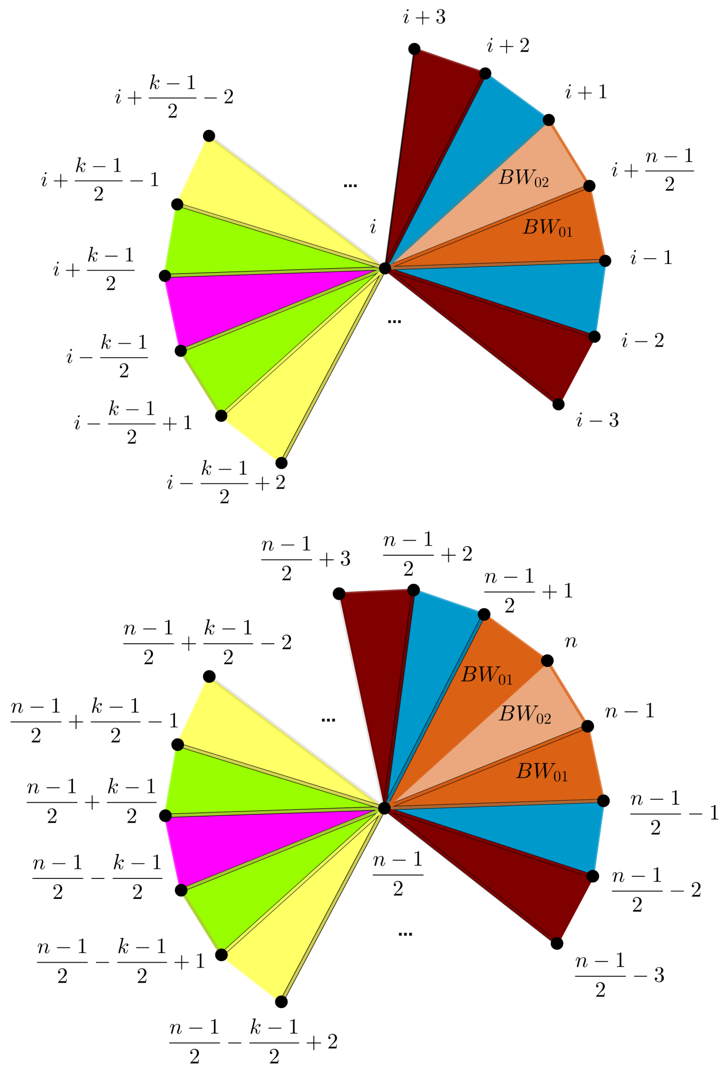

Figure 7.3. Corner Diagrams at Vertex i and $\frac{n-1}{2}$ of $H_{k,n}$, $\frac{n-1}{2}$ even

Theorem 7.4 *The Rotation System of a Maximal Embedding of $H_{k,n}$ (k and n both odd)*

A maximal embedding of $H_{k,n}$, k and n both odd, is given by the rotation system obtained from the standard rotation system with the following swaps:

1. The same swaps made for $H_{k-1,n}$ if $\frac{n-1}{2}$ is odd, with the same outcomes.

2. If $\frac{n-1}{2}$ is even, then the same Type 1 swaps to unite the k-gap boundary walks as for $H_{k-1,n}$, but with a new set of Type 2 swaps, that is,

- *$i(i - \frac{k-1}{2})$ and $i(i - \frac{k-1}{2} + 1)$ at vertex i for all $i \in \mathbb{Z}^+$ such that $i \leq d$*
- *$d(d + \frac{n-1}{2})$ with $d(d-1)$, $d(d-2)$ with $d(d-3)$, $\ldots$ $d(d-j+1)$ with $d(d-j)$ at vertex $d = \gcd(n, \frac{k-1}{2})$ if $k - 1 \geq 8$, with*

$$j = \begin{cases} \frac{k-1}{2} - 3, & \text{if } \frac{k-1}{2} \text{ even,} \\ \frac{k-1}{2} - 2, & \text{if } \frac{k-1}{2} \text{ odd.} \end{cases}$$

The number of boundary walks, F, for the resulting rotation system is given by

1. *$F = 2$, if $\frac{k-1}{2}$ is even.*
2. *$F = 1$, if $\frac{k-1}{2}$ is odd.*

The proof follows the same argument as Theorem 6.4, with the arrangements of boundary walks around all vertices (except $\frac{n-1}{2}$) the same, except in that the positions of BW_{01} and BW_{02} are swapped, as seen in Figure 7.3. The reader may check to see that the slight difference in the arrangement of the boundary walks at vertex $\frac{n-1}{2}$ seen in 7.3 does not affect the argument. Notice that we do not need to specify an additional swap to be made if $\gcd(n, \frac{k-1}{2})$ is even (as we did in Theorem 6.4), because $\gcd(n, \frac{k-1}{2})$ must be odd when n is odd.

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BIOGRAPHY OF THE AUTHOR

Christopher Allen Smith was born in Waldo county Maine to Leon and Amy Smith, growing up in the small town of Lincolnville. He attended Camden Hills Regional High School, graduating with highest honors in 2012. He went on to attend The University of Maine at Orono, where he received a Bachelor of Science in Mathematics upon graduating after the spring semester of 2016. He enrolled the following semester as a graduate student and teaching assistant in the Department of Mathematics at The University of Maine. Christopher Allen Smith is a candidate for the Master of Arts degree in Mathematics from The University of Maine in August 2019.