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Strömbergsson, Andreas; Södergren, Anders

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# ON THE GENERALIZED CIRCLE PROBLEM FOR A RANDOM LATTICE IN LARGE DIMENSION

ANDREAS STRÖMBERGSSON AND ANDERS SÖDERGREN

ABSTRACT. In this note we study the error term  $R_{n,L}(x)$  in the generalized circle problem for a ball of volume  $x$  and a random lattice  $L$  of large dimension  $n$ . Our main result is the following functional central limit theorem: Fix an arbitrary function  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$  satisfying  $\lim_{n \rightarrow \infty} f(n) = \infty$  and  $f(n) = O_\varepsilon(e^{\varepsilon n})$  for every  $\varepsilon > 0$ . Then, the random function

$$t \mapsto \frac{1}{\sqrt{2f(n)}} R_{n,L}(tf(n))$$

on the interval  $[0, 1]$  converges in distribution to one-dimensional Brownian motion as  $n \rightarrow \infty$ . The proof goes via convergence of moments, and for the computations we develop a new version of Rogers' mean value formula from [19]. For the individual  $k$ th moment of the variable  $(2f(n))^{-1/2} R_{n,L}(f(n))$  we prove convergence to the corresponding Gaussian moment more generally for functions  $f$  satisfying  $f(n) = O(e^{c_n})$  for any fixed  $c \in (0, c_k)$ , where  $c_k$  is a constant depending on  $k$  whose optimal value we determine.

## 1. INTRODUCTION

Gauss' circle problem is a classical problem in number theory asking for the number of integer lattice points inside a Euclidean circle of radius  $t$  centered at the origin. Gauss observed that this quantity equals the area  $A(t) = \pi t^2$  enclosed by the circle up to an error term of size at most  $O(t)$ . Hardy conjectured [7] that the error term can be improved to  $O_\varepsilon(t^{1/2+\varepsilon})$ ; a bound which is known to be essentially optimal. Despite efforts of many mathematicians, Hardy's conjecture remains open and the best known bound is  $O_\varepsilon(t^{131/208+\varepsilon})$  due to Huxley [12].

In this paper we will be interested in the circle problem generalized to dimension  $n$  and a general  $n$ -dimensional lattice  $L$  of covolume 1. We denote the space of all such lattices by  $X_n$  and recall that  $X_n$  can be identified with the homogeneous space  $\mathrm{SL}(n, \mathbb{Z}) \backslash \mathrm{SL}(n, \mathbb{R})$  via the correspondence  $\mathbb{Z}^n g \leftrightarrow \mathrm{SL}(n, \mathbb{Z})g$ . As a consequence of this identification,  $X_n$  inherits a right  $\mathrm{SL}(n, \mathbb{R})$ -invariant probability measure  $\mu_n$  originating from a Haar measure on  $\mathrm{SL}(n, \mathbb{R})$ .

Given  $n \geq 2$ , a lattice  $L \in X_n$  and a real number  $x \geq 0$ , we let  $N_{n,L}(x)$  denote the number of non-zero lattice points of  $L$  in the closed ball of volume  $x$  centered at

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the origin in  $\mathbb{R}^n$ , i.e. we let

$$(1.1) \quad N_{n,L}(x) := \#\left\{\mathbf{m} \in L \setminus \{\mathbf{0}\} : |\mathbf{m}| \leq \left(\frac{x}{V_n}\right)^{1/n}\right\},$$

where  $V_n$  denotes the volume of the unit ball in  $\mathbb{R}^n$ . We also define, for  $x \geq 0$ , the function

$$R_{n,L}(x) := N_{n,L}(x) - x,$$

and formulate, for a given  $L \in X_n$ , the generalized circle problem as the problem of giving the best possible upper bound on  $R_{n,L}(x)$  as  $x \rightarrow \infty$ .

In a series of papers Bentkus and Götze [1, 2] and Götze [6] proved strong explicit bounds on  $R_{n,L}(x)$  for an arbitrary given lattice  $L \in X_n$ . In particular, Götze proved in [6] that  $|R_{n,L}(x)| = O(x^{1-2/n})$  holds for every  $L \in X_n$  when  $n \geq 5$ . This result is best possible for all rational lattices  $L \in X_n$ , while for irrational lattices Götze proved the stronger bound  $R_{n,L}(x) = o(x^{1-2/n})$  as  $x \rightarrow \infty$ .<sup>1</sup> However, it turns out that for most lattices (in the measure sense) one can do much better. In fact, Schmidt [25] proved that for any  $n \geq 2$  and  $\mu_n$ -almost every  $L \in X_n$  we have  $R_{n,L}(x) = O_\varepsilon(x^{1/2}(\log x)^{5/2+\varepsilon})$ . This upper bound should be compared to Landau's result  $R_{n,L}(x) = \Omega(x^{1/2-1/(2n)})$  (cf. [16]). Hence, for large  $n$ , Schmidt's bound is close to optimal. In this vein it should also be noted that, for  $n \geq 3$ ,<sup>2</sup>

$$(1.2) \quad \text{Var}(R_{n,L}(x)) = \mathbb{E}(R_{n,L}(x)^2) := \int_{X_n} R_{n,L}(x)^2 d\mu_n(L) \asymp x$$

(cf., e.g., [25, p. 518] or [30, Lemma 3.1]).

In a closely related direction, the second author has recently studied the distribution of lengths of lattice vectors in a  $\mu_n$ -random lattice of large dimension  $n$ . Given a lattice  $L \in X_n$ , we order its non-zero vectors by increasing lengths as  $\pm\mathbf{v}_1, \pm\mathbf{v}_2, \pm\mathbf{v}_3, \dots$  and define, for each  $j \geq 1$ ,

$$\mathcal{V}_j(L) := V_n |\mathbf{v}_j|^n.$$

We stress that the first few vectors in this list, that is, the shortest non-zero vectors in  $L$ , encode important geometric information attached to  $L$ . Indeed, these short vectors play a crucial role in, for example, the lattice sphere packing problem where the quantity  $2^{-n} \sup_{L \in X_n} \mathcal{V}_1(L)$  determines the maximal density of a lattice sphere packing in  $\mathbb{R}^n$ . In [27], by calculating the limits as  $n \rightarrow \infty$  of mixed moments of the form

$$(1.3) \quad \mathbb{E}\left(\prod_{j=1}^k N_{n,L}(x_j)\right)$$

for any fixed  $k \geq 1$  and  $0 < x_1 \leq x_2 \leq \dots \leq x_k$ , the following theorem is established:

**Theorem 1.1** (Södergren). *The sequence  $\{\mathcal{V}_j(\cdot)\}_{j=1}^\infty$  converges in distribution, as  $n \rightarrow \infty$ , to the sequence  $\{T_j\}_{j=1}^\infty$ , where  $0 < T_1 < T_2 < T_3 < \dots$  denote the points of a Poisson process  $\mathcal{P} = \{\mathcal{N}(x), x \geq 0\}$  on  $\mathbb{R}^+$  with constant intensity  $\frac{1}{2}$ .*

<sup>1</sup>Here we call a lattice  $L$  irrational if the Gram matrix for every  $\mathbb{Z}$ -basis of  $L$  is not proportional to a matrix with integer entries only.

<sup>2</sup>Throughout the paper,  $\mathbb{E}$  will denote the expected value with respect to the measure  $\mu_n$  on  $X_n$ .

The convergence in Theorem 1.1 is equivalent to the convergence of all finite dimensional distributions, i.e. to the fact that the truncated sequence  $\{\mathcal{V}_j(\cdot)\}_{j=1}^N$  converges in distribution to the corresponding truncated sequence  $\{T_j\}_{j=1}^N$ , for every fixed  $N \in \mathbb{Z}^+$ . This raises the question whether it is possible to allow for more flexibility in Theorem 1.1 in the sense of allowing  $N = N(n)$  to grow as a function of the dimension  $n$ ? It seems reasonable to expect that for moderately growing  $N$  the Poisson characteristic of the limit sequence should remain intact, but that the Poissonian behavior will eventually disappear as  $N$  is allowed to grow faster. A first result in this direction, indicating a Poissonian behavior for  $N \leq cn$  where  $c > 0$  is a small absolute constant, is proved in a recent paper by Kim [15] using a sieving argument (cf. also [14] where the range  $N \leq (n/2)^{1/2-\varepsilon}$  was obtained). The following result extends this range, giving an indication of Poissonian behavior for any  $N$  growing *sub-exponentially* with respect to  $n$ .

**Theorem 1.2.** *Let  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$  be any function satisfying  $\lim_{n \rightarrow \infty} f(n) = \infty$  and  $f(n) = O_\varepsilon(e^{\varepsilon n})$  for every  $\varepsilon > 0$ . Let  $\mathcal{N}(x)$  be a Poisson distributed random variable with expectation  $x/2$ . Then*

$$(1.4) \quad \text{Prob}_{\mu_n}(N_{n,L}(x) \leq 2N) - \text{Prob}(\mathcal{N}(x) \leq N) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

*uniformly with respect to all  $N, x \geq 0$  satisfying  $\min(x, N) \leq f(n)$ .*

We will deduce Theorem 1.2 from Theorem 1.1 combined with the following result, a central limit theorem for the normalized error term in the generalized circle problem for a random lattice  $L$ .

**Theorem 1.3.** *Let  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$  be any function satisfying  $\lim_{n \rightarrow \infty} f(n) = \infty$  and  $f(n) = O_\varepsilon(e^{\varepsilon n})$  for every  $\varepsilon > 0$ . Let  $Z_n^{(B)}$  be the random variable*

$$(1.5) \quad Z_n^{(B)} := \frac{1}{\sqrt{2f(n)}} R_{n,L}(f(n)),$$

*with  $L$  picked at random in  $(X_n, \mu_n)$ . Then*

$$Z_n^{(B)} \xrightarrow{d} N(0, 1) \quad \text{as } n \rightarrow \infty.$$

The “B” in  $Z_n^{(B)}$  stands for “ball”. In fact, the same convergence holds even if we consider completely general subsets of  $\mathbb{R}^n$  symmetric about the origin.

**Theorem 1.3’.** *Let  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$  be as in Theorem 1.3, and for each  $n$  let  $S_n$  be a Borel measurable subset of  $\mathbb{R}^n$  satisfying  $\text{vol}(S_n) = f(n)$  and  $S_n = -S_n$ . Set*

$$(1.6) \quad Z_n := \frac{\#(L \cap S_n \setminus \{\mathbf{0}\}) - f(n)}{\sqrt{2f(n)}},$$

*with  $L$  picked at random in  $(X_n, \mu_n)$ . Then*

$$Z_n \xrightarrow{d} N(0, 1) \quad \text{as } n \rightarrow \infty.$$

*Remark 1.4.* Theorem 1.3’ remains true if we consider  $L \cap S_n$  instead of  $L \cap S_n \setminus \{\mathbf{0}\}$  in (1.6), since  $f(n) \rightarrow \infty$ . However, the fact that we remove  $\mathbf{0}$  in (1.1) is essential for Theorem 1.2 to hold, namely in the case when  $x$  stays bounded as  $n \rightarrow \infty$ .

In Theorem 4.2 below we generalize Theorem 1.3’ to the case of  $r$  pairwise disjoint subsets of  $\mathbb{R}^n$ , for any fixed  $r \in \mathbb{Z}^+$ , showing that the joint distribution of the normalized counting variables approaches  $r$  independent normal distributions. In

the special case of balls centered at the origin, we also have the following *functional* central limit theorem, generalizing Theorem 1.3:

**Theorem 1.5.** *Let  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$  be any function satisfying  $\lim_{n \rightarrow \infty} f(n) = \infty$  and  $f(n) = O_\varepsilon(e^{\varepsilon n})$  for every  $\varepsilon > 0$ . Consider, for  $n \in \mathbb{Z}^+$  and  $L$  picked at random in  $(X_n, \mu_n)$ , the random function*

$$t \mapsto \tilde{Z}_n^{(B)}(t) := \frac{1}{\sqrt{2f(n)}} R_{n,L}(tf(n))$$

on the interval  $[0, 1]$ . Let  $P_n$  denote the corresponding probability measure on the space  $\mathcal{D}[0, 1]$  of cadlag functions on  $[0, 1]$ . Then  $\tilde{Z}_n^{(B)}(t)$  converges in distribution to one-dimensional Brownian motion, or equivalently,  $P_n$  converges weakly to Wiener measure, as  $n \rightarrow \infty$ .

*Remark 1.6.* In a different direction, for  $n = 2$  and fixed  $L \in X_2$ , a result by Bleher [4] (cf. also Heath-Brown [8] for the case  $L = \mathbb{Z}^2$ ) implies the existence of a limit distribution of  $t^{-1/4} R_{2,L}(t)$  for  $t$  random in  $(0, T)$ , as  $T \rightarrow \infty$ . This limit distribution is non-Gaussian; however the corresponding limit for the number of lattice points in thin annuli is Gaussian in certain situations; cf. [11] and [32]. We are not aware of any similar results in dimension  $n \geq 3$ ; cf. however Peter [17].

It is an interesting question whether the above limit results could be extended to more rapidly growing functions  $f(n)$ . Our proof of Theorem 1.3' goes by establishing convergence of all moments of  $Z_n$ . For any *fixed* moment  $\mathbb{E}(Z_n^k)$ , the method actually yields the desired limit result even for  $f(n)$  of modest exponential growth; however for more rapidly growing  $f(n)$  the moment diverges (if  $k \geq 3$ ). In the case of balls, we have determined the precise growth rate where this transition occurs: Set

$$(1.7) \quad c_2 = +\infty \quad \text{and} \quad c_k = \frac{k-1}{k-2} \log(k-1) - \log k \quad (k \geq 3).$$

Note that  $\{c_k\}_{k \geq 3}$  is a positive, strictly decreasing sequence; its first values are  $c_3 = 0.28768\dots$ ,  $c_4 = 0.26162\dots$ ,  $c_5 = 0.23895\dots$ , and  $c_k \sim k^{-1} \log k$  as  $k \rightarrow \infty$ .

**Theorem 1.7.** *Let  $k \geq 2$  and  $0 < c < c_k$ , and let  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$  be any function satisfying  $\lim_{n \rightarrow \infty} f(n) = \infty$  and  $f(n) = O(e^{cn})$ . For each  $n$  let  $S_n$  be a Borel measurable subset of  $\mathbb{R}^n$  satisfying  $\text{vol}(S_n) = f(n)$  and  $S_n = -S_n$ , and define  $Z_n$  as in Theorem 1.3'. Then*

$$(1.8) \quad \lim_{n \rightarrow \infty} \mathbb{E}(Z_n^k) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ (k-1)!! & \text{if } k \text{ is even.} \end{cases}$$

On the other hand, if  $k \geq 3$  and  $c > c_k$ , and if  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$  is any function satisfying  $f(n) \gg e^{cn}$  as  $n \rightarrow \infty$ , then  $\mathbb{E}((Z_n^{(B)})^k) \rightarrow +\infty$  as  $n \rightarrow \infty$ .

The last result shows in particular that the assumption of sub-exponential growth imposed in Theorem 1.3' is best possible for our method of proof via convergence of moments; however the question remains open whether a limit distribution of  $Z_n$  exists (Gaussian or not) also for more rapidly growing  $f(n)$ . Theorem 1.7 shows in this regard that any limit distribution of any subsequence of  $Z_n$  is necessarily *close* to the Gaussian  $N(0, 1)$  distribution, in the weak topology, so long as  $f(n) = O(e^{cn})$  with  $c > 0$  sufficiently small.

*Remark 1.8.* In the setting of balls as in Theorem 1.3, taking  $f(n) = e^{cn}$  corresponds to counting all lattice vectors  $\mathbf{m} \in L \setminus \{\mathbf{0}\}$  of length  $|\mathbf{m}| \lesssim e^c \sqrt{\frac{n}{2\pi e}}$ . In this connection we note that for any fixed  $N$ , with probability tending to one as  $n \rightarrow \infty$ , the first  $N$  shortest non-zero vectors  $\pm \mathbf{v}_1, \dots, \pm \mathbf{v}_N$  of a random lattice  $L \in X_n$  all have length  $\sqrt{\frac{n}{2\pi e}}(1 + O(\frac{\log n}{n}))$ . This follows e.g. from Theorem 1.1, using the asymptotics  $V_n \sim (\frac{2\pi e}{n})^{n/2}(\pi n)^{-1/2}$ .

*Remark 1.9.* Kelmer has recently obtained a bound on the mean square of  $R_{n,L}(x)$  for fixed  $n \geq 2$  and large  $x$ ; cf. [13, Thm. 2]. This bound supports the conjecture that for almost every  $L \in X_n$ ,  $R_{n,L}(x) \ll x^{\frac{1}{2} - \frac{1}{2n} + \varepsilon}$  holds as  $x \rightarrow \infty$  (cf. also [5], [9]). Kelmer's bound implies that if  $f(n)$  grows *sufficiently rapidly* (the growth condition could be made explicit with further work), then  $Z_n^{(B)}$  converges in distribution to 0 as  $n \rightarrow \infty$ , showing that the normalization in (1.5) is inappropriate in this regime.

Our original motivation for studying the limit distribution of  $Z_n^{(B)}$  comes from questions concerning the Epstein zeta function of a random lattice  $L \in X_n$  as  $n \rightarrow \infty$ ; cf. [23, 29, 30]. Recall that for  $\operatorname{Re} s > \frac{n}{2}$  and  $L \in X_n$  the Epstein zeta function is defined by the absolutely convergent series

$$E_n(L, s) := \sum_{\mathbf{m} \in L \setminus \{\mathbf{0}\}} |\mathbf{m}|^{-2s}.$$

The function  $E_n(L, s)$  can be meromorphically continued to  $\mathbb{C}$  and satisfies a functional equation of "Riemann type" relating  $E_n(L, s)$  and  $E_n(L^*, \frac{n}{2} - s)$ . (Here  $L^*$  denotes the dual lattice of  $L$ .) An outstanding question from [30] is whether  $E_n(L, s)$  for  $s$  on or near the central point  $s = \frac{n}{4}$ , possesses, after appropriate normalization, a limit distribution as  $n \rightarrow \infty$ ? This question turns out to be closely related to the behavior of the random function  $\tilde{Z}_n^{(B)}(t)$ , and we expect that Theorem 1.5 in this paper in combination with the methods of [30] will make it possible to give an answer in the case of  $s = cn$  with  $c > \frac{1}{4}$  tending to  $\frac{1}{4}$  sufficiently slowly as a function of  $n$ . However in order to handle  $c = \frac{1}{4}$  or  $c$  arbitrarily near  $\frac{1}{4}$ , it appears that we need a precise understanding of the limit of  $\tilde{Z}_n^{(B)}(t)$  when the volume  $f(n)$  is allowed to grow as rapidly as  $e^{\frac{1}{2}(1-\log 2)n}$ , and furthermore we need to understand this distribution jointly with the corresponding distribution for the dual lattice of  $L$ . We hope to return to these matters in future work.

The organization of the paper is as follows. As mentioned, Theorem 1.3' is proved by computing the moments of  $Z_n$ ; similarly Theorem 1.5 is proved by computing the mixed moments of the finite dimensional distributions of  $\tilde{Z}_n^{(B)}(t)$ . The standard tool for calculating moments of this form is Rogers' mean value formula [19]; however, the assumption  $\lim_{n \rightarrow \infty} f(n) = \infty$  causes divergence problems. To get around these, we develop, in Section 2, a new version of Rogers' formula suitable for calculating moments of functions that can be represented in the form

$$\sum_{\mathbf{m} \in L \setminus \{\mathbf{0}\}} \rho(V_n |\mathbf{m}|^n) - \int_0^\infty \rho(x) dx$$

for suitable test functions  $\rho$ ; in particular the formula can be applied to calculate moments of  $\tilde{Z}_n^{(B)}(t)$ . The proof of this formula is combinatorial in nature. Using the formula, in Section 3 we prove Theorems 1.3' and 1.2, and in Section 4 we prove Theorem 1.5. Finally in Section 5 we prove Theorem 1.7, by a careful analysis of the

sizes of the various non-leading order terms appearing in the moment computation used to prove Theorem 1.3’.

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## 2. A NEW VERSION OF ROGERS’ MEAN VALUE FORMULA

To begin, we describe Rogers’ original formula. Let  $1 \leq k \leq n - 1$  and let  $\rho : (\mathbb{R}^n)^k \rightarrow \mathbb{R}_{\geq 0}$  be a non-negative Borel measurable function. In [19] Rogers proved the following remarkable identity:

$$(2.1) \quad \int_{X_n} \sum_{\mathbf{m}_1, \dots, \mathbf{m}_k \in L \setminus \{\mathbf{0}\}} \rho(\mathbf{m}_1, \dots, \mathbf{m}_k) d\mu_n(L) \\ = \sum_{q=1}^{\infty} \sum_D \left( \frac{e_1}{q} \cdots \frac{e_m}{q} \right)^n \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \rho \left( \sum_{i=1}^m \frac{d_{i1}}{q} \mathbf{x}_i, \dots, \sum_{i=1}^m \frac{d_{ik}}{q} \mathbf{x}_i \right) d\mathbf{x}_1 \cdots d\mathbf{x}_m.$$

Here the inner sum is over all integer matrices  $D = (d_{ij})$  having size  $m \times k$  for some  $1 \leq m \leq k$ , satisfying the following properties: No column of  $D$  vanishes identically; the entries of  $D$  have greatest common divisor equal to 1; and finally there exists a division  $(\nu; \mu) = (\nu_1, \dots, \nu_m; \mu_1, \dots, \mu_{k-m})$  of the numbers  $1, \dots, k$  into two sequences  $\nu_1, \dots, \nu_m$  and  $\mu_1, \dots, \mu_{k-m}$ , satisfying

$$(2.2) \quad \begin{aligned} 1 &= \nu_1 < \nu_2 < \dots < \nu_m \leq k, \\ 1 &< \mu_1 < \mu_2 < \dots < \mu_{k-m} \leq k, \\ \nu_i &\neq \mu_j, \text{ if } 1 \leq i \leq m, 1 \leq j \leq k - m, \end{aligned}$$

such that

$$(2.3) \quad \begin{aligned} d_{i\nu_j} &= q\delta_{ij}, \quad i = 1, \dots, m, \quad j = 1, \dots, m, \\ d_{i\mu_j} &= 0, \quad \text{if } \mu_j < \nu_i, \quad i = 1, \dots, m, \quad j = 1, \dots, k - m. \end{aligned}$$

We call these matrices  $\langle k, q \rangle$ -admissible.<sup>3</sup> Finally  $e_i = (\varepsilon_i, q)$ ,  $i = 1, \dots, m$ , where  $\varepsilon_1, \dots, \varepsilon_m$  are the elementary divisors of the matrix  $D$ . We stress that the right-hand side of (2.1) is a *positive* infinite linear combination of integrals of  $\rho$  over certain linear subspaces of  $(\mathbb{R}^n)^k$ .

*Remark 2.1.* The formula (2.1) should be understood as an equality in  $\mathbb{R}_{\geq 0} \cup \{+\infty\}$ ; if either side of (2.1) is divergent, then so is the other side. By Schmidt, [24, Thm. 2], if  $\rho$  is bounded and of compact support then both sides of (2.1) are finite. Hence, under this restriction we may remove the assumption that  $\rho$  is non-negative, i.e. the formula (2.1) is in fact valid for any real-valued Borel measurable function  $\rho$  on  $(\mathbb{R}^n)^k$  which is bounded and of compact support, with both sides of (2.1) being nicely absolutely convergent.

*Remark 2.2.* It follows from the conditions on the matrices  $D$  and [10, Thm. 14.5.1] that we always have  $e_1 = 1$ , and hence  $\left(\frac{e_1}{q} \cdots \frac{e_m}{q}\right)^n \leq q^{-n}$ .

We now state our new version of Rogers’ mean value formula.

<sup>3</sup>Note that the only  $\langle k, 1 \rangle$ -admissible matrix with  $m = k$  is the  $k \times k$  identity matrix, and for  $q > 1$  there are no  $\langle k, q \rangle$ -admissible matrices with  $m = k$ .

**Theorem 2.3.** *Let  $n > k > 0$ , and let  $f_1, \dots, f_k$  be real-valued Borel measurable functions on  $\mathbb{R}^n$  which are bounded and of compact support. Define the functions  $F_1, \dots, F_k$  on  $X_n$  by*

$$(2.4) \quad F_j(L) := \sum_{\mathbf{m} \in L \setminus \{\mathbf{0}\}} f_j(\mathbf{m}) - \int_{\mathbb{R}^n} f_j(\mathbf{x}) d\mathbf{x}.$$

Then

$$\begin{aligned} \mathbb{E}\left(\prod_{j=1}^k F_j(L)\right) &= \sum_{q=1}^{\infty} \sum_D' \left(\frac{e_1}{q} \dots \frac{e_m}{q}\right)^n \\ &\quad \times \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} f_1\left(\sum_{i=1}^m \frac{d_{i1}}{q} \mathbf{x}_i\right) \dots f_k\left(\sum_{i=1}^m \frac{d_{ik}}{q} \mathbf{x}_i\right) d\mathbf{x}_1 \dots d\mathbf{x}_m, \end{aligned}$$

where  $'$  indicates that the inner sum is over all  $\langle k, q \rangle$ -admissible matrices  $D$  with the property that there are at least two non-zero entries in each row.

We note that in the simple case  $k = 1$ , Theorem 2.3 states that

$$\mathbb{E}(F_1(L)) = 0.$$

This is in fact an immediate consequence of Siegel's mean value formula; see [26].

*Proof.* Let  $K = \{1, \dots, k\}$ . Using (2.4) and (2.1), we get

$$\begin{aligned} &\mathbb{E}\left(\prod_{j=1}^k F_j(L)\right) \\ &= \sum_{A \subset K} (-1)^{\#(K \setminus A)} \left( \prod_{j \in K \setminus A} \int_{\mathbb{R}^n} f_j(\mathbf{x}) d\mathbf{x} \right) \mathbb{E}\left(\prod_{j \in A} \left(\sum_{\mathbf{m}_j \in L \setminus \{\mathbf{0}\}} f_j(\mathbf{m}_j)\right)\right) \\ (2.5) \quad &= \sum_{A \subset K} (-1)^{\#(K \setminus A)} \left( \prod_{j \in K \setminus A} \int_{\mathbb{R}^n} f_j(\mathbf{x}) d\mathbf{x} \right) \sum_{q=1}^{\infty} \sum_D \left(\frac{e_1}{q} \dots \frac{e_m}{q}\right)^n \\ &\quad \times \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \prod_{\ell=1}^a f_{j_\ell} \left(\sum_{i=1}^m \frac{d_{i\ell}}{q} \mathbf{x}_i\right) d\mathbf{x}_1 \dots d\mathbf{x}_m, \end{aligned}$$

where  $A$  runs through all subsets of  $K$ , we write  $a = \#A$  and  $A = \{j_1, \dots, j_a\}$  with  $j_1 < \dots < j_a$ , and the inner sum is taken over all  $\langle a, q \rangle$ -admissible matrices  $D$ . As usual  $m = m(D)$  denotes the number of rows of  $D$ . Note that all multiple sums and integrals appearing in (2.5) are absolutely convergent, because of our assumptions on  $f_1, \dots, f_k$ ; cf. Remark 2.1.

Given any  $A = \{j_1, \dots, j_a\}$ ,  $q$  and  $D$  appearing in the sum, we set  $m' := m + k - a$  and write  $K \setminus A = \{j'_1, \dots, j'_{k-a}\}$  with  $j'_1 < \dots < j'_{k-a}$ . We then let  $D' = D'(A, D) = (d'_{ij})$  be the  $m' \times k$  matrix which has  $d'_{i,j_\ell} = d_{i,\ell}$  for  $\langle i, \ell \rangle \in \{1, \dots, m\} \times \{1, \dots, a\}$ ,  $d'_{m+\ell, j'_\ell} = q$  for  $\ell = 1, \dots, k - a$ , and all other entries equal to zero. Note that the matrix  $D'$  is typically not  $\langle k, q \rangle$ -admissible. Let  $\varepsilon'_1, \dots, \varepsilon'_{m'}$  be the elementary divisors of  $D'$  and set  $e'_j = (\varepsilon'_j, q)$ . Then  $e'_1 \dots e'_{m'} = q^{\#(K \setminus A)} e_1 \dots e_m$  (cf., e.g., [19, Lemma 1]), and so  $\frac{e_1}{q} \dots \frac{e_m}{q} = \frac{e'_1}{q} \dots \frac{e'_{m'}}{q}$ . We may now rewrite each product of integrals in



the right-hand side of (2.5) in terms of the matrices  $D' = D'(A, D) = (d'_{ij})$ :

$$(2.6) \quad \mathbb{E} \left( \prod_{j=1}^k F_j(L) \right) = \sum_{A \subset K} (-1)^{\#(K \setminus A)} \sum_{q=1}^{\infty} \sum_D \left( \frac{e'_1}{q} \dots \frac{e'_{m'}}{q} \right)^n \\ \times \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \prod_{\ell=1}^k f_{\ell} \left( \sum_{i=1}^{m'} \frac{d'_{i\ell}}{q} \mathbf{x}_i \right) d\mathbf{x}_1 \dots d\mathbf{x}_{m'}.$$

Note that any matrix  $D' = D'(A, D)$  appearing in this sum can be brought, by a unique row permutation, into a  $\langle k, q \rangle$ -admissible matrix (this is easily seen by considering the admissibility conditions column by column, starting from the left). Conversely, given any  $\langle k, q \rangle$ -admissible matrix  $D'$ , let  $S(D')$  be the set of indices of those columns of  $D'$  which have the property that the column has a unique non-zero entry and this entry is also the only non-zero entry in its row. Then the matrix  $D'$  is attained as a row permutation of  $D'(A, D)$  for exactly  $2^{\#S(D')}$  pairs  $\langle A, D \rangle$  appearing in the above sum, namely exactly once for each  $B \subset S(D')$ . Hence

$$(2.7) \quad \mathbb{E} \left( \prod_{j=1}^k F_j(L) \right) = \sum_{q=1}^{\infty} \sum_{D'} \left( \sum_{B \subset S(D')} (-1)^{\#B} \right) \left( \frac{e'_1}{q} \dots \frac{e'_{m'}}{q} \right)^n \\ \times \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \prod_{\ell=1}^k f_{\ell} \left( \sum_{i=1}^{m'} \frac{d'_{i\ell}}{q} \mathbf{x}_i \right) d\mathbf{x}_1 \dots d\mathbf{x}_{m'},$$

where now the sum over  $D'$  is taken over all  $\langle k, q \rangle$ -admissible matrices. But here  $\sum_{B \subset S(D')} (-1)^{\#B}$  equals 1 if  $S(D') = \emptyset$  and equals 0 otherwise. Hence we obtain the formula stated in the theorem.  $\square$

*Remark 2.4.* Note the close connection between the formula in Theorem 2.3 and the formula in [30, Prop. 7.1].

*Remark 2.5.* Clearly the family of functions  $f_j$  admitted in Theorem 2.3 can be extended by approximation arguments. However the present family is more than sufficient for our purposes in this paper.

*Remark 2.6.* The formula in Theorem 2.3 is useful in the study of the Epstein zeta function  $E_n(L, s)$ . Recall from [30, Sect. 4] that, for  $L \in X_n$  and  $s \in \mathbb{C} \setminus \{0, \frac{n}{2}\}$ , we have

$$(2.8) \quad \pi^{-s} \Gamma(s) E_n(L, s) = H_n(L, s) + H_n(L^*, \frac{n}{2} - s),$$

where  $L^*$  is the dual lattice of  $L$ ,

$$H_n(L, s) := -\frac{1}{\frac{n}{2} - s} + \sum_{\mathbf{m} \in L \setminus \{0\}} G(s, \pi |\mathbf{m}|^2),$$

and

$$G(s, x) := \int_1^{\infty} t^{s-1} e^{-xt} dt, \quad \operatorname{Re} x > 0.$$

The connection between  $E_n(L, s)$  and the present discussion comes from the relation

$$H_n(L, s) = \int_0^{\infty} G \left( s, \pi (V_n^{-1} x)^{2/n} \right) dR_{n,L}(x), \quad 0 < s < \frac{n}{2}$$

(cf. [30, Eq. (4.7)]). It follows that Theorem 2.3 can be used to calculate (truncated) moments of  $H_n(L, s)$ . Furthermore, since  $H_n(L, s)$  dominates  $H_n(L^*, \frac{n}{2} - s)$  in the interval  $(\frac{1}{4} + \varepsilon)n < s < \frac{n}{2}$  ( $\varepsilon > 0$  fixed) for most lattices  $L \in X_n$  when  $n$  is large enough, we also find that the (truncated) moments of  $H_n(L, s)$  are of apparent interest in the study of  $E_n(L, s)$  in the limit as  $n \rightarrow \infty$ . We do not pursue this further here since we plan to give a detailed account of this topic elsewhere.

We close this section by giving a generalization of Theorem 2.3 which seems potentially useful, although it will not be used in the present paper.

**Theorem 2.7.** *Let  $k, \ell > 0$  and  $n > k\ell$ . Let  $g_j : (\mathbb{R}^n)^k \rightarrow \mathbb{R}$ ,  $1 \leq j \leq \ell$ , be Borel measurable functions which are bounded and of compact support. Consider the related functions  $G_j : X_n \rightarrow \mathbb{R}$  defined by*

$$G_j(L) := \sum_{\mathbf{m}_1, \dots, \mathbf{m}_k \in L \setminus \{\mathbf{0}\}} g_j(\mathbf{m}_1, \dots, \mathbf{m}_k) - \mathbb{E} \left( \sum_{\mathbf{m}_1, \dots, \mathbf{m}_k \in L \setminus \{\mathbf{0}\}} g_j(\mathbf{m}_1, \dots, \mathbf{m}_k) \right).$$

Then

$$\begin{aligned} \mathbb{E} \left( \prod_{j=1}^{\ell} G_j(L) \right) &= \sum_{q=1}^{\infty} \sum_D^* \left( \frac{e_1}{q} \dots \frac{e_m}{q} \right)^n \\ &\times \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \prod_{j=0}^{\ell-1} g_{j+1} \left( \sum_{i=1}^m \frac{d_{i,jk+1}}{q} \mathbf{x}_i, \sum_{i=1}^m \frac{d_{i,jk+2}}{q} \mathbf{x}_i, \dots, \sum_{i=1}^m \frac{d_{i,jk+k}}{q} \mathbf{x}_i \right) d\mathbf{x}_1 \dots d\mathbf{x}_m, \end{aligned}$$

where  $*$  indicates that the inner sum is over all  $\langle k\ell, q \rangle$ -admissible matrices  $D$  with the property that there do not exist any  $j \in \{0, \dots, \ell-1\}$  and  $1 \leq i_1 \leq i_2 \leq m$  such that the submatrix at rows  $i_1, i_1+1, \dots, i_2$  and columns  $jk+1, jk+2, \dots, jk+k$  of  $D$  is a multiple of a  $\langle k, q' \rangle$ -admissible matrix for some  $q' \mid q$ , and all the remaining entries of these rows and columns of  $D$  are zero.

*Outline of proof.* Mimicking the beginning of the proof of Theorem 2.3, in particular expanding  $\mathbb{E} \left( \prod_{j=1}^{\ell} G_j(L) \right)$  as much as possible using (2.1), we obtain the formula

$$\begin{aligned} \mathbb{E} \left( \prod_{j=1}^{\ell} G_j(L) \right) &= \sum_{A \subset \{1, \dots, \ell\}} (-1)^{\ell - \#A} \sum_{\{q_j\}} \sum_{\{D_j\}} \sum_{q=1}^{\infty} \sum_D \left( \frac{e'_1}{q'} \dots \frac{e'_m}{q'} \right)^n \\ (2.9) \quad &\times \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \prod_{j=0}^{\ell-1} g_{j+1} \left( \sum_{i=1}^{m'} \frac{d'_{i,jk+1}}{q'} \mathbf{x}_i, \dots, \sum_{i=1}^{m'} \frac{d'_{i,jk+k}}{q'} \mathbf{x}_i \right) d\mathbf{x}_1 \dots d\mathbf{x}_{m'}, \end{aligned}$$

where the notation is as follows. As before,  $a = \#A$  and  $A = \{j_1, \dots, j_a\}$  with  $j_1 < \dots < j_a$ . In the sums,  $\{q_j\}$  and  $\{D_j\}$  are short-hands for  $\{q_j\}_{j \in A^c}$  and  $\{D_j\}_{j \in A^c}$ , where  $A^c$  is the complement of  $A$  in  $\{1, \dots, \ell\}$ ; and  $\{q_j\}$  runs through all  $(\ell - a)$ -tuples of positive integers while  $\{D_j\}$  runs through all  $(\ell - a)$ -tuples of matrices such that  $D_j$  is  $\langle k, q_j \rangle$ -admissible for each  $j \in A^c$ . In the innermost sum,  $D$  runs through all  $\langle ak, q \rangle$ -admissible matrices. For any  $A$ ,  $\{q_j\}, \{D_j\}, q, D$  appearing in the multiple sum we let  $q'$  be the least common multiple of  $q$  and all the  $q_j$ 's, and set  $m' = m + \sum_{j \in A^c} m_j$ , where  $m$  is the number of rows of  $D$  and  $m_j$  is the number of rows of  $D_j$ . Writing also  $D_j = (d_{uv}^{(j)})$ ,  $D = (d_{uv})$  and  $\bar{m}_j := m + \sum_{j' < j} m_{j'}$ , we define  $D' = D'(A, \{q_j\}, \{D_j\}, q, D) = (d'_{ij})$  to be the  $m' \times k\ell$  matrix which has

$d'_{i,(ju-1)k+v} = \frac{q'}{q} d_{i,(u-1)k+v}$  for all  $i \in \{1, \dots, m\}$ ,  $u \in \{1, \dots, a\}$ ,  $v \in \{1, \dots, k\}$ , and  $d'_{\bar{m}_j+i,(j-1)k+v} = \frac{q'}{q_j} d_{iv}^{(j)}$  for all  $j \in A^c$ ,  $v \in \{1, \dots, k\}$ ,  $i \in \{1, \dots, m_j\}$ , and all other entries equal to zero. Finally  $e'_j = (\varepsilon'_j, q)$ , where  $\varepsilon'_1, \dots, \varepsilon'_{m'}$  are the elementary divisors of  $D'$ . This completes the description of the notation in (2.9).

One notes that each matrix  $D'$  which appears above can be brought, by a unique row permutation, into a  $\langle k\ell, q' \rangle$ -admissible matrix. The rest of the proof follows closely the proof of Theorem 2.3.  $\square$

### 3. PROOFS OF THEOREM 1.3' AND THEOREM 1.2

Our first goal is to prove Theorem 1.3' (and thus also Theorem 1.3). Let  $f, S_n$  and  $Z_n$  be as in the statement of the theorem. Thus  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$  is a function satisfying  $\lim_{n \rightarrow \infty} f(n) = \infty$  and  $f(n) = O_\varepsilon(e^{\varepsilon n})$  for every  $\varepsilon > 0$ ; for each  $n$ ,  $S_n$  is a Borel measurable subset of  $\mathbb{R}^n$  which has volume  $f(n)$  and which is symmetric about the origin (viz.,  $-S_n = S_n$ ), and finally

$$(3.1) \quad Z_n := \frac{\#(L \cap S_n \setminus \{\mathbf{0}\}) - f(n)}{\sqrt{2f(n)}},$$

with  $L$  picked at random in  $(X_n, \mu_n)$ . It follows from Siegel's formula [26] that for each  $n \geq 2$  we have  $\mathbb{E}(\#(L \cap S_n \setminus \{\mathbf{0}\})) = f(n)$  and thus  $\mathbb{E}(Z_n) = 0$ . Using Theorem 2.3, we now determine the limits as  $n \rightarrow \infty$  of the higher moments of  $Z_n$ .

**Proposition 3.1.** *For any fixed  $k \in \mathbb{Z}^+$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{E}(Z_n^k) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ (k-1)!! & \text{if } k \text{ is even.} \end{cases}$$

*Proof.* Let  $\chi_n$  be the characteristic function of  $S_n$ . For any  $n > k$ , Theorem 2.3 gives

(3.2)

$$\begin{aligned} \mathbb{E}(Z_n^k) &= \frac{1}{(2f(n))^{k/2}} \mathbb{E} \left( \left( \sum_{\mathbf{m} \in L \setminus \{\mathbf{0}\}} \chi_n(\mathbf{m}) - \int_{\mathbb{R}^n} \chi_n(\mathbf{x}) d\mathbf{x} \right)^k \right) \\ &= \frac{1}{(2f(n))^{k/2}} \sum_{q=1}^{\infty} \sum'_D \left( \frac{e_1}{q} \cdots \frac{e_m}{q} \right)^n \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^k \chi_n \left( \sum_{i=1}^m \frac{d_{ij}}{q} \mathbf{x}_i \right) d\mathbf{x}_1 \cdots d\mathbf{x}_m. \end{aligned}$$

We let

$$(3.3) \quad M_{k,n} := \sum''_D \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^k \chi_n \left( \sum_{i=1}^m d_{ij} \mathbf{x}_i \right) d\mathbf{x}_1 \cdots d\mathbf{x}_m,$$

where the sum is taken over all  $\langle k, 1 \rangle$ -admissible matrices  $D$  having entries  $d_{ij} \in \{0, \pm 1\}$ , with at least two non-zero entries in each row and exactly one non-zero entry in each column. Let  $R_{k,n}$  be the sum of all the terms in (3.2) that are not accounted for in  $M_{k,n}$ , so that

$$(3.4) \quad \mathbb{E}(Z_n^k) = (2f(n))^{-k/2} (M_{k,n} + R_{k,n}).$$

Now, let  $\mathcal{P}'(k)$  denote the set of partitions of  $\{1, \dots, k\}$  containing no singleton sets. Using  $S_n = -S_n$  and  $\text{vol}(S_n) = f(n)$ , and then [27, Lemma 3], we have

$$(3.5) \quad M_{k,n} = \sum_D'' f(n)^m = \sum_{P \in \mathcal{P}'(k)} 2^{k-\#P} f(n)^{\#P}.$$

It remains to bound the term  $R_{k,n}$  in (3.4). The summation condition in  $\sum_D'$  implies that all matrices  $D$  appearing in  $R_{k,n}$  have at most  $k-1$  rows. Hence, an easy modification of the arguments in [20, Sect. 9] and [21, Sect. 4] (see also [27, Sect. 3]) gives that, for  $n$  sufficiently large,

$$(3.6) \quad 0 \leq R_{k,n} \ll \left(\frac{3}{4}\right)^{n/2} f(n)^{k-1},$$

where the implied constant depends on  $k$  but not on  $n$ . If  $k$  is odd, then we may assume that  $k \geq 3$  and in this situation we have  $\#P \leq (k-1)/2$  for every  $P \in \mathcal{P}'(k)$ . Recall that we are assuming  $f(n) = O_\varepsilon(e^{\varepsilon n})$ . Hence it follows from (3.4), (3.5) and (3.6) that, for any odd  $k \geq 3$ ,

$$\lim_{n \rightarrow \infty} \mathbb{E}(Z_n^k) = 0.$$

On the other hand, if  $k$  is even, then (3.4), (3.5) and (3.6) imply that

$$\lim_{n \rightarrow \infty} \mathbb{E}(Z_n^k) = \#\{P \in \mathcal{P}'(k) : \#B = 2, \forall B \in P\} = (k-1)!!.$$

This completes the proof of the proposition.  $\square$

*Remark 3.2.* Note that the variance of  $Z_n$  can be controlled for a much larger class of functions  $f$ . Indeed, for *any*  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$  satisfying  $\lim_{n \rightarrow \infty} f(n) = \infty$ , we have

$$\text{Var}(Z_n) = 1 + O\left(\left(\frac{3}{4}\right)^{n/2}\right) \quad \text{as } n \rightarrow \infty.$$

Cf. (3.4) and (3.6) and note that  $k-1 = k/2 = 1$  for  $k = 2$ .

*Proof of Theorem 1.3'.* The desired convergence follows immediately from Proposition 3.1.  $\square$

*Proof of Theorem 1.2.* Let  $\varepsilon > 0$  be given. It follows from Theorem 1.3 that there exist  $x_0 > 0$  and  $n_0 \in \mathbb{Z}^+$  such that for all  $n \geq n_0$ ,  $x \in [x_0, f(n)]$  and  $r \in \mathbb{R}$ ,

$$(3.7) \quad \left| \text{Prob}\left(\frac{N_{n,L}(x) - x}{\sqrt{2x}} \leq r\right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^r e^{-t^2/2} dt \right| < \frac{\varepsilon}{2}.$$

(Indeed, otherwise there is a sequence of positive integers  $n_1 < n_2 < \dots$  and positive numbers  $x_1, x_2, \dots$  with  $x_j \leq f(n_j)$  and  $\lim_{j \rightarrow \infty} x_j = \infty$ , such that for each  $j$ , (3.7) fails for  $n = n_j$ ,  $x = x_j$  and some  $r = r_j \in \mathbb{R}$ . We then obtain a contradiction against Theorem 1.3 applied to the function  $f_1 : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$  given by  $f_1(n_j) = x_j$  and, say,  $f_1(n) = f(n)$  for  $n \notin \{n_1, n_2, \dots\}$ .) Using also the fact that  $\frac{2\mathcal{N}(x)-x}{\sqrt{2x}}$  tends in distribution to  $N(0, 1)$ , and taking  $r = \frac{2N-x}{\sqrt{2x}}$ , it follows that after possibly increasing  $x_0$ , we have

$$(3.8) \quad \left| \text{Prob}(N_{n,L}(x) \leq 2N) - \text{Prob}(\mathcal{N}(x) \leq N) \right| < \varepsilon$$

for all  $n \geq n_0$ ,  $x \in [x_0, f(n)]$ ,  $N \geq 0$ . On the other hand it follows from Theorem 1.1 (or [21, Thm. 3]) that, after possibly increasing  $n_0$ , (3.8) also holds for all  $n \geq n_0$ ,  $x \in [0, x_0]$ ,  $N \geq 0$ .

Hence we have proved that (1.4) holds uniformly with respect to all  $N \geq 0$  and  $0 \leq x \leq f(n)$ . The extension to the remaining case, i.e.  $x > f(n)$  and  $N \leq f(n)$ , is now straightforward: Applying what we have already proved to the function  $n \mapsto 4f(n)$ , it follows that the convergence in (1.4) holds uniformly with respect to all  $N \geq 0$  and  $0 \leq x \leq 4f(n)$ ; thus it only remains to consider the case when  $x > 4f(n)$  and  $N \leq f(n)$ . However, for such  $x$  and  $N$ , we have

$$(3.9) \quad \text{Prob}_{\mu_n}(N_{n,L}(x) \leq 2N) \leq \text{Prob}_{\mu_n}(N_{n,L}(4f(n)) \leq 2f(n))$$

and

$$(3.10) \quad \text{Prob}(\mathcal{N}(x) \leq N) \leq \text{Prob}(\mathcal{N}(4f(n)) \leq f(n)).$$

Here the right-hand side of (3.10) tends to zero as  $n \rightarrow \infty$ , and so by the convergence already established also the right-hand side of (3.9) tends to zero. Hence also the left-hand sides of (3.9) and (3.10) tend to zero as  $n \rightarrow \infty$ , uniformly over all  $x > 4f(n)$  and  $N \leq f(n)$ . This concludes the proof.  $\square$

#### 4. JOINT DISTRIBUTION FOR FAMILIES OF SUBSETS, AND PROOF OF THEOREM 1.5

Our main goal in this section is to prove Theorem 1.5. As a first step, we generalize Proposition 3.1 and Theorem 1.3' to finite families of disjoint subsets of  $\mathbb{R}^n$ . Specifically, let us again fix a function  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$  satisfying  $\lim_{n \rightarrow \infty} f(n) = \infty$  and  $f(n) = O_\varepsilon(e^{\varepsilon n})$  for every  $\varepsilon > 0$ . Fix a positive integer  $r$  and positive real numbers  $c_1, \dots, c_r$ . For each  $n$ , let  $S_{1,n}, \dots, S_{r,n}$  be Borel measurable subsets of  $\mathbb{R}^n$  satisfying  $\text{vol}(S_{j,n}) = c_j f(n)$ ,  $-S_{j,n} = S_{j,n}$ , and  $S_{j,n} \cap S_{j',n} = \emptyset$  for all  $j \neq j'$ . In analogy with (3.1) we set

$$(4.1) \quad Z_{j,n} := \frac{\#(L \cap S_{j,n} \setminus \{\mathbf{0}\}) - c_j f(n)}{\sqrt{2f(n)}},$$

with  $L$  picked at random in  $(X_n, \mu_n)$ .

**Proposition 4.1.** *In this situation, for any fixed  $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}_{\geq 0}^r$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{E}\left(Z_{1,n}^{k_1} \dots Z_{r,n}^{k_r}\right) = \begin{cases} \prod_{j=1}^r (c_j^{k_j/2} (k_j - 1)!!) & \text{if } k_1, \dots, k_r \text{ are all even,} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Set  $\widehat{k} = k_1 + \dots + k_r$ . Let  $\chi_{j,n}$  be the characteristic function of  $S_{j,n}$ . For any  $n > \widehat{k}$ , Theorem 2.3 gives

$$(4.2) \quad \mathbb{E}\left(Z_{1,n}^{k_1} \dots Z_{r,n}^{k_r}\right) = (2f(n))^{-\widehat{k}/2} \sum_{q=1}^{\infty} \sum_D' \left(\frac{e_1}{q} \dots \frac{e_m}{q}\right)^n \\ \times \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \prod_{j=1}^r \prod_{\ell_j=1}^{k_j} \chi_{j,n} \left( \sum_{i=1}^m \frac{d_{i,k_1+\dots+k_{j-1}+\ell_j}}{q} \mathbf{x}_i \right) d\mathbf{x}_1 \dots d\mathbf{x}_m,$$

where the sum over  $D = (d_{ij})$  runs through all  $\langle \widehat{k}, q \rangle$ -admissible matrices with the property that there are at least two non-zero entries in each row. As in the proof of Proposition 3.1, we divide the right-hand side into two parts as

$$\mathbb{E}\left(Z_{1,n}^{k_1} \dots Z_{r,n}^{k_r}\right) = (2f(n))^{-\widehat{k}/2} (\widetilde{M}_{\mathbf{k},n} + \widetilde{R}_{\mathbf{k},n}),$$

where

$$(4.3) \quad \widetilde{M}_{\mathbf{k},n} := \sum''_D \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^r \prod_{\ell_j=1}^{k_j} \chi_{j,n} \left( \sum_{i=1}^m d_{i,k_1+\cdots+k_{j-1}+\ell_j} \mathbf{x}_i \right) d\mathbf{x}_1 \cdots d\mathbf{x}_m,$$

the sum being taken over all  $(\widehat{\mathbf{k}}, 1)$ -admissible matrices having entries  $d_{ij} \in \{0, \pm 1\}$ , with at least two non-zero entries in each row and exactly one non-zero entry in each column. Using the assumption that  $S_{1,n}, \dots, S_{r,n}$  are pairwise disjoint it follows that the terms in the right-hand side of (4.3) are zero unless, for each  $i \in \{1, \dots, m\}$ , there is some  $j \in \{1, \dots, r\}$  such that the  $i$ th row of  $D$  has all its non-zero elements in columns corresponding to the fixed function  $\chi_{j,n}$ . The rest of the proof follows closely that of Proposition 3.1.  $\square$

Note that Proposition 4.1 immediately implies the following theorem, generalizing Theorem 1.3'.

**Theorem 4.2.** *Fix  $r \in \mathbb{Z}^+$ ,  $c_1, \dots, c_r > 0$ , and a function  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$  satisfying  $\lim_{n \rightarrow \infty} f(n) = \infty$  and  $f(n) = O_\varepsilon(e^{\varepsilon n})$  for every  $\varepsilon > 0$ . For each  $n$ , let  $S_{1,n}, \dots, S_{r,n}$  be Borel measurable subsets of  $\mathbb{R}^n$  which are pairwise disjoint, and which satisfy  $\text{vol}(S_{j,n}) = c_j f(n)$  and  $-S_{j,n} = S_{j,n}$ . Set*

$$Z_{j,n} := \frac{\#(L \cap S_{j,n} \setminus \{\mathbf{0}\}) - c_j f(n)}{\sqrt{2f(n)}},$$

with  $L$  picked at random in  $(X_n, \mu_n)$ . Then

$$(Z_{1,n}, \dots, Z_{r,n}) \xrightarrow{d} (N(0, c_1), N(0, c_2), \dots, N(0, c_r)) \quad \text{as } n \rightarrow \infty,$$

where the random vector in the right-hand side has independent coordinates.

We are now in position to complete the proof of Theorem 1.5.

*Proof of Theorem 1.5.* To simplify notation, in this proof we write  $\widetilde{Z}_n(t) := \widetilde{Z}_n^{(B)}(t)$ . Given any fixed numbers  $0 < t_1 < t_2 < \cdots < t_r \leq 1$ , by applying Theorem 4.2 with  $S_{1,n}, \dots, S_{r,n}$  as the annuli

$$S_{j,n} = \left\{ \mathbf{x} \in \mathbb{R}^n : \left( \frac{t_{j-1} f(n)}{V_n} \right)^{1/n} < |\mathbf{x}| \leq \left( \frac{t_j f(n)}{V_n} \right)^{1/n} \right\}, \quad j = 1, \dots, r$$

(with  $t_0 := 0$ ), we conclude that the random vector

$$\left( \widetilde{Z}_n(t_1), \widetilde{Z}_n(t_2) - \widetilde{Z}_n(t_1), \dots, \widetilde{Z}_n(t_r) - \widetilde{Z}_n(t_{r-1}) \right)$$

tends in distribution to

$$(N(0, t_1), N(0, t_2 - t_1), \dots, N(0, t_r - t_{r-1}))$$

as  $n \rightarrow \infty$ . Note also that  $\widetilde{Z}_n(0) = 0$  by definition. We have thus proved that the convergence in Theorem 1.5 holds on the level of finite dimensional distributions, and it now only remains to establish the tightness of the sequence  $P_n$  of probability measures on  $\mathcal{D}[0, 1]$ .

By [3, Thm. 13.5 and (13.14)] (applied with  $F(t) = C\sqrt{t}$  and  $\beta = 1$ ), it suffices to prove that there exist  $\alpha > \frac{1}{2}$  and  $N \in \mathbb{N}$  such that

$$(4.4) \quad \mathbb{E} \left( (\widetilde{Z}_n(s) - \widetilde{Z}_n(r))^2 (\widetilde{Z}_n(t) - \widetilde{Z}_n(s))^2 \right) \ll (\sqrt{t} - \sqrt{r})^{2\alpha},$$

uniformly over all  $0 \leq r \leq s \leq t \leq 1$  and  $n \geq N$ . We begin by noting that Proposition 4.1 implies that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left( (\tilde{Z}_n(s) - \tilde{Z}_n(r))^2 (\tilde{Z}_n(t) - \tilde{Z}_n(s))^2 \right) = (t-s)(s-r) \leq (t-r)^2.$$

Hence, using also the fact that  $t-r \leq 2(\sqrt{t} - \sqrt{r})$  for all  $0 \leq r \leq t \leq 1$ , we see that in the limit of large dimension  $n$ , (4.4) holds with  $\alpha = 1$ . In order to get a more uniform statement, note that by naively modifying Rogers' arguments in [20, Sect. 9] and [21, Sect. 4] as in the proofs of Propositions 3.1 and 4.1, we have

$$(4.5) \quad \mathbb{E} \left( (\tilde{Z}_n(s) - \tilde{Z}_n(r))^2 (\tilde{Z}_n(t) - \tilde{Z}_n(s))^2 \right) \\ \ll (t-r)^2 + \max \left( 2^{-n}(t-r)f(n)^{-1}, \left(\frac{3}{4}\right)^{n/2}(t-r)^2, \left(\frac{3}{4}\right)^{n/2}(t-r)^3f(n) \right)$$

for all  $n \geq 6$ , where the implied constant is absolute.

The bound (4.5) is close but not quite sufficient for our purposes; the problematic term is  $2^{-n}(t-r)f(n)^{-1}$ . This term arises as a bound on the collected contribution of all  $\langle 4, q \rangle$ -admissible matrices  $D$  with  $m = 1$  (and  $q$  arbitrary) in the expression that is obtained by applying Theorem 2.3 to the left-hand side of (4.5) (cf. (4.2)). Recall that  $m = 1$  means that  $D$  has only one row. In order to improve the bound, note that any such matrix  $D = (q, d_1, d_2, d_3)$  gives a contribution

$$(4.6) \quad \frac{1}{4q^n f(n)^2} \int_{\mathbb{R}^n} \chi_1(V_n |\mathbf{x}|^n) \chi_1 \left( V_n \left| \frac{d_1}{q} \mathbf{x} \right|^n \right) \chi_2 \left( V_n \left| \frac{d_2}{q} \mathbf{x} \right|^n \right) \chi_2 \left( V_n \left| \frac{d_3}{q} \mathbf{x} \right|^n \right) d\mathbf{x}$$

to the left-hand side of (4.5), where  $\chi_1$  and  $\chi_2$  are the characteristic functions of the open intervals  $(rf(n), sf(n))$  and  $(sf(n), tf(n))$ , respectively. Let us temporarily assume that  $r > 0$ . Then, for the integral in (4.6) to be non-zero, we must have

$$1 < \left| \frac{d_2}{q} \right|^n, \left| \frac{d_3}{q} \right|^n < \frac{t}{r} = 1 + \frac{t-r}{r}.$$

Hence, since  $d_2$  and  $d_3$  are integers, we conclude that a (crude) necessary condition for (4.6) to be non-zero is

$$q^n > \frac{r}{t-r}.$$

Let  $Q$  be the smallest value of  $q \in \mathbb{Z}^+$  satisfying this inequality. Then, for  $n \geq 6$ , the estimate [20, p. 246 (line 20)] with  $\sum_{q=1}^{\infty}$  replaced by  $\sum_{q=Q}^{\infty}$  gives

$$(4.7) \quad \sum_{q=Q}^{\infty} \sum'_{\substack{D \\ (m=1)}} \frac{1}{4q^n f(n)^2} \int_{\mathbb{R}^n} \chi_1(V_n |\mathbf{x}|^n) \chi_1 \left( V_n \left| \frac{d_1}{q} \mathbf{x} \right|^n \right) \chi_2 \left( V_n \left| \frac{d_2}{q} \mathbf{x} \right|^n \right) \chi_2 \left( V_n \left| \frac{d_3}{q} \mathbf{x} \right|^n \right) d\mathbf{x} \\ \ll Q^{5-n} (t-r) f(n)^{-1}.$$

Replacing the term  $2^{-n}(t-r)f(n)^{-1}$  in (4.5) by the bound in (4.7) and using  $Q \geq \max(1, (r/(t-r))^{1/n})$ , we obtain, allowing now the implied constant to depend on

$f$ :

$$\begin{aligned}
(4.8) \quad & \mathbb{E} \left( (\tilde{Z}_n(s) - \tilde{Z}_n(r))^2 (\tilde{Z}_n(t) - \tilde{Z}_n(s))^2 \right) \\
& \ll (t-r)^2 + (t-r) \min \left( 1, \left( \frac{t-r}{r} \right)^{1-\frac{5}{n}} \right) \\
& \ll (t-r) \min \left( 1, \left( \frac{t-r}{r} \right)^{1-\frac{5}{n}} \right).
\end{aligned}$$

This bound is also valid when  $r = 0$ , with the convention that  $\min(1, \dots)$  then equals 1.

Now fix the constant  $\frac{1}{2} < \alpha < 1$  in an arbitrary manner, and then take  $N \geq 6$  so large that  $1 - \alpha - \frac{5}{N} > 0$ . We then claim that

$$(4.9) \quad (t-r) \min \left( 1, \left( \frac{t-r}{r} \right)^{1-\frac{5}{n}} \right) \ll (\sqrt{t} - \sqrt{r})^{2\alpha},$$

uniformly over all  $n \geq N$  and  $0 \leq r \leq t \leq 1$ . Indeed, if  $t \geq 2r$  then (4.9) is clear from  $(\sqrt{t} - \sqrt{r})^{2\alpha} \asymp (\sqrt{t})^{2\alpha} = t^\alpha$ . In the remaining case, i.e. when  $0 < r \leq t < 2r$ , we have  $\sqrt{t} - \sqrt{r} \asymp (t-r)/\sqrt{r}$  and (4.9) is equivalent to  $t-r \ll r^{(1-\alpha-\frac{5}{n})/(2-2\alpha-\frac{5}{n})}$ , which is true since  $(1-\alpha-\frac{5}{n})/(2-2\alpha-\frac{5}{n}) < 1$  and  $t < 2r$ . This completes the proof of (4.9), and in view of (4.8) we thus obtain (4.4), completing the proof of Theorem 1.5.  $\square$

## 5. MOMENT BOUNDS FOR EXPONENTIALLY GROWING VOLUMES

Our goal in this section is to prove Theorem 1.7. Thus, for each  $n$  we assume given a Borel subset  $S_n$  of  $\mathbb{R}^n$  satisfying  $\text{vol}(S_n) = f(n)$  and  $S_n = -S_n$ . Throughout the section we let  $\chi_n$  denote the characteristic function of  $S_n$ . Our task is to go back to the proof of Proposition 3.1 and improve the bound on  $R_{k,n}$ , i.e. the sum of those terms in (3.2) which come from  $\langle k, q \rangle$ -admissible matrices  $D$  with at least two non-zero entries in each row and such that either  $q \geq 2$ , or some column contains more than one non-zero entry, or some entry has absolute value  $|d_{ij}| \geq 2$ . It will turn out that the dominating contribution to  $R_{k,n}$  comes from  $\langle k, 1 \rangle$ -admissible matrices  $D$  of the form

$$(5.1) \quad D = \begin{pmatrix} 1 & 0 & \cdots & 0 & \pm 1 \\ 0 & 1 & \cdots & 0 & \pm 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \pm 1 \end{pmatrix} \quad (\text{thus } m = m(D) = k-1).$$

**5.1. Auxiliary lemmas.** In our first lemma, by repeated use of an integral inequality of Rogers, [22, Theorem 1], we bound  $\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^k \chi_n \left( \sum_{i=1}^m \frac{d_{ij}}{q} \mathbf{x}_i \right) d\mathbf{x}_1 \cdots d\mathbf{x}_m$  from above by a product of integrals of the following form:

$$(5.2) \quad J_a^{(n)}[c_1, \dots, c_a] := \int_{(\mathbb{R}^n)^a} I \left( |\mathbf{x}_i| < 1 \ (i = 1, \dots, a), \left| \sum_{i=1}^a c_i \mathbf{x}_i \right| < 1 \right) d\mathbf{x}_1 \cdots d\mathbf{x}_a.$$

Here  $n \geq a \geq 1$  and  $c_1, \dots, c_a \in \mathbb{R}_{>0}$ , and  $I(\cdot)$  is the indicator function. We extend the definition to the case  $a = 0$  by setting  $J_0^{(n)}[\ ] := 1$  for all  $n$ .

Let  $D$  be a  $\langle k, q \rangle$ -admissible matrix of size  $m \times k$ , having at least two non-zero entries in each row. Set  $r = k - m$ , let  $(\nu; \mu) = (\nu_1, \dots, \nu_m; \mu_1, \dots, \mu_r)$  be as in



Section 2, and let  $\mu'_1, \dots, \mu'_r$  be an arbitrary permutation of  $\mu_1, \dots, \mu_r$ . For  $j = 1, \dots, r$ , we set

$$\bar{A}_j = \{i \in \{1, \dots, m\} : d_{i, \mu'_j} \neq 0\}; \quad A_j = \bar{A}_j \setminus (\cup_{\ell < j} \bar{A}_\ell), \quad \text{and} \quad a_j = \#A_j.$$

Since  $D$  has at least two non-zero entries in each row, the sets  $A_1, \dots, A_r$  form a partition of  $\{1, \dots, m\}$ , possibly with  $A_j = \emptyset$  for some  $j$ 's. Hence  $\sum_{j=1}^r a_j = m$ .

**Lemma 5.1.** *For  $D$  as above,*

(5.3)

$$\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^k \chi_n \left( \sum_{i=1}^m \frac{d_{ij}}{q} \mathbf{x}_i \right) d\mathbf{x}_1 \dots d\mathbf{x}_m \leq V_n^{-m} f(n)^m \prod_{j=1}^r J_{a_j}^{(n)} [ (|d_{i, \mu'_j}|/q)_{i \in A_j} ].$$

*Proof.* We express the left-hand side of (5.3) as an iterated integral in the following way. For each  $j \in \{1, \dots, r\}$  we write  $\mathbf{x}^{(j)} := (\mathbf{x}_i)_{i \in A_j} \in (\mathbb{R}^n)^{a_j}$  and  $d\mathbf{x}^{(j)} := \prod_{i \in A_j} d\mathbf{x}_i$ . (If  $a_j = 0$  then we understand  $(\mathbb{R}^n)^0$  and  $d\mathbf{x}^{(j)}$  to be the singleton set  $\{\mathbf{0}\}$  with its unique probability measure.) Let  $F_r(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(r)})$  be the constant function 1, and set, iteratively for  $j = r, r-1, \dots, 1$ ,

$$(5.4) \quad F_{j-1}(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(j-1)}) := \int_{(\mathbb{R}^n)^{a_j}} \left( \prod_{i \in A_j} \chi_n(\mathbf{x}_i) \right) \chi_n \left( \sum_{i=1}^m \frac{d_{i, \mu'_j}}{q} \mathbf{x}_i \right) F_j(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(j)}) d\mathbf{x}^{(j)}.$$

Then the left-hand side of (5.3) equals  $F_0$ . (The sum  $\sum_{i=1}^m (d_{i, \mu'_j}/q) \mathbf{x}_i$  appearing in the right-hand side of (5.4) is well-defined since  $d_{i, \mu'_j} = 0$  for all  $i \in \{1, \dots, m\} \setminus (A_1 \cup \dots \cup A_j)$ .)

Now let  $B$  be the closed ball of volume  $f(n)$  centered at the origin in  $\mathbb{R}^n$ , and let  $\chi_B$  be its characteristic function. Using (5.4) and [22, Theorem 1], we have

$$\begin{aligned} F_{j-1}(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(j-1)}) &\leq (\sup F_j) \int_{(\mathbb{R}^n)^{a_j}} \left( \prod_{i \in A_j} \chi_n(\mathbf{x}_i) \right) \chi_n \left( \sum_{i=1}^m \frac{d_{i, \mu'_j}}{q} \mathbf{x}_i \right) d\mathbf{x}^{(j)} \\ &\leq (\sup F_j) \int_{(\mathbb{R}^n)^{a_j}} \left( \prod_{i \in A_j} \chi_B(\mathbf{x}_i) \right) \chi_B \left( \sum_{i=1}^m \frac{d_{i, \mu'_j}}{q} \mathbf{x}_i \right) d\mathbf{x}^{(j)}, \end{aligned}$$

since  $\chi_B$  is the spherical symmetrization both of  $\chi_n$  and of  $\mathbf{y} \mapsto \chi_n(\mathbf{y} + \mathbf{z})$  for any fixed  $\mathbf{z} \in \mathbb{R}^n$ . Hence, since  $B$  has radius  $V_n^{-1/n} f(n)^{1/n}$ , we conclude

$$\sup F_{j-1} \leq V_n^{-a_j} f(n)^{a_j} J_{a_j}^{(n)} [ (|d_{i, \mu'_j}|/q)_{i \in A_j} ] \cdot \sup F_j.$$

Using this bound for  $j = 1, \dots, r$ , together with  $\sum_{j=1}^r a_j = m$ , we obtain (5.3).  $\square$

We say that a function  $F : (\mathbb{R}^n)^m \rightarrow \mathbb{R}$  ( $1 \leq m \leq n$ ) is  $O(n)$ -invariant if  $F(k\mathbf{x}_1, \dots, k\mathbf{x}_m) = F(\mathbf{x}_1, \dots, \mathbf{x}_m)$  for all  $k \in O(n)$ . When this holds, we define  $\bar{F} : (\mathbb{R}^m)^m \rightarrow \mathbb{R}$  through  $\bar{F}(\mathbf{x}_1, \dots, \mathbf{x}_m) = F(\iota(\mathbf{x}_1), \dots, \iota(\mathbf{x}_m))$ , where  $\iota$  is any fixed Euclidean isometry of  $\mathbb{R}^m$  into  $\mathbb{R}^n$ . Note that  $\bar{F}$  is independent of the choice of  $\iota$ . Given any  $\mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{R}^m$ , we denote by  $[\mathbf{x}_1, \dots, \mathbf{x}_m]$  the volume of the parallelepiped in  $\mathbb{R}^m$  spanned by  $\mathbf{x}_1, \dots, \mathbf{x}_m$ . Finally, we write  $\omega_n := nV_n$  for the volume of the  $(n-1)$ -sphere.

**Lemma 5.2.** *Let  $1 \leq m \leq n$  and let  $F : (\mathbb{R}^n)^m \rightarrow \mathbb{R}_{\geq 0}$  be a non-negative Borel measurable function which is  $O(n)$ -invariant. Then*

$$(5.5) \quad \int_{(\mathbb{R}^n)^m} F(\mathbf{x}_1, \dots, \mathbf{x}_m) d\mathbf{x}_1 \cdots d\mathbf{x}_m \\ = \frac{\prod_{j=n-m+1}^n \omega_j}{\prod_{j=1}^m \omega_j} \int_{(\mathbb{R}^m)^m} \overline{F}(\mathbf{x}_1, \dots, \mathbf{x}_m) [\mathbf{x}_1, \dots, \mathbf{x}_m]^{n-m} d\mathbf{x}_1 \cdots d\mathbf{x}_m.$$

*Proof.* Let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  be the standard unit vectors in  $\mathbb{R}^n$ . Passing to polar coordinates and then performing the same substitution as in [28, p. 754], the left-hand side of (5.5) becomes

$$\left( \prod_{j=n-m+1}^n \omega_j \right) \int_{(\mathbb{R}_{>0})^m} \int_{(0,\pi)^M} F(\mathbf{x}_1, \dots, \mathbf{x}_m) \prod_{1 \leq i < j \leq m} (\sin \phi_{i,j})^{n-i-1} \prod_{j=1}^m r_j^{n-1} d\phi d\mathbf{r},$$

where  $M = \binom{m}{2}$ ,  $\mathbf{r} = (r_1, \dots, r_m)$ ,  $\phi = (\phi_{i,j})_{1 \leq i < j \leq m}$ , and  $d\mathbf{r}$  and  $d\phi$  denote Lebesgue measure on  $\mathbb{R}^m$  and  $\mathbb{R}^M$ , respectively, and

$$(5.6) \quad \mathbf{x}_j = r_j \left( \sum_{1 \leq i < j} \left( \prod_{i' < i} \sin \phi_{i',j} \right) (\cos \phi_{i,j}) \mathbf{e}_i + \left( \prod_{i' < j} \sin \phi_{i',j} \right) \mathbf{e}_j \right).$$

(In particular  $\mathbf{x}_1 = r_1 \mathbf{e}_1$ .) We view  $\mathbb{R}^m$  as a subspace of  $\mathbb{R}^n$  through  $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, 0, \dots, 0)$ . Then all the  $\mathbf{x}_j$  in (5.6) lie in  $\mathbb{R}^m$ . The desired formula now follows by performing the same substitutions backwards, in  $\mathbb{R}^m$  instead of in  $\mathbb{R}^n$ , and using  $(\prod_{j=1}^m r_j) \prod_{i < j} \sin \phi_{i,j} = [\mathbf{x}_1, \dots, \mathbf{x}_m]$ .  $\square$

Applying Lemma 5.2 to the integral in (5.2), we see that the asymptotics of  $J_a^{(n)}[c_1, \dots, c_a]$  as  $n \rightarrow \infty$  depends mainly on the quantity

$$(5.7) \quad \mathcal{V}_a[c_1, \dots, c_a] := \sup \left\{ [\mathbf{x}_1, \dots, \mathbf{x}_a] : \mathbf{x}_1, \dots, \mathbf{x}_a \in \mathbb{R}^a, |\mathbf{x}_j| \leq 1 (\forall j), \right. \\ \left. |c_1 \mathbf{x}_1 + \dots + c_a \mathbf{x}_a| \leq 1 \right\}.$$

**Lemma 5.3.** *For any  $1 \leq a \leq n$  and  $c_1, \dots, c_a > 0$ ,*

$$(5.8) \quad J_a^{(n)}[c_1, \dots, c_a] \ll_a n^{a(a+3)/4} V_n^a \mathcal{V}_a[c_1, \dots, c_a]^{n-a}.$$

*On the other hand, for any fixed  $c_1, \dots, c_a \in \mathbb{R}_{>0}$  and  $\mathcal{V} \in (0, \mathcal{V}_a[c_1, \dots, c_a])$ , we have  $\lim_{n \rightarrow \infty} \mathcal{V}^{-n} V_n^{-a} J_a^{(n)}[c_1, \dots, c_a] = \infty$ .*

*Proof.* Let  $B$  be the open unit ball in  $\mathbb{R}^a$  centered at the origin. Then Lemma 5.2 gives

$$(5.9) \quad J_a^{(n)}[c_1, \dots, c_a] = \frac{\prod_{j=n-a+1}^n \omega_j}{\prod_{j=1}^a \omega_j} \int_{B^a} I \left( \left| \sum_{i=1}^a c_i \mathbf{x}_i \right| < 1 \right) [\mathbf{x}_1, \dots, \mathbf{x}_a]^{n-a} d\mathbf{x}_1 \cdots d\mathbf{x}_a \\ \leq \frac{\prod_{j=n-a+1}^n \omega_j}{\prod_{j=1}^a \omega_j} V_a^a \mathcal{V}_a[c_1, \dots, c_a]^{n-a}.$$

Furthermore, by Stirling's formula,

$$(5.10) \quad \omega_j = j V_j = \frac{2\pi^{j/2}}{\Gamma(j/2)} \asymp_a n^{1+(n-j)/2} V_n$$

for all  $j \in \{n-a+1, n-a+2, \dots, n\}$ . These two bounds imply (5.8).

Next, let  $c_1, \dots, c_a \in \mathbb{R}_{>0}$  and  $\mathcal{V}$  be given as in the statement of the lemma. It is clear from (5.7) that there exist non-empty open subsets  $U_1, \dots, U_a$  of  $B$  such that all  $(\mathbf{x}_1, \dots, \mathbf{x}_a) \in U_1 \times \dots \times U_a$  satisfy both  $|c_1 \mathbf{x}_1 + \dots + c_a \mathbf{x}_a| < 1$  and  $[\mathbf{x}_1, \dots, \mathbf{x}_a] > \mathcal{V}$ . Using the first equality in (5.9), it follows that

$$J_a^{(n)}[c_1, \dots, c_a] \geq \frac{\prod_{j=n-a+1}^n \omega_j \prod_{j=1}^a \text{vol}(U_j)}{\prod_{j=1}^a \omega_j} \mathcal{V}^{n-a}.$$

Using this and (5.10), the second claim of the lemma follows.  $\square$

The next lemma gives a bound on the product  $\frac{e_1}{q} \dots \frac{e_m}{q}$  appearing in (3.2). Recall that  $e_i = (\varepsilon_i, q)$ , where  $\varepsilon_1, \dots, \varepsilon_m$  are the elementary divisors of the matrix  $D$ .

**Lemma 5.4.** *For any  $D$  as in Lemma 5.1,*

$$(5.11) \quad \frac{e_1}{q} \dots \frac{e_m}{q} \leq \prod_{j=1}^r \frac{g_j}{q}, \quad \text{with } g_j = \gcd(\{q\} \cup \{d_{i,\mu'_j} : i \in A_j\}).$$

(Note that if  $A_j = \emptyset$  then  $g_j = q$ , giving a factor 1 in the product in (5.11).)

*Proof.* By [19, Lemma 1],

$$e_1 \dots e_m = N(D, q) := \#\left\{ (x_1, \dots, x_m) \in (\mathbb{Z}/q\mathbb{Z})^m : \sum_{i=1}^m d_{ij} x_i \equiv 0 \pmod{q} \ (\forall j) \right\}.$$

As a preliminary step, note that for any integers  $c, d_1, \dots, d_\ell$ ,

$$(5.12) \quad \#\left\{ (x_1, \dots, x_\ell) \in (\mathbb{Z}/q\mathbb{Z})^\ell : \sum_{j=1}^{\ell} d_j x_j \equiv c \pmod{q} \right\} \leq q^{\ell-1} \gcd(q, d_1, \dots, d_\ell).$$

Indeed, this is immediate when  $q$  is a prime power, and the general case can be reduced to this case using the Chinese Remainder Theorem. We now set  $\tilde{A}_0 := \emptyset$  and  $\tilde{A}_j := A_1 \cup \dots \cup A_j = \tilde{A}_1 \cup \dots \cup \tilde{A}_j$  for  $j \geq 1$ . For any  $j \in \{1, \dots, r\}$  and any given  $(x_i)_{i \in \tilde{A}_{j-1}}$  in  $(\mathbb{Z}/q\mathbb{Z})^{\#\tilde{A}_{j-1}}$ , it follows from (5.12) that the number of tuples  $(x_i)_{i \in A_j} \in (\mathbb{Z}/q\mathbb{Z})^{a_j}$  satisfying  $\sum_{i=1}^m d_{i,\mu'_j} x_i \equiv 0 \pmod{q}$  is less than or equal to  $q^{a_j-1} g_j$ . Using this fact for each  $j = 1, \dots, r$ , we obtain

$$N(D, q) \leq \prod_{j=1}^r (q^{a_j-1} g_j) = q^m \prod_{j=1}^r \frac{g_j}{q}.$$

This completes the proof of the lemma.  $\square$

**5.2. Some basic properties of  $\mathcal{V}_a[c_1, \dots, c_a]$ .** Recall that, for any integer  $a \geq 1$  and real numbers  $c_1, \dots, c_a > 0$ ,

$$(5.13) \quad \mathcal{V}_a[c_1, \dots, c_a] := \sup \left\{ [\mathbf{x}_1, \dots, \mathbf{x}_a] : \mathbf{x}_1, \dots, \mathbf{x}_a \in \mathbb{R}^a, |\mathbf{x}_j| \leq 1 \ (\forall j), \right. \\ \left. |c_1 \mathbf{x}_1 + \dots + c_a \mathbf{x}_a| \leq 1 \right\},$$

where  $[\mathbf{x}_1, \dots, \mathbf{x}_a]$  denotes the volume of the parallelotope in  $\mathbb{R}^a$  spanned by  $\mathbf{x}_1, \dots, \mathbf{x}_a$ . Note that  $0 < \mathcal{V}_a[c_1, \dots, c_a] \leq 1$ , and  $\mathcal{V}_a[c_1, \dots, c_a]$  is invariant under any permutation of  $c_1, \dots, c_a$ .

**Lemma 5.5.**  $\mathcal{V}_a[c_1, \dots, c_a] = c_1^{-1} \mathcal{V}_a[c_1^{-1}, c_1^{-1} c_2, \dots, c_1^{-1} c_a]$ , for any  $c_1, \dots, c_a > 0$ .

*Proof.* Set  $\mathbf{d} := c_1 \mathbf{x}_1 + \dots + c_a \mathbf{x}_a$  and note that  $\mathbf{x}_1 = c_1^{-1}(\mathbf{d} - \sum_{j=2}^a c_j \mathbf{x}_j)$  and  $[\mathbf{x}_1, \dots, \mathbf{x}_a] = c_1^{-1}[\mathbf{d}, \mathbf{x}_2, \dots, \mathbf{x}_a]$ . Hence the lemma follows by substituting  $\mathbf{x}_1 = \mathbf{d}^{(old)}$  and  $\mathbf{x}_j = -\mathbf{x}_j^{(old)}$  ( $j \geq 2$ ) in the definition of  $\mathcal{V}_a[c_1^{-1}, c_1^{-1}c_2, \dots, c_1^{-1}c_a]$ .  $\square$

**Lemma 5.6.** *If  $c_1^2 + \dots + c_a^2 \leq 1$ , then  $\mathcal{V}_a[c_1, \dots, c_a] = 1$ . Furthermore, we have  $\mathcal{V}_a[c_1, \dots, c_a] \leq c_\ell^{-1}$  for each  $\ell \in \{1, \dots, a\}$ , and if  $c_\ell^2 \geq 1 + \sum_{j \neq \ell} c_j^2$  then  $\mathcal{V}_a[c_1, \dots, c_a] = c_\ell^{-1}$ .*

*Proof.* The first statement is clear by taking  $\mathbf{x}_1, \dots, \mathbf{x}_a$  to be an ON-basis in the definition of  $\mathcal{V}_a[c_1, \dots, c_a]$ . The remaining statements follow from the first statement of the lemma, combined with the general bound  $\mathcal{V}_a[c_1, \dots, c_a] \leq 1$ , Lemma 5.5, and the invariance of  $\mathcal{V}_a[c_1, \dots, c_a]$  under permutations of  $c_1, \dots, c_a$ .  $\square$

*Remark 5.7.* For  $a = 1$  we have  $\mathcal{V}_1[c] = \min(1, c^{-1})$ . This is clear directly from the definition, or from Lemma 5.6.

**Lemma 5.8.** *For any  $c_1, \dots, c_a > 0$  and  $c'_1, \dots, c'_a > 0$ ,*

$$\mathcal{V}_a[c'_1, \dots, c'_a] \geq \left(1 + \sum_{j=1}^a |c_j - c'_j|\right)^{-a} \mathcal{V}_a[c_1, \dots, c_a].$$

*In particular  $\mathcal{V}_a$  is a continuous function on  $(\mathbb{R}_{>0})^a$ .*

*Proof.* Set  $\delta = (1 + \sum_{j=1}^a |c_j - c'_j|)^{-1} \leq 1$ . Let  $\mathbf{x}_1, \dots, \mathbf{x}_a$  be vectors which achieve the supremum in (5.13). Then

$$|c'_1 \mathbf{x}_1 + \dots + c'_a \mathbf{x}_a| \leq |c_1 \mathbf{x}_1 + \dots + c_a \mathbf{x}_a| + \sum_{j=1}^a |c_j - c'_j| \leq \delta^{-1}.$$

Hence the vectors  $\delta \mathbf{x}_1, \dots, \delta \mathbf{x}_a$  are admissible in the supremum defining  $\mathcal{V}_a[c'_1, \dots, c'_a]$ , so that  $\mathcal{V}_a[c'_1, \dots, c'_a] \geq [\delta \mathbf{x}_1, \dots, \delta \mathbf{x}_a] = \delta^a \mathcal{V}_a[c_1, \dots, c_a]$ .  $\square$

The following technical lemma gives the key input both to a monotonicity property of  $\mathcal{V}_a$  which we will need (Lemma 5.10), and to the explicit determination of  $\mathcal{V}_a[c_1, \dots, c_a]$  in the case  $c_1 = \dots = c_a$  (Lemma 5.11).

**Lemma 5.9.** *Assume  $c_1, \dots, c_a > 0$ ,  $c_1^2 + \dots + c_a^2 > 1$  and  $c_j^2 < 1 + \sum_{\ell \neq j} c_\ell^2$  for each  $j$ . Let  $\mathbf{x}_1, \dots, \mathbf{x}_a$  be vectors which achieve the supremum in (5.13). Let  $\mathbf{d} := c_1 \mathbf{x}_1 + \dots + c_a \mathbf{x}_a$ , and for each  $j \in \{1, \dots, a\}$ , let  $\delta_j$  be the length of the orthogonal projection of  $\mathbf{d}$  onto the subspace  $U_j = \text{Span}\{\mathbf{x}_\ell : \ell \in \{1, \dots, a\} \setminus \{j\}\}$ . Then, for each  $j \in \{1, \dots, a\}$ ,*

- (i) *there is  $\varepsilon > 0$  such that  $c'_j \in (c_j - \varepsilon, c_j) \Rightarrow \mathcal{V}_a[c_1, \dots, c'_j, \dots, c_a] > \mathcal{V}_a[c_1, \dots, c_a]$ ;*
- (ii)  *$\delta_j^2 + c_j^2 > 1$ , and the number  $\delta_j^2 - (\delta_j - \delta_j^3)(\delta_j^2 + c_j^2 - 1)^{-1/2}$  is independent of  $j$ .*

*Proof.* For each  $j$ ,  $\mathbf{x}_j \notin U_j$  since  $\mathcal{V}_a[c_1, \dots, c_a] > 0$ ; we let  $\mathbf{e}_j$  be the unique unit vector in  $\mathbb{R}^a$  which is orthogonal to  $U_j$  and satisfies  $\mathbf{x}_j \cdot \mathbf{e}_j > 0$ . Let  $\mathbf{d}_j$  be the orthogonal projection of  $\mathbf{d}$  onto  $U_j$ ; thus  $\delta_j = |\mathbf{d}_j|$ .

Let us fix  $j$  temporarily, and set  $\mathbf{y} = \mathbf{d} - c_j \mathbf{x}_j \in U_j$  and  $y = |\mathbf{y}|$ . The optimality property of  $\mathbf{x}_1, \dots, \mathbf{x}_a$  implies in particular that among all  $\mathbf{x}'_j \in \mathbb{R}^a$  satisfying  $|\mathbf{x}'_j| \leq 1$  and  $|c_j \mathbf{x}'_j + \mathbf{y}| \leq 1$ , the vector  $\mathbf{x}'_j = \mathbf{x}_j$  has maximal distance from  $U_j$ . By a straightforward analysis one deduces from this fact (and  $\mathbf{x}_j \cdot \mathbf{e}_j > 0$ ) that

$$(5.14) \quad \mathbf{x}_j = -\alpha \mathbf{y} + \beta \mathbf{e}_j,$$

with

$$(5.15) \quad \begin{cases} \alpha = 0 \text{ and } \beta = 1 & \text{if } y^2 \leq 1 - c_j^2, \\ \alpha = \beta = c_j^{-1} & \text{if } y^2 \leq c_j^2 - 1, \\ \alpha = (2c_j y^2)^{-1}(y^2 + c_j^2 - 1) \text{ and } \beta = \sqrt{1 - (\alpha y)^2} & \text{if } y^2 > |c_j^2 - 1|. \end{cases}$$

Let us first assume that  $y^2 < 1 - c_j^2$ . Then  $\mathbf{x}_j = \mathbf{e}_j$  by (5.15) and  $|\mathbf{d}| = |\mathbf{y} + c_j \mathbf{x}_j| = (y^2 + c_j^2)^{1/2} < 1$ , and so the optimality property of  $\mathbf{x}_1, \dots, \mathbf{x}_a$  forces  $\{\mathbf{x}_\ell : \ell \neq j\}$  to be an orthonormal basis of  $U_j$ . Hence  $\sum_{\ell \neq j} c_\ell^2 = y^2 < 1 - c_j^2$ , which contradicts our assumption that  $c_1^2 + \dots + c_a^2 > 1$ . This shows that  $y^2 < 1 - c_j^2$  cannot hold.

Similarly,  $y^2 < c_j^2 - 1$  is impossible. Indeed,  $[\mathbf{x}_1, \dots, \mathbf{x}_a] = c_j^{-1}[\mathbf{d}, \mathbf{x}_1, \dots, \widehat{\mathbf{x}}_j, \dots, \mathbf{x}_a]$  (where  $\widehat{\mathbf{x}}_j$  denotes omission of  $\mathbf{x}_j$  in the list), and hence the optimality property of  $\mathbf{x}_1, \dots, \mathbf{x}_a$  can be rephrased as saying that the  $a$  vectors  $\mathbf{d}, \mathbf{x}_1, \dots, \widehat{\mathbf{x}}_j, \dots, \mathbf{x}_a$  maximize  $[\mathbf{d}, \mathbf{x}_1, \dots, \widehat{\mathbf{x}}_j, \dots, \mathbf{x}_a]$  subject to  $|\mathbf{d}| \leq 1$ ,  $|\mathbf{x}_\ell| \leq 1$  (all  $\ell \neq j$ ) and  $|\mathbf{d} - \sum_{\ell \neq j} c_\ell \mathbf{x}_\ell| \leq c_j$ . Assume now  $y^2 < c_j^2 - 1$ . Then (5.15) gives  $\mathbf{d} = \mathbf{y} + c_j \mathbf{x}_j = \mathbf{e}_j$  and  $|\mathbf{x}_j|^2 = (y/c_j)^2 + (1/c_j)^2 < 1$ , and so the optimality property just noted forces  $\{\mathbf{x}_\ell : \ell \neq j\}$  to again be an orthonormal basis of  $U_j$ . Therefore  $\sum_{\ell \neq j} c_\ell^2 = y^2 < c_j^2 - 1$ , contradicting our assumption that  $c_j^2 < 1 + \sum_{\ell \neq j} c_\ell^2$ .

In conclusion,  $y^2 \geq |c_j^2 - 1|$  must hold. Let us also assume  $y > 0$ . Then one verifies that the formulas for  $\alpha$  and  $\beta$  in the third line of (5.15) hold true (viz., they remain valid even when  $y^2 = |c_j^2 - 1|$ ). These formulas imply  $|\mathbf{x}_j| = |\mathbf{d}| = 1$ . Using  $\mathbf{d}_j = (1 - c_j \alpha) \mathbf{y}$  and the formula for  $\alpha$ , we obtain  $\delta_j = (y^2 + 1 - c_j^2)/(2y)$  and  $0 \leq \delta_j \leq y$ . Solving for  $y$  gives  $\delta_j^2 + c_j^2 \geq 1$  and

$$(5.16) \quad y = \delta_j + \tau_j, \quad \text{with } \tau_j := (\delta_j^2 + c_j^2 - 1)^{1/2}.$$

Eliminating  $\mathbf{y}$  from  $\mathbf{x}_j = -\alpha \mathbf{y} + \beta \mathbf{e}_j$  and  $\mathbf{d} = \mathbf{y} + c_j \mathbf{x}_j$  gives  $(1 - c_j \alpha) \mathbf{x}_j = -\alpha \mathbf{d} + \beta \mathbf{e}_j$ , and here  $1 - c_j \alpha = \delta_j / y$ . Hence  $c_j \delta_j \mathbf{x}_j = c_j y (\beta \mathbf{e}_j - \alpha \mathbf{d})$ . Using (5.16), we obtain  $c_j \alpha y = \tau_j$  and  $c_j \beta = (1 - \delta_j^2)^{1/2}$ . Therefore

$$(5.17) \quad c_j \delta_j \mathbf{x}_j = (\delta_j + \tau_j)(1 - \delta_j^2)^{1/2} \mathbf{e}_j - \tau_j \mathbf{d}.$$

We take note of two more facts. First:

$$(5.18) \quad \mathbf{d} \cdot \mathbf{e}_j = (c_j \mathbf{x}_j + \mathbf{y}) \cdot \mathbf{e}_j = c_j \beta = (1 - \delta_j^2)^{1/2} > 0.$$

Second:

$$(5.19) \quad \tau_j = 0 \Rightarrow \mathbf{x}_j = \mathbf{e}_j.$$

Indeed,  $\tau_j = 0$  implies  $y = \delta_j = (y^2 + 1 - c_j^2)/(2y)$  by (5.16); thus  $y^2 = 1 - c_j^2$ , giving  $\mathbf{x}_j = \mathbf{e}_j$ .

In the remaining case  $y = 0$ , we have  $c_j = 1$  (since  $y^2 \geq |c_j^2 - 1|$ ) and  $\mathbf{x}_j = \mathbf{e}_j$  (by (5.15), (5.14)); thus also  $\mathbf{d} = \mathbf{e}_j$ ,  $\delta_j = \tau_j = 0$ , and all of (5.16)–(5.19) are still valid.

We now prove the first half of (ii), which asserts that in fact  $\tau_j > 0$  must hold for all  $j$ . Assume  $\tau_i = 0$  for some  $i$ . Then  $\mathbf{x}_i = \mathbf{e}_i$  by (5.19), and now for every  $j \neq i$  we have  $\mathbf{e}_j \cdot \mathbf{e}_i = \mathbf{e}_j \cdot \mathbf{x}_i = 0$ , since  $\mathbf{x}_i \in U_j$ . Similarly  $\mathbf{x}_j \cdot \mathbf{e}_i = 0$ . Therefore  $\tau_j \mathbf{d} \cdot \mathbf{e}_i = 0$ , by (5.17); but  $\mathbf{d} \cdot \mathbf{e}_i > 0$  (cf. (5.18)); hence  $\tau_j = 0$ . It follows that  $\tau_j = 0$  and  $\mathbf{x}_j = \mathbf{e}_j$  for all  $j$ ; hence  $\mathbf{x}_1, \dots, \mathbf{x}_a$  is an orthonormal basis of  $\mathbb{R}^a$ . Then  $1 = |\mathbf{d}|^2 = c_1^2 + \dots + c_a^2$ , which contradicts one of our assumptions. Hence indeed  $\tau_j > 0$  for all  $j$ .

Next, for any  $i \neq j$  in  $\{1, \dots, a\}$ , we compute  $c_i c_j \delta_i \delta_j \mathbf{x}_i \cdot \mathbf{x}_j$  in two different ways. On the one hand, using (5.17) and  $\mathbf{x}_i \cdot \mathbf{e}_j = 0$ , we have

$$(5.20) \quad \begin{aligned} c_i c_j \delta_i \delta_j \mathbf{x}_i \cdot \mathbf{x}_j &= c_i \delta_i \mathbf{x}_i \cdot (-\tau_j \mathbf{d}) = ((\delta_i + \tau_i)(1 - \delta_i^2)^{1/2} \mathbf{e}_i - \tau_i \mathbf{d}) \cdot (-\tau_j \mathbf{d}) \\ &= \tau_j (\tau_i \delta_i^2 + \delta_i^3 - \delta_i), \end{aligned}$$

where in the last equality we used  $|\mathbf{d}| = 1$  and  $\mathbf{e}_i \cdot \mathbf{d} = (1 - \delta_i^2)^{1/2}$ . On the other hand, by symmetry, the same formula holds with  $i$  and  $j$  interchanged. Thus

$$(5.21) \quad \tau_j (\tau_i \delta_i^2 + \delta_i^3 - \delta_i) = \tau_i (\tau_j \delta_j^2 + \delta_j^3 - \delta_j).$$

This holds for all  $i \neq j$ , and dividing through with  $\tau_i \tau_j$ , we have proved (ii).

Let  $t$  be the number  $\delta_j^2 - \tau_j^{-1}(\delta_j - \delta_j^3)$ , which is independent of  $j$ . Let us first assume that  $\delta_\ell = 0$  for some  $\ell$ . Then  $t = 0$ , and also  $\mathbf{d} \cdot \mathbf{e}_\ell = 1$  by (5.18), and since  $|\mathbf{d}| = 1$  this forces  $\mathbf{d} = \mathbf{e}_\ell$ . For each  $j \neq \ell$ , we have  $U_j \neq U_\ell$  and thus  $\delta_j > 0$ . For any  $i \neq j$ , the right-hand side of (5.20) vanishes, since  $t = 0$ , and if further  $i, j \neq \ell$  then we may divide through with  $\delta_i \delta_j$  to conclude that  $\mathbf{x}_i \cdot \mathbf{x}_j = 0$ . Hence  $\{\mathbf{e}_\ell\} \cup \{\mathbf{x}_j : j \neq \ell\}$  is an orthonormal basis of  $\mathbb{R}^a$ . Now, from  $c_\ell \mathbf{x}_\ell = \mathbf{d} - \sum_{j \neq \ell} c_j \mathbf{x}_j = \mathbf{e}_\ell - \sum_{j \neq \ell} c_j \mathbf{x}_j$  it follows that  $c_\ell^2 = 1 + \sum_{j \neq \ell} c_j^2$ , which contradicts our assumption that  $c_\ell^2 < 1 + \sum_{j \neq \ell} c_j^2$ . Hence we conclude that  $\delta_j > 0$  must hold for all  $j$ . Expanding  $1 = |\mathbf{d}|^2 = |\sum_j c_j \mathbf{x}_j|^2$  using (5.20), we now obtain

$$1 = \sum_{j=1}^a c_j^2 + 2 \sum_{i < j} \frac{\tau_i \tau_j}{\delta_i \delta_j} t.$$

In view of our assumption  $\sum c_j^2 > 1$ , this forces  $t < 0$ . Hence  $\tau_j \delta_j < 1 - \delta_j^2$ , or equivalently  $c_j^2 > (\delta_j + \tau_j)^2 - 1$ , for all  $j$ .

Now fix  $j$  again, and write  $\mathbf{y} = \mathbf{d} - c_j \mathbf{x}_j$  and  $y = |\mathbf{y}|$  as before; note that  $y > 0$  since  $\tau_j > 0$ . By (5.16),  $c_j^2 > (\delta_j + \tau_j)^2 - 1$  means that  $c_j^2 > y^2 - 1$ , and this is easily seen to imply that there is some  $\varepsilon > 0$  such that the function  $c \mapsto (y^2 + c^2 - 1)/(2yc)$  is strictly increasing in the interval  $c \in [c_j - \varepsilon, c_j]$ . We have  $y^2 > 1 - c_j^2$  since  $\tau_j > 0$ ; hence, by shrinking  $\varepsilon$  if necessary, we may also assume that  $(y^2 + c^2 - 1)/(2yc) > 0$  for all  $c \in [c_j - \varepsilon, c_j]$ . In particular, taking  $\alpha, \beta$  as in (5.15), and setting, for any given  $c'_j \in (c_j - \varepsilon, c_j)$ ,

$$\alpha' = (2c'_j y^2)^{-1} (y^2 + c'^2_j - 1) \quad \text{and} \quad \beta' = \sqrt{1 - (\alpha' y)^2},$$

we have  $0 < y\alpha' < y\alpha < 1$ , and hence  $\beta' > \beta > 0$ . Now set  $\mathbf{x}'_j = -\alpha' \mathbf{y} + \beta' \mathbf{e}_j$ . Then  $|\mathbf{x}'_j| = 1$  since  $(\alpha' y)^2 + \beta'^2 = 1$ , and  $|\sum_{i \neq j} c_i \mathbf{x}_i + c'_j \mathbf{x}'_j| = |\mathbf{y} + c'_j \mathbf{x}'_j| = 1$  since  $(1 - c'_j \alpha')^2 y^2 + c'^2_j \beta'^2 = 1$ . Hence

$$\begin{aligned} \mathcal{V}_a[c_1, \dots, c'_j, \dots, c_a] &\geq [\mathbf{x}_1, \dots, \mathbf{x}'_j, \dots, \mathbf{x}_a] = \frac{\beta'}{\beta} [\mathbf{x}_1, \dots, \mathbf{x}_a] > [\mathbf{x}_1, \dots, \mathbf{x}_a] \\ &= \mathcal{V}_a[c_1, \dots, c_a], \end{aligned}$$

which concludes the proof of (i).  $\square$

We next establish a monotonicity property of the function  $\mathcal{V}_a[c_1, \dots, c_a]$ .

**Lemma 5.10.** *If  $c_j \geq c'_j > 0$  for  $j = 1, \dots, a$ , then  $\mathcal{V}_a[c_1, \dots, c_a] \leq \mathcal{V}_a[c'_1, \dots, c'_a]$ .*

*Proof.* It suffices to prove that for any fixed  $c_2, \dots, c_a > 0$ ,  $\mathcal{V}_a[c_1, c_2, \dots, c_a]$  is a decreasing function of  $c_1 > 0$ . Without loss of generality, we assume that  $c_2 \geq c_j$  for  $j \geq 3$ . Set

$$\alpha = \max\left(0, 1 - \sum_{j \geq 2} c_j^2, c_2^2 - 1 - \sum_{j \geq 3} c_j^2\right)^{1/2} \quad \text{and} \quad \beta = \left(1 + \sum_{j \geq 2} c_j^2\right)^{1/2}.$$

Then for  $\alpha < c_1 < \beta$ , Lemma 5.9 applies, and part (i) of that lemma, together with the continuity of  $\mathcal{V}_a$  (cf. Lemma 5.8), implies that  $c_1 \mapsto \mathcal{V}_a[c_1, c_2, \dots, c_a]$  is strictly decreasing for  $\alpha < c_1 < \beta$ . In fact this is valid for  $\alpha \leq c_1 \leq \beta$ , again by continuity. Finally, Lemma 5.6 implies that  $c_1 \mapsto \mathcal{V}_a[c_1, c_2, \dots, c_a]$  is decreasing for  $0 < c_1 \leq \alpha$  and for  $c_1 \geq \beta$ , and the proof is complete.  $\square$

The following lemma gives the exact value of  $\mathcal{V}_a[c_1, \dots, c_a]$  when  $c_1 = \dots = c_a$ .

**Lemma 5.11.** *For  $a \geq 1$  and  $0 < c \leq 1$ ,*

$$\tilde{\mathcal{V}}_{a,c} := \mathcal{V}_a[c, \dots, c] = \begin{cases} \sqrt{\frac{c^{-2}(a^2 - c^{-2})^{a-1}}{a^a(a-1)^{a-1}}} & \text{if } c > a^{-1/2}, \\ 1 & \text{if } c \leq a^{-1/2}. \end{cases}$$

*Proof.* The case  $c \leq a^{-1/2}$  follows from Lemma 5.6; hence we now assume  $c > a^{-1/2}$  (and  $a \geq 2$ ). Then Lemma 5.9 applies. Let  $\mathbf{x}_1, \dots, \mathbf{x}_a$  and  $\delta_1, \dots, \delta_a$  be as in the statement of that lemma. Set  $\gamma := 1 - c^2 \in [0, 1)$ . One verifies by differentiation that  $\frac{\delta - \delta^3}{(\delta^2 - \gamma)^{1/2}}$  is a strictly decreasing function of  $\delta$  in the interval  $\sqrt{\gamma} < \delta \leq 1$ ; hence, a fortiori,  $\delta^2 - \frac{\delta - \delta^3}{(\delta^2 - \gamma)^{1/2}}$  is strictly increasing in that interval. Hence Lemma 5.9(ii) implies  $\delta_1 = \dots = \delta_a > \sqrt{\gamma}$ . Using this in the formula (5.20) (wherein  $\tau_i = (\delta_i^2 + c^2 - 1)^{1/2}$ ), it follows that the scalar product  $\mathbf{x}_i \cdot \mathbf{x}_j$  takes one and the same value for all choices of  $i \neq j$ . Call this value  $s$ . It was also seen in the proof of Lemma 5.9 that  $|\mathbf{x}_j| = 1$  for all  $j$ , and  $|\sum_{j=1}^a c\mathbf{x}_j| = 1$ . Squaring and expanding the last relation gives  $c^2(a + a(a-1)s) = 1$ . We have thus proved

$$\mathbf{x}_i \cdot \mathbf{x}_j = s = \frac{c^{-2} - a}{a(a-1)}, \quad \text{for all } i \neq j.$$

Hence

$$\mathcal{V}_a[c, \dots, c] = [\mathbf{x}_1, \dots, \mathbf{x}_a] = \sqrt{D_{a,s}}, \quad \text{with } D_{a,s} := \begin{vmatrix} 1 & s & \cdots & s \\ s & 1 & \cdots & s \\ \vdots & & \ddots & \vdots \\ s & s & \cdots & 1 \end{vmatrix}.$$

Subtracting  $s$  times the first row from each of the other rows, we get

$$D_{a,s} = \begin{vmatrix} 1 - s^2 & s - s^2 & \cdots & s - s^2 \\ s - s^2 & 1 - s^2 & \cdots & s - s^2 \\ \vdots & & \ddots & \vdots \\ s - s^2 & s - s^2 & \cdots & 1 - s^2 \end{vmatrix} = (1 - s^2)^{a-1} D_{a-1, s/(1+s)},$$

and from this one proves by induction that  $D_{a,s} = (as - s + 1)(1 - s)^{a-1}$ . This gives the formula stated in the lemma.  $\square$

The case  $c = 1$  will turn out to be of special importance, and we set

$$(5.22) \quad \tilde{\mathcal{V}}_a := \tilde{\mathcal{V}}_{a,1} = \sqrt{\frac{(a+1)^{a-1}}{a^a}} \quad (a \geq 1); \quad \tilde{\mathcal{V}}_0 := 1.$$

**5.3. Proof of Theorem 1.7.** For  $k = 2$ , the statement of Theorem 1.7 follows from Remark 3.2. Hence, from now on we fix  $k$  to be an integer  $\geq 3$ . We also fix  $c$  and  $f$  as in Theorem 1.7; thus  $0 < c < c_k$ ,  $\lim_{n \rightarrow \infty} f(n) = \infty$  and  $f(n) = O(e^{cn})$ .

The following lemma takes care of all except finitely many terms in (3.2); it is proved using the same bounds as in Rogers, [20, pp. 245–246], which were also used in the proof of Proposition 3.1 above.

**Lemma 5.12.** *The total contribution to (3.2) from all  $D$  which satisfy  $\max\{|d_{ij}|\} \geq \tilde{\mathcal{V}}_{k-1}^{-1}$  (the maximum being taken over all entries of  $D$ ) tends to zero as  $n \rightarrow \infty$ .*

*Remark 5.13.* If  $k \leq 10$  then  $\tilde{\mathcal{V}}_{k-1}^{-1} < 2$ , so that Lemma 5.12 in fact takes care of all  $D$  except those which have  $q = 1$  and all entries  $d_{ij} \in \{-1, 0, 1\}$ .

*Proof.* We fix  $m \in \{1, \dots, k-1\}$ , and consider the contribution from all  $D$  as in the lemma with the further requirement that  $D$  is of size  $m \times k$ . Set  $\Delta := \max\{|d_{ij}|\}$ . Then, by [27, Remark 1] and [20, (72)],

$$\left(\frac{e_1}{q} \cdots \frac{e_m}{q}\right)^n \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^k \chi_n\left(\sum_{i=1}^m \frac{d_{ij}}{q} \mathbf{x}_i\right) d\mathbf{x}_1 \cdots d\mathbf{x}_m \leq f(n)^m \Delta^{-n}.$$

Note that the number of  $\langle k, q \rangle$ -admissible matrices of size  $m \times k$  and with given values of  $q$  and  $\Delta$ , is less than  $\binom{k-1}{m-1} (3\Delta)^{m(k-m)}$ , and there are *no* such matrices with  $\Delta < q$ . Hence, if we let  $v_k$  be the smallest integer  $\geq \tilde{\mathcal{V}}_{k-1}^{-1}$  (thus  $v_k \geq 2$ ), and assume that  $n \geq m(k-m) + 3$ , then the total contribution to (3.2) from all  $D$  with  $q \geq \tilde{\mathcal{V}}_{k-1}^{-1}$  is

$$\leq \binom{k-1}{m-1} f(n)^{m-k/2} \sum_{q=v_k}^{\infty} \sum_{\Delta \geq q} (3\Delta)^{m(k-m)} \Delta^{-n} \ll_k f(n)^{m-k/2} v_k^{-n}.$$

Similarly, assuming  $n \geq m(k-m) + 2$ , the total contribution to (3.2) from all  $D$  satisfying  $q < \tilde{\mathcal{V}}_{k-1}^{-1}$  and  $\Delta \geq \tilde{\mathcal{V}}_{k-1}^{-1}$  (viz.,  $q < v_k$  and  $\Delta \geq v_k$ ) is

$$\leq \binom{k-1}{m-1} f(n)^{m-k/2} \sum_{q=1}^{v_k-1} \sum_{\Delta=v_k}^{\infty} (3\Delta)^{m(k-m)} \Delta^{-n} \ll_k f(n)^{m-k/2} v_k^{-n}.$$

Finally, using  $\lim_{n \rightarrow \infty} f(n) = \infty$  and  $f(n) = O(e^{cn})$  with  $0 < c < c_k$ , the desired convergence is seen to follow from the fact that

$$c\left(m - \frac{k}{2}\right) - \log v_k < c_k\left(\frac{k}{2} - 1\right) + \log \tilde{\mathcal{V}}_{k-1} = 0,$$

cf. (1.7) and (5.22).  $\square$

In the next three lemmas, we let  $D$  be any fixed  $\langle k, q \rangle$ -admissible matrix appearing in the sum in (3.2). (We could assume that  $D$  does not satisfy the condition in Lemma 5.12, but we won't need this.) Let  $m, r, (\mu'_j)_{j=1}^r, (\bar{A}_j)_{j=1}^r, (A_j)_{j=1}^r, (a_j)_{j=1}^r$  be as in Section 5.1.



**Lemma 5.14.** *If  $n \geq \max(a_1, \dots, a_r)$ , then*

$$(5.23) \quad \left(\frac{e_1}{q} \cdots \frac{e_m}{q}\right)^n \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \prod_{j=1}^k \chi_n \left( \sum_{i=1}^m \frac{d_{ij}}{q} \mathbf{x}_i \right) d\mathbf{x}_1 \cdots d\mathbf{x}_m \\ \ll_m n^{m(m+3)/4} f(n)^m \left( \prod_{j=1}^r \tilde{\mathcal{V}}_{a_j} \right)^n.$$

*Proof.* By Lemmas 5.1, 5.3, 5.4, and using  $\sum_{j=1}^r a_j = m$  (thus  $\sum_{j=1}^r a_j^2 \leq m^2$ ), the left-hand side of (5.23) is

$$(5.24) \quad \ll_m n^{m(m+3)/4} f(n)^m \prod_{j=1}^r \left( c_j^n \mathcal{V}_{a_j} [ (|d_{i,\mu'_j}|/q)_{i \in A_j} ]^{n-a_j} \right),$$

where  $c_j := q^{-1} \gcd(\{q\} \cup \{d_{i,\mu'_j} : i \in A_j\})$ , and we use the convention that  $\mathcal{V}_0[\cdot] := 1$ . Using Lemma 5.10 and the fact that  $|d_{i,\mu'_j}| \geq qc_j$  for all  $i \in A_j$ , we have  $\mathcal{V}_{a_j} [ (|d_{i,\mu'_j}|/q)_{i \in A_j} ] \leq \tilde{\mathcal{V}}_{a_j, c_j}$  for each  $j$ . Note that  $0 < c_j \leq 1$  by definition, and thus  $\tilde{\mathcal{V}}_{a_j, c_j} \geq \tilde{\mathcal{V}}_{a_j} \gg_m 1$ . Also, inspecting the formula in Lemma 5.11, one notes that for any fixed  $a \geq 1$ ,  $c \tilde{\mathcal{V}}_{a,c}$  is a strictly increasing function of  $c \in (0, 1]$ ; on the other hand, for each  $j$  with  $a_j = 0$  we have  $c_j = 1$  and  $\tilde{\mathcal{V}}_{a_j, c_j} = 1$ . Using these facts, we see that for each  $j \in \{1, \dots, r\}$ ,

$$(5.25) \quad c_j^n \mathcal{V}_{a_j} [ (|d_{i,\mu'_j}|/q)_{i \in A_j} ]^{n-a_j} \ll_m (c_j \tilde{\mathcal{V}}_{a_j, c_j})^n \leq \tilde{\mathcal{V}}_{a_j}^n.$$

Now (5.23) follows from (5.24) and (5.25).  $\square$

**Lemma 5.15.** *Let  $D$  be as above, and assume furthermore that  $D$  has some column containing more than one non-zero element. Then the contribution from  $D$  to (3.2) tends to zero as  $n \rightarrow \infty$ .*

*Proof.* Recall that Lemma 5.14 is valid for  $\mu'_1, \dots, \mu'_r$  an arbitrary permutation of  $\mu_1, \dots, \mu_r$ . We now fix the choice of  $\mu'_1, \dots, \mu'_r$  so that the number of non-zero elements in column number  $\mu'_1$  is as large as possible. Then  $a_1 = \#A_1 = \#\bar{A}_1 \geq \#\bar{A}_j \geq a_j$  for all  $j \in \{1, \dots, r\}$ , and  $a_1 \geq 2$  by our assumption on  $D$ .

Now note that  $\log(\tilde{\mathcal{V}}_x)$ , which we take to be defined for arbitrary real  $x \geq 1$  through the formula (5.22), is a strictly decreasing and strictly convex function of  $x \geq 1$ . This is easily verified by differentiation. It follows that for any  $j \geq 2$ , if  $a_j \geq 2$  (and thus  $a_1 \geq a_j \geq 2$ ), the product  $\prod_{j=1}^r \tilde{\mathcal{V}}_{a_j}$  increases if we simultaneously replace  $a_1$  by  $a_1 + 1$  and  $a_j$  by  $a_j - 1$ . Repeating this operation for as long as possible, and recalling  $\tilde{\mathcal{V}}_1 = \tilde{\mathcal{V}}_0 = 1$ , we conclude that  $\prod_{j=1}^r \tilde{\mathcal{V}}_{a_j} \leq \tilde{\mathcal{V}}_a$  for some integer  $a \geq a_1 \geq 2$  satisfying  $a + r - 1 \geq m$ , i.e.  $a \geq 2m - k + 1$ . Hence, applying Lemma 5.14 and dividing through by  $f(n)^{k/2}$ , we conclude that the contribution from  $D$  to (3.2) is

$$\ll_m n^{m(m+3)/4} f(n)^{m-k/2} \tilde{\mathcal{V}}_a^n.$$

If  $m \leq k/2$ , then this bound obviously tends to zero as  $n \rightarrow \infty$ , since  $\tilde{\mathcal{V}}_a < 1$  and  $f(n) \rightarrow \infty$ ; hence from now on we assume that  $m > k/2$ . Then, using the assumption  $f(n) = O(e^{cn})$  and the fact that  $\tilde{\mathcal{V}}_a$  is a decreasing function of  $a$ , we see that our term is  $\ll_m n^{m(m+3)/4} \exp((c(m-k/2) + \log \tilde{\mathcal{V}}_{2m-k+1})n)$ , and hence to complete the

proof of the lemma it suffices to prove that

$$(5.26) \quad c < \frac{-2 \log \tilde{\mathcal{V}}_{2m-k+1}}{2m-k}.$$

However, by what we noted above,  $-2 \log \tilde{\mathcal{V}}_{x+1}$  is a strictly concave function of  $x \geq 0$ , taking the value 0 at  $x = 0$ . Also  $2m - k \leq k - 2$ . Hence

$$\frac{-2 \log \tilde{\mathcal{V}}_{2m-k+1}}{2m-k} \geq \frac{-2 \log \tilde{\mathcal{V}}_{k-1}}{k-2} = c_k$$

(cf. (1.7) and (5.22)), and so (5.26) follows from the assumption that  $c < c_k$ .  $\square$

The matrices  $D$  not covered by Lemma 5.15 are very easy to handle:

**Lemma 5.16.** *Let  $D$  be a matrix appearing in (3.2) with exactly one non-zero element in each column. Then either  $D$  is accounted for in  $M_{k,n}$  (cf. (3.3)) or else the contribution from  $D$  to (3.2) tends to zero as  $n \rightarrow \infty$ .*

*Proof.* Let  $\Delta_i := \max(|d_{i1}|, |d_{i2}|, \dots, |d_{ik}|)$  for  $i = 1, \dots, m$ . Then, using [27, Remark 1], we obtain

$$(5.27) \quad \begin{aligned} & \frac{1}{(2f(n))^{k/2}} \left(\frac{e_1}{q} \dots \frac{e_m}{q}\right)^n \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \prod_{j=1}^k \chi_n \left( \sum_{i=1}^m \frac{d_{ij}}{q} \mathbf{x}_i \right) d\mathbf{x}_1 \dots d\mathbf{x}_m \\ & \leq f(n)^{-k/2} q^{-n} \prod_{i=1}^m \left( \int_{\mathbb{R}^n} \chi_n \left( \frac{\Delta_i}{q} \mathbf{x}_i \right) d\mathbf{x}_i \right) = q^{-n} f(n)^{m-k/2} \prod_{i=1}^m \left( \frac{q}{\Delta_i} \right)^n. \end{aligned}$$

Now note that  $k \geq 2m$ , since  $D$  has exactly one non-zero element in each column but at least two non-zero entries in each row. Hence, if we keep  $n$  so large that  $f(n) \geq 1$ , we have  $f(n)^{m-k/2} \leq 1$ . Note also that  $\Delta_i \geq q$  for each  $i$ , since  $D$  is  $\langle k, q \rangle$ -admissible. Furthermore, assuming that  $D$  is *not* accounted for in  $M_{k,n}$ , we have either  $q \geq 2$  or  $q = 1$  at the same time as  $\Delta_i > 1$  for some  $i$ . Hence the bound in (5.27) is  $\leq 2^{-n}$ , and the lemma is proved.  $\square$

*Proof of Theorem 1.7.* Taken together, Lemma 5.12 and Lemmas 5.14–5.16 show that the total contribution from all  $D$  in (3.2) which are not accounted for in  $M_{k,n}$  tends to zero as  $n \rightarrow \infty$ . On the other hand, the treatment of  $M_{k,n}$  in the proof of Proposition 3.1 applies verbatim in the present situation with a more general function  $f$ , and shows that  $\lim_{n \rightarrow \infty} (2f(n))^{-k/2} M_{k,n}$  exists and equals 0 for  $k$  odd and  $(k-1)!!$  for  $k$  even. Hence (1.8) holds.

We now turn to the second statement of Theorem 1.7. Thus assume that  $k \geq 3$  and  $c > c_k$ ; let  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$  be a function satisfying  $f(n) \gg e^{cn}$  as  $n \rightarrow \infty$ , and consider (3.2) with  $\chi_n$  being the characteristic function of the closed ball of volume  $f(n)$  centered at the origin. Then the contribution from any matrix  $D$  as in (5.1) to the sum in (3.2) equals

$$(5.28) \quad 2^{-k/2} f(n)^{\frac{k}{2}-1} V_n^{1-k} J_{k-1}^{(n)}[1, \dots, 1].$$

Now  $c > c_k$  implies that  $e^{(1-\frac{k}{2})c} < \tilde{\mathcal{V}}_{k-1}$  (cf. (1.7) and (5.22)); hence by the second part of Lemma 5.3, the expression in (5.28) tends to  $\infty$  as  $n \rightarrow \infty$ . This completes the proof of Theorem 1.7.  $\square$

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DEPARTMENT OF MATHEMATICS, BOX 480, UPPSALA UNIVERSITY, 751 06 UPPSALA, SWEDEN  
[astrombe@math.uu.se](mailto:astrombe@math.uu.se)

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN, UNIVERSITETSPARKEN  
5, 2100 COPENHAGEN Ø, DENMARK  
*Present address:* SCHOOL OF SCIENCE AND TECHNOLOGY, ÖREBRO UNIVERSITY, 701 82 ÖREBRO,  
SWEDEN  
[anders.sodergren@oru.se](mailto:anders.sodergren@oru.se)