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Projectable Hořava Gravity in 2+1 Dimensions: Phases and Flows

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Abstract. We study the phase structure of projectable Hořava gravity in 2+1 dimensions and consider the implications of recent renormalization group calculations on the phase diagram.

1. Introduction

Next-generation experiments promise to test the assumptions of general relativity (GR) to ever greater precision putting ever tighter bounds on Lorentz-violating effects in gravity [1]. Above and beyond the theoretical aspirations of a potential renormalizable ultraviolet (UV) completion of GR, practicality dictates that we study nonrelativistic theories of quantum gravity in preparation for their collision with data. As a first step, in this work we study a simplified model of Hořava gravity.

Hořava gravity is a Lifshitz-type quantum field theory of gravity, in which time and space scale differently from one another [2]. Unlike the theory of General Relativity, Hořava's theory is power-counting renormalizable and is thus a candidate ultraviolet-complete theory of quantum gravity. Comparisons have also been drawn between Hořava gravity and the lattice approach of Causal Dynamical Triangulations (CDT) [3], which is a Monte Carlo method of simulating quantum gravity on a computer. As a point of comparison to CDT, one natural first observable to consider in Hořava gravity is the ground state, which, at the classical level, is described just by the geometry of the solutions to the equations of motion. We will examine the phases of these solutions as functions of the various parameters in the theory.

While a full-fledged calculation of the renormalization group flow of the theory in general remains a significant challenge, results have been obtained recently for the so-called projectable Hořava gravity theory in $2+1$ dimensions [4, 5]. We describe these results below and comment on their effect on the phase diagram.

2. Hořava Gravity

In Hořava gravity, spacetime is equipped with a global time coordinate t and a foliation structure by leaves of constant time. The natural geometric fields are the lapse function N, the shift vector N_i and the spatial metric g_{ij}. The theory is invariant under diffeomorphisms which preserve the foliation structure. The building blocks with which the action is constructed are the spatial Riemann tensor $R_{ijk\ell}$, the acceleration vector a_i and the extrinsic curvature tensor K_{ij}. The

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latter two are defined as

$$a_i = \frac{\nabla_i N}{N}, \tag{1}$$

$$K_{ij} = \frac{1}{2N}(\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i), \tag{2}$$

where ∇_i is the covariant derivative with respect to g_{ij} and Latin indices are raised and lowered by the spatial metric. Here, we will immediately specialize to the projectable case, in which $N(t, \mathbf{x}) = N(t)$ or $a_i = 0$. In this case, the action is given by

$$S = \frac{1}{2\kappa^2}\int dt\, d^D\mathbf{x}\, N\sqrt{g}\Big(K_{ij}K^{ij} - \lambda K^2 - V(g_{ij}, \nabla_i)\Big), \tag{3}$$

where D is the space dimension and V is a potential function of the metric and connection.

In the vicinity of a Gaussian fixed point, the theory gains an anisotropic scaling symmetry with the dynamical critical exponent z, whereby time and space scale as

$$\begin{aligned} t &\to b^z t, \\ \mathbf{x} &\to b\mathbf{x}. \end{aligned} \tag{4}$$

In the ultraviolet (UV), z is classically the maximum number of pairs of derivatives in a term in V (assuming parity invariance). With classical dimensions adapted to this scaling, terms in V with greater than, equal to, and less than D pairs of derivatives are irrelevant, marginal, and relevant, respectively.

Firstly, in $D = 2$, the potential can only be a function of the Ricci scalar R since the Riemann tensor is is in fact proportional to R. Secondly, considering only marginal and relevant terms,

$$V_{D=2} = \frac{\alpha}{2}R^2 - \beta R + 2\gamma. \tag{5}$$

In $D = 3$, the Riemann tensor depends on both the Ricci scalar and the Ricci tensor. In addition to the terms above, there are three marginal terms of R^3 type and two of $(\nabla R)^2$ type, and one additional relevant term $R_{ij}R^{ij}$. To avoid this proliferation of terms, we will consider only projectable Hořava gravity in $2 + 1$ dimensions given by (3) and (5).

Theories like this, which contain multiple terms quadratic in fields but with different powers of spatial derivative, naturally give rise to multiple types of observers, which measure different scaling dimensions. In our case, the UV observer sets $\alpha = 1$, whereas the IR observer sets $\beta = 1$. The UV observer sees $z = 2$ scaling between time and space, whereas the IR observer sees $z = 1$ scaling and effectively sets $\kappa^2 = 8\pi G_N$, where G_N is Newton's constant, so that the action resembles GR. Note that full diffeomorphism invariance would further require $\lambda = 1$.

3. The Phases

We take the following Friedmann-Lemaître-Robertson-Walker (FLRW) ansatz for the metric,

$$\begin{aligned} N &= 1, \\ N_i &= 0, \\ g_{ij} &= f(t)\hat{g}_{ij}, \end{aligned} \tag{6}$$

where $\hat{g}_{ij}$ is a constant curvature metric with

$$\hat{R}_{ij} = k\hat{g}_{ij}, \tag{7}$$

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and k can be normalized to $k = \pm 1, 0$.

Evaluated on the FLRW ansatz, the Friedmann equation (the N equation of motion) is

$$\kappa^2 f^2 \mathcal{H} = -\varepsilon \dot{f}^2 + \alpha - \beta k f + \gamma f^2 = 0, \tag{8}$$

where $\mathcal{H}$ is the Hamiltonian density and

$$\varepsilon \equiv \frac{2\lambda - 1}{4}. \tag{9}$$

The g_{ij} equation of motion reduces to $\dot{\mathcal{H}}/\dot{f} = 0$, or simply $\dot{\mathcal{H}} = 0$, so long as f is not constant.

Before solving these equations, let us first determine an appropriate set of parameters with which to write the solution. Note that α and κ^2 do not have an independent meaning and only the combination

$$\mathcal{G} \equiv \sqrt{\frac{2\kappa^4}{\alpha}}, \tag{10}$$

matters, where $\mathcal{G}$ is the notation used in [5]. Even if we measure time dimensions T and space dimensions L separately, $\mathcal{G}$ remains dimensionless. The other key dimensionless parameter is ε. Similarly, the appropriate quantity parametrizing the Einstein-Hilbert term in the action is

$$\eta \equiv \frac{\beta}{2\kappa^2}, \tag{11}$$

which has dimensions of $[\eta] = T^{-1}$. A convenient measure of the cosmological constant is

$$\Omega \equiv \frac{2\gamma}{\mathcal{G}^2} = \frac{\alpha\gamma}{\kappa^4}, \tag{12}$$

which has dimensions of $[\Omega] = T^{-2}$. It also turns out to be convenient to define

$$\begin{aligned} \tau &\equiv \frac{\mathcal{G}}{\sqrt{2\varepsilon}} t, \\ \zeta(\tau) &\equiv \frac{\mathcal{G}^2}{2\kappa^2} f(t). \end{aligned} \tag{13}$$

With this definition, $\zeta(\tau)$ has dimension $[\zeta] = T$.

Firstly, when $\varepsilon = 0$, the solution is constant

$$\zeta(\tau)|_{\varepsilon=0} = \frac{k}{\eta}, \tag{14}$$

as long as this is positive and $\eta^2 = \Omega$. Henceforth, we consider $\varepsilon \neq 0$.

3.1. Real Time

Real time corresponds to $\mathcal{G}^2$ and ε having the same sign. We assume that they are both positive since the case when they are both negative can easily be retrieved by analytic continuation.

Consider the three regions:

$$\begin{aligned} &\text{Region 1: } \Omega < 0, \\ &\text{Region 2: } \eta^2 > \Omega > 0, \\ &\text{Region 3: } \Omega > \eta^2 > 0. \end{aligned} \tag{15}$$

In region 1, the solution for $\zeta(\tau)$ reads

$$\zeta_1(\tau) = \frac{1}{|\Omega|}\left[-\eta k + |k|\sqrt{\eta^2 + |\Omega|}\,\cos\left(\sqrt{|\Omega|}\,\tau\right)\right]. \tag{16}$$

In region 2, there are two potential solutions

$$\zeta_2(\tau) = \frac{1}{\Omega}\left[\eta k \pm |k|\sqrt{\eta^2 - \Omega}\,\cosh\left(\sqrt{\Omega}\,\tau\right)\right]. \tag{17}$$

If $\eta k > 0$, then the + solution is an eternal universe that is infinite flat space in the infinite past, curves into a sphere ($k > 0$) or hyperbolic plane ($k < 0$) of decreasing size up to a minimum at $\tau = 0$, and then grows infinitely again towards the future. In contrast, the − solution is the direct analytic continuation of the big bang/big crunch solution (16). On the other hand, if $\eta k < 0$, the − solution is not permissible since it is always negative, and the + solution consists of two disconnected pieces: a big crunch, then a big bang.

In region 3, two solutions exist: one is an ever-growing big bang and the other is the time-reversed version (big crunch),

$$\zeta_3(\tau) = \frac{1}{\Omega}\left[\eta k \pm |k|\sqrt{\Omega - \eta^2}\,\sinh\left(\sqrt{\Omega}\,\tau\right)\right]. \tag{18}$$

3.2. Imaginary Time

Imaginary time corresponds to $\mathcal{G}^2$ and ε having opposite sign. Suppose $\mathcal{G}^2 > 0$ and $\varepsilon < 0$ (the other choice is related by analytic continuation).

Writing $\tau = i\tau_E$, the solution in region 1 is the same as (16) but with τ replaced with τ_E and cos replaced with cosh. This is similar to the eternal universe solution in region 2 in the real time case (17). The solution in region 2 is the same as (17) but with τ again replaced with τ_E and with a cos instead of cosh. The two signs ± are actually identical now and there is only one solution, which is oscillating forever between a maximum and minimum radius. Additionally, only spherical solutions ($k > 0$) are allowed when $\eta > 0$ and only hyperbolic solutions ($k < 0$) are allowed when $\eta < 0$. Finally, and perhaps most interestingly, no solution exists in region 3. In this case, we call region 3 the "Forbidden Region".

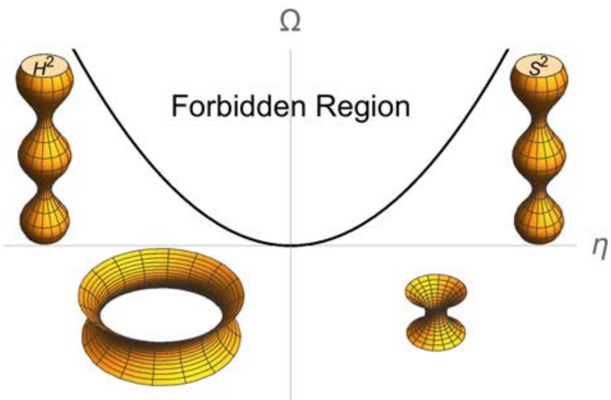

Figure 1. Imaginary time phase diagram. Region 1 ($\Omega < 0$) is flat space at $\tau = \pm\infty$ and curves to a minimum radius at $\tau = 0$. Region 2 ($\eta^2 > \Omega > 0$) is oscillating spheres for $\eta > 0$ and hyperbolic disks for $\eta < 0$. No solution of the form (6) exists in the Forbidden Region.

4. The Flows

The renormalization group flow within the subspace $(\mathcal{G}, \lambda)$ of dimensionless couplings was computed in [5] with the results

$$\begin{aligned} \beta_{\mathcal{G}} &= -\frac{16 - 33\lambda + 18\lambda^2}{64\pi(1-\lambda)^2}\sqrt{\frac{1-\lambda}{1-2\lambda}}\,\mathcal{G}^2, \\ \beta_\lambda &= \frac{15 - 14\lambda}{64\pi}\sqrt{\frac{1-2\lambda}{1-\lambda}}\,\mathcal{G}. \end{aligned} \tag{19}$$

There is an asymptotically free ($\mathcal{G} = 0$) UV fixed at $\lambda = \frac{15}{14}$. For $1 < \lambda < \frac{15}{14}$, the theory flows towards strong coupling at the GR value of $\lambda = 1$. For $\lambda > \frac{15}{14}$, the theory flows towards ever increasing λ and $\mathcal{G}$. There is also an interesting family of UV fixed points at the value $\lambda = \frac{1}{2}$ parametrized by the value of the coupling $\tilde{\mathcal{G}} = \mathcal{G}/\sqrt{1-2\lambda}$.

At each point on this two-dimensional RG flow diagram sits a two-dimensional phase diagram in the dimensionful couplings (η, Ω). The effect of the values of $\mathcal{G}$ and λ are parametrized in the definitions of τ and $\zeta(\tau)$ in (13). At the mean-field level, the RG flows in the (η, Ω) plane are controlled by the dimensions of η and Ω and, since $[\Omega] = T^{-2} = [\eta^2]$, the flow lines are simply parabolas from the origin. Taking one-loop effects into account, it was shown in [4] that η does not flow, but Ω gains an anomalous dimension (measured in units of momentum)

$$\delta_\Omega = \frac{1}{32\pi}\left(\frac{1-2\lambda}{1-\lambda}\right)^{3/2}\mathcal{G}. \tag{20}$$

Therefore, instead of parabolas $\Omega \propto \eta^2$, the flow lines are altered to

$$\Omega \propto \eta^{2-\frac{\delta_\Omega}{2}}. \tag{21}$$

5. Discussion

There is a striking similarity between the phase diagram in imaginary time of projectable Hořava gravity in 3 dimensions and that of CDT in 4 dimensions (after Wick rotation). The eternal universe in region 1 corresponds to the de Sitter-like phase (phase C) of CDT. The oscillating phase in region 2 with $\eta > 0$ corresponds to the oscillating phase (phase A) of CDT.

The forbidden region would correspond to the "pancake" phase (phase B) of CDT in which the entire universe lives inside one time step. Thus, one may conjecture that the forbidden region represents a "topological phase of lapse" in which $N \to 0$ and the spatial metric is static. Indeed, when measured with respect to any finite lapse, the history of this static universe collapses to zero time. In this case, the Hamiltonian is no longer forced to vanish and should, in principle, be minimized in the ground state. Why the Hamiltonian should be deconfined away from 0 as one crosses the phase transition line into the forbidden region is an outstanding mystery.

The identification of a fourth phase (the so-called "bifurcation" phase or phase D) in CDT between phases B and C naturally leads one to identify this with the oscillating hyperbolic phase in region 2 with $\eta < 0$. Of course, CDT cannot produce non-compact universes, but it instead seems to observe a metric signature change [6].

This comparison begs the question of the difference in spacetime dimension between our analysis herein and the simulations of CDT. If CDT were indeed simulating Hořava gravity in $3 + 1$ dimensions, either a zoo of phases have eluded observation or CDT is seeing signs of a fixed point at which many of the couplings vanish. To examine this possibility further, we would need the RG flow of the full theory in $3 + 1$ dimensions, which is hopefully forthcoming.

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References
[1] Williams J, Chiow S, Mueller H and Yu N 2016 *New J. Phys.* **18** 025018
[2] Hořava P 2009 *J. High Energy Phys.* JHEP03(2009)020
[3] Ambjørn J, Görlich A, Jordan S, Jurkiewicz J and Loll R 2010 *Phys. Lett.* B **690** 413-9
[4] Griffin T, Grosvenor K T, Melby-Thompson C M and Yan Z 2017 *J. High Energy Phys.* JHEP06(2017)004
[5] Barvinsky A O, Blas D, Herrero-Valea M, Sibiryakov S M and Steinwachs C F 2016 *Phys. Rev.* D **93** 064022
[6] Ambjørn J, Coumbe D N, Gizbert-Studnicki J and Jurkiewicz J 2015 *J. High Energy Phys.* JHEP08(2015)033