INVESTIGATING THE SOLUTIONS TO NON-LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

Jose Rolando Gonzalez, Keshav R. Acharya

Embry-Riddle Aeronautical University

Abstract

The main focus of this paper is that in which the nonlinearity does not occur in the highest differentiated term.This paper will further discuss one observed method. The method to be presented has the particular advantage that its wide scope and application, yet it is constrained to differential equations that are associated with nonconservative systems.We wish to extend the results of [2] to all real numbers in the domain of x. The Ermakov-Pinney method of lineraization was employed to obtain solvable form of the equation. An implicit solution of the nonlinear differential equation $y'' + P(x)y = q_m(x)/y^m$ is found to be $y = |w[(C_1 \int \frac{dx}{w^2} + C_2)^2 + C_3]^{|\frac{1}{1-m}|}| + C_4$ if $q_m(x) = w(x)^{m-3}$. Where w is the combination of two linearly independent solutions u and v, such the $w(x) = au(x) + bv(x)$, as well as $\varphi = \int w(x)^{-2}dx$. Where C_1,C_2,C_3, and C_4 are arbitrary constants.

Introduction

Only a few nonlinear differential equations are known to have exact solutions, many of the important differential equations that model the real world do not have such solutions. Some of these equations could be linearized by some substitution in which the nonlinear term is discarded. There exists times in which the nonlinear term cannot be negated or discarded due to its vital contribution to the solution. A differential equation can be approximated by another equation with small nonlinearities. The small changes give rise to solutions that are valid over different ranges of its parameters. When tackling these small nonlinearities there are two main types focused on. The first "boundary layer problems"[1] are nonlinear differential equations in which the nonlinearity occurs in the highest differentiated term. These equations are quite crucial to explaining several physical theories. They include important nonlinear partial differential equations problems, as well as some ordinary nonlinear differential equations. Boundary layer problems are very closely tied to their applications, and little research has been conducted into their development

Method

Given a nonlinear second order differential equation in the form

$$y'' + P(x)y = \frac{q_m(x)}{y^m}$$

It can be proven that the solution takes on the form

$$y = |w[(C_1 \int \frac{dx}{w^2} + C_2)^2 + C_3]^{|\frac{1}{1-m}|}| + C_4$$

The proof is as follows: Let w be defined as the linear combination of the two linearly independent solutions to the homogeneous equation u and v, such that

$$w(x) = au(x) + bv(x)$$

Setting

$$y(x) = w(x)z(x), \qquad \varphi = \int \frac{dx}{w^2}$$

upon differentiation

$$y'' = w''z + 2w'z' + wz''$$

Applying the fact that $w'' + P(x)w = 0$ and simplifying the equation becomes

$$y'' + P(x)y = z''w + 2z'w'$$

Substituting back into the original equation

$$z''w + 2z'w' = \frac{w^{m-3}}{z^m w^m}$$

Performing a change of variable and substituting back into the equation

$$\frac{d^2z}{d\varphi^2} = z''w^4 + 2z'w^3$$

Applying another change of variable and setting the value of $m \neq 1$ the equation becomes a simple ODE with the form

$$\frac{dz}{d\varphi} = \sqrt{\frac{2z^{1-m}}{1-m} + C_1}$$

Results

Solving the first order differential equation, and substituting back in the change of variables we arrive at the result

$$y = \left| w \left[(C_1 \int \frac{dx}{w^2} + C_2)^2 + C_3 \right]^{|\frac{1}{1-m}|} \right| + C_4$$

where C_1,C_2,C_3 ,and C_4 are constants

Verification

To verify the equation, it shall be analyzed in the same context as E.Pinney [2] that is

$$y'' + w^2 y = \frac{q_m(t)}{y^3}$$

Edmund Pinney showed that is equation has solution in the form

$$y = [u^2 - cW^{-2}v^2]^{\frac{1}{2}}$$

when setting the constant $w^2 = 1$ and substituting in the solution to the homogeneous equation the solutions become

$$y = [sin^2(x) - kcos^2(x)]^{\frac{1}{2}} \qquad (Black\ Line)$$

$$y = |(asin(x) + bcos(x))[(C_1\varphi + C_2)^2 + C_3]^{\frac{1}{2}}| + C_4 \qquad (Dotted\ Line)$$

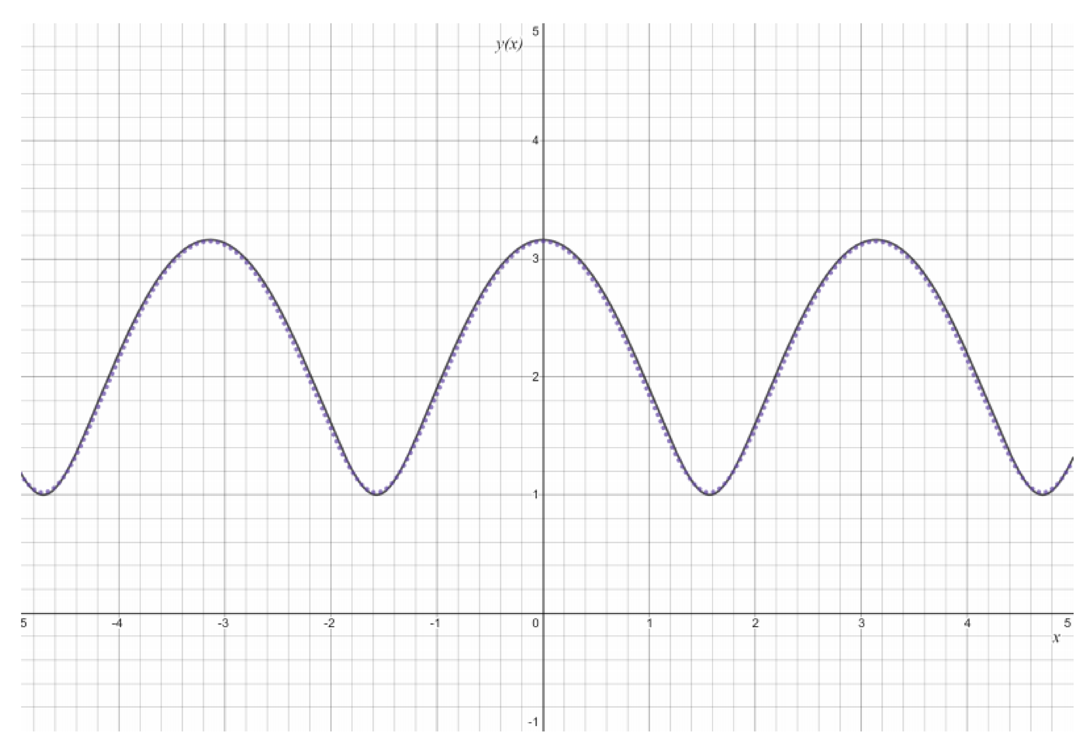

Fig. 1: Plots of solutions to $y'' + y = \frac{c}{y^3}$.

To further verify the validity of the findings the equation is evaluated at an arbitrary value of m for instance, $m = 5$ has solution

$$y'' + y - \frac{(asinx + bcosx)^2}{y^3} = 0 \qquad (Black\ Line)$$

$$y = |(asin(x) + bcos(x))[(C_1\varphi + C_2)^2 + C_3]^{\frac{1}{4}}| + C_4 \qquad (Dotted\ Line)$$

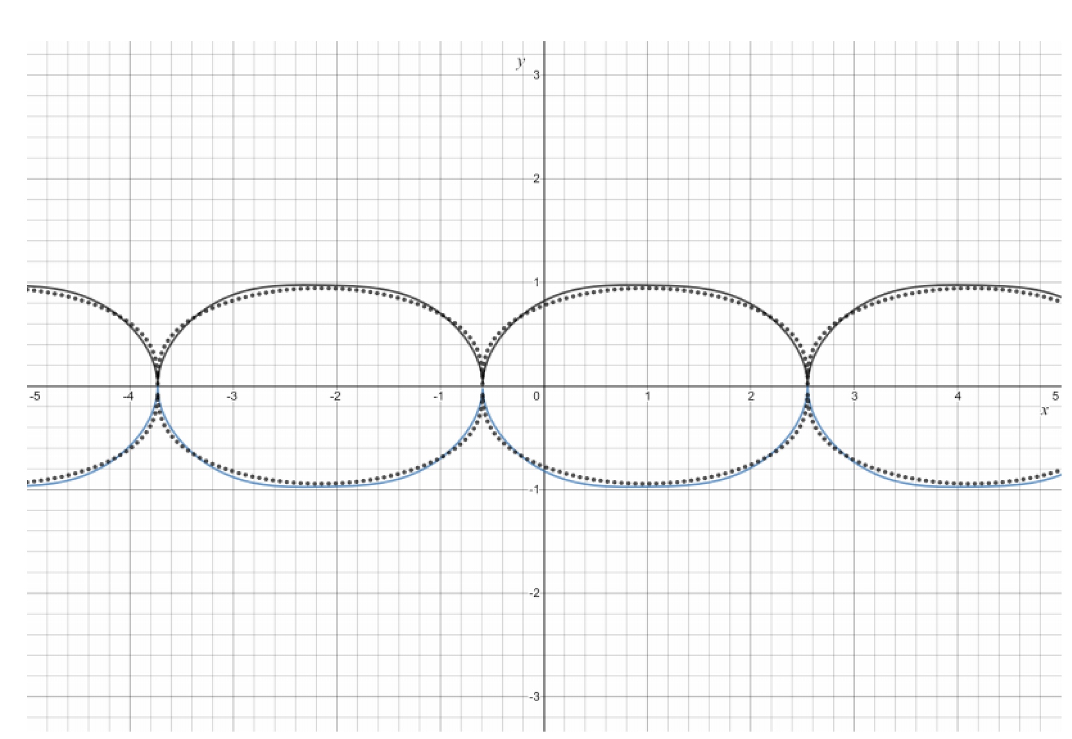

Fig. 2: Plots of solutions to $y'' + y = \frac{(asinx+bcosx)^2}{y^5}$

Applications

Nonlinear second order differential equations can be seen all around the world from the small interactions of particles with their surroundings to oscillations of a system. One of the principle applications of this paper would be the Duffing Oscillator problem [1]

$$x'' + \delta x' + \beta x + \alpha x^3 = 0 \qquad , where\ \delta \geq 0$$

in this case a rigid metal beam oscillates back and forth between two main poles, all the while the metal rod not being elastically deformed.

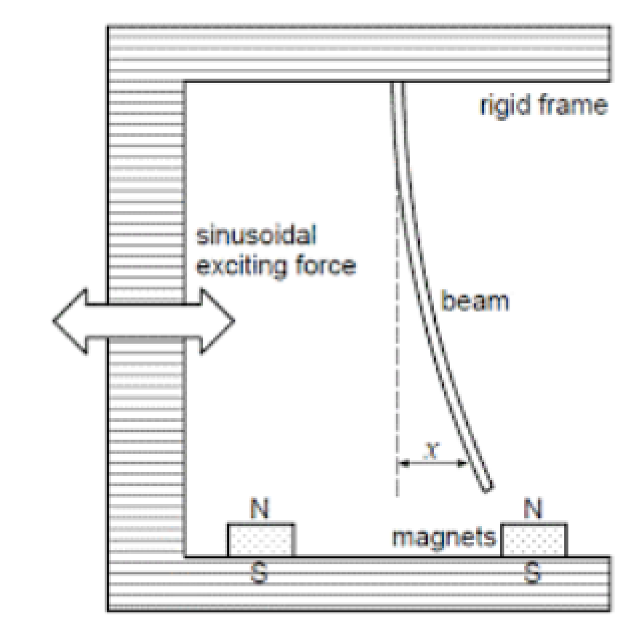

Fig. 3: Standard Duffing Oscillator.

this equation to take the form

$$x'' + \beta x = \frac{cx^3}{(asin\beta t + bcos\beta t)^6}$$

Graphically observing the behavior of the right hand sides it can be proven that

$$\frac{cx^3}{(asin\beta t + bcos\beta t)^6} = constant$$

Thus the results of the paper offers a viable solution to the differential equation; that solution is

$$x = |w[(C_1\varphi + C_2)^2 + C_3]^{1/4}| + C_4$$

This can physically be interpreted as the displacement of the oscillating body. What was considered above was the ideal case where $\delta = 0$; that is to say the oscillator was undamped.

Remarks

This research paper aimed at providing an alternative to the Reid equation [1] by formulating a more generalized method. Following the Ermakov method to solving a particular form of differential equations a general solution was created.

$$y = w[(C_1\varphi + C_2)^2 + C_3]^{|\frac{1}{1-m}|} + C_4$$

This solution is of course not perfect, it is limited by $m = 1$ a value which does not apply . With further study and research in this field of study new and more generalized equations can be formulated. Further investigation into the special case provided in this paper could result in an equation that could change the way real world problems are approached.

References

References

[1] G Duffing. “Erzwungene Schwingungen bei Veranderlicher Eigenfrequenz”. In: (1918).

[2] E. Pinney. “The nonlinear differential equation $y'' + P(x)y - cy^{-3}$”. In: *Proceeds of American Mathematical Society* (1950).