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A note on triangulated monads and categories of module spectra

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A NOTE ON TRIANGULATED MONADS AND CATEGORIES OF MODULE SPECTRA

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ABSTRACT. Consider a monad on an idempotent complete triangulated category with the property that its Eilenberg-Moore category of modules is also triangulated, in the unique compatible way. We show that any other triangulated adjunction realizing this monad is 'essentially monadic', i.e. becomes monadic after performing the two evident necessary operations of taking the Verdier quotient by the kernel of the right adjoint and idempotent completion. In this sense, the monad itself is 'intrinsically monadic'. It follows that for any highly structured ring spectrum, its category of homotopy (a.k.a. naïve) modules is triangulated if and only if it is equivalent to its category of highly structured (a.k.a. strict) modules.

1. TRIANGULATED MONADS AND THEIR REALIZATIONS

Let $\mathcal{C}$ be an idempotent complete triangulated category. Given a monad $\mathbb{A}$ on $\mathcal{C}$, we can ask whether the Eilenberg-Moore category $\mathbb{A}\text{-}\mathrm{Mod}_{\mathcal{C}}$ of $\mathbb{A}$-modules in $\mathcal{C}$ (a.k.a. $\mathbb{A}$-algebras) is also triangulated, in a way that makes the forgetful functor $U_{\mathbb{A}}\colon \mathbb{A}\text{-}\mathrm{Mod}_{\mathcal{C}} \to \mathcal{C}$ exact. As the following lemma shows, there can be at most one such triangulation on $\mathbb{A}\text{-}\mathrm{Mod}_{\mathcal{C}}$, so we can regard it as a property of the monad $\mathbb{A}$:

1.1. **Lemma.** *Let* $U\colon \mathcal{M} \to \mathcal{C}$ *be a faithful functor to a triangulated category* $\mathcal{C}$. *There exists at most one triangulation on the category* $\mathcal{M}$ *making* U *exact, and it must coincide with the triangulation created by* U, *i.e. a triangle in* $\mathcal{M}$ *is exact if and only if it is mapped by* U *to an exact triangle in* $\mathcal{C}$.

Proof. Assume $\mathcal{M}$ has a triangulation such that U preserves exact triangles. Let $x \to y \to z \to \Sigma x$ be a diagram in $\mathcal{M}$ such that $U(x \to y \to z \to \Sigma x)$ is exact in $\mathcal{C}$. In particular $U(x \to y \to z)$ is zero, hence so is the composite $x \to y \to z$ because U is faithful. Therefore, making use of the given triangulation on $\mathcal{M}$, we find a commutative diagram in $\mathcal{M}$ as follows

$$
\begin{array}{ccccccc}
x & \longrightarrow & y & \longrightarrow & c & \longrightarrow & \Sigma x \\
\mathrm{id}_x \downarrow & & \mathrm{id}_y \downarrow & & \downarrow & & \downarrow \Sigma\,\mathrm{id}_x \\
x & \longrightarrow & y & \longrightarrow & z & \longrightarrow & \Sigma x \\
& & & & \downarrow & & \\
& & & & d & & \\
& & & & \downarrow & & \\
& & & & \Sigma c & &
\end{array}
$$

where the first row and the third column are exact triangles. After applying U both rows and the third column become exact, hence $U(c \to z)$ is an isomorphism, hence $U(d) \cong 0$. As U is faithful, this implies that $d \cong 0$ and thus that $c \to z$ is an isomorphism. Therefore the given triangle $x \to y \to z \to \Sigma x$ is distinguised in $\mathcal{M}$, and we conclude that U not only preserves but also reflects exact triangles. $\square$

Although it is rare for $\mathbb{A}\text{-}\mathrm{Mod}_{\mathcal{C}}$ to be triangulated, there are notable cases where it is. This happens, for example, if $\mathbb{A}$ is idempotent (i.e. a Bousfield localization). More generally, Balmer [Bal11] proved that this is the case when the monad $\mathbb{A}$ is separable (provided that $\mathcal{C}$ is endowed with an ∞-triangulation, which is always the case when it admits an underlying model or derivator). Some non-separable examples are also known (see [Gut05] and Example 1.12).

In this note, we consider the *consequences* of $\mathbb{A}\text{-}\mathrm{Mod}_{\mathcal{C}}$ being triangulated. We prove that if this is the case then *any* triangulated realization of $\mathbb{A}$ is essentially monadic, i.e. monadic after applying two necessary operations: a Verdier quotient and an idempotent completion. In a slogan:

> Triangulated monads which have triangulated Eilenberg-Moore adjunctions are intrinsically monadic.

The proof of this amusing fact is easy and will be given in Proposition 1.7 below (see also Corollary 1.9). Applications to categories of module spectra will be discussed at the end (e.g. Corollary 1.11).

1.2. *Terminology.* We recall some basic facts about monads from [ML98, Chap. VI], mostly to fix notation. Every adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ with unit $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ and counit $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ defines a monad $\mathbb{A}$ on $\mathcal{C}$ consisting of the endofunctor $\mathbb{A} := GF\colon \mathcal{C} \to \mathcal{C}$ equipped with the multiplication map $G\varepsilon F\colon \mathbb{A}^2 \to \mathbb{A}$ and unit map $\eta\colon \mathrm{id}_{\mathcal{C}} \to \mathbb{A}$. We say the adjunction $F \dashv G$ *realizes* the monad $\mathbb{A}$. Given any monad $\mathbb{A}$ on $\mathcal{C}$, there always exists an initial and a final adjunction realizing $\mathbb{A}$:

(1.3)

The final one is provided by the *Eilenberg-Moore* category $\mathbb{A}\text{-}\mathrm{Mod}_{\mathcal{C}}$, whose objects are $\mathbb{A}$-modules $(x, \rho\colon \mathbb{A}x \to x)$ in $\mathcal{C}$, together with the forgetful functor $U_{\mathbb{A}}\colon (x, \rho) \mapsto x$ and its left adjoint *free-module* functor $F_{\mathbb{A}}$. The full image of $F_{\mathbb{A}}$, together with the restricted adjunction, provides the initial realization $\mathbb{A}\text{-}\mathrm{Free}_{\mathcal{C}}$ (often called the *Kleisli category*). For any adjunction $F \dashv G$ realizing the monad, the fully faithful inclusion $\mathbb{A}\text{-}\mathrm{Free}_{\mathcal{C}} \to \mathbb{A}\text{-}\mathrm{Mod}_{\mathcal{C}}$ uniquely factors as a composite $E \circ K$ of two comparison functors satisfying $KF_{\mathbb{A}} = F, GK = U_{\mathbb{A}}$ and $EF = F_{\mathbb{A}}, U_{\mathbb{A}}E = G$. The functor K is always automatically fully faithful. Finally, an adjunction $F \dashv G$ is *monadic* if the associated Eilenberg-Moore comparison E is an equivalence.

1.4. *Remark.* If $\mathcal{C}$ is a triangulated category then we can also consider *triangulated realizations* of $\mathbb{A}$, i.e. realizations by an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ of exact functors between triangulated categories. (Of course, $\mathbb{A} : \mathcal{C} \to \mathcal{C}$ must be exact for a triangulated realization to exist.) Note that if $\mathbb{A}\text{-}\mathrm{Mod}_{\mathcal{C}}$ is triangulated such that $U_{\mathbb{A}}$ is exact, then the free module functor $F_{\mathbb{A}}\colon \mathcal{C} \to \mathbb{A}\text{-}\mathrm{Mod}_{\mathcal{C}}$ is also automatically exact, hence the adjunction $F_{\mathbb{A}} \dashv U_{\mathbb{A}}$ is a triangulated realization of $\mathbb{A}$. However, $\mathbb{A}\text{-}\mathrm{Mod}_{\mathcal{C}}$ need not admit a triangulation in general.

1.5. *Remark.* As $\mathcal{C}$ is assumed to be idempotent complete, one easily checks that the Eilenberg-Moore category $\mathbb{A}\text{-}\mathrm{Mod}_{\mathcal{C}}$ is also idempotent complete. Moreover $U_{\mathbb{A}}$ is faithful, so in particular it detects the vanishing of objects; since $U_{\mathbb{A}}$ is an exact functor, the latter is equivalent to being *conservative*, i.e. reflecting isomorphisms.

1.6. *Remark.* If $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is any triangulated realization of $\mathbb{A}$, then we can always canonically modify it to a triangulated adjunction $\widetilde{F} : \mathcal{C} \rightleftarrows \widetilde{\mathcal{D}} : \widetilde{G}$ where, as with the Eilenberg-Moore adjunction, the target category $\widetilde{\mathcal{D}}$ is idempotent complete and the right adjoint $\widetilde{G}$ is conservative. Indeed, construct the Verdier quotient $\mathcal{D}/\operatorname{Ker} G$, embed it into its idempotent completion $(\mathcal{D}/\operatorname{Ker} G)^{\natural}$ (see [BS01]), and let $\widetilde{F}$ be the composite $\mathcal{D} \to \mathcal{D}/\operatorname{Ker} G \to (\mathcal{D}/\operatorname{Ker} G)^{\natural} =: \widetilde{\mathcal{D}}$. Since $\mathcal{C}$ is idempotent complete, G extends to a functor $\widetilde{G} \colon \widetilde{\mathcal{D}} \to \mathcal{C}$ right adjoint to $\widetilde{F}$ and one easily verifies that the adjunctions $\widetilde{F} \dashv \widetilde{G}$ and $F \dashv G$ realize the same monad.

Our aim is to study all possible triangulated realizations of $\mathbb{A}$ under the hypothesis that the Eilenberg-Moore adjunction is triangulated. By Remarks 1.5 and 1.6, this problem reduces to the case where the target category is idempotent complete and the right adjoint is conservative. But surprisingly, such an adjunction is necessarily monadic:

1.7. **Proposition.** *Let $\mathcal{C}$ be an idempotent complete triangulated category equipped with a monad $\mathbb{A}$ such that $\mathbb{A}\text{-}\operatorname{Mod}_{\mathcal{C}}$ is compatibly triangulated, i.e. such that the free-forgetful adjunction $F_{\mathbb{A}} \dashv U_{\mathbb{A}}$ is triangulated. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be any triangulated realization of the same monad $\mathbb{A} = GF$. If $\mathcal{D}$ is idempotent complete and G is conservative, then $F \dashv G$ is monadic, i.e. the comparison functor $\mathcal{D} \xrightarrow{\sim} \mathbb{A}\text{-}\operatorname{Mod}_{\mathcal{C}}$ is a triangulated equivalence.*

Proof. Keep in mind (1.3) throughout and consult [ML98, Chap. VI] if necessary. Let $d \in \mathcal{D}$ be an arbitrary object. By definition, the $\mathbb{A}$-module Ed consists of the object $Gd \in \mathcal{C}$ equipped with the action

$$G\varepsilon_d \colon \mathbb{A}(Gd) = GFGd \longrightarrow Gd$$

where ε denotes the counit of the adjunction $F \dashv G$. As for any module, its action map can also be seen as a map in $\mathbb{A}\text{-}\operatorname{Mod}_{\mathcal{C}}$

$$G\varepsilon_d \colon F_{\mathbb{A}}U_{\mathbb{A}}Ed = (GFGd, G\varepsilon_{FGd}) \longrightarrow (Gd, G\varepsilon_d) = Ed$$

providing the counit at the object Ed for the Eilenberg-Moore adjunction. By hypothesis, the right adjoint $U_{\mathbb{A}} \colon \mathbb{A}\text{-}\operatorname{Mod}_{\mathcal{C}} \to \mathcal{C}$ of $F_{\mathbb{A}}$ is a faithful exact functor between triangulated categories, hence the counit $G\varepsilon_d$ admits a section $\sigma \colon Gd \to GFGd = F_{\mathbb{A}}U_{\mathbb{A}}Ed$ in $\mathbb{A}\text{-}\operatorname{Mod}_{\mathcal{C}}$ (see e.g. [BDS16, Lemma 4.2]). Thus we have in $\mathbb{A}\text{-}\operatorname{Mod}_{\mathcal{C}}$ the split idempotent $p^2 = p := \sigma \circ G\varepsilon_d$ on the object $F_{\mathbb{A}}U_{\mathbb{A}}Ed$ with image Ed:

$$F_{\mathbb{A}}U_{\mathbb{A}}Ed \xrightarrow{\quad p \quad} F_{\mathbb{A}}U_{\mathbb{A}}Ed$$

with $G\epsilon_d$ down to Ed and σ up from Ed.

Since the composite functor EK is fully faithful and the free module $F_{\mathbb{A}}U_{\mathbb{A}}Ed$ belongs to its image, we must have $p = EKq$ for an idempotent q in $\mathbb{A}\text{-}\operatorname{Free}_{\mathcal{C}}$, hence $p = Er$ for the idempotent $r := Kq$ in $\mathcal{D}$ on the object FGd. As $\mathcal{D}$ is idempotent complete by hypothesis, r must split:

(1.8)
$$\exists\, d' $$
$$FGd \xrightarrow[\quad r \quad]{} FGd.$$
with α up to d' and β down from d'.

Applying E to (1.8), we see that the idempotent p of $\mathbb{A}\text{-}\mathrm{Mod}_{\mathcal{C}}$ splits in two ways:

Applying the functor $U_{\mathbb{A}}$ (i.e. forgetting actions), this yields in $\mathcal{C}$ the two splittings

of the idempotent p on $GFGd$. It follows that the composite $G(\varepsilon_d) \circ G(\beta) = G(\varepsilon_d \circ \beta)$ is an isomorphism $Gd \cong Gd'$ of the two images. Since G is assumed to be conservative, this implies that $\varepsilon_d \circ \beta$ is already an isomorphism $d' \cong d$ in $\mathcal{D}$. We conclude from (1.8) that d is a retract of an object $FGd = K(Gd)$ in the image of the functor K.

As $d \in \mathcal{D}$ was arbitrary, we have proved that the fully faithful functor K is surjective up to direct summands. Consider now the idempotent completions

together with the induced functors $K^{\natural}$ and $E^{\natural}$. The two rightmost canonical inclusions are equivalences, since $\mathcal{D}$ and $\mathbb{A}\text{-}\mathrm{Mod}_{\mathcal{C}}$ are idempotent complete. What we have just proved amounts to $K^{\natural}$ being an equivalence too, i.e. the Kleisi comparison functor induces an equivalence

$$K^{\natural} \colon (\mathbb{A}\text{-}\mathrm{Free}_{\mathcal{C}})^{\natural} \xrightarrow{\sim} \mathcal{D}^{\natural} \cong \mathcal{D}$$

after idempotent completion.

We can now easily see that E is an equivalence by chasing the above diagram: as $(EK)^{\natural} = E^{\natural}K^{\natural}$ is fully faithful and $K^{\natural}$ is an equivalence, $E^{\natural}$ is fully faithful, hence so is E. As already observed, the faithfulness of $U_{\mathbb{A}}$ implies that $F_{\mathbb{A}} = EF$ is surjective up to summands, so E must be too. But E is fully faithful and its domain $\mathcal{D}$ is idempotent complete, hence it must be essentially surjective. $\qquad\square$

1.9. **Corollary.** *Let $\mathbb{A}$ be a monad on an idempotent complete triangulated category $\mathcal{C}$. If the Eilenberg-Moore adjunction $\mathcal{C} \rightleftarrows \mathbb{A}\text{-}\mathrm{Mod}_{\mathcal{C}}$ is triangulated, then any triangulated realization $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ of $\mathbb{A}$ induces canonical equivalences:*

$$(\mathbb{A}\text{-}\mathrm{Free}_{\mathcal{C}})^{\natural} \xrightarrow{\sim} (\mathcal{D}/\ker G)^{\natural} \xrightarrow{\sim} \mathbb{A}\text{-}\mathrm{Mod}_{\mathcal{C}} .$$

1.10. *Example.* Let $\mathcal{C} := \mathrm{SH}$ denote the stable homotopy category of spectra. If A is a highly structured ring spectrum (S-algebra, E_{∞}-ring spectrum, brave new ring,...), then we may consider its *derived category* $\mathrm{D}(A)$, defined to be the homotopy category of highly structured A-modules; see [EKMM97]. The unit map

$f \colon S \to A$ induces a triangulated adjunction $f^* = A \wedge - \colon \mathrm{SH} = \mathrm{D}(S) \rightleftarrows \mathrm{D}(A) \colon$ $\mathrm{Hom}_A(A_S, -) = f_*$. On the other hand, by forgetting structure, A is also a monoid in SH and therefore we may consider modules over it in SH. The resulting category $A\text{-}\mathrm{Mod}_{\mathrm{SH}}$ of *naïve* or *homotopy* A-modules is nothing else but the Eilenberg-Moore category for the monad associated with the adjunction $f^* \dashv f_*$. Thus we obtain a comparison functor as in (1.2), which can be thought of as forgetting the higher structure of an A-module:

$$
\begin{array}{ccc}
\mathrm{SH} & & \\
{\scriptstyle f^*} \big\Vert {\scriptstyle f_*} & \overset{U_A}{\nwarrow} & \\
 & \underset{F_A}{\searrow} & \\
\mathrm{D}(A) & \xrightarrow[\exists! \, E]{} & A\text{-}\mathrm{Mod}_{\mathrm{SH}}
\end{array}
$$

Note that both triangulated categories SH and $\mathrm{D}(A)$ are idempotent complete; e.g. because they admit infinite coproducts. Moreover, the right adjoint f_* is conservative by construction; this is equivalent to $A = f^*S \in \mathrm{D}(A)$ weakly generating $\mathrm{D}(A)$. Hence Proposition 1.7 immediately implies the following result.

1.11. Corollary. *Let A be any highly structured ring spectrum. Then the category $A\text{-}\mathrm{Mod}_{\mathrm{SH}}$ of naïve A-modules is triangulated if and only if the canonical comparison functor $\mathrm{D}(A) \to A\text{-}\mathrm{Mod}_{\mathrm{SH}}$ is a (necessarily exact) equivalence.* $\qquad\square$

1.12. *Example.* The comparison between strict and naïve modules was studied by Gutiérrez [Gut05] in the special case where $A = HR$ is the Eilenberg-Mac Lane spectrum of an ordinary associative and unital ring R. He showed that the comparison map is an equivalence if R is a field or a subring of $\mathbb{Q}$, for instance the ring of integers $\mathbb{Z}$.

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