UNIVERSITY OF COPENHAGEN

Complex action suggests future-included theory

Nagao, Keiichi; Nielsen, Holger Bech

Letter

Complex action suggests future-included theory

Keiichi Nagao[1,*] and Holger Bech Nielsen[2,*]

[1]*Faculty of Education, Ibaraki University, Bunkyo 2-1-1, Mito 310-8512 Japan*
[2]*Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark*
*E-mail: keiichi.nagao.phys@vc.ibaraki.ac.jp; hbech@nbi.dk

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In quantum theory its action is usually taken to be real, but we can consider another theory whose action is complex. In addition, in the Feynman path integral, the time integration is usually performed over the period between the initial time T_A and some specific time, say, the present time t. Besides such a future-not-included theory, we can consider the future-included theory, in which not only the past state $|A(T_A)\rangle$ at the initial time T_A but also the future state $|B(T_B)\rangle$ at the final time T_B is given at first, and the time integration is performed over the whole period from the past to the future. Thus quantum theory can be classified into four types, according to whether its action is real or not, and whether the future is included or not. We argue that, if a theory is described with a complex action, then such a theory is suggested to be the future-included theory, rather than the future-not-included theory. Otherwise persons living at different times would see different histories of the universe.
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Subject Index   A60, B30

1. *Introduction*   Quantum theory is usually described by using the the Feynman path integral (FPI), where the time integration is performed over the period between the initial time T_A and some specific time, say, the present time t. In addition to this future-not-included theory, we can consider another formulation, the future-included theory, in which not only the past state $|A(T_A)\rangle$ at the initial time T_A but also the future state $|B(T_B)\rangle$ at the final time T_B is given at first, and the time integration is performed over the whole period from the past to the future. In addition, in quantum theory its action is usually taken to be real. Let us call this the real action theory (RAT). We can consider another theory whose action is complex at the fundamental level. If we pursue a fundamental theory, it is better to require fewer conditions to be imposed on it at first. In this sense such a complex action theory (CAT) is preferable to the RAT, because the former has fewer conditions by at least one: there is no reality condition on the action. Thus quantum theory can be classified into four types, according to whether its action is real or not, and whether the future is included or not, as summarized in Table 1.

We have studied various properties of both the future-included and future-not-included CAT. In particular, the future-included CAT has been investigated with the expectation that the imaginary part of the action would give some falsifiable predictions [1–4], and various interesting suggestions have been made for the Higgs mass [5], quantum-mechanical philosophy [6–8], some fine-tuning problems [9,10], black holes [11], de Broglie–Bohm particles, and a cut-off in loop diagrams [12].

Table 1. Four types of quantum theory.

	Real action	Complex action
Future is not included	Future-not-included RAT	Future-not-included CAT
Future is included	Future-included RAT	Future-included CAT

In addition, in Ref. [13], introducing the proper inner product I_Q for the Hamiltonian $\hat{H}$[1], where a Hermitian operator Q[2] is chosen so that the eigenstates of $\hat{H}$ become orthogonal to each other with respect to I_Q[3], we showed that we can effectively obtain a Hamiltonian that is Q-Hermitian, i.e., Hermitian with respect to I_Q, after a long time development. Furthermore, using the complex coordinate formalism [20], we explicitly derived the momentum relation $p = m\dot{q}$, where m is a complex mass, via the FPI [21].

In the future-included CAT, the normalized matrix element [1][4] $\langle\hat{\mathcal{O}}\rangle^{BA} \equiv \frac{\langle B(t)|\hat{\mathcal{O}}|A(t)\rangle}{\langle B(t)|A(t)\rangle}$, where t is an arbitrary time ($T_A \leq t \leq T_B$), is a strong candidate for an expectation value of the operator $\hat{\mathcal{O}}$. Indeed, if we regard $\langle\hat{\mathcal{O}}\rangle^{BA}$ as the expectation value in the future-included CAT, we can obtain the Heisenberg equation, Ehrenfest's theorem, and a conserved probability current density [24,25]. Utilizing the mechanism for effectively obtaining a Q-Hermitian Hamiltonian [13], we proposed the correspondence principle, which claims that, if we regard $\langle\hat{\mathcal{O}}\rangle^{BA}$ as an expectation value in the future-included CAT, the expectation value at the present time t for large $T_B - t$ and large $t - T_A$ corresponds to that of the future-not-included theory with the proper inner product for large $t - T_A$ [24,25]. Therefore, the future-included CAT, which influences the past in principle, is not excluded phenomenologically, though it looks very exotic.

As for the future-not-included CAT, an expectation value of an operator $\hat{\mathcal{O}}$ is given by $\langle\hat{\mathcal{O}}\rangle^{AA} \equiv \frac{\langle A(t)|\hat{\mathcal{O}}|A(t)\rangle}{\langle A(t)|A(t)\rangle}$. In Ref. [26], we studied the various properties of $\langle O\rangle^{AA}$, and pointed out that the momentum relation $p = m\dot{q}$, which was shown to be correct in the future-included CAT [21], is not valid in the future-not-included CAT. Looking at the time development of $\langle O\rangle^{AA}$, we obtained the correct momentum relation in the future-not-included CAT, $p = \left(m_R + m_I^2/m_R\right)\dot{q}$, where m_R and m_I are the real and imaginary parts of m respectively. We also argued that its classical theory is described by a certain real action S_{eff}. In addition, we provided another way to understand the time development of the future-not-included theory by making use of the future-included theory. Furthermore, applying the method of deriving the momentum relation via the FPI [21] to the future-not-included theory properly by introducing a formal Lagrangian, we derived the correct momentum relation in the future-not-included theory, which is consistent with that mentioned above.

Thus the future-not-included CAT has very intriguing properties, so it seems to be worthwhile to study it more. However, in this letter, we point out that, if we adopt a theory whose action is complex, then it is suggested that the theory has to be the future-included CAT, rather than the future-not-included CAT. We encounter a philosophical discrepancy in the future-not-included CAT. We

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[1] $\hat{H}$ is generically non-normal. Hence the set of the Hamiltonians that we considered is much larger than that of the PT-symmetric non-Hermitian Hamiltonians, which has been intensively studied [14–18].

[2] In the special case of the Hamiltonian $\hat{H}$ being normal, Q is just a unit operator.

[3] Similar inner products are also studied in Refs. [17–19].

[4] The normalized matrix element $\langle\hat{\mathcal{O}}\rangle^{BA}$ is called the weak value [22] in the context of the future-included RAT, and it has been intensively studied. For details of the weak value, see Refs. [22,23] and references therein.

illustrate this suggestion with a couple of simple examples after briefly reviewing the future-included and future-not-included CAT.

2. *Review of the future-included and future-not-included CAT*   In a system defined with a single degree of freedom, we consider the CAT, in which the FPI is described with the Lagrangian $L(q(t), \dot{q}(t)) = \frac{1}{2} m \dot{q}^2 - V(q)$, where m is a complex mass, and $V(q)$ is a complex potential term.

Following Refs. [24,25,27], we briefly review the future-included theory. In the future-included theory, not only the past state $|A(T_A)\rangle$ at the initial time T_A but also the future state $|B(T_B)\rangle$ at the final time T_B are given at first, and $|A(t)\rangle$ and $|B(t)\rangle$ are supposed to time-develop according to the Schrödinger equations

$$i\hbar \frac{d}{dt} |A(t)\rangle = \hat{H} |A(t)\rangle, \tag{1}$$

$$i\hbar \frac{d}{dt} |B(t)\rangle = \hat{H}^\dagger |B(t)\rangle. \tag{2}$$

In Refs. [24,25] we investigated the normalized matrix element $\langle \hat{\mathcal{O}} \rangle^{BA} \equiv \frac{\langle B(t)|\hat{\mathcal{O}}|A(t)\rangle}{\langle B(t)|A(t)\rangle}$[1], which is a strong candidate for an expectation value in the future-included theory. Indeed, this $\langle \hat{\mathcal{O}} \rangle^{BA}$ obeys $\frac{d}{dt} \langle \hat{\mathcal{O}} \rangle^{BA} = \langle \frac{i}{\hbar} [\hat{H}, \hat{\mathcal{O}}] \rangle^{BA}$. Substituting $\hat{q}_{\text{new}}$ and $\hat{p}_{\text{new}}$[5] for $\hat{\mathcal{O}}$, we obtain

$$\frac{d}{dt} \langle \hat{q}_{\text{new}} \rangle^{BA} = \frac{1}{m} \langle \hat{p}_{\text{new}} \rangle^{BA}, \tag{3}$$

$$\frac{d}{dt} \langle \hat{p}_{\text{new}} \rangle^{BA} = -\langle V'(\hat{q}_{\text{new}}) \rangle^{BA}, \tag{4}$$

and Ehrenfest's theorem, $m \frac{d^2}{dt^2} \langle \hat{q}_{\text{new}} \rangle^{BA} = -\langle V'(\hat{q}_{\text{new}}) \rangle^{BA}$. Also, Eq. (3) leads to the momentum relation $p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$. Thus, $\langle \hat{\mathcal{O}} \rangle^{BA}$ provides the simple time development of the saddle point for $\exp(\frac{i}{\hbar} S)$. In addition, using both the complex coordinate formalism [20] and the automatic hermiticity mechanism [13,20], i.e., the mechanism to obtain a Hermitian Hamiltonian after a long time development, we obtained a correspondence principle that $\langle \hat{\mathcal{O}} \rangle^{BA}$ for large $T_B - t$ and large $t - T_A$ is almost equivalent to $\langle \hat{\mathcal{O}} \rangle^{AA}_{Q'} \equiv \frac{\langle A(t)|_{Q'} \hat{\mathcal{O}} |A(t)\rangle}{\langle A(t)|_{Q'} A(t)\rangle}$ for large $t - T_A$, where Q' is a Hermitian operator that is used to define the proper inner product so that the eigenstates of the Hamiltonian become orthogonal to each other with regard to it. Thus the future-included theory is not excluded phenomenologically, though it looks very exotic.

Following Refs. [13,20,26,27], we briefly review the future-not-included theory. In the future-not-included theory, only the past state $|A(T_A)\rangle$ at the initial time T_A is given at first, and $|A(t)\rangle$ is supposed to time-develop according to Eq. (1). The expectation value in the future-not-included theory is given by $\langle \hat{\mathcal{O}} \rangle^{AA} \equiv \frac{\langle A(t)|\hat{\mathcal{O}}|A(t)\rangle}{\langle A(t)|A(t)\rangle} = {}_N\langle A(t)|\hat{\mathcal{O}}|A(t)\rangle_N$, where we have introduced a normalized

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[5] $\hat{q}_{\text{new}}$ and $\hat{p}_{\text{new}}$ are generalized coordinate and momentum operators that are constructed in the context of the complex coordinate formalism [20,27] so that they are non-Hermitian and have complex eigenvalues q and p. The complex coordinate formalism is not relevant for the purposes of this letter, so we do not discuss it. The details are referred to in Refs. [20,27].

Table 2. Comparison between the future-included and future-not-included theories.

	future-included theory	future-not-included theory
action	$S = \int_{T_A}^{T_B} dtL$	$S = \int_{T_A}^{t} dtL$
"expectation value"	$\langle \hat{\mathcal{O}} \rangle^{BA} = \frac{\langle B(t)|\hat{\mathcal{O}}|A(t)\rangle}{\langle B(t)|A(t)\rangle}$	$\langle \hat{\mathcal{O}} \rangle^{AA} = \frac{\langle A(t)|\hat{\mathcal{O}}|A(t)\rangle}{\langle A(t)|A(t)\rangle}$
time development	$i\hbar \frac{d}{dt}\langle \hat{\mathcal{O}} \rangle^{BA}$	$i\hbar \frac{d}{dt}\langle \hat{\mathcal{O}} \rangle^{AA}$
	$= \langle[\hat{\mathcal{O}}, \hat{H}]\rangle^{BA}$	$= \langle[\hat{\mathcal{O}}, \hat{H}_h]\rangle^{AA} + \left\langle \left\{ \hat{\mathcal{O}} - \langle\hat{\mathcal{O}}\rangle^{AA}, \hat{H}_a \right\} \right\rangle^{AA}$
		$\simeq \langle[\hat{\mathcal{O}}, \hat{H}_h]\rangle^{AA}$
classical theory	$\delta S = 0$	$\delta S_{\text{eff}} = 0, \ S_{\text{eff}} = \int_{T_A}^{t} dtL_{\text{eff}}$
momentum relation	$p = m\dot{q}$	$p = m_{\text{eff}}\dot{q}$

state $|A(t)\rangle_N \equiv \frac{1}{\sqrt{\langle A(t)| \ A(t)\rangle}}|A(t)\rangle$. Then, $|A(t)\rangle_N$ obeys the slightly modified Schrödinger equation,

$$i\hbar \frac{d}{dt}|A(t)\rangle_N = \hat{H}|A(t)\rangle_N - {}_N\langle A(t)|\hat{H}_a|A(t)\rangle_N|A(t)\rangle_N$$

$$= \hat{H}_h|A(t)\rangle_N + \left(\hat{H}_a - {}_N\langle A(t)|\hat{H}_a|A(t)\rangle_N \right)|A(t)\rangle_N, \tag{5}$$

where $\hat{H}_h$ and $\hat{H}_a$ are the Hermitian and anti-Hermitian parts of $\hat{H}$ respectively. In Eq. (5) we see that the effect of the anti-Hermitian part of $\hat{H}$ disappears in the classical limit, though the theory is defined with $\hat{H}$ at the quantum level. In addition, we find the time development of $\langle \hat{\mathcal{O}} \rangle^{AA}$ as follows:

$$i\hbar \frac{d}{dt}\langle \hat{\mathcal{O}} \rangle^{AA} = \langle[\hat{\mathcal{O}}, \hat{H}_h]\rangle^{AA} + \langle F(\hat{\mathcal{O}}, \hat{H}_a)\rangle^{AA} \simeq \langle[\hat{\mathcal{O}}, \hat{H}_h]\rangle^{A(t)A(t)}, \tag{6}$$

where $F(\hat{\mathcal{O}}, \hat{H}_a)(t)$, a quantum fluctuation term given by $F(\hat{\mathcal{O}}, \hat{H}_a)(t) = \left\{ \hat{\mathcal{O}}, \hat{H}_a - \langle\hat{H}_a\rangle^{AA} \right\} = \left\{ \hat{\mathcal{O}} - \langle\hat{\mathcal{O}}\rangle^{AA}, \hat{H}_a \right\}$, disappears in the classical limit. Substituting $\hat{q}_{\text{new}}$ and $\hat{p}_{\text{new}}$ for $\hat{\mathcal{O}}$ in Eq. (6), we obtain

$$\frac{d}{dt}\langle \hat{q}_{\text{new}} \rangle^{AA} \simeq \frac{1}{i\hbar}\langle[\hat{q}_{\text{new}}, \hat{H}_h]\rangle^{AA} \simeq \frac{1}{m_{\text{eff}}}\langle \hat{p}_{\text{new}} \rangle^{AA}, \tag{7}$$

$$\frac{d}{dt}\langle \hat{p}_{\text{new}} \rangle^{AA} \simeq \langle[\hat{p}_{\text{new}}, \hat{H}_h]\rangle^{AA} \simeq -\langle V_R'(\hat{q}_{\text{new}}) \rangle^{AA}, \tag{8}$$

where $m_{\text{eff}} \equiv m_R + \frac{m_I^2}{m_R}$, and V_R is the real part of the potential term V. Combining Eq. (7) with Eq. (8), we obtain Ehrenfest's theorem, $m_{\text{eff}}\frac{d^2}{dt^2}\langle \hat{q}_{\text{new}} \rangle^{AA} \simeq -\langle V_R'(\hat{q}_{\text{new}}) \rangle^{AA}$, which suggests that the classical theory of the future-not-included theory is described not by the full action S, but $S_{\text{eff}} \equiv \int_{T_A}^{t} dtL_{\text{eff}}$, where $L_{\text{eff}}(\dot{q}, q) \equiv \frac{1}{2}m_{\text{eff}}\dot{q}^2 - V_R(q)$. Thus the classical theory of the future-not-included theory is described by $\delta S_{\text{eff}} = 0$. We also find that Eq. (7) leads to the momentum relation $p = \frac{\partial L_{\text{eff}}}{\partial \dot{q}} = m_{\text{eff}}\dot{q}$.

We give a brief summary of the future-included and future-not-included theories in Table 2 [26]. We see that the classical theory of the future-included theory is quite in contrast to that of the future-not-included theory.

3. *Complex action suggests a future-included theory*   In the FPI $\int \mathcal{D}\text{path} \ \psi_B^* \psi_A e^{\frac{i}{\hbar}S[\text{path}]}$, the integrand that includes the action S is expressed as $e^{\frac{i}{\hbar}S[\text{path}]} = e^{\frac{i}{\hbar}S_R[\text{path}]}e^{-\frac{1}{\hbar}S_I[\text{path}]}$, where S_R and S_I are real and imaginary parts of S, respectively. Since $e^{-\frac{1}{\hbar}S_I[\text{path}]}$ can have higher orders of magnitude
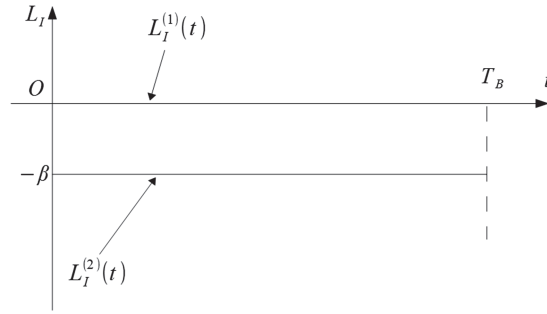
Fig. 1. The first example of L_I.

than $e^{\frac{i}{\hbar}S_R[\text{path}]}$ and the boundary wave functions $\psi_B^* \psi_A$, it is $S_I[\text{path}]$ that has the greatest influence on the selection of paths. Paths realizing smaller $S_I[\text{path}]$ are essentially favored and chosen[6]. $S_I[\text{path}]$ is obtained by the time integration of the imaginary part of the Lagrangian $L_I(q,\dot{q}) \equiv \frac{1}{2}m_I\dot{q}^2 - V_I(q)$, where m_I and V_I are imaginary parts of m and V, respectively. We symbolically write $L_I(q,\dot{q})$ as the function of t, $L_I^{\text{path}}(t)$. $S_I([T_A,t]) \equiv \int_{T_A}^t L_I^{\text{path}}(t')dt'$ and $S_I([T_A,T_B]) = \int_{T_A}^{T_B} L_I^{\text{path}}(t')dt'$ are used in the future-not-included and future-included theories, respectively. If $L_I(q,\dot{q})$ varies greatly in time[7], paths are nontrivially chosen in the FPI. We give a couple of simple examples of two paths, and discuss which path is chosen by comparing $S_I[\text{path}]$ in each example.

In the following, taking the initial time T_A as $T_A = 0$ for simplicity, we consider a pair of constant L_I as the first example of two paths for pedagogical reasons. Next we present the second example, where one of the two L_I is constant, but the other is time-dependent. In this second example, we show that, if we stand on the future-not-included theory and respect objectivity, then we encounter a philosophical contradiction, and thus we are led to the future-included theory.

Let us begin with the first example, a pair of constant L_I as two paths. Such a pair of L_I is defined as follows: $L_I^{(1)}(t) = 0$, $L_I^{(2)}(t) = -\beta$, where $\beta > 0$. $L_I^{(1)}$ and $L_I^{(2)}$ are drawn in Fig. 1. Each $S_I^{(j)}$ for $L_I^{(j)}$ $(j = 1, 2)$ in the future-not-included theory is given by $S_I^{(1)}([0,t]) = \int_0^t L_I^{(1)}(t')dt' = 0$ and $S_I^{(2)}([0,t]) = \int_0^t L_I^{(2)}(t')dt' = -\beta t$. Since $S_I^{(2)}([0,t]) < S_I^{(1)}([0,t])$, a person living in the time t who believes that our universe is described by the future-not-included theory judges that path 2 is favored, and thinks that our universe is determined by path 2. If another person believes the future-included theory, he compares $S_I^{(1)}([0,T_B]) = 0$ and $S_I^{(2)}([0,T_B]) = -\beta T_B$. Since $S_I^{(2)}([0,T_B]) < S_I^{(1)}([0,T_B])$, he judges that path 2 is favored, and thinks that our universe is determined by path 2. This is a very simple example, so we do not encounter any problems. Both interpretations, the future-included and future-not-included theories, can stand. However, if we consider a slightly more nontrivial example, then we could easily encounter difficulties. We see this in the next example.

Let us consider the second example such that one of L_I varies in time. We take the following pair of L_I as two paths:

$$L_I^{(1)}(t) = \alpha \left\{ \cos\left(\frac{\pi}{T_B}t\right) - 1 \right\}, \tag{9}$$

$$L_I^{(2)}(t) = -\beta, \tag{10}$$

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[6] In other words, paths with larger imaginary parts of the eigenvalues of the Hamiltonian $\hat{H}$ are favored and chosen.

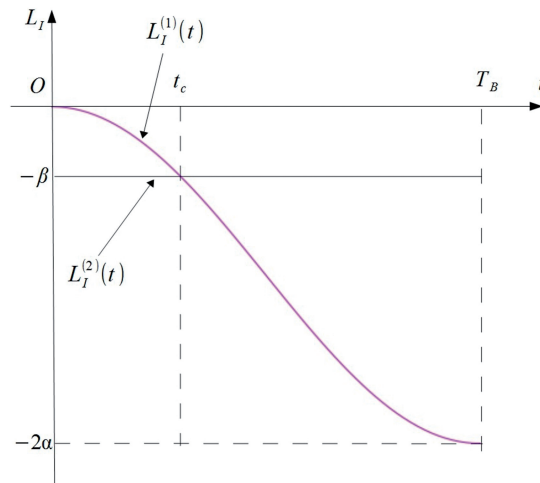[7] A time-dependent non-Hermitian Hamiltonian is studied in Ref. [28].

Fig. 2. The second example of L_I.

where α and β are constants such that $\alpha > \beta > 0$. $L_I^{(1)}$ and $L_I^{(2)}$ are drawn in Fig. 2, where t_c is the solution to $L_I^{(1)}(t_c) = L_I^{(2)}(t_c)$, and found to be $t_c = \frac{T_B}{\pi} \cos^{-1}\left(1 - \frac{\beta}{\alpha}\right)$. Let us suppose that a person living in the time t believes the future-not-included theory. Each $S_I^{(j)}$ for $L_I^{(j)}$ ($j = 1, 2$) is expressed as $S_I^{(1)}([0,t]) = \alpha \left\{ \frac{T_B}{\pi} \sin\left(\frac{\pi}{T_B}t\right) - t \right\}$, $S_I^{(2)}([0,t]) = -\beta t$.

At a glance, for $t < t_c$, we easily see that $S_I^{(2)}([0,t]) < S_I^{(1)}([0,t])$, because $L_I^{(2)}(t) < L_I^{(1)}(t)$. So, for $t < t_c$, he judges that path 2 is favored. Then how does he judge for $t > t_c$? We can answer this question by knowing the time t_d such that $S_I^{(1)}([0, t_d])$ balances with $S_I^{(2)}([0, t_d])$. That is, t_d is defined as the solution to $S_I^{(1)}([0, t_d]) = S_I^{(2)}([0, t_d])$, which is reduced to $\sin\left(\frac{\pi}{T_B}t_d\right) = \left(1 - \frac{\beta}{\alpha}\right)\frac{\pi}{T_B}t_d$. In Fig. 3, t_d is determined so that each area of the two domains with slanted lines is equal to each other. Using this t_d, we find the following relations:

$$S_I^{(1)}([0,t]) > S_I^{(2)}([0,t]) \quad \text{for } 0 \le t < t_d, \tag{11}$$

$$S_I^{(1)}([0,t]) < S_I^{(2)}([0,t]) \quad \text{for } t_d < t \le T_B. \tag{12}$$

In the future-not-included theory only what happened in the past can matter. Therefore, the person living at the earlier time $0 \le t < t_d$ judges that path 2 is chosen, but in the later time $t_d < t \le T_B$ he will judge that path 1 is chosen. Thus we have encountered a strange situation. We usually want to have objectivity for any theory to be reasonable, but the mentioned property indicates that the future-not-included theory is subjective. Such a scenario in which what happened should depend on whom you ask, which lacks objectivity, reminds us of the so-called Mandela effect[8], which was named by the blogger Fiona Broome. If in the later time path 1 is chosen, then even in the earlier time path 1 should have been chosen, as long as we respect objectivity. Looking at the history, we will effectively find influence from the future looking back even in the future-not-included theory. This is a philosophical contradiction. To avoid this discrepancy, the person is led to the future-included theory, rather than the future-not-included theory.

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[8] That is, a large part of the population believed that deceased former South African President Nelson Mandela had already been dead a couple of decades before he really died [29].
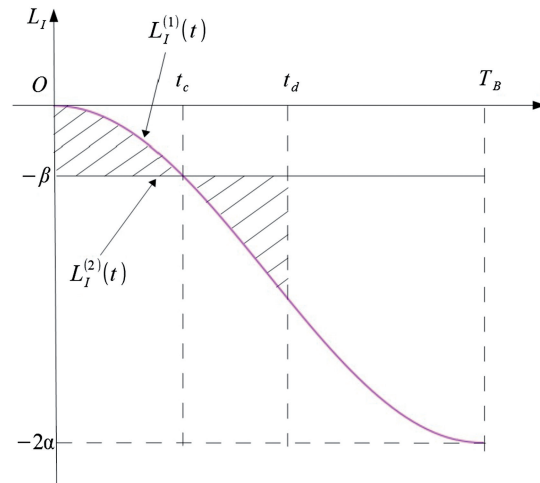
Fig. 3. The second example of L_I with t_d included.

Indeed, if he believes the future-included theory, then he compares $S_I^{(1)}([0, T_B]) = -\alpha T_B$ and $S_I^{(2)}([0, T_B]) = -\beta T_B$. Since $S_I^{(1)}([0, T_B]) < S_I^{(2)}([0, T_B])$, he judges that path 1 is favored at any time t ($0 \leq t \leq T_B$). We do not encounter any contradiction in the future-included theory. Therefore, if an action is allowed to be complex, then such an action has to be described in the future-included theory. It is very interesting that complex action suggests the future-included theory.

If the person persists in believing the future-not-included theory, how does he feel in the earlier time $0 \leq t < t_d$? In the earlier time $0 \leq t < t_d$, since $S_I^{(2)}([0, t]) < S_I^{(1)}([0, t])$, he thinks that it is a miraculous phenomenon that path 1 is chosen. This story implies that, if the action of our universe is allowed to be complex, then we could see miraculous phenomena. Oppositely, if we see miraculous phenomena in the usual theory, i.e., the future-not-included RAT, then we have a possibility that our universe is described by the future-included CAT. If so, such phenomena can be understood reasonably well. The future-included CAT gives similar effects to the anthropic principle.

4.   *Discussion*   In this letter, after briefly reviewing the future-included and future-not-included CAT, we have given a couple of examples of imaginary parts of Lagrangians L_I as two paths, and discussed which path is favored and chosen by comparing imaginary parts of actions S_I. In one of the examples we have encountered a philosophical contradiction in the future-not-included CAT as long as we respect objectivity. In the future-not-included theory, as future becomes past, the influence of L_I in such time intervals becomes relevant for the relative probability for various states in the FPI. This would lead to a strange re-choosing of initial states in the perspective of determinism so as to have had the smallest S_I until the present time. Such changing of initial states would be exceedingly strange, at least classically. Indeed, in Ref. [26], we reported such a complicated aspect of the future-not-included theory. We showed that time derivatives of $\langle \hat{q}_{\text{new}} \rangle_{AA}$ and $\langle \hat{p}_{\text{new}} \rangle_{AA}$ have complicated anticommutation terms, and provided an unusual way to understand the time development by using such re-choosing of the initial states. If a historian sees that people in the past were governed by their future, then it would be strange if we were not governed by the future. If we are to be governed by the future, then the future should exist. The historical people would have the happening leading to low L_I in their future because $e^{-\frac{1}{\hbar} S_I[\text{path}]}$ promotes it so. This means that they are influenced by the future. Thus we are led to the future-included CAT. If we stand on the future-included CAT, we do not

see any contradiction. It is much stabler for the predictions and consistent with determinism to have influence from an always or ever-existing future. Therefore, if an action is allowed to be complex, then such an action has to be described in the future-included theory. Agreeing with determinism, at least crudely, is a major benefit of the future-included CAT. Also, the future-included CAT can yield a simpler classical equation of motion for $\langle \hat{q}_{\text{new}} \rangle_{BA}$ and $\langle \hat{p}_{\text{new}} \rangle_{BA}$ than the future-not-included CAT.

In the future-included theories we need a final condition analogous to an initial condition to deliver the final state $|B(T_B)\rangle$. In the future-included RAT we need two boundary conditions $|B(T_B)\rangle$ and $|A(T_A)\rangle$. So the future-included RAT is a bit more complicated than the future-not-included RAT that needs only one boundary condition. In the future-included CAT we obtain the boundaries unified with the dynamics; both $|B(T_B)\rangle$ and $|A(T_A)\rangle$ are effectively obtained from S_I. The future-included CAT makes such an initial or final condition automatically. Indeed, in Refs. [27,30–32], introducing a slightly modified normalized matrix element $\langle \hat{\mathcal{O}} \rangle_Q^{BA} \equiv \frac{\langle B(t)|_Q \hat{\mathcal{O}}|A(t)\rangle}{\langle B(t)|_Q A(t)\rangle}$, which is obtained just by changing the notation of $\langle B(t)|$ as $\langle B(t)| \to \langle B(t)|_Q \equiv \langle B(t)|Q$ in $\langle \hat{\mathcal{O}} \rangle^{BA}$, we presented a theorem that states that, provided that an operator $\hat{\mathcal{O}}$ is Q-Hermitian, the normalized matrix element $\langle \hat{\mathcal{O}} \rangle_Q^{BA}$ becomes real and time-develops under a Q-Hermitian Hamiltonian for $|B(t)\rangle$ and $|A(t)\rangle$ selected such that the absolute value of the transition amplitude $|\langle B(t)|_Q A(t)\rangle|$ is maximized. We call this way of thinking the maximization principle. This provides us both reality of $\langle \hat{\mathcal{O}} \rangle_Q^{BA}$ and Q-hermiticity of the Hamiltonian, even though $\langle \hat{\mathcal{O}} \rangle_Q^{BA}$ is generically complex by definition and the given Hamiltonian $\hat{H}$ is non-normal at first[9]. We found that in the case of the CAT a unique class of paths is chosen by the maximization principle. Besides this fact, since the functional integral expression is simpler in the future-included theories than the future-not-included theories, we argued that the future-included CAT is the most elegant. The study in this letter partly supports this speculation.

In this letter we have argued that the existence of an imaginary part of the action suggests the future-included theory. Then, can we say the reverse, i.e., does the future-included theory suggest the existence of an imaginary part of the action? It is not clear, but it would be interesting if we could say something about it. If we show that the effects of the imaginary part turn out to be unobservable in practice in a good approximation, then we can argue that there is no strong reason to assume the action to be real in nature. The reality of the action can be regarded as a restriction on parameters in the action, and thus really an extra—and according to our argument—unnecessary assumption. So the real benefit from our CAT would be that we can have a more general action by getting rid of the restriction.

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