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Kristjansen, C.

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Three-spin strings on $AdS_5 \times S^5$ from $\mathcal{N} = 4$ SYM

C. Kristjansen

NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

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Abstract

Using the integrable spin chain picture we study the one-loop anomalous dimension of certain single trace scalar operators of $\mathcal{N} = 4$ SYM expected to correspond to semi-classical string states on $AdS_5 \times S^5$ with three large angular momenta (J_1, J_2, J_3) on S^5. In particular, we investigate the analyticity structure encoded in the Bethe equations for various distributions of Bethe roots. In a certain region of the parameter space our operators reduce to the gauge theory duals of the *folded* string with two large angular momenta and in another region to the duals of the *circular* string with angular momentum assignment (J, J', J'), $J > J'$. In between we locate a critical line. We propose that the operators above the critical line are the gauge theory duals of the circular elliptic string with three different spins and support this by a perturbative calculation.

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1. Introduction

Recent development, triggered by the pp-wave/BMN correspondence [1], has led to new insights on the duality between string theory and gauge theory and has in particular revealed interesting novel integrability structures in both kind of theories [2–11]. The progress has been made by focusing on a simple set of observables which, according to the original AdS/CFT correspondence [12], should be closely related, namely the energy spectrum of single string states on $AdS_5 \times S^5$ and anomalous dimensions of local single trace operators of $\mathcal{N} = 4$ SYM. Here the string states are characterized by various quantum numbers such as angular momenta and these should match the representation labels of the corresponding operators. Following the formulation of the pp-wave/BMN correspondence efficient techniques for evaluating anomalous dimensions of $\mathcal{N} = 4$ SYM operators were developed [13]. A further crucial step was the discovery of Minahan and Zarembo that the planar one-loop dilatation operator in the scalar sector of $\mathcal{N} = 4$ SYM could be identified as the Hamiltonian of an integrable $SO(6)$ spin chain [2]. The spin chain picture was later extended to the set of all operators in $\mathcal{N} = 4$ SYM and yielded an integrable $PSU(2, 2|4)$ super spin chain [4,5]. On the string theory side it was realized that the classical

E-mail address: kristjan@nbi.dk (C. Kristjansen).

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energy of certain string states with several large angular momenta on S^5 exhibited an analytical dependence on the parameter $\lambda' = \lambda/J^2$ where J is the total angular momentum and where λ is the squared string tension which, via the AdS/CFT dictionary, is mapped onto the 't Hooft coupling constant $g_{\mathrm{YM}}^2 N$. Furthermore, for these states string quantum corrections were suppressed as $1/J$ in the limit $J \to \infty$, $\lambda' = \lambda/J^2$ fixed [14,15].[1] This led to the suggestion that taking the limit $J \to \infty$ for fixed λ', the term linear in λ' in the small λ' expansion of the classical string energy would match the one-loop anomalous dimension of a gauge theory operator carrying the same (large) $SO(6)$ representation labels as the string state [14,15]. In the spin chain picture, determining the one-loop anomalous dimension amounts to solving a set of Bethe equations [18] and considering large representation labels (i.e., long operators) corresponds to going to the thermodynamical limit.

Comparison of classical string energies and one-loop anomalous dimensions has been successfully carried out in a number of cases, the prime example involving strings carrying two large angular momenta (J_1, J_2) on S^5. On the gauge theory side two types of solutions of the Bethe equations were found [19] and these were identified as the gauge theory dual of respectively a folded and a circular string [19,20] with the folded string being the one of lowest energy. Expressions giving the one-loop anomalous dimension, respectively the $\mathcal{O}(\lambda')$ contribution to the classical energy as a function of the representation labels in a parametric form were found. These parametrizations involved elliptic integrals and were shown to match at a functional level [10,21]. The situation where three spins (J_1, J_2, J_3) on S^5 are non-vanishing is less well understood. One particularly simple three spin circular string solution was found a while ago [14,15]. It has two of its three spins equal, i.e., $J_2 = J_3$ and is stable for large enough values of J_1. This solution is again parametrized in terms of elliptic functions and its gauge theory dual was identified in [17]. The generic case of (rigid) strings with three *different* S^5 spin quantum numbers was studied in [10] where it was shown that the relevant sub-sector of the string σ-model could be mapped onto an integrable Neumann model. Further generalizations and relations to integrable models were found in [11]. The parallel gauge theory analysis is so far lacking. A characteristic feature which distinguishes the three-spin solutions from the two-spin ones is that whereas the latter are conveniently parametrized in terms of elliptic integrals the former generically require the use of *hyper*-elliptic integrals. There does, however, exist a class of three-spin solutions which are still elliptic [10]. In reference [10] particular attention was paid to hyper-elliptic three spin solutions generalizing respectively the folded and circular two-spin string. Of these three-spin solutions the circular one exists in an elliptic version whereas the folded one does not [10]. Here, we shall study a class of holomorphic gauge theory operators carrying generic $SO(6)$ representation labels (J_1, J_2, J_3). In a certain region of the parameter space (corresponding to $J_3 = 0$) the operators reduce to the gauge theory duals of the *folded* two-spin string whereas in another one they constitute the duals of the *circular* string with spin assignment (J, J', J'), $J > J'$ [17]. We will show that these two different manifestations of the dual string are made possible through the existence of a line of critical points in the parameter space. Furthermore, we propose that above the critical line the gauge theory operators studied are the duals of the circular elliptic three-spin states of [10]. The proposal is supported by a perturbative calculation.

2. The general gauge theory set-up

Gauge theory operators dual to rigid strings with three non-vanishing angular momenta, (J_1, J_2, J_3), on S^5 are expected to be operators of the type $\mathrm{Tr}((\chi\chi)^k X^{J_1-k} Y^{J_2-k} Z^{J_3-k} + \text{perm's})$, $k < \min\{J_1, J_2, J_3\}$, where X, Y and Z are the three complex scalars of $\mathcal{N} = 4$ SYM with $SO(6)$ weights $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ and where χ is the fermion with $SO(6)$ weight $(1/2, 1/2, 1/2)$. In the present Letter we shall work at one-loop order, i.e., at $\mathcal{O}(\lambda)$, where the dilatation generator only mixes the operators without fermionic constituents. We shall thus be interested in diagonalizing the one-loop dilatation generator in the sub-set of operators of the type

[1] Several other types of string states with similar properties have been found. These include string states with non-vanishing angular momenta on AdS_5 [10,11,14] as well as a class of so-called pulsating string solutions [16,17].

$\mathrm{Tr}(X^{J_1}Y^{J_2}Z^{J_3} + \text{perm's})$ or equivalently diagonalizing the Hamiltonian of the integrable $SO(6)$ spin chain in the appropriate sub-set of spin states. The spin chain picture is particularly convenient when considering operators for which $L \equiv J_1 + J_2 + J_3 \to \infty$. Finding an eigenstate and corresponding eigenvalue of the $SO(6)$ spin chain Hamiltonian consists in solving a set of algebraic equations for the Bethe roots. For the $SO(6)$ spin chain there are three different types of Bethe roots reflecting the fact that the Lie algebra $SO(6)$ has three simple roots. However, for holomorphic operators only two of the three types of roots can be excited. Denoting the number of roots of the two relevant types as n_1 and n_2 and the roots themselves as $\{u_{1,j}\}_{j=1}^{n_1}$ and $\{u_{2,j}\}_{j=1}^{n_2}$ the Bethe equations read

$$\left(\frac{u_{1,j}+i/2}{u_{1,j}-i/2}\right)^L = \prod_{k\neq j}^{n_1}\frac{u_{1,j}-u_{1,k}+i}{u_{1,j}-u_{1,k}-i}\prod_{k=1}^{n_2}\frac{u_{1,j}-u_{2,k}-i/2}{u_{1,j}-u_{2,k}+i/2}, \tag{1}$$

$$1 = \prod_{k\neq j}^{n_2}\frac{u_{2,j}-u_{2,k}+i}{u_{2,j}-u_{2,k}-i}\prod_{k=1}^{n_1}\frac{u_{2,j}-u_{1,k}-i/2}{u_{2,j}-u_{1,k}+i/2}. \tag{2}$$

We shall assume that $n_1 \leqslant L/2$, $n_2 \leqslant n_1/2$. The $SO(6)$ representation implied by this choice of Bethe roots is given by the Dynkin labels $[n_1 - 2n_2, L - 2n_1 + n_2, n_1]$. In terms of the spin quantum numbers, assuming $J_1 \geqslant J_2 \geqslant J_3$ this corresponds to $[J_2 - J_3, J_1 - J_2, J_2 + J_3]$ or $J_1 = L - n_1$, $J_2 = n_1 - n_2$, $J_3 = n_2$. A given solution of the Bethe equations gives rise to an eigenvalue of the spin chain Hamiltonian, i.e., a one loop anomalous dimension which is

$$\gamma = \frac{\lambda}{8\pi^2}\sum_{j=1}^{n_1}\frac{1}{(u_{1,j})^2 + 1/4}. \tag{3}$$

To enforce the cyclicity of the trace we have in addition to Eqs. (1) and (2) the following constraint

$$1 = \prod_{j=1}^{n_1}\left(\frac{u_{1,j}+i/2}{u_{1,j}-i/2}\right). \tag{4}$$

In the thermodynamical limit $L \to \infty$ all roots are $\mathcal{O}(L)$ and it is convenient to re-scale them accordingly. Doing so, taking the logarithm of the Bethe equations and imposing the limit $L \to \infty$ one is left with a set of integral equations.

3. The present gauge theory set-up

Let us define

$$\alpha = \frac{n_1}{L}, \qquad \beta = \frac{n_2}{L}. \tag{5}$$

Then the spin quantum numbers are given by $(J_1, J_2, J_3) = ((1-\alpha)L, (\alpha-\beta)L, \beta L)$. We shall assume that the Bethe roots $\{u_{1,j}\}_{j=1}^{n_1}$ are distributed as in the case of the folded two spin string solution of reference [19], i.e., they live on two arches in the complex plane, $\mathcal{C}_+$ and $\mathcal{C}_-$, which are each others mirror images with respect to zero. Each arch is symmetric around the real axis and neither one intersects the imaginary axis. For this configuration the constraint (4) is fulfilled (but n_1 is required to be even). Furthermore, let us assume that the roots $\{u_{2,j}\}_{j=1}^{n_2}$ live on some curve $\mathcal{C}_2$ not intersecting $\mathcal{C}_+$ or $\mathcal{C}_-$.

Performing the above mentioned manipulations relevant for the thermodynamical limit we can write the two Bethe equations as in [17]

$$\frac{1}{u} - 2\pi m = 2\,\fint_{\mathcal{C}_+} du'\,\frac{\sigma(u')}{u-u'} + 2\int_{\mathcal{C}_+} du'\,\frac{\sigma(u')}{u+u'} - \int_{\mathcal{C}_2} du'\,\frac{\rho_2(u')}{u-u'}, \quad u \in \mathcal{C}_+, \tag{6}$$

$$2\pi m_2 = 2 \fint_{\mathcal{C}_2} du' \frac{\rho_2(u')}{u-u'} - \int_{\mathcal{C}_+} du' \frac{\sigma(u')}{u-u'} - \int_{\mathcal{C}_+} du' \frac{\sigma(u')}{u+u'}, \quad u \in \mathcal{C}_2, \tag{7}$$

where m and m_2 are integers which reflect the ambiguities coming from the different possible choices of branches for the logarithm and where $\fint$ means that the integral has to be understood in the principal value sense. Furthermore, $\rho_2(u)$ and $\sigma(u)$ are root densities describing the continuum distribution of $\{u_{2,j}\}_{j=1}^{n_2}$ and the subset of $\{u_{1,j}\}_{j=1}^{n_1}$ with positive real part, respectively. The densities are normalized as

$$\frac{\alpha}{2} = \int_{\mathcal{C}_+} \sigma(u)\, du, \qquad \beta = \int_{\mathcal{C}_2} \rho_2(u)\, du. \tag{8}$$

We shall shortly see that the mode number m_2 actually has to vanish.[2] Rather than working with the densities we prefer to work with the resolvents $W(u)$ and $W_2(u)$ defined by

$$W(u) = \int_{\mathcal{C}_+} du' \frac{\sigma(u')}{u-u'}, \qquad W_2(u) = \int_{\mathcal{C}_2} du' \frac{\rho_2(u')}{u-u'}. \tag{9}$$

The resolvents are analytic in the complex plane except for a cut respectively along $\mathcal{C}_+$ and $\mathcal{C}_2$. In the continuum language the one-loop anomalous dimension, γ, is given by

$$\gamma = \frac{\lambda}{4\pi^2 L} \int_{\mathcal{C}_+} du \frac{\sigma(u)}{u^2} = -\frac{\lambda}{4\pi^2 L} W'(0). \tag{10}$$

Not only are the resolvents technically more convenient. It appears that they are indeed objects with a direct physical interpretation. For instance, $W(u)$ is the generating function of all the higher conserved charges of the spin chain [22]. It would be interesting to gain a similar understanding of $W_2(u)$.

One possible configuration for the roots $\{u_{2,j}\}_{j=1}^{n_2}$ is that they lie in an interval $[-ic, ic]$ on the imaginary axis [17]. In Ref. [17] the case $c \to \infty$ was studied and the corresponding string state was identified as the circular string of [14] with spin assignment (J, J', J'), $J > J'$. Here we shall analyze the generic c case. Our strategy when solving the Bethe equations will be the same as that of reference [17]. We will express $\rho_2(u)$ in terms of $\sigma(u)$ by means of Eq. (7) and use the resulting expression to eliminate $\rho_2(u)$ from Eq. (6). We see that $\rho_2(u)$ only enters Eq. (6) via the corresponding resolvent. Thus we do not need to determine $\rho_2(u)$ itself. Rewriting Eq. (7) as

$$\fint_{\mathcal{C}_2} du' \frac{\rho_2(u')}{u-u'} = \pi m_2 + \int_{\mathcal{C}_+} du' \frac{\sigma(u')u}{u^2 - u'^2}, \quad u \in \mathcal{C}_2, \tag{11}$$

we recognize the saddle point equation of the Hermitian one-matrix model with the terms on the right-hand side playing the role of the derivative of the potential. Thus we can immediately write down a contour integral expression for the resolvent, see, for instance [23]

$$W_2(u) = \oint_{\mathcal{C}} \frac{d\omega}{2\pi i} \frac{1}{u-\omega} \sqrt{\frac{u^2+c^2}{\omega^2+c^2}} \left\{ \pi m_2 + \int_{\mathcal{C}_+} du' \frac{\sigma(u')\omega}{\omega^2 - u'^2} \right\}, \tag{12}$$

[2] This is natural from the spin chain point of view as m_2 can be interpreted as a discrete momentum associated with the roots $\{u_{2,j}\}_{j=1}^{n_2}$ and all momentum is known to be carried by the roots $\{u_{1,j}\}_{j=1}^{n_1}$ (cf. Eq. (3)).

where the contour $\mathcal{C}$ encircles the interval $[-ic, ic]$ but not the various other singularities of the integrand. Interchanging the order of integrations in the last term we can write this as

$$W_2(u) = m_2\pi + \fint_{\mathcal{C}_+} du' \frac{\sigma(u')u}{u^2 - u'^2} - \fint_{\mathcal{C}_+} du' \frac{u'\sigma(u')}{u^2 - u'^2}\sqrt{\frac{u^2 + c^2}{u'^2 + c^2}}. \tag{13}$$

The parameter c can be expressed in terms of α and β by making use of the asymptotic behaviour of $W_2(u)$ as $u \to \infty$. One has

$$W_2(u) \sim \frac{\beta}{u}, \quad \text{as } u \to \infty, \tag{14}$$

which immediately gives

$$0 = m_2\pi, \tag{15}$$

$$\beta = \frac{\alpha}{2} - \int_{\mathcal{C}_+} du \frac{\sigma(u)u}{\sqrt{u^2 + c^2}}. \tag{16}$$

We notice the following two limiting cases of Eq. (13) which serve as a consistency check of our solution

$$\lim_{c \to 0} W_2(u) = 0, \tag{17}$$

$$\lim_{c \to \infty} W_2(u) = \fint_{\mathcal{C}_+} du' \frac{\sigma(u')}{u + u'}. \tag{18}$$

Here the last expression coincides with the one obtained in reference [17]. As noted in respectively [19] and [17] the integral equation (6) reduces to that of the $O(n)$ model on a random lattice [24] with $n = -2$ for $c \to 0$ and $n = -1$ for $c \to \infty$. The $O(n)$ model on a random lattice can be solved exactly for any value of n and the solution is for generic n parametrized in terms of elliptic functions [25]. However, a simplification occurs at the so-called rational points where $n = 2\cos(\pi p/q)$ with p and q co-prime integers [26,27]. The reason why elliptic integrals appear can most easily be understood by rewriting the integral equation of the $O(n)$ model in terms of the resolvent $W(u)$ which, as mentioned above, is analytic in the complex plane except for a cut along the contour $\mathcal{C}_+$. The relevant integral equation involves $W(u)$ as well as $W(-u)$. Effectively, one thus has *two* cuts and that is what leads to the elliptic structure for generic values of n. For details we refer to [25].

We can conveniently rewrite the expression (13) for $W_2(u)$ as

$$W_2(u) = \frac{1}{2}\fint_{\mathcal{C}_+} du' \frac{\sigma(u')}{u - u'}\left(1 - \sqrt{\frac{u^2 + c^2}{u'^2 + c^2}}\right) + \frac{1}{2}\fint_{\mathcal{C}_+} du' \frac{\sigma(u')}{u + u'}\left(1 + \sqrt{\frac{u^2 + c^2}{u'^2 + c^2}}\right).$$

Inserting this expression for $W_2(u)$ in Eq. (6) we get the following integral equation for general c

$$\begin{aligned} \frac{1}{u} - 2\pi m = {} & \frac{1}{2}\fint_{\mathcal{C}_+} du' \frac{\sigma(u')}{u - u'}\left(3 + \sqrt{\frac{u^2 + c^2}{u'^2 + c^2}}\right) \\ & + \frac{1}{2}\int_{\mathcal{C}_+} du' \frac{\sigma(u')}{u + u'}\left(3 - \sqrt{\frac{u^2 + c^2}{u'^2 + c^2}}\right), \end{aligned} \tag{19}$$

with $u \in \mathcal{C}_+$. We can trade the square roots in Eq. (19) for extra poles (or rather cuts) by performing a change of variables, obtaining an integral equation which exposes the analyticity structure of the problem in a simpler manner.

The relevant changes of variables are different for small and for large c and the resulting integral equations show that there is a phase transition taking place at some intermediate value of c. This explains why the string state dual to the operator considered does not need to be of the same type for $c \to 0$ (folded) as for $c \to \infty$ (circular).

4. The case of small c

For c small a convenient change of variables is

$$u = \frac{p^2 + c^2}{2ip}, \qquad u' = \frac{q^2 + c^2}{2iq}, \tag{20}$$

which is well-defined as $c \to 0$ but not as $c \to \infty$. With this change of variables we get

$$\sqrt{\frac{c^2 + u^2}{c^2 + u'^2}} = \frac{q(p^2 - c^2)}{p(q^2 - c^2)}, \tag{21}$$

and we see that the limit $c \to 0$ is as we wish. Inserting the change of variables (20) into the integral equation (19) we get with $du\,\sigma(u) \equiv dq\,\rho(q)$

$$\frac{p}{p^2 + c^2} + i\pi m = \fint_{\tilde{C}_+} dq\,\rho(q) \frac{q^2}{q^2 - c^2} \left\{ \frac{p}{c^2 - qp} + \frac{2}{p - q} + \frac{2}{p + q} + \frac{p}{c^2 + qp} \right\}, \tag{22}$$

with $p \in \tilde{C}_+$ where $\tilde{C}_+$ is the contour for the transformed roots q. The boundary equation (16) turns into

$$\beta = \frac{\alpha}{2} - \int_{\tilde{C}_+} dq\,\rho(q) \frac{q^2 + c^2}{q^2 - c^2}, \tag{23}$$

and the expression for γ becomes

$$\gamma = -\frac{\lambda}{\pi^2 L} \int_{\tilde{C}_+} dq \frac{\rho(q) q^2}{(q^2 + c^2)^2}. \tag{24}$$

Here it is convenient to define a resolvent by

$$W(p) = \int_{\tilde{C}_+} dq\,\rho(q) \frac{q^2}{q^2 - c^2} \frac{1}{p - q}. \tag{25}$$

Again, $W(p)$ is analytic in the complex plane except for a cut along the contour $\tilde{C}_+$ and we can express the anomalous dimension, γ through $W(p)$ as

$$\gamma = -\frac{\partial}{\partial p^2} \left(\frac{p^2 - c^2}{2p} \big(W(p) - W(-p) \big) \right) \bigg|_{p = ic}. \tag{26}$$

Apart from the function $W(p)$ the integral equation (22) involves $W(-p)$, $W(c^2/p)$ and $W(-c^2/p)$. This integral equation can be viewed as a "super-position" of that of the usual $O(n)$ model on a random lattice [24] and that of the plaquette model studied in [28]. In particular, we see that we effectively have four different cuts. In other words, the presence of the Bethe roots $\{u_{2,j}\}_{j=1}^{n_2}$ has the effect of introducing an extra pair of "mirror" cuts in the integral equation for the Bethe roots $\{u_{1,j}\}_{j=1}^{n_1}$. Denoting the end points of the cut $\tilde{C}_+$ as a and $b = -a^*$ and writing symbolically $\tilde{\mathcal{C}}_+ = [a, b]$ (knowing that $\tilde{\mathcal{C}}_+$ is not a straight line) the other cuts are $[-b, -a]$, $[c^2/a, c^2/b]$ and $[-c^2/b, -c^2/a]$. Such a 4-cut integral equation generically has a solution in terms of hyper-elliptic integrals. However, since the weight of the additional cuts can be written in the form $n = 2\cos(p\pi/q)$ with p and q co-prime integers ($p = 1$, $q = 3$) we expect to have a situation which generalizes the above mentioned rational points of the $O(n)$ model on a random lattice. This indicates that the solution can be at most elliptic. As the present parametrization is designed to study the system for small values of c we can assume that $|c| < |a| = |b|$. Then the cuts $[c^2/a, c^2/b]$ and $[-c^2/b, -c^2/a]$ are “inside” (i.e., closer to the origin than) the cuts $[a, b]$ and $[-b, -a]$. When $c \to 0$ these inner cuts shrink to zero and disappear. In this limit we recover the $O(n = -2)$ model of reference [19]. As $|c| \to |a|$ (or $\beta \to (\beta_c(\alpha))_-$) the two sets of cuts approach each other and a singularity occurs. Eq. (22) looses its meaning, an obvious sign being the divergence of the pre-factor $q^2/(q^2 - c^2)$. As mentioned above, this explains why the string state dual to the gauge theory operator considered does not need to be of the same type for small and for large c.

5. The case of large c

To study the case where c is large, let us return to Eq. (19) and choose another change of variables. In this case we set

$$u = \frac{2ip}{1 + p^2/c^2}, \qquad u' = \frac{2iq}{1 + q^2/c^2}, \tag{27}$$

which we notice is well-behaved as $c \to \infty$ but singular as $c \to 0$. Now, we find

$$\sqrt{\frac{c^2 + u^2}{c^2 + u'^2}} = \frac{(1 - p^2/c^2)(1 + q^2/c^2)}{(1 + p^2/c^2)(1 - q^2/c^2)}. \tag{28}$$

In accordance with the remark just above, this formula gives rise to the correct asymptotic expansion as $c \to \infty$ but not as $c \to 0$. In the new variables the integral equation (19) reads, with $du\,\sigma(u) \equiv dq\,\rho(q)$

$$\begin{aligned} &\frac{1 + p^2/c^2}{2p} + 2\pi m i \\ &\quad = \frac{1}{2} \oint_{\tilde{\mathcal{C}}_+} dq\,\rho(q) \left(\frac{1 + q^2/c^2}{1 - q^2/c^2}\right) \left\{\frac{1 - qp/c^2}{p + q} + \frac{2/c^2(p - q)}{1 + qp/c^2} + \frac{1/c^2(p + q)}{1 - qp/c^2} + \frac{2(1 + qp/c^2)}{p - q}\right\}, \end{aligned} \tag{29}$$

where $p \in \tilde{\mathcal{C}}_+$ with $\tilde{\mathcal{C}}_+$ being the contour for the transformed roots q. The boundary equation (16) turns into

$$\beta = \frac{\alpha}{2} - \frac{1}{c} \int_{\tilde{\mathcal{C}}_+} dq\,\rho(q) \frac{2iq}{1 - q^2/c^2}, \tag{30}$$

and the expression for γ reads

$$\gamma = -\frac{\lambda}{16\pi^2 L} \int_{\tilde{\mathcal{C}}_+} dq\,\rho(q) \frac{(1 + q^2/c^2)^2}{q^2}. \tag{31}$$

This time a natural definition of the resolvent is

$$W(p) = \int_{\tilde{\mathcal{C}}_+} dq\,\rho(q) \left(\frac{1 + q^2/c^2}{1 - q^2/c^2}\right) \frac{1 + qp/c^2}{p - q}, \tag{32}$$

and γ can be expressed as

$$\gamma = \frac{\lambda}{16\pi^2 L}\frac{\partial}{\partial p}\left(W(p) - W\left(-\frac{c^2}{p}\right)\right)\bigg|_{p=0}. \tag{33}$$

Once again, apart from $W(p)$ the integral equation involves $W(-p)$, $W(c^2/p)$ and $W(-c^2/p)$. Hence, we discover that the effect of the Bethe roots $\{u_{2,j}\}_{j=1}^{n_2}$ has been to introduce an extra pair of "mirror cuts" in the integral equation for $\{u_{1,j}\}_{j=1}^{n_1}$ so that the density $\rho(q)$ now effectively obeys a 4-cut integral equation. Also in this case the integral equation shares some features with both the one of the $O(n)$ model on a random lattice [24] and the one of the plaquette model of [28]. Furthermore, due to the weights of the various cuts we expect to have a situation which generalizes the rational points of the $O(n)$ model and thus a solution which is at most elliptic. Denoting the end points of the cut $\tilde{\mathcal{C}}_+$ as a and $b = -a^*$ and writing symbolically $\tilde{\mathcal{C}}_+ = [a, b]$ (still knowing that $\tilde{\mathcal{C}}_+$ is not a straight line) the other cuts are $[-b, -a]$, $[c^2/a, c^2/b]$ and $[-c^2/b, -c^2/a]$. The present parametrization is designed to study the case where c is large. Therefore, let us consider $|c| > |a| = |b|$. In this case the cuts $[c^2/a, c^2/b]$ and $[-c^2/b, -c^2/a]$ are "outside" (i.e., further from the origin than) the cuts $[a, b]$ and $[-b, -a]$. When $c \to \infty$ the two outer cuts move out to infinity and disappear. In this limit we recover the simple $O(n = -1)$ integral equation studied in Ref. [17]. When $|c| \to |a|$ (or $\beta \to (\beta_c(\alpha))_+$) the two sets of cuts approach each other and for $|c| = |a|$ a singularity occurs. This coincides with the divergence of the pre-factor $(1 + q^2/c^2)/(1 - q^2/c^2)$.

6. Perturbative expansion for $\beta \approx \alpha/2$

Let us define

$$\epsilon = \frac{\alpha}{2} - \beta, \tag{34}$$

and let us consider $\epsilon \ll \alpha, \beta$. In terms of angular momenta we have $(J_1, J_2, J_3) = ((1-\alpha)L, (\alpha/2 + \epsilon)L, (\alpha/2 - \epsilon)L)$ or

$$\epsilon = \frac{1}{2L}(J_2 - J_3), \qquad J_1 > J_2, J_3. \tag{35}$$

The operator in question is expected to be the gauge theory dual of a slightly perturbed version of the circular three-spin state of [14,15] which has angular momenta (J, J', J'), $J > J'$. Obviously, a small value of ϵ corresponds to a large value of c. Expanding the expression (13) for large c we get

$$W_2(u) = \fint_{\mathcal{C}_+} du' \frac{\sigma(u')}{u + u'} - \frac{1}{2c^2}\int_{\mathcal{C}_+} du'\, u'\sigma(u'). \tag{36}$$

Inserting this into the integral equation (6) and making use of the boundary equation (16) gives

$$\frac{1}{u} - 2\pi m - \frac{\epsilon}{2c} = 2\fint_{\mathcal{C}_+} du' \frac{\sigma(u')}{u - u'} + \int_{\mathcal{C}_+} du' \frac{\sigma(u')}{u + u'}, \quad u \in \mathcal{C}_+. \tag{37}$$

This equation can again be recognized as the saddle point equation of the $O(n)$ model on a random lattice for $n = -1$, with the terms on the left-hand side playing the role of the derivative of the potential. In terms of the resolvent of Eq. (9) the equation reads

$$W(u + i0) + W(u - i0) - W(-u) = V'(u), \quad u \in \mathcal{C}_+, \tag{38}$$

where

$$V'(u) = \frac{1}{u} - 2\pi\left(m + \frac{\epsilon}{4\pi c}\right). \tag{39}$$

The asymptotic behaviour of $W(u)$ is

$$W(u) \sim \frac{\alpha}{2u} + \frac{\epsilon c}{u^2}, \quad \text{as } u \to \infty. \tag{40}$$

Defining

$$W(u) = W_r(u) + W_s(u), \tag{41}$$

where $W_r(u)$ and $W_s(u)$ are respectively the regular and the singular part of $W(u)$, we have

$$W_r(u) = \frac{1}{3}\bigl(2V'(u) + V'(-u)\bigr). \tag{42}$$

Furthermore, by analyticity considerations [27] (see also [17]) one can show that $W_s(u)$ has to fulfill the following cubic equation

$$\bigl(W_s(u)\bigr)^3 - R_1(u)W_s(u) - R_2(u) = 0, \tag{43}$$

where

$$R_1(u) = 4\pi^2\left(m + \frac{\epsilon}{4\pi c}\right)^2 + \frac{1}{3u^2},$$

$$R_2(u) = \frac{2}{27u^3} + \left(\frac{\alpha}{2} - \frac{1}{3}\right)8\pi^2\left(m + \frac{\epsilon}{4\pi c}\right)^2\frac{1}{u}.$$

Solving Eq. (43) perturbatively for large u we get a relation between ϵ and c. It reads

$$\frac{1}{4\pi c} = \frac{m\epsilon}{\alpha(1 - 3\alpha/4)}. \tag{44}$$

Next, solving Eq. (43) perturbatively for small u we get an expression for γ (cf. Eq. (10))

$$\gamma = \frac{\lambda\alpha}{2L}\left(m + \frac{\epsilon}{4\pi c}\right)^2 \approx \frac{\lambda\alpha m^2}{2L}\left(1 + \frac{2\epsilon^2}{\alpha(1 - 3\alpha/4)}\right). \tag{45}$$

We can express α as

$$\alpha = 1 - \frac{J_1}{L} \equiv 1 - j_1, \tag{46}$$

which leads to the following expression for γ

$$\gamma = \frac{\lambda m^2}{2L}\left(1 - j_1 + 8\epsilon^2\frac{1}{1 + 3j_1} + \mathcal{O}\bigl(\epsilon^4\bigr)\right). \tag{47}$$

Using the formalism of [10] one can derive in parametric form an expression for the semi-classical energy of a three-spin circular string of elliptic type with winding number m. Using the same notation for the angular momenta as above the result reads [29]

$$E = L + \frac{\lambda m^2}{2L}\left[\frac{4}{\pi^2}\frac{\mathrm{K}(t)}{\mathrm{E}(t)}\Bigl(\bigl(\mathrm{E}(t)\bigr)^2 + j_1(t-1)\bigl(\mathrm{K}(t)\bigr)^2\Bigr)\right], \tag{48}$$

where t is determined as a function of ϵ and j_1 from the following equation

$$\epsilon = \frac{1}{t} - \frac{1}{2} - \frac{\mathrm{E}(t)}{t\,\mathrm{K}(t)} + j_1\left[\frac{1}{t} - \frac{1}{2} - \frac{\mathrm{K}(t)}{t\,\mathrm{E}(t)} + \frac{\mathrm{K}(t)}{\mathrm{E}(t)}\right], \tag{49}$$

with $\mathrm{K}(t)$ and $\mathrm{E}(t)$ being the elliptic integrals of the first and the second kind, respectively. Solving Eq. (49) for t in terms of j_1 to leading order in ϵ and inserting the solution in Eq. (48) one finds that to the given order in ϵ the λ-dependent part of E precisely agrees with the expression for γ in Eq. (47), i.e.,

$$E = L + \frac{\lambda m^2}{2L}\left(1 - j_1 + 8\epsilon^2 \frac{1}{1+3j_1} + \mathcal{O}\big(\epsilon^4\big)\right). \tag{50}$$

Thus, we propose that the dual of the operator considered here is the three-spin circular elliptic string of [10]. It would of course be interesting to reproduce Eqs. (48) and (49) from an exact solution of the integral equation (29).

7. Conclusion

We have studied a class of single trace scalar, holomorphic gauge theory operators with general R-charge assignment $(J_1, J_2, J_3) = ((1-\alpha)L, (\alpha-\beta)L, \beta L)$ in the limit $L \to \infty$ with $\alpha \in [0, 1/2]$ and $\beta \in [0, \alpha/2]$. Analyzing the relevant Bethe equations we have exposed the analyticity structure of the problem of determining the one-loop anomalous dimension of these operators. In particular, we have located a line of critical points in the parameter space, $\beta = \beta_c(\alpha)$, which explains why the nature of the dual string, as observed, does not need to be the same for $\beta \to 0$ and $\beta \to \alpha/2$. Furthermore, we have proposed that for $\beta > \beta_c(\alpha)$ the gauge theory operators studied are the duals of the circular elliptic three-spin string of [10] and supported this by a perturbative calculation. It would of course be interesting to identify the dual string state also for $\beta < \beta_c(\alpha)$. The only candidate available at the moment seems to be the hyper-elliptic three-spin state of [10] which generalizes the two-spin folded string of [15]. As we have seen there exists a mechanism encoded in the Bethe equations which effectively leads to the appearance of extra cuts but it seems that the Bethe root configurations studied here are still not general enough to lead to a true hyper-elliptic structure. In the integrable Neumann model the hyper-elliptic structure is reflected by the appearance of two integer winding number like parameters. The corresponding (but not identical) degrees of freedom of the folded string are the number of foldings and the number of so-called bend points. The folded three-spin rigid string of [10] needs to have at least one bend-point. In the case of the two-spin folded string it is known that the parameter m in Eq. (6) counts the number of foldings [20,21] but it is not obvious how bend points would manifest themselves on the gauge theory side. A detailed understanding of the nature of the operators studied for $\beta < \beta_c(\alpha)$ and their relation to semi-classical string states requires an exact solution of the integral equation (22) and we hope to report on this in the future [30]. An exact expression for the resolvent associated with the density of Bethe roots $\{u_{1,j}\}_{j=1}^{n_1}$ would not only give us access to the one-loop anomalous dimension of our gauge theory operators but also to the infinite set of conserved higher charges [22]. In this connection it should be mentioned that one might envisage a more direct way of comparing gauge theory and string theory results, namely by directly deriving the relevant string sigma model from the spin chain. So far this has only been accomplished for the simple case of the $SU(2)$ sub-sector of the $SO(6)$ integrable spin chain [31]. Another interesting line of investigation which has also only been pursued in a sub-sector not including the operators considered here is the derivation of the dilatation operator to higher loop orders [3,32,33] and the formulation of the corresponding Bethe ansatz [34].

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