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Published in:
Chicago Journal of Theoretical Computer Science

DOI:
10.4086/cjtcs.2018.005

Publication date:
2018

Document version
Publisher's PDF, also known as Version of record

CHICAGO JOURNAL OF THEORETICAL COMPUTER SCIENCE 2018, Article 05, pages 1–16
http://cjtcs.cs.uchicago.edu/

The border support rank of two-by-two matrix multiplication is seven

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Received June 14, 2017; Revised September 25, 2017, and in final form August 21, 2018; Published August 25, 2018

Abstract: We show that the border support rank of the tensor corresponding to two-by-two matrix multiplication is seven over the complex numbers. We do this by constructing two polynomials that vanish on all complex tensors with format four-by-four-by-four and border rank at most six, but that do not vanish simultaneously on any tensor with the same support as the two-by-two matrix multiplication tensor. This extends the work of Hauenstein, Ikenmeyer, and Landsberg. We also give two proofs that the support rank of the two-by-two matrix multiplication tensor is seven over any field: one proof using a result of De Groote saying that the decomposition of this tensor is unique up to sandwiching, and another proof via the substitution method. These results answer a question asked by Cohn and Umans. Studying the border support rank of the matrix multiplication tensor is relevant for the design of matrix multiplication algorithms, because upper bounds on the border support rank of the matrix multiplication tensor lead to upper bounds on the computational complexity of matrix multiplication, via a construction of Cohn and Umans. Moreover, support rank may be interpreted as a quantum communication complexity measure.

Key words and phrases: matrix multiplication, border support rank, algebraic complexity theory

1 Introduction

Multiplication of two $n \times n$ matrices over a field $\mathbb{F}$ is an $\mathbb{F}$-bilinear map $\mathbb{F}^{n\times n} \times \mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n}$ called the matrix multiplication map. The matrix multiplication map corresponds naturally to the following

*Supported by European Research Council (ERC Grant Agreement no. 337603), the Danish Council for Independent Research (Sapere Aude), and VILLUM FONDEN via the QMATH Centre of Excellence (Grant no. 10059)
†Supported by NWO (617.023.116)

 DOI: 10.4086/cjtcs.2018.005

structure tensor. Let $[n]$ be the set $\{1,2,\dots,n\}$ and let $\{e_{ij} : i,j \in [n]\}$ be the standard basis for the vector space $\mathbb{F}^{n\times n}$ of $n\times n$ matrices. Define the structure tensor of the matrix multiplication map as

$$\langle n,n,n\rangle := \sum_{i,j,k\in[n]} e_{ij}\otimes e_{jk}\otimes e_{ki} \in \mathbb{F}^{n\times n}\otimes\mathbb{F}^{n\times n}\otimes\mathbb{F}^{n\times n}.$$

(Technically, this is the structure tensor of the trilinear map that computes the trace of a product of three matrices.) Let V_1, V_2, and V_3 be vector spaces. The tensor rank of a tensor $t\in V_1\otimes V_2\otimes V_3$ is the smallest number r such that t can be written as a sum of r simple tensors $v_1\otimes v_2\otimes v_3\in V_1\otimes V_2\otimes V_3$. The computational complexity of matrix multiplication is tightly related to the tensor rank of the tensor $\langle n,n,n\rangle$ (see e.g. [5]). Strassen showed that the tensor rank of $\langle 2,2,2\rangle$ is at most seven over any field [23]; Hopcroft and Kerr [13] showed that the tensor rank is at least seven over the finite field $\mathbb{F}_2$, and Winograd [25] showed that the tensor rank is at least seven over any field. Over an algebraically closed field, the border rank of a tensor $t\in V_1\otimes V_2\otimes V_3$ is the smallest number r such that t is in the Zariski closure of all tensors of rank at most r in $V_1\otimes V_2\otimes V_3$. Landsberg proved that the border rank of $\langle 2,2,2\rangle$ is seven over the field $\mathbb{C}$ of complex numbers [16], and a different proof for this based on highest-weight vectors was later given by Hauenstein, Ikenmeyer and Landsberg [12].

We extend the above results. Let $t\in V_1\otimes V_2\otimes V_3$ be a tensor in a fixed basis, a hypermatrix. The *support* of t is the set of coordinates where t has a nonzero coefficient. The *support rank* of t is the minimal rank of a tensor with the same support as t. This has also been called s-rank [7], nondeterministic rank [9], zero-one rank [24] and minimum rank of a nonzero pattern [2] in the literature. The *border support rank* of t is the minimal border rank of a tensor with the same support as t. We prove the following.

Theorem 1.1. *The support rank of $\langle 2,2,2\rangle$ is seven over any field $\mathbb{F}$.*

Theorem 1.2. *The border support rank of $\langle 2,2,2\rangle$ is seven over $\mathbb{C}$.*

Theorem 1.1 and Theorem 1.2 answer a question of Cohn and Umans [7], that was also posed as an open problem during the Algorithms and Complexity in Algebraic Geometry programme at the Simons Institute [1]. We note that, in general, computing the tensor rank or support rank of a tensor is a computationally hard task. Namely, given a 3-tensor t and a natural number r, deciding whether the tensor rank of t is at most r is NP-complete over any finite field [11] and NP-hard over any integral domain [22]. Moreover, given a 2-tensor (that is, a matrix) A and a natural number r, deciding whether the support rank of A is at most r is NP-hard over the real numbers [3].

Previously, it was known that the border support rank of the matrix multiplication tensor $\langle n,n,n\rangle$ is at least $2n^2-n$ [4], so in particular that the border support rank of $\langle 2,2,2\rangle$ is at least six. This result was obtained using Young flattenings.

Studying (border) support rank is interesting for two reasons. The first reason comes from algebraic complexity theory. As mentioned above, the tensor rank of the matrix multiplication tensor is tightly related to the computational complexity of matrix multiplication. It turns out that asymptotically, the border support rank of matrix multiplication gives an upper bound on the tensor rank of matrix multiplication, as follows. The exponent of matrix multiplication ω is defined as the smallest number β such that for any $\varepsilon>0$ the tensor rank of $\langle n,n,n\rangle$ is in $\mathcal{O}(n^{\beta+\varepsilon})$. The number ω is between 2 and 2.3728639 [18] and it is a major open problem in algebraic complexity theory to decide whether ω equals 2. One can define an analogous quantity ω_s for the support rank of $\langle n,n,n\rangle$. One can show with Strassen's laser method that $\omega \leq (3\omega_s - 2)/2$ [7]. To show that $\omega = 2$, it therefore suffices to show that $\omega_s = 2$. Cohn and Umans aim to obtain upper bounds on ω_s by realizing the algebra of $n \times n$ matrices inside some cleverly chosen group algebra.

The second reason, which was our original motivation, comes from quantum communication complexity. Let $f : X \times Y \times Z \to \{0,1\}$ be a function on a product of finite sets X, Y and Z. Alice, Bob and Charlie have to compute f in the following sense. Alice receives an $x \in X$, Bob receives a $y \in Y$ and Charlie receives a $z \in Z$. Moreover, the players share a so-called Greenberger–Horne–Zeilinger (GHZ) state of rank r, which is described by the tensor $\mathrm{GHZ}_r = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i \in (\mathbb{C}^r)^{\otimes 3}$. The players apply local quantum operations. After this, each player has to output a bit such that if $f(x,y,z) = 1$, then with some nonzero probability all players output 1 and if $f(x,y,z) = 0$, then with probability zero all players output 1. The complexity of such a protocol is the logarithm of the rank r of the GHZ-state used, and the minimum complexity of all quantum protocols for f is the *nondeterministic communication complexity* of f. This number equals the logarithm of the support rank of the tensor with support given by f, that is $\sum_{x,y,z} f(x,y,z)\, e_x \otimes e_y \otimes e_z$ [4]. Similarly, the logarithm of the border support rank of the tensor with support given by f equals the *approximate* nondeterministic communication complexity of f. Since tensor rank and border rank are natural measures of entanglement, our result may also be of interest to the quantum information theory community.

Notation. For any tensor t, we will denote tensor rank by $\mathrm{R}(t)$, border rank by $\underline{\mathrm{R}}(t)$, support rank by $\mathrm{R}_{\mathrm{s}}(t)$ and border support rank by $\underline{\mathrm{R}}_{\mathrm{s}}(t)$.

Paper outline. This paper is structured as follows. In Section 2 we give two proofs for Theorem 1.1. In Section 3 we give a short introduction to border rank lower bounds by highest-weight vectors and then apply this theory to prove Theorem 1.2.

2 Support rank

We will give two proofs for Theorem 1.1. Both proofs use the following lemma that reduces the 8-parameter minimization problem at hand to a 1-parameter minimization problem. Let $\mathbb{F}$ be a field. Let $e_{11} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $e_{12} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $e_{21} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $e_{22} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ be the standard basis of the space of 2×2 matrices $\mathbb{F}^{2\times 2}$ over $\mathbb{F}$. Let e_1, e_2, e_3, e_4 be the standard basis of $\mathbb{F}^4$. We naturally identify $\mathbb{F}^{2\times 2}$ with $\mathbb{F}^4$ by $e_{11} \mapsto e_1$, $e_{12} \mapsto e_2$, $e_{21} \mapsto e_3$, $e_{22} \mapsto e_4$. Let $\mathrm{GL}_4(\mathbb{F})^{\times 3}$ act on the tensor space $\mathbb{F}^{2\times 2} \otimes \mathbb{F}^{2\times 2} \otimes \mathbb{F}^{2\times 2}$ accordingly.

Lemma 2.1 (Parameter reduction)**.** *Let $t \in \mathbb{F}^{2\times 2} \otimes \mathbb{F}^{2\times 2} \otimes \mathbb{F}^{2\times 2}$ be a tensor with the same support as the matrix multiplication tensor $\langle 2,2,2\rangle$. There is a tensor s in the $\mathrm{GL}_4(\mathbb{F})^{\times 3}$-orbit of t, with the same support as t, such that all nonzero entries of s are 1 except possibly for the coefficient of $e_{11} \otimes e_{11} \otimes e_{11}$.*

Proof. Identify the tensor $\langle 2,2,2\rangle = \sum_{i,j,k\in[2]} e_{ij} \otimes e_{jk} \otimes e_{k\ell}$ with the tensor

$$e_{111} + e_{123} + e_{231} + e_{243} + e_{312} + e_{324} + e_{432} + e_{444} \in \mathbb{F}^4 \otimes \mathbb{F}^4 \otimes \mathbb{F}^4,$$

where $e_{ijk} = e_i \otimes e_j \otimes e_k$. We can view this tensor as as a $4 \times 4 \times 4$ cube filled with elements 0 and 1 from $\mathbb{F}$. Let t be a tensor in $\mathbb{F}^4 \otimes \mathbb{F}^4 \otimes \mathbb{F}^4$ with the same support as $\langle 2,2,2 \rangle$, so, in 1-slices,

$$t = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & d & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & e & 0 & 0 \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & 0 & h \end{bmatrix}$$

where a,b,c,d,e,f,g,h are nonzero elements in $\mathbb{F}$. Here we index the 1-slices by the first tensor leg, the rows of the slices by the second tensor leg and columns of the slices by the third tensor leg. Scaling the 1-slices of t according to $\operatorname{diag}(1/b,1/d,1/f,1/h)$, that is, applying $\operatorname{diag}(1/b,1/d,1/f,1/h) \otimes \mathbf{1}_4 \otimes \mathbf{1}_4$ to t, yields a tensor of the form

$$t' = \begin{bmatrix} a' & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c' & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & e' & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & g' & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling the rows of t' according to $\operatorname{diag}(1/e',1,1/g',1)$, that is, applying $\mathbf{1}_4 \otimes \operatorname{diag}(1/e',1,1/g',1) \otimes \mathbf{1}_4$ to t', yields a tensor of the form

$$t'' = \begin{bmatrix} a'' & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c'' & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, scaling the columns of t'' according to $\operatorname{diag}(1/c'',1,1,1)$, i.e., applying $\mathbf{1}_4 \otimes \mathbf{1}_4 \otimes \operatorname{diag}(1/c'',1,1,1)$ to t'', yields a tensor of the required form. □

Our first proof of Theorem 1.1 uses a corollary of a result of De Groote on the uniqueness of the decomposition of $\langle 2,2,2 \rangle$ into simple tensors.

Theorem 2.2 ([8, Remark 4.2]). *Let $v := v_1 \otimes v_2 \otimes v_3 \in \mathbb{F}^{2\times 2} \otimes \mathbb{F}^{2\times 2} \otimes \mathbb{F}^{2\times 2}$ be an element of an arbitrary optimal decomposition of $\langle 2,2,2 \rangle$ into simple tensors over $\mathbb{F}$ such that the rank of each v_i as an element of $\mathbb{F}^2 \otimes \mathbb{F}^2$ is one. Then there exist invertible matrices $A,B,C \in \mathrm{GL}_2(\mathbb{F})$ such that we have $v = (A^{-1}e_{11}B) \otimes (B^{-1}e_{11}C) \otimes (C^{-1}e_{11}A)$, where A, B and C act by matrix multiplication from the left and right on $\mathbb{F}^{2\times 2}$.*

Definition 2.3. For any number $q \in \mathbb{F}$, define the perturbed matrix multiplication tensor

$$\langle 2,2,2 \rangle_q := \langle 2,2,2 \rangle + (q-1)\, e_{11} \otimes e_{11} \otimes e_{11}.$$

This is the tensor obtained from $\langle 2,2,2 \rangle$ by replacing the coefficient of $e_{11} \otimes e_{11} \otimes e_{11}$ by q.

We now give our first proof of Theorem 1.1 using the above uniqueness statement.

Proof of Theorem 1.1; uniqueness argument. As was already observed by De Groote, Theorem 2.2 gives the upper bound $\mathrm{R}(\langle 2,2,2\rangle_0) \le 6$ and thus $\mathrm{R}(\langle 2,2,2\rangle_q) \le 7$ for all $q \in \mathbb{F}$. We claim $\mathrm{R}(\langle 2,2,2\rangle_q) \ge 7$ for all nonzero $q \in \mathbb{F}$. Suppose q is a number in $\mathbb{F}$ such that $\mathrm{R}(\langle 2,2,2\rangle_q) = 6$. Suppose we have a decomposition $\langle 2,2,2\rangle_q = \sum_{i=1}^{6} u_i \otimes v_i \otimes w_i$ into simple tensors. Then

$$\langle 2,2,2\rangle = \langle 2,2,2\rangle_q + (1-q)\, e_{11} \otimes e_{11} \otimes e_{11} = \sum_{i=1}^{6} u_i \otimes v_i \otimes w_i + (1-q)\, e_{11} \otimes e_{11} \otimes e_{11}$$

is an optimal decomposition of $\langle 2,2,2\rangle$ into simple tensors. Therefore, by Theorem 2.2, there exist A, B, C in $\mathrm{GL}_2(\mathbb{F})$ such that

$$A^{-1} e_{11} B \otimes B^{-1} e_{11} C \otimes C^{-1} e_{11} A = (1-q)\, e_{11} \otimes e_{11} \otimes e_{11}.$$

Let f_1, f_2 be the standard basis of $\mathbb{F}^2$. Then by taking appropriate transposes the previous equation is equivalent to

$$A^{-1} f_1 \otimes B^T f_1 \otimes B^{-1} f_1 \otimes C^T f_1 \otimes C^{-1} f_1 \otimes A^T f_1 = (1-q)\, f_1^{\otimes 6},$$

which implies that A^T, B^T, C^T each have eigenvector f_1. Let α, β, γ be the respective eigenvalues. Then A^{-1}, B^{-1}, C^{-1} have eigenvalues $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$. This yields the equation $\alpha^{-1} \beta \beta^{-1} \gamma \alpha \gamma^{-1} = 1-q$. We conclude that $q = 0$. By Theorem 2.1 we can conclude that $\mathrm{R_s}(\langle 2,2,2\rangle) = 7$. □

Our second proof of Theorem 1.1 uses a method called the substitution method. Let x_{ij}, y_{ij}, z_{ij} ($i, j \in [2]$) be variables. Let X, Y, Z be the corresponding 2×2 variable matrices. For $q \in \mathbb{F}$, define the function

$$f_q(X,Y,Z) \coloneqq \sum_{i,j,k \in [2]} x_{ij}\, y_{jk}\, z_{ki} + (q-1)\, x_{22}\, y_{22}\, z_{22}.$$

The tensor rank of $\langle 2,2,2\rangle_q$ is equal to the smallest number r such that $f_q(X,Y,Z)$ can be written as a sum $\sum_{\rho=1}^{r} u_\rho(X) v_\rho(Y) w_\rho(Z)$, where u_ρ is a linear form in the x_{ij}, similarly for v_ρ and w_ρ.

Proof of Theorem 1.1; substitution method. Suppose that the function $f_q(X,Y,Z)$ has rank r in the sense that it has a decomposition into a sum of r products of three linear forms as described above. If $U = (u_{ij})_{ij \in [2]}$ is any upper triangular matrix, then $f_q(X, YU^{-1}, UZ)$ has rank at most r and by direct computation

$$f_q(X, YU^{-1}, UZ) = f_q(X,Y,Z) + \frac{u_{12}}{u_{11}} (q-1)\, x_{22}\, y_{21}\, z_{22}.$$

There exists an upper triangular matrix U such that the function $g_q(X,Y,Z) \coloneqq f_q(X, YU^{-1}, UZ)$ has a decomposition

$$g_q(X,Y,Z) = \sum_{\rho=1}^{r} u_\rho(X) v_\rho(Y) w_\rho(Z) \tag{2.1}$$

in which $w_r(Z)$ is of the form $z_{21} + a_{12}z_{12} + a_{22}z_{22}$ for some $a_{12}, a_{22} \in \mathbb{F}$. Apply the substitution $z_{21} \mapsto \tilde{w}(Z) := -a_{12}z_{12} - a_{22}z_{22}$ to (2.1) to see that

$$\sum_{j\in[2]} \big(x_{1j}\,y_{j1}\,z_{11} + x_{2j}\,y_{j1}\,z_{12} + x_{1j}\,y_{j2}\,\tilde{w}(Z)\big) + x_{21}\,y_{12}\,z_{22} + q\,x_{22}\,y_{22}\,z_{22} + \frac{u_{12}}{u_{11}}(q-1)\,x_{22}\,y_{21}\,z_{22} \\ = \sum_{\rho=1}^{r-1} u_\rho(X)v_\rho(Y)w_\rho\Big(\Big[\begin{smallmatrix} z_{11} & z_{12} \\ \tilde{w}(Z) & z_{22}\end{smallmatrix}\Big]\Big). \tag{2.2}$$

We can test that y_{22} occurs in the obtained decomposition of (2.2) by setting x_{22}, z_{22} to 1 and y_{21} to 0 and the other x_{ij}, z_{ij} to 0. We can test that y_{12} occurs in the obtained decomposition of (2.2) by setting x_{21}, z_{22} to 1 and the other x_{ij}, z_{ij} to 0. Say y_{22} occurs in v_{r-1} and y_{12} occurs in v_{r-2}. Then, there is a substitution $y_{12} \mapsto \tilde{v}_{12}(Y)$, $y_{22} \mapsto \tilde{v}_{22}(Y)$, which, applied to (2.2) yields

$$\sum_{j\in[2]} \big(x_{1j}\,y_{j1}\,z_{11} + x_{2j}\,y_{j1}\,z_{12} + x_{1j}\,\tilde{v}_{j2}(Y)\,\tilde{w}(Z)\big) + x_{21}\,\tilde{v}_{12}(Y)\,z_{22} + q\,x_{22}\,\tilde{v}_{22}(Y)\,z_{22} + \frac{u_{12}}{u_{11}}(q-1)\,x_{12}\,y_{21}\,z_{22} \\ = \sum_{\rho=1}^{r-3} u_\rho(X)v_\rho\Big(\Big[\begin{smallmatrix} y_{11} & \tilde{v}_{12}(Y) \\ y_{21} & \tilde{v}_{22}(Y)\end{smallmatrix}\Big]\Big)w_\rho\Big(\Big[\begin{smallmatrix} z_{11} & z_{12} \\ \tilde{w}(Z) & z_{22}\end{smallmatrix}\Big]\Big). \tag{2.3}$$

To clean up, setting $z_{22} \mapsto 0$ in (2.3) shows that

$$\sum_{j\in[2]} x_{1j}\,y_{j1}\,z_{11} \;+\; x_{2j}\,y_{j1}\,z_{12} \;+\; x_{1j}\,\tilde{v}_{j2}(Y)\,\tilde{w}(Z) \;=\; \sum_{\rho=1}^{r-3} u_\rho(X)v_\rho\Big(\Big[\begin{smallmatrix} y_{11} & \tilde{v}_{12}(Y) \\ y_{21} & \tilde{v}_{22}(Y)\end{smallmatrix}\Big]\Big)w_\rho\Big(\Big[\begin{smallmatrix} z_{11} & z_{12} \\ \tilde{w}(Z) & 0\end{smallmatrix}\Big]\Big). \tag{2.4}$$

We can test that x_{21} occurs in the obtained decomposition (2.4) by setting y_{11}, z_{12} to 1 and the other x_{ij}, z_{ij} to 0. Similarly, we can test that x_{22} occurs in the obtained decomposition (2.4) by setting y_{21}, z_{12} to 1 and the other x_{ij}, z_{ij} to 0. Say x_{21} occurs in u_{r-3} and x_{22} occurs in u_{r-4}. We apply a substitution $x_{21} \mapsto \tilde{u}_{21}(X)$, $x_{22} \mapsto \tilde{u}_{22}(X)$ to see that

$$\sum_{j\in[2]} x_{1j}\,y_{j1}\,z_{11} + \tilde{u}_{2j}(X)\,y_{j1}\,z_{12} + x_{1j}\,\tilde{v}_{j2}(Y)\,\tilde{w}(Z) \\ = \sum_{\rho=1}^{r-5} u_\rho\Big(\Big[\begin{smallmatrix} x_{11} & x_{12} \\ \tilde{u}_{21}(X) & \tilde{u}_{22}(X)\end{smallmatrix}\Big]\Big)v_\rho\Big(\Big[\begin{smallmatrix} y_{11} & \tilde{v}_{12}(Y) \\ y_{21} & \tilde{v}_{22}(Y)\end{smallmatrix}\Big]\Big)w_\rho\Big(\Big[\begin{smallmatrix} z_{11} & z_{12} \\ \tilde{w}(Z) & 0\end{smallmatrix}\Big]\Big). \tag{2.5}$$

Apply the substitution $z_{12} \mapsto 0$ to (2.5) to get

$$\sum_{j\in[2]} x_{1j}\,y_{j1}\,z_{11} = \sum_{\rho=1}^{r-5} u_\rho\Big(\Big[\begin{smallmatrix} x_{11} & x_{12} \\ \tilde{u}_{21}(X) & \tilde{u}_{22}(X)\end{smallmatrix}\Big]\Big)v_\rho\Big(\Big[\begin{smallmatrix} y_{11} & \tilde{v}_{12}(Y) \\ y_{21} & \tilde{v}_{22}(Y)\end{smallmatrix}\Big]\Big)w_\rho\Big(\Big[\begin{smallmatrix} z_{11} & 0 \\ \tilde{w}(Z) & 0\end{smallmatrix}\Big]\Big). \tag{2.6}$$

which clearly has rank 2. Therefore, $r \geq 7$. By Theorem 2.1 we are done. □

3 Border support rank

In this section all vector spaces are complex vector spaces. We will review a method that was introduced in [17] to study equations for border rank and that was later used in [12] to give a proof that $\underline{\mathrm{R}}(\langle 2,2,2\rangle) \geq 7$. Then, we will use this method to show that the border support rank of $\langle 2,2,2\rangle$ equals seven. Our Python code is included as an ancillary file with the arXiv submission.

View the space $\otimes^3\mathbb{C}^n$ as an affine variety, and let $\mathbb{C}[\otimes^3\mathbb{C}^n]$ be its coordinate ring. Define $\sigma_r \subseteq \otimes^3\mathbb{C}^n$ as the subset of tensors with border rank at most r,

$$\sigma_r := \{s \in \otimes^3\mathbb{C}^n : \underline{\mathrm{R}}(s) \leq r\}.$$

This is called the rth secant variety of the Segre variety of $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n$. The set σ_r is Zariski closed in $\otimes^3\mathbb{C}^n$ by definition of border rank. In other words, if we let $I(\sigma_r) \subseteq \mathbb{C}[\otimes^3\mathbb{C}^n]$ be the ideal of polynomials on $\otimes^3\mathbb{C}^n$ that vanish identically on σ_r, then $Z(I(\sigma_r)) = \sigma_r$.

3.1 Lower bounds by polynomials

By definition, if $\underline{\mathrm{R}}(t) > r$ then there exists a polynomial in $I(\sigma_r)$ that does not vanish on t. The following standard proposition says that we may in fact assume that this polynomial is homogeneous.

Proposition 3.1. *Let $t \in \otimes^3\mathbb{C}^n$. If $\underline{\mathrm{R}}(t) > r$, then there exists a homogeneous polynomial f in $I(\sigma_r)$ such that $f(t) \neq 0$.*

Proof. We give a proof for the convenience of the reader. If $f(t) = 0$ for all $f \in I(\sigma_r)$, then we have $t \in Z(I(\sigma_r)) = \sigma_r$, which is a contradiction. Let f be a polynomial in $I(\sigma_r)$ such that $f(t) \neq 0$. Let $f = \sum_d f_d$ be the decomposition of f into homogeneous parts. There is a d such that $f_d(t) \neq 0$.

Let $v \in \sigma_r$. For any $\alpha \in \mathbb{C}$, define $g(\alpha) := f(\alpha v)$. This is a polynomial in α. We have that $g(\alpha) = \sum_d \alpha^d f_d(v)$. Since σ_r is closed under scaling and $f(v) = 0$, we have $g(\alpha) = 0$ for any $\alpha \in \mathbb{C}$, so g is the zero polynomial. Therefore, each coefficient $f_d(v)$ is 0. This argument holds for any $v \in \sigma_r$, so $f_d \in I(\sigma_r)$ for each d. □

The polynomial ring $\mathbb{C}[\otimes^3\mathbb{C}^n]$ decomposes into a direct sum of homogeneous parts $\mathbb{C}[\otimes^3\mathbb{C}^n]_d$ and, by the above argument, the vanishing ideal $I(\sigma_r)$ decomposes accordingly as $I(\sigma_r) = \oplus_d I(\sigma_r)_d$ with $I(\sigma_r)_d \subseteq \mathbb{C}[\otimes^3\mathbb{C}^n]_d$.

The space $\otimes^3\mathbb{C}^n$ has a natural action of $G := \mathrm{GL}_n^{\times 3}$ and σ_r is a G-submodule. Thus $\mathbb{C}[\otimes^3\mathbb{C}^n]_d \cong \mathrm{Sym}^d(\otimes^3(\mathbb{C}^n)^*)$ has a natural action of G and $I(\sigma_r)_d$ is a G-submodule. We will use the well-known theory of highest-weight vectors to exploit this symmetry. The theory of highest-weight vectors holds in a much more general setting than we need here. We refer to [15, III.1.5] and [10] for the general theory, and focus on a description of the theory for the group $\mathrm{GL}_n^{\times 3}$.

Let W be a finite-dimensional G-module. Choose a basis so that G becomes the group of triples of invertible matrices. Let $T \subseteq G$ be the subgroup of triples of diagonal matrices. For

$$t = (\mathrm{diag}(a_1,\ldots,a_n), \mathrm{diag}(b_1,\ldots,b_n), \mathrm{diag}(c_1,\ldots,c_n)) \in T$$

and $z = (u, v, w) \in (\mathbb{Z}^n)^3$ define $t^z := \prod_{i=1}^n a_i^{u_i} b_i^{v_i} c_i^{w_i}$. As a T-module, W decomposes into weight spaces,

$$W = \bigoplus_{z \in (\mathbb{Z}^n)^3} W_z \qquad \text{where} \quad W_z = \{w \in W : t \cdot w = t^z w \;\forall t \in T\}.$$

The vectors in W_z are said to have *weight* z. Let $U \subseteq G$ be the subgroup of triples of unipotent matrices, that is, upper triangular matrices with ones on the diagonal. A nonzero vector $v \in W_z$ is a *highest-weight vector* if $u \cdot v = v$ for all $u \in U$.

A finite-dimensional (rational) representation W of $\mathrm{GL}_n^{\times 3}$ is irreducible if and only if it has a unique highest-weight vector v, up to multiplication by a scalar, that is, $[W]^U = \mathrm{Span}_{\mathbb{C}}\, v$. If W is irreducible and v is a highest-weight vector, then one has $W = \mathrm{Span}_{\mathbb{C}}(Gv)$. Moreover, two irreducible representations are isomorphic if and only if their highest-weight vectors have the same weight. We call a sequence of n nonincreasing integers a generalized partition. It turns out that the weight of a highest-weight vector is a triple of generalized partitions. For any triple of generalized partitions λ, we will denote an abstract realisation of the G-module with highest-weight λ by V_λ. For any finite-dimensional G-module W, the highest-weight vectors in W of weight λ form a vector space, which we denote by $[W_\lambda]^U$.

For a generalized partition λ, define the dual partition λ^* as the generalized partition obtained from λ by negating every entry and reversing the order. Then $V_{\lambda^*} = V_\lambda^*$, the dual module. We note that the polynomial irreducible representations are precisely the ones that are isomorphic to V_λ with λ a partition.

Recall that σ_r is the variety of tensor in $\otimes^3\mathbb{C}^n$ of border rank at most r. Consider the isotypic decomposition of $W := \mathrm{Sym}^d(\otimes^3(\mathbb{C}^n)^*)$ and $I(\sigma_r)_d$ under the action of $\mathrm{GL}_n^{\times 3}$,

$$\begin{aligned} W &= \bigoplus_{\lambda \vdash d} W_{\lambda^*} &&= \bigoplus_{\lambda \vdash d} k(\lambda)\, V_\lambda^*, \\ I(\sigma_r)_d &= \bigoplus_{\lambda \vdash d} I(\sigma_r)_{\lambda^*} &&= \bigoplus_{\lambda \vdash d} m(\lambda)\, V_\lambda^*, \end{aligned}$$

where λ runs over all *triples* of partitions of d with at most n parts, and $k(\lambda)\, V_\lambda^*$ denotes an isotypic component consisting of a direct sum of $k(\lambda)$ copies of the irreducible G-representation V_λ^*, similarly for $m(\lambda)\, V_\lambda^*$. Note that, although the direct sums run over triples of partitions λ, the representations W and $I(\sigma_r)$ are not polynomial since we take duals. The number $k(\lambda)$ is exactly the dimension of the highest-weight vector space $[W_{\lambda^*}]^U$, and the number $m(\lambda)$ is the dimension of the highest-weight vector space $[I(\sigma_r)_{\lambda^*}]^U$. The following proposition extends Theorem 3.1 by saying that we may assume that the polynomial we are looking for is a highest-weight vector, if we replace t by a random point in its G-orbit.

Proposition 3.2. *Let $t \in \otimes^3\mathbb{C}^n$. If $\underline{\mathrm{R}}(t) > r$, then there exists a highest-weight vector $f \in I(\sigma_r)$ and a group element $g \in G$ such that $f(gt) \neq 0$.*

Proof. We provide the proof for the convenience of the reader. By Theorem 3.1, there exists a homogeneous polynomial $f \in I(\sigma_r)$ such that we have $f(t) \neq 0$. By highest-weight theory, the polynomial f can be written as a sum $\sum_{\lambda,i} g_{\lambda,i} f_{\lambda,i}$, where $f_{\lambda,i}$ is a highest-weight vector of type λ in $I(\sigma_r)$ and $g_{\lambda,i} \in G$. Since $f(t) \neq 0$, there exists a λ and an i so that $f_{\lambda,i}(g_{\lambda,i}^{-1} t) \neq 0$. □

3.2 Highest-weight vector method

The following method was first proposed in [17] to study equations for border rank and was later used in [12] to give a proof that $\underline{\mathrm{R}}(\langle 2,2,2\rangle) \geq 7$. Let $t \in \otimes^3\mathbb{C}^n$ be a tensor for which we want to show $\underline{\mathrm{R}}(t) > r$.

1. Choose a degree $d \in \mathbb{N}$. Let W be the space $\mathrm{Sym}^d(\otimes^3(\mathbb{C}^n)^*)$. Choose a partition triple $\lambda \vdash d$ such that the highest-weight vector space $[W_{\lambda^*}]^U$ is nonzero.

2. Construct a basis $b_1,\ldots,b_k$ for $[W_{\lambda^*}]^U$.

3. Find a linear combination f of the basis elements $b_1,\ldots,b_k$ that vanishes on all tensors of border rank at most r, that is, $f \in [I(\sigma_r)_{\lambda^*}]^U$ where σ_r is the variety of tensors with border rank at most r.

4. Show that f does not vanish on gt for some $g \in G$.

The above method is guaranteed to work by Theorem 3.2. Before applying the method, we will consider each step in more detail.

Step 1. Kronecker coefficient. The dimension of the space of U-invariants $[(\mathrm{Sym}^d(\otimes^3(\mathbb{C}^n)^*))_{\lambda^*}]^U$ is the so-called *Kronecker coefficient* $k(\lambda)$. We pick a partition triple λ such that the number $k := k(\lambda)$ is nonzero. Algorithms for computing Kronecker coefficients have been implemented in for example Schur [26], Sage [20] and the Python package Kronecker [6].

Step 2. Las Vegas construction of basis. For any natural number $\ell \leq n$, let $\phi_\ell := e_1^* \wedge \cdots \wedge e_\ell^*$ be the Slater determinant living in $\wedge^\ell(\mathbb{C}^n)^*$. For any partition $\mu \vdash d$ with at most n parts, we let ϕ_μ denote the tensor $\phi_{\nu_1} \otimes \cdots \otimes \phi_{\nu_{\mu_1}}$ living in $\otimes^d(\mathbb{C}^n)^*$, where ν denotes the transpose of μ. Let $\lambda = (\lambda^{(1)},\lambda^{(2)},\lambda^{(3)})$ be a triple of partitions of d. We define $\phi_\lambda := \phi_{\lambda^{(1)}} \otimes \phi_{\lambda^{(2)}} \otimes \phi_{\lambda^{(3)}}$. This tensor lives in $\otimes^3 \otimes^d (\mathbb{C}^n)^*$, but we view it as a tensor in $\otimes^d \otimes^3 (\mathbb{C}^n)^*$ via the canonical reordering. Let P_d be the canonical symmetrizer $\otimes^d \otimes^3 (\mathbb{C}^n)^* \to \mathrm{Sym}^d(\otimes^3(\mathbb{C}^n)^*)$ acting from the right. The group $S_d^{\times 3}$ has a natural right action on $\otimes^3 \otimes^d (\mathbb{C}^n)^*$ and via the reordering also on $\otimes^d \otimes^3 (\mathbb{C}^n)^*$. Let λ be a triple of partitions of d. The tensors $\{\phi_\lambda \pi P_d : \pi \in S_d^{\times 3}\}$ span the vector space $[(\mathrm{Sym}^d(\otimes^3(\mathbb{C}^n)^*))_{\lambda^*}]^U$, see [14, 4.2.17].

We construct a basis of $[(\mathrm{Sym}^d(\otimes^3(\mathbb{C}^n)^*))_{\lambda^*}]^U$ as follows. Randomly pick k permutation pairs $\tau_1,\ldots,\tau_k \in S_d^{\times 2}$. Let $e \in S_d$ be the identity permutation. Let $\pi_i = (e, \tau_i^{(1)}, \tau_i^{(2)})$ and let $b_i := \phi_\lambda \pi_i P_d$. Pick k random tensors $w_1,\ldots,w_k$ in $\otimes^3\mathbb{C}^n$ and evaluate every b_i in every w_j, giving a k-by-k evaluation matrix M. If M has full rank, then $(b_1,\ldots,b_k)$ is the desired basis.

Before going to the next step we discuss how to efficiently implement the evaluation of a polynomial represented by a pair of permutations, as was already described in [12]. Let $f = \phi_\lambda \pi P_d$ and let t be the tensor $\sum_{i=1}^r t_i^1 \otimes t_i^2 \otimes t_i^3$ in $\otimes^3\mathbb{C}^n$. The evaluation of the polynomial f at t is equal to the contraction

$$\begin{aligned}
\phi_\lambda \pi P_d t^{\otimes d} &= \phi_\lambda \pi t^{\otimes d} \\
&= \sum_{j \in [r]^d} \phi_\lambda \pi (t_{j_1}^1 \otimes t_{j_1}^2 \otimes t_{j_1}^3) \otimes \cdots \otimes (t_{j_d}^1 \otimes t_{j_d}^2 \otimes t_{j_d}^3) \\
&= \sum_{j \in [r]^d} \phi_{\lambda^{(1)}}(t_{j_1}^1 \otimes \cdots \otimes t_{j_d}^1) \\
&\qquad \cdot \phi_{\lambda^{(2)}}\, \tau^{(1)}(t_{j_1}^2 \otimes \cdots \otimes t_{j_d}^2) \\
&\qquad \cdot \phi_{\lambda^{(3)}}\, \tau^{(2)}(t_{j_1}^3 \otimes \cdots \otimes t_{j_d}^3).
\end{aligned}$$

Note that the last expression is a sum of a product of determinants. Let us study the first factor of a summand. Let ν denote the transpose of $\lambda^{(1)}$. We have

$$\begin{aligned}\phi_{\lambda^{(1)}}(t^1_{j_1} \otimes \cdots \otimes t^1_{j_d}) &= (\phi_{\nu_1} \otimes \cdots \otimes \phi_{\nu_{\mu_1}})(t^1_{j_1} \otimes \cdots \otimes t^1_{j_d}) \\ &= \det_{\nu_1}(t^1_{j_1}, \ldots, t^1_{j_{\nu_1}}) \det_{\nu_2}(t^1_{j_{\nu_1}}, \ldots, t^1_{j_{\nu_1+\nu_2}}) \\ &\quad \cdots \det_{\nu_{\mu_1}}(t^1_{j_{d-\nu_{\mu_1}}}, \ldots, t^1_{j_d}),\end{aligned}$$

where $\det_m(v_1, \ldots, v_m)$ denotes top m-by-m minor of the matrix with columns $v_1, \ldots, v_m$. Suppose that, in our evaluation of $\sum_j$, we have chosen values for $j_1, \ldots, j_{\nu_1}$ and suppose $\det_{\nu_1}(t^1_{j_1}, \ldots, t^1_{j_{\nu_1}})$ is 0. Then whatever choices we make for $j_{\nu_1+1}, \ldots, j_d$, the summand at hand will be zero. Recognizing this situation early is crucial.

Step 3. Construction of a vector in $I(\sigma_r)$. Pick k random tensors $t_1, \ldots, t_k$ of rank r. Evaluate each basis element b_i in each random tensor t_j. If the resulting matrix $(b_i(t_j))_{i,j\in[k]}$ has a nontrivial kernel, then we find a candidate highest-weight vector f in $I(\sigma_r)$. We can verify the correctness of the candidate by evaluating f at a symbolic tensor of rank r. This evaluation should be zero. The way we do this symbolic evaluation is by working in $\otimes^3\mathbb{C}^6$ and using the straightening algorithm, see e.g. the SchurFunctors package in Macaulay2 [21]. We used multi-prolongation to split up the computation in order to save memory. We refer to [17, 19] for a discussion of multi-prolongation.

Step 4. Evaluating at gt. Evaluate f at gt for a random $g \in G$. (In our case, it turns out that taking g to be the identity is good enough.)

3.3 The matrix multiplication tensor

We will now prove that $\underline{\mathrm{R}}_s(\langle 2,2,2\rangle) = 7$.

Proof of Theorem 1.2. The upper bound follows from Theorem 1.1, so it remains to prove the lower bound. Let σ_6 be the variety of tensors in $\otimes^3\mathbb{C}^4$ of border rank at most 6. We will apply the method described above to the tensor $\langle 2,2,2\rangle_q$, see Theorem 2.3.

Let $d = 20$ and let λ be the partition triple $(5,5,5,5)^3$. The Kronecker coefficient $k(\lambda)$ equals 4. Let $W := \mathrm{Sym}^{20}(\otimes^3(\mathbb{C}^4)^*)$ and denote by W_{λ^*} the isotypic component of type λ^*. Writing permutations in the one-line notation, the following pairs of permutations define a basis (b_1, b_2, b_3, b_4) for the highest-weight vector space $[W_{\lambda^*}]^U$:

$$\begin{aligned}\pi_1 = (&[5,14,8,2,12,0,1,15,6,11,18,13,4,3,9,17,7,10,16,19],\\ &[14,5,9,0,6,13,16,15,4,11,3,10,12,8,2,17,7,19,18,1]),\end{aligned}$$

$$\begin{aligned}\pi_2 = (&[11,18,2,12,10,5,1,17,19,9,3,4,7,6,13,0,14,16,15,8],\\ &[19,1,2,7,8,3,13,6,17,10,18,12,15,4,5,11,16,0,14,9]),\end{aligned}$$

$$\begin{aligned}\pi_3 = (&[2,16,17,1,4,0,7,5,10,14,11,6,18,15,9,12,19,13,3,8],\\ &[15,9,0,11,19,16,18,7,2,13,5,6,17,14,8,1,12,4,10,3]),\end{aligned}$$

$$\begin{aligned}\pi_4 = (&[9,12,14,2,6,19,18,3,15,0,1,5,11,17,7,16,8,4,13,10],\\ &[14,4,18,3,11,16,15,12,5,0,17,2,10,9,13,19,7,6,1,8]).\end{aligned}$$

The polynomial $f_{20} = 11832g_1 + 233074g_2 + 34117g_3 - 32732g_4$ is the only linear combination of the basis elements that is in $I(\sigma_6)$, up to scaling. We verified that f_{20} is indeed in $I(\sigma_6)$ with the straightening algorithm. Evaluating f_{20} on $\langle 2,2,2\rangle_q$ yields

$$f_{20}(\langle 2,2,2\rangle_q) = -730140480(q+1)q^2.$$

Let $d = 19$ and let λ be the partition triple $(5,5,5,4)^3$. The Kronecker coefficient $k(\lambda)$ equals 31. Let $W := \mathrm{Sym}^{19}(\otimes^3(\mathbb{C}^4)^*)$ and denote by W_{λ^*} the isotypic component of type λ^*. The following pairs of permutations define a basis $(b_1,\ldots,b_{31})$ for the highest-weight vector space $[W_{\lambda^*}]^U$:

$$\begin{aligned}
\pi_1 = (&[4,8,13,3,1,12,5,11,9,15,2,7,0,17,14,6,10,18,16],\\
&[2,18,5,7,9,13,0,12,1,15,10,8,4,11,16,3,17,6,14]),
\end{aligned}$$

$$\begin{aligned}
\pi_2 = (&[12,15,11,7,2,6,8,17,9,1,16,13,4,0,3,10,18,14,5],\\
&[11,9,14,0,15,13,16,3,6,8,17,7,10,5,18,2,12,1,4]),
\end{aligned}$$

$$\begin{aligned}
\pi_3 = (&[14,1,2,15,6,3,7,13,4,18,8,9,12,10,16,5,17,0,11],\\
&[7,18,2,10,4,12,0,9,15,6,5,13,1,17,14,16,8,3,11]),
\end{aligned}$$

$$\begin{aligned}
\pi_4 = (&[4,1,0,12,7,13,9,16,6,8,18,15,17,11,14,2,10,3,5],\\
&[5,13,17,14,3,4,6,11,8,18,1,15,2,0,9,16,7,10,12]),
\end{aligned}$$

$$\begin{aligned}
\pi_5 = (&[11,14,5,0,15,8,2,17,1,13,4,9,16,6,7,10,18,3,12],\\
&[8,18,4,14,6,16,10,2,11,9,5,0,13,12,1,7,3,17,15]),
\end{aligned}$$

$$\begin{aligned}
\pi_6 = (&[10,5,18,8,15,2,16,1,0,13,3,4,7,14,11,6,12,17,9],\\
&[0,8,12,2,3,9,11,13,5,1,14,7,4,16,17,18,15,10,6]),
\end{aligned}$$

$$\begin{aligned}
\pi_7 = (&[12,1,11,16,13,7,2,17,10,15,3,0,5,4,14,6,9,8,18],\\
&[8,1,4,2,12,14,18,15,7,9,0,11,3,10,6,17,13,5,16]),
\end{aligned}$$

$$\begin{aligned}
\pi_8 = (&[17,18,6,11,4,2,1,9,15,16,5,8,10,0,12,13,3,14,7],\\
&[14,1,18,6,10,15,3,5,11,16,12,9,13,7,0,17,8,4,2]),
\end{aligned}$$

$$\begin{aligned}
\pi_9 = (&[8,2,10,3,6,4,11,18,13,0,5,1,15,17,12,16,14,7,9],\\
&[2,5,13,16,1,10,3,14,4,17,18,12,0,11,9,6,7,8,15]),
\end{aligned}$$

$$\begin{aligned}
\pi_{10} = (&[13,17,15,1,12,0,9,10,6,18,7,16,14,5,2,4,11,8,3],\\
&[6,12,11,10,2,14,13,0,9,15,16,17,5,8,3,7,1,18,4]),
\end{aligned}$$

$$\begin{aligned}
\pi_{11} = (&[14,5,4,1,16,8,3,7,10,13,18,6,2,17,11,9,15,12,0],\\
&[5,9,10,1,2,4,14,18,8,11,7,6,15,17,16,3,0,13,12]),
\end{aligned}$$

$$\begin{aligned}
\pi_{12} = (&[1,5,4,13,15,2,17,16,8,10,11,6,7,3,12,14,9,0,18],\\
&[9,5,7,8,6,11,18,3,10,4,14,17,13,0,12,15,16,1,2]),
\end{aligned}$$

$$\begin{aligned}
\pi_{13} = (&[16,13,4,3,5,2,1,15,18,6,12,0,14,8,17,7,10,11,9],\\
&[2,7,8,18,16,4,6,14,0,15,9,5,1,12,10,13,17,11,3]),
\end{aligned}$$

$$\pi_{14} = ([5,12,0,9,3,7,17,2,6,14,11,8,15,4,1,10,13,18,16],\\ [5,15,18,8,17,11,9,4,13,1,16,2,0,14,7,10,12,3,6]),$$

$$\pi_{15} = ([12,6,9,14,18,5,17,2,1,4,3,11,0,10,15,7,16,13,8],\\ [9,1,16,18,14,5,6,0,10,13,3,7,15,4,11,17,12,2,8]),$$

$$\pi_{16} = ([1,18,4,8,5,3,0,16,6,10,11,2,17,7,9,12,14,13,15],\\ [8,2,15,12,18,6,0,11,13,5,9,4,16,7,10,17,14,1,3]),$$

$$\pi_{17} = ([18,8,16,6,5,7,2,13,0,4,12,11,14,15,3,17,1,10,9],\\ [12,9,14,2,18,5,0,13,4,16,8,7,1,10,6,3,17,11,15]),$$

$$\pi_{18} = ([7,5,16,15,1,0,8,11,14,17,12,6,9,3,10,18,13,4,2],\\ [8,9,0,4,2,3,5,13,18,12,6,1,16,11,17,10,14,7,15]),$$

$$\pi_{19} = ([2,17,0,14,15,8,1,9,12,5,10,3,7,11,4,16,6,13,18],\\ [13,3,0,15,7,17,18,10,6,16,1,8,9,14,12,4,5,2,11]),$$

$$\pi_{20} = ([0,16,9,3,15,1,4,14,7,2,18,10,12,11,17,8,6,5,13],\\ [3,2,13,11,8,1,5,4,0,16,7,17,6,12,14,9,18,15,10]),$$

$$\pi_{21} = ([17,3,5,14,0,16,2,8,1,11,7,18,12,6,9,15,4,13,10],\\ [7,2,17,8,0,13,6,1,4,5,18,9,15,10,16,11,3,14,12]),$$

$$\pi_{22} = ([5,4,1,14,16,3,9,17,12,8,2,6,11,7,18,15,13,0,10],\\ [6,14,8,7,9,18,3,12,15,2,0,1,13,5,10,16,4,11,17]),$$

$$\pi_{23} = ([17,4,10,13,14,1,6,8,5,15,9,2,0,11,18,7,3,12,16],\\ [6,3,11,12,15,17,10,2,8,5,1,0,14,7,9,18,13,4,16]),$$

$$\pi_{24} = ([3,9,0,15,14,7,1,16,2,8,11,4,17,12,10,6,18,13,5],\\ [10,11,3,2,1,9,14,13,18,16,0,4,15,8,5,12,6,7,17]),$$

$$\pi_{25} = ([12,2,8,6,16,1,15,9,11,14,10,3,5,17,0,13,18,4,7],\\ [8,2,14,1,6,17,16,3,7,9,11,12,18,0,5,13,15,10,4]),$$

$$\pi_{26} = ([2,16,14,6,9,0,11,12,3,15,1,18,17,7,4,8,13,5,10],\\ [10,15,13,12,17,0,16,7,4,11,1,2,6,14,8,5,9,3,18]),$$

$$\pi_{27} = ([10,7,6,0,12,11,16,13,1,3,17,14,8,18,4,2,9,5,15],\\ [3,17,11,12,6,5,2,13,18,14,9,1,7,16,4,8,10,15,0]),$$

$$\pi_{28} = ([16,6,8,4,7,5,9,1,0,2,14,13,17,10,18,15,11,3,12],\\ [6,11,1,12,2,8,5,9,3,16,15,18,4,7,14,0,10,17,13]),$$

$$\pi_{29} = ([8,13,7,0,17,4,2,15,16,1,18,3,5,11,12,10,6,14,9],\\ [4,13,1,10,18,12,2,5,17,7,6,15,8,9,0,11,16,14,3]),$$

$$
\begin{aligned}
\pi_{30} = (&[1,6,12,0,3,10,9,13,17,4,7,8,18,14,2,5,15,16,11],\\
&[16,6,10,11,15,8,17,13,14,4,5,1,3,12,2,7,0,18,9]),\\
\pi_{31} = (&[5,10,11,8,17,16,2,15,12,14,0,18,3,1,7,9,6,4,13],\\
&[10,15,4,12,18,3,16,6,0,13,11,7,1,8,9,2,14,17,5]).
\end{aligned}
$$

Let

$$
\begin{aligned}
c_1 &= 289082199568614200505625810989998081122378290025627334\\
c_2 &= 41448548699164679707399349100915823812613974963005402\\
c_3 &= 211649838021887426162677078824519293749517217920047823\\
c_4 &= -118150576713220917823141541211872001702845422153137763\\
c_5 &= -71972591371289085208000082313759547126396087856917092\\
c_6 &= -148042611712972282129069557835544665097810271759437007\\
c_7 &= -20671385701071233448917086723379921457752823704368686\\
c_8 &= -41700697565765737458921317121977791710351222967960389\\
c_9 &= 89818454969459149830510070194701368406615458716738371\\
c_{10} &= -33389561951163547125931836395846743479037338582546746\\
c_{11} &= -55953034618025281839233784369005651793756337420914611\\
c_{12} &= 99436050816695444459576518293215696786461418941439932\\
c_{13} &= -30608800079918651823012662681016076665421200200986429\\
c_{14} &= 62322369796163233078186315204176712499710334162812978\\
c_{15} &= 71531123200873494604907676681446086219352685074695096\\
c_{16} &= 11103950876950753893392891180499777390516447716768874\\
c_{17} &= -18170416924354926777786745151805158474424942420073625\\
c_{18} &= 56636600557844043196391811853778001287738236566321291\\
c_{19} &= -49475697236538461568207568070821224602714314684182556\\
c_{20} &= -58897567946922439319826816178640661508235201647724834\\
c_{21} &= -29789369352552042959878217935401203848547004115080562\\
c_{22} &= 42553086095082787553533988614363448520647296308373860\\
c_{23} &= -10584947869810207513601472123471095674362492708851758\\
c_{24} &= -155536179226293398590182659612811187764949236460651258\\
c_{25} &= -15163630056597008306009257387099740416829146255166469\\
c_{26} &= 152468055855066906135282920200590542819196123610118125\\
c_{27} &= -170101205621738870358375711649013594303036219144235962\\
c_{28} &= -36619800006361115328892590783407206736313224654320560\\
c_{29} &= 63636824324804825079032794300460871506246849887804488
\end{aligned}
$$

$$c_{30} = -11442265501801519315039163142435000064529397796113574 0$$
$$c_{31} = \quad 99270978701207213884119395668714341424298017907910144$$

and define $f_{19} = c_1 g_1 + \cdots + c_{31} g_{31}$. This is the only linear combination that is in $I(\sigma_6)$, up to scaling. We verified that f_{19} is in $I(\sigma_6)$ by straightening. Evaluating f_{19} at $\langle 2,2,2\rangle_q$ yields

$$69332245782016022615247261570208505413020193878724712262(3q+2)q.$$

We have thus found two highest-weight vectors

$$f_{19} \in [I(\sigma_6)_{(5,5,5,4)^{3*}}]^U \subseteq \mathrm{Sym}^{19}(\otimes^3(\mathbb{C}^4)^*)$$
$$f_{20} \in [I(\sigma_6)_{(5,5,5,5)^{3*}}]^U \subseteq \mathrm{Sym}^{20}(\otimes^3(\mathbb{C}^4)^*)$$

such that $f_{19}(\langle 2,2,2\rangle_q) = \alpha\, q(3q+2)$ and $f_{20}(\langle 2,2,2\rangle_q) = \beta\, q^2(q+1)$, where α and β are nonzero constants. The only simultaneous root of these polynomials occurs at $q = 0$. This means that for any nonzero q, the point $\langle 2,2,2\rangle_q$ is not contained in σ_6. From Theorem 2.1 we conclude that the border rank of any tensor with the same support as $\langle 2,2,2\rangle$ is at least seven, which proves the theorem. □

Remark 3.3. The lower bound $\underline{\mathrm{R}}(\langle 2,2,2\rangle) \geq 7$ in [12] was also obtained by showing on the one hand that the highest-weight vector space $[I(\sigma_6)_{(5,5,5,5)^{3*}}]^U$ is nonzero, and on the other hand that the evaluation of a nonzero element $v \in [I(\sigma_6)_{(5,5,5,5)^{3*}}]^U$ at $\langle 2,2,2\rangle$ is nonzero.

Acknowledgements. The authors are grateful to Christian Ikenmeyer for helpful discussions.

References

[1] Algorithms and complexity in algebraic geometry, Simons Institute, fall 2014: Computational open questions. https://wiki.simons.berkeley.edu/doku.php?id=ag14:start. 2

[2] AVI BERMAN, SHMUEL FRIEDLAND, LESLIE HOGBEN, URIEL G. ROTHBLUM, AND BRYAN SHADER: Minimum rank of matrices described by a graph or pattern over the rational, real and complex numbers. *Electron. J. Combin.*, 15(1):Research Paper 25, 19, 2008. 2

[3] AMEY BHANGALE AND SWASTIK KOPPARTY: The complexity of computing the minimum rank of a sign pattern matrix. *arXiv*, 2015. [arXiv:1503.04486] 2

[4] HARRY BUHRMAN, MATTHIAS CHRISTANDL, AND JEROEN ZUIDDAM: Nondeterministic quantum communication complexity: the cyclic equality game and iterated matrix multiplication. *Proceedings of the 2017 ACM Conference on Innovations in Theoretical Computer Science*, 2017. [doi:10.4230/LIPIcs.ITCS.2017.24, arXiv:1603.03757] 2, 3

[5] PETER BÜRGISSER, MICHAEL CLAUSEN, AND M. AMIN SHOKROLLAHI: Algebraic complexity theory, volume 315 of Grundlehren der Mathematischen Wissenschaften, 1997. 2

[6] MATTHIAS CHRISTANDL, BRENT DORAN, AND MICHAEL WALTER: Computing Multiplicities of Lie Group Representations. In *2012 IEEE 53rd Annual Symposium on Foundations of Computer Science (FOCS)*, pp. 639–648, 2012. Software available at `http://www.leetspeak.org/kronecker/`. [doi:10.1109/FOCS.2012.43, arXiv:1204.4379] 9

[7] HENRY COHN AND CHRISTOPHER UMANS: Fast matrix multiplication using coherent configurations. In *Proceedings of the Twenty-fourth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '13, pp. 1074–1086, Philadelphia, PA, USA, 2013. Society for Industrial and Applied Mathematics. [arXiv:1207.6528] 2, 3

[8] HANS F. DE GROOTE: On varieties of optimal algorithms for the computation of bilinear mappings II. optimal algorithms for 2×2-matrix multiplication. *Theoret. Comput. Sci.*, 7(2):127 – 148, 1978. [doi:10.1016/0304-3975(78)90045-2] 4

[9] RONALD DE WOLF: Nondeterministic quantum query and communication complexities. *SIAM J. Comput.*, 32(3):681–699, 2003. [doi:10.1137/S0097539702407345, arXiv:cs/0001014] 2

[10] BRIAN HALL: *Lie groups, Lie algebras, and representations*. Volume 222 of *Graduate Texts in Mathematics*. Springer, second edition, 2015. [doi:10.1007/978-3-319-13467-3] 7

[11] JOHAN HÅSTAD: Tensor rank is NP-complete. *J. Algorithms*, 11(4):644–654, 1990. [doi:10.1016/0196-6774(90)90014-6] 2

[12] JONATHAN D. HAUENSTEIN, CHRISTIAN IKENMEYER, AND JOSEPH M. LANDSBERG: Equations for lower bounds on border rank. *Exp. Math.*, 22(4):372–383, 2013. [doi:10.1080/10586458.2013.825892, arXiv:1305.0779] 2, 7, 9, 14

[13] JOHN E. HOPCROFT AND LESLIE R. KERR: On minimizing the number of multiplications necessary for matrix multiplication. *SIAM J. Appl. Math.*, 20(1):30–36, 1971. [doi:10.1137/0120004] 2

[14] CHRISTIAN IKENMEYER: *Geometric complexity theory, tensor rank, and Littlewood-Richardson coefficients*. Ph. D. thesis, Universität Paderborn, 2013. 9

[15] HANSPETER KRAFT: *Geometrische Methoden in der Invariantentheorie*. Springer, 1984. [doi:10.1007/978-3-663-10143-7] 7

[16] JOSEPH M. LANDSBERG: The border rank of the multiplication of 2×2 matrices is seven. *J. Amer. Math. Soc.*, 19(2):447–459, 2006. [doi:10.1090/S0894-0347-05-00506-0, arXiv:math/0407224] 2

[17] JOSEPH M. LANDSBERG AND LAURENT MANIVEL: On the ideals of secant varieties of Segre varieties. *Found. Comput. Math.*, 4(4):397–422, 2004. [doi:10.1007/s10208-003-0115-9, arXiv:math/0311388] 7, 9, 10

[18] FRANÇOIS LE GALL: Powers of tensors and fast matrix multiplication. In *Proceedings of the 39th International Symposium on Symbolic and Algebraic Computation*, ISSAC '14, pp. 296–303, New York, NY, USA, 2014. ACM. [doi:10.1145/2608628.2608664, arXiv:1401.7714] 2

[19] CLAUDIU RAICU: Secant varieties of Segre–Veronese varieties. *Algebra Number Theory*, 6(8):1817–1868, 2012. [doi:10.2140/ant.2012.6.1817] 10

[20] Sagemath, the Sage Mathematics Software System (Version 7.2). Software. `http://www.sagemath.org`. 9

[21] Schurfunctors – for computing Schur functors. Macaulay2 package. `http://www.math.uiuc.edu/Macaulay2/doc/Macaulay2-1.9.2/share/doc/Macaulay2/SchurFunctors/html/index.html`. 10

[22] YAROSLAV SHITOV: How hard is the tensor rank? *arXiv*, 2016. [arXiv:1611.01559] 2

[23] VOLKER STRASSEN: Gaussian elimination is not optimal. *Numer. Math.*, 13(4):354–356, 1969. [doi:10.1007/BF02165411] 2

[24] AVI WIGDERSON: Talk: Matrix rank – extensions, applications and open problems, 2014. Geometric complexity theory workshop, Simons Institute. `https://simons.berkeley.edu/talks/avi-wigderson-2014-09-17`. 2

[25] SHMUEL WINOGRAD: On multiplication of 2×2 matrices. *Linear Algebra Appl.*, 4:381–388, 1971. [doi:10.1016/0024-3795(71)90009-7] 2

[26] BRIAN G. WYBOURNE, FRANCK BUTELLE, RONALD KING, AND FRÉDÉRIC TOUMAZET: *Schur Group Theory Software (Version 6.02)*, 2014. Software. `http://schur.sourceforge.net`. 9

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