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Fixed Point Algebras for Easy Quantum Groups

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Abstract. Compact matrix quantum groups act naturally on Cuntz algebras. The first author isolated certain conditions under which the fixed point algebras under this action are Kirchberg algebras. Hence they are completely determined by their K-groups. Building on prior work by the second author, we prove that free easy quantum groups satisfy these conditions and we compute the K-groups of their fixed point algebras in a general form. We then turn to examples such as the quantum permutation group S_n^+, the free orthogonal quantum group O_n^+ and the quantum reflection groups H_n^{s+}. Our fixed point-algebra construction provides concrete examples of free actions of free orthogonal easy quantum groups, which are related to Hopf–Galois extensions.

Key words: K-theory; Kirchberg algebras; easy quantum groups; noncrossing partitions; fusion rules; free actions; free orthogonal quantum groups; quantum permutation groups; quantum reflection groups

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In memory of the late Professor John E. Roberts.

1 Introduction

This article was initiated from the meeting of the two authors and their respective interests in easy quantum groups and the fixed point algebra construction. Let us start by reminders on the setting of the present article.

Compact quantum groups (CQGs) were defined by Woronowicz and further studied in a series of papers [41, 42, 44]. Following the paradigm of noncommutative geometry, the general idea is to describe all properties of a compact group G in terms of its algebra $C(G)$ of (continuous) functions, using in particular a *comultiplication* $\Delta\colon C(G) \to C(G \times G) \simeq C(G) \otimes C(G)$ to realise the group law $\mu\colon G \times G \to G$. If we then consider (possibly noncommutative) C^*-algebras with such a comultiplication, we get CQGs as an extension of compact groups. Of course, additional assumptions are needed to make the above rigorous (see Section 2.1 below). To be more precise, we will mainly deal with *compact matrix quantum groups* (CMQGs).

Among CMQGs, there is a class of particular examples, called *easy quantum groups*. Categories of partitions and easy quantum groups were first defined by Banica and Speicher in [5] in the orthogonal case. Tarrago and the second author extended their approach to the unitary setting, see [37]. To each easy quantum group is associated a category of partitions, which provides a way to "visualise" it. The basic idea of easy quantum groups is that they should form a tractable sub-class of CMQGs, since they can be described and manipulated *via* their category of partitions, by a Tannaka–Krein type argument [43]. This line of argument is illustrated by the article [18], where Freslon and the second author provided a description of fusion rules for easy quantum groups based on their categories of partitions.

On another note, the *Cuntz algebra* $\mathcal{O}_n$ has been defined as a universal C^*-algebra by Cuntz in his paper [11] and has evolved over time into one of the most important examples of C^*-algebras, with applications to classification theory and physics. An example of an application is provided by Doplicher and Roberts's abstract, Tannaka–Krein like duality results in a series of articles (see, e.g., [15, 16]) for actions of (ordinary) compact groups on C^*-algebras. This discovery motivated a considerable interest (see for instance [8, 33, 34]). A basic step of Doplicher–Roberts's duality theory is to consider so-called "canonical actions" of compact groups on Cuntz algebras. A source of inspiration for further research in this direction is the article [32], where Pinzari introduces a fixed point algebra $\mathcal{O}_{\lambda(G)}$ from the regular representation λ acting on $\mathcal{O}_{L^2(G)}$ and proves that given two compact groups G, G', the fixed point algebras $\mathcal{O}_{\lambda(G)}$ and $\mathcal{O}_{\lambda(G')}$ are isomorphic as $\mathbb{Z}$-algebras if and only if $C^*(G) \simeq C^*(G')$.

Motivated by the desire of generalising Doplicher–Roberts theory and following the articles [10, 27, 31], the first author considered an action of a CQG $\mathbb{G}$ on a Cuntz algebra and described its fixed point algebra. More precisely, two conditions (C1) and (C2) were introduced, which ensure that the fixed point algebra is actually a Kirchberg algebra in the UCT class $\mathscr{N}$. Kirchberg–Phillips's classification theory (see [23, 24]) then proves that up to $*$-isomorphism, the fixed point algebra is characterised by its K-theory. In [19], examples of computations of the K-theory of the fixed point algebra are given – they only depend on the fusion rules of $\mathbb{G}$.

In the present article, we combine these two directions of research to describe the fixed point algebras of actions of the free orthogonal quantum group O_n^+, the quantum permutation group S_n^+ and the quantum reflection group H_n^{s+}. An interesting feature of the present fixed point algebra construction is that it provides a very concrete realisation of the intertwiner spaces defining the easy quantum group (see Proposition 3.2 below).

The main results of this paper are the reformulation and characterisation of the hypotheses (C1) and (C2) of the fixed point algebra construction theory of [19] in terms of partition categories (see Theorems 3.6 and 3.7 below), together with the identifications of K-theory for the fixed point algebras associated to the natural representations of O_n^+, S_n^+ and H_n^{s+} (see Theorems 4.1 and 5.11 below). The fixed point algebras for O_n^+ and S_n^+ are isomorphic while the one for H_n^{s+} is very different. Thus, in some sense, the actions of S_n^+ and O_n^+ are somehow "similar" while H_n^{s+} acts very differently. Moreover, since these fixed point algebras depend only on the fusion rules of the CQGs at hand, this phenomenon manifests concretely that the fusion rules of H_n^{s+} are very different from those of O_n^+ and S_n^+.

This article is organised as follows: in Section 2, we start by a review of the notions of CQGs and CMQGs, before presenting the notions of categories of partitions and easy quantum groups and discussing Cuntz algebras and actions of CMQGs on these. Section 3 is devoted to the fixed point algebra construction properly speaking, while Sections 4 and 5 are detailed studies of examples, namely the free orthogonal quantum group O_n^+ and the quantum permutation group S_n^+ on the one hand, and the quantum reflection group H_n^{s+} on the other hand.

2 Reminders and review

2.1 Compact matrix quantum groups

In this article, we consider only *minimal* tensor products of C^*-algebras. We will deal with *compact quantum groups* (CQGs) which we denote by $\mathbb{G}$. They are defined by a separable unital C^*-algebra $C(\mathbb{G})$ together with a unital $*$-algebra homomorphism $\Delta\colon C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ which satisfies coassociativity and cancellation properties – for more details on these objects and their representations, see [41, 44].

These compact quantum groups admit (unitary) representations or, equivalently *actions* on Hilbert spaces and C^*-algebras. A *unitary representation* u of $\mathbb{G}$ on a Hilbert space $\mathscr{H}$ of finite dimension d is a $C(\mathbb{G})$-valued $d \times d$ matrix $U = (u_{ij}) \in M_d(C(\mathbb{G}))$ which is unitary and satisfies the coassociativity property

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}.$$

In particular, for any $\mathbb{G}$, we have a trivial representation denoted by ϵ, defined by the (1×1)-matrix $1 \in C(\mathbb{G})$.

For two fixed representations $u \in M_{d_1}(C(\mathbb{G}))$ and $v \in M_{d_2}(C(\mathbb{G}))$ acting on the Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, a linear map $T\colon \mathscr{H}_1 \to \mathscr{H}_2$ is an *intertwiner* (see [41]) if

$$Tu = vT.$$

We denote by $\mathrm{Hom}(u, v)$ the set of interwiners between u and v. If $\mathrm{Hom}(u, v)$ includes an invertible map, then we say that u and v are *equivalent*. Just like for ordinary compact groups, any representation of $\mathbb{G}$ is equivalent to a unitary one, therefore we will only consider unitary representations and we will refer to these as "representations". A representation u is called *irreducible* if $\mathrm{Hom}(u, u) \simeq \mathbb{C}$. Representations of CQGs admit notions of direct sum – with the above notations, $u \oplus v$ is a representation on $\mathscr{H}_1 \oplus \mathscr{H}_2$ – and tensor product – $u \otimes v$ is then a representation on $\mathscr{H}_1 \otimes \mathscr{H}_2$.

It is a well-known property of CQGs (see, e.g., [44, Theorem 3.4]) that every unitary representation of a CQG is unitarily equivalent to a direct sum of irreducible unitary representations and any irreducible representation is finite-dimensional. Given two representations u and v, we use the notation $u \leqslant v$ to express that the representation u is *included* in v (i.e., there is an isometry in $\mathrm{Hom}(u, v)$).

Of particular interest for our investigations are *compact matrix quantum groups* (CMQGs), a particular class of CQGs. A CMQG is given by a so-called *fundamental representation* u, whose coefficients generate a dense subalgebra of $C(\mathbb{G})$. One can define CMQGs in the following way:

Definition 2.1. A *compact matrix quantum group* is defined by a unital C^*-algebra A generated by elements u_{ij}, $1 \leq i, j \leq n$ such that $u = (u_{ij})$ and $u^t = (u_{ji})$ are invertible and the map

$$\Delta\colon\ A \to A \otimes A, \qquad u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}$$

is a $*$-homomorphism. A CMQG (A, u) is a *quantum subgroup* of (B, v), if there is a surjective $*$-homomorphism from B to A mapping v_{ij} to u_{ij}.

Of course, in the definition of a CMQG (A, u), we should think of A as the functions on the "quantum space" $\mathbb{G}_A$ (and (B, v) as associated to $\mathbb{G}_B$). Gelfand duality, which is contravariant, therefore explains why the "inclusion $\mathbb{G}_A \hookrightarrow \mathbb{G}_B$" is represented by a *surjective* morphism $B \to A$.

In this paper, we use the notations $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $\mathbb{N} = \{1, 2, \ldots\}$ (as opposed to [19], where $\mathbb{N} = \{0, 1, \ldots\}$).

2.2 Categories of partitions and easy quantum groups

Colored partitions are key tools for the introduction of unitary easy quantum groups, as done in [37]. They generalize the orthogonal easy quantum groups of Banica and Speicher [5]. Let $k, l \in \mathbb{N}_0$ and consider a finite ordered set with $k + l$ elements each being colored either in white or in black. A *partition* is a decomposition of this set into disjoint subsets, the *blocks*. Let

$P^{\circ\bullet}(k,l)$ denote the set of all such partitions and put $P^{\circ\bullet} := \bigcup_{k,l\in\mathbb{N}_0} P^{\circ\bullet}(k,l)$. We usually use a pictorial representation of a partition involving lines representing the block structure, and we assume k of these points to be placed on an upper row and l on a lower row, see [36]. If these lines may be drawn in a way such that they do not cross, we call the partition *noncrossing*. Let $NC^{\circ\bullet}$ be the set of all noncrossing partitions. Some examples of partitions are the *singleton partitions* $\uparrow_{\circ} \in P^{\circ\bullet}(0,1)$ consisting of a single white lower point or $\uparrow_{\bullet} \in P^{\circ\bullet}(0,1)$, the *pair partitions* $\sqcap_{\circ\bullet}$ and $\sqcap_{\bullet\circ}$ in $P(0,2)$ consisting of two lower points of different colors which are in the same block, the *identity partitions* $\vert^{\circ}_{\circ}, \vert^{\bullet}_{\bullet} \in P(1,1)$ consisting of one upper and one lower point both of the same color and both in the same block, or the partitions $b_s \in P^{\circ\bullet}(0,s)$ consisting of s lower white points in a single block.

We have the following operations on the set $P^{\circ\bullet}$ of partitions. The *tensor product* of $p \in P^{\circ\bullet}(k,l)$ and $q \in P^{\circ\bullet}(k',l')$ is $p \otimes q \in P^{\circ\bullet}(k+k', l+l')$ obtained by placing p and q side by side. The *composition* of $p \in P^{\circ\bullet}(k,l)$ and $q \in P^{\circ\bullet}(l,m)$ is $qp \in P^{\circ\bullet}(k,m)$ obtained by placing p above q. We may only perform it when the color pattern of the lower points of p matches the upper color pattern of q. The *involution* of $p \in P^{\circ\bullet}(k,l)$ is $p^* \in P^{\circ\bullet}(l,k)$ obtained by reflecting p at the horizontal axis. The *rotation* of the left upper point of $p \in P^{\circ\bullet}(k,l)$ to the lower row is a partition in $P^{\circ\bullet}(k-1, l+1)$. When rotating a point, its color is inverted but its membership to a block remains untouched. Likewise we have a rotation of the left lower or right upper/lower points. These operations (tensor product, composition, involution and rotation) are called the *category operations*.

A collection $\mathcal{C}$ of subsets $\mathcal{C}(k,l) \subseteq P^{\circ\bullet}(k,l)$ (for every $k,l \in \mathbb{N}_0$) is a *category of partitions* if it is invariant under the category operations and if the identity partitions $\vert^{\circ}_{\circ}, \vert^{\bullet}_{\bullet} \in P^{\circ\bullet}(1,1)$ and the pair partitions $\sqcap_{\circ\bullet}, \sqcap_{\bullet\circ} \in P^{\circ\bullet}(0,2)$ are in $\mathcal{C}$. Note that rotation may be deduced from the other category operations. We write $\mathcal{C} = \langle p_1, \dots, p_m \rangle$, if $\mathcal{C}$ is the smallest category of partitions containing the partitions $p_1, \dots, p_m$. We say that $\mathcal{C}$ is *generated* by $p_1, \dots, p_m$.

Let $n \in \mathbb{N}$. Given $p \in P^{\circ\bullet}(k,l)$ and two multi-indices $(i(1), \dots, i(k))$, $(j(1), \dots, j(l))$ with entries in $\{1, \dots, n\}$, we can label the diagram of p with these numbers (the upper and the lower row both are labelled from left to right, respectively) and we put

$$\delta_p(i,j) = \begin{cases} 1, & \text{if each block of } p \text{ connects only equal indices,} \\ 0, & \text{otherwise.} \end{cases}$$

We fix a basis $e_1, \dots, e_n$ of $\mathbb{C}^n$ and define a map $T_p \colon (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$ associated to p by

$$T_p(e_{i(1)} \otimes \dots \otimes e_{i(k)}) = \sum_{1 \le j(1), \dots, j(l) \le n} \delta_p(i,j) \cdot e_{j(1)} \otimes \dots \otimes e_{j(l)}.$$

We use the convention that $(\mathbb{C}^n)^{\otimes 0} = \mathbb{C}$, i.e., for a partition $p \in P^{\circ\bullet}(0,l)$ with no upper points, $T_p(1)$ is actually a vector in $(\mathbb{C}^n)^{\otimes l}$. Note that the colors of the points of p do not play a role in the definition of T_p. The operations on the partitions match nicely with canonical operations of the linear maps T_p, namely we have $T_{p\otimes q} = T_p \otimes T_q$, $T_{qp} = n^{-b(p,q)} T_q T_p$ and $T_{p^*} = (T_p)^*$. Here, $b(p,q)$ denotes the number of removed blocks when composing p and q. The maps T_p can be normalized in such a way that they become partial isometries, see [18].

From a category $\mathcal{C}$ and the realisation $p \mapsto T_p$, we may construct a concrete monoidal W^*-category with a distinguished object u, and we thus may assign a CMQG (A,u) to it. We call it the *easy quantum group* associated to $\mathcal{C}$. If $\mathcal{C} \subset NC^{\circ\bullet}$, then it is called a *free easy quantum group*. Given a color string $r \in \{\circ, \bullet\}^k$, we may define u^r as the tensor product of copies u and $\bar{u}$. Here $\circ$ corresponds to u while $\bullet$ corresponds to $\bar{u}$. We then may say: a CMQG $S_n \subset \mathbb{G} \subset U_n^+$ is easy, if there is a category of partitions $\mathcal{C}$ such that the intertwiner spaces of $\mathbb{G}$ are of the form

$$\{T \,|\, Tu^r = u^s T\} = \text{span}\{T_p \,|\, p \in \mathcal{C}(k,l) \text{ with color strings } r \text{ (upper) and } s \text{ (lower)}\}.$$

Easy quantum groups are a class of CMQGs with a quite intrinsic combinatorial structure. They are completely determined by their associated category of partitions and in many cases certain quantum algebraic properties of an easy quantum group may be traced back to certain combinatorial properties of its category of partitions. A simple criterion for verifying that a given CMQG is easy is contained in the following lemma.

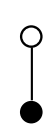
Let $n \in \mathbb{N}$ and let A be a C^*-algebra generated by n^2 elements u_{ij}, $1 \leq i,j \leq n$. Let $p \in P^{\circ\bullet}(k,l)$ be a partition with upper color string $r \in \{\circ,\bullet\}^k$ and lower color string $s \in \{\circ,\bullet\}^l$. We say that the generators u_{ij} *fulfill the relations* $R(p)$, if for all $\beta(1),\dots,\beta(l) \in \{1,\dots,n\}$ and for all $i(1),\dots,i(k) \in \{1,\dots,n\}$, we have

$$\sum_{\alpha(1),\dots,\alpha(k)=1}^{n} \delta_p(\alpha,\beta) u^{r_1}_{\alpha(1)i(1)} \cdots u^{r_k}_{\alpha(k)i(k)} = \sum_{\gamma(1),\dots,\gamma(l)=1}^{n} \delta_p(i,\gamma) u^{s_1}_{\beta(1)\gamma(1)} \cdots u^{s_l}_{\beta(l)\gamma(l)}.$$

The left-hand side of the equation is $\delta_p(\varnothing,\beta)$ if $k=0$ and analogously $\delta_p(i,\varnothing)$ for the right-hand side if $l=0$. Furthermore, $u^\circ_{ij} := u_{ij}$ and $u^\bullet_{ij} := u^*_{ij}$.

Lemma 2.2 ([37, Corollary 3.12])**.** *Let $p_1,\dots,p_m \in P^{\circ\bullet}$ be partitions and let A be the universal C^*-algebra generated by elements u_{ij}, $1 \leq i,j \leq n$ such that $u=(u_{ij})$ and $\bar{u}=(u^*_{ij})$ are unitary (i.e., $\sum_k u^*_{ik}u_{jk} = \sum_k u^*_{ki}u_{kj} = \sum_k u_{ik}u^*_{jk} = \sum_k u_{ki}u^*_{kj} = \delta_{ij}$) and u_{ij} satisfy relations $R(p_l)$ for $l=1,\dots,m$. Then A is an easy quantum group with associated category $\mathcal{C} = \langle p_1,\dots,p_m\rangle$.*

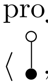
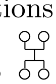
Proof. The proof relies on the fact that the relations $R(p)$ are fulfilled if and only if T_p intertwines u^r and u^s. See [37, Lemma 3.9, Corollary 3.12] for details. ■

Note that for $p =$  the relation $R(p)$ effects that all u_{ij} are selfadjoint. On the combinatorial level this amounts to having partitions whose points have no colors. In this case, we recover the orthogonal easy quantum groups of Banica and Speicher [5].

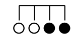
Example 2.3.

(a) The *free orthogonal quantum group* O_n^+ is given by the universal unital C^*-algebra $C(O_n^+)$ generated by selfadjoint elements u_{ij}, $1 \leq i,j \leq n$ such that $u=(u_{ij})$ is an orthogonal matrix. It is easy with category $\mathcal{C} = \langle$  $\rangle = NC_2$, the set of all (noncolored) noncrossing pair partitions (each block consists of exactly two elements). We omit to write down the generating partitions ,  and ,  since they are contained in every category, by definition.

(b) The *free unitary quantum group* U_n^+ is given by the universal unital C^*-algebra $C(U_n^+)$ generated by elements u_{ij}, $1 \leq i,j \leq n$ such that $u=(u_{ij})$ and $\bar{u}=(u^*_{ij})$ are unitary matrices. It is an easy quantum group with $\mathcal{C} = \langle\varnothing\rangle$.

(c) The *quantum permutation group* S_n^+ is given by the universal unital C^*-algebra $C(S_n^+)$ generated by projections u_{ij}, $1 \leq i,j \leq n$ such that $\sum_k u_{ik} = \sum_k u_{kj} = 1$. It is easy with category $\mathcal{C} = \langle$ , ,  $\rangle$.

The quantum groups in (a), (b) and (c) were introduced by Wang [39, 40].

(d) For $s \in \mathbb{N}$, the *quantum reflection group* H_n^{s+} is given by the universal unital C^*-algebra $C(H_n^{s+})$ generated by elements u_{ij}, $1 \le i,j \le n$ such that $u = (u_{ij})$ and $\bar{u} = (u_{ij}^*)$ are unitaries, all u_{ij} are partial isometries and we have $u_{ij}^s = u_{ij}u_{ij}^* = u_{ij}^*u_{ij}$. We have $H_n^{1+} = S_n^+$ and $H_n^{2+} = H_n^+$, the latter one being Banica, Bichon and Collins's hyperoctahedral quantum group [4]. The quantum reflection groups H_n^{s+} were studied by Banica, Belinschi, Capitaine and Collins in [3]. They are easy with $\mathcal{C} = \langle b_s,$  $\rangle$, see [37]. Both H_n^+ and H_n^{s+} can be traced back to Bichon's work on free wreath product [9].

2.3 Cuntz algebra

The Cuntz algebra $\mathcal{O}_n$ is the universal unital C^*-algebra generated by isometries $S_1,\dots,S_n$ such that $\sum_i S_iS_i^* = 1$. It has been defined and studied by Cuntz in [11]. He first proved that $\mathcal{O}_n$ is a crossed product by an endomorphism and then introduced a criterion for (what we call today) purely infinite and simple algebras. He used it to show that all $\mathcal{O}_n$ are simple.

By an *action* of a CQG on a C^*-algebra A we mean a faithful unital $*$-homomorphism $\alpha\colon A \to A \otimes C(\mathbb{G})$ such that $(\alpha \otimes \mathrm{Id}) \circ \alpha = (\mathrm{Id} \otimes \Delta) \circ \alpha$ holds and $(1 \otimes C(\mathbb{G}))\alpha(A)$ is linearly dense in $A \otimes C(\mathbb{G})$. In our case, we have the following action on the Cuntz algebra, as observed by Cuntz [12] (see also [25]).

Proposition 2.4. *Let $\mathbb{G}$ be a CMQG such that $\mathbb{G} \subset U_n^+$ (i.e., the matrices u and $\bar{u}$ are unitaries). It acts on $\mathcal{O}_n$ by*

$$\alpha(S_i) = \sum_{j=1}^n S_j \otimes u_{ji}.$$

Proof. For the convenience of the reader, we give the proof since it is a short argument. By the universal property of $\mathcal{O}_n$, the unital $*$-homomorphism α exists, and it is faithful since $\mathcal{O}_n$ is simple. It is straightforward to check that $(\alpha \otimes \mathrm{Id}) \circ \alpha = (\mathrm{Id} \otimes \Delta) \circ \alpha$ is satisfied, and the computation (using orthogonality of u)

$$\sum_k (1 \otimes u_{ik}^*)\left(\sum_l S_l \otimes u_{lk}\right) = S_i \otimes 1$$

shows that the linear span of $(1 \otimes C(\mathbb{G}))\alpha(A)$ equals $A \otimes C(\mathbb{G})$. ■

It is a well-known fact that we may find copies of matrix algebras inside the Cuntz algebra. Indeed, the monomials $S_{j(1)}\cdots S_{j(k)}S_{i(k)}^*\cdots S_{i(1)}^*$ satisfy the relations of matrix units, thus we have

$$\mathrm{span}\{S_{j(1)}\cdots S_{j(k)}S_{i(k)}^*\cdots S_{i(1)}^* \,|\, 1 \le i(t), j(t) \le d\} \cong B\big((\mathbb{C}^n)^{\otimes k}\big).$$

Thus, $S_{j(1)}\cdots S_{j(k)}S_{i(k)}^*\cdots S_{i(1)}^*$ corresponds exactly to the rank one operator mapping $e_{i(1)} \otimes \cdots \otimes e_{i(k)}$ to $e_{j(1)} \otimes \cdots \otimes e_{j(k)}$. More generally, we may identify linear maps

$$T\colon\ (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l},$$
$$e_{i(1)} \otimes \cdots \otimes e_{i(k)} \mapsto \sum_{j(1),\dots,j(l)} a(i(1),\dots,i(k),j(1),\dots,j(l))e_{j(1)} \otimes \cdots \otimes e_{j(l)}$$

(where $a(i(1),\dots,i(k),j(1),\dots,j(l)) \in \mathbb{C}$) with elements

$$\sum_{i(1),\dots,i(k),j(1),\dots,j(l)=1} a(i(1),\dots,i(k),j(1),\dots,j(l))S_{j(1)}\cdots S_{j(l)}S_{i(k)}^*\cdots S_{i(1)}^*$$

in $\mathcal{O}_n$. Hence, we may view the maps $T_p : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$ indexed by partitions $p \in P(k,l)$ as elements in $\mathcal{O}_n$ via

$$T_p \leftrightarrow \sum_{i(1),\dots,i(k),j(1),\dots,j(l)=1}^{n} \delta_p(i,j) S_{j(1)} \cdots S_{j(l)} S_{i(k)}^* \cdots S_{i(1)}^*.$$

3 Actions of easy quantum groups on the Cuntz algebra

3.1 The fixed point algebra

As explained in Section 2.3, any given CQMG acts naturally on a Cuntz algebra $\mathcal{O}_n$ for a suitable choice of n. Our aim is to understand the fixed point algebra in the case of easy quantum groups.

Definition 3.1. Let $\mathbb{G}$ be a CMQG with fundamental representation $u = (u_{ij})$ and let α be the action as in Proposition 2.4. The *fixed point algebra* $\mathcal{O}^\alpha$ is defined as

$$\mathcal{O}^\alpha := \{x \in \mathcal{O}_n \,|\, \alpha(x) = x \otimes 1\}.$$

In the case of easy quantum groups, we may read the fixed point algebras directly from the categories of partitions. The following statement appeared in [27, Proposition 3.4], see also [19, Lemma 2.5].

Proposition 3.2. *Given $\mathcal{C}$, let $\mathbb{G}$ be the associated easy QG. The intersections of the fixed point algebra $\mathcal{O}^\alpha$ with the copies of $B((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$ in $\mathcal{O}_n$ (as described in Section 2.3) are given by*

$$\mathcal{O}^\alpha \cap B\big((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l}\big) = \operatorname{span}\{T_p \in \mathcal{O}_n \,|\, p \in \mathcal{C}(k,l) \text{ all points are white}\}.$$

Proof. The general proof may be found in [27, Proposition 3.4], but we give a direct argument here. Let $p \in \mathcal{C}(k,l)$ be a partition with only white points. Hence, in $C(\mathbb{G})$ the relations $R(p)$ hold. Moreover, we use the fact that u is unitary. Now

$$\begin{aligned}
\alpha(T_p) &= \sum_{i,i',j,j'} \delta_p(i,j) S_{j'(1)} \cdots S_{j'(l)} S_{i'(k)}^* \cdots S_{i'(1)}^* \otimes u_{j'(1)j(1)} \cdots u_{j'(l)j(l)} u_{i'(k)i(k)}^* \cdots u_{i'(1)i(1)}^* \\
&= \sum_{i',j'} S_{j'(1)} \cdots S_{j'(l)} S_{i'(k)}^* \cdots S_{i'(1)}^* \\
&\quad \otimes \left(\sum_i \left(\sum_j \delta_p(i,j) u_{j'(1)j(1)} \cdots u_{j'(l)j(l)} \right) u_{i'(k)i(k)}^* \cdots u_{i'(1)i(1)}^* \right) \\
&= \sum_{i',j'} S_{j'(1)} \cdots S_{j'(l)} S_{i'(k)}^* \cdots S_{i'(1)}^* \\
&\quad \otimes \left(\sum_i \left(\sum_s \delta_p(s,j') u_{s(1)i(1)} \cdots u_{s(k)i(k)} \right) u_{i'(k)i(k)}^* \cdots u_{i'(1)i(1)}^* \right) \\
&= \sum_{i',j'} S_{j'(1)} \cdots S_{j'(l)} S_{i'(k)}^* \cdots S_{i'(1)}^* \\
&\quad \otimes \left(\sum_s \delta_p(s,j') \left(\sum_i u_{s(1)i(1)} \cdots u_{s(k)i(k)} u_{i'(k)i(k)}^* \cdots u_{i'(1)i(1)}^* \right) \right) \\
&= \sum_{i',j'} S_{j'(1)} \cdots S_{j'(l)} S_{i'(k)}^* \cdots S_{i'(1)}^* \otimes \left(\sum_s \delta_p(s,j') \delta_{si'} \right) \\
&= \sum_{j'} \delta_p(i',j') S_{j'(1)} \cdots S_{j'(l)} S_{i'(k)}^* \cdots S_{i'(1)}^* \otimes 1 = T_p \otimes 1.
\end{aligned}$$

Conversely, let $T \in \mathcal{O}^\alpha \cap B((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$. Reversing the above computation, we see that T intertwines $u^{\otimes k}$ and $u^{\otimes l}$. But $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l})$ consists exactly of the linear span of all $T_p \in \mathcal{O}_n$ such that $p \in \mathcal{C}(k,l)$ has only white points. ■

Proposition 3.3. *The algebraic result of Proposition 3.2 extends into a topological one, namely,*

$$\mathcal{O}^\alpha = \overline{\mathrm{span}\{T_p \in \mathcal{O}_n \,|\, p \in \mathcal{C}(k,l) \text{ all points are white}, \forall\, k,l \in \mathbb{N}_0\}}.$$

Proof. Lemma 2.7 of [19, p. 1017] (see also [25, Lemma 6]) ensures that "algebraic elements" (i.e., those obtained by (finite) polynomial combination of S_j) of $\mathcal{O}^\alpha$ are dense in $\mathcal{O}^\alpha$. Proposition 3.2 above ensures that any such algebraic element is in the linear span of T_p, so the conclusion follows. ■

3.2 Obtaining Kirchberg algebras

In [19], the first author isolated two conditions which turn $\mathcal{O}^\alpha$ into a Kirchberg algebra. This class of algebras plays a central role in Kirchberg–Phillips classification theory, which proves that Kirchberg algebras are completely classified by their K-groups – see [23, 24] for the original papers, [35] for an overview of classification for nuclear simple C^*-algebras and [38] for the latest developments in this area.

We denote by $\mathcal{T}_u$ the set of (classes of) irreducible representations appearing in the iterated tensor products $u^{\otimes l}$ for $l \in \mathbb{N}_0$.

(C1) For any $v \in \mathcal{T}_u$, we can find $v' \in \mathcal{T}_u$ such that the representation $v \otimes v'$ possesses a nonzero invariant vector.

(C2) There are integers $N, k_0 \in \mathbb{N}_0$ such that $u^{\otimes N}$ is contained in $u^{\otimes (N+k_0)}$ and for all integers t, l with $0 < t < k_0$, $\mathrm{Hom}(u^{\otimes l}, u^{\otimes (l+t)}) = 0$.

Condition (C1) is actually satisfied for all finite-dimensional semisimple Hopf algebras over $\mathbb{C}$, see [19, Remark 7.5] and [22, Theorem of Section 4.2]. The following result has been proven by the first author, see [19, Lemma 2.10, Corollary 4.7, Lemma 6.4]. It holds true in a much more general setting but we restrict it to CMQG.

Proposition 3.4. *Let $\mathbb{G}$ be a CMQG and let α be its action on $\mathcal{O}_n$ as described in Proposition 2.4. If the conditions (C1) and (C2) are satisfied, then the fixed point algebra $\mathcal{O}^\alpha$ is a Kirchberg algebra, i.e., it is purely infinite, simple, separable, unital and nuclear (satisfying the UCT).*

We are now going to study which easy quantum groups satisfy conditions (C1) and (C2). In order to do so, let us rephrase these conditions in the language of partitions. The representation theory of easy quantum groups – i.e., the set $\mathcal{T}_u$ – is completely understood and can be given in terms of partitions, see [18]. We review some facts from [18].

We say that a partition $p \in P^{\circ\bullet}(k,k)$ is *projective*, if $p = p^* = p^2$. It follows that T_p is a projection up to normalization. Moreover, the upper points of p are colored exactly like the lower points of p. If $q \in P^{\circ\bullet}(k,k)$ is another projective partition, we write $q \prec p$ if $pq = qp = q$ and $p \neq q$. In this case, T_q is a subprojection of T_p. Given a category $\mathcal{C}$ of partitions, we write $\mathrm{Proj}_{\mathcal{C}}(k)$ for the set of all projective partitions in $\mathcal{C}(k,k)$. For $p \in \mathrm{Proj}_{\mathcal{C}}(k)$ with upper (and equivalently lower) color string s, we put

$$R_p := \bigvee_{q \in \mathrm{Proj}_{\mathcal{C}}(k),\, q \prec p} T_q, \qquad \text{and} \qquad P_p := T_p - R_p \in \mathrm{Hom}(u^s, u^s).$$

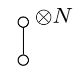
We denote by u_p the subrepresentation $(\mathrm{Id} \otimes P_p)(u^s)$ of u^s. Two such subrepresentations u_p and u_q are unitarily equivalent if and only if there is a partition $r \in \mathcal{C}$ such that $p = r^*r$ and $q = rr^*$.

Proposition 3.5 ([18, Theorem 5.5]). *If $\mathcal{C} \subset NC^{\circ\bullet}$, then u_p is irreducible. Furthermore, any irreducible representation of the associated $\mathbb{G}$ is unitarily equivalent to some u_p for some $p \in \mathrm{Proj}_{\mathcal{C}}$, thereby inducing a one-to-one correspondence.*

Since only tensor powers of u play a role for conditions (C1) and (C2), let us denote by $\mathrm{Proj}_{\mathcal{C}}^{\mathrm{white}}$ and $\mathcal{C}^{\mathrm{white}}$ the restrictions of $\mathrm{Proj}_{\mathcal{C}}$ and $\mathcal{C}$ to those partitions consisting only of white points.

Theorem 3.6. *Let $\mathbb{G}$ be a free easy quantum group, namely the associated category of partitions $\mathcal{C}$ contains only noncrossing partitions. The following two conditions imply the conditions (C1) and (C2).*

(C_P1) *For any projective partition $p \in \mathrm{Proj}_{\mathcal{C}}^{\mathrm{white}}(a)$, we can find a projective partition $q \in \mathrm{Proj}_{\mathcal{C}}^{\mathrm{white}}(b)$ and a partition $r \in \mathcal{C}^{\mathrm{white}}(0, a+b)$ with no upper points such that $(P_p \otimes P_q)T_r \neq 0$.*

(C_P2) *There are integers $k_0, N \in \mathbb{N}_0$ and a partition $r \in \mathcal{C}^{\mathrm{white}}(N+k_0, N)$ such that $rr^* =$*  *. Moreover, for all $t \in \mathbb{N}_0$ with $0 < t < k_0$ and for all $l \in \mathbb{N}_0$ we have $\mathcal{C}^{\mathrm{white}}(l, l+t) = \varnothing$.*

Proof. Due to Proposition 3.5, the set $\mathcal{T}_u$ is in bijection with equivalence classes of projective partitions in $\mathcal{C}$. Let us first prove that (C_P1) implies (C1). Let $p \in \mathrm{Proj}_{\mathcal{C}}$ be a projective partition with only white points and let $u_p \in \mathcal{T}_u$ be the associated irreducible representation. Let $q \in \mathrm{Proj}_{\mathcal{C}}$ and $r \in \mathcal{C}(0, a+b)$ be partitions according to (C_P1). Then $T_r(1)$ is a vector in $(\mathbb{C}^n)^{\otimes a+b}$ as described in Section 2.2 and hence $(P_p \otimes P_q)T_r(1)$ is a non-zero vector. Now, since P_p, P_q and T_r are intertwiners for tensor powers of u, we have:

$$\begin{aligned}(u_p \otimes u_q)(P_p \otimes P_q)T_r &= (\mathrm{Id} \otimes (P_p \otimes P_q))\big(u^{\otimes a+b}\big)(P_p \otimes P_q)T_r \\ &= (\mathrm{Id} \otimes (P_p \otimes P_q))\big(u^{\otimes a+b}\big)T_r = (P_p \otimes P_q)T_r.\end{aligned}$$

This proves (C1). As for deducing (C2) from (C_P2), observe that $u^{\otimes N}T_r = T_r u^{\otimes N+k_0}$ implies $u^{\otimes N}T_rT_r^* = T_r u^{\otimes N+k_0}T_r^*$. Since $T_rT_r^* = 1$ up to normalization, this proves that $u^{\otimes N}$ is a subrepresentation of $u^{\otimes N+k_0}$. Moreover, $\mathrm{Hom}(u^{\otimes l}, u^{\otimes (l+t)})$ is spanned by all T_p with $p \in \mathcal{C}^{\mathrm{white}}(l, l+t)$. We have proved (C2). ■

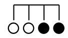
The advantage of dealing with free easy quantum groups is that they are all known: the class of categories $\mathcal{C} \subset NC^{\circ\bullet}$ – and hence the class of free easy quantum groups – is completely classified. The complete list may be found in [36, Section 7].

Throughout the classification process, a parameter $k(\mathcal{C}) \in \mathbb{N}_0$ is assigned to any category of partitions $\mathcal{C}$, see [36, Definition 2.5]. It is given by the following. For a partition $p \in P^{\circ\bullet}$ denote by $c_\circ(p)$ the sum of the number of white points on the lower line of p and the number of black points on its upper line. Likewise put $c_\bullet(p)$ to be the number of lower black points plus the number of upper white points. Let $c(p) := c_\circ(p) - c_\bullet(p)$. The number $c(p)$ is designed in such a way that it yields the difference of the number of white and black points, if p has no upper points; moreover $c(p)$ is invariant under rotation. Now, let $k(\mathcal{C})$ be the minimum of all numbers $c(p) > 0$ for $p \in \mathcal{C}$ if such a number exists, and $k(\mathcal{C}) := 0$ otherwise. One can show that for every partition $p \in \mathcal{C}$ the number $c(p)$ is a multiple (from the integers) of $k(\mathcal{C})$. One of the main results in [37] is then to detect the cyclic group of order $k(\mathcal{C})$ as a building block in the easy quantum group associated to $\mathcal{C}$. The parameter $k(\mathcal{C})$ also appears in connection with conditions (C_P1) and (C_P2).

Theorem 3.7. *Let $\mathbb{G}$ be an easy quantum group with $\mathcal{C} \subset NC^{\circ\bullet}$.*

(*a*) *If $k(\mathcal{C}) = 0$, then neither condition (C_P1) nor (C_P2) are satisfied.*

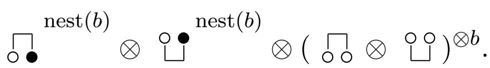
(b) *If* $k(\mathcal{C}) \neq 0$ *and*  $\otimes$  $\in \mathcal{C}$, *then condition* (C_P1) *holds.*

(c) *If* $k(\mathcal{C}) \neq 0$ *and*  $\in \mathcal{C}$, *then condition* (C_P1) *holds.*

(d) *If* $k(\mathcal{C}) \neq 0$, *then condition* (C_P2) *holds.*

Proof. (a) Observe that if a partition p is in $\mathcal{C}^{\mathrm{white}}(l, l+t)$, then $c(p) = t$. Thus, if $k(\mathcal{C}) = 0$, all sets $\mathcal{C}^{\mathrm{white}}(l, l+t)$ are empty for all $l \in \mathbb{N}_0$ and all $t \neq 0$, using [36, Proposition 2.7]. Therefore neither condition (C_P1) nor condition (C_P2) hold.

(b) Let $p \in \mathrm{Proj}_{\mathcal{C}}^{\mathrm{white}}(a)$ be a projective partition. We basically want to show that the contragredient representation of u_p does the job for choosing q, but the colorization of the points turns this into a nontrivial problem (see also the next Examples 3.8 and the remarks in [19, p. 1019] on condition (C1)). Let q_1 be the partition obtained by reflecting p about the vertical axis (without inverting the colors). It is in $\mathcal{C}$, since the verticolor reflected partition $\tilde{p}$ is in $\mathcal{C}$ [36, Lemma 1.1(a)], and $p \otimes p \otimes \tilde{p}$ is in $\mathcal{C}$; using color permutation [36, Lemmas 1.3(a) and 1.1(b)], we infer $q_1 \in \mathcal{C}$. Now, let $b \in \mathbb{N}_0$ be such that $2(a+b)$ is a multiple of $k(\mathcal{C})$. Let $q_0 =$  $^{\mathrm{nest}(b)} \otimes$  $^{\mathrm{nest}(b)}$ be the partition obtained by nesting b copies of the pair partition  into each other, both on the upper and the lower line respectively, see [37, Lemma 2.4]. It is in $\mathcal{C}$ since we may apply [36, Lemmas 1.3(a) and 1.1(b)] on the following partition which is in $\mathcal{C}$:



We then put $q := q_0 \otimes q_1 \in \mathrm{Proj}_{\mathcal{C}}^{\mathrm{white}}(a+2b)$ and $r :=$  $^{\mathrm{nest}(a+b)}$. We have $r \in \mathcal{C}^{\mathrm{white}}(0, 2(a+b))$, see for instance Step 3 in the proof of [37, Theorem 4.13].

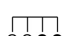
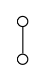
If now $s \in \mathrm{Proj}_{\mathcal{C}}^{\mathrm{white}}(a)$ is a projective partition with $s \prec p$, then $(s \otimes q)r \neq (p \otimes q)r$ by [18, Lemma 2.23]. Hence $T_{(s\otimes q)r}$ is linearly independent from $T_{(p\otimes q)r}$ by [18, Lemma 4.16]. Likewise $T_{(p\otimes t)r}$ is linearly independent from $T_{(p\otimes q)r}$ for $t \prec q$, $t \in \mathrm{Proj}_{\mathcal{C}}^{\mathrm{white}}(a+2b)$. This proves that $(T_p \otimes T_q)T_r$ is linearly independent from $(R_p \otimes T_q)T_r$, $(T_p \otimes R_q)T_r$ and $(R_p \otimes R_q)T_r$. Thus, $(P_p \otimes P_q)T_r \neq 0$.

(c) Let $p \in \mathrm{Proj}_{\mathcal{C}}^{\mathrm{white}}(a)$ be a projective partition. Using the through-block decomposition of p, we may bring it into the following form (see Proposition 2.9 and the remarks after Proposition 2.12 in [18]):

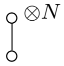
$$p = p_u^* p_u, \qquad p_u = s_0 \otimes t_1 \otimes s_1 \otimes \cdots \otimes t_l \otimes s_l.$$

Here, s_i are partitions with no upper points while each t_i has exactly one upper point. Let α_i be the sum of the number of points of s_i and t_i, with α_1 being the sum of the number of points of s_0, t_1 and s_1. Let β_i be numbers such that $\alpha_i + \beta_i$ is a multiple of $k(\mathcal{C})$, for all i. Let q be the partition

$$q := q_1 \otimes \cdots \otimes q_l,$$

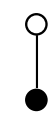
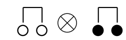
where each q_i is the partition consisting of a single block on β_i upper white points and β_i lower white points. Since  $\in \mathcal{C}$, we have $q \in \mathcal{C}$, applying [36, Lemma 1.3(c)] on  $^{\otimes \beta_i}$. Let $r \in \mathcal{C}^{\mathrm{white}}(0, \sum_i(\alpha_i + \beta_i))$ be the partition obtained from nesting l blocks b_i into each other, each of size $\alpha_i + \beta_i$, such that block b_{i+1} has α_i legs of b_i to its left and β_i legs of b_i to its right. We then conclude $(P_p \otimes P_q)T_r \neq 0$ similarly to (b).

(d) Put $N := 1$ and $k_0 := k(\mathcal{C}) > 0$. We find a partition $p \in \mathcal{C}$ such that $c(p) = k_0$, by definition of $k(\mathcal{C})$. Using rotation and [36, Lemma 2.6(d)], we may assume that p consists only of lower points, hence $p \in P^{\circ\bullet}(0, m)$ for some $m \geqslant k_0$. Then $m = c_\circ(p) + c_\bullet(p)$ and $k_0 = c(p) = c_\circ(p) - c_\bullet(p)$, thus p has $\frac{m+k_0}{2}$ white points and $\frac{m-k_0}{2}$ black points.

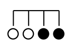
Using [36, Lemma 1.1(b)], we may erase $\frac{m-k_0}{2}$ pairs of a white and a black point and we obtain a partition $p_0 \in \mathcal{C}^{\mathrm{white}}(0, k_0)$ on k_0 white points. Put $r :=$  $\otimes\, p_0^* \in \mathcal{C}^{\mathrm{white}}(1 + k_0, 1)$. It satisfies $rr^* =$  for $N = 1$.

Moreover, $\mathcal{C}^{\mathrm{white}}(l, l + t) = \varnothing$ for $0 < t < k(\mathcal{C})$, since if we had $p \in \mathcal{C}^{\mathrm{white}}(l, l + t)$, then $c(p) = t$, but $c(p)$ is an integer multiple of $k(\mathcal{C})$ by [36, Proposition 2.7]. ■

Example 3.8.

(a) If $\mathbb{G}$ is free orthogonal easy ($u = \bar{u}$) with category $\mathcal{C} \subset NC^{\circ\bullet}$, then $k(\mathcal{C}) = 1$ if  $\in \mathcal{C}$ and $k(\mathcal{C}) = 2$ otherwise, see [36, Section 7]. Moreover, as  is in $\mathcal{C}$, we also have  $\in \mathcal{C}$. Hence, conditions (C_P1) and (C_P2) hold for S_n^+, O_n^+ and the other five free orthogonal easy quantum groups.

(b) For U_n^+, we have $k(\mathcal{C}) = 0$, thus conditions (C_P1) and (C_P2) are violated.

(c) The quantum reflection groups H_n^{s+} of Example 2.3 have the parameter $k(\mathcal{C}) = s$ and  $\in \mathcal{C}$, thus conditions (C_P1) and (C_P2) hold.

Theorem 3.7 is about as far as we can go for free easy quantum groups in general: indeed, if Theorem 3.7 together with Proposition 3.4 provide a way to prove that some fixed point algebras are Kirchberg algebras, the exact identification of the fixed point algebra requires a computation of K-theory that can only be performed for definite fusion rules. For this reason, we focus on examples in the rest of the paper.

3.3 Free actions

In this subsection, we investigate the relation between our fixed point construction and free actions. Free actions appear in articles such as [7, 13, 14]. However, there are not that many concrete examples available, and for this reason we prove that our construction generates new examples.

We remind the reader of the following Definition 2.4 of [17] (see also [14]):

Definition 3.9. Given a CQG $\mathbb{G}$, an action $\underline{\delta}\colon A \to A \otimes C(\mathbb{G})$ on a C^*-algebra A is called *free* if $(A \otimes 1)\underline{\delta}(A)$ is dense in $A \otimes C(\mathbb{G})$.

In order to discuss more easily the above definition, we follow the terminology and notations of [7, 13, 21] and introduce the *canonical map* $\mathrm{can}\colon A \otimes A \to A \otimes C(\mathbb{G})$ given by $\mathrm{can}(a \otimes a') := (a \otimes 1)\underline{\delta}(a')$. The condition above is therefore that the map can has a dense image.

Remark 3.10. The (C^*-)freeness property defined above has strong ties with the notion of *Hopf–Galois extension* (first defined in [26] – see for instance [28]). Indeed, Woronowicz proved [44] that any CQG contains a canonical dense Hopf $*$-algebra $\mathcal{O}(\mathbb{G})$. Using the action of $\mathbb{G}$ on A, we can in turn define the *Peter–Weyl subalgebra* $\mathcal{P}_{\mathbb{G}}(A)$ of A (see, e.g., [7]) – which is not a C^*-algebra but just a $*$-algebra. The preprint [7, p. 3] proves in its Theorem 0.4 that the action $\underline{\delta}$ is (C^*-)free if and only if $\mathcal{P}_{\mathbb{G}}(A)$ is a Hopf–Galois extension over its fixed point algebra – compare the *Peter–Weyl–Galois condition* of [7, Definition 0.2, p. 3] with Definition 2.2 of Hopf–Galois extensions in [28, p. 372].

Going back to our initial motivations concerning free actions, we prove:

Proposition 3.11. *Let $\mathbb{G}$ be a compact matrix quantum group and u its fundamental representation. If $\mathbb{G}$ satisfies condition (C1) then the action $\underline{\delta}$ induced from Proposition 2.4 is free.*

Remark 3.12. At this point, it may seem plausible that condition (C1) is equivalent to freeness of the action of $\mathbb{G}$ on $\mathcal{O}_n$. However, as we have seen in [19, Notation 3.1], (C1) is not satisfied for the natural representation of $\mathbb{G} = \mathrm{U}(1)$ (or multiple thereof). Using the same kind of argument as in the proof below, it appears that the action of $\mathrm{U}(1)$ on $\mathcal{O}_2$ is free.

Of course, this action is not strictly induced from $\mathrm{U}(1)$ by Proposition 2.4. But we can build on this example by considering $\mathrm{SU}(2) \times \mathrm{U}(1)$. This is a CMQG which does not satisfy condition (C1), but which still acts freely on $\mathcal{O}_2$.

Proof of Proposition 3.11. For our proof, we will consider the set

$$\mathcal{S} := \left\{ s \in C(\mathbb{G}) \,|\, \exists\, k_m \in \mathbb{N}_0,\ v_m, w_m \in \mathscr{H}^{\otimes k_m},\ \mathrm{can}\left(\sum_m^{\text{finite}} v_m^* \otimes w_m \right) = 1 \otimes s \right\}.$$

It is clear from the definition of $\underline{\delta}$ that for any i, j,

$$\mathrm{can}(S_j^* \otimes S_i) = (S_j^* \otimes 1) \left(\sum_{k=1}^n S_k \otimes u_{ki} \right) = 1 \otimes u_{ji}.$$

This means that $\mathcal{S}$ contains all u_{ji}. Now, if $v_{i,m}$, $w_{i,m}$ are elements in $\mathscr{H}^{\otimes k_i}$ for $i = 1, 2$ and such that $\mathrm{can}\left(\sum_m v_{i,m}^* \otimes w_{i,m} \right) = 1 \otimes s_i$, then

$$\begin{aligned}
\mathrm{can}\left(\sum_{m,l} v_{2,m}^* v_{1,l}^* \otimes w_{1,l} w_{2,m} \right) &= \left(\sum_{m,l} v_{2,m}^* v_{1,l}^* \otimes 1 \right) \underline{\delta}(w_{1,l} w_{2,m}) \\
&= \left(\sum_m v_{2,m}^* \otimes 1 \right) \left(\sum_l v_{1,l}^* \otimes 1 \right) \underline{\delta}(w_{1,l}) \underline{\delta}(w_{2,m}) \\
&= \left(\sum_m v_{2,m}^* \otimes 1 \right) (1 \otimes s_1) \underline{\delta}(w_{2,m}) \\
&= (1 \otimes s_1) \left(\sum_m v_{2,m}^* \otimes 1 \right) \underline{\delta}(w_{2,m}) \\
&= (1 \otimes s_1)(1 \otimes s_2) = 1 \otimes s_1 s_2.
\end{aligned}$$

This in turn proves that $\mathcal{S}$ is stable under multiplication. It follows that $\mathcal{S}$ is an algebra, which contains all polynomials in u_{ij}.

If the generators of $\mathbb{G}$ are selfadjoint, the polynomials in u_{ij} coincide with the $*$-polynomials in u_{ij} and they are all contained in $\mathcal{S}$, thus $\mathcal{S}$ is dense in $C(\mathbb{G})$.

In general, we can always decompose u into irreducible representations, which all correspond to unitary matrices with coefficients in $C(\mathbb{G})$. Up to a conjugation by a *scalar-valued* unitary matrix, the algebra generated by the entries of u and their adjoints is the same as the algebra generated by the irreducible components and their adjoints. So without loss of generality, we assume that u is an irreducible representation. It is well known (see, e.g., [30, Definition 1.3.8, p. 11]) that the contragredient of (u_{ij}) is (u_{ij}^*) in a suitable basis. u is irreducible, so condition (C1) implies that up to equivalence, we can recover the coefficients (u_{ij}^*) through a subrepresentation of a high enough power $\mathscr{H}^{\otimes k}$. It follows that $\mathcal{S}$ contains the $*$-polynomials in u_{ij} and thus is dense in $C(\mathbb{G})$.

Given any $T \in \mathcal{O}_n$, for any $\mathrm{can}\left(\sum v_j^* \otimes w_j \right) = 1 \otimes s$ we have

$$\mathrm{can}\left(\sum T v_j^* \otimes w_j \right) = (T \otimes 1)\, \mathrm{can}\left(\sum v_j^* \otimes w_j \right) = T \otimes s.$$

It follows that the image of can is dense in $\mathcal{O}_n \otimes C(\mathbb{G})$ and thus the action $\underline{\delta}$ is free. ∎

Remark 3.13. On the one hand, it follows from the proof above that assumption (C1) is not needed for Proposition 3.11 in the orthogonal case (i.e., if all the entries of the fundamental representation are selfadjoint). On the other hand, the argument above also shows that in this orthogonal case condition (C1) is automatically satisfied.

Theorem 3.14. *The actions of S_n^+, O_n^+ and H_n^{s+} are free.*

4 Examples – O_n^+ and S_n^+

To illustrate our previous results, we consider three cases, namely those of O_n^+, S_n^+ and of the quantum reflection groups H_n^{s+} – which we are going to consider separately in the next section.

We start with the case of O_n^+. This free easy quantum group was introduced by S. Wang in [40] and its fusion rules were studied by Banica in [1]. It appears that it shares the same fusion rules as SU(2), i.e., its irreducible representations are denoted by u_k for $k \in \mathbb{N}_0$ – u_0 being the trivial representation – and the tensor products decompose into

$$u_k \otimes u_l = u_{|k-l|} \oplus u_{|k-l|+2} \oplus \cdots \oplus u_{k+l}. \tag{4.1}$$

It is clear from the statement of conditions (C1) and (C2) that they only depend on the fusion rules of the quantum group and not on the quantum group itself. This provides an elementary way to recover the result of Example 3.8(a) in this case. In [19], the case of $\mathrm{SU}_q(2)$ was studied in detail (see Section 7.1, p. 17, therein). The computation of the K-theory of the fixed point algebra also depends only on the fusion rules. Since the fundamental representation of O_n^+ corresponds to the natural representation of SU(2), the proof of Proposition 7.10 [19, p. 1031] applies *verbatim* and we get Theorem 4.1 below for $\mathbb{G} = O_n^+$.

The case of S_n^+ is very similar: this CMQG was introduced by S. Wang in [39] and its fusion rules were computed by Banica in [2]. The fusion rules are the same as those of SO(3), i.e., we take the fusion rules (4.1) of SU(2) but consider only *even* representations u_{2k}, $k \in \mathbb{N}_0$. By Example 3.8(a), conditions (C1) and (C2) are satisfied. The fundamental representation u of S_n^+ decomposes into $u = u_0 \oplus u_2 = u_1 \otimes u_1$ and this shows that the proof of Proposition 7.10 of [19] applies again:

Theorem 4.1. *For $\mathbb{G} = O_n^+$ and $\mathbb{G} = S_n^+$, the fixed point algebra $\mathcal{O}^\alpha$ obtained from Proposition 2.4 via the fundamental representation u is a Kirchberg algebra in the UCT class $\mathcal{N}$ whose K-theory is*

$$K_0(\mathcal{O}^\alpha) = \mathbb{Z}, \qquad K_1(\mathcal{O}^\alpha) = 0.$$

Moreover, $[1_{\mathcal{O}^\alpha}]_0 = 1$ and therefore $\mathcal{O}^\alpha$ is C^*-isomorphic to the infinite Cuntz algebra $\mathcal{O}_\infty$.

5 Examples – quantum reflection groups

As mentioned in Example 2.3(d), the *quantum reflection groups* H_n^{s+} were studied by Banica, Belinschi, Capitaine and Collins in [3]. Their fusion rules were computed by Banica and Vergnioux in their article [6] – see in particular Theorem 7.3, p. 348, therein. We follow the notations used in their article up to a point: for a reason that will be clear later on, we write r_s for the representation denoted by r_0 in the original article. For the reader's convenience, we reproduce here the fusion rules of these quantum groups, as described in [6, Theorem 7.3]. The monoid $F = \langle \mathbb{Z}/s\mathbb{Z} \rangle$ of words over $\mathbb{Z}/s\mathbb{Z}$ is equipped with an involution and a fusion operation:

(1) Involution: $(i_1 \ldots i_k)^- = (-i_k) \cdots (-i_1)$.

(2) Fusion: $(i_1 \ldots i_k) \cdot (j_1 \ldots j_l) = i_1 \ldots i_{k-1}(i_k + j_1)j_2 \ldots j_l$.

Using these relations, the fusion rules can be written

$$r_x \otimes r_y = \sum_{x=vz,\, y=\bar{z}w} r_{vw} + r_{v\cdot w}, \tag{5.1}$$

where $v \cdot w$ is not defined when v or w is the empty word. In the case $z = \varnothing$ ("leading terms of the fusion rule"), we distinguish between the term r_{xy} – that we call the *concatenation term* – and the term $r_{x\cdot y}$ that we call the *product term*. We are going to compute the K-theory of the fixed point algebra generated from the natural representation (indexed by r_1).

In the lemma below, we gather useful results:

Lemma 5.1. *For any quantum reflection group $\mathbb{G} = H_n^{s+}$,*

- *for all $0 < \ell \leqslant s$, the representation r_ℓ appears in $(r_1)^\ell$, i.e., $r_\ell \leqslant (r_1)^\ell$;*
- *moreover, r_{s-1} and r_1 are contragredient to one another, i.e., $1 \leqslant (r_1)^s$;*
- *all irreducible representations of $\mathbb{G}$ appear as irreducible components of some r_1^ℓ for ℓ large enough. Actually,*

$$r_{\sigma_1\dots\sigma_k} \leqslant (r_1)^{\sigma_1+\dots+\sigma_k}. \tag{5.2}$$

Remark 5.2. It follows immediately from this lemma together with Woronowicz's abstract existence of a contragredient for any irreducible representation that condition (C1) is satisfied for H_n^{s+} and its representation r_1.

Proof of Lemma 5.1. We prove the first point by induction on ℓ: the result is true and obvious for $\ell = 1$. For $\ell = 2$,

- if $s = 2$ then $r_1 \cdot r_1 = r_{11} + r_2 + 1$ and the result is true;
- if $s > 2$, then $r_1 \cdot r_1 = r_{11} + r_2$, i.e., $r_2 \leqslant (r_1)^2$ and thus the property is true for $\ell = 2$.

Let us now assume the property for $\ell > 1$, then (provided $\ell+1 < s$), $r_\ell \cdot r_1 = r_{\ell 1} + r_{\ell+1} \leqslant (r_1)^{\ell+1}$, which shows the result for $\ell + 1$.

If $\ell + 1 = s$, then the product becomes

$$r_\ell \cdot r_1 = r_{\ell 1} + r_{\ell+1} + 1.$$

The first two terms above correspond to $z = \varnothing$, while the third one corresponds to $z = \ell$, i.e., $\bar{z} = -\ell = 1$ (equality in $\mathbb{Z}/s\mathbb{Z}$).

Let $r_{\sigma_1\dots\sigma_k}$ be any irreducible representation in H_n^{s+} (where all σ_j are taken between 1 and s), then it appears as an irreducible component of $r_1^{\sigma_1+\dots+\sigma_k}$: our first point proved that r_σ appears in r_1^σ. Therefore, in the decomposition into irreducible components of $r_1^{\sigma_1+\dots+\sigma_k} = (r_1)^{\sigma_1} \cdots (r_1)^{\sigma_k}$, there appears a product of $r_{\sigma_1} \cdot r_{\sigma_2} \cdots r_{\sigma_k}$, which in turn produces a copy of $r_{\sigma_1\dots\sigma_k}$ – by using iteratively only the first (concatenation) term of the fusion rules, for $z = \varnothing$. ■

This lemma enables us to set:

Definition 5.3 (degree function for H_n^{s+})**.** Given an irreducible representation p of $\mathbb{G}$, we denote by $\delta(p)$ the smallest integer ℓ such $p \leqslant r_1^\ell$.

We can actually give an explicit estimate of δ:

Proposition 5.4. *The degree of the irreducible representation $r_{\sigma_1\dots\sigma_k}$, where the σ_js are chosen with $0 < \sigma_j \leqslant s$, is $\delta(r_{\sigma_1\dots\sigma_k}) = \sigma_1 + \dots + \sigma_k$.*

Proof. Given any irreducible representation $r_{x_1 \dots x_k}$ with for all i, $0 < x_i \leqslant s$, we define $\sigma(r_{x_1 \dots x_k}) = x_1 + \dots + x_k$. It follows by direct examination of (5.1) that if $\gamma \leqslant \alpha \cdot \beta$, then

$$\sigma(\gamma) \leqslant \sigma(\alpha) + \sigma(\beta). \tag{5.3}$$

Iterating the argument and combining it with (5.2), it appears that for all irreducible representation, $\gamma \leqslant (r_1)^{\sigma(\gamma)}$, i.e., $\delta(\gamma) \leqslant \sigma(\gamma)$.

Conversely, if $\gamma \leqslant (r_1)^{\ell}$, then iterating the "subadditivity property" (5.3) of σ and using $\sigma(r_1) = 1$, we get: $\sigma(\gamma) \leqslant \ell$. Since this is valid for all ℓ, we get $\sigma(\gamma) \leqslant \delta(\ell)$. This proves that $\delta(\gamma) = \sigma(\gamma)$ for all irreducible γ and concludes the proof. ■

Lemma 5.5. *For all irreducible representations $\alpha = r_x$, $\beta = r_y$ and γ with $\gamma \leqslant \alpha \cdot \beta$,*

$$\delta(\gamma) \leqslant \delta(\alpha) + \delta(\beta). \tag{5.4}$$

For $\mathbb{G} = H_n^{s+}$, the cases of equality in (5.4) can only occur for the terms $z = \varnothing$ of (5.1). For those terms, the equality is true unconditionally for r_{xy}, and only if $x_k + y_1 \leqslant s$ for $r_{x \cdot y}$.

Remark 5.6. The above lemma is the reason why we use the notation r_s instead of r_0.

Proof of Lemma 5.5. The inequality (5.4) follows from

$$\gamma \leqslant \alpha \cdot \beta \leqslant (r_1)^{\delta(\alpha)} (r_1)^{\delta(\beta)} = (r_1)^{\delta(\alpha)+\delta(\beta)}.$$

This is just a variation on the proof of Lemma 5.1 above.

The equality requires to study the behavior of the total degree in the fusion rules, starting from two irreducible representations $\alpha = r_x = r_{x_1 \dots x_k}$ and $\beta = r_y = r_{y_1 \dots y_l}$ with finite sequences (x_i) and (y_j) taking their values in $\{1, \dots, s\}$.

If γ arises from a term in (5.1) with $z \neq \varnothing$, then the inequality (5.4) is strict. Indeed, γ could then arise from the $z = \varnothing$ term of v and w, where $x = vz$ and $y = \bar{z}w$.

Assuming now that $z = \varnothing$, using the estimate of the degree of Proposition 5.4, the term $r_{x_1 \dots x_k y_1 \dots y_l}$ yields an equality case for (5.4). The same is true of the term $r_{x_1 \dots (x_k + y_1) \dots y_l}$ provided $x_k + y_1 \leqslant s$. It remains to treat the case of $x_k + y_1 > s$, but such a term corresponds to a strict inequality in (5.4) and this completes the proof. ■

We will use the following notations extensively: let R_ℓ (resp. ∂R_ℓ) be the $\mathbb{Z}$-free module constructed on irreducible representations appearing in $(r_1)^{\ell}$ (resp. appearing in $(r_1)^{\ell}$ and not in any $(r_1)^k$ for $0 \leqslant k < \ell$).

It follows immediately from the definition of R_ℓ that $R_\ell \cdot R_k \subseteq R_{\ell+k}$, where the product $R_\ell \cdot R_k$ is taking place in the fusion ring of $\mathbb{G}$.

Lemma 5.7. *The fusion rules (5.1) actually ensure that in $\mathbb{Z}/s\mathbb{Z}$:*

$$[\delta(\gamma)] = [\delta(\alpha)] + [\delta(\beta)]. \tag{5.5}$$

Proof. Direct examination and Proposition 5.4 show that if $z = \varnothing$, then $r_{xy} = r_{i_1 \dots i_k j_1 \dots j_l}$ has degree $\delta(r_{xy}) = i_1 + \dots + i_k + j_1 + \dots + j_l = \delta(r_x) + \delta(r_y)$. The degree of $r_{x \cdot y}$ is the same in $\mathbb{Z}/s\mathbb{Z}$ (since a simplification in $\mathbb{Z}/s\mathbb{Z}$ may happen in the fusion $x \cdot y$).

More generally, if $z \neq \varnothing$, then the definition of the involution on the monoid F ensures that taking out both z and $\bar{z}$ do not change $[\delta(r_{vw})]$ in $\mathbb{Z}/s\mathbb{Z}$. However, simplifying by z and $\bar{z}$ lessen the total degree of the expression, i.e., the degree $\delta(r_{vw})$ has to be strictly less than $\delta(r_x) + \delta(r_y)$. The same argument applies to r_{v+w}. ■

We are now in position to compute the *chain group* $\mathfrak{C}(\mathbb{G})$ as introduced in [8, 29]. This object is also known as *universal grading group*, see, e.g., [20].

Proposition 5.8. *The chain group of* $\mathbb{G} = H_n^{s+}$ *is* $\mathfrak{C}(\mathbb{G}) = \mathbb{Z}/s\mathbb{Z}$.

Proof. The equation (5.5) shows that if $\tau_1, \dots, \tau_k, p, q$ are irreducible representations and that they satisfy $p, q \leqslant \tau_1 \cdot \tau_2 \cdots \tau_k$ (both p and q appear in the fusion product), then (in the group $\mathbb{Z}/s\mathbb{Z}$)

$$[\delta(p)] = [\delta(\tau_1)] + \cdots + [\delta(\tau_k)] = [\delta(q)].$$

In other words, if p and q have the same class in the chain group $\mathfrak{C}(\mathbb{G})$, then $[\delta(p)] = [\delta(q)]$.

Conversely, take $[\delta(p)] = [\delta(q)]$ in $\mathbb{Z}/s\mathbb{Z}$. If $\delta(p) = \delta(q) = \ell$, then both representations appear in $(r_1)^\ell$ and they have the same class in the chain group. Otherwise, without loss of generality, we can assume that $\delta(p) > \delta(q)$ and thus there is an integer k such that $\delta(p) = \delta(q) + ks$. By definition of $\delta(q)$, $q \leqslant (r_1)^{\delta(q)}$. We can then use Lemma 5.1 (and especially the part $1 \leqslant (r_1)^s$) to show

$$q = q \cdot 1^k \leqslant (r_1)^{\delta(q)} (r_1)^s \cdots (r_1)^s = (r_1)^{\delta(q)+ks}.$$

This in turn proves that p and q share the same class in $\mathfrak{C}(\mathbb{G})$. The equation (5.5) then ensures that the group law in $\mathfrak{C}(\mathbb{G})$ and $\mathbb{Z}/s\mathbb{Z}$ coincide. ■

Remark 5.9. It follows from this evaluation of the chain group $\mathfrak{C}(\mathbb{G})$ together with the property $1 \leqslant (r_1)^s$ of Lemma 5.1 that condition (C2) is satisfied for $\mathbb{G} = H_n^{s+}$ equipped with its representation r_1, for the integers $k_0 = s$ and $N = 1$.

Corollary 5.10. *There is a decomposition*

$$R_\ell = \bigoplus_{0 \leqslant k \leqslant \ell, [k]=[\ell]} \partial R_k, \tag{5.6}$$

where the equality $[k] = [\ell]$ *takes place in* $\mathbb{Z}/s\mathbb{Z}$.

Proof. This is a consequence of $p \leqslant (r_1)^\ell \implies [\delta(p)] = [\ell]$ in $\mathbb{Z}/s\mathbb{Z}$ together with $1 \leqslant (r_1)^s$. ■

Finally, we will need the notion of *length* $\lambda(r_s)$ of an irreducible representation r_s, which is just the length (number of letters) of its indexing sequence s. It is clear from (5.4) that for all $\gamma \leqslant \alpha \cdot \beta$, $\lambda(\gamma) \leqslant \lambda(\alpha) + \lambda(\beta)$ with equality *only* for the concatenation term of $z = \varnothing$.

Theorem 5.11. *For* $\mathbb{G} = H_n^{s+}$ *and its representation* $\alpha = r_1$, *condition* (C2) *is satisfied for the integer* $k_0 = s$ *and the computation of* K*-theory yields*

$$K_0(\mathcal{O}^\alpha) = \bigoplus_{\mathbb{N}} \mathbb{Z}, \qquad K_1(\mathcal{O}^\alpha) = 0.$$

The proof of this theorem is going to require a few intermediate lemmas and a restatement of the problem.

Indeed, Theorem 5.11 above is stated in terms of the representation $\alpha = r_1$, but the computations below will be easier if we consider the case of $\alpha = r_1^s$ – which yields isomorphic results, according to Proposition 5.8 above and [19, Proposition 7.8, p. 1029].

Consider the maps $\varphi\colon R \to R$ and $\psi\colon R \to R$ given on all $a \in R$ by

$$\varphi(a) = a(r_1^s - 1), \qquad \psi(a) = a r_1^s.$$

The previous properties of degree show that these maps induce $\varphi_\ell\colon R_\ell \to R_{\ell+s}$ and $\psi_\ell\colon R_\ell \to R_{\ell+s}$. The K-theory of the fixed point algebra $\mathcal{F}^\alpha$ is given by the inductive limit of the system

$$\cdots \to R_\ell \xrightarrow{\psi_\ell} R_{\ell+s} \xrightarrow{\psi_{\ell+s}} R_{\ell+2s} \to \cdots.$$

The general theory presented in [19] (see in particular Theorem 5.4, p. 1025) shows that $K_0(\mathcal{O}^\alpha)$ is obtained as the cokernel of the map $\varphi\colon \varinjlim R_\ell \to \varinjlim R_\ell$ defined from the system

$$\begin{array}{ccccccccccccc} \cdots \longrightarrow & R_\ell & \xrightarrow{\psi_\ell} & R_{\ell+s} & \xrightarrow{\psi_{\ell+s}} & R_{\ell+2s} & \xrightarrow{\psi_{\ell+2s}} & R_{\ell+3s} & \xrightarrow{\psi_{\ell+3s}} & R_{\ell+4s} & \longrightarrow \cdots \\ & & \nearrow^{\varphi_\ell} & & \nearrow^{\varphi_{\ell+s}} & & \nearrow^{\varphi_{\ell+2s}} & & \nearrow^{\varphi_{\ell+3s}} & & \\ \cdots \longrightarrow & R_\ell & \xrightarrow[\psi_\ell]{} & R_{\ell+s} & \xrightarrow[\psi_{\ell+s}]{} & R_{\ell+2s} & \xrightarrow[\psi_{\ell+2s}]{} & R_{\ell+3s} & \xrightarrow[\psi_{\ell+3s}]{} & R_{\ell+4s} & \longrightarrow \cdots . \end{array} \tag{5.7}$$

All the squares in the diagram above are commutative – indeed, it amounts to proving that for any $a \in R$, $ar_1^s(r_1^s - 1) = a(r_1^s - 1)r_1^s$. Thus, the map $\varphi\colon \varinjlim R_\ell \to \varinjlim R_\ell$ is well-defined and we can compute its cokernel. To this end, we start by computing the cokernels at each finite level ℓ and it is a well-known property that we will obtain the overall cokernel as inductive limit of those finite cokernels.

The evaluation below is the corner stone of our argument:

Lemma 5.12. *For any irreducible representation r_x with $\delta(r_x) = \ell$, we have*

$$\varphi(r_x) = r_x(r_1^s - 1) = r_{x\underbrace{1\ldots1}_{s\ terms}} + r_{xs} + m, \tag{5.8}$$

where m is a $\mathbb{Z}$-linear combination of irreducible representations in $R_{\ell+s}$ which do not *contain any term $r_{x1\ldots1}$ and r_{xs}.*

Proof. It is clear from the fusion rules that by taking only the concatenation term for $z = \varnothing$ in the s successive fusion products of r_x with r_1, we obtain a term $r_\mu = r_{x\underbrace{1\ldots1}_{s\ \text{terms}}}$. Moreover, this irreducible irrepresentation has maximal length (namely $\lambda(r_\mu) = \lambda(r_x) + s$) among those appearing in the product $r_x(r_1^s - 1)$. Given $\gamma \leqslant \alpha\cdot\beta$, we know that $\lambda(\gamma) \leqslant \lambda(\alpha) + \lambda(\beta)$ is actually an *equality* only for the concatenation term of $z = \varnothing$. It follows that there is only one way to obtain a representation of such length. Thus, no further term involving r_μ appear in $\varphi(r_x)$.

For r_{xs}, the argument is slightly different: first, we remark that it has maximum degree ($\delta(r_{xs}) = \delta(r_x) + s$). This implies that it was obtained by taking only terms with $z = \varnothing$ in the successive fusion products. We then remark that its length is minimal among those terms obtained by taking only *leading terms* ($z = \varnothing$) in the fusion. This in turn ensures that it is (and can only be) obtained by taking *product terms* in the s successive fusion products. Thus, no further term involving r_{xs} appear in $\varphi(r_x)$. ■

Lemma 5.13. *For all ℓ, there is free $\mathbb{Z}$-module $C_\ell \subseteq R_\ell$ such that*

$$R_{\ell+s} = C_{\ell+s} \oplus \varphi_\ell(R_\ell). \tag{5.9}$$

Moreover, this free $\mathbb{Z}$-module decomposes according to the degree into

$$C_\ell = \bigoplus_{0 \leqslant k \leqslant \ell,\, [k]=[\ell]} \partial C_k. \tag{5.10}$$

Remark 5.14. The statement above calls for several remarks:

- The notation $R_{\ell+s} = C_{\ell+s} \oplus \varphi_\ell(R_\ell)$ indicates that any element of $R_{\ell+s}$ can be written *in a unique way* as a sum of an element of $C_{\ell+s}$ and an element of $\varphi_\ell(R_\ell)$.
- An immediate consequence of (5.9) is that we can thus identify $C_{\ell+s}$ with the cokernel $R_{\ell+s}/\varphi(R_\ell)$.
- In the decomposition (5.10), we use obvious notations similar to those of (5.6).

Proof of Lemma 5.13. We proceed by induction: for a minimal level $0 \leqslant \ell < 2s$, the decomposition (5.9) shows that we just have to find C_ℓ s.t. $R_\ell = C_\ell \oplus \varphi_{\ell-s}(R_{\ell-s})$. Given any $a \in R_\ell$, we can use relation (5.8) to cancel any term of the form $r_{x\underbrace{1\dots1}_{s\text{ terms}}}$ appearing in a. If we then define C_ℓ as the free $\mathbb{Z}$-module generated on all irreducible representations appearing in R_ℓ which are *not* of the form $r_{x\underbrace{1\dots1}_{s\text{ terms}}}$, then clearly

$$R_\ell = C_\ell \oplus \varphi_{\ell-s}(R_{\ell-s}).$$

Let us now assume that at level ℓ, we have a decomposition

$$R_\ell = C_\ell \oplus \varphi(R_\ell),$$

where $C_\ell = \bigoplus_{0\leqslant k\leqslant \ell,[k]=[\ell]} \partial C_k$, we want to prove that $R_{\ell+s}$ admits a similar decomposition.

A consequence of the decomposition (5.6), is that $\varphi(R_\ell) = \bigoplus_{0\leqslant k\leqslant \ell,[k]=[\ell]} \varphi(\partial R_k)$.

Let us now introduce $\partial C_{\ell+2}$ as the free $\mathbb{Z}$-module generated by all irreducible representation of degree *exactly* $\ell + s$ which are *not* of the form $r_{x\underbrace{1\dots1}_{s\text{ terms}}}$, then

$$R_{\ell+s} = \partial C_{\ell+s} \oplus \varphi(\partial R_\ell) \oplus R_\ell = \partial C_{\ell+s} \oplus \bigoplus_{0\leqslant k\leqslant \ell,[k]=[\ell]} \partial C_k \oplus \varphi(R_{\ell+s}).$$

This completes the proof of the existence of the $\mathbb{Z}$-free module $C_\ell = \bigoplus_{0\leqslant k\leqslant \ell,[k]=[\ell]} \partial C_k$ which implements the cokernel in R_ℓ. ∎

Let us now study the connecting maps between these cokernels. Remember from the commutation relations appearing in (5.7) that all connecting maps $\psi_\ell\colon R_\ell \to R_{\ell+s}$ induce quotient maps at the level of cokernels, which we denote by

$$\widetilde{\psi_\ell}\colon\ R_\ell/\varphi(R_{\ell-s}) \to R_{\ell+s}/\varphi(R_\ell).$$

Lemma 5.15. *The connecting maps between the cokernels are the identity: for any $a \in R_\ell$, $\widetilde{\psi_\ell}([a]_\ell) = [a]_{\ell+s}$, where $[a]_\ell$ and $[a]_{\ell+s}$ are the class of a in $R_\ell/\varphi(R_{\ell-s})$ and $R_{\ell+s}/\varphi(R_\ell)$, respectively.*

Proof. Indeed, take any $a \in R_\ell$, then $\psi(a) = \varphi(a) + a$. We know that in the cokernel $R_{\ell+s}/\varphi(R_\ell)$, $[\varphi(a)]_{\ell+s} = 0$, thus $[\psi(a)]_{\ell+s} = [a]_{\ell+s}$. ∎

Remark 5.16. A consequence of the above Lemma 5.15 is that the inductive limit $\varinjlim C_\ell$ is simply the increasing union of free $\mathbb{Z}$-modules and it suffices to estimate the number of irreducible representations of H_n^{s+} of degree $\ell + s$ which are not of the form $r_{x\underbrace{1\dots1}_{s\text{ terms}}}$. We do precisely this in the next lemma.

Lemma 5.17. *Let $m_{\ell+s}$ be the number of irreducible representations of H_n^{s+} of degree $\ell + s$ which are not of the form $r_{x\underbrace{1\dots1}_{s\text{ terms}}}$, then $m_{\ell+s} \to \infty$.*

Proof. Let us introduce the number n_ℓ of irreducible representations of degree *exactly* ℓ, then relation (5.8) ensures that

$$n_{\ell+s} \geqslant 2n_\ell. \tag{5.11}$$

Indeed, for each irreducible representation r_x of degree ℓ, there are at least two irreducible representations of degree $\ell + 2$, namely $r_{x\underbrace{1\dots1}_{s\text{ terms}}}$ and r_{xs}. This also forces $m_{\ell+s} \geqslant n_\ell$. The relation (5.11), together with the equality $n_0 = 1$, shows that n_ℓ (and therefore m_ℓ) tends to infinity when ℓ tends to infinity. ∎

Proof of Theorem 5.11. It follows from Example 3.8(a) that conditions (C1) and (C2) are satisfied for the fusion rules of the quantum reflection group $\mathbb{G} = H_n^{s+}$. Consequently, Proposition 3.4 applies to the fixed point algebra – which is therefore determined up to $*$-isomorphism by its K-theory.

To compute $K_*(\mathcal{O}^\alpha)$, we use the inductive system (5.7). We first evaluate $K_1(\mathcal{O}^\alpha)$: according to [19, Theorem 5.4, p. 1025] this K-group is the kernel of the map φ defined by the inductive system (5.7). If c is a nonzero element in R, it can be realised on a finite level ℓ. Let us consider the top length nonvanishing irreducible representations appearing in $c \in R_\ell$ and write $c = \sum \alpha_j r_{x_1^j \dots x_\lambda^j} + m$ where $\alpha_j \in \mathbb{Z} \setminus \{0\}$, λ is the maximum length of irreducible representations in c and m is a combination of irreducible representations with lower length. Following Lemma 5.12,

$$\varphi(c) = \sum \alpha_j r_{x_1^j \dots x_\lambda^j 1 \dots 1} + m',$$

where m' is a linear combination of irreducible representations which *do not* contain any term $r_{x_1^j \dots x_\lambda^j 1 \dots 1}$ (i.e., maximal length terms). It follows that $\varphi(c) - c \neq 0$ (since irreducible representations in c have length at most λ and $r_{x_1^j \dots x_\lambda^j 1 \dots 1}$ has length $\lambda + s$, no cancellation can occur). Essentially the same argument proves that if $c \in R_\ell$ is nonvanishing, then $\psi_\ell(c) \in R_{\ell+s}$ is also nonvanishing. Consequently, $\ker(1 - \varphi_*) = K_1(\mathcal{O}^\alpha) = \{0\}$.

The computation of $K_0(\mathcal{O}^\alpha)$ is an easy consequence of Lemma 5.13, Remark 5.14 and Lemmas 5.15 and 5.17. The proof of Theorem 5.11 is thus complete. ∎

As a final comment, this paper shows how techniques from classification theory for C^*-algebras and a thorough understanding of fusion rules can be combined to identify free actions of compact quantum groups on C^*-algebras. The characterisation of the fixed point algebra requires a concrete computation of K-theory, and explains why we restricted ourselves to examples in the second part of the paper. Similar results should however be possible for other classes of CQGs, as soon as we have a fine comprehension of their fusion rules.

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