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LARGE EXCURSIONS AND CONDITIONED LAWS FOR RECURSIVE SEQUENCES GENERATED BY RANDOM MATRICES

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We study the large exceedance probabilities and large exceedance paths of the recursive sequence $V_n = M_n V_{n-1} + Q_n$, where $\{(M_n, Q_n)\}$ is an i.i.d. sequence, and M_1 is a $d \times d$ random matrix and Q_1 is a random vector, both with nonnegative entries. We impose conditions which guarantee the existence of a unique stationary distribution for $\{V_n\}$ and a Cramér-type condition for $\{M_n\}$. Under these assumptions, we characterize the distribution of the first passage time $T_u^A := \inf\{n : V_n \in uA\}$, where A is a general subset of $\mathbb{R}^d$, exhibiting that T_u^A/u^α converges to an exponential law for a certain $\alpha > 0$. In the process, we revisit and refine classical estimates for $\mathbb{P}(V \in uA)$, where V possesses the stationary law of $\{V_n\}$. Namely, for $A \subset \mathbb{R}^d$, we show that $\mathbb{P}(V \in uA) \sim C_A u^{-\alpha}$ as $u \to \infty$, providing, most importantly, a new characterization of the constant C_A. As a simple consequence of these estimates, we also obtain an expression for the extremal index of $\{|V_n|\}$. Finally, we describe the large exceedance paths via two conditioned limit theorems showing, roughly, that $\{V_n\}$ follows an exponentially-shifted Markov random walk, which we identify. We thereby generalize results from the theory of classical random walk to multivariate recursive sequences.

1. Introduction. The goal of this paper is to describe the extremal behavior and tail asymptotics, and to develop certain conditioned limit theorems, for the multivariate recursive sequence

$$V_n = M_n V_{n-1} + Q_n, \quad n = 1, 2, \ldots, \qquad V_0 \sim \gamma, \tag{1.1}$$

where $\{(M_n, Q_n)\}$ is an i.i.d. sequence, M_1 is a $d \times d$ random matrix with nonnegative entries, and is Q_1 a nonnegative random vector, and the initial measure γ in (1.1) is supported on the nonnegative orthant and independent

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of $\{(M_n, Q_n)\}$ (typically taken to be point mass at $v \in [0,\infty)^d$). We allow for an arbitrary dependence structure between M_1 and Q_1.

Motivated by branching processes in random environments with immigration, as considered by Solomon [47, 48], the recursive sequence (1.1) was originally studied in the fundamental paper of Kesten [28]. Assuming that the top Lyapunov exponent for $\{M_n\}$ is negative, then the Markov chain $\{V_n\}$ has a unique stationary distribution; and if V is a random variable possessing the stationary law of $\{V_n\}$, then it is shown in [28] that under appropriate moment and irreducibility conditions,

$$\mathbb{P}\left(\langle w, V\rangle > u\right) \sim \mathscr{C}_w u^{-\alpha} \quad \text{as} \quad u \to \infty, \tag{1.2}$$

for any vector $w \in (0,\infty)^d$ and some constant $\mathscr{C}_w > 0$.

Recently, there has been a renewed interest in Kesten's estimate. For example, the asymptotics in (1.2) have been shown to characterize the stationary tail decay in the GARCH(p,q) financial time series model or, analogously, the ARMA(p,q) process with random coefficients; cf. [19, 38]. The process (1.1) is also relevant for the study of random walk in random environment (cf., e.g., [30, 50]), and in a variety of other problems related to branching processes and Mandelbrot cascades; cf. [11, 24, 33] and references therein. Furthermore, in recent years, the scope of Kesten's method has broadened to include more general fixed point equations in $\mathbb{R}$; namely equations of the form $V \stackrel{d}{=} F(V)$, where $F : \mathbb{R} \to \mathbb{R}$ is a random function independent of V, and $F(v) \approx Mv$, for large v, where M is a random variable in $\mathbb{R}$; cf. [4, 18, 23, 39]. (Here $\stackrel{d}{=}$ denotes equality in distribution.) Moreover, generalizations to Markov-dependent recursive sequences (satisfying different assumptions from those we consider here) have been obtained in [14, 17, 44].

It is natural to ask whether this theory may be extended to reveal more refined path properties of the process $\{V_n\}$. In fact, some characteristics of $\{V_n\}$ over large excursions can essentially be inferred from those of the Markov random walk $\{(X_n, S_n) : n = 0, 1, \ldots\}$, defined by

$$X_n = \frac{M_n \cdots M_1 X_0}{|M_n \cdots M_1 X_0|}, \qquad S_n = \log |M_n \cdots M_1 X_0|, \tag{1.3}$$

where $|\cdot|$ denotes a norm in $\mathbb{R}^d$, and X_0 can be taken to be the projection of V_0 onto the unit sphere. While the rough equivalence between $\{V_n\}$ and $\{e^{S_n} X_n\}$ has been utilized by numerous authors, including Kesten [28], the correspondence between these processes has typically only been employed to obtain estimates such as (1.2), and not to characterize more detailed path properties. In contrast, our approach will be to *quantify* this

discrepancy using Markov nonlinear renewal theory, as developed in Melfi [34, 35], yielding—after accounting for the small-time behavior—estimates which show that $\{V_n\}$ is closely approximated by $\{e^{S_n}X_n\}$ in a manner which we characterize mathematically. Consequently, it is natural to expect that, over a large excursion, the random walk structure inherent in $\{(X_n, S_n)\}$ may be exploited to yield deeper properties of $\{V_n\}$ which mimic known attributes of Markov random walk. Following this approach, we shall reexamine Kesten's estimate, then extend the approach to obtain related asymptotic results relevant in extreme value theory, and, ultimately, derive certain path estimates conditioned on a large excursion, showing quantitatively that the path of $\{V_n\}$ under a large excursion resembles a Markov random walk, but in an exponentially-tilted measure (which we will identify as the "α-shifted measure" below).

We start by revisiting (1.2), establishing under appropriate conditions that, for any set $A \subset [0,\infty)^d$ with positive distance to the origin,

$$\mathbb{P}\left(V \in uA\right) \sim \frac{C}{\lambda'(\alpha)} \mathfrak{L}_\alpha(A)\, u^{-\alpha} \quad \text{as} \quad u \to \infty, \tag{1.4}$$

for a universal constant C and a measure $\mathfrak{L}_\alpha$. In particular, we obtain a new representation of the constant C as the α^{th} moment of a certain power series derived from $\{(M_n, Q_n)\}$ and the time-reversed products of $\{M_n\}$; see (2.9) and (2.10) below. The formula we derive can be viewed as a multidimensional extension of a central result in [18]. (For related one-dimensional estimates, see also [12, 21] and the discussion in Section 2.3 below.) From (1.4), we immediately conclude that V is *multivariate* regularly varying, as could only be deduced from (1.2) with the help of the Cramér-Wold device; cf. [6, 8]. We emphasize that this additional step is not needed in our method.

Following a similar approach, we then examine the extremal behavior of $\{V_n\}$. Specifically, letting $A \subset (0,\infty)^d$ have a positive distance to the origin and setting $T_u^A = \inf\{n : V_n \in uA\}$, we study the growth rate of T_u^A as $u \to \infty$. We show that

$$\lim_{u\to\infty} \mathbb{P}\left(\frac{T_u^A}{u^\alpha} \le z \,\middle|\, V_0 = v \right) = e^{-K_A z}, \quad z \ge 0, \tag{1.5}$$

where α is given as in (1.2) and K_A is a constant which we also characterize, relating this constant explicitly to C and to the pre-factor appearing in the asymptotic expression, as $u \to \infty$, for the hitting probability of the set uA by $\{e^{S_n}X_n\}$. As a special case, setting $A = \{x : |x| > 1\}$, we then conclude that $\{|V_n|\}$ belongs to the maximum domain of attraction of the Fréchet distribution. However, it should be emphasized that (1.5) is actually

a stronger result, yielding the directional dependence of $\{V_n\}$ and suggesting a natural extension of classical extreme value theory to this multidimensional setting. Note that (1.5) characterizes the first passage times of the "forward" iterates $\{V_n\}$ (in the sense of Letac [32]), which are qualitatively different from the "backward" iterates. In one dimension, the backward iterates are perpetuities, and the first passage times of these sequences have recently been studied in in [9], yielding very different results from those we obtain here. In contrast, (1.5) is qualitatively similar to *reflected* random walk, and (1.5) can be viewed as an extension, to our setting, of a classical result due to Iglehart [27] and some of its extensions, e.g. [20]. In particular, (1.5) sharpens earlier work, largely restricted to one-dimensional recursions, in [11, 19, 41, 42]; cf. Remark 2.9 below.

The key to establishing (1.4) and (1.5) is a proposition, where we study the behavior of $\{V_n\}$ over *cycles* emanating from, and then returning to, a given set $\mathbb{D} \subset [0,\infty)^d$. Drawing an analogy with reflected random walk, these returns to $\mathbb{D}$ play the role of Iglehart's [27] returns of a reflected random walk to the origin. Letting τ denote the first return time to $\mathbb{D}$, then for any suitable function g and any $m \in \{1,2,\ldots\}$, we consider in Proposition 4.1 the limit behavior of

$$u^{\alpha}\mathbb{E}\left[g\left(\frac{V_{T_u^A}}{u},\ldots,\frac{V_{T_u^A+m}}{u}\right)\mathbf{1}_{\{T_u^A<\tau\}}\,\Big|\,V_0=v\right] \quad \text{as} \quad u\to\infty.$$

If $g=1$, then this quantity represents the rescaled probability that $\{V_n\}$ enters the set uA before returning to $\mathbb{D}$. Moreover, for general g, we show that the post-T_u^A-process behaves as $\{e^{S_n}X_n\}$, but starting with the *stationary overjump distribution*. This idea is then extended in the final section of the article to include the path behavior *prior* to time T_u^A, drawing a close analogy to the trajectory of $\{e^{S_n}X_n\}$ in the α-shifted measure.

Namely, we develop two conditioned limit theorems. In the first, we study the empirical law of $\{\log|V_n| - \log|V_{n-1}|\}$ conditioned on $\{T_u^A<\tau\}$, showing that this empirical law converges weakly in $\mathbb{P}\left(\cdot\,|T_u^A<\tau\right)$-probability to the distribution, under stationarity, of S_1 in the α-shifted measure. We also establish a result concerning the joint distribution of $\{V_{I_u}, V_{I_u+1},\ldots\}$ conditioned on $\{T_u^A<\tau\}$, where I_u grows "slowly" compared wtih u.

We emphasize that we shall develop our limit theorems *without* assuming that the process $\{V_n\}$ is Harris recurrent, and, thus—while we shall often draw upon the theory of Harris recurrent chains and these methods will play an important role in our analysis—our appoach will ultimately not require this standard assumption from Markov chain theory, which is unnatural in our setting. We circumvent this requirement by introducing a smoothing

technique, where the sequence $\{Q_{kn}\}$ is "smoothed" for some $k \in \{1, 2, \ldots\}$, thereby ensuring that the resulting process is Harris recurrent, yet the effect of this smoothing is negligible in an asymptotic limit. This technique could also be adapted to other recursive sequences satisfying a stochastic fixed point equation (as considered in the one-dimensional setting in [18]). To obtain a general theory without Harris recurrence, we shall, instead, rely throughout the article on the recently-developed theory of Guivarc'h and Le Page [25], which exploits spectral gap properties on special function spaces for matrix products under weak regularity conditions. While the theory in [25] is developed for invertible matrices, a formulation for matrices with nonnegative entries, as we consider here, has recently been given in [10].

We now turn to a precise statement of our main results.

2. Statement of results.

2.1. *Notation.* Let $\mathbb{N}_+ := \{1, 2, \ldots\}$ denote the positive integers. For given $d \in \mathbb{N}_+$, assume that $\mathbb{R}^d$ is endowed with the scalar product $\langle \cdot, \cdot \rangle$ and canonical orthonormal basis $\{e_i\}$. Set $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : \langle x, e_i \rangle \geq 0,\ 1 \leq i \leq d\}$.

Let $|\cdot|$ denote a norm in $\mathbb{R}^d_+$, and assume throughout the article that $|\cdot|$ is monotone, i.e., if $x, y \in \mathbb{R}^d_+$ satisfy $y - x \in \mathbb{R}^d_+$, then $|x| \leq |y|$. Let $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ denote the unit sphere and $\mathbb{S}^{d-1}_+ := \mathbb{R}^d_+ \cap \mathbb{S}^{d-1}$; and for any $x \in \mathbb{R}^d \setminus \{0\}$, let $\widetilde{x}$ denote its projection onto the unit sphere, namely

$$\widetilde{x} \equiv (x)^{\sim} := |x|^{-1} x.$$

Set $B_r(y) = \{x \in \mathbb{R}^d : |x - y| < r\}$, $r > 0$; and $B^+_r(y) = B_r(y) \cap \mathbb{R}^d_+$.

For any subspace $\mathscr{S}$ of $\mathbb{R}^d_+$, let $\mathscr{B}(\mathscr{S})$ denote the collection of Borel sets on $\mathscr{S}$; and let E°, $\bar{E}$, E^c, and ∂E denote the interior, closure, complement, and boundary of $E \in \mathscr{B}(\mathscr{S})$, respectively. For any measure ν on $\mathscr{S} \subset \mathbb{R}^d_+$, denote the support of ν by $\operatorname{supp} \nu$. Also, denote the set of bounded continuous real-valued functions on a space E by $\mathscr{C}_b(E)$, equipped with the norm $|f|_\infty := \sup\{|f(x)| : x \in E\}$.

Let $\mathfrak{M}$ denote the collection of $d \times d$ matrices with nonnegative coefficients, and let $\|\mathfrak{m}\|$ denote operator norm, i.e., $\|\mathfrak{m}\| := \sup_{x \in \mathbb{S}^{d-1}} |\mathfrak{m}x|$, $\mathfrak{m} \in \mathfrak{M}$.

Now suppose that $\{V_n\}$ and $\{(M_n, Q_n) : n \in \mathbb{N}_+\}$ are defined as in the previous section; in particular, each (M_n, Q_n) is an i.i.d. copy of (M, Q), where the random matrix M takes values in $\mathfrak{M}$ a.s., Q takes values in $\mathbb{R}^d_+$ a.s., and we allow the pair (M, Q) to have an arbitrary dependence structure. Denote the probability laws of (M, Q), M, and Q by μ, μ_M, and μ_Q, respectively. Assume that $\{(M_n, Q_n) : n = 1, \ldots, n\}$ is adapted to a given filtration $\{\mathscr{F}_n : n = 1, 2, \ldots\}$.

2.2. *Basic assumptions.* We first introduce certain restrictions on $\{M_n\}$.

Allowable and positively regular matrices. We say that a matrix $\mathfrak{m} \in \mathfrak{M}$ is *allowable* if it has no zero row or column. Moreover, if the coefficients of a given matrix $\mathfrak{m} \in \mathfrak{M}$ are *strictly* positive, then we write $\mathfrak{m} \succ 0$ and say that $\mathfrak{m}$ is *positively regular*. Also write $\mathfrak{M}^\circ = \{\mathfrak{m} \in \mathfrak{M} : \mathfrak{m} \succ 0\}$. *As a standing assumption, we shall always assume that there exists an* $n \in \mathbb{N}_+$ *such that*

$$\mathfrak{N} := \inf\{n \in \mathbb{N}_+ : M_n \cdots M_1 \succ 0\} < \infty \quad a.s.;$$

thus, ultimately, the product $M_n \cdots M_1$ *is positively regular with probability one.* This assumption will be subsumed in the stronger Hypothesis (H_1), given below (cf. [26, Lemma 3.1] or [10, Lemma 6.3]).

Non-arithmetic distributions for random matrices. Next, we shall need a generalization of the notion of a non-arithmetic distribution to the setting of random matrices. To this end, let Γ_M denote the smallest closed subsemigroup of $\mathfrak{M}$ which contains $\operatorname{supp}\mu_M$.

Definition 2.1. We say that μ_M is *non-arithmetic* if the additive group generated by $\{\log\|\mathfrak{m}\| : \mathfrak{m} \in \Gamma_M \cap \mathfrak{M}^\circ\}$ is dense in $\mathbb{R}$.

It is shown in [13, Lemma 2.7] that this condition implies that of Shurenkov [45], which is closer to the condition imposed on one-dimensional Markov random walks, but not easily verified in the setting of random matrices. It is worth observing that, alternatively, we could replace $\log\|\mathfrak{m}\|$ with the Frobenius eigenvalue of $\mathfrak{m}$ in Definition 2.1; thus, our definition is in agreement with the one given by Kesten in [28].

We are now prepared to introduce our basic assumptions on the distribution function μ_M of M.

Hypothesis (H_1). μ_M is non-arithmetic and $\mu_M\{\mathfrak{m} : \mathfrak{m} \text{ is allowable}\} = 1$.

Next, we turn to certain moment conditions that will be imposed on the pair (M, Q). Let

$$\mathfrak{D} = \left\{\theta \geq 0 \,:\, \int_{\mathfrak{M}} \|\mathfrak{m}\|^\theta \,\mu_M(d\mathfrak{m}) < \infty\right\} = \left\{\theta \geq 0 \,:\, \mathbb{E}\left[\|M\|^\theta\right] < \infty\right\};$$

and let $\mathfrak{m}^T$ denote the transpose of $\mathfrak{m}$. Then for any $\theta \in \mathfrak{D}$ and any $f \in \mathscr{C}_b(\mathbb{S}_+^{d-1})$, set:

$$P_\theta f(x) = \mathbb{E}\left[|Mx|^\theta\, f\big(\widetilde{Mx}\big)\right]; \quad P_\theta^* f(x) = \mathbb{E}\left[\big|M^T x\big|^\theta\, f\big(\widetilde{M^T x}\big)\right];$$

$$\lambda(\theta) = \lim_{n\to\infty} \left(\mathbb{E}\left[\|M_n \cdots M_1\|^\theta \right] \right)^{1/n}; \quad \Lambda(\theta) = \log \lambda(\theta).$$

In the following lemma, we describe the left-invariant measures and right-invariant functions associated with the operators P_θ and P_θ^*.

LEMMA 2.2. *Assume $\theta \in \mathfrak{D}$ and $\mu_M\{\mathfrak{m} : \mathfrak{m}$ is allowable$\} = 1$. Then $\lambda(\theta)$ is the spectral radius of P_θ, and there is a unique probability measure l_θ on $\mathbb{S}_+^{d-1}$ and a unique, strictly positive function $r_\theta \in \mathscr{C}_b(\mathbb{S}_+^{d-1})$ with $\int r_\theta(x) l_\theta(dx) = 1$ such that*

$$l_\theta P_\theta = \lambda(\theta) l_\theta \quad \text{and} \quad P_\theta r_\theta = \lambda(\theta) r_\theta. \tag{2.1}$$

Furthermore, the function r_θ is $\max\{\theta, 1\}$-Hölder continuous; and thus, r_θ is bounded from above and below by finite positive constants.

Similarly, the spectral radius of P_θ^* equals $\lambda(\theta)$, and there exist a pair (l_θ^*, r_θ^*) which has the equivalent properties, relative to P_θ^*, as those possessed by (l_θ, r_θ) relative to P_θ. Moreover,

$$r_\theta(x) = c \int_{\mathbb{S}_+^{d-1}} \langle x, y\rangle^\theta l_\theta^*(dy), \quad \forall x \in \mathbb{S}_+^{d-1}, \tag{2.2}$$

for $c = \left(\int \langle x, y\rangle^\alpha l_\theta^*(dx) l_\theta(dy) \right)^{-1}$; and likewise, (2.2) also holds if (r_θ, l_θ^*) is replaced with (r_θ^*, l_θ).

In the above lemma, we have written $l_\theta P_\theta$ for the application of the adjoint operator P_θ' to the measure l_θ, i.e. $l_\theta P_\theta$ is the unique measure satisfying

$$\int_{\mathbb{S}_+^{d-1}} f(x) \big(l_\theta P_\theta\big)(dx) = \int_{\mathbb{S}_+^{d-1}} \big(P_\theta f(x)\big)\, l_\theta(dx), \quad \text{for all } f \in \mathscr{C}_b(\mathbb{S}_+^{d-1}).$$

The proof of Lemma 2.1 can be found in [10], Proposition 3.1; see also [25, Theorem 2.16] for an analogous result in the setting of invertible matrices.

For any allowable matrix $\mathfrak{m}$, now define $\boldsymbol{i}(\mathfrak{m}) := \inf_{x\in\mathbb{S}_+^{d-1}} |\mathfrak{m}x|$.

HYPOTHESIS (H_2). There exists an $\alpha > 0$ such that $\lambda(\alpha) = 1$, and the following moment conditions hold:

$$\mathbb{E}\Big[\|M\|^\alpha \max\{|\log \|M\||, |\log \boldsymbol{i}(M)|\} \Big] < \infty; \quad \text{and} \quad \mathbb{E}\big[|Q|^\alpha\big] < \infty.$$

The shifted distribution. We shall utilize the constant α in (H_2) to employ a change of measure, as developed in the multidimensional framework by Kesten [28]. Namely for $\mathfrak{m} \in \mathfrak{M}$, $\theta \in \mathfrak{D}$ and any $n \in \mathbb{N}_+$, define

$$p_n^\theta(x, \mathfrak{m}) = \frac{|\mathfrak{m}x|^\theta}{(\lambda(\theta))^n} \frac{r_\theta\big(\widetilde{\mathfrak{m}x}\big)}{r_\theta(x)}, \qquad x \in \mathbb{S}_+^{d-1}.$$

Note by an application of Lemma 2.2 that

$$\int p_n^\theta(x, \mathfrak{m}_n \cdots \mathfrak{m}_1)\ \mu^{\otimes n}\left(\{d\mathfrak{m}_i, dq_i\}_{i=1}^n\right) = 1, \quad x \in \mathbb{S}_+^{d-1}.$$

Moreover, the system of probability measures $\mu_{n,x}^\theta = p_n^\theta(x,\cdot)\mu^{\otimes n}$ is a projective system; hence by the Kolmogorov extension theorem, there exists a unique probability measure $\mathbb{P}_x^\theta$ on $(\mathfrak{M} \times \mathbb{R}_+^d)^{\mathbb{N}_+}$ having marginals $\mu_{n,x}^\theta$. When the random variables $\{(M_n, Q_n) : n = 1, 2, \ldots\}$ are generated by the measure $\mathbb{P}_x^\theta$ rather than the true underlying probability measure, we write $\mathbb{E}_x^\theta[\cdot]$. *We shall refer to this measure as the "θ-shifted measure."*

It is worth observing that, although $\{(M_n, Q_n) : n = 1, 2, \ldots\}$ is assumed to be i.i.d. in the unshifted measure, this sequence will be *Markov-dependent* in the θ-shifted measure, for any $\theta > 0$. However,

$$\eta_\theta(E) := \int_E r_\theta(x) l_\theta(dx) \quad \text{yields that} \quad \widehat{\mathbb{P}}^\theta := \int_{\mathbb{S}_+^{d-1}} \mathbb{P}_x^\theta\ \eta_\theta(dx) \tag{2.3}$$

is shift-invariant; i.e., the sequence $\{(M_n, Q_n)\}$ is stationary under $\widehat{\mathbb{P}}^\theta$; cf. Section 3.1 of [10]. This is an important observation, as it will allow us to apply the results of Hennion [26] on products of random matrices; cf. Section 4 below. Furthermore, by Lemma 6.2 of [10], $\mathbb{P}_x^\theta \ll \widehat{\mathbb{P}}^\theta$ for all $x \in \mathbb{S}_+^{d-1}$; and we shall use this result frequently to infer convergence $\mathbb{P}_x^\alpha$-a.s., for arbitrary $x \in \mathbb{S}_+^{d-1}$, by proving $\widehat{\mathbb{P}}^\theta$-a.s. convergence.

In the θ-shifted measure, the limit behavior is described through the following generalization of the Furstenberg-Kesten theorem, which may be deduced from [26, Theorem 2] together with [10, Theorem 6.1].

LEMMA 2.3. *Assume that (H_1) is satisfied and let $\theta \in \mathfrak{D}$, and suppose that (H_2) holds with θ in place of α. Then for $x, y \in \mathbb{S}_+^{d-1}$, we have $\mathbb{P}^\theta$-a.s. that:*

$$\lim_{n\to\infty} \frac{1}{n} \log |M_n \cdots M_1 x| = \lim_{n\to\infty} \frac{1}{n} \log \|M_n \cdots M_1\| = \Lambda'(\theta) = \hat{\mathbb{E}}^\theta[S_1];$$

$$\lim_{n\to\infty} \sup \left\{ \left| \frac{1}{n} \mathbf{1}_{\{\mathfrak{N} \le n\}} \log\langle y, M_n \ldots M_1 x\rangle - \Lambda'(\theta) \right| \ :\ x, y \in \mathbb{S}_+^{d-1} \right\} = 0.$$

Here, $\Lambda'(\theta)$ is interpreted as a one-sided derivative if $\theta \in \partial\mathfrak{D}$. Note $\Lambda'(0) < 0$, since Λ is convex and $\Lambda(0) = \Lambda(\alpha) = 1$; thus, the top Lyapunov exponent associated with $\{M_n\}$ is negative. Together with the moment assumptions in (H_2), this guarantees the existence of a unique stationary distribution for $\{V_n\}$ (cf. [28]), which is given by the law of

$$V \ :=\ Q_1 + \sum_{k=2}^{\infty} M_1 \cdots M_{k-1} Q_k. \tag{2.4}$$

The Markov random walk. The process $\{M_n\}$ induces a Markov random walk on $\mathbb{S}_+^{d-1} \times \mathbb{R}$, obtained by setting

$$X_n = (M_n \cdots M_1 X_0)^{\sim}, \quad S_n = \log |M_n \cdots M_1 X_0|, \quad n = 1, 2, \ldots, \tag{2.5}$$

for some initial state $X_0 \in \mathbb{S}_+^{d-1}$ and $S_0 = 0$. *In contrast to* (1.3)*, in some contexts we will need to take* X_0 *to be different from* V_0*,* but still independent of $\{(M_n, Q_n)\}$. This process will play an important role in the sequel. Note that in the θ-shifted measure, $\{X_n\}$ has a unique stationary distribution given by the measure η_θ in (2.3); see [10, Theorem 4.11].

Probability measures. We introduce the following conventions to describe conditional probabilities which depend on the initial values of X_0 and V_0. Write:

$$\mathbb{P}_v(\cdot) = \mathbb{P}(\cdot | V_0 = v), \quad \mathbb{P}_x^\theta(\cdot) = \mathbb{P}(\cdot | X_0 = x), \quad \mathbb{P}_{x,v}^\theta(\cdot) = \mathbb{P}_x^\theta(\cdot | V_0 = v),$$

and use the same notation for the corresponding expectations. When conditioning on an initial distribution $V_0 \sim \gamma$, write $\mathbb{P}_\gamma(\cdot) = \int \mathbb{P}_v(\cdot)\, \gamma(dv)$, $\mathbb{P}_\gamma^\theta(\cdot) = \int \mathbb{P}_{\tilde{v},v}^\theta(\cdot)\gamma(dv)$, and finally set $\mathbb{P}_{\boldsymbol{\delta}_v}^\theta(\cdot) = \mathbb{P}_{\tilde{v},v}^\theta(\cdot)$. We note that while working in the θ-shifted measure, we will generally need to specify *both* X_0 and V_0 in these equations, and we will typically take $X_0 = \widetilde{V_0}$. The reason for the asymmetry comes from the observation that, due to the Markov dependence in the θ-shifted measure, the initial state *does* affect the law of $\{M_n\}$ and hence that of $\{V_n\}$ under $\mathbb{P}^\theta$. Finally, we note that we will sometimes suppress the dependence on (x, v) when these values are clear and simply write $\mathbb{P}^\alpha$-a.s.

In this terminology, the change of measure can be written as follows: for all $n \in \mathbb{N}_+$, $x \in \mathbb{S}_+^{d-1}$, and any bounded measurable function $f : \mathbb{S}_+^{d-1} \times (\mathfrak{M} \times \mathbb{R}_+^d)^n$,

$$\begin{aligned} r_\alpha(x)\mathbb{E}_{x,v}^\alpha\Big[\frac{e^{-\alpha S_n}}{r_\alpha(X_n)} f(X_0, V_0, M_1, Q_1, \ldots, M_n, Q_n)\Big] \\ = \mathbb{E}\Big[f(x, v, M_1, Q_1, \cdots, M_n, Q_n)\Big]. \end{aligned} \tag{2.6}$$

2.3. *Tail estimates for* $\{V_n\}$. We now turn to our first main result, where we revisit and extend Kesten's well known theorem in [28].

Let π denote the stationary distribution of $\{V_n\}$, which is given by the law of the random variable V defined in (2.4). Now fix a set $\mathbb{D} \subset \mathbb{R}_+^d$ where $\pi(\mathbb{D}) > 0$, and let $\pi_{\mathbb{D}}$ denote the stationary distribution of $\{V_n\}$ restricted to $\mathbb{D}$; that is,

$$\pi_{\mathbb{D}}(E) = \frac{\pi(E \cap \mathbb{D})}{\pi(\mathbb{D})}, \quad E \in \mathscr{B}(\mathbb{R}_+^d). \tag{2.7}$$

Also let τ denote the first return time of $\{V_n\}$ to $\mathbb{D}$; namely,

$$\tau = \inf\{n \in \mathbb{N}_+ : V_n \in \mathbb{D}\}.$$

Next, let $\vec{1} = (1, \dots, 1)^T$, and define

$$Y_i = \lim_{n\to\infty} \left(M_i^\top \cdots M_n^\top \vec{1}\right)^\sim, \quad n = 1, 2, \dots. \tag{2.8}$$

Note that if $\theta \in \mathfrak{D}$, then the limit on the right-hand side exists $\mathbb{P}^\theta$-a.s., since this product constitutes a backward sequence of an iterated function system and the maps $\{M_n\}$ act as contractions on $\mathbb{S}_+^{d-1}$; cf. [26, Section 3]. Moreover, the law of Y_i is given by

$$\eta_\theta^*(E) := \int_E r_\theta^*(x) l_\theta^*(dx), \quad E \in \mathscr{B}(\mathbb{S}_+^{d-1}),$$

where r_θ^* and l_θ^* are given as in Lemma 2.2 (cf. [25], Theorem 3.2; [10], Proposition 3.1).

***The condition* ($\mathfrak{K}$).** Recall that under (H_1), the measure μ_M is non-arithmetic and hence $M_n \cdots M_1$ is positively regular for sufficiently large n w.p.1, implying that for some positive integer k and some $s > 0$,

$$M_k \cdots M_2 Q_1 \succ s\vec{1} \quad \text{with positive probability.} \tag{$\mathfrak{K}$}$$

Now if $k > 1$, then it is natural to introduce the k-step process; namely, fix $k \in \mathbb{N}_+$, and for all $n \in \mathbb{N}_+$, set

$$\widehat{M}_n := M_{kn} \cdots M_{k(n-1)+1} \quad \text{and} \quad \widehat{Q}_n = \sum_{i=k(n-1)+1}^{kn} M_{kn} \cdots M_{i+1} Q_i.$$

Note as a consequence of these definitions that

$$V_{kn} = \widehat{M}_n V_{k(n-1)} + \widehat{Q}_n, \quad n = 1, 2, \dots,$$

where $\widehat{Q}_n - s\vec{1} \succ 0$ with positive probability. It is worth observing here that the stationary distributions of $\{V_{kn}\}$ and $\{V_n\}$ are, of course, identical.

Finally, let $\mathscr{C}_0\big(\mathbb{R}_+^d \setminus \{0\}\big)$ denote the set of bounded continuous functions on $\mathbb{R}_+^d \setminus \{0\}$ which are supported on $\mathbb{R}_+^d \setminus B_r(0)$, for some $r > 0$.

THEOREM 2.4. *Assume that Hypotheses (H_1) and (H_2) are satisfied, and suppose that $\mathbb{D} = B_r^+(0)$, where r has been chosen sufficiently large such that $\pi(\mathbb{D}) > 0$. If $f \in \mathscr{C}_0\big(\mathbb{R}_+^d \setminus \{0\}\big)$ and $k = 1$ in $(\mathfrak{K})$, then*

$$\lim_{u\to\infty} u^\alpha \mathbb{E}\left[f\left(\frac{V}{u}\right)\right] = \frac{C}{\lambda'(\alpha)} \int_{\mathbb{S}_+^{d-1}\times\mathbb{R}} e^{-\alpha s} f(e^s x) l_\alpha(dx) ds, \tag{2.9}$$

where

$$C = \int_{\mathbb{D}} r_\alpha(\widetilde{v}) \mathbb{E}^\alpha_{\boldsymbol{\delta}_v} \left[\left(|v| + \sum_{i=1}^{\infty} \frac{\langle Y_i, \widetilde{Q}_i \rangle}{\langle Y_i, X_i \rangle} \frac{|Q_i|}{|M_i \cdots M_1 \widetilde{v}|} \right)^\alpha \mathbf{1}_{\{\tau=\infty\}} \right] \pi(dv). \tag{2.10}$$

If $k > 1$ in $(\mathfrak{K})$, then the theorem still holds, but the constant C is then computed with respect to the k-step chain $\{V_{kn}\}$ generated by $\{(\widehat{M}_i, \widehat{Q}_i)\}$ rather than with respect to the 1-step chain $\{V_n\}$.

If $\{V_n\}$ is a Harris recurrent chain, then we may always take $k = 1$; see Proposition 5.2 below. Moreover, if $Q \succ 0$ with positive probability, then we may again take $k = 1$.

More generally, when dealing with the k-step chain, we observe that the stopping time τ in (2.10) must now be computed with respect to that chain (rather than the 1-step chain), and the drift factor $\lambda'(\alpha)$ in (2.9) must be replaced with the drift of the k-step chain, namely $k\lambda'(\alpha)$; cf. Remark 5.3 below.

Remark 2.5. For another representation of (2.9), let $\mathfrak{L}_\alpha$ be the measure on $\mathbb{R}^d_+ \setminus \{0\}$ defined by the equation

$$\int_{\mathbb{S}^{d-1}_+ \times \mathbb{R}} e^{-\alpha s} f(e^s x) l_\alpha(dx) ds = \int_{\mathbb{R}^d_+ \setminus \{0\}} f(x) \mathfrak{L}_\alpha(dx).$$

Then (2.9) gives the vague convergence (of measures on $\mathbb{R}^d_+ \setminus \{0\}$) toward $C/\lambda'(\alpha)$. In particular, for any measurable set $A \subset \mathbb{R}^d_+$ which is bounded away from zero and satisfies $\mathfrak{L}_\alpha(\partial A) = 0$, it follows from the Portmanteau theorem that

$$\lim_{u\to\infty} u^\alpha \mathbb{P}\left(V \in uA\right) = \frac{C}{\lambda'(\alpha)} \mathfrak{L}_\alpha(A). \tag{2.11}$$

Furthermore, note that for any $t > 0$ and any measurable $E \subset \mathbb{S}^{d-1}_+$ with $l_\alpha(\partial E) = 0$, the sets $E^t := \{x \in \mathbb{R}^d_+ \,:\, |x| > t, x/|x| \in E\}$ are $\mathfrak{L}_\alpha$-continuous. Hence, for all $E \subset \mathbb{S}^{d-1}_+$ with $l_\alpha(\partial E) = 0$,

$$\lim_{u\to\infty} u^\alpha \mathbb{P}\left(|V| > tu,\, \frac{V}{|V|} \in E\right) = \frac{C}{\alpha\lambda'(\alpha)} t^{-\alpha} l_\alpha(E). \tag{2.12}$$

Thus we infer the weak convergence

$$\lim_{u\to\infty} \mathbb{P}\left(\frac{V}{|V|} \in \cdot \;\Big|\; |V| > u\right) \Rightarrow l_\alpha(\cdot). \tag{2.13}$$

REMARK 2.6. Let $\widetilde{C}(v)$ denote the expectation in (2.10); that is, $C = \int_{\mathbb{D}} r_\alpha(\tilde{v})\widetilde{C}(v)\pi(dv)$. Then we have two further representations for $\widetilde{C}(v)$. First, by Lemma 3.6 below, it will follow that

$$\widetilde{C}(v) = \lim_{n\to\infty} \mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\left[\left(\frac{|V_n|}{|M_n\cdots M_1\tilde{V}_0|}\right)^\alpha \mathbf{1}_{\{\tau\geq n\}}\right]. \tag{2.14}$$

Moreover, it will follow by combining Corollary 4.2 (noting $C(v) = r_\alpha(\tilde{v})\widetilde{C}(v)$) with Lemma 6.1 that

$$\widetilde{C}(v) = \lim_{u\to\infty} \frac{\mathbb{P}\big(|V_n| > u,\ 0\leq n<\tau \,|\, V_0 = v\big)}{\mathbb{P}\big(S_n > \log u, \text{ for some } n\in\mathbb{N}\,|\, X_0 = \tilde{v}\big)}. \tag{2.15}$$

The latter expression shows that $\widetilde{C}(v)$ describes the discrepancy between the probability of a large exceedance of $\{V_n\}$ occuring over a cycle, and the probability of ruin for the corresponding Markov random walk.

We conclude this section with a brief comparison of our result to some recent one-dimensional representations. As noted in the introduction, (2.10) can be viewed as a generalization of a result of Collamore and Vidyashankar [18] to the multidimensional setting. Alternatively, building upon Goldie [23], it is shown in Buraczewski et al. [12] that in the one-dimensional setting (and its generalization to the class of similarities described there), we have $C = (\alpha\lambda'(\alpha))^{-1}\lim_{n\to\infty} n^{-1}\mathbb{E}\left[|V_n|^\alpha\right]$. Finally, a further one-dimensional representation was derived in Enriquez et al. [21], expressed in terms of expectations of perpetuity sequences under a delicate conditioning on the process. The proofs of these one-dimensional results are all quite different, and therefore it is not transparent how they can be easily unified.

2.4. *Extremal estimates for maxima and first passage times.* Our next objective is to study the probability of a large exceedance occuring over a single cycle emanating from, and then returning to, a given set $\mathbb{D}\subset\mathbb{R}^d_+$, and, in this way, to characterize the distribution of the first passage time

$$T_u := \inf\{n\in\mathbb{N}_+ : |V_n| > u\}, \tag{2.16}$$

or more generally,

$$T^A_u := \inf\{n\in\mathbb{N}_+ : V_n\in uA\}, \quad \text{where} \quad A\subset\{x\in\mathbb{R}^d_+ : |x|>1\}, \tag{2.17}$$

and we assume that the set A satisfies the following regularity property.

DEFINITION 2.7. We say that a set $A\in\mathscr{B}(\mathbb{R}^d)$ is a *semi-cone* if $A\subset\mathbb{R}^d_+\setminus B_1(0)$ and $x\in\partial A\Rightarrow\{tx : t>1\}\subset A$.

Now suppose that A is a semi-cone, let $\{S_n\}$ be defined as in (2.5), and set

$$\begin{aligned} &d_A(x) = \inf\{t > 1 : tx \in A\}, \quad x \in \mathbb{S}_+^{d-1}; \\ &S_n^A = S_n - \log d_A(X_n), \quad n = 0, 1, 2, \ldots; \\ &r_\alpha^A(x) = r_\alpha(x)\big(d_A(x)\big)^\alpha, \quad x \in \mathbb{S}_+^{d-1}; \quad \mathfrak{P}_A = \Big\{x \in \mathbb{S}_+^{d-1} : d_A(x) < \infty\Big\}. \end{aligned} \tag{2.18}$$

As a consequence of Kesten's renewal theorem, it will be shown in Lemma 6.1 below that if $\mathfrak{P}_A = \mathbb{S}_+^{d-1}$, then

$$\mathbb{P}\left(M_n \cdots M_1 \widetilde{V}_0 \in uA, \text{ for some } n \in \mathbb{N}_+ \,\middle|\, V_0 = v\right) \sim r_\alpha(\widetilde{v}) D_A u^{-\alpha} \tag{2.19}$$

as $u \to \infty$, where

$$D_A := \int_{\mathbb{S}_+^{d-1} \times \mathbb{R}_+} \frac{e^{-\alpha s}}{r_\alpha^A(x)} \varrho^A(dx, ds) \tag{2.20}$$

and the measure ϱ^A will be specified below in Section 3.3. Essentially, (2.19) is the *ruin estimate* for the Markov random walk $\{(X_n, S_n^A) : n = 0, 1, \ldots\}$ under the initial state $X_0 = \tilde{v}$, and ϱ^A corresponds to the stationary excess distribution for this process. Indeed, if A is a semi-cone and d_A is continuous, then it follows immediately from the definitions that $e^{S_n} X_n \in uA \Leftrightarrow e^{S_n} > u \cdot d_A(X_n)$ and hence, on the left-hand side of (2.19),

$$M_n \cdots M_1 \widetilde{V}_0 \in uA \;\Leftrightarrow\; e^{S_n} X_n \in uA \;\Leftrightarrow\; S_n^A > \log u.$$

Now if $\mathfrak{P}_A \neq \mathbb{S}_+^{d-1}$, then (2.19) will still hold and this *defines* the constant D_A, although the identification of D_A is less explicit in that case (i.e., there is no equivalent of (2.20)). However, D_A can nonetheless be interpreted as the ruin constant for the Markov random walk; see Section 6 below.

Finally, let C be defined as in (2.10), and set

$$C(v) = r_\alpha(\widetilde{v}) \mathbb{E}_{\boldsymbol{\delta}_v}^\alpha \left[\left(|v| + \sum_{i=1}^\infty \frac{\langle Y_i, \widetilde{Q}_i\rangle}{\langle Y_i, X_i\rangle} \frac{|Q_i|}{|M_i \cdots M_1 \widetilde{v}|}\right)^\alpha \mathbf{1}_{\{\tau = \infty\}}\right]. \tag{2.21}$$

THEOREM 2.8. *Suppose that Hypotheses (H_1) and (H_2) are satisfied and $\mathbb{D} = B_r^+(0)$, where r has been chosen sufficiently large such that $\pi(\mathbb{D}) > 0$. Assume that A is a semi-cone, and d_A is continuous. Then for any $v \in \mathbb{R}_+^d \setminus \{0\}$,*

$$\lim_{u \to \infty} u^\alpha \mathbb{P}\left(T_u^A < \tau \,\middle|\, V_0 = v\right) = D_A C(v). \tag{2.22}$$

Furthermore, assuming that $k = 1$ in $(\mathfrak{K})$, we have that the normalized sequence $\{T_u^A/u^\alpha\}$ converges in distribution; more precisely,

$$\lim_{u\to\infty} \mathbb{P}\left(\frac{T_u^A}{u^\alpha} \le z \,\middle|\, V_0 = v\right) = 1 - e^{-K_A z}, \quad z \ge 0, \tag{2.23}$$

for all $v \in \mathbb{R}_+^d \setminus \{0\}$, where $K_A = CD_A$.

As in Theorem 2.4, the assumption $k = 1$ is *not* necessary if $\{V_n\}$ is Harris recurrent, or if $Q \succ 0$ with positive probability.

Remark 2.9. For one-dimensional recursions, related estimates have previously been given for the distribution of $\max\{V_i : 1 \le i \le n^{1/\alpha}u\}$ as $n \to \infty$; cf. [19, Theorem 2.1] or [41]. However, in the multidimensional setting, the only result we are aware of is that of Perfekt [42], who studies the componentwise maxima, namely

$$\left(\max_{1\le i\le n} V_1, \dots, \max_{1\le i\le n} V_d\right) \quad \text{as} \quad n \to \infty.$$

Note that the componentwise maxima need not be achieved simultaneously; hence Perfekt's results do not coincide with ours. Moreover, in all of these references, additional conditions are assumed which we do not impose here; in particular, in their formulations it must be assumed that $V_0 \sim \pi$.

Remark 2.10. As a particular application of the previous theorem, we now determine the extremal index of $\{|V_n|\}$. Integrating with respect to the measure π in (2.23), we obtain that

$$\lim_{u\to\infty} \mathbb{P}\left(\frac{T_u^A}{u^\alpha} \le z \,\middle|\, V_0 \sim \pi\right) = 1 - e^{-K_A z}, \quad z \ge 0.$$

Set $A = \{x : |x| > 1\}$. Then it easily follows with $u = n^{1/\alpha}w$ and $z = w^{-\alpha}$ that

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{1\le i\le n} |V_i| \le n^{1/\alpha}w \,\middle|\, V_0 \sim \pi\right) = e^{-K_A w^{-\alpha}}. \tag{2.24}$$

Moreover, for this choice of A, it follows by Theorem 2.4 that

$$\lim_{n\to\infty} n\mathbb{P}\left(|V| > n^{1/\alpha}w\right) = \frac{C}{\alpha\lambda'(\alpha)} w^{-\alpha}. \tag{2.25}$$

Then reasoning as in [31, Section 2.2], we conclude from (2.24) and (2.25) that the extremal index of $\{|V_n|\}$ is given by

$$\Theta = \alpha\lambda'(\alpha)D_A. \tag{2.26}$$

For a related result in the one-dimensional setting, see [18, Proposition 2.2].

2.5. *The path and empirical law under a large exceedance.* We conclude by examining the path behavior of $\{V_n\}$ prior to a large exceedance. Motivated by classical results for random walk (e.g. Section XII.6.(d) of [22] or more recent work in [5, 7]), it is natural to expect that, conditioned on $\{T_u^A < \tau\}$ (where τ is the return time to any π-positive set $\mathbb{D}$), $\{V_n\}$ should behave as its "associate," which, in our setting, translates to the process $\{e^{S_n} X_n\}$ under the α-shifted measure.

However, in our problem, we cannot anticipate that the behavior of $\{V_n\}$ will mimic that of $\{e^{S_n} X_n\}$ over the *entire* trajectory. For this reason, we introduce an "initial" level ε_u, where $\varepsilon_u = o(u)$ and $\varepsilon_u \uparrow \infty$ as $u \to \infty$, and study the trajectory of $\{V_n\}$ subsequent to its exceedance over the level ε_u.

THEOREM 2.11. *Suppose that Hypotheses (H_1) and (H_2) are satisfied, and assume that A is a semi-cone and the function d_A is bounded and continuous on $\mathbb{S}_+^{d-1}$. Let $m \in \mathbb{N}_+$, and let $g : (\mathbb{R}_+^d)^{m+1} \to \mathbb{R}$ be θ-Hölder continuous, for some $\theta \leq \min\{1, \alpha\}$, and also bounded. Set*

$$I_u = T_{\varepsilon_u}, \quad \text{where} \quad \varepsilon_u = o(u) \quad \text{and} \quad \varepsilon_u \nearrow \infty \quad \text{as} \quad u \to \infty.$$

Then for all $v \in \mathbb{R}_+^d$,

$$\begin{aligned} \lim_{u\to\infty} \mathbb{E}_v \left[g\left(\frac{V_{I_u}}{|V_{I_u}|}, \ldots, \frac{V_{I_u+m}}{|V_{I_u}|} \right) \middle| T_u^A < \tau \right] \\ = \int_{\mathbb{S}_+^{d-1}\times\mathbb{R}_+} \mathbb{E}_x^\alpha \Big[g\big(X_0, e^{S_1} X_1, \ldots, e^{S_m} X_m\big) \Big] \varrho(dx, ds). \end{aligned} \tag{2.27}$$

The class of θ-Hölder continuous functions is a separating class, and thus for all $m \in \mathbb{N}_+$, we then deduce the weak convergence

$$\begin{aligned} \mathbb{P}\left(\left(\frac{V_{I_u}}{|V_{I_u}|}, \ldots, \frac{V_{I_u+m}}{|V_{I_u}|} \right) \in \cdot \,\middle|\, T_u^A < \tau \right) \\ \Rightarrow \int_{\mathbb{S}_+^{d-1}\times\mathbb{R}_+} \mathbb{P}_x^\alpha \left(\big(X_0, e^{S_1} X_1, \ldots, e^{S_m} X_m\big) \in \cdot \right) \varrho(dx, ds), \end{aligned}$$

for any given V_0.

Finally, we conclude by studying the empirical law of $\{\log|V_n| - \log|V_{n-1}|\}$.

THEOREM 2.12. *Suppose that Hypotheses (H_1) and (H_2) are satisfied, and assume that A is a semi-cone and d_A is bounded and continuous on $\mathbb{S}_+^{d-1}$. Then for any $v \in \mathbb{R}_+^d$ and any bounded Lipschitz continuous function $g : \mathbb{R} \to \mathbb{R}$,*

$$\lim_{u\to\infty} \mathbb{E}_v \left[\left| \frac{1}{T_u^A} \sum_{n=1}^{T_u^A} g\left(\log\left(\frac{|V_n|}{|V_{n-1}|} \right) \right) - \widehat{\mathbb{E}}^\alpha \left[g(S_1) \right] \right| \middle| T_u^A < \tau \right] = 0. \tag{2.28}$$

Thus the empirical law of $\{(\log|V_n| - \log|V_{n-1}|)\}$ converges weakly, in $\mathbb{P}_v(\cdot|T_u^A < \tau)$-probability, to $\widehat{\mathbb{P}}^\alpha\big(S_1 \in \cdot\big)$.

By comparing with [10], Theorem 6.1, we see that (2.28) agrees precisely with the empirical law, without conditioning, of the Markov random walk $\{(X_n, S_n)\}$ under the α-shifted measure.

3. Background. 3.1. *Preliminary results from Markov chain theory.* We start by deriving an analog of the drift condition from Markov chain theory.

LEMMA 3.1. *Assume that* (H_1) *and* (H_2) *are satisfied. Then for any* $0 < \theta < \min\{1, \alpha\}$*, there exist positive constants* $t < 1$ *and* $L < \infty$ *such that for* $\mathbb{D}^\dagger := \{v \in \mathbb{R}_+^d : |v| \le L\}$*,*

$$\mathbb{E}\left[|V_n|^\theta r_\theta(\widetilde{V}_n)\,\middle|\,\mathscr{F}_{n-1}\right] \le t|V_{n-1}|^\theta r_\theta(\widetilde{V}_{n-1}), \quad \textit{for all } V_{n-1} \in \mathbb{R}_+^d \setminus \mathbb{D}^\dagger. \tag{3.1}$$

In particular, for $\tau^\dagger := \inf\{n \in \mathbb{N}_+ : V_n \in \mathbb{D}^\dagger\}$*, there exists* $B < \infty$ *such that*

$$\mathbb{E}\left[|V_n|^\theta \mathbf{1}_{\{\tau^\dagger > n\}}\middle| V_0 = v\right] \le Bt^n|v|^\theta, \quad \textit{for all } v \in \mathbb{R}_+^d \setminus \mathbb{D}^\dagger. \tag{3.2}$$

PROOF. Let $\theta \in (0,1)$. By applying Eq. (2.2) of Lemma 2.2, then using subadditivity and a further application of (2.2), we obtain that for some $c \in (0,\infty)$,

$$\begin{aligned}\mathbb{E}\left[|V_n|^\theta r_\theta(\widetilde{V}_n)\middle|\mathscr{F}_{n-1}\right] &= c\,\mathbb{E}\left[\int_{\mathbb{S}_+^{d-1}} \langle y, V_n\rangle^\theta l_\theta^*(dy)\middle|\mathscr{F}_{n-1}\right] \\ &\le c\,\mathbb{E}\left[\int_{\mathbb{S}_+^{d-1}} \left(\langle y, M_nV_{n-1}\rangle^\theta + \langle y, Q_n\rangle^\theta\right) l_\theta^*(dy)\middle|V_{n-1}\right] \\ &\le \mathbb{E}\left[|M_nV_{n-1}|^\theta r_\theta\big((M_nV_{n-1})^\sim\big)\middle|V_{n-1}\right] + \mathbb{E}\left[|Q_n|^\theta\right].\end{aligned} \tag{3.3}$$

The first term on the right-hand side equals

$$|V_{n-1}|^\theta P_\theta r_\theta(\widetilde{V}_{n-1}) = |V_{n-1}|^\theta \lambda(\theta) r_\theta(\widetilde{V}_{n-1}),$$

and so the required estimate follows under Hypothesis (H_2), choosing θ such that $0 < \theta < \min\{\alpha, 1\}$.

Using that r_θ is bounded from above and below by finite positive constants (by Lemma 2.2), (3.2) is then obtained by iterating (3.1). □

LEMMA 3.2. *Suppose* (H_1) *and* (H_2) *are satisfied, and let* $\mathbb{D} = B_r^+(0)$ *for some* $r > 0$ *such that* $\pi(\mathbb{D}) > 0$. *Let* $\mathbb{D}^\dagger = \{v \in \mathbb{R}_+^d : |v| \le L\}$, *where* L *is chosen such that* (3.1) *is satisfied and such that* $\mathbb{D}^\dagger \supset \mathbb{D}$. *Then there exist constants* $t \in (0,1)$ *and* $B < \infty$ *such that for* $\tau = \inf\{n \in \mathbb{N}_+ : V_n \in \mathbb{D}\}$,

$$\sup_{v \in \mathbb{D}^\dagger} \mathbb{P}\left(\tau > n \,|\, V_0 = v\right) \le Bt^n, \quad \text{for all } n \in \mathbb{N}_+. \tag{3.4}$$

PROOF. From (3.1), it follows that, starting from an initial state $V_0 \notin \mathbb{D}^\dagger$, $\{V_n\}$ returns to $\mathbb{D}^\dagger$ at a geometric rate; for a proof, see [37, Theorem 15.2.5]. Thus it suffices to show that

$$\sup_{v \in \mathbb{D}^\dagger} \mathbb{P}\left(\tau > n \,|\, V_0 = v\right) \le (1 - s) \tag{3.5}$$

for some $s > 0$ and $n \in \mathbb{N}_+$. To establish (3.5), we use Proposition 4.3.1 of [11], which precisely describes $\operatorname{supp} \pi$. Namely, there exists a set $\mathscr{S}$ with $\overline{\mathscr{S}} = \operatorname{supp} \pi$ such that the following holds: For each $v_0 \in \mathscr{S}$, there exists $l \in \mathbb{N}_+$ and $\mathfrak{m}_1, \ldots, \mathfrak{m}_l \in \operatorname{supp} \mu_M$, $q_1, \ldots, q_l \in \operatorname{supp} \mu_Q$ such that

$$h : v \mapsto \mathfrak{m}_l \cdots \mathfrak{m}_1 v + \sum_{i=1}^{l} \mathfrak{m}_l \cdots \mathfrak{m}_{i+1} q_i$$

is a contraction on $\mathbb{R}_+^d$ with v_0 as the unique fixed point. Hence, using that $\mathbb{D}^\dagger$ is compact, we obtain that for any $\delta > 0$, there exists $j \in \mathbb{N}_+$ such that $|h^j(v) - v_0| < \delta/2$ for all $v \in \mathbb{D}^\dagger$. Then, from continuity and the definition of the support, we conclude that

$$\inf_{v \in \mathbb{D}^\dagger} \mathbb{P}\left(|V_{lj} - v_0| < \delta \,|\, V_0 = v\right) > 0. \tag{3.6}$$

Since $\mathbb{D}$ is open in $\mathbb{R}_+^d$ and $\pi(\mathbb{D}) > 0$, and hence $\mathbb{D} \cap \operatorname{supp}\ \pi \neq \emptyset$, it follows that $\mathbb{D} \cap \mathscr{S} \neq \emptyset$ as well. Now let $v_0 \in \mathbb{D} \cap \mathscr{S}$ and choose $\delta > 0$ such that $B_\delta(v_0) \in \mathbb{D}$. Then (3.5) follows from (3.6) with $k = lj$. □

For an arbitrary π-positive set $\mathbb{D}$, define the return times $\kappa_0 = 0$ and

$$\kappa_i = \inf\left\{n > \kappa_{i-1} : V_n \in \mathbb{D}\right\}, \quad i = 1, 2, \ldots,$$

and let $\tau_i := \kappa_i - \kappa_{i-1}$. Set $N_{\mathbb{D}}(n) = \sum_{k=1}^n \mathbf{1}_{\mathbb{D}}(V_k)$. Using that $\{V_n\}$ is stationary and ergodic when $V_0 \sim \pi$, we infer the following strong law of large numbers for $\{\kappa_i\}$, which is standard.

LEMMA 3.3. *Suppose that* (H_1) *and* (H_2) *are satisfied and* $\pi(\mathbb{D}) > 0$. *Then for* π*-a.e.* $v \in \mathbb{R}^d_+$,

$$\lim_{i\to\infty} \frac{\kappa_i}{i} = \lim_{n\to\infty} \left(\frac{N_{\mathbb{D}}(n)}{n}\right)^{-1} = \frac{1}{\pi(\mathbb{D})} = \mathbb{E}_{\pi_{\mathbb{D}}}[\tau_1] \qquad \mathbb{P}_v\text{-a.s.}, \tag{3.7}$$

and $\pi_{\mathbb{D}}(\cdot) := \pi(\cdot)/\pi(\mathbb{D})$ *is invariant for the process* $\{V_{\kappa_i} : i = 0, 1, \ldots\}$.

Now let P denote the transition kernel of $\{V_n\}$. We conclude this section with two results which hold under the following additional Hypothesis (H_3) (which will ultimately be dropped in our main theorems by utilizing a smoothing argument).

HYPOTHESIS (H_3). Assume the following conditions.
(i) There exists a π-positive set F such that, for each $v \in F$, $P(v, \cdot)$ has an absolutely continuous component with respect to some σ-finite non-null measure Φ.
(ii) $(\text{supp } \pi)^\circ \neq \emptyset$.

Note that under (H_3), it follows from [3], Theorem 2.1 (b) and Theorem 2.2 (b) that $\{V_n\}$ is an aperiodic, positive Harris chain on $\mathbb{R}^d_+$. Once this is observed, the following result is also standard; cf. [37, Theorem 15.0.1].

LEMMA 3.4. *Assume that* (H_1), (H_2), *and* (H_3) *are satisfied. Then* $\{V_n\}$ *is an aperiodic, positive Harris chain on* $\mathbb{R}^d_+$. *Moreover,* $\{V_n\}$ *is* ψ*-irreducible, regular, and geometrically recurrent.*

LEMMA 3.5. *Suppose that* (H_1), (H_2), *and* (H_3) *are satisfied, and let* $\mathbb{D} \subset \mathbb{R}^d_+$ *be chosen such that* $\pi(\mathbb{D}) > 0$. *Let* $\tau := \inf\{n \in \mathbb{N}_+ : V_n \in \mathbb{D}\}$ *denote the first return time of* $\mathbb{D}$. *Then for any* π*-integrable function* h,

$$\int h(v)\pi(dv) = \mathbb{E}\big[h(V)\big] = \frac{1}{\mathbb{E}_{\pi_{\mathbb{D}}}[\tau]} \mathbb{E}_{\pi_{\mathbb{D}}}\left[\sum_{i=0}^{\tau-1} h(V_i)\right]. \tag{3.8}$$

PROOF. See [40], Proposition 5.9 and the discussion just prior to [40], Corollary 5.3. For a closely related result, see the proof of [15, Thm. 2.1]. □

3.2. *Quantifying the discrepancy between* $\{V_n\}$ *and* $\{e^{S_n}X_n\}$. Set

$$Z_n = \frac{V_n}{|M_n \cdots M_1 X_0|} \quad \text{and} \quad Z_n^{(0)} = \frac{\sum_{i=1}^n M_n \cdots M_{i+1} Q_i}{|M_n \cdots M_1 X_0|}, \tag{3.9}$$

for all $n \in \mathbb{N}$. (Thus $Z_n^{(0)} = (V_n - V_0)/|M_n \cdots M_1 X_0|$.) Also introduce the shorthand notation

$$\Pi_n := M_n \cdots M_1 \quad \text{and} \quad \Pi_i^n := M_n \cdots M_i.$$

LEMMA 3.6. *Assume* (H_1) *and* (H_2). *Then*:

(i) $\sup_{n\in\mathbb{N}} |Z_n| < \infty \quad \mathbb{P}^\alpha$-a.s. and $\sup_{n\in\mathbb{N}} \left|Z_n^{(0)}\right| < \infty \quad \mathbb{P}^\alpha$-a.s.

(ii) *Suppose* $v \in \mathbb{R}^d_+ \setminus \{0\}$. *Then in* $\mathbb{P}^\alpha_{\boldsymbol{\delta}_v}$*-measure, the sequence* $\{Z_n\}$ *converges in law to a random variable* Z, *and* $|Z_n| \Rightarrow |Z|$ *a.s., where*

$$|Z| = |v| + \sum_{i=1}^{\infty} \frac{\langle Y_i, \widetilde{Q}_i \rangle}{\langle Y_i, X_i \rangle} \frac{|Q_i|}{|\Pi_i \widetilde{v}|} \qquad \mathbb{P}^\alpha_{\boldsymbol{\delta}_v}\text{-a.s.} \tag{3.10}$$

Moreover, $|Z|$ *is strictly positive and finite* $\mathbb{P}^\alpha_{\boldsymbol{\delta}_v}$*-a.s. Similarly,*

$$\lim_{n\to\infty} \left|Z_n^{(0)}\right| = \sum_{i=1}^{\infty} \frac{\langle Y_i, \widetilde{Q}_i \rangle}{\langle Y_i, X_i \rangle} \frac{|Q_i|}{|\Pi_i \widetilde{v}|} \qquad \mathbb{P}^\alpha_{\boldsymbol{\delta}_v}\text{-a.s.} \tag{3.11}$$

(iii) *Let* $F \subset \mathbb{R}^d_+ \setminus \{0\}$ *be a bounded set and let* τ' *be any* $\{\mathscr{F}_n\}$*-stopping time such that* $\sup_{v\in F} \mathbb{P}\left(\tau' > k | V_0 = v\right) \le Bt^k$, $k \in \mathbb{N}$, *for some finite constant* B *and* $t \in (0,1)$. *Then for any* $v \in \mathbb{R}^d_+ \setminus \{0\}$,

(3.12)

$$\sup_{v\in F} \mathbb{E}^\alpha_{\boldsymbol{\delta}_v} \left[\sup_{n\in\mathbb{N}} |Z_n|^\alpha \mathbf{1}_{\{\tau'\ge n\}}\right] < \infty \quad \text{and} \quad \sup_{v\in F} \mathbb{E}^\alpha_{\boldsymbol{\delta}_v} \left[|Z|^\alpha \mathbf{1}_{\{\tau'=\infty\}}\right] < \infty.$$

(iv) *For* $v \in \mathbb{R}^d_+ \setminus \{0\}$, *we have the* L^1*-convergence*

$$\lim_{n\to\infty} \mathbb{E}^\alpha_{\boldsymbol{\delta}_v} \left[\left| |Z_n|^\alpha \mathbf{1}_{\{\tau'\ge n\}} - |Z|^\alpha \mathbf{1}_{\{\tau'=\infty\}} \right|\right] = 0. \tag{3.13}$$

Note that by Lemma 3.2, the condition in (iii) holds, in particular, for $\tau' = \tau := \inf\{n \in \mathbb{N}_+ : V_n \in \mathbb{D}\}$ with $F = \mathbb{D} \setminus \{0\}$.

PROOF. For any vector $x \in \mathbb{R}^d$, let $x^{(i)} = \langle e_i, x \rangle$ denote the i^{th} component of x, and set $\vec{1} = (1, \ldots, 1)^T$. Also, except in part (iii), fix $V_0 = v$ throughout the proof.

First recall that any $\mathbb{P}^\alpha_{x,v}$ is absolutely continuous with respect to $\widehat{\mathbb{P}}^\alpha$ ([10], Lemma 6.2), and hence the convergence of $\{Z_n\}$ in law, or the convergence of $\{|Z_n|\}$ $\widehat{\mathbb{P}}^\alpha$-a.s., implies the respective convergence under $\mathbb{P}^\alpha_{x,v}$. Thus, it is sufficient to prove the convergence results in part (i) and (ii) with respect to the measure $\widehat{\mathbb{P}}^\alpha$, under which the sequence $\{(M_n, Q_n) : n = 1, 2, \ldots\}$ is *stationary* (cf. Section 2 above), thus allowing us to apply Hennion [26].

(i) Suppose $\mathfrak{m} \in \mathfrak{M}$, and let $x_{\mathfrak{m}}$ be chosen such that $\|\mathfrak{m}\| = |\mathfrak{m}x_{\mathfrak{m}}|$. Since $\mathfrak{m}$ is nonnegative, an elementary argument shows that $x_{\mathfrak{m}}$ can, in fact, be chosen such that $x_{\mathfrak{m}}^{(i)} \geq 0$ for all i. Then for any $x \in \mathbb{S}_+^{d-1}$,

$$|\mathfrak{m}x| \geq \big(\min_j x^{(j)}\big)\left|\mathfrak{m}\vec{1}\right| \geq \big(\min_j x^{(j)}\big)\,|\mathfrak{m}x_{\mathfrak{m}}| = \big(\min_j x^{(j)}\big)\|\mathfrak{m}\|.$$

Thus

$$\frac{\|\mathfrak{m}\|}{|\mathfrak{m}x|} \leq \frac{1}{\min_j x^{(j)}}, \quad \text{for all } x \in \mathbb{S}_+^{d-1} \text{ and all } \mathfrak{m} \in \mathfrak{M}.$$

Recall the stopping time $\mathfrak{N} := \inf\big\{n \in \mathbb{Z}_+ : \Pi_n \succ 0\big\}$, which is finite $\widehat{\mathbb{P}}^\alpha$-a.s. by (H_1). [Since μ_M is equivalent to $\widehat{\mathbb{P}}^\alpha(M_1 \in \cdot)$, (H_1) holds equally well for $\widehat{\mathbb{P}}^\alpha(M_1 \in \cdot)$. Then Lemma 3.1 of [26] yields finiteness of $\mathfrak{N}$.] Identifying $Q_0 := V_0 = v$ yields

$$\begin{aligned}|Z_n| &\leq \sum_{i=0}^{n} \frac{\big|\Pi_{i+1}^n Q_i\big|}{|\Pi_n X_0|} \leq \sum_{i=0}^{n} \frac{\big\|\Pi_{i+1}^n\big\|\,|Q_i|}{\big|\Pi_{i+1}^n X_i\big|\,|\Pi_i X_0|} \\ &\leq \sum_{i=0}^{\mathfrak{N}\wedge n} \frac{\big\|\Pi_{i+1}^n\big\|\,|Q_i|}{\big|\Pi_{i+1}^n X_i\big|\,|\Pi_i X_0|} \;+\; \sum_{i=\mathfrak{N}\wedge n}^{n} \frac{1}{\min_j X_i^{(j)}} \frac{|Q_i|}{|\Pi_i X_0|}.\end{aligned} \tag{3.14}$$

By [10, Lemma 6.3], $\mathfrak{C}_i(x) := \inf_{n\in\mathbb{N}}\big(|\Pi_{i+1}^n x|/\big\|\Pi_{i+1}^n\big\|\big) > 0$ $\widehat{\mathbb{P}}^\alpha$-a.s., $\forall x \in \mathbb{S}_+^{d-1}$. Also $X_i^{(j)} = (\Pi_i X_0)^{(j)}/|\Pi_i X_0|$, implying $X_i^{(j)}|\Pi_i X_0| = (\Pi_i X_0)^{(j)} = \langle e_j, \Pi_i X_0\rangle$. This identifies the denominator in the second sum of (3.14), and shows that this denominator is positive for $i \geq \mathfrak{N}$. Hence

$$\begin{aligned}\sup_{n\in\mathbb{N}}|Z_n| &\leq \sum_{i=0}^{\mathfrak{N}} \frac{|Q_i|}{\mathfrak{C}_i(X_i)\,|\Pi_i X_0|} + \sum_{i=\mathfrak{N}}^{\infty} \frac{|Q_i|}{\min_j\langle e_j, \Pi_i X_0\rangle} \\ &\leq \sum_{i=0}^{\mathfrak{N}} \frac{|Q_i|}{\mathfrak{C}_i(X_i)\,|\Pi_i X_0|} + \sum_{i=\mathfrak{N}}^{\infty}\sum_{j=1}^{d} \frac{|Q_i|}{\langle e_j, \Pi_i X_0\rangle}.\end{aligned} \tag{3.15}$$

Since $\mathfrak{N} < \infty$ $\widehat{\mathbb{P}}^\alpha$-a.s., it suffices to focus on the second sum. By Lemma 2.3, we have that $\widehat{\mathbb{P}}^\alpha$-a.s.,

$$\lim_{n\to\infty} \sup\left\{\left|\frac{1}{n}\mathbf{1}_{\{\mathfrak{N}\leq n\}}\log\langle y, \Pi_n x\rangle - \Lambda'(\alpha)\right| : x, y \in \mathbb{S}_+^{d-1}\right\} = 0. \tag{3.16}$$

Furthermore, by a Borel-Cantelli argument, $\widehat{\mathbb{P}}^\alpha\left(\log|Q_i| > \delta i \text{ i.o.}\right) = 0$, for all $\delta > 0$. Thus, given $\varepsilon \in (0, \Lambda'(\alpha))$, there exists a finite integer k_0 such that, for all $i \geq k_0$ and all $j \in \{1, \ldots, d\}$,

$$\log|Q_i| - \log\langle e_j, \Pi_i X_0\rangle \;\leq\; -\Big(\Lambda'(\alpha) - \varepsilon\Big)i \quad \widehat{\mathbb{P}}^\alpha\text{-a.s.} \tag{3.17}$$

Since (3.17) holds *uniformly* in j, substituting (3.17) into (3.15) establishes part (i) of the lemma, where we also use that $\left|Z_n^{(0)}\right| \le |Z_n|$ for all $n \in \mathbb{N}$.

(ii) Following [26], let $\varrho(\Pi_i^n)$ denote the spectral radius of Π_i^n, and let R_n^i and L_n^i denote the right and left eigenvectors corresponding to the maximal eigenvalue in modulus; that is,

$$\Pi_i^n R_n^i = \varrho(\Pi_i^n) R_n^i \quad \text{and} \quad (\Pi_i^n)^T L_n^i = \varrho(\Pi_i^n) L_n^i, \qquad 1 \le i \le n.$$

Note that the Perron-Frobenius theorem assures that R_n^i and L_n^i have nonnegative entries. We further assume the following normalization: $\left|L_n^i\right| = 1$, $\langle L_n^i, R_n^i \rangle = 1$, $1 \le i \le n$. Let $\{Y_i\}$ be defined as in (2.8). Then we will show

$$\lim_{n\to\infty} \left| \langle e_j, Z_n \rangle - \langle e_j, \widetilde{R}_n^1 \rangle \sum_{i=0}^{n} \frac{\langle Y_i, \widetilde{Q}_i \rangle}{\langle Y_i, X_i \rangle} \frac{|Q_i|}{|\Pi_i X_0|} \right| = 0 \quad \widehat{\mathbb{P}}^\alpha\text{-a.s.}, \tag{3.18}$$

for all $1 \le j \le d$. The sequence $\{\widetilde{R}_n^1\}$ converges *in distribution* as $n \to \infty$ ([26], Theorem 1 (ii) (b)); hence we obtain the convergence, in distribution, of $\{Z_n\}$ to

$$Z := \lim_{n\to\infty} \widetilde{R}_n^1 \cdot \lim_{n\to\infty} \sum_{i=0}^{n} \frac{\langle Y_i, \widetilde{Q}_i \rangle}{\langle Y_i, X_i \rangle} \frac{|Q_i|}{|\Pi_i X_0|}.$$

Moreover, since $\left|\widetilde{R}_n^1\right| = 1$, (3.18) yields (3.10), i.e. $\lim_{n\to\infty} |Z_n| = |Z|$ $\widehat{\mathbb{P}}^\alpha$-a.s. In the same way, (3.11) is obtained by setting $Q_0 = 0$.

To establish (3.18), first recall (with the identification $Q_0 := V_0 = v$) that $Z_n = \sum_{i=0}^{n} \left(\Pi_{i+1}^n Q_i / |\Pi_n X_0|\right)$, and observe that

$$\left| \sum_{i=\lfloor n/2 \rfloor + 1}^{n} \frac{\langle e_j, \Pi_{i+1}^n Q_i \rangle}{|\Pi_n X_0|} \right| \le \sum_{i=\lfloor n/2 \rfloor + 1}^{\infty} \frac{\left\|\Pi_{i+1}^n\right\| |Q_i|}{\left|\Pi_{i+1}^n X_i\right| |\Pi_i X_0|},$$

and the right-hand side tends to zero as $n \to \infty$, by the proof of part (i) (in particular, (3.15)). Since Y_i is a unit vector with nonnegative entries, $\langle Y_i, X_i \rangle \ge d^{-1} \min_j X_i^{(j)}$. Hence we also have

$$\left| \langle e_j, \widetilde{R}_n^1 \rangle \sum_{i=\lfloor n/2 \rfloor + 1}^{n} \frac{\langle Y_i, \widetilde{Q}_i \rangle}{\langle Y_i, X_i \rangle} \frac{|Q_i|}{|\Pi_i X_0|} \right| \le \sum_{i=\lfloor n/2 \rfloor + 1}^{n} \frac{d}{\min_j X_i^{(j)}} \frac{|Q_i|}{|\Pi_i X_0|}.$$

Thus, to establish (3.18) (and part (ii)), it is enough to show that $\widehat{\mathbb{P}}^\alpha$-a.s.,

$$\lim_{n\to\infty} \left| \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{\langle e_j, \Pi_{i+1}^n Q_i \rangle}{|\Pi_n X_0|} - \langle e_j, \widetilde{R}_n^1 \rangle \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{\langle Y_i, \widetilde{Q}_i \rangle}{\langle Y_i, X_i \rangle} \frac{|Q_i|}{|\Pi_i X_0|} \right| = 0. \tag{3.19}$$

Then by the triangle inequality, it is sufficient to establish the following.

SUBLEMMA 3.7. *The following limits hold* $\widehat{\mathbb{P}}^\alpha$-a.s. :

$$\lim_{n\to\infty} \sum_{i=0}^{\lfloor n/2\rfloor} \left| \frac{\langle e_j, \Pi_{i+1}^n Q_i\rangle}{|\Pi_n X_0|} - \langle e_j, \widetilde{R}_n^{i+1}\rangle \frac{\langle L_n^{i+1}, Q_i\rangle}{\langle L_n^{i+1}, X_i\rangle} \frac{1}{|\Pi_i X_0|} \right| = 0; \tag{3.20}$$

$$\lim_{n\to\infty} \sum_{i=0}^{\lfloor n/2\rfloor} \left| \frac{\langle e_j, \widetilde{R}_n^{i+1}\rangle}{|\Pi_i X_0|} \left(\frac{\langle L_n^{i+1}, Q_i\rangle}{\langle L_n^{i+1}, X_i\rangle} - \frac{\langle Y_{i+1}, Q_i\rangle}{\langle Y_{i+1}, X_i\rangle} \right) \right| = 0; \tag{3.21}$$

$$\lim_{n\to\infty} \sum_{i=0}^{\lfloor n/2\rfloor} \left| \frac{1}{|\Pi_i X_0|} \frac{\langle Y_{i+1}, Q_i\rangle}{\langle Y_{i+1}, X_i\rangle} \left(\langle e_j, \widetilde{R}_n^{i+1}\rangle - \langle e_j, \widetilde{R}_n^{1}\rangle \right) \right| = 0. \tag{3.22}$$

PROOF OF THE SUBLEMMA. For (3.20), observe by [26, Corollary 1] that

$$\lim_{n\to\infty} \left(\frac{\Pi_{i+1}^n}{\left\|\Pi_{i+1}^n\right\|} - \frac{R_n^{i+1} \otimes L_n^{i+1}}{\left\|R_n^{i+1} \otimes L_n^{i+1}\right\|} \right) = 0 \quad \widehat{\mathbb{P}}^\alpha\text{-a.s.}, \tag{3.23}$$

where $a \otimes b$ is the rank-one matrix with $\langle e_i, (a\otimes b) e_j\rangle = \langle e_i, a\rangle\langle b, e_j\rangle$. From (3.23) we infer the asymptotic identities

$$\lim_{n\to\infty} \left(\frac{\langle e_j, \Pi_{i+1}^n Q_i\rangle}{\left\|\Pi_{i+1}^n\right\|} - \frac{\langle e_j, R_n^{i+1}\rangle\langle L_n^{i+1}, Q_i\rangle}{\left\|R_n^{i+1} \otimes L_n^{i+1}\right\|} \right) = 0; \tag{3.24}$$

$$\lim_{n\to\infty} \left(\frac{\left|\Pi_{i+1}^n X_i\right|}{\left\|\Pi_{i+1}^n\right\|} - \frac{\left|R_n^{i+1}\right| \langle L_n^{i+1}, X_i\rangle}{\left\|R_n^{i+1} \otimes L_n^{i+1}\right\|} \right) = 0. \tag{3.25}$$

Combining (3.24) and (3.25), we conclude that

$$\begin{aligned} \lim_{n\to\infty} \frac{\langle e_j, \Pi_{i+1}^n Q_i\rangle}{|\Pi_n X_0|} &= \lim_{n\to\infty} \frac{\langle e_j, \Pi_{i+1}^n Q_i\rangle}{|\Pi_{i+1}^n X_i||\Pi_i X_0|} \\ &= \lim_{n\to\infty} \langle e_j, \widetilde{R}_n^{i+1}\rangle \frac{\langle L_n^{i+1}, Q_i\rangle}{\langle L_n^{i+1}, X_i\rangle} \frac{1}{|\Pi_i X_0|} \quad \widehat{\mathbb{P}}^\alpha\text{-a.s.}, \end{aligned} \tag{3.26}$$

showing, in particular, that the individual terms in (3.20) $\to 0$ $\widehat{\mathbb{P}}^\alpha$-a.s.

To prove that the sum in (3.20) converges to zero, we now invoke a dominated convergence argument. Since $\mathfrak{N}$ is finite a.s., it suffices to focus on summands with $i \geq \mathfrak{N}$, where we can assume that all components of X_i are positive, as the remaining terms form a finite sum. Observe that

$$\frac{\langle L_n^{i+1}, \vec{1}\rangle \max_j Q_i^{(j)}}{\langle L_n^{i+1}, \vec{1}\rangle \min_j X_i^{(j)}} \leq \frac{|Q_i|}{\min_j X_i^{(j)}}, \tag{3.27}$$

and therefore

$$\sup_n \sum_{i=\mathfrak{N}}^{\lfloor n/2\rfloor} \left| \frac{\langle e_j, \Pi_{i+1}^n Q_i\rangle}{|\Pi_n X_0|} - \langle e_j, \widetilde{R}_n^{i+1}\rangle \frac{\langle L_n^{i+1}, Q_i\rangle}{\langle L_n^{i+1}, X_i\rangle} \frac{1}{|\Pi_i X_0|} \right| \leq 2 \sup_n \sum_{i=\mathfrak{N}}^{\lfloor n/2\rfloor} \frac{1}{\min_i X_i^{(j)}} \frac{|Q_i|}{|\Pi_i X_0|} < \infty \quad \widehat{\mathbb{P}}^\alpha\text{-a.s.}, \tag{3.28}$$

by part (i) (where we have used the calculation in (3.14) to handle the first term on the left-hand side). Thus, using a dominated convergence argument (applied pointwise on the space where (3.25) and (3.28) hold), we deduce that (3.20) follows from (3.25).

Next we turn to (3.21). It follows by Lemma 3.3 of [26] that, under $\widehat{\mathbb{P}}^\alpha$, the sequence $\{L_n^{i+1}\}$ converges a.s. as $n \to \infty$ to Y_{i+1}. Hence, by a dominated convergence argument, we conclude that (3.21) holds.

Finally, to establish (3.22), note by Proposition 3.1 of [26] that

$$\left|\widetilde{R}_n^{i+1} - \widetilde{R}_n^1\right| = \left|(\Pi_{i+1}^n R_n^{i+1})^\sim - (\Pi_n R_n^1)^\sim\right| \leq 2\mathbf{c}(\Pi_{i+1}^n), \tag{3.29}$$

where $\mathbf{c}(\cdot)$ is bounded above by one and tends to zero $\widehat{\mathbb{P}}^\alpha$-a.s. as $(n-i) \to \infty$ ([26, Lemma 3.2]). Then (3.22) follows, once again, by the dominated convergence theorem. This completes the proof of the sublemma and, consequently, part (ii) of Lemma 3.6. □

Proof of Lemma 3.6 (continued). We now return to the proof of main lemma, where it remains to verify that (iii) and (iv) hold.

(iii) Let $m \in \mathbb{N}$ and $B_1 = \max_{x,y}\big(r_\alpha(x)/r_\alpha(y)\big) \in (0,\infty)$. Then for $\alpha > 0$,

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{\delta}_v}^\alpha\Big[\Big(\sup_{n\leq m}|Z_n|\mathbf{1}_{\{\tau'\geq n\}}\Big)^\alpha\Big] \leq \mathbb{E}_{\boldsymbol{\delta}_v}^\alpha\Big[\Big(\sup_{n\leq m}\Big(|v| + \sum_{k=1}^n \frac{|\Pi_{k+1}^n Q_k|}{|\Pi_n X_0|}\mathbf{1}_{\{\tau'\geq k-1\}}\Big)\Big)^\alpha\Big] \\
&= (r_\alpha(\widetilde{v}))^{-1}\,\mathbb{E}_{\boldsymbol{\delta}_v}\Big[r_\alpha(X_m)|\Pi_m X_0|^\alpha\Big(\sup_{n\leq m}\Big(|v| + \sum_{k=1}^n \frac{|\Pi_{k+1}^n Q_k|}{|\Pi_n X_0|}\mathbf{1}_{\{\tau'\geq k-1\}}\Big)\Big)^\alpha\Big] \\
&\leq B_1\mathbb{E}_{\boldsymbol{\delta}_v}\Big[\Big(\sup_{n\leq m}\Big(|\Pi_m X_0||v| + \sum_{k=1}^n |\Pi_{n+1}^m X_n|\cdot|\Pi_{k+1}^n Q_k|\mathbf{1}_{\{\tau'\geq k-1\}}\Big)\Big)^\alpha\Big] \\
&= B_1\mathbb{E}_v\Big[\Big(\sum_{k=0}^m \big\|\Pi_{n+1}^m\big\|\cdot\big\|\Pi_{k+1}^n\big\|\cdot|Q_k|\mathbf{1}_{\{\tau'\geq k-1\}}\Big)^\alpha\Big], \quad \text{where } Q_0 := v.
\end{aligned}$$

Now suppose that $\alpha \geq 1$. Then by Minkowski's inequality,

$$\Big(\mathbb{E}_v\Big[\Big(\sum_{k=0}^m \big\|\Pi_{n+1}^m\big\|\cdot\big\|\Pi_{k+1}^n\big\|\cdot|Q_k|\mathbf{1}_{\{\tau'\geq k-1\}}\Big)^\alpha\Big]\Big)^{1/\alpha}$$

$$\leq \sum_{k=0}^{m} \Big(\mathbb{E}\Big[\,\big\|\Pi_{n+1}^{m}\big\|^{\alpha}\,\Big]\Big)^{1/\alpha}\Big(\mathbb{E}\Big[\,\big\|\Pi_{k+1}^{n}\big\|^{\alpha}\,\Big]\Big)^{1/\alpha}\Big(|v|+\mathbb{E}\big[|Q_1|^{\alpha}\big]\Big)^{1/\alpha} p_k^{\,1/\alpha},$$

where $p_k := \mathbb{P}_v\,(\tau' \geq k-1)$. Now by [10, Corollary 4.6], $\mathbb{E}\,[\|\Pi_n\|^{\alpha}] \leq B_2 \in (0,\infty)$, for all n. Moreover, $\mathbb{E}[|Q_i|^{\alpha}] < \infty$ by (H_2); and by the assumption of part (iii), $p_k \leq B_3 t^k$ for some $t \in (0,1)$ (uniformly in v). Combining these estimates yields

$$\mathbb{E}_{\boldsymbol{\delta}_v}^{\alpha}\left[\sup_{n\in\mathbb{N}} |Z_n|^{\alpha}\mathbf{1}_{\{\tau'\geq n\}}\right] \;\leq\; B_1 B_2^2\Big(|v|+\mathbb{E}[|Q_1|^{\alpha}]\Big)\left(\sum_{k=0}^{\infty}(B_3 t^k)^{1/\alpha}\right)^{\alpha} < \infty,$$

and this bound is uniform over $v \in F$, for any bounded set $F \subset \mathbb{R}_+^d \setminus \{0\}$.

If $\alpha \leq 1$, then we use the subadditivity, namely the inequality $|x+y|^{\alpha} \leq |x|^{\alpha} + |y|^{\alpha}$ in place of Minkowski's inequality, and then proceed as before.

Now it follows from part (ii) that $|Z_n|^{\alpha}\,\mathbf{1}_{\{\tau'\geq n\}} \to |Z|^{\alpha}\,\mathbf{1}_{\{\tau'=\infty\}}$ $\mathbb{P}^{\alpha}$-a.s. as $n \to \infty$. Consequently,

$$\sup_{v\in F}\mathbb{E}_{\boldsymbol{\delta}_v}^{\alpha}\Big[\,|Z|^{\alpha}\,\mathbf{1}_{\{\tau'=\infty\}}\Big] \;\leq\; \sup_{v\in F}\mathbb{E}_{\boldsymbol{\delta}_v}^{\alpha}\left[\sup_{n\in\mathbb{N}}|Z_n|^{\alpha}\,\mathbf{1}_{\{\tau'\geq n\}}\right] < \infty.$$

(iv) The almost sure convergence $|Z_n|^{\alpha}\,\mathbf{1}_{\{\tau'\geq n\}} \to |Z|^{\alpha}\,\mathbf{1}_{\{\tau'=\infty\}}$ was obtained in part (ii), and it was shown in part (iii) that $\big\{\,|Z_n|^{\alpha}\,\mathbf{1}_{\{\tau'\geq n\}}\big\}_{n\in\mathbb{N}}$ is uniformly integrable, and the L^1-convergence follows. □

3.3. *Markov nonlinear renewal theory.* Set $\mathfrak{T}_u = \inf\{n \in \mathbb{N}_+ : S_n > \log u\}$. Assuming (H_1) and (H_2), then Kesten [28, Theorem 2] proved that there is a probability measure ϱ on $\mathbb{S}_+^{d-1} \times (0,\infty)$, namely the *asymptotic overjump distribution*, for which we have the weak convergence

$$(X_{\mathfrak{T}_u}, S_{\mathfrak{T}_u} - \log u) \;\Rightarrow\; \varrho(\cdot) \quad \text{as} \quad u \to \infty. \tag{3.30}$$

If the function d_A is bounded and continuous, and we define $S_n^A := S_n - \log d_A(X_n)$ and $\mathfrak{T}_u^A = \inf\{n \in \mathbb{N}_+ : S_n^A > \log u\}$, then only minor modifications are needed to deduce that for a certain probability measure ϱ^A,

$$(X_{\mathfrak{T}_u}, S_{\mathfrak{T}_u}^A - \log u) \;\Rightarrow\; \varrho^A(\cdot) \quad \text{as} \quad u \to \infty. \tag{3.31}$$

In this section, we apply the Markov nonlinear renewal theory developed by Melfi [34, 35] to obtain the asymptotic overjump distributions for the processes $\{V_n\}$ and $\{V_n^A\}$, where

$$V_n^A := \frac{V_n}{d_A(\widetilde{V}_n)}, \qquad n \in \mathbb{N}. \tag{3.32}$$

First recall the definitions of T_u, T_u^A in (2.16), (2.17), and note that it follows from the definitions that $T_u^A = \inf\{n \in \mathbb{N}_+ : |V_n^A| > u\}$.

THEOREM 3.8. *Assume* (H_1) *and* (H_2). *Let* $d_A \in \mathscr{C}_b\big(\mathbb{S}_+^{d-1}\big)$. *Then for all* $f \in \mathscr{C}_b\big(\mathbb{S}_+^{d-1} \times (0,\infty)\big)$ *and all* $x \in \mathbb{S}_+^{d-1}$ *and* $v \in \mathbb{R}_+^d \setminus \{0\}$,

$$\lim_{u\to\infty} \mathbb{E}_{x,v}^{\alpha}\left[f\left(\widetilde{V}_{T_u}, \log\frac{|V_{T_u}|}{u}\right)\right] = \int_{\mathbb{S}_+^{d-1}\times\mathbb{R}_+} f(y,s)\,\varrho(dy,ds), \tag{3.33}$$

where ϱ *is given as in* (3.30). *Moreover, the same result also holds if* (ϱ, V_{T_u}) *is replaced with* $(\varrho^A, V^A_{T_u^A})$.

PROOF. We need to verify conditions (I′), (II), and (III) of [35, Theorem 3] for the process $\{(W_{\log u}, Z_{\log u}, R_{\log u}) = (\widetilde{V}_{T_u}, \log|V_{T_u}|, \log|V_{T_u}| - \log u)\}$.

Condition (III), namely tightness of $\{\widetilde{V}_{T_u}\}$, is satisfied since $\mathbb{S}_+^{d-1}$ is compact. The validity of Conditions (I) and (II) is proved below in Lemmas 3.9 and 3.11, respectively. Then (3.33) follows from [35, Theorem 3].

Turning to the case where $(\varrho, \widetilde{V}_{T_u})$ is replaced with $(\varrho^A, \widetilde{V}^A_{T_u^A})$, we need to check the validity of Conditions (I′) and (II) for $\{(\widetilde{V}_n^A, \log|V_n^A|)\}$. By (3.32), $\widetilde{V}_n^A = \widetilde{V}_n$. Thus, for $f(x,s) = (x, s - \log d_A(x))$, we have that $\{(\widetilde{V}_n^A, \log|V_n^A|)\} = \{f(\widetilde{V}_n, \log|V_n|)\}$ and $\{(X_n, S_n^A)\} = \{f(X_n, S_n)\}$ (using (2.18)). Hence (I′) can be deduced from Lemma 3.9 below. Finally, since d_A is bounded, the tightness of $\{\log|V^A_{T_u^A}| - \log u\} = \{\log|V_{T_u^A}| - \log d_A(V_{T_u^A}) - \log u\}$ follows from Lemma 3.11. □

Write $\mathfrak{d}_m$ for the Prokhorov distance on the space of probability measures on $(\mathbb{S}_+^{d-1} \times [0,\infty))^m$.

LEMMA 3.9. *Assume* (H_1) *and* (H_2). *Then for all* $m \in \mathbb{N}_+$,

$$\mathfrak{d}_m\left(\mathbb{P}^{\alpha}\left(\left(\widetilde{V}_{T_u+k}, \log\frac{|V_{T_u+k}|}{|V_{T_u}|}\right)_{k=1}^m \in \cdot \,\middle|\, \mathscr{F}_{T_u}\right),\ \mathbb{P}^{\alpha}_{Y_{T_u}}\left((X_k,S_k)_{k=1}^m \in \cdot\right)\right)$$

converges to zero as $u \to \infty$ *in* $\mathbb{P}^{\alpha}$*-probability.*

PROOF. Using the Markov property,

$$\begin{aligned}&\mathbb{P}^{\alpha}\left((\widetilde{V}_{T_u+k}, \log|V_{T_u+k}| - \log|V_{T_u}|)_{1\le k\le m} \in \cdot \,\middle|\, \mathscr{F}_{T_u}\right)\\ &\quad = \mathbb{P}^{\alpha}_{X_{T_u}, V_{T_u}}\left((\widetilde{V}_k, \log|V_k| - \log|V_0|)_{1\le k\le m} \in \cdot\right).\end{aligned}$$

By [25, Lemma 3.5], the total variation distance between $\mathbb{P}^{\alpha}_{x,v}$ and $\mathbb{P}^{\alpha}_{y,v}$ is bounded above by $B\,|x-y|^{\bar\alpha}$ for some $B < \infty$, where $\bar\alpha = \min\{\alpha, 1\}$. [The

proof in [25] is for invertible matrices, but carries over to nonnegative matrices.] Then, as total variation distance is an upper bound for the Prokhorov distance,

$$\begin{aligned}
&\mathfrak{d}_m\Big(\mathbb{P}^\alpha_{X_{T_u},V_{T_u}}\Big((\widetilde{V}_k,\log|V_k|-\log|V_0|)_{k=1}^m\in\cdot\Big)\,,\,\mathbb{P}^\alpha_{\widetilde{V}_{T_u}}\Big((X_k,S_k)_{k=1}^m\in\cdot\Big)\Big)\\
&\le\ B\Big|X_{T_u}-\widetilde{V}_{T_u}\Big|^{\bar\alpha}\\
&\quad+\mathfrak{d}_m\Big(\mathbb{P}^\alpha_{\widetilde{V}_{T_u},V_{T_u}}\Big((\widetilde{V}_k,\log|V_k|-\log|V_0|)_{k=1}^m\in\cdot\Big),\mathbb{P}^\alpha_{\widetilde{V}_{T_u}}\left((X_k,S_k)_{k=1}^m\in\cdot\right)\Big).
\end{aligned}$$

It will be proved in Lemma 3.10 below that $\big|X_{T_u}-\widetilde{V}_{T_u}\big|$ tends to zero in $\mathbb{P}^\alpha$-probability. We thus consider only the last term. Fix the initial values $(\widetilde{V}_{T_u},V_{T_u})=(\widetilde{v},v)$, and introduce the notation $V_1^{(0)}=Q_1$ and $V_k^{(0)}:=\sum_{j=1}^k M_k\cdots M_{j+1}Q_j$. Then standard estimates yield, for all $v\in\mathbb{R}^d_+\setminus\{0\}$,

$$\begin{aligned}
&\mathbb{P}^\alpha_{\tilde v}\Big(\Big|(\widetilde{V}_k,\log|V_k|-\log|V_0|)_{1\le k\le m}-(X_k,S_k)_{1\le k\le m}\Big|_\infty\ge\varepsilon\,\Big|\,V_0=v\Big)\\
&\quad\le\ \mathbb{P}^\alpha_{\tilde v}\left(2\sum_{k=1}^m\frac{|V_k^{(0)}|}{|\Pi_k v|}\ge\varepsilon\right)\ \le\ \frac{B}{\varepsilon^\alpha|v|^\alpha}\sum_{k=1}^m\mathbb{E}\Big[\big|V_k^{(0)}\big|^\alpha\Big],
\end{aligned}$$

for some universal constant B, where we used Chebyshev's inequality and boundedness of r_α in the last inequality. Hence

$$\lim_{u\to\infty}\sup_{v\,:\,|v|\ge u}\mathbb{P}^\alpha_{\tilde v}\Big(\Big|\Big(\widetilde{V}_k,\log\frac{|V_k|}{|V_0|}\Big)_{k=1}^m-\big(X_k,S_k\big)_{k=1}^m\Big|_\infty\ge\varepsilon\,\Big|\,V_0=v\Big)=0.$$

Recall that convergence in probability implies convergence in the Prokhorov metric. Since $\mathbb{P}^\alpha(T_u<\infty)=1$ and $|V_{T_u}|\ge u$, we infer the $\mathbb{P}^\alpha$-a.s. convergence; namely, as $u\to\infty$,

$$\mathfrak{d}_m\Big(\mathbb{P}^\alpha_{\widetilde{V}_{T_u},V_{T_u}}\Big(\Big(\widetilde{V}_k,\log\frac{|V_k|}{|V_0|}\Big)_{k=1}^m\in\cdot\Big),\mathbb{P}^\alpha_{\widetilde{V}_{T_u}}\left((X_k,S_k)_{k=1}^m\in\cdot\right)\Big)\to0.\quad\square$$

LEMMA 3.10. *For all $x\in\mathbb{S}^{d-1}_+$ and $V_0=v\in\mathbb{R}^d_+\setminus\{0\}$,*

$$\lim_{u\to\infty}\Big|X_{T_u}-\widetilde{V}_{T_u}\Big|=0\quad\text{in }\mathbb{P}^\alpha_{x,v}\text{-probability.}\tag{3.34}$$

PROOF. Let $w=u/2$, and decompose the process based on its behavior prior and subsequent to the time T_w. Recalling that $\Pi_i^n:=M_n\cdots M_i$ and using the triangle inequality, we see that $\mathbb{P}^\alpha_x\left(\left|\widetilde{V}_{T_u}-X_{T_u}\right|>\varepsilon\right)$ can be written as

$$\mathbb{P}^\alpha_x\left(\left|\frac{\Pi^{T_u}_{T_w+1}V_{T_w}+\sum_{i=T_w+1}^{T_u}\Pi^{T_u}_{i+1}Q_i}{|V_{T_u}|}-\Big(\Pi^{T_u}_{T_w+1}X_{T_w}\Big)^\sim\right|>\varepsilon\right)$$

$$\leq \mathbb{E}_x^\alpha \left[\mathbb{P}_x^\alpha \left(\left| \frac{\Pi_{T_w+1}^{T_u} V_{T_w} + \sum_{i=T_w+1}^{T_u} \Pi_{i+1}^{T_u} Q_i}{|V_{T_u}|} - \frac{\Pi_{T_w+1}^{T_u} V_{T_w}}{\left|\Pi_{T_w+1}^{T_u} V_{T_w}\right|} \right| > \frac{\varepsilon}{2} \,\middle|\, \mathscr{F}_{T_w} \right) \right]$$
$$+ \mathbb{P}_x^\alpha \left(\left| \left(\Pi_{T_w+1}^{T_u} V_{T_w} \right)^\sim - \left(\Pi_{T_w+1}^{T_u} X_{T_w} \right)^\sim \right| > \frac{\varepsilon}{2} \right) := \mathbb{I}_1(u) + \mathbb{I}_2(u).$$

To compute $\mathbb{I}_2(u)$ as $u \to \infty$, we apply Proposition 3.1 of [26], which yields

$$\sup_{x,y \in \mathbb{S}_+^{d-1}} \left| (\Pi_{i+1}^n x)^\sim - (\Pi_{i+1}^n y)^\sim \right| \leq 2\mathbf{c}(\Pi_{i+1}^n)$$

for a function $\mathbf{c}(\cdot)$ which is bounded above by one and tends to zero $\widehat{\mathbb{P}}^\alpha$-a.s. as $(n-i) \to \infty$ ([26, Lemma 3.2]). Since $\mathbb{P}_x^\alpha$ is absolutely continuous with respect to the measure $\widehat{\mathbb{P}}^\alpha$ ([10, Lemma 6.2]), it follows that $\mathbf{c}\big(\Pi_{T_{u/2}+1}^{T_u}\big) \to 0$ $\mathbb{P}_x^\alpha$-a.s. for all $x \in \mathbb{S}_+^{d-1}$, and hence

$$\mathbb{I}_2(u) \leq \mathbb{P}_x^\alpha \left(2\mathbf{c} \left(\Pi_{T_w+1}^{T_u} \right) > \frac{\varepsilon}{2} \right) \searrow 0 \quad \text{as} \quad u \to \infty. \tag{3.35}$$

Now consider $\mathbb{I}_1(u)$ as $u \to \infty$. Standard estimates yield

$$\mathbb{I}_1(u) \leq \mathbb{E}_x^\alpha \left[\mathbb{P}_{\widetilde{V}_{T_w}}^\alpha \left(\frac{4|V_{T_u}^{(0)}|}{u\left|\Pi_{T_u} \widetilde{V}_0\right|} > \frac{\varepsilon}{2} \right) \right]. \tag{3.36}$$

Next recall by Lemma 3.6 (i) that $\mathscr{Z}^0 := \sup_{n \in \mathbb{N}} \Big(|V_n^{(0)}| / |\Pi_n X_0| \Big) < \infty$ $\widehat{\mathbb{P}}^\alpha$-a.s., and this quantity does *not* depend on the initial value, V_0. Using that $\mathbb{P}_{\widetilde{V}_{T_w}}^\alpha(\cdot) \leq B\, \widehat{\mathbb{P}}^\alpha(\cdot)$ for some universal constant B ([10, Lemma 6.2]), we then obtain that

$$\mathbb{I}_1(u) \leq B \lim_{u \to \infty} \widehat{\mathbb{P}}^\alpha \left(\frac{\mathscr{Z}^0}{u} > \frac{\varepsilon}{8} \right) = 0. \tag{3.37}$$

□

The following lemma concludes the proof of Theorem 3.8.

LEMMA 3.11. *$\{\widetilde{V}_{T_u} - \log u\}_{u \geq 1}$ is tight under $\mathbb{P}^\alpha$.*

PROOF. A sufficient condition is given in [35], Section 5.2: Letting $\xi_n := \log|V_n| - S_n$ and supposing that $\{\xi_{T_u}\}_{u \geq 1}$ and $\{\xi_{\mathfrak{T}_u}\}_{u \geq 1}$ are tight under $\mathbb{P}^\alpha$, then it follows that $\{W_{T_u} - \log u\}_{u \geq 1}$ is tight.

Now by Lemma 3.6,

$$\xi_n = \log \frac{|V_n|}{|\Pi_n V_0|} \to \log Z \quad \mathbb{P}^\alpha\text{-a.s.},$$

for a finite random variable Z. Since T_u and $\mathfrak{T}_u$ are stopping times with respect to the filtration $\{\mathscr{F}_n\}$ and tend to infinity as $u \to \infty$, we deduce that

$$\lim_{u\to\infty} \xi_{T_u} = \log Z \quad \mathbb{P}^\alpha\text{-a.s.} \qquad \text{and} \qquad \lim_{u\to\infty} \xi_{\mathfrak{T}_u} = \log Z \quad \mathbb{P}^\alpha\text{-a.s.}$$

Thus, in particular, the families $\{\xi_{T_u}\}_{u\geq 1}$ and $\{\xi_{\mathfrak{T}_u}\}_{u\geq 1}$ converge in distribution under $\mathbb{P}^\alpha$ and are consequently tight. □

The last result concerns the first passage times in the α-shifted measure.

LEMMA 3.12. *Assume that d_A is bounded and continuous. Then*

$$\lim_{u\to\infty} \frac{T_u^A}{\log u} = \frac{1}{\lambda'(\alpha)} \quad \text{in } \mathbb{P}^\alpha\text{-probability.} \tag{3.38}$$

PROOF. By definition, $V_n = Z_n e^{S_n}$ and $V_n^A = V_n/d_A(\widetilde{V}_n)$, and consequently

$$\log|V_n^A| = S_n + |Z_n| - \log d_A(\widetilde{V}_n) := S_n + \xi_n. \tag{3.39}$$

Recall that $\sup_{n\in\mathbb{N}} Z_n$ is finite a.s., by Lemma 3.6. Since d_A is bounded, it follows that the sequence $\{\xi_i\}$ in (3.39) is *slowly changing* (as defined in [46], Eq. (9.5)). Now by Lemma 2.3, $S_n/n \to \lambda'(\alpha)$ a.s., and hence $\log|V_n^A|/n \to \lambda'(\alpha)$ a.s. The result then follows by reasoning as in [46, Lemma 9.13]. □

4. Characterizing the large exceedances over cycles.

4.1. *Proposition 4.1 and its consequences.* Recall that τ denotes the return time to a set $\mathbb{D} = B_r^+(0)$, where $\pi(\mathbb{D}) > 0$, and $T_u^A := \inf\{n : V_n \in uA\} = \inf\{n : |V_n^A| > u\}$, where $V_n^A := V_n/d_A(\widetilde{V}_n)$ (cf. (3.32)). Also recall that $r_\alpha^A(x) := r_\alpha(x)\,(d_A(x))^\alpha$, $x \in \mathbb{S}_+^{d-1}$.

We say that a function $g : (\mathbb{R}_+^d)^{m+1} \to \mathbb{R}$ is *almost θ-Hölder continuous* if

$$g(v_0,\dots,v_m) = \hat{g}(v_0,\dots,v_m)\mathbf{1}_{\{|v_m|\geq\delta\}} \tag{4.1}$$

for some $\delta \geq 0$ and θ-Hölder continuous function $\hat{g}$.

PROPOSITION 4.1. *Assume (H_1) and (H_2) are satisfied. Let $m \in \mathbb{N}$ and $g : (\mathbb{R}_+^d)^{m+1} \to \mathbb{R}$ be a bounded almost θ-Hölder continuous function for $\theta \leq \min\{1,\alpha\}$, and assume that the function d_A is bounded and continuous on $\mathbb{S}_+^{d-1}$. Then for any $v \in \mathbb{R}_+^d \setminus \{0\}$,*

$$\lim_{u\to\infty} u^\alpha \mathbb{E}\left[g\left(\frac{V_{T_u^A}}{u},\dots,\frac{V_{T_u^A+m}}{u}\right)\mathbf{1}_{\{T_u^A<\tau\}} \,\middle|\, V_0 = v\right]$$

$$= r_\alpha(\widetilde{v})\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\left[|Z|^\alpha \mathbf{1}_{\{\tau=\infty\}}\right]$$
$$\times \int \frac{e^{-\alpha s}}{r^A_\alpha(x)}\mathbb{E}\Big[g\big(e^{S_0}X_0,\dots,e^{S_m}X_m\big)\,\Big|\,X_0=x,\,S_0=s+\log d_A(x)\Big]\,\varrho^A(dx,ds).$$

Recall that ϱ^A is the asymptotic overjump distribution related to $\{V^A_{T^A_u}\}$, while on the left-hand side of the above equation, we evaluate g for the process $\{V_n\}$ (not $\{V^A_n\}$) at a sequence of times commencing at the time T^A_u. This explains the additional term "$\log d_A(x)$" in the expression for S_0; namely, it arises when transforming $V^A_{T^A_u}$ to $V_{T^A_u}$.

As a corollary, we specialize to the case where $g=1$ (in (4.2)), and then to the case where we also have $d_A=1$ (in (4.3)). We use the shorthand notation $C(v)=r_\alpha(\widetilde{v})\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\left[|Z|^\alpha\mathbf{1}_{\{\tau=\infty\}}\right]$ (which is equivalent to the definition given in Section 2) and employ the change of measure in the second identity (namely (4.3)).

COROLLARY 4.2. *Under the assumptions of Proposition 4.1, we have that for any $v\in\mathbb{R}^d_+\setminus\{0\}$,*

$$\lim_{u\to\infty} u^\alpha\mathbb{P}\left(T^A_u<\tau\,\middle|\,V_0=v\right)=C(v)\int\frac{e^{-\alpha s}}{r^A_\alpha(x)}\,\varrho^A(dx,ds), \tag{4.2}$$

and

$$\lim_{u\to\infty} u^\alpha\mathbb{E}\left[g\left(\frac{V_{T_u}}{u},\dots,\frac{V_{T_u+m}}{u}\right)\mathbf{1}_{\{T_u<\tau\}}\,\middle|\,V_0=v\right] \tag{4.3}$$
$$= C(v)\int\mathbb{E}^\alpha_x\left[\frac{e^{-\alpha(S_m+s)}}{r_\alpha(X_m)}g\big(e^sX_0,\dots,e^{S_m+s}X_m\big)\right]\varrho(dx,ds).$$

4.2. *Proof of Proposition 4.1.* We will rely on the following.

LEMMA 4.3. *Assume the conditions of Proposition 4.1. Then:*
(i) *For all $v\in\mathbb{R}^d_+\setminus\{0\}$, we have the L^1-convergence*

$$\lim_{n\to\infty}\lim_{u\to\infty}\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\left[\left|\left|Z_{T^A_u}\right|^\alpha\mathbf{1}_{\{T^A_u<\tau\}}-|Z_n|^\alpha\,\mathbf{1}_{\{n\le T^A_u\}}\mathbf{1}_{\{n\le\tau\}}\right|\right]=0. \tag{4.4}$$

(ii) *For $u>0$, define*

$$\mathfrak{G}_u=\frac{1}{r_\alpha(X_{T^A_u})}\left(\frac{\left|V_{T^A_u}\right|}{u}\right)^{-\alpha}\mathbb{E}\left[g\left(\frac{V_{T^A_u}}{u},\dots,\frac{V_{T^A_u+m}}{u}\right)\,\middle|\,\mathscr{F}_{T^A_u}\right].$$

Then, independent of n, we have $\mathbb{P}^\alpha$-a.s. that

$$\lim_{u\to\infty} \mathbb{E}^\alpha\left[\mathfrak{G}_u|\mathscr{F}_n\right]\mathbf{1}_{\{n\le T_u^A\}} \tag{4.5}$$
$$= \int \frac{e^{-\alpha s}}{r_\alpha^A(x)}\mathbb{E}\Big[g\big((e^{S_n}X_n)_{n=0}^m\big)\,\Big|\,X_0=x,\,S_0=s+\log d_A(x)\Big]\,\varrho^A(dx,ds).$$

Proof of Lemma 4.3. (i) By Lemma 3.2, τ satisfies the assumptions in Lemma 3.6 (iii). Thus, this result is a direct consequence of Lemma 3.6 (iv), where the L^1-convergence $|Z_n|^\alpha\,\mathbf{1}_{\{n\le\tau\}} \to |Z|^\alpha\,\mathbf{1}_{\{\tau=\infty\}}$ is proved. It follows that $|Z_n|^\alpha\,\mathbf{1}_{\{n\le\tau\}}$ constitutes a Cauchy sequence in L^1, yielding the assertion.

(ii) Let $n\in\mathbb{N}_+$. Then by the Markov property,

$$\mathbb{E}^\alpha\left[\mathfrak{G}_u|\mathscr{F}_n\right]\mathbf{1}_{\{n\le T_u^A\}} = \mathbb{E}^\alpha_{X_n,V_n}\big[\mathfrak{G}_u\big]\mathbf{1}_{\{n\le T_u^A\}} \qquad \mathbb{P}^\alpha\text{-a.s.}$$

As $\lim_{u\to\infty}\mathbf{1}_{\{n\le T_u^A\}}=1$ $\mathbb{P}^\alpha$-a.s., it suffices to determine $\lim_{u\to\infty}\mathbb{E}^\alpha_{x,v}\big[\mathfrak{G}_u\big]$ and show that this quantity is independent of x and v.

For all $v\in\mathbb{R}^d_+$ and $u>0$, set

$$G_u(v)=\mathbb{E}\bigg[g\bigg(v,\ldots,\Pi_m v+\frac{V_m^{(0)}}{u}\bigg)\bigg],\quad G(v)=\mathbb{E}\bigg[g\bigg(v,\Pi_1 v,\ldots,\Pi_m v\bigg)\bigg],$$

where $V_m^{(0)}:=\sum_{i=1}^m\Pi_{i+1}^m Q_i$ for $m\ge 2$ and $V_1^{(0)}:=Q_1$. Now consider the decomposition:

$$\begin{aligned}\mathbb{E}^\alpha_{x,v}\big[\mathfrak{G}_u\big] \;&=\; \mathbb{E}^\alpha_{x,v}\bigg[\frac{1}{r_\alpha(X_{T_u^A})}\left(\frac{|V_{T_u^A}|}{u}\right)^{-\alpha}\bigg(G_u\Big(\frac{V_{T_u^A}}{u}\Big)-G\Big(\frac{V_{T_u^A}}{u}\Big)\bigg)\bigg]\\ &\quad+\;\mathbb{E}^\alpha_{x,v}\bigg[\frac{r_\alpha(\widetilde{V}_{T_u^A})}{r_\alpha(X_{T_u^A})}\frac{1}{r_\alpha(\widetilde{V}_{T_u^A})}\left(\frac{|V_{T_u^A}|}{u}\right)^{-\alpha}G\Big(\frac{V_{T_u^A}}{u}\Big)\bigg]:=\mathbb{I}_1(u)+\mathbb{I}_2(u).\end{aligned}$$

Step 1. We begin by showing that $\mathbb{I}_1(u)\to 0$ as $u\to\infty$. Let $\hat g$ be a θ-Hölder continuous function with $g(v_0,\ldots,v_m)=\hat g(v_0,\ldots,v_m)\mathbf{1}_{\{|v_m|\ge\delta\}}$. Then

$$\begin{aligned}&\Big|g\Big(\Big(\Pi_n v+\big(V_n^{(0)}/u\big)\Big)_{n=0}^m\Big)-g\Big(\big(\Pi_n v\big)_{n=0}^m\Big)\Big|\\ &\le\Big|\hat g\Big(\Big(\Pi_n v+\big(V_n^{(0)}/u\big)\Big)_{n=0}^m\Big)-\hat g\Big(\big(\Pi_n v\big)_{n=0}^m\Big)\Big|\,\mathbf{1}_{\{|\Pi_m v|\ge\delta\}}\\ &\qquad+|g|_\infty\left(\mathbf{1}_{[\delta,\infty)}\Big(\Big|\Pi_m v+\big(V_m^{(0)}/u\big)\Big|\Big)-\mathbf{1}_{[\delta,\infty)}\big(|\Pi_m v|\big)\right)\end{aligned}$$

$$\leq \frac{1}{u^\theta} B_1 \sum_{n=1}^{m} \left|V_n^{(0)}\right|^\theta + |g|_\infty \left(\mathbf{1}_{[\delta,\infty)}\left(\left| \Pi_m v + \left(V_m^{(0)}/u\right) \right| \right) - \mathbf{1}_{[\delta,\infty)}\left(|\Pi_m v|\right) \right)$$

for some constant B_1 arising from the θ-Hölder continuity of $\hat{g}$. Let (M^*, Q^*) be a pair of random variables that is independent of the sequence $\{(M_n, Q_n)\}$, where the $\mathbb{P}^\alpha$-law of (M^*, Q^*) is given by $\mathbb{P}\big((\Pi_m, V_m^{(0)}) \in \cdot\big)$. Upon setting $B_2 = \max_{y \in \mathbb{S}_+^{d-1}} (r_\alpha(y))^{-1}$ and using that $\left(|V_{T_u^A}|/u\right)^{-\alpha} < 1$, we obtain that

$$\mathbb{I}_1(u) \leq \frac{1}{u^\theta}\, B_1 B_2\, \mathbb{E}\Big[\sum_{n=1}^{m} \left|V_n^{(0)}\right|^\theta \Big] + B_2\, |g|_\infty \left(\mathbb{P}^\alpha_{x,v}\left(\left| M^* \frac{V_{T_u^A}}{u} + \frac{Q^*}{u} \right| \geq \delta \right) - \mathbb{P}^\alpha_{x,v}\left(\left| M^* \frac{V_{T_u^A}}{u} \right| \geq \delta \right) \right). \tag{4.6}$$

Since the θ-moment of $V_n^{(0)}$ is finite, the first term tends to zero as $u \to \infty$. For the second term, use the $\mathbb{P}^\alpha$-convergence $(M^*, Q^*/u) \Rightarrow (M^*, 0)$ and $\big(\widetilde{V}^A_{T_u^A}, \log|V^A_{T_u^A}| - \log u\big) \Rightarrow \varrho^A$ (by Theorem 3.8). Let $(X, S) \sim \varrho^A$ be a random vector independent of (M^*, Q^*) under $\mathbb{P}^\alpha$. Solving (3.32) for V_n, we have that $V_{T_u^A}/u \Rightarrow d_A(X) e^S X$. [Here X describes the limiting direction of $V^A_{T_u^A}/u$ and S the limiting logarithmic overjump, as $\log|V^A_{T_u^A}| - \log u \Rightarrow S$.] Since the sequences $\{(M^*, Q^*/u)\}$ and $\{V_{T_u^A}/u\}$ are independent, they converge jointly in distribution. Hence, under $\mathbb{P}^\alpha$,

$$M^* \frac{V_{T_u^A}}{u} + \frac{Q^*}{u} \;\Rightarrow\; d_A(X) e^S M^* X \quad \text{and} \quad M^* \frac{V_{T_u^A}}{u} \;\Rightarrow\; d_A(X) e^S M^* X.$$

Thus, the second term in (4.6) vanishes if $[\delta, \infty)$ is a continuity set for $d_A(X) e^S |M^* X|$.

We now show that $[\delta, \infty)$ is a continuity set. Since M^* is independent of (X, S), it suffices to show that for any allowable matrix $\mathfrak{m}$, the event $\{d_A(X) e^S |\mathfrak{m} X| = \delta\}$ has probability 0. Now for each fixed $y \in \mathbb{S}_+^{d-1}$, the equation $\mathfrak{h}(s) := d_A(y) e^s |\mathfrak{m} y| = \delta$ has a unique solution $s_y \in \mathbb{R}$. Hence

$$\mathbb{P}^\alpha\big(d_A(X) e^S |\mathfrak{m} X| = \delta\big) = \int_{\mathbb{S}_+^{d-1} \times \mathbb{R}} \mathbf{1}_{\{s = s_y\}}\, \varrho^A(dy, ds) = 0,$$

since the radial component of the overjump distribution is absolutely continuous with respect to Lebesgue measure (as can be seen from the representation of ϱ^A in Eq. (1.16) of [29]).

Thus, having shown that $[\delta, \infty)$ is a continuity set, we conclude by the Portmanteau theorem that for all $x \in \mathbb{S}_+^{d-1}$ and $v \in \mathbb{R}_+^d \setminus \{0\}$,

$$\mathbb{P}^\alpha_{x,v}\left(\left| M^* \frac{V_{T_u^A}}{u} + \frac{Q^*}{u} \right| \geq \delta \right) - \mathbb{P}^\alpha_{x,v}\left(\left| M^* \frac{V_{T_u^A}}{u} \right| \geq \delta \right)$$

$$\to\ \mathbb{P}^\alpha\Big(d_A(X)e^S|M^*X|\geq\delta\Big)-\mathbb{P}^\alpha\Big(d_A(X)e^S|M^*X|\geq\delta\Big).$$

Hence also the second member of (4.6) vanishes as $u\to\infty$. Thus $\mathbb{I}_1(u)\to 0$ as $u\to\infty$.

STEP 2. Now turn to $\mathbb{I}_2(u)$. Using Theorem 3.8, again invoke the convergence $\big((V^A_{T^A_u})^\sim,\log|V^A_{T^A_u}|-\log u\big)\Rightarrow\varrho^A$ under $\mathbb{P}^\alpha$. Moreover, by Lemma 3.10, using the continuity and boundedness of r_α, we have that $r_\alpha(\widetilde{V}_{T_u})/r_\alpha(X_{T_u})$ tends to one in $\mathbb{P}^\alpha$-probability. Hence by Slutsky's theorem, the quantity inside $\mathbb{I}_2(u)$ converges in law, and identifying this limit distribution, we deduce that

$$\begin{aligned}\lim_{u\to\infty}\mathbb{I}_2(u)&=\lim_{u\to\infty}\mathbb{E}^\alpha_{x,v}\bigg[\frac{r_\alpha(\widetilde{V}_{T^A_u})}{r_\alpha(X_{T^A_u})}\frac{1}{r^A_\alpha(\widetilde{V}^A_{T^A_u})}\left(\frac{|V^A_{T^A_u}|}{u}\right)^{-\alpha}G\Big(\frac{V_{T^A_u}}{u}\Big)\bigg]\\&=\int_{\mathbb{S}^{d-1}_+\times\mathbb{R}_+}\frac{e^{-\alpha s}}{r^A_\alpha(y)}G\Big(d_A(y)e^sy\Big)\varrho^A(dy,ds).\end{aligned}$$

Recalling that $G(v)=\mathbb{E}\big[g\big(v,\Pi_1v,\ldots,\Pi_mv\big)\big]$, the assertion follows. □

PROOF OF PROPOSITION 4.1. Note that $\{V_n\}$ is transient in the α-shifted measure and thus $T^A_u<\infty$ a.s.; cf. Lemma 3.12. Hence, employing the change of measure *only* over the random time interval $[0,T^A_u]$ (namely the "dual" change of measure of [18], Section 4, which we denote by the superscript $\mathscr{D}$), we obtain that

$$\begin{aligned}&u^\alpha\mathbb{E}\left[g\left(\frac{V_{T^A_u}}{u},\ldots,\frac{V_{T^A_u+m}}{u}\right)\mathbf{1}_{\{T^A_u<\tau\}}\,\middle|\,V_0=v\right]\\&\quad=u^\alpha r_\alpha(\widetilde{v})\mathbb{E}^{\mathscr{D}}_{\boldsymbol{\delta}_v}\left[\frac{e^{-\alpha S_{T^A_u}}}{r_\alpha(X_{T^A_u})}\mathbf{1}_{\{T^A_u<\tau\}}\mathbb{E}\left[g\left(\frac{V_{T^A_u}}{u},\ldots,\frac{V_{T^A_u+m}}{u}\right)\,\middle|\,\mathscr{F}_{T^A_u}\right]\right].\end{aligned}$$

Now substitute the quantity $\mathfrak{G}_u$ of Lemma 4.3 (ii) into the previous equation. Noting that $Z_n=V_n/|\Pi_nX_0|=\big(V_n/e^{S_n}\big)$, $n\in\mathbb{N}_+$, we obtain after a little algebra that

$$\begin{aligned}u^\alpha\mathbb{E}\left[g\left(\frac{V_{T^A_u}}{u},\ldots,\frac{V_{T^A_u+m}}{u}\right)\mathbf{1}_{\{T^A_u<\tau\}}\,\middle|\,X_0=\widetilde{v},\,V_0=v\right]&\\=r_\alpha(\widetilde{v})\mathbb{E}^{\mathscr{D}}_{\boldsymbol{\delta}_v}\Big[|Z_{T^A_u}|^\alpha\mathfrak{G}_u\mathbf{1}_{\{T^A_u<\tau\}}\Big].&\end{aligned}\tag{4.7}$$

For $n\in\mathbb{N}_+$, the right-hand side can be further equated to

$$r_\alpha(\widetilde{v})\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\left[\Big(\big|Z_{T^A_u}\big|^\alpha\mathbf{1}_{\{T^A_u<\tau\}}-|Z_n|^\alpha\,\mathbf{1}_{\{n\leq T^A_u\}}\mathbf{1}_{\{n\leq\tau\}}\Big)\mathfrak{G}_u\right]\tag{4.8}$$

$$+ r_\alpha(\widetilde{v})\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\Big[\,|Z_n|^\alpha\,\mathbf{1}_{\{n\le T^A_u\}}\mathbf{1}_{\{n\le\tau\}}\,\mathbb{E}^\alpha\big[\mathfrak{G}_u\,|\mathscr{F}_n\big]\Big],$$

where we have replaced $\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}[\cdot|\mathscr{F}_n]$ with $\mathbb{E}^\alpha[\cdot|\mathscr{F}_n]$ in the last expectation, since this conditional expectation depends only on (X_n, V_n), and not on the initial values (X_0, V_0) once (X_n, V_n) has been specified. Moreover, the superscript $\mathscr{D}$ can now be dropped, since the change of measure over the random time interval $[0, T^A_u]$ coincides with the usual α-shifted measure for $\mathscr{F}_{T^A_u}$-measurable random variables.

To analyze the quantity in (4.8), we first take the limit as $u \to \infty$ and then as $n \to \infty$. By part (i) of Lemma 4.3 and the boundedness of $\mathfrak{G}_u$, we deduce from (4.7) and (4.8) that

$$\begin{aligned}(4.9)\qquad &\lim_{u\to\infty} u^\alpha\mathbb{E}\left[g\left(\frac{V_{T^A_u}}{u},\dots,\frac{V_{T^A_u+m}}{u}\right)\mathbf{1}_{\{T^A_u<\tau\}}\,\middle|\,V_0=v\right]\\ &\quad=\lim_{n\to\infty}\lim_{u\to\infty} r_\alpha(\widetilde{v})\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\Big[\,|Z_n|^\alpha\,\mathbf{1}_{\{n\le T^A_u\}}\mathbf{1}_{\{n\le\tau\}}\,\mathbb{E}^\alpha\big[\mathfrak{G}_u\,\big|\,\mathscr{F}_n\big]\Big].\end{aligned}$$

Now by Lemma 3.6 (iii), $\big\{|Z_n|^\alpha\mathbf{1}_{\{n\le\tau\}}\big\}$ is uniformly integrable. Denote by $\mathfrak{G}$ the right-hand side of (4.5). Since $\mathfrak{G}_u$ is bounded by $b^{-1}\,|g|_\infty$ and $T^A_u \uparrow \infty$ $\mathbb{P}^\alpha$-a.s., it follows by Lemma 4.3 (ii) that

$$\begin{aligned}(4.10)\qquad &\lim_{u\to\infty} u^\alpha\mathbb{E}\left[g\left(\frac{V_{T^A_u}}{u},\dots,\frac{V_{T^A_u+m}}{u}\right)\mathbf{1}_{\{T^A_u<\tau\}}\,\middle|\,V_0=v\right]\\ &\quad=\lim_{n\to\infty} r_\alpha(\widetilde{v})\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\Big[\,|Z_n|^\alpha\,\mathbf{1}_{\{n\le\tau\}}\lim_{u\to\infty}\mathbf{1}_{\{n\le T^A_u\}}\mathbb{E}^\alpha\big[\mathfrak{G}_u\,|\mathscr{F}_n\big]\Big]\\ &\quad=r_\alpha(\widetilde{v})\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\Big[\lim_{n\to\infty}|Z_n|^\alpha\,\mathbf{1}_{\{n\le\tau\}}\,\mathfrak{G}\Big]=r_\alpha(\widetilde{v})\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\big[\,|Z|^\alpha\,\mathbf{1}_{\{\tau=\infty\}}\big]\,\mathfrak{G}.\ \square\end{aligned}$$

In some cases, it is useful to consider functions g which depend on the *infinite* path $(V_{T^A_u}, V_{T^A_u+1}, \dots)$, or to consider functions g which need not be bounded. Moreover, it is also useful to have *uniform* upper bounds. In these situations, a variant of the above proposition is useful.

PROPOSITION 4.4. *Suppose that $g : (\mathbb{R}^d_+)^{\mathbb{N}} \to [0,\infty)$ is a nonnegative measurable function, and set*

$$\bar{\mathfrak{G}}_u = \frac{1}{r_\alpha\big(X_{T^A_u}\big)}\left(\frac{\big|V_{T^A_u}\big|}{u}\right)^{-\alpha}\mathbb{E}\left[g\left(\left(\frac{V_{T^A_u+k}}{u}\right)_{k\ge0}\right)\,\middle|\,\mathscr{F}_{T^A_u}\right].$$

Further, assume that for some finite constant B and some $\mathscr{U} \ge 0$,

$$(4.11)\qquad \sup_{u\ge\mathscr{U}}\bar{\mathfrak{G}}_u \le B \quad \mathbb{P}^\alpha\text{-a.s.}$$

Then for any bounded set $F \subset \mathbb{R}^d_+ \setminus \{0\}$, there exists a finite constant L, not depending on B, such that

$$0 \;\le\; \sup_{u\ge\mathscr{U}} \sup_{v\in F} u^\alpha \mathbb{E}\left[g\left(\left(\frac{V_{T_u^A+k}}{u} \right)_{k\ge 0} \right) \mathbf{1}_{\{T_u^A<\tau\}} \middle| V_0 = v \right] \;\le\; BL. \tag{4.12}$$

Moreover, if (4.11) *holds and* $\limsup_{u\to\infty} \bar{\mathfrak{G}}_u = 0$ $\mathbb{P}^\alpha$*-a.s., then*

$$\lim_{u\to\infty} u^\alpha \mathbb{E}\left[g\left(\left(\frac{V_{T_u^A+k}}{u} \right)_{k\ge 0} \right) \mathbf{1}_{\{T_u^A<\tau\}} \middle| V_0 = v \right] = 0. \tag{4.13}$$

Proof. Repeating the argument in the proof of Proposition 4.1 leading to (4.9), we obtain that

$$\begin{aligned} 0 \;&\le\; \sup_{u\ge\mathscr{U}} \sup_{v\in F} u^\alpha \mathbb{E}\left[g\left(\left(\frac{V_{T_u^A+k}}{u} \right)_{k\ge 0} \right) \mathbf{1}_{\{T_u^A<\tau\}} \middle| V_0 = v \right] \\ &\le\; B r_\alpha(\widetilde{v}) \sup_{v\in F} \mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\left[\sup_{n\in\mathbb{N}} |Z_n|^\alpha \mathbf{1}_{\{n\le\tau\}} \right], \end{aligned}$$

which is finite by Lemma 3.6 (iii) and the boundedness of r_α. The boundedness of $\bar{\mathfrak{G}}_u$ then allows us to use the dominated convergence theorem in order to deduce (4.13) from (4.9). □

4.3. *Toward the proof of Theorem 2.4.* We now restrict our attention to the case where $d_A = 1$; thus $T_u^A = T_u$, $r_\alpha^A = r_\alpha$, and $\varrho^A = \varrho$.

In order to establish Theorem 2.4 in the subsequent section, we first prove a proposition which, together with Lemma 3.5, will link the tail properties of V to the renewal measure associated with $\{(X_n, S_n)\}$. As before, we use the shorthand $C(v) = r_\alpha(\widetilde{v})\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\left[|Z|^\alpha \mathbf{1}_{\{\tau=\infty\}}\right]$.

Proposition 4.5. *Assume (H_1) and (H_2) are satisfied. Let $\theta \le \min\{1,\alpha\}$ and let f be a nonnegative bounded θ-Hölder continuous function. Then for all $v \in \mathbb{R}^d_+ \setminus \{0\}$,*

$$\begin{aligned} &\lim_{u\to\infty} u^\alpha \mathbb{E}\left[\sum_{i=0}^{\tau-1} f\left(\frac{V_i}{u}\right) \mathbf{1}_{\{|V_i|\ge u\}} \middle| V_0 = v \right] \\ &\qquad = C(v) \int_{\mathbb{S}^{d-1}_+\times\mathbb{R}_+} \mathbb{E}^\alpha_x \left[\sum_{i=0}^{\infty} F(X_i, S_i + s) \right] \varrho(dx, ds), \end{aligned} \tag{4.14}$$

where $F(x,s) := \big(e^{-\alpha s} f(e^s x)/r_\alpha(x)\big)\mathbf{1}_{[0,\infty)}(s)$ is directly Riemann integrable.

We note by [36, Section 6.1] that, if F directly Riemann integrable (as defined in [10], Eq. (7.1)), it follows that on the right-hand side of (4.14),

$$\sup_{x\in\mathbb{S}_+^{d-1}}\sup_{s\in\mathbb{R}}\mathbb{E}_x^\alpha\left[\sum_{i=0}^{\infty}|F(X_i,S_i+s)|\right]<\infty. \tag{4.15}$$

We shall deduce Proposition 4.5 from Corollary 4.2. However, to do so, we need to handle the remainder terms, which we study in the following.

LEMMA 4.6. *Let h be a bounded measurable function such that $h(x)=0$ for all $x\in B_\delta^+(0)$, for some $\delta>0$. Then for all $v\in\mathbb{R}_+^d\setminus\{0\}$ and all $m\in\mathbb{N}$,*

$$\lim_{m\to\infty}\lim_{u\to\infty}u^\alpha\mathbb{E}\left[\sum_{k=m}^{\tau-1-T_u}h\Big(\frac{V_{T_u+k}}{u}\Big)\mathbf{1}_{\{T_u+m<\tau\}}\Big|V_0=v\right]=0. \tag{4.16}$$

Moreover, if $F\subset\mathbb{R}_+^d\setminus\{0\}$ is bounded, then

$$\limsup_{u\to\infty}\sup_{v\in F}u^\alpha\mathbb{E}\left[\sum_{i=0}^{\tau-1}h\left(\frac{V_i}{u}\right)\mathbf{1}_{\{|V_i|>u\}}\Big|V_0=v\right]<\infty. \tag{4.17}$$

Furthermore, for all $v\in\mathbb{R}_+^d\setminus\{0\}$,

$$\lim_{m\to\infty}\lim_{u\to\infty}u^\alpha\mathbb{E}\left[\sum_{i=\tau}^{T_u+m}h\Big(\frac{V_i}{u}\Big)\mathbf{1}_{\{T_u<\tau\le T_u+m\}}\Big|V_0=v\right]=0. \tag{4.18}$$

PROOF. STEP 1. First we establish (4.16).

By Eq. (4.12) in Proposition 4.4, it suffices to prove that

$$\sup_{u\ge\mathscr{U}}\bar{\mathfrak{G}}_u:=\sup_{u\ge\mathscr{U}}\frac{1}{r_\alpha(X_{T_u})}\left(\frac{|V_{T_u}|}{u}\right)^{-\alpha}\mathbb{E}\Big[\sum_{k=m}^{\tau-1-T_u}\Big|h\Big(\frac{V_{T_u+k}}{u}\Big)\Big|\,\mathbf{1}_{\{T_u+m<\tau\}}\Big|\mathscr{F}_{T_u}\Big]$$

is bounded above by $B(m,\mathscr{U})$, where the sequence $\{B(m,\mathscr{U})\}$ tends to zero as we first let $\mathscr{U}\to\infty$ and then let $m\to\infty$. By employing the Markov property and the boundedness of r_α, we see that it is enough to show that, for a suitable sequence $B(m,\mathscr{U})$,

$$\begin{aligned}\sup_{u\ge\mathscr{U}}\sup_{v:|v|\ge u}H_u(v)&:=\sup_{u\ge\mathscr{U}}\sup_{v:|v|\ge u}\mathbb{E}\Big[\left(\frac{|V_0|}{u}\right)^{-\alpha}\sum_{k=m}^{\tau-1}h\Big(\frac{V_k}{u}\Big)\mathbf{1}_{\{m<\tau\}}\Big|V_0=v\Big]\\&\le\ B(m,\mathscr{U}).\end{aligned} \tag{4.19}$$

Now let $\mathbb{D}^\dagger = \{v \in \mathbb{R}^d_+ : |v| \le L\}$ be defined as in Lemma 3.1, and set $\tau^\dagger := \inf\{n \in \mathbb{N}_+ : V_n \in \mathbb{D}^\dagger\}$. Recall that $h(x) = 0$ for all $x \in B^+_\delta(0)$. Hence, for $0 < \theta < \min\{1, \alpha\}$ and $|v| > u$,

$$H_u(v) \;\le\; |h|_\infty \mathbb{E}\Big[\left(\frac{|V_0|}{u}\right)^{-\alpha} \left(\sum_{k=m}^{\tau^\dagger - 1} \mathbf{1}_{\{|V_k|>\delta u\}} + \sum_{k=\tau^\dagger}^{\tau-1} \mathbf{1}_{\{|V_k|>\delta u\}} \right) \Big| V_0 = v \Big]$$

$$\le |h|_\infty \left(\frac{|v|}{u}\right)^{-\alpha} \sum_{k=m}^{\infty} (\delta u)^{-\theta} \mathbb{E}_v\big[\, |V_k|^\theta \, \mathbf{1}_{\{\tau^\dagger > k\}}\big] + |h|_\infty \sup_{w \in \mathbb{D}^\dagger} \mathbb{E}_w \left[\sum_{k=0}^{\tau} \mathbf{1}_{\{|V_k|>\delta u\}} \right].$$

The first sum can be estimated further by employing Lemma 3.1; namely,

$$\sup_{v:|v|\ge u} \left(\frac{|v|}{u}\right)^{-\alpha} \sum_{k=m}^{\infty} (\delta u)^{-\theta} \mathbb{E}_v\big[\, |V_k|^\theta \, \mathbf{1}_{\{\tau^\dagger > k\}}\big]$$

$$\le \; \frac{B}{\delta^\theta} \left(\sup_{v:|v|\ge u} \left(\frac{|v|}{u}\right)^{\theta-\alpha} \right) \frac{t^m}{1-t} = \frac{B}{\delta^\theta} \frac{t^m}{1-t},$$

and this last term tends to zero as $m \to \infty$. For the second term, note that (3.4) implies that $\sup_{w\in\mathbb{D}^\dagger} \mathbb{E}[\tau \,|\, V_0 = w] < \infty$. Hence we can apply a dominated convergence argument to infer that

$$\sup_{u \ge \mathscr{U}} \sup_{w \in \mathbb{D}^\dagger} \mathbb{E}_w \left[\sum_{k=0}^{\tau} \mathbf{1}_{\{|V_k|>\delta u\}} \right] \le \sup_{w \in \mathbb{D}^\dagger} \mathbb{E}_w \left[\sum_{k=0}^{\tau} \mathbf{1}_{\{|V_k|>\delta \mathscr{U}\}} \right] \;\to\; 0 \text{ as } \mathscr{U} \to \infty.$$

Combining these estimates establishes (4.19), and (4.16) follows.

Finally, (4.17) is a direct consequence of (4.19) (with $m = 0$ and $\delta = 1$) combined with (4.12).

Step 2. Turning to (4.18), we apply the second part of Proposition 4.4. Using that $h = 0$ on $B_\delta(0)$, it is now sufficient to show that for any $m \in \mathbb{N}$,

$$\bar{\mathfrak{G}}_u \;:=\; \frac{1}{r_\alpha(X_{T_u})} \left(\frac{|V_{T_u}|}{u}\right)^{-\alpha} \mathbb{E}\left[\sum_{i=\tau}^{T_u+m} \left| h\Big(\frac{V_i}{u}\Big) \right| \mathbf{1}_{\{|V_i|>\delta u\}} \mathbf{1}_{\{T_u < \tau \le T_u+m\}} \Big| \mathscr{F}_{T_u} \right]$$

is bounded uniformly in u and tends to zero $\mathbb{P}^\alpha$-a.s. as $u \to \infty$. As the prefactors are bounded, it suffices to estimate

$$\mathbb{E}\Big[\sum_{i=\tau}^{T_u+m} \left| h\Big(\frac{V_i}{u}\Big) \right| \mathbf{1}_{\{|V_i|>\delta u\}} \mathbf{1}_{\{T_u < \tau \le T_u+m\}} \Big| \mathscr{F}_{T_u} \Big] \tag{4.20}$$

$$\le \; |h|_\infty \, \mathbb{E}\left[\mathbb{E}\left[\sum_{i=\tau}^{\tau+m} \mathbf{1}_{\{|V_i|>\delta u\}} \Big| \mathscr{F}_\tau \right] \mathbf{1}_{\{T_u<\tau\}} \Big| \mathscr{F}_{T_u} \right].$$

Let $\theta \in (0, \alpha)$. Then for all $k = 0, \dots, m$,

$$\begin{aligned}\mathbb{P}\left(|V_{\tau+k}| > \delta u | V_\tau = v\right) &\leq \sup_{v\in\mathbb{D}} (\delta u)^{-\theta} \mathbb{E}\left[|V_k|^\theta \Big| V_0 = v\right] \\ &\leq \delta^{-\theta} u^{-\theta} \left(\sup_{v\in\mathbb{D}} \mathbb{E}\big[\|\Pi_k\|^\theta \big] \cdot |v|^\theta + \sum_{j=1}^{k} \mathbb{E}\left[\left| \Pi_{j+1}^k Q_j \right|^\theta \right] \right) \leq B_1 u^{-\theta}\end{aligned}$$

for some finite constant B_1 (dependent on m). Substituting this estimate into (4.20) and then into the definition of $\bar{\mathfrak{G}}_u$ above (4.20), we obtain that $\bar{\mathfrak{G}}_u \leq B_2 m u^{-\theta} \downarrow 0$ as $u \to \infty$, some $B_2 < \infty$. Thus, by Proposition 4.4,

$$\lim_{u\to\infty} u^\alpha \mathbb{E}\left[\sum_{i=\tau}^{T_u+m} h\Big(\frac{V_i}{u}\Big) \mathbf{1}_{\{T_u < \tau \leq T_u+m\}} \Big| V_0 = v \right] = 0, \quad \forall m \in \mathbb{N},$$

and (4.18) follows. □

Proof of Proposition 4.5. Let $g(v) := f(v)\mathbf{1}_{\{|v|\geq 1\}}$, and note that g is an almost θ-Hölder-continuous function. Now

$$u^\alpha \mathbb{E}_v \left[\sum_{i=0}^{\tau-1} f\left(\frac{V_i}{u}\right) \mathbf{1}_{\{|V_i|\geq u\}} \right] = u^\alpha \mathbb{E}_v \left[\sum_{i=T_u}^{\tau-1} f\left(\frac{V_i}{u}\right) \mathbf{1}_{\{|V_i|\geq u\}} \mathbf{1}_{\{T_u<\tau\}} \right],$$

and the right-hand side can be decomposed into three terms, namely

$$\begin{aligned} &u^\alpha \sum_{k=0}^{m} \mathbb{E}_v \left[g\left(\frac{V_{T_u+k}}{u}\right) \mathbf{1}_{\{T_u<\tau\}} \right] + u^\alpha \mathbb{E}_v \left[\sum_{k=m+1}^{\tau-1-T_u} g\left(\frac{V_{T_u+k}}{u}\right) \mathbf{1}_{\{T_u+m<\tau\}} \right] \\ &\qquad - u^\alpha \mathbb{E}_v \left[\sum_{i=\tau}^{T_u+m} g\left(\frac{V_i}{u}\right) \mathbf{1}_{\{T_u<\tau\leq T_u+m\}} \right]. \end{aligned} \tag{4.21}$$

On the right-hand side of (4.21), the last two terms tend to zero, by Lemma 4.6, when taking first the limit $u \to \infty$ and then $m \to \infty$. Next, by Corollary 4.2, Eq. (4.3), we obtain for the remaining term that

$$\begin{aligned} &\lim_{u\to\infty} u^\alpha \sum_{k=0}^{m} \mathbb{E}\left[g\left(\frac{V_{T_u+k}}{u}\right) \mathbf{1}_{\{T_u<\tau\}} \Big| V_0 = v \right] \\ &= C(v) \int_{\mathbb{S}_+^{d-1}\times\mathbb{R}_+} \mathbb{E}_x^\alpha \left[\sum_{i=0}^{m} \frac{e^{-\alpha(S_i+s)}}{r_\alpha(X_i)} f(e^{S_i+s} X_i) \mathbf{1}_{\{S_i+s\geq 0\}} \right] \varrho(dx, ds). \end{aligned}$$

It remains to show that $F(x,s) := \big(e^{-\alpha s} f(e^s x)/r_\alpha(x)\big)\mathbf{1}_{[0,\infty)}(s)$ is directly Riemann integrable, which, by (4.15), will allows us to take the limit as

$m \to \infty$. Since r_α is bounded from below, it follows that for some positive constant b,

$$\overline{F}(s) := \sup_{x \in \mathbb{S}_+^{d-1}} |F(x,s)| \leq \frac{1}{b} |f|_\infty e^{-\alpha s} \mathbf{1}_{[0,\infty)}(s).$$

Since the right-hand side is a decreasing integrable function, we conclude that $\overline{F}$ is (univariate) directly Riemann integrable. But $\overline{F}$ is obtained from F by taking the supremum over all $x \in \mathbb{S}_+^{d-1}$, so it follows immediately that F is (multivariate) Riemann integrable. □

5. Proof of Theorem 2.4. In this section, we provide the proof of Theorem 2.4, first under the additional hypothesis (H_3) of Section 3, which is then removed by approximating $\{V_n\}$ from above and below by smoothed processes for which (H_3) is satisfied.

To establish Theorem 2.4, we apply Proposition 4.5 directly, except that we must identify the integral in (4.14). This is done in the following lemma.

LEMMA 5.1. *Let $g : \mathbb{S}_+^{d-1} \times \mathbb{R} \to \mathbb{R}$ be a directly Riemann integrable function. Then*

$$\int \mathbb{E}_x^\alpha \left[\sum_{i=0}^{\infty} g(X_i, S_i + s) \mathbf{1}_{\{S_i + s \geq 0\}} \right] \varrho(dx, ds) = \frac{1}{\lambda'(\alpha)} \int g(x,s)\, \eta_\alpha(dx)\, ds.$$

The crucial point is to identify ϱ as the *stationary Markov delay* distribution, i.e., the initial distribution for $\{(X_n, S_n)\}$ under which the renewal measure (restricted to $\mathbb{S}_+^{d-1} \times \mathbb{R}_+$) *equals* $\eta_\alpha \otimes ds$. This identification can be done along identical lines to the proofs of Theorem 3 and Corollary 3 in [2]. For this reason, we omit the proof and refer the reader to the arXiv version [16] of our article for the details.

We now establish Theorem 2.4 under the additional Hypothesis (H_3) of Section 3.

PROPOSITION 5.2. *Assume that Hypotheses (H_1), (H_2) and (H_3) are satisfied, and suppose that $\mathbb{D} \in \mathscr{B}(\mathbb{R}_+^d)$ is bounded and $\pi(\mathbb{D}) > 0$. Then for any $f \in \mathscr{C}_0\big(\mathbb{R}_+^d \setminus \{0\}\big)$,*

$$\lim_{u \to \infty} u^\alpha \mathbb{E}\left[f\left(\frac{V}{u} \right) \right] = \frac{C}{\lambda'(\alpha)} \int_{\mathbb{S}_+^{d-1} \times \mathbb{R}} e^{-\alpha s} f(e^s x) l_\alpha(dx) ds, \tag{5.1}$$

where C is given as in (2.10). *Equivalently, we have the weak convergence*

$$\lim_{u \to \infty} u^\alpha \mathbb{P}\left(|V| > tu, \frac{V}{|V|} \in \cdot \right) \;\Rightarrow\; \frac{C}{\alpha \lambda'(\alpha)} t^{-\alpha} l_\alpha(\cdot), \quad \text{for all } t > 0. \tag{5.2}$$

PROOF. We first prove the result under the additional assumption that f satisfies $f(x) = \hat{f}(x)\mathbf{1}_{\{|x|\geq r\}}$ for some $r > 0$, where $\hat{f}$ is a θ-Hölder continuous function with $\theta \leq \min\{1, \alpha\}$, i.e., f is almost θ-Hölder continuous.

STEP 1. First assume $r = 1$, i.e., $f(x) = \hat{f}(x)\mathbf{1}_{\{|x|\geq 1\}}$. Since (H_3) is satisfied, it follows from Lemma 3.5 that

$$\begin{aligned}\mathbb{E}\left[\hat{f}\left(\frac{V}{u}\right)\mathbf{1}_{\{|V|\geq u\}}\right] &= \frac{1}{\mathbb{E}_{\pi_{\mathbb{D}}}[\tau]}\int_{\mathbb{D}}\mathbb{E}\left[\sum_{i=0}^{\tau-1}\hat{f}\left(\frac{V_i}{u}\right)\mathbf{1}_{\{|V_i|\geq u\}}\middle| V_0 = v\right]\pi_{\mathbb{D}}(dv)\\ &= \int_{\mathbb{D}}\mathbb{E}\left[\sum_{i=0}^{\tau-1}\hat{f}\left(\frac{V_i}{u}\right)\mathbf{1}_{\{|V_i|\geq u\}}\middle| V_0 = v\right]\pi_{\mathbb{D}}(dv), \qquad (5.3)\end{aligned}$$

where τ denotes the first return time of $\{V_n\}$ to $\mathbb{D}$. Moreover, from Lemma 3.3 we have that $\pi(\mathbb{D}) = (\mathbb{E}_{\pi_{\mathbb{D}}}[\tau])^{-1}$, where $\pi_{\mathbb{D}}(\cdot) = \pi(\cdot \cap \mathbb{D})/\pi(\mathbb{D})$.

Now apply Proposition 4.5 and the identity (5.3) separately to the positive and negative parts of $\hat{f}$ to obtain that

$$\begin{aligned}&\lim_{u\to\infty} u^{\alpha}\mathbb{E}\left[\hat{f}\left(\frac{V}{u}\right)\mathbf{1}_{\{|V|\geq u\}}\right] \qquad (5.4)\\ &\quad = \int_{\mathbb{D}} C(v)\left(\int_{\mathbb{S}_+^{d-1}\times\mathbb{R}_+}\mathbb{E}_x^{\alpha}\left[\sum_{i=0}^{\infty}F(X_i, S_i + s)\right]\varrho(dx, ds)\right)\pi(dv),\end{aligned}$$

where $F(x, s) = \big(e^{-\alpha s}f(e^s x)/r_\alpha(x)\big)\mathbf{1}_{[0,\infty)}$. Note that Proposition 4.5 actually holds conditional on $\{V_0 = v\}$, where $v \in \mathbb{R}_+^d \setminus \{0\}$; and to extend this result so that it holds conditional on $\{V_0 \sim \pi_{\mathbb{D}}\}$, we have applied a dominated convergence argument together with the bound provided by (4.17) of Lemma 4.6. Moreover, we have used that $\pi(\{0\}) = 0$, which follows since π is the law of $V := \sum_{k=1}^{\infty} M_1 \cdots M_{k-1}Q_{k-1} \neq 0$ w.p.1. Next, observe by Lemma 5.1 that

$$\begin{aligned}&\int \mathbb{E}_x^{\alpha}\left[\sum_{i=0}^{\infty}F(X_i, S_i + s)\right]\varrho(dx, ds) = \frac{1}{\lambda'(\alpha)}\int F(x, s)\eta_\alpha(dx)ds \qquad (5.5)\\ &\quad = \frac{1}{\lambda'(\alpha)}\int \frac{e^{-\alpha s}}{r_\alpha(x)}f(e^s x)\,\eta_\alpha(dx)\,ds = \frac{1}{\lambda'(\alpha)}\int e^{-\alpha s}f(e^s x)\,l_\alpha(dx)\,ds,\end{aligned}$$

using that $\eta_\alpha(dx) = r_\alpha(x)l_\alpha(dx)$ (cf. (2.3) and the discussion there).

Also, by applying Lemma 3.6 (ii), we obtain that

$$\int_{\mathbb{D}} C(v)\,\pi(dv) := \int_{\mathbb{D}} r_\alpha(\widetilde{v})\mathbb{E}_{\boldsymbol{\delta}_v}^{\alpha}\left[|Z|^{\alpha}\mathbf{1}_{\{\tau=\infty\}}\right]\pi(dv) = C, \qquad (5.6)$$

where C is given as in (2.10). Then (5.4), (5.5), and (5.6) imply that

$$\lim_{u\to\infty} u^\alpha \mathbb{E}\left[f\left(\frac{V}{u}\right)\right] = \lim_{u\to\infty} u^\alpha \mathbb{E}\left[\hat{f}\left(\frac{V}{u}\right)\mathbf{1}_{\{|V|\geq u\}}\right] = \frac{C}{\lambda'(\alpha)}\int_{\mathbb{S}_+^{d-1}\times\mathbb{R}_+} e^{-\alpha s} f(e^s x) l_\alpha(dx)ds \tag{5.7}$$

for any bounded, almost θ-Hölder continuous function f satisfying the representation $f(x) = \hat{f}(x)\mathbf{1}_{\{|x|\geq 1\}}$, where $\hat{f}$ is θ-Hölder continuous.

The validity of (5.1) for f with $f(x) = \hat{f}(x)\mathbf{1}_{\{|x|\geq r\}}$, for general $r > 0$, then follows by applying (5.7) to the function $\hat{f}_r(x) = r^{-\alpha} f(rv)$.

Step 2. It remains to remove the assumption that f is almost θ-Hölder continuous, needed to apply Proposition 4.5 in the above argument. To this end, observe that for all $r > 0$,

$$\Upsilon_u^{(r)} := u^\alpha \mathbb{P}\left(\frac{V}{u} \in \cdot, \frac{|V|}{u} \geq r\right)$$

defines a family of uniformly bounded measures on $\mathbb{R}_+^d \setminus B_r^+(0)$, where the boundedness follows by employing (5.1) with $f(x) = \mathbf{1}_{\{|x|\geq r\}}$, which is an almost θ-Hölder continuous function. The Fourier characters $x \mapsto e^{i\langle x,y\rangle}$ are bounded Lipschitz continuous functions for any $y \in \mathbb{R}^d$; then $f_y(x) := e^{i\langle x,y\rangle}\mathbf{1}_{\{|x|\geq r\}}$ is almost θ-Hölder continuous for any $\theta \leq \min\{1, \alpha\}$. Let $\mathfrak{L}_\alpha$ be the measure on $\mathbb{R}_+^d \setminus \{0\}$ defined by the equation

$$\int_{\mathbb{S}_+^{d-1}\times\mathbb{R}} e^{-\alpha s} f(e^s x) l_\alpha(dx)ds = \int_{\mathbb{R}_+^d\setminus\{0\}} f(x)\mathfrak{L}_\alpha(dx),$$

and let $\mathfrak{L}_\alpha^{(r)}$ denote its restriction to a measure on $\mathbb{R}_+^d \setminus B_r^+(0)$. Then, based on what we have proved so far, by considering real and imaginary parts separately we may infer the convergence, as $u \to \infty$, of

$$\int e^{i\langle x,y\rangle}\Upsilon_u^{(r)}(dx) = u^\alpha \mathbb{E}\left[e^{i\langle u^{-1}V,y\rangle}\mathbf{1}_{\{|V|\geq ru\}}\right] \;\to\; \frac{C}{\lambda'(\alpha)}\int e^{i\langle x,y\rangle}\mathfrak{L}_\alpha^{(r)}(dx),$$

for all $y \in \mathbb{R}^d$. Then the Lévy continuity theorem yields the weak convergence $\Upsilon_u^{(r)} \Rightarrow \frac{C}{\lambda'(\alpha)}\mathfrak{L}_\alpha^{(r)}$, for any $r > 0$. Now if $f \in \mathscr{C}_0\big(\mathbb{R}_+^d \setminus \{0\}\big)$, then there exists $r > 0$ such that f is supported on $\big(B_r^+(0)\big)^c$. Hence

$$\lim_{u\to\infty} u^\alpha \mathbb{E}\left[f\Big(\frac{V}{u}\Big)\right] = \lim_{u\to\infty}\int f(x)\Upsilon_u^{(r)}(dx) = \frac{C}{\lambda'(\alpha)}\int f(x)\mathfrak{L}_\alpha(dx),$$

i.e. (5.1) holds. Finally, the equivalence of (5.1) to (5.2) follows from Theorem 2 in [43]. □

5.1. *Smoothing.* To remove Hypothesis (H_3), we employ a lower and upper approximation, where the approximating sequences are smoothed so that (H_3) is satisfied by these sequences.

We begin by constructing the lower approximating sequence. First recall the condition $(\mathfrak{K})$ introduced just prior to the statement of Thereom 2.4. Also, from this discussion in Section 2, recall the definitions

$$\widehat{M}_n := M_{kn} \cdots M_{k(n-1)+1}, \quad \widehat{Q}_n := \sum_{i=k(n-1)+1}^{kn} M_{kn} \cdots M_{i+1} Q_i, \tag{5.8}$$

for all $n \in \mathbb{N}_+$. Now let $k \in \mathbb{N}_+$ be chosen such that $(\mathfrak{K})$ holds. Then $\{(\widehat{M}_n, \widehat{Q}_n)\}$ is an i.i.d. sequence under $\mathbb{P}$, and with positive probability, $\widehat{Q}_1 - s\vec{1} \succ 0$ for some $s > 0$. Let $\mathbb{B}_n = \{\widehat{Q}_n - s\vec{1} \succ 0\}$, and let $\chi_{n,\varepsilon} := (-\varepsilon)\chi_n$ for an i.i.d. sequence $\{\chi_n\}$, independent of $\{(\widehat{M}_n, \widehat{Q}_n)\}$, such that χ_1 has a nondegenerate absolutely continuous distribution concentrated on $[0,1]^d$ (and thus $\chi_{1,\varepsilon}$ is concentrated on $[-\varepsilon, 0]^d$).

For each n, set $\widehat{Q}_{n,\varepsilon} := \widehat{Q}_n + \mathbf{1}_{\mathbb{B}_n}\chi_{n,\varepsilon}$ and note that, conditioned on the event $\mathbb{B}_n$, $\widehat{Q}_{n,\varepsilon}$ has a continuous distribution function. Then, since the event $\mathbb{B}_n$ occurs with positive probability, the distribution function of $\widehat{Q}_{n,\varepsilon}$ has an absolutely continuous component with respect to Lebesgue measure.

Now set

$$V_{n,\varepsilon} = \widehat{M}_n V_{n-1,\varepsilon} + \widehat{Q}_{n,\varepsilon}, \quad n = 1, 2, \ldots; \qquad V_{0,\varepsilon} = V_0. \tag{5.9}$$

Then $\{V_{n,\varepsilon}\}$ forms the *smoothed lower sequence*. Let

$$V_\varepsilon := \widehat{Q}_{1,\varepsilon} + \sum_{k=2}^{\infty} \widehat{M}_1 \cdots \widehat{M}_{k-1} \widehat{Q}_{k,\varepsilon}, \tag{5.10}$$

and note that the law of V_ε is the stationary distribution of the process $\{V_{n,\varepsilon}\}$, which we denote by π_ε.

A *smoothed upper sequence* is constructed analogously, now choosing $\chi_n^\varepsilon := \varepsilon\chi_n$, so that this random variable is concentrated on the interval $[0,\varepsilon]^d$. For each n, let $\widehat{Q}_n^\varepsilon = \widehat{Q}_n + \mathbf{1}_{\mathbb{B}_n}\chi_n^\varepsilon$. Then set $V_0^\varepsilon = V_0$ and

$$V_n^\varepsilon = \widehat{M}_n V_{n-1}^\varepsilon + \widehat{Q}_n^\varepsilon, \quad n = 1, 2, \ldots; \qquad V^\varepsilon = \widehat{Q}_1^\varepsilon + \sum_{k=2}^{\infty} \widehat{M}_1 \cdots \widehat{M}_{k-1} \widehat{Q}_k^\varepsilon.$$

Let π^ε denote the distribution of V^ε.

REMARK 5.3. At this stage, it should be emphasized that this smoothing construction only affects the random quantity $\widehat{Q}_n$, and not $\widehat{M}_n$, and so the function Λ is unchanged. Thus, in particular, the solution α to the equation $\Lambda(\alpha) = 0$ and the corresponding invariant function r_α and invariant measure l_α are the *same* as for the unsmoothed process. Moreover, since $\widehat{M}_1 = M_k \cdots M_1$ and $\lambda(\alpha) = 1$, the factor $\lambda'(\alpha)$ must now be replaced with $k\lambda'(\alpha)$; cf. Lemma 2.3.

REMARK 5.4. Observe that if $k > 1$ in $(\mathfrak{K})$, then the evolution of the lower and upper smoothed sequences *cannot* be compared to the dynamics of the process $\{V_n\}$, but to that of the k-step chain $\{V_{kn} : n \in \mathbb{N}\}$, which at time n is equal to

$$\widehat{V}_n := \widehat{M}_n \cdots \widehat{M}_1 V_0 + \sum_{i=1}^{n} \widehat{M}_n \cdots \widehat{M}_{i+1} \widehat{Q}_i.$$

We then have the sandwich inequality

$$V_{n,\varepsilon} \le \widehat{V}_n \le V_n^\varepsilon, \quad \text{where} \quad \widehat{V}_n = V_{kn}.$$

For the remainder of this section, we consider the k-step chain $\{\widehat{V}_n\}$, defined in terms of $\{(\widehat{M}_i, \widehat{Q}_i)\}$. This k-step chain has the same stationary law, but different dynamics, than the 1-step chain $\{V_n\}$.

For any $x \in \mathbb{S}_+^{d-1}$ and $F \subset \mathbb{S}_+^{d-1}$, let $d(x, F) = \inf\{|x - y| : y \in F\}$; and for a given set $E \subset \mathbb{S}_+^{d-1}$, let

$$E^\varepsilon = \left\{ x \in \mathbb{S}_+^{d-1} : d(x, E) \le \frac{2\varepsilon}{s} \right\} \text{ and } E_\varepsilon = \left\{ x \in \mathbb{S}_+^{d-1} : d(x, E^c) > \frac{2\varepsilon}{s} \right\}.$$

LEMMA 5.5. *Let $\varepsilon > 0$. Then under the assumptions of Theorem 2.4:*

(i) *The approximating sequences $\{V_{n,\varepsilon}\}_{n\in\mathbb{N}}$ and $\{V_n^\varepsilon\}_{n\in\mathbb{N}}$ each satisfy Hypothesis (H_3).*

(ii) *For all $u > 0$, we have the sandwich inequality*

$$\mathbb{P}\left(|V_\varepsilon| > u, \frac{V_\varepsilon}{|V_\varepsilon|} \in E_\varepsilon \right) \le \mathbb{P}\left(|V| > u, \frac{V}{|V|} \in E \right) \le \mathbb{P}\left(|V^\varepsilon| > u, \frac{V^\varepsilon}{|V^\varepsilon|} \in E^\varepsilon \right).$$

PROOF. (i) To verify part (i) of (H_3), let P_ε denote the transition kernel of the process $\{V_{n,\varepsilon}\}$ in (5.9). Recall that χ_ε is independent of $\widehat{M}$ and $\widehat{Q}$. Hence, by construction,

$$P_\varepsilon(v, E) = \mathbb{P}\left(\widehat{M}v + \widehat{Q} \in E, \mathbb{B}^c \right) + \int_{[-\varepsilon,0]^d} \mathbb{P}\left(\widehat{M}v + \widehat{Q} + y \in E, \mathbb{B} \right) \mathbb{P}\left(\chi_\varepsilon \in dy \right)$$

$$:= P_{1,\varepsilon}(v, E) + P_{2,\varepsilon}(v, E).$$

The kernel $P_{2,\varepsilon}$ is obtained by the convolution of $\mathbb{P}\left(\widehat{M}v + \widehat{Q} \in \cdot, \mathbb{B}\right)$ with the probability measure $\mathbb{P}(\chi_\varepsilon \in \cdot)$, which, by assumption, is smooth; thus $P_{2,\varepsilon}(v, \cdot)$ itself has a Lebesgue density for all $v \in \mathbb{R}^d_+$. Hence part (i) of (H_3) is satisfied with Φ taken to be Lebesgue measure and $F = \mathbb{R}^d_+$.

Since π_ε is the stationary distribution of the Markov chain with transition kernel P_ε, it follows that π_ε has a continuous component with respect to Lebesgue measure. Hence $(\operatorname{supp} \pi_\varepsilon)^\circ \neq \emptyset$ and part (ii) of (H_3) is satisfied.

The verification for the process $\{V^\varepsilon_n\}$ is analogous.

(ii) By construction,

$$V - V_\varepsilon = -\mathbf{1}_{\mathbb{B}_1}\chi_{1,\varepsilon} - \sum_{k=1}^{\infty} \widehat{M}_1 \cdots \widehat{M}_k \mathbf{1}_{\mathbb{B}_{k+1}}\chi_{k+1,\varepsilon}, \tag{5.11}$$

since $\widehat{Q}_k - \widehat{Q}_{k,\varepsilon} = \mathbf{1}_{\mathbb{B}_k}\chi_\varepsilon$. [Here we define $\mathbb{B}_k$ in the same way as $\mathbb{B}$, but with respect to the pair $(\widehat{M}_k, \widehat{Q}_k)$.] Consequently, setting $\widehat{M}_0$ to be equal to the identity matrix and recalling that χ_ε is supported on $[-\varepsilon, 0]^d$, we obtain that

$$|V - V_\varepsilon| \;\leq\; \varepsilon \Big| \sum_{k=0}^{\infty} \left(\widehat{M}_0 \cdots \widehat{M}_k \vec{1}\right) \mathbf{1}_{\mathbb{B}_{k+1}} \Big|. \tag{5.12}$$

Moreover,

$$|V| = \Big| \sum_{k=0}^{\infty} \widehat{M}_0 \cdots \widehat{M}_k \widehat{Q}_{k+1} \Big| \geq s \Big| \sum_{k=0}^{\infty} \left(\widehat{M}_0 \cdots \widehat{M}_k \vec{1}\right) \mathbf{1}_{\mathbb{B}_{k+1}} \Big|. \tag{5.13}$$

This implies that

$$\left| \frac{V}{|V|} - \frac{V^\varepsilon}{|V^\varepsilon|} \right| \leq \frac{1}{|V|} |V - V^\varepsilon| + |V^\varepsilon| \left| \frac{1}{|V|} - \frac{1}{|V^\varepsilon|} \right| \leq 2\frac{|V - V^\varepsilon|}{|V|} \leq \frac{2\varepsilon}{s}. \tag{5.14}$$

Hence

$$\frac{V_\varepsilon}{|V_\varepsilon|} \in E_\varepsilon \Rightarrow d\left(\frac{V_\varepsilon}{|V_\varepsilon|}, E^c\right) > \frac{2\varepsilon}{s} \Rightarrow d\left(\frac{V}{|V|}, E^c\right) > 0 \Rightarrow \frac{V}{|V|} \in E.$$

Furthermore, by (5.11), we also have that $|V_\varepsilon| > u \Rightarrow |V| > u$. Consequently,

$$\mathbb{P}\left(|V_\varepsilon| > u,\ \frac{V_\varepsilon}{|V_\varepsilon|} \in E_\varepsilon\right) \;\leq\; \mathbb{P}\left(|V| > u,\ \frac{V}{|V|} \in E\right). \tag{5.15}$$

The remaining inequality of part (ii) is established analogously. □

Since Hypothesis (H_3) is satisfied for the two approximating sequences in Lemma 5.5, it is natural to apply Proposition 5.2 to these sequences, yielding upper and lower bounds for $\mathbb{P}\left(|V| > u,\, V/|V| \in E\right)$ as $u \to \infty$.

Let $\left|\widehat{Z}_\varepsilon\right|$ be defined the same as the random variable $|Z|$ in Section 3.2, but with respect to the process $\left\{(\widehat{M}_i, \widehat{Q}_{i,\varepsilon}) : i = 1, 2, \dots\right\}$; namely,

$$\left|\widehat{Z}_\varepsilon\right| = |v| + \sum_{i=1}^{\infty} \frac{\langle \widehat{Y}_i, (\widehat{Q}_{i,\varepsilon})^\sim \rangle}{\langle \widehat{Y}_i, \widehat{X}_i \rangle} \frac{\left|\widehat{Q}_{i,\varepsilon}\right|}{\left|\widehat{M}_i \cdots \widehat{M}_1 \widetilde{v}\right|} \qquad \mathbb{P}^\alpha_{\boldsymbol{\delta}_v}\text{-a.s.}, \tag{5.16}$$

where

$$\widehat{Y}_i := \lim_{n\to\infty} \left(\widehat{M}_i^\top \cdots \widehat{M}_n^\top \vec{1}\right)^\sim, \quad n = 1, 2, \dots,$$

and $\widehat{X}_i = \left(\widehat{M}_i \cdots \widehat{M}_1 v\right)^\sim$. Then, with $E = \mathbb{S}_+^{d-1}$, we obtain by Proposition 5.2, Lemma 5.5, and Remark 5.3 that

$$\frac{C_\varepsilon}{\alpha k \lambda'(\alpha)} \le u^\alpha \liminf_{u\to\infty} \mathbb{P}\left(|V| > u\right) \;\le\; \limsup_{u\to\infty} u^\alpha \mathbb{P}\left(|V| > u\right) \;\le\; \frac{C^\varepsilon}{\alpha k \lambda'(\alpha)},$$

where, in view of Lemma 3.6 (ii), we have

$$C_\varepsilon = \int_{\mathbb{D}} r_\alpha(\widetilde{v}) \mathbb{E}^\alpha_{\boldsymbol{\delta}_v} \left[|\widehat{Z}_\varepsilon|^\alpha \mathbf{1}_{\{\tau_\varepsilon = \infty\}} \right] \pi_\varepsilon(dv)$$

and

$$C^\varepsilon = \int_{\mathbb{D}} r_\alpha(\widetilde{v}) \mathbb{E}^\alpha_{\boldsymbol{\delta}_v} \left[|\widehat{Z}^\varepsilon|^\alpha \mathbf{1}_{\{\tau^\varepsilon = \infty\}} \right] \pi^\varepsilon(dv).$$

In what follows, we will generally write $\tau \equiv \tau(\mathbb{D})$ to emphasize the dependence of this quantity on the choice of $\mathbb{D}$. *However, it is important to observe from Proposition 5.2 that C_ε and C^ε are universal constants, not dependent on the choice of $\mathbb{D}$.*

The next lemma shows that these constants converge to the required constant C in (2.10) as $\varepsilon \downarrow 0$.

Lemma 5.6. *Assume the conditions of Theorem 2.4. Then for any set $\mathbb{D} = B_r^+(0)$ with $\pi(\mathbb{D}) > 0$,*

$$C_\varepsilon \nearrow C \quad \text{and} \quad C^\varepsilon \searrow C \quad \text{as} \quad \varepsilon \to 0, \tag{5.17}$$

where C is independent *of the choice of $\mathbb{D}$ and has the representation*

$$C = \int_{\mathbb{D}} r_\alpha(\widetilde{v}) \mathbb{E}^\alpha_{\boldsymbol{\delta}_v} \left[|\widehat{Z}|^\alpha \mathbf{1}_{\{\tau(\mathbb{D}) = \infty\}} \right] \pi(dv). \tag{5.18}$$

PROOF. To establish the result, we will show

$$\lim_{\varepsilon\to 0} C_\varepsilon \geq C(\mathbb{D}) \quad \text{and} \quad \lim_{\varepsilon\to 0} C^\varepsilon \leq C(\mathbb{D}), \tag{5.19}$$

where $C(\mathbb{D})$ represents the quantity on the right-hand side of (5.18). Note that these limits necessarily exist, since C_ε, $-C^\varepsilon$ are monotonically increasing (as follows from the monotonicity, in ε, of V_ε and V^ε and (5.2)). Then (5.19) yields $\lim_{\varepsilon\to 0} C_\varepsilon = C(\mathbb{D}) = \lim_{\varepsilon\to 0} C^\varepsilon$.

STEP 1. We begin by establishing that $\lim_{\varepsilon\to 0} C_\varepsilon \geq C(\bar{\mathbb{D}})$. Set

$$H_\varepsilon(v) = r_\alpha(v)\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\left[|\widehat{Z}_\varepsilon|^\alpha \mathbf{1}_{\{\tau_\varepsilon(\bar{\mathbb{D}})=\infty\}}\right]; \quad H(v) = r_\alpha(v)\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\left[|\widehat{Z}|^\alpha \mathbf{1}_{\{\tau(\bar{\mathbb{D}})=\infty\}}\right].$$

Then we need to show that

$$\liminf_{\varepsilon\to 0} \int_{\bar{\mathbb{D}}} H_\varepsilon(v)\pi_\varepsilon(dv) \;\geq\; \int_{\bar{\mathbb{D}}} H(v)\pi(dv). \tag{5.20}$$

We will prove below that: (i) $H_\varepsilon(v) \uparrow H(v)$ as $\varepsilon \to 0$; and (ii) for all $\varepsilon \geq 0$, the function $v \mapsto H_\varepsilon(v)$ is lower semicontinuous.

Assume that (i) and (ii) hold, and fix $\varepsilon_0 > 0$. Since $H_\varepsilon(v)$ is a monotone increasing sequence as $\varepsilon \downarrow 0$,

$$C_\varepsilon \geq \int_{\bar{\mathbb{D}}} H_{\varepsilon_0}(v)\pi_\varepsilon(dv), \quad \text{for all } \varepsilon \leq \varepsilon_0.$$

As the function $v \mapsto H_{\varepsilon_0}(v)$ is lower semicontinous and bounded from below by 0, and $\pi_\varepsilon \Rightarrow \pi$ (cf. (5.11)), in then follows from the Portmanteau theorem ([49, Theorem 1.3.4 (iv)]) that

$$\liminf_{\varepsilon\to 0} C_\varepsilon \;\geq\; \liminf_{\varepsilon\to 0} \int_{\bar{\mathbb{D}}} H_{\varepsilon_0}(v)\pi_\varepsilon(dv) \;\geq\; \int_{\bar{\mathbb{D}}} H_{\varepsilon_0}(v)\pi(dv).$$

Now let $\varepsilon_0 \to 0$ and use the monotone convergence $H_{\varepsilon_0} \uparrow H$ to infer by the monotone convergence theorem that

$$\liminf_{\varepsilon\to 0} C_\varepsilon \;\geq\; \lim_{\varepsilon_0\to 0} \int_{\bar{\mathbb{D}}} H_{\varepsilon_0}(v)\pi(dv) = \int_{\bar{\mathbb{D}}} H(v)\pi(dv) = C(\bar{\mathbb{D}}).$$

It remains to prove (i) and (ii). In order to obtain (i), observe that $|V_{n,\varepsilon}|$ increases monotonically to $|\widehat{V}_n|$ as $\varepsilon \to 0$. Thus, if the process $\{\widehat{V}_n\}$ enters $\bar{\mathbb{D}}$, then so does $\{V_{n,\varepsilon}\}$, for all $\varepsilon > 0$. Hence, we trivially obtain that $\mathbf{1}_{\{\tau(\bar{\mathbb{D}})=\infty\}} \geq \mathbf{1}_{\{\tau_\varepsilon(\bar{\mathbb{D}})=\infty\}}$, where $\tau(\bar{\mathbb{D}})$, $\tau_\varepsilon(\bar{\mathbb{D}})$ are the first passage times of $\{\widehat{V}_n\}$, $\{V_{n,\varepsilon}\}$ into $\bar{\mathbb{D}}$, respectively. Conversely, observe that if $\tau(\bar{\mathbb{D}}) = \infty$, then

$\mathbf{V} := (\widehat{V}_1, \widehat{V}_2, \ldots) \in \left(\bar{\mathbb{D}}^c\right)^{\mathbb{N}}$, which is open. Now $\mathbf{V}_\varepsilon := (V_{1,\varepsilon}, V_{2,\varepsilon}, \ldots)$ converges to $\mathbf{V}$ a.s. in the product topology (as χ_ε is supported on $[-\varepsilon, 0]^d$). It follows that $\mathbf{V}_\varepsilon \in \left(\bar{\mathbb{D}}^c\right)^{\mathbb{N}}$ for sufficiently small ε. Consequently, $\mathbf{1}_{\{\tau(\bar{\mathbb{D}})=\infty\}} \leq \liminf_{\varepsilon\to 0} \mathbf{1}_{\{\tau_\varepsilon(\bar{\mathbb{D}})=\infty\}}$. Thus we conclude that $\mathbf{1}_{\{\tau(\bar{\mathbb{D}})=\infty\}} = \lim_{\varepsilon\to 0} \mathbf{1}_{\{\tau_\varepsilon(\bar{\mathbb{D}})=\infty\}}$ and, moreover, the convergence is monotone, i.e. $\mathbf{1}_{\{\tau_\varepsilon(\bar{\mathbb{D}})=\infty\}} \uparrow \mathbf{1}_{\{\tau(\bar{\mathbb{D}})=\infty\}}$ as $\varepsilon \to 0$. Furthermore, as $\widehat{Q}_\varepsilon$ increases componentwise to $\widehat{Q}$, we deduce from (5.16) and Lemma 3.6 (ii) that $|\widehat{Z}_\varepsilon| \uparrow |\widehat{Z}|$ as $\varepsilon \to 0$. Also, by Lemma 3.6 (iii), $|\widehat{Z}|^\alpha \mathbf{1}_{\{\tau(\bar{\mathbb{D}})=\infty\}}$ is an integrable upper bound for the family $\left\{|\widehat{Z}_\varepsilon|^\alpha \mathbf{1}_{\{\tau_\varepsilon(\bar{\mathbb{D}})=\infty\}}\right\}_{\varepsilon>0}$, and thus we obtain the monotone convergence $H_\varepsilon(v) \uparrow H(v)$ as $\varepsilon \to 0$, for all $v \in \bar{\mathbb{D}}$.

To obtain (ii), observe that if $v \to \hat{v}$, then $\mathbf{V}(v)$ converges to $\mathbf{V}(\hat{v})$, where $\mathbf{V} := (V_{1,\varepsilon}, V_{2,\varepsilon}, \ldots)$, and by writing $\mathbf{V}(v)$, we emphasize the dependence of this quantity on its initial state. Then by repeating the argument given above, we obtain that $\bar{\mathbb{D}}$ closed $\Rightarrow$ $\mathbf{1}_{\{\tau(\bar{\mathbb{D}},\hat{v})=\infty\}} \leq \liminf_{v\to\hat{v}} \mathbf{1}_{\{\tau(\bar{\mathbb{D}},\hat{v})=\infty\}}$, where, once again, $\tau(\bar{\mathbb{D}}, \cdot)$ denotes the dependence on the initial state. From the representation (5.16), we deduce that the function $v \mapsto \widehat{Z}_\varepsilon(v)$ is continuous a.s. (namely, this series converges a.s. by Lemma 3.6 (i)). Then we may apply Fatou's Lemma and use the continuity of r_α to infer that H_ε is lower semicontinuous.

Step 2. To establish the second inequality in (5.19), we proceed as before, now using (i′) the convergence

$$H^\varepsilon(v) := r_\alpha(v)\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\left[|\widehat{Z}^\varepsilon|^\alpha \mathbf{1}_{\{\tau^\varepsilon(\mathbb{D})=\infty\}}\right] \searrow r_\alpha(v)\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\left[|\widehat{Z}|^\alpha \mathbf{1}_{\{\tau(\mathbb{D})=\infty\}}\right] := H^\circ(v)$$

and (ii′) the upper semicontinuity of $H^\varepsilon(v)$, which follows since we consider now the hitting time of an open set. Furthermore, Lemma 3.6 (iii) yields that $\sup_{v\in\mathbb{D}} H_{\varepsilon_0}(v) \leq B$, for some finite constant B. Then we may apply the Portmanteau theorem ([49, Theorem 1.3.4 (v)]) to infer that

$$\limsup_{\varepsilon\to 0} C^\varepsilon \;\leq\; \limsup_{\varepsilon\to 0} \int_{\mathbb{D}} H^{\varepsilon_0}(v)\pi^\varepsilon(dv) = \int_{\mathbb{D}} H^{\varepsilon_0}(v)\pi(dv),$$

for all $\varepsilon_0 > 0$, and thus, letting $\varepsilon_0 \to 0$ and using (i′),

$$\limsup_{\varepsilon\to 0} C^\varepsilon \;\leq\; \int_{\mathbb{D}} H^\circ(v)\pi(dv) = C(\mathbb{D}).$$

Step 3. It remains to show that if $\mathbb{D} = B_r^+(0)$, where $\pi(\mathbb{D}) > 0$, then, in the first equation in (5.19), we in fact have that $\lim_{\varepsilon\to 0} C_\varepsilon \geq C(\mathbb{D})$. To this end, let $\{r_i\}$ be chosen such that $r_i \uparrow r$ as $i \to \infty$, and set $\mathbb{D}_i = B^+_{r_i}(0)$. If $\{\widehat{V}_n\}$ avoids $\mathbb{D}$, then it also avoids each $\mathbb{D}_i$, so we trivially obtain that

$\mathbf{1}_{\{\tau(\bar{\mathbb{D}}_i)=\infty\}} \geq \mathbf{1}_{\{\tau(\mathbb{D})=\infty\}}$. Conversely, $V_n \in \mathbb{D} \Rightarrow V_n \in \bar{\mathbb{D}}_i$ for sufficiently large i. Thus $\lim_{i\to\infty} \mathbf{1}_{\{\tau(\bar{\mathbb{D}}_i)=\infty\}} = \mathbf{1}_{\{\tau(\mathbb{D})=\infty\}}$. Now $\lim_{\varepsilon\to 0} C_\varepsilon$ is a universal constant, independent of the choice of the set $\bar{\mathbb{D}}$ in (5.19). Consequently, we conclude by (5.19) that

$$\begin{aligned}(5.21)\qquad \lim_{\varepsilon\to 0} C_\varepsilon \geq \lim_{i\to\infty} C(\bar{\mathbb{D}}_i) &= \lim_{i\to\infty} \int_{\bar{\mathbb{D}}_i} r_\alpha(\widetilde{v})\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\left[|\widehat{Z}|^\alpha \mathbf{1}_{\{\tau(\bar{\mathbb{D}}_i)=\infty\}}\right]\pi(dv)\\ &= \int_{\mathbb{D}} r_\alpha(\widetilde{v})\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\left[|\widehat{Z}|^\alpha \mathbf{1}_{\{\tau(\mathbb{D})=\infty\}}\right]\pi(dv) = C(\mathbb{D}),\end{aligned}$$

as required. □

Proof of Theorem 2.4. It follows directly from Proposition 5.2 and Lemmas 5.5 and 5.6 that for any $E \in \mathscr{B}(\mathbb{S}_+^{d-1})$,

$$(5.22)\qquad \liminf_{u\to\infty} u^\alpha \mathbb{P}\left(|V| > tu,\ \frac{V}{|V|} \in E\right) \geq \frac{C}{\alpha k \lambda'(\alpha)} t^{-\alpha} \limsup_{\varepsilon\to 0} l_\alpha(E_\varepsilon)$$

and

$$(5.23)\qquad \limsup_{u\to\infty} u^\alpha \mathbb{P}\left(|V| > tu,\ \frac{V}{|V|} \in E\right) \leq \frac{C}{\alpha k \lambda'(\alpha)} t^{-\alpha} \liminf_{\varepsilon\to 0} l_\alpha(E^\varepsilon).$$

Now if $l_\alpha(\partial E) = 0$, then

$$\limsup_{\varepsilon\to 0} l_\alpha(E_\varepsilon) = \liminf_{\varepsilon\to 0} l_\alpha(E^\varepsilon) = l_\alpha(E).$$

Hence the two bounds coincide and, thus, for all measurable $E \subset \mathbb{S}_+^{d-1}$ with $l_\alpha(\partial E) = 0$,

$$\lim_{u\to\infty} u^\alpha \mathbb{P}\left(|V| > tu,\ \frac{V}{|V|} \in E\right) = \frac{C}{\alpha k \lambda'(\alpha)} t^{-\alpha} l_\alpha(E).$$

By the Portmanteau Theorem, this implies the weak convergence

$$(5.24)\qquad u^\alpha \mathbb{P}\left(|V| > tu,\ \frac{V}{|V|} \in \cdot\right) \ \Rightarrow\ \frac{C}{\alpha\lambda'(\alpha)} t^{-\alpha} l_\alpha(\cdot) \quad \text{as } u \to \infty,$$

for all $t > 0$, which is equivalent to (2.9) by Theorem 2 of [43]. □

6. Proof of Theorem 2.8. Next we turn to the proof of Theorem 2.8. Throughout this section, assume that the set A is a semi-cone.

We begin by identifying the constant appearing in the ruin problem for the random walk $\{(X_n, S_n^A)\}$. For this purpose, define

$$\mathfrak{T}_u^A = \inf\left\{n \in \mathbb{N} : M_n \cdots M_1 \widetilde{V}_0 \in uA\right\}.$$

LEMMA 6.1. *Suppose that* (H_1) *and* (H_2) *are satisfied, and assume that* d_A *is bounded and continuous on* $\mathbb{S}_+^{d-1}$. *Then for all* $v \in \mathbb{R}_+^d \setminus \{0\}$,

$$\lim_{u\to\infty} u^\alpha \mathbb{P}\left(\mathfrak{T}_u^A < \infty \middle| V_0 = v\right) = r_\alpha(\widetilde{v}) \int \frac{e^{-\alpha s}}{r_\alpha^A(x)} \varrho^A(dx, ds) := r_\alpha(\widetilde{v}) D_A, \tag{6.1}$$

where ϱ^A *is the asymptotic overjump distribution of the process* $\{(X_n, S_n^A)\}$.

PROOF. Converting to the α-shifted measure, we obtain that for any $v \in \mathbb{R}_+^d \setminus \{0\}$,

$$u^\alpha \mathbb{P}\big(\mathfrak{T}_u^A < \infty \,|\, V_0 = v\big) = r_\alpha(\widetilde{v}) \mathbb{E}_{\widetilde{v}}^\alpha \left[e^{-\alpha\left(S^A_{\mathfrak{T}_u^A} - \log u\right)} \left(r_\alpha^A\big(X_{\mathfrak{T}_u^A}\big)\right)^{-1} \mathbf{1}_{\{\mathfrak{T}_u^A < \infty\}} \right],$$

using the definitions of S_n^A and r_α^A. To characterize the limit on the right-hand side, use the weak convergence (3.31), which holds due to Kesten's renewal theorem. □

To establish the weak convergence of $\{T_u^A/u^\alpha\}$, the main idea will be to study the excursions of $\{V_n\}$ over cycles eminating from the set $\mathbb{D}$. For this purpose, we introduce the random variables

$$U_i := \max_{\kappa_{i-1} < n \le \kappa_i} V_n^A, \quad i = 1, 2, \ldots,$$

where $V_n^A := V_n/d_A(\widetilde{V}_n)$ and $\{\kappa_i\}$ denote the successive return times to $\mathbb{D}$; that is, $\kappa_i = \inf\{n > \kappa_{i-1} : V_n \in \mathbb{D}\}$, $i \in \mathbb{N}_+$; $\kappa_0 = 0$. For $n \in \mathbb{N}_+$, also set

$$\mathscr{M}_n^U = \max\{U_1, \ldots, U_n\}; \quad \mathscr{M}_n = \max\left\{V_1^A, \ldots, V_n^A\right\}.$$

Recall that $\{T_u^A \le N\} = \{V_n^A > u, \text{ some } n \le N\}$. Thus, $\{\mathscr{M}_n^U > u\}$ describes the event that T_u^A occurs by the random time κ_n, while $\{\mathscr{M}_n > u\}$ describes the event that T_u^A occurs by the deterministic time n.

PROPOSITION 6.2. *Suppose that* (H_1) *and* (H_2) *are satisfied, and suppose that there exists* $m \in \mathbb{N}_+$ *such that* (H_3) *holds for the* m*-skeleton* $\{V_{mn} : n \in \mathbb{N}\}$. *Assume that* $\mathbb{D} \in \mathscr{B}(\mathbb{R}_+^d)$ *is bounded and* $\pi(\mathbb{D}) > 0$, *and suppose that the function* d_A *is bounded and continuous on* $\mathbb{S}_+^{d-1}$. *Then for all* $v \in \mathbb{R}_+^d \setminus \{0\}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\mathscr{M}_n^U \le n^{1/\alpha} u \,\middle|\, V_0 = v\right) = \exp\left\{ - K_A\, \mathbb{E}_{\pi_{\mathbb{D}}}[\tau]\, u^{-\alpha}\right\}, \tag{6.2}$$

where $K_A = C D_A$ *and* C *is given as in* (2.10).

Unless explicitly noted, we assume throughout the rest of this section that $V_0 = v$ for a fixed $v \in \mathbb{R}_+^d \setminus \{0\}$, i.e., $\mathbb{P} = \mathbb{P}_v$.

PROOF. Set $u_n = n^{1/\alpha}u$. Then for any $l \in \mathbb{N}_+$,

$$\sum_{i=1}^{l} \mathbb{P}\left(U_i > u_n\right) - \sum_{1\le i<j\le l} \mathbb{P}\left(U_i > u_n,\, U_j > u_n\right) \le \mathbb{P}\left(\mathscr{M}_l^U > u_n\right) \le \sum_{i=1}^{l} \mathbb{P}\left(U_i > u_n\right). \tag{6.3}$$

Now fix $k \in \mathbb{N}_+$. We begin by calculating $\sum_{i=1}^{l(n)} \mathbb{P}\left(U_i > u_n\right)$, as $n \to \infty$, for the sequence $l(n) = \lfloor n/k \rfloor$. By Eq. (4.2) of Corollary 4.2 and the Markov property, we have that for all $i \in \mathbb{N}_+$ and $w \in \mathbb{R}_+^d \setminus \{0\}$,

$$\begin{aligned}\lim_{n\to\infty} nu^\alpha \mathbb{P}\left(U_i > u_n \,\middle|\, V_{\kappa_{i-1}} = w\right) &= \lim_{n\to\infty} nu^\alpha \mathbb{P}\left(T_u^A < \tau \,|\, V_0 = w\right) \\ &= r_\alpha(\widetilde{w})\mathbb{E}_{\boldsymbol{\delta}_w}^\alpha\left[|Z|^\alpha \mathbf{1}_{\{\tau=\infty\}}\right] \int \frac{e^{-\alpha s}}{r_\alpha^A(x)} \varrho^A(dx, ds) := H(w) = C(w)D_A.\end{aligned} \tag{6.4}$$

Under Hypotheses (H_3), $\{V_{mn} : n \ge 0\}$ is a positive aperiodic Harris chain (Lemma 3.4). Then $\{V_n\}$, and hence $\{V_{\kappa_i}\}$, are positive m-periodic Harris chains (cf. [1, Theorem 8.3.7]), and the invariant measure of $\{V_{\kappa_i}\}$ is $\pi_{\mathbb{D}}$ (cf. Lemma 3.3). If γ_i denotes the law of V_{κ_i}, $i \in \mathbb{N}_+$, then Harris recurrence gives that $|n^{-1}\sum_{i=1}^n \gamma_i - \pi_{\mathbb{D}}|_{\mathrm{TV}} \to 0$ as $n \to \infty$, where $|\cdot|_{\mathrm{TV}}$ denotes the total variation distance; see [37, Theorem 13.3.4]. Set

$$H_n(w) = nu^\alpha \mathbb{P}\left(U_1 > u_n \,|\, V_0 = w\right).$$

Using (4.17) with $h \equiv 1$, we have $\sup\{H_n(w) : w \in \mathbb{D} \setminus \{0\},\, n \in \mathbb{N}\} \le B < \infty$. Now $nu^\alpha \mathbb{P}\left(U_i > u_n\right) = \int_{\mathbb{D}} H_n(w)\gamma_{i-1}(dw)$, and

$$\begin{aligned}&\left|\frac{k}{n}\sum_{i=1}^{\lfloor n/k \rfloor} nu^\alpha \mathbb{P}\left(U_i > u_n\right) - \int_{\mathbb{D}} H(w)\pi_{\mathbb{D}}(dw)\right| \\ &\le \left|\int_{\mathbb{D}} H_n(w)\left(\frac{k}{n}\sum_{i=1}^{\lfloor n/k \rfloor} \gamma_{i-1} - \pi_{\mathbb{D}}\right)(dw) - \int_{\mathbb{D}} \big(H_n(w) - H(w)\big)\pi_{\mathbb{D}}(dw)\right| \\ &\le B\left|\frac{k}{n}\sum_{i=1}^{\lfloor n/k \rfloor} \gamma_{i-1} - \pi_{\mathbb{D}}\right|_{\mathrm{TV}} + \int_{\mathbb{D}} \left|H_n(w) - H(w)\right| \pi_{\mathbb{D}}(dw).\end{aligned} \tag{6.5}$$

The second term tends to zero as $n \to \infty$ by dominated convergence and the fact that $H_n(w) \to H(w)$, by (6.4). Thus the left-hand side of (6.5) tends to zero as $n \to \infty$, and hence, using (6.4),

$$\lim_{n\to\infty} \sum_{i=1}^{\lfloor n/k \rfloor} u^{\alpha} \mathbb{P}\left(U_i > u_n\right) = \frac{1}{k} \int_{\mathbb{D}} H(w) \pi_{\mathbb{D}}(dw) \tag{6.6}$$
$$= \frac{D_A}{k} \int_{\mathbb{D}} C(w) \, \frac{\pi(dw)}{\pi(\mathbb{D})} = \frac{K_A \mathbb{E}_{\pi_{\mathbb{D}}}[\tau]}{k},$$

since $C = \int_{\mathbb{D}} C(w) \pi(dw)$ and $\mathbb{E}_{\pi_{\mathbb{D}}}[\tau] = (\pi(\mathbb{D}))^{-1}$ (by Lemma 3.3). Substituting this equation into (6.3), we deduce that for any $k \in \mathbb{N}_+$,

$$\limsup_{n\to\infty} \mathbb{P}\left(\mathscr{M}^U_{\lfloor n/k \rfloor} > u_n\right) \leq \frac{K_A \mathbb{E}_{\pi_{\mathbb{D}}}[\tau]}{k} u^{-\alpha}. \tag{6.7}$$

Note that the right-hand side is *independent* of $V_0 \in \mathbb{R}^d_+ \setminus \{0\}$, and hence the same calculation yields the asymptotic behavior of the maximum over any block of comparable length; in particular, for $\limsup_{n\to\infty} \mathbb{P}\big(U_{\lfloor jn/k \rfloor + 1}, \ldots, U_{\lfloor (j+1)n/k \rfloor} > u_n \,\big|\, \mathscr{F}_{\kappa_{\lfloor jn/k \rfloor}}\big)$, $j = 0, \ldots, k-1$. Hence, letting $k \to \infty$, we conclude by (6.7) that

$$\limsup_{n\to\infty} \mathbb{P}\left(\mathscr{M}^U_n \leq u_n\right) \leq \left(1 - \frac{K_A \mathbb{E}_{\pi_{\mathbb{D}}}[\tau] u^{-\alpha}}{k}\right)^k \to \exp\Big\{ - K_A \mathbb{E}_{\pi_{\mathbb{D}}}[\tau] u^{-\alpha} \Big\}$$

as $k \to \infty$. Moreover, again using the upper bound provided by Lemma 4.6 (uniform in the initial state), we obtain that for any positive integer k,

$$\limsup_{n\to\infty} \sum_{1 \leq i < j \leq \lfloor n/k \rfloor} \mathbb{P}\left(U_i > u_n, \, U_j > u_n\right) = o\left(\frac{1}{k}\right) \quad \text{as} \quad n \to \infty. \tag{6.8}$$

Finally, using (6.6) and (6.8) in (6.3), we see that we also have

$$\liminf_{n\to\infty} \mathbb{P}\left(\mathscr{M}^U_n \leq u_n\right) \geq \exp\Big\{ - K_A \mathbb{E}_{\pi_{\mathbb{D}}}[\tau] u^{-\alpha} \Big\}. \qquad \square$$

LEMMA 6.3. *Suppose that* (H_1), (H_2), *and* (H_3) *are satisfied and* d_A *is bounded and continuous. Then for any* $\Delta > 0$, *there exists a constant* $\delta > 0$ *such that*

$$\limsup_{n\to\infty} \mathbb{P}\left(\max_{|m-n| < n\delta} |\mathscr{M}_m - \mathscr{M}_n| > n^{1/\alpha} \Delta \right) \leq \Delta. \tag{6.9}$$

PROOF. Let $k \in \mathbb{N}_+$. Then $\mathscr{M}_{n+k} = \max\{\mathscr{M}_n, V_{n+1}^A, \ldots, V_{n+k}^A\}$, and hence

$$\mathscr{M}_n \le \mathscr{M}_{n+k} \le \mathscr{M}_n + \max\{V_{n+1}^A, \ldots, V_{n+k}^A\}.$$

Since $V_n^A := V_n/d_A(\widetilde{V}_n)$, it follows that

$$\max_{|m-n|<n\delta} |\mathscr{M}_m - \mathscr{M}_n| \le b \max\{|V_{\lfloor n-n\delta\rfloor+1}|, \ldots, |V_{\lfloor n+n\delta\rfloor}|\}, \tag{6.10}$$

where $b = \max\{(d_A(x))^{-1} : x \in \mathbb{S}_+^{d-1}\} < \infty$. We now determine the maximum on the right-hand side, conditioned on $\{V_{\lfloor n-n\delta\rfloor} = v\}$. Set

$$m_n = \lfloor n + n\delta\rfloor - (\lfloor n - n\delta\rfloor + 1) \le 2n\delta,$$

and observe that as an upper bound, it is sufficient to study $\mathscr{M}_{m_n}$ conditioned on $\{V_0 = v\}$.

Let $\mathbb{D} \subset \mathbb{R}_+^d$ be chosen such that $\pi(\mathbb{D}^c) \le \Delta/2$, and let $v \in \mathbb{D} \setminus \{0\}$. Since $\mathscr{M}_n \le \mathscr{M}_n^U$ and $m_n \le 2n\delta$, we obtain from Proposition 6.2 (with $A = \{x \in \mathbb{R}_+^d : |x| > 1\}$) that

$$\begin{aligned}\limsup_{n\to\infty}\ \mathbb{P}\big(\mathscr{M}_{m_n} > n^{1/\alpha}\Delta \,\big|\, V_0 = v\big) &\le 1 - \exp\{-K_A\,\mathbb{E}_{\pi_{\mathbb{D}}}[\tau]\cdot 2\delta\Delta^{-\alpha}\}\\ &= 2K_A\,\mathbb{E}_{\pi_{\mathbb{D}}}[\tau]\,\Delta^{-\alpha}t, \quad \text{where } t \in (0,\delta),\end{aligned} \tag{6.11}$$

and the right-hand side is $\le \Delta/2$ when δ is chosen sufficiently small. Note that (6.11) holds for all $v \in \mathbb{D} \setminus \{0\}$. Finally, let γ_n denote the distribution function of $V_{\lfloor n-n\delta\rfloor}$. By the positive Harris recurrence of $\{V_n\}$ (Lemma 3.4), we have that $|\gamma_n - \pi|_{\mathrm{TV}} \to 0$ as $n \to \infty$. Then, using Fatou's lemma, we deduce that

$$\begin{aligned}&\limsup_{n\to\infty} \mathbb{P}\left(\max_{|m-n|<n\delta} |\mathscr{M}_m - \mathscr{M}_n| > n^{1/\alpha}\Delta\right)\\ &\le \limsup_{n\to\infty}\left(\gamma_n(\mathbb{D}^c) + |\gamma_n - \pi|_{\mathrm{TV}} + \int_{\mathbb{D}} \mathbb{P}\left(\mathscr{M}_{m_n} > n^{1/\alpha}\Delta \,\middle|\, V_0 = v\right)\,\pi(dv)\right)\\ &\le \pi(\mathbb{D}^c) + \frac{\Delta}{2}\pi(\mathbb{D}) \ \le\ \Delta.\end{aligned}$$

□

PROOF OF THEOREM 2.8. Assuming that d_A is bounded, the first assertion follows from Corollary 4.2, Eq. (4.2), and the uniformity provided by Lemma 4.6. To remove the assumption that d_A is bounded, see Step 4 below. To establish the remaining assertion, we proceed in four steps.

STEP 1. First assume that d_A is bounded and continuous and that (H_3) is satisfied. We claim that

$$\lim_{n\to\infty} \mathbb{P}\left(\mathscr{M}_n \le n^{1/\alpha}u\right) = e^{-K_A u^{-\alpha}}; \tag{6.12}$$

that is to say, we can transfer the result for maxima over cycles (Proposition 6.2) to the process of running maxima, namely to $\mathscr{M}_n$.

To establish an upper bound for $\limsup_{n\to\infty} \mathbb{P}\left(\mathscr{M}_n \le n^{1/\alpha}u\right)$, observe that, by definition, $\mathscr{M}^U_{N_{\mathbb{D}}(n)}$ corresponds to the value of $\{\mathscr{M}_j\}$ during its last visit to $\mathbb{D}$ in the time interval $[0,n]$. Thus

$$\mathbb{P}\left(\mathscr{M}_n > n^{1/\alpha}u\right) \ge \mathbb{P}\left(\mathscr{M}^U_{N_{\mathbb{D}}(n)} > n^{1/\alpha}u\right). \tag{6.13}$$

To replace the random time $N_{\mathbb{D}}(n)$ by a fixed time, observe by Lemma 3.3 that for all $\delta > 0$,

$$\mathbb{P}\left(\left|\frac{N_{\mathbb{D}}(n)}{n} - \pi(\mathbb{D})\right| \ge \delta\right) \to 0 \quad \text{as } n \to \infty. \tag{6.14}$$

Set $t_n = n\left(\pi(\mathbb{D}) - \delta\right)$ and $\Omega_n = \left\{\left|\left(N_{\mathbb{D}}(n)/n\right) - \pi(\mathbb{D})\right| < \delta\right\}$, and note that $N_{\mathbb{D}}(n) \ge \lfloor t_n \rfloor$ on Ω_n. Then

$$\mathbb{P}\left(\mathscr{M}^U_{N_{\mathbb{D}}(n)} > n^{1/\alpha}u\right) \ge \mathbb{P}\left(\mathscr{M}^U_{\lfloor t_n \rfloor} > n^{1/\alpha}u\right) - \mathbb{P}\left(\Omega_n^c\right). \tag{6.15}$$

Then combining (6.13), (6.14), and (6.15) and applying Proposition 6.2, we conclude that for all $\delta > 0$,

$$\begin{aligned}\liminf_{n\to\infty} \mathbb{P}\left(\mathscr{M}_n > n^{1/\alpha}u\right) &\ge \lim_{n\to\infty} \mathbb{P}\left(\mathscr{M}^U_{\lfloor t_n \rfloor} > n^{1/\alpha}u\right) \\ &= 1 - \exp\Big\{ - K_A \mathbb{E}_{\pi_{\mathbb{D}}}[\tau]\left(\pi(\mathbb{D}) - \delta\right) u^{-\alpha}\Big\}.\end{aligned}$$

Hence, letting $\delta \downarrow 0$ and recalling that $\pi(\mathbb{D}) = \left(\mathbb{E}_{\pi_{\mathbb{D}}}[\tau]\right)^{-1}$ (Lemma 3.3), we obtain that

$$\limsup_{n\to\infty} \mathbb{P}\left(\mathscr{M}_n \le n^{1/\alpha}u\right) \le \exp\{-K_A u^{-\alpha}\}. \tag{6.16}$$

To establish the corresponding lower bound for $\mathbb{P}\left(\mathscr{M}_n \le n^{1/\alpha}u\right)$, observe that for any $\Delta > 0$,

$$\begin{aligned}\mathbb{P}\left(\mathscr{M}_n > n^{1/\alpha}u\right) \le{}& \mathbb{P}\left(\mathscr{M}^U_{N_{\mathbb{D}}(n)} > n^{1/\alpha}(u-\Delta)\right) \\ &+ \mathbb{P}\left(\left|\mathscr{M}_n - \mathscr{M}^U_{N_{\mathbb{D}}(n)}\right| > n^{1/\alpha}\Delta\right).\end{aligned} \tag{6.17}$$

Reasoning as before, we see that the first term on the right-hand side satisfies

$$\limsup_{n\to\infty} \mathbb{P}\left(\mathscr{M}^U_{N_{\mathbb{D}}(n)} > n^{1/\alpha}(u-\Delta)\right) \le 1 - \exp\left\{-K_A(u-\Delta)^{-\alpha}\right\}. \tag{6.18}$$

Now to quantify the second term on the right-hand side of (6.17), recall that $\mathscr{M}^U_{N_{\mathbb{D}}(n)}$ is the value of the process $\{\mathscr{M}_j\}$ during its last visit to $\mathbb{D}$ in the interval $[0,n]$. Since κ_i denotes the time of the i^{th} visit to $\mathbb{D}$, it follows by definition that $\mathscr{M}^U_{N_{\mathbb{D}}(n)} = \mathscr{M}_{\kappa_{N_{\mathbb{D}}(n)}}$. Moreover, for any $\delta > 0$, we obtain by Lemma 3.3 that

$$\mathbb{P}\left(\Big|\frac{\kappa_{N_{\mathbb{D}}(n)}}{n} - 1\Big| \geq \delta\right) \to 0 \quad \text{as } n \to \infty.$$

Set $\widehat{\Omega}_n = \big\{\big|\big(\kappa_{N_{\mathbb{D}}(n)}/n\big) - 1\big| < \delta\big\}$. Then

$$\begin{aligned}\mathbb{P}\left(\Big|\mathscr{M}_n - \mathscr{M}^U_{N_{\mathbb{D}}(n)}\Big| > n^{1/\alpha}\Delta\right) &\leq \mathbb{P}\left(\Big|\mathscr{M}_n - \mathscr{M}^U_{N_{\mathbb{D}}(n)}\Big| > n^{1/\alpha}\Delta;\, \widehat{\Omega}_n\right) + \mathbb{P}\big(\widehat{\Omega}_n^c\big)\\ &\leq \mathbb{P}\left(\max_{|m-n|<n\delta} |\mathscr{M}_m - \mathscr{M}_n| > n^{1/\alpha}\Delta\right) + o(1)\end{aligned}$$

as $n \to \infty$. Hence by Lemma 6.3,

$$\limsup_{n\to\infty} \mathbb{P}\left(\Big|\mathscr{M}_n - \mathscr{M}^U_{N_{\mathbb{D}}(n)}\Big| > n^{1/\alpha}\Delta\right) \leq \Delta. \tag{6.19}$$

Finally, substituting (6.18) and (6.19) into (6.17) and letting $\Delta \to 0$, we conclude that $\liminf_{n\to\infty} \mathbb{P}\left(\mathscr{M}_n \leq n^{1/\alpha}u\right) \;\geq\; \exp\big\{-K_A u^{-\alpha}\big\}$. Together with (6.16), the assertion follows.

Step 2. Next we remove the additional assumption (H_3), but still assume that the function d_A is bounded and continuous.

To remove (H_3), we employ the smoothing argument introduced in Section 5. Let $\{(\widehat{M}_n, \widehat{Q}_n : n = 0,1,\ldots\}$ be defined as in (5.8). Then, since we are assuming here that $k = 1$ in $(\mathfrak{K})$, it follows that $(\widehat{M}_n, \widehat{Q}_n) = (M_n, Q_n)$ for all $n \in \mathbb{N}_+$. This yields $V_{n,\varepsilon} \leq V_n \leq V_n^\varepsilon$ for all $n \in \mathbb{N}_+$.

By repeating the computation leading to (5.14), we obtain

$$\Big|\widetilde{V}_{n,\varepsilon} - \widetilde{V}_n\Big| \leq \frac{2\varepsilon}{s} \quad \text{and} \quad \Big|\widetilde{V}_n^\varepsilon - \widetilde{V}_n\Big| \leq \frac{2\varepsilon}{s} \qquad \text{for all } n \in \mathbb{N}_+.$$

Since d_A is assumed to be continuous on the compact set $\mathbb{S}_+^{d-1}$, it is, in fact, equicontinous. Hence there is a sequence $\delta(\varepsilon)$, tending to zero as $\varepsilon \to 0$, such that

$$\Big|d_A\big(\widetilde{V}_{n,\varepsilon}\big) - d_A\big(\widetilde{V}_n\big)\Big| \leq \delta(\varepsilon) \text{ and } \Big|d_A\big(\widetilde{V}_n^\varepsilon\big) - d_A\big(\widetilde{V}_n\big)\Big| \leq \delta(\varepsilon) \quad \text{for all } n \in \mathbb{N}_+.$$

Since $A \subset \{v : |v| > 1\} \Rightarrow d_A > 1$, we have $|V_n| \leq u d_A\big(\widetilde{V}_n\big) \Rightarrow$

$$|V_{n,\varepsilon}| \leq u\, d_A\big(\widetilde{V}_n\big) \leq u\, d_A\big(\widetilde{V}_{n,\varepsilon}\big) + u\,\delta(\varepsilon) \leq u\, d_A\big(\widetilde{V}_{n,\varepsilon}\big)\big(1 + \delta(\varepsilon)\big).$$

Thus $\mathbb{P}(\mathscr{M}_n \le u) \le \mathbb{P}(\mathscr{M}_{n,\varepsilon} \le u(1+\delta(\varepsilon)))$. Similarly, for all $n \in \mathbb{N}_+$,

$$|V_n^\varepsilon| \le u\, d_A\big(\widetilde{V}_n^\varepsilon\big)\big(1-\delta(\varepsilon)\big) \Rightarrow |V_n| \le u\, d_A\big(\widetilde{V}_n^\varepsilon\big) - u\,\delta(\varepsilon) \le u\, d_A\big(\widetilde{V}_n\big),$$

and we obtain that $\mathbb{P}(\mathscr{M}_n \le u) \ge \mathbb{P}(\mathscr{M}_n^\varepsilon \le u(1-\delta(\varepsilon)))$. Using these upper and lower bounds together with Step 1, we conclude that

$$\begin{aligned}(6.20)\qquad \exp\big\{-K^\varepsilon(u-\delta(\varepsilon))^{-\alpha}\big\} &\le \liminf_{n\to\infty} \mathbb{P}\left(\mathscr{M}_n \le n^{1/\alpha}u\right) \\ &\le \limsup_{n\to\infty} \mathbb{P}\left(\mathscr{M}_n \le n^{1/\alpha}u\right) \le \exp\big\{-K_\varepsilon(u+\delta(\varepsilon))^{-\alpha}\big\},\end{aligned}$$

for constants $K_\varepsilon := C_\varepsilon D_A$ and $K^\varepsilon := C^\varepsilon D_A$, where C_ε and C^ε are given as in Section 5. By Lemma 5.6, this yields (6.12).

Step 3. We now relate the behavior of the maxima to the behavior of the first passage times. Recall that $V_n \in uA \Leftrightarrow |V_n| > u\, d_A(\widetilde{V}_n) \Leftrightarrow |V_n^A| > u$. Hence for all $n \in \mathbb{N}_+$ and all $u > 0$,

$$(6.21)\qquad \mathbb{P}\left(T_u^A \le n\right) = \mathbb{P}\left(|V_i^A| > u, \text{ some } 1 \le i \le n\right) = \mathbb{P}\left(\mathscr{M}_n > u\right).$$

Then by (6.12) and (6.21), $\lim_{n\to\infty} \mathbb{P}\Big(T^A_{n^{1/\alpha}w} \le n\Big) = 1 - e^{-K_A\, w^{-\alpha}}$; and setting $u = n^{1/\alpha}w$ and $z = w^{-\alpha}$ yields

$$(6.22)\qquad \lim_{u\to\infty} \mathbb{P}\left(\frac{T_u^A}{u^\alpha} \le z\right) = 1 - e^{-K_A\, z}, \quad z \ge 0.$$

Step 4. Finally suppose that $\mathfrak{P}_A := \{x \in \mathbb{S}_+^{d-1} : d_A(x) < \infty\} \ne \mathbb{S}_+^{d-1}$. For any $L \ge 1$, set

$$\mathscr{K}_L = \{w \in \mathbb{R}_+^d : |w| \ge L\} \quad \text{and} \quad A_L = A \cup \mathscr{K}_L.$$

First observe that $d_{\mathscr{K}_L}(x) := \inf\{t : tx \in \mathscr{K}_L\} = L$, $\forall x \in \mathbb{S}_+^{d-1}$. Hence, letting $r_\alpha^{\mathscr{K}_L}$ be defined as in (2.18), we have that $r_\alpha^{\mathscr{K}_L}(x) = L^\alpha r_\alpha(x) \uparrow \infty$ as $L \to \infty$ (uniformly in x, by Lemma 2.2). Now in general, the constant K_A is proportional to D_A, where the latter constant was characterized in Lemma 6.1. Using this characterization, we see that $r_\alpha^{\mathscr{K}_L}(x) \uparrow \infty$, $\forall x \Rightarrow D_A^{\mathscr{K}_L} \downarrow 0$ as $L \to \infty$. Consequently,

$$(6.23)\qquad \Delta(L) := \lim_{u\to\infty} \mathbb{P}\left(\frac{T_u^{\mathscr{K}_L}}{u^\alpha} \le z\right) \searrow 0 \quad \text{as} \quad L \to \infty.$$

Since

$$\left|\mathbb{P}\left(\frac{T_u^{A_L}}{u^\alpha} \le z\right) - \mathbb{P}\left(\frac{T_u^{A}}{u^\alpha} \le z\right)\right| \le \mathbb{P}\left(\frac{T_u^{\mathscr{K}_L}}{u^\alpha} \le z\right),$$

we conclude that for all $z \geq 0$,

$$\lim_{u\to\infty} \mathbb{P}\left(\frac{T_u^{A_L}}{u^\alpha} \leq z\right) - \Delta(L) \leq \liminf_{u\to\infty} \mathbb{P}\left(\frac{T_u^{A}}{u^\alpha} \leq z\right)$$
$$\leq \limsup_{u\to\infty} \mathbb{P}\left(\frac{T_u^{A}}{u^\alpha} \leq z\right) \leq \lim_{u\to\infty} \mathbb{P}\left(\frac{T_u^{A_L}}{u^\alpha} \leq z\right) + \Delta(L).$$

Thus, by (6.23) and Step 3,

$$\lim_{u\to\infty} \mathbb{P}\left(\frac{T_u^{A}}{u^\alpha} \leq z\right) = 1 - \lim_{L\to\infty} \exp\big\{-(CD_{A_L})z\big\} := 1 - \exp\big\{-(CD_A)z\big\}.$$

Observe that $D_A := \lim_{L\to\infty} D_{A_L}$ exists, since $D_{A_L} = u^\alpha \mathbb{P}\left(\mathfrak{T}_u^{A_L} < \infty \,|\, V_0 \sim \pi_{\mathbb{D}}\right)$ is a decreasing sequence; namely, it represents the hitting probability of a decreasing sequence of sets.

It remains to identify D_A as the ruin constant in this case. Arguing as before, we have that for all $u > 0$,

$$\left| u^\alpha \mathbb{P}\left(\mathfrak{T}_u^{A_L} < \infty \,|\, V_0 = v\right) - u^\alpha \mathbb{P}\left(\mathfrak{T}_u^{A} < \infty \,|\, V_0 = v\right) \right| \leq u^\alpha \mathbb{P}\left(\mathfrak{T}_u^{\mathscr{K}_L} < \infty \,|\, V_0 = v\right),$$

which tends to zero as $L \to \infty$. Thus, by another sandwich argument,

$$\lim_{u\to\infty} u^\alpha \mathbb{P}\left(\mathfrak{T}_u^{A} < \infty \,|\, V_0 = v\right) = \lim_{L\to\infty} \left(\lim_{u\to\infty} u^\alpha \mathbb{P}\left(\mathfrak{T}_u^{A_L} < \infty \,|\, V_0 = v\right) \right)$$
$$= r_\alpha(\widetilde{v}) \lim_{L\to\infty} D_{A_L} = r_\alpha(\widetilde{v}) D_A,$$

which gives the required identification of D_A as the constant in the ruin problem for the Markov random walk; cf. Lemma 6.1.

To conclude the proof, observe that the same reasoning yields (2.22) for unbounded functions d_A; namely, one can again introduce the family $A_L = A \cup \mathscr{K}_L$ for $L \geq 1$, and argue that the hitting probability of the set $\mathscr{K}_L$—now prior to the return time τ—becomes asymptotically negligible as $L \to \infty$. The argument is entirely identical, so we omit the details. □

7. Determining the path of large exceedance. We conclude by providing the proofs of Theorems 2.11 and 2.12. [For a stronger version of Theorem 2.11, allowing paths of infinite length, see the arXiv version [16].]

Proof of Theorem 2.11. It follows by Theorem 2.8 that

$$\lim_{u\to\infty} u^\alpha \mathbb{E}_v\Big[\mathbf{1}_{\{T_u^A < \tau\}}\Big] = C(v) \int \frac{e^{-\alpha s}}{r_\alpha^A(x)} \varrho^A(dx, ds) = C(v) D_A,$$

where $C(v) = r_\alpha(\widetilde{v})\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\left[|Z|^\alpha \mathbf{1}_{\{\tau=\infty\}}\right]$ and D_A is given as in (2.20). Thus, to establish the result, it suffices to show that

$$\lim_{u\to\infty} u^\alpha \mathbb{E}_v\left[g\Big(\frac{V_{I_u}}{|V_{I_u}|},\dots,\frac{V_{I_u+m}}{|V_{I_u}|}\Big)\mathbf{1}_{\{T^A_u<\tau\}}\right] = C(v)D_A\int \mathbb{E}^\alpha_x\Big[g\big(X_0,e^{S_1}X_1,\dots,e^{S_m}X_m\big)\Big]\varrho(dx,ds), \tag{7.1}$$

To verify (7.1), proceed as in the proof of Proposition 4.1, first converting to the α-shifted measure to obtain that

$$\begin{aligned} u^\alpha \mathbb{E}_v\Big[g\Big(\frac{V_{I_u}}{|V_{I_u}|},\dots,\frac{V_{I_u+m}}{|V_{I_u}|}\Big)\mathbf{1}_{\{T^A_u<\tau\}}\Big] &= u^\alpha r_\alpha(\widetilde{v})\,\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\left[\frac{e^{-\alpha S_{T^A_u}}}{r_\alpha(X_{T^A_u})}g\Big(\frac{V_{I_u}}{|V_{I_u}|},\dots,\frac{V_{I_u+m}}{|V_{I_u}|}\Big)\mathbf{1}_{\{T^A_u<\tau\}}\right] \\ &= r_\alpha(\widetilde{v})\,\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\Big[|Z_{T^A_u}|^\alpha \mathfrak{G}_u \mathbf{1}_{\{T^A_u<\tau\}}\Big], \end{aligned} \tag{7.2}$$

where $Z_n := |V_n|/e^{S_n}$, $n=0,1,\dots$, and

$$\mathfrak{G}_u := \frac{1}{r^A_\alpha(X_{T^A_u})}\left(\frac{|V^A_{T^A_u}|}{u}\right)^{-\alpha}\left(\frac{d_A(X_{T^A_u})}{d_A(\widetilde{V}_{T^A_u})}\right)^{\alpha} g\Big(\frac{V_{I_u}}{|V_{I_u}|},\dots,\frac{V_{I_u+m}}{|V_{I_u}|}\Big). \tag{7.3}$$

[The term $\Big(d_A(X_{T^A_u})/d_A(\widetilde{V}_{T^A_u})\Big)^\alpha$ arises when replacing $|V_{T^A_u}|^{-\alpha}/r_\alpha(X_{T^A_u})$ with $|V^A_{T^A_u}|^{-\alpha}/r^A_\alpha(X_{T^A_u})$, as can be seen directly from the definitions (2.18) and (3.32).] The right-hand side of (7.2) can be written as the difference of two terms, namely

$$\begin{aligned} r_\alpha(\widetilde{v})\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\left[\Big(\big|Z_{T^A_u}\big|^\alpha \mathbf{1}_{\{T^A_u<\tau\}} - |Z_n|^\alpha \mathbf{1}_{\{n\le T^A_u\}}\mathbf{1}_{\{n\le\tau\}}\Big)\mathfrak{G}_u\right] \\ +\ r_\alpha(\widetilde{v})\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\Big[|Z_n|^\alpha \mathbf{1}_{\{n\le T^A_u\}}\mathbf{1}_{\{n\le\tau\}}\,\mathbb{E}^\alpha\big[\mathfrak{G}_u\,|\mathscr{F}_n\big]\Big]. \end{aligned} \tag{7.4}$$

As in the proof of Proposition 4.1, we may then apply Lemma 4.3 (i) and use the uniform boundedness of $\{\mathfrak{G}_u\}$ to conclude that the first term in (7.4) tends to zero as $u\to\infty$ and then $n\to\infty$.

Thus, it suffices to analyze the second term in (7.4). Reasoning again as in the proof of Proposition 4.1, it suffices to show that

$$\lim_{u\to\infty}\mathbb{E}^\alpha\left[\mathfrak{G}_u|\mathscr{F}_n\right] = D_A\lim_{u\to\infty}\mathbb{E}^\alpha_{X_n,V_n}\left[g\Big(\frac{V_{I_u}}{|V_{I_u}|},\dots,\frac{V_{I_u+m}}{|V_{I_u}|}\Big)\right]\ \ \mathbb{P}^\alpha\text{-a.s.}. \tag{7.5}$$

Now to establish (7.5), introduce a further conditioning on $\mathscr{F}_{I_u+m}$ inside $\mathbb{E}^\alpha[\mathfrak{G}_u|\mathscr{F}_n]$. Then by the nonlinear renewal theorem (Theorem 3.8) together with Lemma 3.10,

$$D(u) := \mathbb{E}^\alpha_{X_{I_u+m},V_{I_u+m}}\left[\frac{1}{r^A_\alpha(\widetilde{V}_{T^A_u})}\left(\frac{|V^A_{T^A_u}|}{u}\right)^{-\alpha}\left(\frac{d_A(X_{T^A_u})}{d_A(\widetilde{V}_{T^A_u})}\right)^{\alpha}\frac{r^A_\alpha(\widetilde{V}_{T^A_u})}{r^A_\alpha(X_{T^A_u})}\right]\mathbf{1}_{\{I_u+m<T^A_u\}}$$

converges to D_A in $\mathbb{P}^\alpha$-probability. Then on $\{n \le T^A_u\}$,

$$\begin{aligned}\lim_{u\to\infty}\mathbb{E}^\alpha[\mathfrak{G}_u|\mathscr{F}_n] &= \lim_{u\to\infty}\mathbb{E}^\alpha_{X_n,V_n}\left[(D(u)-D_A)\ g\Big(\frac{V_{I_u}}{|V_{I_u}|},\dots,\frac{V_{I_u+m}}{|V_{I_u}|}\Big)\right]\\ &\quad+\lim_{u\to\infty}D_A\,\mathbb{E}^\alpha_{X_n,V_n}\left[g\Big(\frac{V_{I_u}}{|V_{I_u}|},\dots,\frac{V_{I_u+m}}{|V_{I_u}|}\Big)\right]\quad\mathbb{P}^\alpha\text{-a.s.},\end{aligned}$$

and the first term on the right-hand side vanishes by dominated convergence. This completes the proof of (7.5) and hence the theorem. □

Proof of Theorem 2.12. It suffices to show that

$$\limsup_{u\to\infty}u^\alpha\mathbb{E}_v\left[\left|\frac{1}{T^A_u}\sum_{n=1}^{T^A_u}g\left(\log\left(\frac{|V_n|}{|V_{n-1}|}\right)\right)\ -\ \widehat{\mathbb{E}}^\alpha[g(S_1)]\right|\mathbf{1}_{\{T^A_u<\tau\}}\right]=0.$$

For simplicity, introduce the shorthand notation $\mu_g := \widehat{\mathbb{E}}^\alpha[g(S_1)]$ and $\Sigma^n_i := \sum_{j=i}^n g\big(\log|V_j|-\log|V_{j-1}|\big)$.

Let $\{\varepsilon_u\}_{u>0}$ be a sequence such that $\varepsilon_u = o(u)$ and $\varepsilon_u\uparrow\infty$ as $u\to\infty$. Let $\gamma_u = u-\varepsilon_u$ and $J_u = T^A_{\gamma_u}$, and set $B_1 = \max_{x,y}(r_\alpha(x)/r_\alpha(y))$. Then from a change of measure argument, we infer that

$$\begin{aligned}&u^\alpha\mathbb{E}\left[\Big|\frac{1}{T^A_u}\Sigma_1^{T^A_u}-\mu_g\Big|\mathbf{1}_{\{T^A_u<\tau\}}\ \Big|\ V_0=v\right]\\ &= r_\alpha(\widetilde{v})\,\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\left[\frac{e^{-\alpha(S_{T^A_u}-\log u)}}{r_\alpha(X_{T^A_u})}\Big|\frac{1}{T^A_u}\Sigma_1^{T^A_u}-\mu_g\Big|\mathbf{1}_{\{T^A_u<\tau\}}\right]\\ &\le B_1\,\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\left[\Big|\frac{1}{J_u}\Sigma_1^{J_u}-\mu_g\Big|\right]+B_1\,\mathbb{E}^\alpha_{\boldsymbol{\delta}_v}\left[\Big|\frac{1}{J_u}\Sigma_1^{J_u}-\frac{1}{T^A_u}\Sigma_1^{T^A_u}\Big|\right]:=\mathbb{I}_1(u)+\mathbb{I}_2(u).\end{aligned}$$

First consider $\mathbb{I}_2(u)$. Note $|J_u^{-1}\Sigma_1^{J_u}-(T^A_u)^{-1}\Sigma_1^{T^A_u}|$ is bounded above by

$$\left|\frac{1}{J_u}\left(\Sigma_1^{J_u}-\Sigma_1^{T^A_u}\right)\right|+\left|\left(\frac{1}{J_u}-\frac{1}{T^A_u}\right)\Sigma_1^{T^A_u}\right|\ \le\ 2\left|\frac{T^A_u-J_u}{J_u}\right|\cdot|g|_\infty.$$

By Lemma 3.12, $(J_u/\log u) \to (\lambda'(\alpha))^{-1}$ and $\big((T_u^A - J_u)/\log u\big) \to 0$ in $\mathbb{P}^\alpha$-probability. Hence, the term inside the expectation in $\mathbb{I}_2(u)$ tends to zero in $\mathbb{P}^\alpha$-probability and furthermore is bounded above by $2|g|_\infty$. Thus $\limsup_{u\to\infty} \mathbb{I}_2(u) = 0$.

To study $\mathbb{I}_1(u)$, observe that $|J_u^{-1}\Sigma_1^{J_u} - \mu_g|$ is bounded above by

$$\frac{1}{J_u}\sum_{n=1}^{J_u}\left|g\left(\log\left(\frac{|V_n|}{|V_{n-1}|}\right)\right) - g\big(S_n - S_{n-1}\big)\right| \;+\; \left|\frac{1}{J_u}\sum_{n=1}^{J_u} g\left(S_n - S_{n-1}\right) - \mu_g\right|.$$

By Lemma 3.12, $J_u \uparrow \infty$ a.s. as $u \to \infty$. The second term tends to zero $\mathbb{P}^\alpha$-a.s., by [10, Lemma 6.1]. Next, use the Lipschitz continuity of g to infer that for some finite constant B_g, the first term is bounded above by

$$\frac{B_g}{J_u}\sum_{n=1}^{J_u}\left(\Big|\log|V_n| - \log|V_{n-1}| - (S_n - S_{n-1})\Big|\right).$$

Also, it follows directly from the definitions (as given in (2.5) and (3.9)) that

$$\Big|\log|V_n| - \log|V_{n-1}| - (S_n - S_{n-1})\Big| := \Big|\log|Z_n| - \log|Z_{n-1}|\Big|. \tag{7.6}$$

Now by Lemma 3.6, $\{|Z_n|\}$ converges a.s. to the proper random variable $|Z|$ (and thus forms a Cauchy sequence). Then by Césaro's theorem,

$$\limsup_{u\to\infty}\frac{1}{J_u}\sum_{n=1}^{J_u}\Big|\log|Z_n| - \log|Z_{n-1}|\Big| = 0 \quad \text{a.s.}, \tag{7.7}$$

and we conclude by an application of Fatou's lemma that $\limsup_{u\to\infty} \mathbb{I}_1(u) = 0$ $\mathbb{P}^\alpha$-a.s. This establishes (2.28). □

References.

[1] Alsmeyer, G. (1991). *Erneuerungstheorie: Analyse stochastischer Regenerationsschemata*. B.G. Teubner Stuttgart.

[2] Alsmeyer, G. (1997). The Markov renewal theorem and related results. *Markov Proc. Rel. Fields* **3**, 103–127.

[3] Alsmeyer, G. (2003). On the Harris recurrence of iterated random Lipschitz functions and related convergence rate results. *J. Theoret. Probab.* **16**, 217–247.

[4] Alsmeyer, G. (2016). On the stationary tail index of iterated random Lipschitz functions. *Stochastic Process. Appl.* **126**, 209–233.

[5] Asmussen, S. (1982). Conditioned limit theorems relating a random walk to its associate, with applications to risk reserve processes and the GI/G/1 queue. *Adv. Appl. Probab.* **14**, 143–170.

[6] Basrak, B., R. A. Davis, and T. Mikosch (2002). A characterization of multivariate regular variation. *Ann. Appl. Probab.* **12**, 908–920.

[7] Bertoin, J. and R. A. Doney (1994). On conditioning a random walk to stay nonnegative. *Ann. Probab.* **22**, 2152–2167.

[8] Boman, J. and F. Lindskog (2009). Support theorems for the Radon transform and Cramér-Wold theorems. *J. Theoret. Probab.* **22**, 683–710.

[9] Buraczewski, D., J. F. Collamore, E. Damek, and J. Zienkiewicz (2016, 11). Large deviation estimates for exceedance times of perpetuity sequences and their dual processes. *Ann. Probab. 44*(6), 3688–3739.

[10] Buraczewski, D., E. Damek, Y. Guivarc'h, and S. Mentemeier (2014). On multidimensional Mandelbrot cascades. *J. Difference Equ. Appl.* **20**, 1523–1567.

[11] Buraczewski, D., E. Damek, and T. Mikosch (2016). *Stochastic Models with Power Law Tails: the Equation $X = AX + B$*. Springer Berlin.

[12] Buraczewski, D., E. Damek, and J. Zienkiewicz (2016). On the kesten-goldie constant. *Journal of Difference Equations and Applications 22*(11), 1646–1662.

[13] Buraczewski, D. and S. Mentemeier (2016). Precise large deviation results for products of random matrices. *Ann. Inst. H. Poincaré Probab. Statist. 52*(3), 1474–1513.

[14] Collamore, J. F. (2009). Random recurrence equations and ruin in a Markov-dependent stochastic economic environment. *Ann. Appl. Probab.* **19**, 1404–1458.

[15] Collamore, J. F., G. Diao, and A. N. Vidyashankar (2014). Rare event simulation for processes generated via stochastic fixed point equations. *Ann. Appl. Probab.* **24**, 2143–2175.

[16] Collamore, J. F. and S. Mentemeier (2016, August). Large excursions and conditioned laws for recursive sequences generated by random matrices. *ArXiv e-prints*.

[17] Collamore, J. F. and A. N. Vidyashankar (2013a). Large deviation tail estimates and related limit laws for stochastic fixed point equations. In G. Alsmeyer and M. Löwe (Eds.), *Random Matrices and Iterated Random Functions*, pp. 91–117. Berlin: Springer.

[18] Collamore, J. F. and A. N. Vidyashankar (2013b). Tail estimates for stochastic fixed point equations via nonlinear renewal theory. *Stochastic Process. Appl.* **123**, 3378–3429.

[19] de Haan, L., S. Resnick, H. Rootzén, and C. G. de Vries (1989). Extremal behaviour of solutions to a stochastic difference equation with applications to ARCH processes. *Stoch. Process. Appl.* **32**, 213–224.

[20] Dembo, A., S. Karlin, and O. Zeitouni (1994). Large exceedances for multidimensional Lévy processes. *Ann. Appl. Probab.* **4**, 432–447.

[21] Enriquez, N., C. Sabot, and O. Zindy (2009). A probabilistic representation of constants in Kesten's renewal theorem. *Probab. Theory Relat. Fields* **144**, 581–613.

[22] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications* (2nd ed.), Volume II. New York: John Wiley and Sons.

[23] Goldie, C. M. (1991). Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.* **1**, 126–166.

[24] Guivarc'h, Y. (1990). Sur une extension de la notion de loi semi-stable. *Ann. Inst. H. Poincaré Probab. Statist.* **26**, 261–285.

[25] Guivarc'h, Y. and É. Le Page (2016). Spectral gap properties for linear random walks and Pareto's asymptotics for affine stochastic recursions. *Ann. Inst. H. Poincaré Probab. Statist.* **52**, 503–574.

[26] Hennion, H. (1997). Limit theorems for products of positive random matrices. *Ann. Probab.* **25**, 1545–1587.

[27] Iglehart, D. L. (1972). Extreme values in the GI/G/1 queue. *Ann. Math. Statist.* **43**, 627–635.

[28] Kesten, H. (1973). Random difference equations and renewal theory for products of random matrices. *Acta Math.* **131**, 207–248.

[29] Kesten, H. (1974). Renewal theory for functionals of a Markov chain with general

state space. *Ann. Probab.* **2**, 355–386.
[30] Kesten, H., M. V. Kozlov, and F. Spitzer (1975). A limit law for random walk in a random environment. *Compositio Math.* **30**, 145–168.
[31] Leadbetter, M. R. and H. Rootzén (1988). Extremal theory for stochastic processes. *Ann. Probab.* **16**, 431–478.
[32] Letac, G. (1986). A contraction principle for certain Markov chains and its applications. random matrices and their applications. *Proceedings of AMS-IMS-SIAM Joint Summer Research Conference 1984. Contemporary Mathematics* **50**, 263–273.
[33] Liu, Q. (2000). On generalised multiplicative cascades. *Stochastic Process. Appl.* **86**, 263–286.
[34] Melfi, V. F. (1992). Nonlinear Markov renewal theory with statistical applications. *Ann. Probab.* **20**, 753–771.
[35] Melfi, V. F. (1994). Nonlinear renewal theory for Markov random walks. *Stochastic Process. Appl.* **54**, 71–93.
[36] Mentemeier, S. (2013). *On Multivariate Stochastic Fixed Point Equations: The Smoothing Transform and Random Difference Equations.* Ph. D. thesis, Westfälische Wilhelms-Universität Münster.
[37] Meyn, S. and R. Tweedie (1993). *Markov Chains and Stochastic Stability.* Berlin: Springer-Verlag.
[38] Mikosch, T. (2003). Modeling dependence and tails of financial time series. In B. Finkenstädt and H. Rootzén (Eds.), *Extreme Values in Finance, Telecommunications, and the Environment*, pp. 185–286. Boca Raton: Chapman and Hall.
[39] Mirek, M. (2011). Heavy tail phenomenon and convergence to stable laws for iterated Lipschitz maps. *Probab. Theory Relat. Fields* **151**, 705–734.
[40] Nummelin, E. (1984). *General Irreducible Markov Chains and Non–negative Operators.* Cambridge: Cambridge University Press.
[41] Perfekt, R. (1994). Extremal behaviour of stationary Markov chains with applications. *Ann. Appl. Probab.* **4**, 529–548.
[42] Perfekt, R. (1997). Extreme value theory for a class of Markov chains with values in $\mathbb{R}^d$. *Adv. in Appl. Probab.* **29**, 138–164.
[43] Resnick, S. (2004). On the foundations of multivariate heavy-tail analysis. *J. Appl. Probab.* **41A**, 191–212.
[44] Roitershtein, A. (2007). One-dimensional linear recursions with Markov-dependent coefficients. *Ann. Appl. Probab.* **17**, 572–608.
[45] Shurenkov, V. M. (1984). On Markov renewal theory. *Theory Probab. Appl.* **29**, 247–265.
[46] Siegmund, D. (1985). *Sequential Analysis: Tests and Confidence Intervals.* New York: Springer-Verlag.
[47] Solomon, F. (1972). *Random Walks in Random Environment.* Ph.D. dissertation, Cornell University.
[48] Solomon, F. (1975). Random walks in random environment. *Ann. Probab.* **3**, 1–31.
[49] van der Vaart, A. W. and J. A. Wellner (1996). *Weak Convergence and Empirical Processes with Applications to Statistics.* Heidelberg: Springer-Verlag.
[50] Wang, H. M. (2013). A note on multitype branching process with bounded immigration in random environment. *Acta Math. Sin. (Engl. Ser.)* **29**, 1095–1110.

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