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The first-passage time distribution for the diffusion model with variable drift

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Abstract

The Ratcliff diffusion model is now arguably the most widely applied model for response time data. Its major advantage is its description of both response times and the probabilities for correct as well as incorrect responses. The model assumes a Wiener process with drift between two constant absorbing barriers. The first-passage times at the upper and lower boundary describe the responses in simple two-choice decision tasks, for example, in experiments with perceptual discrimination or memory search. In applications of the model, a usual assumption is a varying drift of the Wiener process across trials. This extra flexibility allows accounting for slow errors that often occur in response time experiments. So far, the predicted response time distributions were obtained by numerical evaluation as analytical solutions were not available. Here, we present an analytical expression for the cumulative first-passage time distribution in the diffusion model with normally distributed trial-to-trial variability in the drift. The solution is obtained with predefined precision, and its evaluation turns out to be extremely fast.

Keywords

Diffusion model; Response time modeling

Background

The diffusion model for response times was proposed about 40 years ago (Ratcliff, 1978) as a continuous-time, continuous-state generalization of earlier discrete-time random walk models (Laming, 1968; Link & Heath, 1975). One of its major advantages over standard response time (RT) analyses (i.e., comparison of mean RTs) is the simultaneous analysis of both response time and accuracy. This avoids problems of speed-accuracy trade-offs that are possible confounders of the results and generally difficult to interpret (e.g., Pachella, 1974).

The standard diffusion model assumes a Wiener process with drift v and diffusion coefficient σ^2 (typically fixed either at $\sigma^2 = 1$ or $\sigma^2 = 0.01$ because it only scales the other parameters) evolving over time in the presence of two absorbing barriers (located at 0 and $a > 0$). Each barrier is associated with one response alternative. The barriers can be viewed as response criteria, that is, the distribution of the first passage time to either barrier produces the predicted response times distribution for the response alternative associated with the barrier.

Although the model is well motivated and the approach is appealing, two issues remain that are often seen as major obstacles for a wider application of the model. Firstly, there is no closed-form solution available for the partial differential equation (PDE) of a diffusion process with the necessary boundary conditions. The available solutions (e.g., Feller, 1968) all require the evaluation of infinite series. These series can be shown to converge quite quickly (Navarro & Fuss, 2009; Blurton, Kesselmeier, & Gondan, 2012; Gondan, Blurton, & Kesselmeier, 2014). However, when fitting the model to data, the series has to be evaluated over and over again, which may take a considerable amount of time. This is especially true if more general versions of the model are fitted to data (see next section). In that case, several numerical integrations have to be carried out that are associated with their own (possibly unknown) approximation errors. However, for parameter estimation it is useful to have an exact result to avoid numerical problems during estimation (e.g., rough likelihood surfaces).

Secondly, the available solutions only cover the standard Wiener process with constant drift across trials. By analogy to the signal detection model (Tanner & Swets, 1954) and based on common sense arguments (the "resonance" metaphor), Ratcliff (1978) argued that the drift rate v shows inter-trial variability that can be described by a normal distribution: $v \sim N(\nu, \eta^2)$. For example, one direct consequence of this assumption is that in a response signal paradigm, perceptual sensitivity d' asymptotes and does not reach infinity with signal time t (Ratcliff, 1978, Eq. 10). However, this extra variability comes at the cost of a missing analytical form for the model predictions. Hence, model predictions must be obtained by numerical evaluation instead (Ratcliff & Tuerlinckx, 2002). Interestingly, the *density* function[1] is known for the case of normally distributed drift rates (e.g., Horrocks & Thompson, 2004) and it has been used in the past for fitting the diffusion model to response time data (Ratcliff & Tuerlinckx, 2002; Wiecki, Sofer, & Frank, 2013). For the lower barrier, it is

$$g(t \mid \nu, \eta^2, a, w) = \frac{1}{\sqrt{t^3(1+\eta^2 t)}} \exp\left[\frac{-\nu^2 t - 2\nu a w + \eta^2 (aw)^2}{2(1+\eta^2 t)}\right] \sum_{j=0}^{\infty} (-1)^j \, r_j \, \phi\left(\frac{r_j}{\sqrt{t}}\right) \quad (1)$$

where $r_j = ja + aw$ for even j or $r_j = ja + a(1-w)$ for odd j, and $\phi(x)$ denotes the standard normal density function evaluated at x, and $0 < w < 1$ is the relative starting point of the Wiener process between the two barriers. Without loss of generality the diffusion coefficient σ^2

[1] Note that the distribution (density) is technically not a probability distribution (density) but a defective distribution (density) because it does not integrate to unity. One obtains a proper distribution (density) by summing the distributions (densities) from the upper and lower criteria or by normalizing through the respective absorption probability.

has been omitted in (1), as $g'(t \mid \nu, \eta^2, \sigma^2, a, w) = g(t \mid \nu/\sigma, \eta^2/\sigma^2, a/\sigma, w)$. The density function is useful if maximum likelihood estimation is desired. However, if parameter estimates are to be obtained from binned data, for example by chi-square methods (e.g., Ratcliff & Smith, 2004) or by the quantile maximum likelihood method (Heathcote, Brown, & Mewhort, 2002) one must rely on numerical integration of the first-passage time density to obtain the distribution function.

Since its introduction additional parameters for inter-trial variability have been added to the model (Ratcliff & Rouder, 1998; Ratcliff & Tuerlinckx, 2002). Thus, the “full” Ratcliff diffusion model fit now requires the numerical evaluation of three integrals (see Tuerlinckx, 2004, Eq. 3). This can become time consuming as the computational complexity raises exponentially (Tuerlinckx, 2004) and all these integrals must be evaluated on infinite series.

Here, we present an analytical solution for the first-passage time distribution of the Ratcliff (1978) model with drift variation. The solution is of theoretical interest and especially for applications of the model. For the application, it increases speed and establishes a pre-defined accuracy of the fitting procedure. It is readily available for use in existing software packages like DMAT (Vandekerckhove & Tuerlinckx, 2008). Researchers that have implemented or seek to implement their own fitting routines will also benefit from the solution as it guarantees a computationally efficient computation with accuracy up to some pre-defined level.

The cumulative distribution function for the Ratcliff diffusion model

Recently, Gondan and colleagues (2014) reported a solution of the PDE for a Wiener process with constant drift between two absorbing barriers that is using a representation stated in terms of the Mills ratio (Hall, 1997). We would like to remind the reader of some of the favorable properties of this representation. Firstly, it is numerically very stable and no numerical problems arise during the calculation of the infinite series. Secondly, and contrasting its related representation (e.g., Blurton et al., 2012), it is defined for all real drift rates and does not suffer from a singularity at zero drift. Clearly, this is very important when integrating over drift rates. Thirdly, it gives the distribution function and not the survivor function so that the separate calculation of the overall absorption probability at a specific barrier is not necessary. In the most widely adapted representation of the first-passage time cumulative distribution, the survivor function is used. In that case, the series must be subtracted from the probability of terminating at the associated barrier to obtain the cumulative distribution (see Ratcliff, 1978, Eq. A12 and p. 105f, for the motivation of this approach). Obtaining the cumulative directly avoids problems in the derivation regarding this probability with drift variation over trials (see Tuerlinckx, 2004). Apart from the latter issue, these points also hold for the alternative solution that is available and usually used in fitting the diffusion model (Ratcliff, 1978; Ratcliff & Tuerlinckx, 2002). However, the analytic solution for this CDF with inter-trial variability in drift rates is yet unknown.

Using the aforementioned representation (1), the cumulative distribution function $F(t)$ of the first-passage time of a Wiener process with drift v between two absorbing barriers placed at 0 and $a > 0$ and starting at aw $(0 < w < 1)$ to the lower boundary can be expressed by the infinite series (Hall, 1997)

$$F(t \mid v, a, w) = \exp\left(-vaw - \frac{v^2 t}{2}\right) \sum_{j=0}^{\infty} (-1)^j \phi\left(\frac{r_j}{\sqrt{t}}\right) \left[M\left(\frac{r_j - vt}{\sqrt{t}}\right) + M\left(\frac{r_j + vt}{\sqrt{t}}\right)\right] \quad (2)$$

with r_j and $\phi(x)$ as defined above, and $M(x) = \frac{1-\Phi(x)}{\phi(x)}$ denoting the inverse hazard function (the "Mills ratio") for the standard normal distribution.

In order to obtain a solution for the more general process with trial-to-trial variability in drift rate v, one must seek a solution of the integral $\int \psi(x) \cdot F(t \mid x, a, w)\, dx$, that is, one must integrate over the density $\psi(x)$ of the assumed drift distribution and the first-passage time distribution $F(t)$. Because drift rates can take any real value and due to the correspondence with the signal detection model (Tanner & Swets, 1954), the normal distribution is usually chosen as a possible distribution for the drift rates (Ratcliff, 1978, Eqs. 8, A24, & A25). Thus, we replace $\psi(x)$ by the normal density $\phi(x \mid v, \eta^2)$ with mean v and variance η^2. Let $G(t \mid v, \eta^2, a, w)$ be the first-passage time distribution of such a process,

$$G(t \mid v, \eta^2, a, w) := \int_{-\infty}^{\infty} \phi(x \mid v, \eta^2) \cdot F(t \mid x, a, w)\, dx$$

$$= \int_{-\infty}^{\infty} \phi(x \mid v, \eta^2) \exp\left(-xaw - \frac{x^2 t}{2}\right) \sum_{j=0}^{\infty} (-1)^j \phi\left(\frac{r_j}{\sqrt{t}}\right) \left[M\left(\frac{r_j - xt}{\sqrt{t}}\right) + M\left(\frac{r_j + xt}{\sqrt{t}}\right)\right] dx$$

The series is absolutely convergent (see Appendix A) so that summation and integration can be exchanged and we may write

$$G(t \mid v, \eta^2, a, w) = \sum_{j=0}^{\infty} g_j(t \mid v, \eta^2, a, w)$$

with

$$g_j := (-1)^j \phi\left(\frac{r_j}{\sqrt{t}}\right) \int_{-\infty}^{\infty} \exp\left(-xaw - \frac{x^2 t}{2}\right) \phi(x \mid v, \eta^2) \left[M\left(\frac{r_j - xt}{\sqrt{t}}\right) + M\left(\frac{r_j + xt}{\sqrt{t}}\right)\right] dx.$$

Each term of the series is composed of two summands, so for simplicity let us define

$$g_j^- := (-1)^j \phi\left(\frac{r_j}{\sqrt{t}}\right) \int_{-\infty}^{\infty} \exp\left(-xaw - \frac{x^2 t}{2}\right) \phi(x \mid \nu, \eta^2)\, M\left(\frac{r_j - xt}{\sqrt{t}}\right) dx$$

and

$$g_j^+ := (-1)^j \phi\left(\frac{r_j}{\sqrt{t}}\right) \int_{-\infty}^{\infty} \exp\left(-xaw - \frac{x^2 t}{2}\right) \phi(x \mid \nu, \eta^2)\, M\left(\frac{r_j + xt}{\sqrt{t}}\right) dx.$$

with $g_j = g_j^- + g_j^+$ (we omitted the arguments for notational compactness). We first derive g_j^-. Replacement of Mills ratio and application of $1 - \Phi(x) = \Phi(-x)$ leads to

$$g_j^- = \frac{(-1)^j}{\sqrt{2\pi\eta^2}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-\nu)^2}{2\eta^2} - xaw - \frac{x^2 t}{2}\right] \exp\left(-\frac{r_j^2}{2t}\right) \Phi\left(\frac{xt - r_j}{\sqrt{t}}\right) \exp\left[\frac{\left(r_j - xt\right)^2}{2t}\right] dx.$$

Then, simplification and rearrangement according to powers of x results in

$$g_j^- = \frac{(-1)^j}{\sqrt{2\pi\eta^2}} \int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{2\eta^2} + \left(\frac{\nu}{\eta^2} - aw - r_j\right) x - \frac{\nu^2}{2\eta^2}\right] \Phi\left(x\sqrt{t} - \frac{r_j}{\sqrt{t}}\right)\, dx.$$

For convenience, we define $p := \frac{\nu}{\eta^2} - aw - r_j$. Next, by completing the square one obtains

$$g_j^- = \frac{(-1)^j}{\sqrt{2\pi\eta^2}} \exp\left(-\frac{\nu^2}{2\eta^2} + \frac{\eta^2}{2} p^2\right) \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{x}{\eta} - \eta p\right)^2\right] \Phi\left(x\sqrt{t} - \frac{r_j}{\sqrt{t}}\right)\, dx$$

The required integral is of the form $\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(\delta x-\gamma)^2\right]\Phi(\beta x-\alpha)\,dx$, to which the solution is $\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(\delta x-\gamma)^2\right]\Phi(\beta x-\alpha)\,dx = \frac{\sqrt{2\pi}}{\delta}\left[1-\Phi\left(\frac{\alpha\delta-\gamma\beta}{\sqrt{\beta^2+\delta^2}}\right)\right]$ (see Appendix B). With the obvious correspondence of α, β, γ, and δ, this leads to

$$g_j^- = (-1)^j \exp\left(-\frac{\nu^2}{2\eta^2}+\frac{\eta^2}{2}p^2\right)\Phi\left(\frac{\eta p\sqrt{t}-r_j/\sqrt{\eta^2 t}}{\sqrt{t+1/\eta^2}}\right)$$

$$= (-1)^j \exp\left[\frac{\eta^2}{2}\left(aw+r_j\right)^2-\nu\left(aw+r_j\right)\right]\Phi\left[\frac{\nu t-\eta^2(aw+r_j)t-r_j}{\sqrt{t(1+\eta^2 t)}}\right]$$

Similarly,

$$g_j^+ = (-1)^j \exp\left[\frac{\eta^2}{2}\left(aw-r_j\right)^2-\nu\left(aw-r_j\right)\right]\Phi\left[\frac{\nu t-\eta^2(aw-r_j)t+r_j}{\sqrt{t(1+\eta^2 t)}}\right].$$

By combining the results for g_j^- and g_j^+, we get $g_j(t \mid \nu, \eta^2, a, w)$ of the series $G(t \mid \nu, \eta^2, a, w)$ as the required analytical solution. However, we further develop the result to obtain a representation using the Mills ratio again because of its favorable numerical properties (see above).

$$g_j^- = (-1)^j \exp\left[\frac{\eta^2}{2}\left(aw+r_j\right)^2-\nu\left(aw+r_j\right)\right]\left\{1-\Phi\left[-\frac{\nu t-\eta^2(aw+r_j)t-r_j}{\sqrt{t(1+\eta^2 t)}}\right]\right\}$$

$$= \frac{(-1)^j}{\sqrt{2\pi}}\exp\left[\frac{-\nu^2 t-2\nu a w+\eta^2(aw)^2}{2(1+t\eta^2)}\right]\exp\left[-\frac{r_j^2+\eta^2 t r_j^2}{2t(1+\eta^2 t)}\right]M\left[\frac{r_j-\nu t+\eta^2(aw+r_j)t}{\sqrt{t(1+\eta^2 t)}}\right]$$

$$= (-1)^j \exp\left[\frac{-\nu^2 t-2\nu a w+\eta^2(aw)^2}{2(1+t\eta^2)}\right]\phi\left(\frac{r_j}{\sqrt{t}}\right)M\left[\frac{r_j-\nu t+\eta^2(aw+r_j)t}{\sqrt{t(1+\eta^2 t)}}\right].$$

Similarly, we have

$$g_j^{+} = (-1)^j \exp\left[\frac{-\nu^2 t - 2\nu a w + \eta^2 (aw)^2}{2(1+t\eta^2)}\right] \phi\left(\frac{r_j}{\sqrt{t}}\right) M\left[\frac{r_j + \nu t + \eta^2 (r_j - aw) t}{\sqrt{t(1+\eta^2 t)}}\right].$$

The cumulative distribution function then reads as

$$G(t \mid \nu, \eta^2, a, w) = \exp\left[\frac{-\nu^2 t - 2\nu a w + \eta^2 (aw)^2}{2(1+\eta^2 t)}\right] \times$$

$$\sum_{j=0}^{\infty} (-1)^j \phi\left(\frac{r_j}{\sqrt{t}}\right) \left\{ M\left[\frac{r_j - \nu t + \eta^2 (r_j + aw) t}{\sqrt{t(1+\eta^2 t)}}\right] + M\left[\frac{r_j + \nu t + \eta^2 (r_j - aw) t}{\sqrt{t(1+\eta^2 t)}}\right] \right\}. \qquad (3)$$

This is the analytic result of the model proposed by Ratcliff (1978). The absorption probability at the upper barrier is obtained by $G(t \mid -\nu, \eta^2, a, 1-w)$. For non-unit variance σ^2, $G'(t \mid \nu, \eta^2, \sigma^2, a, w) = G(t \mid \nu/\sigma, \eta^2/\sigma^2, a/\sigma, w)$. The above solution is interesting in several aspects. Firstly, it bears similarities with the already known density function (Eq. 1) and the solution for an unrestricted Wiener process with normally distributed drift (Ratcliff, 1978, Eq. 8). Secondly, for $\eta^2 = 0$, it simplifies to the distribution function $F(t \mid \nu, a, w)$ of a standard Wiener process (Eq. 2) with constant drift ν. In other words, it can be safely used in a fitting routine, regardless of the (empirical) question, whether there is inter-trial variability in the data or not. If no such variation is observed, the function safely converges to the no-variation case.

Convergence

Because the r_j are strictly increasing, and the Mills ratio is strictly decreasing in its argument, the function $F(t \mid \nu, a, w)$ in (2) is a strictly decreasing alternating series (Gondan et al., 2014). A similar argument can be made for (3): Because $G(t \mid \nu, \eta^2, a, w)$ is a weighted sum of different $F(t \mid \nu, a, w)$, it is a strictly decreasing alternating series as well, so that its evaluation can be stopped as soon as the first summand g_J is below some pre-defined error tolerance $\varepsilon > 0$. Then, it is guaranteed that the truncation error—that is, the difference between the true distribution (3) and the truncated series evaluated up to some J—is not greater than the pre-defined tolerance level.

If a reasonable estimate for the number of required terms is known, the precision of the truncated solution is improved (e.g., by aggregating terms in increasing order). The number of required terms can be obtained by solving, for example, $g_{2K} \leq \varepsilon$ for even $J = 2K$. We first note that for sufficiently large r_{2K} (such that the argument of ϕ is positive), a simple upper bound $h_{2K} \geq g_{2K}$ is found with

$$h_{2K} = 2\exp\left[\frac{-\nu^2 t - 2\nu a w + \eta^2 (aw)^2}{2(1+\eta^2 t)}\right] \times \phi\left[\frac{r_{2K} - |\nu| t}{\sqrt{t(1+\eta^2 t)}}\right] M\left[\frac{r_{2K} - |\nu| t}{\sqrt{t(1+\eta^2 t)}}\right]$$

$$= 2\exp\left[\frac{-\nu^2 t - 2\nu a w + \eta^2 (aw)^2}{2(1+\eta^2 t)}\right]\left\{1 - \Phi\left[\frac{r_{2K} - |\nu| t}{\sqrt{t(1+\eta^2 t)}}\right]\right\}$$

The inequality $h_{2K} \leq \varepsilon$ is then solved for $J = 2K$,

$$J \geq \frac{\sqrt{t(1+\eta^2 t)}}{a} \cdot \Phi^{-1}\left\{1 - \frac{1}{2}\exp\left[\frac{\nu^2 t + 2\nu a w - \eta^2 (aw)^2}{2(1+\eta^2 t)} + \log\varepsilon\right]\right\} + \frac{|\nu| t}{a} - w. \quad (4)$$

Positivity of the arguments of ϕ is given for $J \geq \frac{|\nu| t}{a} - w$.

Efficiency

The CDF in (3) can readily be used for parameter estimation in combination with a fitting function that relies on the CDF—such as chi-square methods or the quantile maximum likelihood estimation (Heathcote et al., 2002). Our first analyses using the solution on simulated data showed that it can be readily used with reasonable computational effort (Table 1): The number of terms needed for convergence up to a pre-defined tolerance ε is generally very low. The number of terms mainly depends on the barrier separation parameter a and the time t at which the function is evaluated: Similar to the constant drift case (Eq. 2), larger t and smaller a lead to slower convergence of the series. The other parameters ν, η^2, and w have hardly any influence on the convergence behavior. Because no numerical integration is required, a tolerance of ε of approximately 1.5×10^{-8} seems appropriate (i.e., around the square root of the smallest positive 32 bit floating-point number ε for which 1 is distinguishable from $1 + \varepsilon$). With this tolerance, none of the calculations shown in Table 1 needed more than ten terms to converge. It is also turned out that the upper bound for J (Eq. 4) is overly conservative. In any case, the scenario in Table 1 is rather pessimistic as we assumed decision times up to 1200 ms and $G(t \mid \nu, \eta^2, a, w)$ converges even quicker for lower values of t.

Table 1

Number of terms needed to achieve pre-defined accuracy.

Parameter			Number of terms	
η^2	a	w	J from Eq. 4	Needed
0.01	0.08	.375	15	8
		.500	15	8
	0.11	.375	11	5
		.500	11	6
	0.14	.375	9	5
		.500	9	4
0.04	0.08	.375	23	7
		.500	23	8
	0.11	.375	17	5
		.500	17	5
	0.14	.375	13	4
		.500	13	4
0.09	0.08	.375	31	7
		.500	31	7
	0.11	.375	23	5
		.500	23	5
	0.14	.375	18	4
		.500	18	4

Note—Scaling parameter was set to $\sigma^2 = 0.01$. The table shows the number of terms needed to achieve accuracy $\varepsilon = 1.5 \times 10^{-8}$ at the lower barrier. The mean drift rate was also varied, $\nu \in \{0, \pm 0.1, \pm 0.2, \pm 0.3\}$, and the highest number was chosen. Time t was varied between 0.1 and 1.2 s; the values presented are for evaluation at 1.2 s as lower t generally lead to faster convergence.

Discussion

In this note we presented an analytical solution to the two-barrier diffusion model proposed by Ratcliff (1978). The solution is easily implemented (see online appendix) and allows for efficient and accurate calculation of the first-passage time CDF of a Wiener process with normally distributed drift rates across trials. The accuracy benefits of an analytic solution and except for the truncation error which can be controlled for, no further inaccuracies occur in the calculation of model predictions. With regard to the efficiency of the calculation we consider the provided solution to lie between the computationally very efficient, but theoretically limited EZ-Diffusion model (Wagenmakers, van der Maas, & Grasman, 2007) and packages like fastDM (Voss & Voss, 2007) and DMAT (Vandekerckhove & Tuerlinckx, 2007, 2008) which allow for a fit of the “full” Ratcliff diffusion model with all the other mixture parameters (variable starting point, variable residual component). The EZ-Diffusion model is computationally very efficient but uses only small portions of the data; namely, mean and variance as well as the proportion of correct responses. But it is computationally extremely efficient as explicit formulae of method of moment estimators exist for the standard case without inter-trial variability. The solution offered in this paper utilizes the full distribution and allows for trial-to-trial variation in drift rates. The additional assumptions of trial-to-trial variation in residual (i.e., non-decision) time (T_{er}) and starting point $z = aw$ could be added based the solution presented in this paper. This additional variation requires numerical evaluation of two integrals—which should be considerably faster than three integrals. Our solution is thereby fully compatible with the DMAT toolbox (Vandekerckhove & Tuerlinckx, 2007). It would be interesting to see how performance of DMAT improved if the provided solution was implemented. In any case, it should greatly improve the accuracy and the speed of the estimation in self-written implementations (e.g., a hierarchical Bayesian model of the Ratcliff diffusion model, see Wiecki et al., 2013).

Conclusions

Despite the obvious advantages of employing a computational model for response time and response accuracy (Smith & Ratcliff, 2004), psychologists have—for a long time—only reluctantly employed formal models (e.g., the two-barrier diffusion model). Recently, there has been a surge in interest for the diffusion model and this article aims at further improving its computational and numerical basis. The analytical solution guarantees a fast and accurate calculation of model predictions for a diffusion model with normally distributed drift rates.

Supplementary material

The online supplement includes R (R core team, 2016) and Matlab code for Equation (3).

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Appendix A: Exchangeability of summation and integration

To exchange the integration and summation operators, one must show the absolute convergence of the series. First, consider the series

$$H(t, x \mid \nu, \eta^2, a, w) = \sum_{j=0}^{\infty} h_j(t, x \mid \nu, \eta^2, a, w)$$

with

$$h_j := (-1)^j \exp\left(-xaw - \frac{x^2 t}{2}\right) \phi(x \mid \nu, \eta^2) \phi\left(\frac{r_j}{\sqrt{t}}\right) \left[M\left(\frac{r_j - xt}{\sqrt{t}}\right) + M\left(\frac{r_j + xt}{\sqrt{t}}\right)\right].$$

That is, $G(t \mid \nu, \eta^2, a, w) = \int_{-\infty}^{\infty} H(t, x \mid \nu, \eta^2, a, w)\, dx$. To establish exchangeability of integration and summation, we use the ratio test to prove absolute convergence. The ratio of consecutive terms in the series is:

$$\left|\frac{h_{j+1}}{h_j}\right| = \left|\frac{\phi\left(\frac{r_{j+1}}{\sqrt{t}}\right)\left[M\left(\frac{r_{j+1} - xt}{\sqrt{t}}\right) + M\left(\frac{r_{j+1} + xt}{\sqrt{t}}\right)\right]}{\phi\left(\frac{r_j}{\sqrt{t}}\right)\left[M\left(\frac{r_j - xt}{\sqrt{t}}\right) + M\left(\frac{r_j + xt}{\sqrt{t}}\right)\right]}\right|.$$

Then, $\lim_{j\to\infty} \left|\frac{h_{j+1}}{h_j}\right| = Q < 1$ is a sufficient condition for absolute convergence. Hence, we must show that

$$\lim_{j\to\infty} \left|\frac{h_{j+1}}{h_j}\right| = \lim_{j\to\infty} \left[\frac{\phi\left(\frac{r_{j+1}}{\sqrt{t}}\right)}{\phi\left(\frac{r_j}{\sqrt{t}}\right)} \cdot \frac{M\left(\frac{r_{j+1} - xt}{\sqrt{t}}\right) + M\left(\frac{r_{j+1} + xt}{\sqrt{t}}\right)}{M\left(\frac{r_j - xt}{\sqrt{t}}\right) + M\left(\frac{r_j + xt}{\sqrt{t}}\right)}\right] < 1.$$

because $\phi(u)$, $M(u) > 0$ for $u \in \mathbb{R}$. The arguments of the Mills ratio depend on x which may range from positive to negative infinity. Hence, we will not seek an explicit solution for the second factor. However, we know from the (log-) convexity of the Mills ratio for the standard normal distribution (Baricz, 2008) that $M(u)$ is strictly decreasing in $u \in \mathbb{R}$. As $r_{j+1} > r_j$ for all j, we can conclude that the limit exists and that it is between zero and one:

$$0 \leq \lim_{j\to\infty} \frac{M\left(\frac{r_{j+1}-xt}{\sqrt{t}}\right)+M\left(\frac{r_{j+1}+xt}{\sqrt{t}}\right)}{M\left(\frac{r_{j}-xt}{\sqrt{t}}\right)+M\left(\frac{r_{j}+xt}{\sqrt{t}}\right)} \leq 1.$$

It remains to show convergence of the ratio of normal densities:

$$\lim_{j\to\infty} \frac{\phi\left(\frac{r_{j+1}}{\sqrt{t}}\right)}{\phi\left(\frac{r_{j}}{\sqrt{t}}\right)} = \lim_{j\to\infty} \frac{\exp\left(-r_{j+1}^2\right)}{\exp\left(-r_j^2\right)}$$

The arguments of the normal density function do not depend on x. Because the r_j are differently defined for odd and even *j*, we must derive the limit for both cases. For simplification, use the compact notation $w' = 1 - w$. Assume that *j* is even and that $j + 1$ is odd, thus, $r_j = ja + aw$ and $r_{j+1} = (j + 1)a + aw'$. Then,

$$\lim_{j\to\infty} \frac{\exp\left(-[(j+1)a+aw']^2\right)}{\exp(-(ja+aw)^2)} = \lim_{j\to\infty} \frac{\exp(-[a^2(j+1)^2+2(j+1)a^2w'+a^2w'^2])}{\exp(-[(ja)^2+2ja^2w+(aw)^2])}$$

$$= \lim_{j\to\infty} \exp(-2ja^2(1 + w' - w) - a^2[1 + w' + w'^2 - w^2])$$

$$= 0,$$

since $w, w' \in (0,1)$. For the alternative case, that is, j is odd and $j+1$ is even, exchange w' with w which does not change the result. Consequently, $\lim_{j\to\infty}\left|\frac{h_{j+1}}{h_j}\right| = 0 < 1$.

Appendix B: Derivation of the definite integral

As stated in the text, we seek a solution of the integral

$$I(\alpha,\beta,\gamma,\delta) \coloneqq \int_{-\infty}^{\infty} \exp[-(\delta x-\gamma)^2/2]\Phi(\beta x-\alpha)\ dx,$$

that is, a parametric function which suggests a solution by differentiation under the integral sign:

$$\frac{d}{d\alpha}\, I(\alpha,\beta,\gamma,\delta) = \int_{-\infty}^{\infty} \frac{\partial}{\partial\alpha} \exp[-(\delta x-\gamma)^2/2]\Phi(\beta x-\alpha)\, dx =$$
$$(-1)\int_{-\infty}^{\infty} \exp[-(\delta x-\gamma)^2/2]\phi(\beta x-\alpha)\, dx$$

Replacing $\phi(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$ and simplification yields:

$$I_{d\alpha}(\alpha,\beta,\gamma,\delta) = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left[-\frac{\beta^2+\delta^2}{2}\left(x-\frac{\gamma\delta+\alpha\beta}{\beta^2+\delta^2}\right)^2 - \frac{(\alpha\delta-\gamma\beta)^2}{2(\beta^2+\delta^2)}\right] dx$$

Integration with respect to x gives

$$I_{d\alpha}(\alpha,\beta,\gamma,\delta) = -\frac{1}{\sqrt{\beta^2+\delta^2}}\exp\left[-\frac{(\alpha\delta-\gamma\beta)^2}{2(\beta^2+\delta^2)}\right]\int_{-\infty}^{\infty}\phi\left(\frac{x-\frac{\gamma\delta+\alpha\beta}{\beta^2+\delta^2}}{\frac{1}{\beta^2+\delta^2}}\right)dx$$

$$= -\frac{1}{\sqrt{\beta^2+\delta^2}}\exp\left[-\frac{(\alpha\delta-\gamma\beta)^2}{2(\beta^2+\delta^2)}\right] = -\sqrt{2\pi}\,\phi\left(\frac{\alpha\delta-\gamma\beta}{\sqrt{\beta^2+\delta^2}}\right).$$

Then, the indefinite integral with respect to α is given by:

$$I(\alpha,\beta,\gamma,\delta) = -\sqrt{2\pi}\int\phi\left(\frac{\alpha\delta-\gamma\beta}{\sqrt{\beta^2+\delta^2}}\right)d\alpha = -\frac{\sqrt{2\pi}}{\delta}\Phi\left(\frac{\alpha\delta-\gamma\beta}{\sqrt{\beta^2+\delta^2}}\right)+C.$$

To obtain C, we may note that by definition of $I(\alpha,\beta,\gamma,\delta)$ it holds that $\lim_{\alpha\to\infty} I(\alpha,\beta,\gamma,\delta) = 0$. Thus,

$$\lim_{\alpha\to\infty}\left[-\frac{\sqrt{2\pi}}{\delta}\Phi\left(\frac{\alpha\delta-\gamma\beta}{\sqrt{\beta^2+\delta^2}}\right)+C\right] = 0 \Leftrightarrow -\frac{\sqrt{2\pi}}{\delta}+C = 0 \Leftrightarrow C = \frac{\sqrt{2\pi}}{\delta}$$

Finally, we have that

$$I(\alpha,\beta,\gamma,\delta) = \frac{\sqrt{2\pi}}{\delta}\left[1-\Phi\left(\frac{\alpha\delta-\gamma\beta}{\sqrt{\beta^2+\delta^2}}\right)\right].$$

This solution is a more general version of a known result (for $\delta = 1$ and $\gamma = 0$, the above solution corresponds to Eq. 10,010.8 in Owen, 1980).

Online supplement: R code

Tested with R version 3.3.1

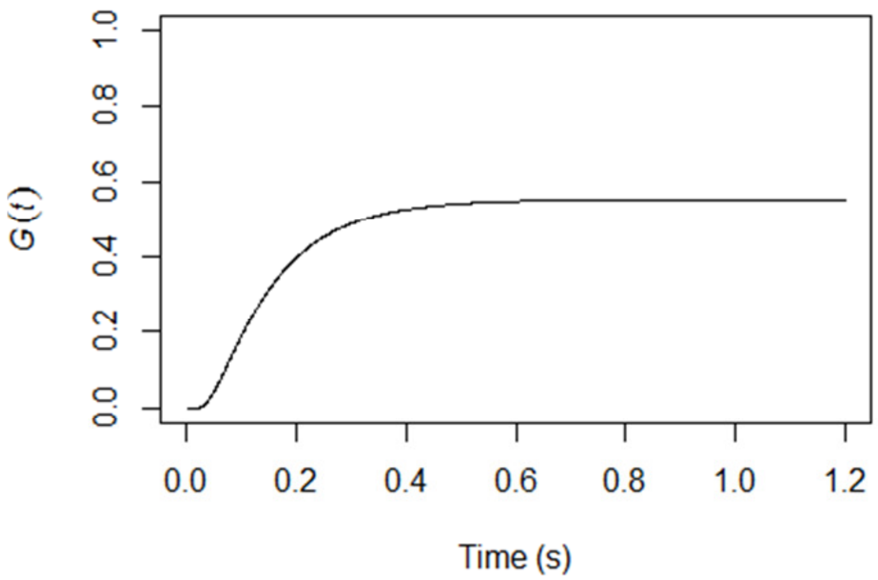
```
# Distribution at lower barrier (Eq. 3 of the article)
#   t: time (vector)
#   nu: average drift
#   eta2: variance of the drift distribution
#   sigma2: variance of Wiener process
#   a: upper barrier
#   w: relative position of X(0) = z, w = z/a
#   eps: required precision
#
G_0 = function(t=1.2, nu=0.1, eta2=0.01, sigma2=0.01, a=0.08, w=.375,
  eps=sqrt(.Machine$double.eps))
{
  nu   = nu / sqrt(sigma2)
  a    = a / sqrt(sigma2)
  eta2 = eta2 / sigma2
  sqt  = sqrt(t)
  sqet = sqt * sqrt(1 + eta2*t)
  G = numeric(length(t))
  j = 0
  repeat
  {
    rj = j*a + a*w
    logphi = dnorm(rj/sqt, log=TRUE)
    logM1  = logMill((rj - nu*t + eta2*(rj + a*w)*t) / sqet)
    logM2  = logMill((rj + nu*t + eta2*(rj - a*w)*t) / sqet)
    gj = exp(logphi + logM1) + exp(logphi + logM2)
    G = G + gj
    if(all(gj < eps))
      return(exp((-nu*nu*t - 2*nu*a*w + eta2*a*a*w*w)/2/(1 + eta2*t)) * G)

    j = j + 1
    rj = j*a + a*(1-w)
    logphi = dnorm(rj/sqt, log=TRUE)
    logM1  = logMill((rj - nu*t + eta2*t*(rj + a*w)) / sqet)
    logM2  = logMill((rj + nu*t + eta2*t*(rj - a*w)) / sqet)
    gj = exp(logphi + logM1) + exp(logphi + logM2)
    G = G - gj
    j = j + 1
  }
}
```

```
# Distribution at upper barrier
#
G_a = function(t=1.2, nu=0.1, eta2=0.01, sigma2=0.01, a=0.08, w=.375,
  eps=sqrt(.Machine$double.eps))
{
  G_0(t, -nu, eta2, sigma2, a, 1-w, eps)
}

# log of Mill's ratio for the normal distribution
#
logMill = function(x) # log of Mill's ratio
{
  m = numeric(length(x))
  m[x >= 10000] = -log(x[x >= 10000]) # limiting case for x -> Inf
  m[x <  10000] = pnorm(x[x < 10000], lower=FALSE, log=TRUE) -
    dnorm(x[x < 10000], log=TRUE)
  m
}

# Example
#
plot(seq(0.001, 1.200, 0.001),
     G_a(t=seq(.001, 1.200, .001), nu=0.1, eta2=.01, sigma2=.01, a=0.08, w=.375),
     type='l', xlab='Time (s)', ylab=expression(italic(G)(italic(t))),
     main='', ylim=c(0, 1))
```



```matlab
function F = ratcliff_cdf(t, v, a, w, eta2, sigma2, err)
%ratcliff_cdf: calculate CDF of FPT in a Ratcliff DDM to the lower barrier
%  v is mean drift rate
%  a is barrier separation
%  w is relative starting point
%  eta2 is drift rate variance;
%  sigma2 is diffusion constant (usually 0.01)
%  err is error tolerance of the infinite series truncation

  F = zeros(1, length(t));
  if(nargin < 7); err = sqrt(eps); end

  if(any(t>0))
    sigma = sqrt(sigma2);
    F(t>0) = ratcliff_cdf1(t(t>0), v/sigma, a/sigma, w, eta2/sigma2, err);
  end
return

function F = ratcliff_cdf1(t, v, a, w, eta2, err)

  F = zeros(1, length(t));
  sqt = sqrt(t);
  denomMR = sqt.*sqrt(1+t*eta2);

  j = 0;
  while true %loop through pairs of even and odd j

    %even j
    rj = j*a + a*w;
    S1 = normpdf(rj./sqt) .* (M((rj - t*v + t*eta2*(rj + a*w)) ./ denomMR) + ...
      M((rj + t*v + t*eta2*(rj - a*w)) ./ denomMR));

    if(all(abs(S1) < err)); break; end
    j = j + 1;

    %odd j
    rj = j*a + a*(1-w);
    S2 = normpdf(rj./sqt) .* (M((rj - t*v + t*eta2*(rj + a*w)) ./ denomMR) + ...
      M((rj + t*v + t*eta2*(rj - a*w)) ./ denomMR));
    F = F + S1 - S2;

    if(all(abs(S2) < err)); break; end
    j = j + 1;
  end
  F = F .* exp((-t*v^2-2*v*a*w+eta2*a^2*w^2) ./ (2+2*t*eta2)); %prefactor
return

%calculate Mill's ratio
function M = M(x)
  M = erfcx(x/sqrt(2)) / sqrt(2) * sqrt(pi);
return
```