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A Neighbourhood Semantics for the Logic TK

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Abstract. The logic **TK** was introduced as a propositional logic extending the classical propositional calculus with a new unary operator which interprets some conceptions of Tarski's consequence operator. TK-algebras were introduced as models to **TK**. Thus, by using algebraic tools, the adequacy (soundness and completeness) of **TK** relatively to the TK-algebras was proved. This work presents a neighbourhood semantics for **TK**, which turns out to be deductively equivalent to the non-normal modal logic **EMT4**.

Keywords: Consequence operator; TK algebra; TK logic; neighbourhood semantics.

Introduction

Considering algebraic aspects of the notion of Tarski's consequence operator, Nascimento and Feitosa (2005) defined an algebra that rescues these conceptions in an algebraic context, the TK-algebra. So Feitosa, Grácio and Nascimento (2007) introduced a propositional logic which has as models exactly these TK-algebras. This logical system was presented in the Hilbert-style, with axioms and rules of inference, and the adequacy between the axiomatic system and the TK-models was proved. As the new operator was introduced to interpret the characteristics of the Tarski's operator, this propositional logic turns out to be a modal logic. In this paper, we present a neighbourhood semantics for this new logical system.

1. Tarski's consequence operator

In what follows, we consider the concept of a consequence operator in a way a little more general than was introduced by Tarski, in 1935.

Definition 1.1. A *consequence operator* on S is a function $C : \mathscr{P}(S) \to \mathscr{P}(S)$ such that, for every $A, B \subseteq S$:

(C_1) $A \subseteq C(A)$;
(C_2) $A \subseteq B \Rightarrow C(A) \subseteq C(B)$;
(C_3) $C(C(A)) \subseteq C(A)$.

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Of course, in view of (C_1) and (C_3), for every $A \subseteq S$, the equality $C(C(A)) = C(A)$ holds.

Definition 1.2. A consequence operator C on S is *finitary* when, for every $A \subseteq S$:

$$C(A) = \cup\{C(A_0) : A_0 \text{ is a finite subset of } A\}.$$

Definition 1.3. A *Tarski space* (a *deductive system* or a *closure space*) is a pair (S, C) such that S is a set and C is a consequence operator on S.

Definition 1.4. Let C be a consequence operator on S. The set A is *closed* in (S, C) if $C(A) = A$; and A is *open* in (S, C) if its complement relative to S, denoted by A', is closed.

2. TK-algebras

The definition of a TK-algebra puts in the context of algebraic structures the notions of consequence operator.

Definition 2.1. A *TK-algebra* is a sextuple $\mathscr{A} = (A, 0, 1, \vee, \sim, \bullet)$ such that $(A, 0, 1, \vee, \sim)$ is a Boolean algebra and $\bullet$ is a new operator, the *Tarski operator*, such that:

(i) $a \vee \bullet a = \bullet a$;
(ii) $\bullet a \vee \bullet(a \vee b) = \bullet(a \vee b)$;
(iii) $\bullet(\bullet a) = \bullet a$.

Since we are working with a Boolean algebra, the item (i) of the above definition asserts that, for every $a \in A$, $a \leq \bullet a$ and we can define in a TK-algebra:

$$a \rightarrowtail b =_{\text{df}} \sim a \vee b.$$

Proposition 2.2. *In any TK-algebra the following conditions are valid:*

(i) $\sim \bullet a \leq \sim a \leq \bullet \sim a$;

(ii) $a \leq b \Rightarrow \bullet a \leq \bullet b$.

(iii) $\bullet(a \wedge b) \leq \bullet a \wedge \bullet b$;

(iv) $\bullet a \vee \bullet b \leq \bullet(a \vee b)$.

(v) $\bullet(\bullet a \wedge \bullet b) = \bullet a \wedge \bullet b$.

Naturally we can define a dual operation of $\bullet$ in any TK-algebra:

$$\circ a =_{\text{df}} \sim \bullet \sim a.$$

Proposition 2.3. *In any TK-algebra, the following conditions hold:*

(i) $\circ a \leq a$;

(ii) $\circ(a \wedge b) \leq \circ a$;

(iii) $\circ a \leq \circ \circ a$;

(iv) $a \leq b \Rightarrow \circ a \leq \circ b$.

An element $a \in \mathscr{A}$ is *closed* when $\bullet a = a$ and $a \in \mathscr{A}$ is *open* when $\circ a = a$.

Proposition 2.4.

(i) *If a is open, then $a \leq b \Leftrightarrow a \leq \circ b$;*

(ii) *If b is closed, then $a \leq b \Leftrightarrow \bullet a \leq b$.*

3. TK Logic

The propositional logic **TK** is the logical system associated to the TK-algebras. **TK** is determined over a propositional language $L(\neg, \vee, \rightarrow, \blacklozenge, p_1, p_2, p_3, \ldots)$ as follows:

Axiom Schemas:

(CPC) φ, if φ is a tautology;

(TK_1) $\varphi \rightarrow \blacklozenge\varphi$;

(TK_2) $\blacklozenge\blacklozenge\varphi \rightarrow \blacklozenge\varphi$.

Inference Rules:

(MP) $\varphi \rightarrow \psi,\ \varphi \ /\ \psi$;

($RM^{\blacklozenge}$) $\varphi \rightarrow \psi \ /\ \blacklozenge\varphi \rightarrow \blacklozenge\psi$.

As usual, we write $\vdash_{\mathbf{S}} \varphi$ to indicate that φ is a theorem of some axiomatic system **S**, and we drop the subscript if there is no danger of confusion.

Definition 3.1. Let $\Gamma \cup \{\varphi\}$ a set of formulas of some system **S**. We say that Γ *deduces* φ, what is denoted by $\Gamma \vdash_{\mathbf{S}} \varphi$, if there is a finite subset $\psi_1, \ldots, \psi_n$ of Γ such that $\vdash (\psi_1 \wedge \ldots \wedge \psi_n) \rightarrow \varphi$.

Notice that with the notion of syntactic consequence here presented the Deduction Theorem holds; the inference rules are understood as rules of proof.

Proposition 3.2. $\vdash \blacklozenge\varphi \rightarrow \blacklozenge(\varphi \vee \psi)$.

Proposition 3.3. $\varphi \vdash \blacklozenge\varphi$.

As in the case of a TK-algebra, we can define the dual operator of $\blacklozenge$ in the following way:

$$\blacksquare\varphi =_{\text{df}} \neg\blacklozenge\neg\varphi.$$

Proposition 3.4. $\varphi \to \psi \vdash \blacksquare\varphi \to \blacksquare\psi$.

Corollary 3.5. $\varphi \leftrightarrow \psi \vdash \blacksquare\varphi \leftrightarrow \blacksquare\psi$.

Proposition 3.6. $\vdash \blacksquare\varphi \to \varphi$.

Proposition 3.7. $\vdash \blacksquare\varphi \to \blacksquare\blacksquare\varphi$.

Proposition 3.8. $\vdash \blacksquare(\varphi \land \psi) \to \blacksquare\varphi$.

Corollary 3.9. $\vdash \blacksquare(\varphi \land \psi) \to (\blacksquare\varphi \land \blacksquare\psi)$.

We could, alternatively, consider the operator $\blacksquare$ as primitive and substitute the axioms TK_1 and TK_2 by the following ones:

(TK_1^*) $\blacksquare\varphi \to \varphi$,
(TK_2^*) $\blacksquare\varphi \to \blacksquare\blacksquare\varphi$,

and the rule $RM^{\blacklozenge}$ by the rule $RM^{\blacksquare}$:

($RM^{\blacksquare}$) $\varphi \to \psi \,/\, \blacksquare\varphi \to \blacksquare\psi$.

Feitosa, Grácio and Nascimento (2007) showed the adequacy of **TK** relative to TK-algebras.

4. A neighbourhood semantics for TK

In this section we introduce a new semantic for **TK** and prove, in later section, its adequacy.

We can show that **TK** is deductively equivalent to the classical modal system **EMT4** when considering the operators $\blacksquare$ and $\blacklozenge$ to be identical to the necessity and possibility operators $\Box$ and $\Diamond$. Taking $\Box$ as primitive, $\Diamond$ can be defined in the usual way:

(Df$\Diamond$) $\Diamond\varphi =_{\text{df}} \neg\Box\neg\varphi$.

EMT4 can be axiomatized by adding to the classical propositional calculus the following axiom schemes and rule of inference:

(M) $\Box(\varphi \land \psi) \to (\Box\varphi \land \Box\psi)$;

(T) $\Box\varphi \to \varphi$;
(4) $\Box\varphi \to \Box\Box\varphi$;
(RE) $\varphi \leftrightarrow \psi \,/\, \Box\varphi \leftrightarrow \Box\psi$.

Proposition 4.1. *Every theorem of **EMT4** is a theorem of **TK**.*

Proof. It follows directly from the definition of $\blacksquare$, TK_1^*, TK_2^*, and Corollaries 3.5 and 3.9. □

Proposition 4.2. *Every theorem of **TK** is a theorem of **EMT4**.*

Proof. We only need to show that **EMT4** provides $\mathrm{RM}^{\blacksquare}$.

1.	$\varphi \to \psi$	hypothesis
2.	$\varphi \to (\varphi \wedge \psi)$	CPC in 1
3.	$(\varphi \wedge \psi) \to \varphi$	CPC
4.	$\varphi \leftrightarrow (\varphi \wedge \psi)$	CPC in 2 and 3
5.	$\Box\varphi \leftrightarrow \Box(\varphi \wedge \psi)$	RE in 4
6.	$\Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$	M
7.	$\Box\varphi \to (\Box\varphi \wedge \Box\psi)$	CPC in 5 and 6
8.	$(\Box\varphi \wedge \Box\psi) \to \Box\psi$	CPC
9.	$\Box\varphi \to \Box\psi$	CPC in 7 and 8.

□

Definition 4.3. A *frame* for **TK** is a structure $\mathfrak{F} = \langle U, S\rangle$ such that U is a nonempty set of possible worlds and S is a function that associates to each $x \in U$ a set of subsets of U (that is, $S(x) \subseteq \mathscr{P}(U)$) that satisfies the following conditions:

(m) $X \cap Y \in S(x) \Rightarrow X \in S(x)$ and $Y \in S(x)$;

(t) $X \in S(x) \Rightarrow x \in X$;

(4) $X \in S(x) \Rightarrow \{y \in U : X \in S(y)\} \in S(x)$.

Definition 4.4. A *valuation* V on U is a function from the set of atomic formulas to $\mathscr{P}(U)$.

Definition 4.5. Let $\mathfrak{F} = \langle U, S\rangle$ be a frame and V a valuation in U. A *model* for **TK** is a pair $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ or, equivalently, a triple $\mathfrak{M} = \langle U, S, V\rangle$.

Definition 4.6. Let $\mathfrak{M} = \langle U, S, V\rangle$ be a model and x an element of U. A formula φ is *true* in the world x, what is denoted by $(\mathfrak{M}, x) \vDash \varphi$, when:

$(\mathfrak{M}, x) \vDash p_i \Leftrightarrow x \in V(p_i)$, if p_i is a propositional variable;

$(\mathfrak{M}, x) \vDash \neg\varphi \Leftrightarrow (\mathfrak{M}, x) \nvDash \varphi$;

$(\mathfrak{M}, x) \vDash \varphi \rightarrow \psi \Leftrightarrow (\mathfrak{M}, x) \nvDash \varphi$ or $(\mathfrak{M}, x) \vDash \psi$;

$(\mathfrak{M}, x) \vDash \Box\varphi \Leftrightarrow \|\varphi\|^{\mathfrak{M}} \in S(x)$, with $\|\varphi\|^{\mathfrak{M}} = \{x \in U : (\mathfrak{M}, x) \vDash \varphi\}$.

Definition 4.7. The set $\|\varphi\|^{\mathfrak{M}}$ from the above definition is called the *truth set* of φ in $\mathfrak{M}$.

When there is no risk of confusion, we will drop the superscript and write simply $\|\varphi\|$.

Definition 4.8. A formula φ is *valid in a model* $\mathfrak{M} = \langle U, S, V \rangle$ when it is true in every $x \in U$, and it is *valid* if it is true in any model $\mathfrak{M}$. We denote that a formula φ is valid in a model $\mathfrak{M}$ by $\mathfrak{M} \vDash \varphi$, and that φ is valid by $\vDash \varphi$.

If Γ is a set of formulas and $\mathfrak{M} = \langle U, S, V \rangle$ a model, then we write $\mathfrak{M} \vDash \Gamma$ if and only if $\mathfrak{M} \vDash \varphi$, for each $\varphi \in \Gamma$. For every $x \in U$, we say that $(\mathfrak{M}, x) \vDash \Gamma$ if and only if $(\mathfrak{M}, x) \vDash \varphi$, for each $\varphi \in \Gamma$.

Definition 4.9. Let $\Gamma \cup \{\varphi\}$ be a set of formulas. We say that Γ *implies* φ, or that φ is a *local semantic consequence* of Γ, what is denoted by $\Gamma \vDash \varphi$, when, for every model $\mathfrak{M} = \langle U, S, V \rangle$ and every $x \in U$, we have: if $(\mathfrak{M}, x) \vDash \Gamma$ then $(\mathfrak{M}, x) \vDash \varphi$.

5. Soundness

Since we have shown that **TK** and **EMT4** are the same logic, we will work, in what follows, with the **EMT4** axiomatization.

Lemma 5.1. *Let $\mathfrak{M} = \langle U, S, V \rangle$ be a TK-model, and φ and ψ any formulas. Then:*

(i) $\|\neg\varphi\| = -\|\varphi\|$;

(ii) $\|\varphi \wedge \psi\| = \|\varphi\| \cap \|\psi\|$;

(iii) $\|\varphi \vee \psi\| = \|\varphi\| \cup \|\psi\|$;

(iv) $\|\varphi \rightarrow \psi\| = -\|\varphi\| \cup \|\psi\|$;

(v) $\|\varphi \leftrightarrow \psi\| = (-\|\varphi\| \cup \|\psi\|) \cap (-\|\psi\| \cup \|\varphi\|)$;

(vi) $\|\Box\varphi\| = \{x \in U : \|\varphi\| \in S(x)\}$.

Proof. Items (i) to (v) are straightforward; we only show (vi). Now, for every $x \in U$, $x \in \{x \in U : \|\varphi\| \in S(x)\}$ iff $\|\varphi\| \in S(x)$ iff $x \vDash \Box\varphi$ iff $x \in \|\Box\varphi\|$. It follows that $\{x \in U : \|\varphi\| \in S(x)\} = \|\Box\varphi\|$. $\Box$

Theorem 5.2. *If $\vdash \varphi$, then $\vDash \varphi$.*

Proof. By induction on theorems. Let $\mathfrak{M} = \langle U, S, V \rangle$ be a TK-model.

(A) Let φ be an axiom. If it is a tautology, the proof is straightforward owing to the fact that every tautology is true in every state of a model, and thus in the model. So suppose φ is one of the modal axioms.

For M: Let x be an element of U such that $x \vDash \Box(\varphi \wedge \psi)$. It follows that $\|\varphi \wedge \psi\| \in S(x)$ and, since $\|\varphi \wedge \psi\| = \|\varphi\| \cap \|\psi\|$ by the preceding lemma, then $\|\varphi\| \cap \|\psi\| \in S(x)$. Given that (m) holds in $\mathfrak{M}$, it follows that $S(x)$ contains $\|\varphi\|$ and $\|\psi\|$. But then $x \vDash \Box\varphi$ and $x \vDash \Box\psi$, from what it follows that M holds.

For T: Let x be an element of U such that $x \vDash \Box\varphi$. By definition, $S(x)$ contains $\|\varphi\|$ and thus, because (t) holds, $x \in \|\varphi\|$. But if x belongs to the truth set of φ, we have that $x \vDash \varphi$, and it follows that T is valid.

For 4: Let x be an element of U such that $x \vDash \Box\varphi$. By definition, $S(x)$ contains $\|\varphi\|$ and thus, because (4) holds, $\{y \in U : \|\varphi\| \in S(y)\} \in S(x)$. By Lemma 5.1 (vi), $\{y \in U : \|\varphi\| \in S(y)\} = \|\Box\varphi\|$. Thus, $\|\Box\varphi\| \in S(x)$, so $x \vDash \Box\Box\varphi$, and it follows that 4 is valid.

(B) If φ was obtained by MP, the proof is immediate, since *modus ponens* is validity-preserving. So let us consider RE, and suppose $\vdash \varphi \leftrightarrow \psi$. Then (inductive hypothesis) $\varphi \leftrightarrow \psi$ is valid. So φ and ψ are equivalent, hence $\|\varphi\| = \|\psi\|$. It follows that, for every $x \in U$, $\|\varphi\| \in S(x)$ iff $\|\psi\| \in S(x)$. Thus $x \vDash \Box\varphi$ iff $x \vDash \Box\psi$, from what it follows that $x \vDash \Box\varphi \leftrightarrow \Box\psi$. Hence RE preserves validity. □

Corollary 5.3. *If* $\Gamma \vdash \varphi$*, then* $\Gamma \vDash \varphi$*.*

Proof. Suppose $\Gamma \vdash \varphi$, and let $\mathfrak{M}$ be some model, and x a world in $\mathfrak{M}$, such that $(\mathfrak{M}, x) \vDash \Gamma$. Since $\Gamma \vdash \varphi$, by definition there is a finite subset $\psi_1, \ldots, \psi_n$ of Γ such that $\vdash (\psi_1 \wedge \ldots \wedge \psi_n) \to \varphi$. By the preceding theorem, $(\psi_1 \wedge \ldots \wedge \psi_n) \to \varphi$ is valid and so true at x. Since $(\mathfrak{M}, x) \vDash \Gamma$, and every $\psi_i \in \Gamma$, it follows that $(\mathfrak{M}, x) \vDash \varphi$ and, thus, that $\Gamma \vDash \varphi$. □

6. Completeness

Definition 6.1. A set of formulas Δ is *maximal consistent* if Δ is consistent, and no proper extension of it is consistent.

Theorem 6.2 (Lindenbaum)**.** *Every consistent set of formulas* Γ *can be extended to a maximally consistent set* Δ*.*

Proof. The proof is standard; see, for instance, Fitting and Mendelsohn 1998, p. 76. □

Completeness will be proved using canonical models. Let $\mathfrak{S}$ be the set of all TK-maximal consistent sets of formulas (TK-MCS).

Definition 6.3. The *proof set* of φ is the set $|\varphi| = \{\Gamma \in \mathfrak{S} : \varphi \in \Gamma\}$.

Lemma 6.4. *Let φ and ψ any formulas. Then:*

(i) $|\neg\varphi| = -|\varphi|$;

(ii) $|\varphi \wedge \psi| = |\varphi| \cap |\psi|$;

(iii) $|\varphi \vee \psi| = |\varphi| \cup |\psi|$;

(iv) $|\varphi \to \psi| = -|\varphi| \cup |\psi|$;

(v) $|\varphi \leftrightarrow \psi| = (-|\varphi| \cup |\psi|) \cap (-|\psi| \cup |\varphi|)$;

(vi) $|\varphi| \subseteq |\psi| \Leftrightarrow\ \vdash \varphi \to \psi$;

(vii) $|\varphi| = |\psi| \Leftrightarrow\ \vdash \varphi \leftrightarrow \psi$.

Definition 6.5. $\mathfrak{M} = \langle U, S, V\rangle$ is a *canonical model* for **TK** if it satisfies the following conditions:

(i) $U = \mathfrak{S}$;

(ii) $|\varphi| \in S(\Gamma) \Leftrightarrow \Box\varphi \in \Gamma$, for all $\Gamma \in U$;

(iii) $V(p_i) = |p_i|$, for every propositional variable p_i.

Lemma 6.6. *Let $\mathfrak{M}$ be a canonical model. Then, for every formula φ and every $\Gamma \in U$: $(\mathfrak{M}, \Gamma) \vDash \varphi \Leftrightarrow \varphi \in \Gamma$.*

Proof. The proof proceeds by induction on formulas. Let Γ be some element of U:

(a) $\varphi = p_i$, for some $i \in \mathbb{N}$. By definition, $(\mathfrak{M}, \Gamma) \vDash p_i$ iff $\Gamma \in V(p_i)$ iff $\Gamma \in |p_i|$. By construction of $|p_i|$, Γ is a set in $|p_i|$ iff $p_i \in \Gamma$.

(b) $\varphi = \neg\psi$. $(\mathfrak{M}, \Gamma) \vDash \varphi$ iff $(\mathfrak{M}, \Gamma) \nvDash \psi$ iff (by induction hypothesis) $\psi \notin \Gamma$ iff $\varphi \in \Gamma$.

(c) $\varphi = \psi \to \sigma$. By definition, $\Gamma \vDash \psi \to \sigma$ iff $(\mathfrak{M}, \Gamma) \nvDash \psi$ or $(\mathfrak{M}, x) \vDash \sigma$. By the inductive hypothesis, $(\mathfrak{M}, \Gamma) \nvDash \psi$ iff $\psi \notin \Gamma$, and $(\mathfrak{M}, x) \vDash \sigma$ iff $\sigma \in \Gamma$. Now $\psi \notin \Gamma$ or $\sigma \in \Gamma$ iff $\psi \to \sigma \in \Gamma$. Thus $(\mathfrak{M}, \Gamma) \vDash \psi \to \sigma$ iff $\psi \to \sigma \in \Gamma$.

(d) $\varphi = \Box\psi$. By definition, $(\mathfrak{M}, \Gamma) \vDash \Box\psi$ iff $\|\psi\| \in S(\Gamma)$. By the inductive hypothesis, for every $\Delta \in U$ we have that $(\mathfrak{M}, \Delta) \vDash \psi$ iff $\psi \in \Delta$, that is, $\|\psi\| = |\psi|$. So $\|\psi\| \in S(\Gamma)$ iff $|\psi| \in S(\Gamma)$. Now, by definition of S, $|\psi| \in S(\Gamma)$ iff $\Box\psi \in \Gamma$. Hence, $(\mathfrak{M}, \Gamma) \vDash \Box\psi$ iff $\Box\psi \in \Gamma$. □

It is well known with regard to monotonic logics that the smallest canonical model — that is, the model where, for every Γ, $S(\Gamma)$ contains only proof sets — does not satisfy condition (m). Fortunately we can show that there are other canonical models in which this condition holds.

Definition 6.7. The *supplementation* of $\mathfrak{M}$ is the model $\mathfrak{M}^+ = \langle U, S^+, V\rangle$ such that for every $\Gamma \in U$ and every $X \subseteq U$:

$$X \in S^+(\Gamma) \Leftrightarrow Y \subseteq X \text{ for some } Y \in S(\Gamma).$$

It follows from this definition that $S^+(\Gamma) = \{X \subseteq U : |\varphi| \subseteq X \text{ for some } \Box\varphi \in \Gamma\}$ and, obviously, for every $\Gamma \in U$, $S(\Gamma) \subseteq S^+(\Gamma)$.

We need to prove that $\mathfrak{M}^+$ is a canonical model for **TK**.

Lemma 6.8. *$\mathfrak{M}^+ = \langle U, S^+, V\rangle$ is a canonical model for **TK**.*

Proof. It is enough to show that condition (ii) of the definition is satisfied, that is, for every φ and every $\Gamma \in U$:

$$|\varphi| \in S^+(\Gamma) \Leftrightarrow \Box\varphi \in \Gamma.$$

($\Leftarrow$) If $\Box\varphi \in \Gamma$, then $|\varphi| \in S(\Gamma)$ and since $\mathfrak{M}$ is a canonical for **TK** so $|\varphi| \in S^+(\Gamma)$.

($\Rightarrow$) Let $|\varphi| \in S^+(\Gamma)$. Thus, for some $Y \subseteq |\varphi|$, $Y \in S(\Gamma)$. Since $\mathfrak{M}$ is the smallest canonical model, this means that $Y = |\psi|$, for some ψ. It follows that $|\psi| \subseteq |\varphi|$, and $\Box\psi \in \Gamma$. By Lemma 6.4 we have that $\vdash \psi \to \varphi$, and from RM that $\vdash \Box\psi \to \Box\varphi$. Hence, $\Box\varphi \in \Gamma$.

So $\mathfrak{M}^+$ is a canonical model for **TK**. □

Lemma 6.9. *Let $\mathfrak{M}$ be the smallest canonical model for **TK**, and $\mathfrak{M}^+$ its supplementation. Then the conditions (m), (t) and (4) hold in $\mathfrak{M}^+$.*

Proof. (a) For (m): We have from the previous lemma that $\mathfrak{M}^+$ is a canonical model for **TK**. Let Γ be an element of U, and X and Y be subsets of U such that $X \cap Y \in S^+(\Gamma)$. By construction, there must be some Z such that $Z \subseteq X \cap Y$ and $Z \in S(\Gamma)$. It follows that $Z \subseteq X$ and $Z \subseteq Y$ and, again by construction, $X \in S^+(\Gamma)$ and $Y \in S^+(\Gamma)$.

(b) For (t): Let Γ be an element of U, and X a subset of U such that $X \in S^+(\Gamma)$. Suppose that X is a proof set, that is, $X = |\varphi|$, for some φ. By definition, we have that $\Box\varphi \in \Gamma$. Since Γ is an MCS, and **TK** has T, it follows that $\varphi \in \Gamma$. But then $\Gamma \in |\varphi|$, and (t) holds. Suppose now that X is not a proof set. By construction, for some φ, $|\varphi| \in S(\Gamma)$, $|\varphi| \subseteq X$. But if $|\varphi| \in S(\Gamma)$, $\Box\varphi \in \Gamma$, $\Gamma \vdash \varphi$, $\Gamma \in |\varphi|$, $\Gamma \in X$, and again (t) holds.

(c) For (4): Let Γ be an element of U, and X a subset of U such that $X \in S^+(\Gamma)$. We have to show that $\{\Delta \in U : X \in S^+(\Delta)\} \in S^+(\Gamma)$.

Suppose first that X is a proof set, that is, $X = |\varphi|$, for some φ. By definition, we have that $\Box\varphi \in \Gamma$. Since Γ is an MCS, and **TK** has 4, it follows that $\Gamma \vdash \Box\Box\varphi$ and that $\Box\Box\varphi \in \Gamma$. By canonicity of the model, $|\Box\varphi| \in S(\Gamma)$ and, by construction of $\mathfrak{M}^+$, $|\Box\varphi| \in S^+(\Gamma)$. We must now show that $|\Box\varphi| = \{\Delta \in U : \Box\varphi \in S^+(\Delta)\}$. Now, $|\Box\varphi| = \{\Delta \in U : \Box\varphi \in \Delta\}$. Since the model is canonical, $\Box\varphi \in \Delta$ iff $|\varphi| \in S(\Delta)$ iff (by construction) $|\varphi| \in S^+(\Delta)$. So $|\Box\varphi| = \{\Delta \in U : |\varphi| \in S^+(\Delta)\}$. It follows that $\{\Delta \in U : |\varphi| \in S^+(\Delta)\} \in S^+(\Gamma)$, and (4) holds.

Suppose now that X is not a proof set. By construction of $\mathfrak{M}^+$, however, there is some formula φ such that $|\varphi| \subseteq X$ and $|\varphi| \in S(\Gamma)$. As above, we can show that $|\Box\varphi| \in S^+(\Gamma)$, and that $\{\Delta \in U : |\varphi| \in S^+(\Delta)\} \in S^+(\Gamma)$. Now, for every $\Delta \in U$, if $\varphi \in S^+(\Delta)$, then $X \in S^+(\Delta)$. So $\{\Delta \in U : |\varphi| \in S^+(\Delta)\} \subseteq \{\Delta \in U : X \in S^+(\Delta)\}$. But $\{\Delta \in U : |\varphi| \in S^+(\Delta)\} = |\Box\varphi|$, so it is a proof set. By construction of $\mathfrak{M}^+$, $\{\Delta \in U : X \in S^+(\Delta)\} \in S^+(\Gamma)$, and (4) holds. □

Theorem 6.10 (Completeness). *If* $\Gamma \vDash \varphi$, *then* $\Gamma \vdash \varphi$.

Proof. Suppose that $\Gamma \nvdash \varphi$. Thus, $\Gamma \nvdash \neg\neg\varphi$, and it follows that $\Gamma \cup \{\neg\varphi\}$ is consistent. By Lindenbaum's Theorem, there exists an **TK**-MCS Δ such that $\Gamma \cup \{\neg\varphi\} \subseteq \Delta$, that is, $\neg\varphi \in \Delta$, and $\varphi \notin \Delta$. Let now $\mathfrak{M}$ be the smallest canonical model for **TK**, and $\mathfrak{M}^+$ its supplementation. By the preceding lemma, conditions (m), (t) and (4) hold in $\mathfrak{M}^+$, so it is a model for **TK**. Now Δ is a **TK**-MCS and $\Gamma \subseteq \Delta$, so Δ is a state in $\mathfrak{M}^+$ such that, by Lemma 6.6, $(\mathfrak{M}^+, \Delta) \vDash \Gamma$ and $(\mathfrak{M}^+, \Delta) \nvDash \varphi$; hence $\Gamma \nvDash \varphi$. □

7. Decidability

We show the decidability of **TK** using filtrations.

Definition 7.1. Let Γ be a set of formulas closed under subformulas, and $\mathfrak{M}$ a model. For any states x and y in $\mathfrak{M}$, we say that

$$x \equiv_\Gamma y \text{ iff for every } \varphi \in \Gamma, (\mathfrak{M}, x) \vDash \varphi \text{ iff } (\mathfrak{M}, y) \vDash \varphi.$$

In other words, if $x \equiv_\Gamma y$ then x and y are equivalent with regard to the formulas in Γ. We can easily show that $\equiv_\Gamma$ is indeed an equivalence relation, partitioning the set U of states into disjoint equivalence classes.

Definition 7.2. Let Γ be a set of formulas closed under subformulas and $\mathfrak{M} = \langle U, S, V\rangle$ a model. Then:

(i) if $x \in U$, $[x]_\Gamma = \{y \in U : x \equiv_\Gamma y\}$;

(ii) if $X \subseteq U$, $[X]_\Gamma = \{[x] : x \in X\}$.

Again, we will usually drop the subscript and write $\equiv$, $[x]$, and $[X]$.

Definition 7.3. Let $\mathfrak{M} = \langle U, S, V\rangle$ be some model, and Γ a set of formulas closed under subformulas. A *filtration of* $\mathfrak{M}$ *through* Γ is any model $\mathfrak{M}^* = \langle U^*, S^*, V^*\rangle$ such that:

(a) $W^* = [W]$;

(b) for every $x \in U$, and every formula $\Box\varphi \in \Gamma$,

  (i) $\|\varphi\|^{\mathfrak{M}} \in S(x) \Leftrightarrow [\|\varphi\|^{\mathfrak{M}}] \in S^*([x])$;

  (ii) $\|\Box\varphi\|^{\mathfrak{M}} \in S(x) \Leftrightarrow [\|\Box\varphi\|^{\mathfrak{M}}] \in S^*([x])$;

(c) $V^*(p_i) = [V(p_i)]$, for every $i \in \mathbb{N}$ such that $p_i \in \Gamma$.

Notice that the above definition leaves room for different filtrations of a model.

Definition 7.4. A filtration is the *finest filtration* if, for every $x \in U$, $S^*([x])$ contains only the sets $[\|\varphi\|^{\mathfrak{M}}]$ and $[\|\Box\varphi\|^{\mathfrak{M}}]$ such that, respectively, $\|\varphi\|^{\mathfrak{M}} \in S(x)$ and $\|\Box\varphi\|^{\mathfrak{M}} \in S(x)$, for every $\Box\varphi \in \Gamma$.

This is what is needed for a model to be a filtration. Coarser filtrations will allow $S^*([x])$ to contain other sets besides the minimum required.

Theorem 7.5. *Let $\mathfrak{M}^* = \langle U^*, S^*, V^*\rangle$ be a Γ-filtration of a model $\mathfrak{M} = \langle U, S, V\rangle$. Then, for every $\varphi \in \Gamma$ and every $x \in U$:*

$$(\mathfrak{M}, x) \vDash \varphi \Leftrightarrow (\mathfrak{M}^*, [x]) \vDash \varphi,$$

that is, $[\|\varphi\|^{\mathfrak{M}}] = \|\varphi\|^{\mathfrak{M}^*}$.

Proof. The proof is by induction on formulas. Let x be any state in $\mathfrak{M}$, and p some variable in Γ:

$(\mathfrak{M}, x) \vDash p$	iff	$x \in V(p)$	(by 4.6)
	iff	$[x] \in [V(p)]$	(by 7.2(ii))
	iff	$[x] \in V^*(p)$	(by 7.3(c))
	iff	$(\mathfrak{M}^*, [x]) \vDash p$	(by 4.6).

The boolean cases are straightforward; we show only the case in which $\varphi = \Box\psi$, for some ψ. Let again x be any state in $\mathfrak{M}$:

$(\mathfrak{M}, x) \vDash \Box\psi$	iff	$\|\psi\|^{\mathfrak{M}} \in S(x)$	(by 4.6)
	iff	$[\|\psi\|^{\mathfrak{M}}] \in S^*([x])$	(by 7.3(b))
	iff	$\|\psi\|^{\mathfrak{M}^*} \in S^*([x])$	(inductive hypothesis)
	iff	$(\mathfrak{M}^*, [x]) \vDash \Box\psi$	(by 4.6).

□

Corollary 7.6. *Let $\mathfrak{M}^*$ be a Γ-filtration of a model $\mathfrak{M}$. Then $\mathfrak{M}$ and $\mathfrak{M}^*$ are equivalent modulo Γ, that is, for every $\varphi \in \Gamma$, $\mathfrak{M} \vDash \varphi$ iff $\mathfrak{M}^* \vDash \varphi$.*

A well-known result says that if a logic is axiomatizable and has the finite model property—that is, every nontheorem fails in some finite model—, then it is decidable. **TK** is axiomatizable, as we have shown before. All we need to show is that **TK** is determined by the class of finite models satisfying conditions (m), (t) and (4).

Lemma 7.7. *Let $\mathfrak{M}$ be a model, Γ a set of formulas closed under subformulas, and $\mathfrak{M}^*$ a Γ-filtration of $\mathfrak{M}$. Then, for every $\varphi \in \Gamma$:*

(i) $\|\Box\varphi\|^{\mathfrak{M}} = \{x \in U : \|\varphi\|^{\mathfrak{M}} \in S(x)\}$;

(ii) $[\|\Box\varphi\|^{\mathfrak{M}}] = \{[x] \in U^* : [\|\varphi\|^{\mathfrak{M}}] \in S^*([x])\}$.

Proof.

(i) $x \in \|\Box\varphi\|^{\mathfrak{M}}$ iff $x \vDash \Box\varphi$; [Def. truth-set]
iff $\|\varphi\|^{\mathfrak{M}} \in S(x)$; [Def. 4.6]
iff $x \in \{x \in U : \|\varphi\|^{\mathfrak{M}} \in S(x)\}$.

(ii) $[x] \in [\|\Box\varphi\|^{\mathfrak{M}}]$ iff $[x] \in \|\Box\varphi\|^{\mathfrak{M}^*}$;
iff $[x] \vDash \Box\varphi$;
iff $\|\varphi\|^{\mathfrak{M}^*} \in S^*([x])$;
iff $[\|\varphi\|^{\mathfrak{M}}] \in S^*([x])$;
iff $[x] \in \{[x] \in U^* : [\|\varphi\|^{\mathfrak{M}}] \in S^*([x])\}$. □

Theorem 7.8. *Let $\mathfrak{M}$ be a model satifying conditions (m), (t) and (4), and, for some set of formulas Γ closed under subformulas, let $\mathfrak{M}^*$ be the finest Γ-filtration of $\mathfrak{M}$. Then its supplementation, $\mathfrak{M}^{*+}$, is a Γ-filtration of $\mathfrak{M}$ and satisfies (m), (t) and (4).*

Proof. We first show that $\mathfrak{M}^{*+}$ is a Γ-filtration of $\mathfrak{M}$. That is, we must show that, for every $x \in U$ and every $\Box\varphi \in \Gamma$,

(i) $\|\varphi\|^{\mathfrak{M}} \in S(x) \Leftrightarrow [\|\varphi\|^{\mathfrak{M}}] \in S^{*+}([x])$, and

(ii) $\|\Box\varphi\|^{\mathfrak{M}} \in S(x) \Leftrightarrow [\|\Box\varphi\|^{\mathfrak{M}}] \in S^{*+}([x])$.

(i). Suppose first that $\|\varphi\|^{\mathfrak{M}} \in S(x)$. Then $[\|\varphi\|^{\mathfrak{M}}] \in S^*([x])$, since $\mathfrak{M}^*$ is a Γ-filtration of $\mathfrak{M}$. Thus $[\|\varphi\|^{\mathfrak{M}}] \in S^{*+}([x])$ by supplementation.

Suppose now that $[\|\varphi\|^{\mathfrak{M}}] \in S^{*+}([x])$. By the definition of a supplementation, there must be some ψ (eventually $\psi = \varphi$) such that $\Box\psi \in \Gamma$, $[\|\psi\|^{\mathfrak{M}}] \in S^*([x])$ and $[\|\psi\|^{\mathfrak{M}}] \subseteq [\|\varphi\|^{\mathfrak{M}}]$. It follows that $\|\psi\|^{\mathfrak{M}} \in S(x)$, since $\mathfrak{M}^*$ is the finest Γ-filtration of $\mathfrak{M}$. Now $\mathfrak{M}$ satisfies condition (m), which means that every superset of $\|\psi\|^{\mathfrak{M}}$ belongs to $S(x)$. We just need to show that $\|\psi\|^{\mathfrak{M}} \subseteq \|\varphi\|^{\mathfrak{M}}$.

Now, $[\|\psi\|^{\mathfrak{M}}] \subseteq [\|\varphi\|^{\mathfrak{M}}]$ means that $\{[x] : x \in \|\psi\|^{\mathfrak{M}}\} \subseteq \{[x] : x \in \|\varphi\|^{\mathfrak{M}}\}$. Let $y \in \|\psi\|^{\mathfrak{M}}$. Hence $[y] \in \{[x] : x \in \|\psi\|^{\mathfrak{M}}\}$, and $[y] \in \{[x] : x \in \|\varphi\|^{\mathfrak{M}}\}$. It follows that $y \in \|\varphi\|^{\mathfrak{M}}$. So $\|\psi\|^{\mathfrak{M}} \subseteq \|\varphi\|^{\mathfrak{M}}$, and from this we have that $\|\varphi\|^{\mathfrak{M}} \in S(x)$.

(ii). If $\|\Box\varphi\|^{\mathfrak{M}} \in S(x)$, then $[\|\Box\varphi\|^{\mathfrak{M}}] \in S^*([x])$ because $\mathfrak{M}^*$ is a Γ-filtration of $\mathfrak{M}$, and, by the definition of a supplementation, $[\|\Box\varphi\|^{\mathfrak{M}}] \in S^{*+}([x])$.

Suppose now that $[\|\Box\varphi\|^{\mathfrak{M}}] \in S^{*+}([x])$. By the definition of a supplementation, there must be some ψ such that $\Box\psi \in \Gamma$ and

(a) $[\|\psi\|^{\mathfrak{M}}] \in S^*([x])$ and $[\|\psi\|^{\mathfrak{M}}] \subseteq [\|\Box\varphi\|^{\mathfrak{M}}]$, or

(b) $[\|\Box\psi\|^{\mathfrak{M}}] \in S^*([x])$ and $[\|\Box\psi\|^{\mathfrak{M}}] \subseteq [\|\Box\varphi\|^{\mathfrak{M}}]$.

(a). Since $\mathfrak{M}^*$ is a Γ-filtration of $\mathfrak{M}$, we have $\|\psi\|^{\mathfrak{M}} \in S([x])$. Suppose now $y \in \|\psi\|^{\mathfrak{M}}$. Hence $[y] \in \{[x] : x \in \|\psi\|^{\mathfrak{M}}\}$, and $[y] \in \{[x] : x \in \|\Box\varphi\|^{\mathfrak{M}}\}$ by the second part of (a) above. But then $y \in \|\Box\varphi\|^{\mathfrak{M}}$. Thus $\|\psi\|^{\mathfrak{M}} \subseteq \|\Box\varphi\|^{\mathfrak{M}}$. Since $\mathfrak{M}$ is supplemented, $\|\Box\varphi\|^{\mathfrak{M}} \in S([x])$.

(b). Since $\mathfrak{M}^*$ is a Γ-filtration of $\mathfrak{M}$, we have $\|\Box\psi\|^{\mathfrak{M}} \in S([x])$. Suppose now $y \in \|\Box\psi\|^{\mathfrak{M}}$. Hence $[y] \in \{[x] : x \in \|\Box\psi\|^{\mathfrak{M}}\}$, and $[y] \in \{[x] : x \in \|\Box\varphi\|^{\mathfrak{M}}\}$ by the second part of (b) above. But then $y \in \|\Box\varphi\|^{\mathfrak{M}}$. Thus $\|\Box\psi\|^{\mathfrak{M}} \subseteq \|\Box\varphi\|^{\mathfrak{M}}$. Since $\mathfrak{M}$ is supplemented, $\|\Box\varphi\|^{\mathfrak{M}} \in S([x])$.

Thus, $\mathfrak{M}^{*+}$ is a Γ-filtration of $\mathfrak{M}$. Does it satisfy conditions (m), (t) and (4)? Since $\mathfrak{M}^{*+}$ is a supplementation, it automatically satisfies (m).

Let us consider (t). We need to show, for any $[x] \in U^*$ and any subset X of U^*, that $[x] \in X$, if $X \in S^{*+}([x])$. Suppose first that $X \in S^*([x])$. Since $\mathfrak{M}^*$ is the finest filtration, $X = [\|\varphi\|^{\mathfrak{M}}]$, for some formula φ such that $\Box\varphi \in \Gamma$ and $\|\varphi\|^{\mathfrak{M}} \in S(x)$. Now condition (t) holds in $\mathfrak{M}$, so $x \in \|\varphi\|^{\mathfrak{M}}$, and $[x] \in [\|\varphi\|^{\mathfrak{M}}] = X$.

If now $X \notin S^*([x])$ then, by the definition of a supplementation, X is a superset of some $[\|\varphi\|^{\mathfrak{M}}]$ such that $\Box\varphi \in \Gamma$ and $\|\varphi\|^{\mathfrak{M}} \in S(x)$. As above, it follows that $[x] \in [\|\varphi\|^{\mathfrak{M}}]$, and, thus, that $[x] \in X$. Thus (t) holds in $\mathfrak{M}^{*+}$.

With regard to condition (4), we show first that $\mathfrak{M}^*$ has (4). Let $[x]$ be an element of U^*, and X a subset of U^* such that $X \in S^*([x])$. We have to show that

$$\{[y] \in U^* : X \in S^*([y])\} \in S^*([x]).$$

Since $\mathfrak{M}^*$ is the finest filtration, there must be some formula $\Box\varphi \in \Gamma$ such that

(i) $X = [\|\varphi\|^{\mathfrak{M}}]$ and $\|\varphi\|^{\mathfrak{M}} \in S(x)$, or
(ii) $X = [\|\Box\varphi\|^{\mathfrak{M}}]$ and $\|\Box\varphi\|^{\mathfrak{M}} \in S(x)$.

Suppose it is (i). Since $[\|\Box\varphi\|^{\mathfrak{M}}] = \{[y] \in U^* : [\|\varphi\|^{\mathfrak{M}}] \in S^*([y])\}$ (Lemma 7.7), what we have to show is that $[\|\Box\varphi\|^{\mathfrak{M}}] \in S^*([x])$.

Now condition (4) holds in $\mathfrak{M}$, so $\{y \in U : \|\varphi\|^{\mathfrak{M}} \in S(y)\} \in S(x)$. But (Lemma 7.7) $\|\Box\varphi\|^{\mathfrak{M}} = \{y \in U : \|\varphi\|^{\mathfrak{M}} \in S(y)\}$; thus $\|\Box\varphi\|^{\mathfrak{M}} \in S(x)$. By condition (b.ii) of the definiton of filtration, we immediately have $[\|\Box\varphi\|^{\mathfrak{M}}] \in S^*([x])$.

Now consider (ii). We now have to show that

$$\{[y] \in U^* : [\|\Box\varphi\|^{\mathfrak{M}}] \in S^*([y])\} \in S^*([x]).$$

By Lemma 7.7, $[\|\Box\Box\varphi\|^{\mathfrak{M}}] = \{[y] \in U^* : [\|\Box\varphi\|^{\mathfrak{M}}] \in S^*([y])\}$. So what we have to show is that $[\|\Box\Box\varphi\|^{\mathfrak{M}}] \in S^*([x])$.

Since condition (4) holds in $\mathfrak{M}$, $\{y \in U : \|\Box\varphi\|^{\mathfrak{M}} \in S(y)\} \in S(x)$. By Lemma 7.7, $\|\Box\Box\varphi\|^{\mathfrak{M}} = \{y \in U : \|\Box\varphi\|^{\mathfrak{M}} \in S(y)\}$; thus $\|\Box\Box\varphi\|^{\mathfrak{M}} \in S(x)$.

Now $\vdash \Box\varphi \leftrightarrow \Box\Box\varphi$ (it follows from T and 4), so $\vDash \Box\varphi \leftrightarrow \Box\Box\varphi$. But then, for every $x \in U$, $x \vDash \Box\varphi$ iff $x \vDash \Box\Box\varphi$. Thus $\|\Box\varphi\|^{\mathfrak{M}} = \|\Box\Box\varphi\|^{\mathfrak{M}}$.

From this it follows that $\|\Box\varphi\|^{\mathfrak{M}} \in S(x)$, and also that $[\|\Box\varphi\|^{\mathfrak{M}}] = [\|\Box\Box\varphi\|^{\mathfrak{M}}]$. By conditon b.ii of the definition of filtration, we immediately have $[\|\Box\varphi\|^{\mathfrak{M}}] \in S^*([x])$, and $[\|\Box\Box\varphi\|^{\mathfrak{M}}] \in S^*([x])$.

It follows from (i) and (ii) above that $\mathfrak{M}^*$ has (4). We now show that $\mathfrak{M}^{*+}$ has (4).

Let $[x]$ be an element of U^*, and X a subset of U^* such that $X \in S^{*+}([x])$. We thus have to show that $\{[y] \in U^* : X \in S^{*+}([y])\} \in S^{*+}([x])$.

Since $\mathfrak{M}^{*+}$ is the supplementation of $\mathfrak{M}^*$, there must be some formula φ such that $\Box\varphi \in \Gamma$, $[\|\varphi\|^{\mathfrak{M}}] \in S^*(x)$ and $[\|\varphi\|^{\mathfrak{M}}] \subseteq X$ (eventually $[\|\varphi\|^{\mathfrak{M}}] = X$, of course). But as we have shown above, $\mathfrak{M}^*$ has (4), so $\{[y] \in U^* : [\|\varphi\|^{\mathfrak{M}}] \in S^*([y])\} \in S^*([x])$. Now, for every $[y] \in U^*$, if $[\|\varphi\|^{\mathfrak{M}}] \in S^*([y])$ then $X \in S^{*+}([y])$ by supplementation. So we have:

$$\{[y] \in U^* : [\|\varphi\|^{\mathfrak{M}}] \in S^*([y])\} \subseteq \{[y] \in U^* : X \in S^{*+}([y])\}.$$

Finally, since $\{[y] \in U^* : [\|\varphi\|^{\mathfrak{M}}] \in S^*([y])\}$ belongs to $S^*([x])$, $\{[y] \in U^* : X \in S^{*+}([y])\} \in S^{*+}([x])$ by supplementation and we are done. □

Theorem 7.9. ***TK** is determined by the class of finite models satisfying conditions (m), (t) and (4).*

Proof. If φ is a theorem of **TK**, then it is valid in the class of all models satisfying conditions (m), (t) and (4); in particular, in the class of finite models satisfying these conditions.

For the other direction, suppose φ is not a theorem of **TK**. Then φ fails in some world x of some model $\mathfrak{M}$ for **TK**. Let Γ be any finite set of formulas closed under subformulas such that $\varphi \in \Gamma$, and let $\mathfrak{M}^{*+}$ be the supplementation of a finest Γ-filtration $\mathfrak{M}^*$ of $\mathfrak{M}$. By Theorem 7.8, $\mathfrak{M}^{*+}$, is a Γ-filtration of $\mathfrak{M}$ and satisfies (m),

(t) and (4). Since Γ is a finite set, $\mathfrak{M}^{*+}$ is a finite model. By Theorem 7.5, x and $[x]$ agree on every formula in Γ; thus $(\mathfrak{M}^{*+}, [x]) \nvDash \varphi$. □

In view of the preceding result, every nontheorem of **TK** fails in some finite model, from what it follows that **TK** has the finite model property. Since it is also axiomatizable, **TK** is decidable.

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Resumo. A lógica **TK** foi introduzida como uma lógica proposicional estendendo o cálculo proposicional clássico com um novo operador unário que interpreta algumas concepções do operador de consequência de Tarski. TK-álgebras foram introduzidas como modelos para **TK**. Assim, usando ferramentas algébricas, foi demonstrada a adequação (correção e completude) de **TK** relativamente às TK-álgebras. Este trabalho apresenta uma semântica de vizinhanças para **TK**, lógica que resulta ser dedutivamente equivalente à lógica modal não normal **EMT4**.

Palavras-chave: Operador de consequência; álgebra TK; lógica TK; semântica de vizinhanças.

Notes

[1] Of course, one can also define deduction *locally*: we say that $\Gamma \vdash \varphi$ if there is a finite subset $\psi_1, \ldots, \psi_n$ of Γ such that $\vdash (\psi_1 \wedge \ldots \wedge \psi_n) \rightarrow \varphi$. With this definition, obviously, the Deduction Theorem holds.