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Published in:
Educational design in math and science: The collective aspect

Publication date:
2016

Document version
Early version, also known as pre-print

Citation for published version (APA):
Putra, Z. H. (2016). Collective aspects of pre-service lower secondary teachers' knowledge on density of rational numbers. In M. Achiam, & C. Winsløw (Eds.), *Educational design in math and science: The collective aspect* (pp. 53-61). [7] Copenhagen.

Collective aspects of pre-service lower secondary teachers' knowledge on density of rational numbers

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This study is about the design of hypothetical teacher tasks (HTTs) on density of rational numbers, developed based on the anthropological theory of the didactic (ATD), and used to investigate pre-service lower secondary teachers (PLSTs)' mathematical and didactical knowledge. The PLSTs' knowledge considered in this paper concerns mathematical and didactical techniques to solve the specific tasks on the HTTs. The collective aspects, mutual engagement, joint enterprise, and shared repertoire, are considered during their discussion. The findings show that there is a link between mathematical and didactical techniques, and some didactical techniques purposed by PLSTs are too general.

Introduction

Research on pre-service teachers' knowledge on rational numbers has been studied by various researches during the last decades. Some studies focus on testing their competences related to content knowledge, such as fraction arithmetic (Bradshaw et al, 2014). Other studies give more attention to their competences on pedagogical content knowledge related to problem posing (Toluk-Uçar, 2009). There are also studies looking for relationship and differences between these areas, as known by pre-service teachers (Depaepe et al, 2015). These studies use a similar approach to access teachers' knowledge through diagnostic tests.

I consider a different approach to investigate pre-service teachers' knowledge on rational numbers. The idea is designing hypothetical teacher tasks (HTTs) (Durand-Guerrier et al, 2010) that is used to investigate pre-service mathematical and didactical knowledge, here specifically pre-service teachers' knowledge on density of rational numbers.

In this study, I do not only consider mathematical and didactical techniques used by pre-service teachers but also study the collective nature of pre-service teachers' knowledge on rational numbers. Teachers as a part of a community have a chance to work and act collectively for instance to develop common teaching resources (Gueudet & Trouche, 2012), and the collective organisation of teacher work turns out to hold important potential for improving the learning of students, as shown in comparative studies of East Asian education (Ma, 1999; Winsløw, 2012). Hence, I try to answer two questions for this study in this paper. The first question is how can HTTs be used to investigate pre-service lower secondary teachers (PLSTs)' mathematical and didactical techniques related to the density of rational numbers? and the second one, what shared praxeologies can be observed during collaborative works of PLSTs to solve an HTT about density of rational numbers?

The anthropological theory of the didactic and the collective aspects

There are two main frameworks used for this case study. The first one is the *anthropological theory of the didactic* (ATD) that is used to design the HTT about density of rational numbers and to analyse the result. The second one is the collective aspects finding during the implementation of the HTT.

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The ATD is known as a general epistemological model of mathematical knowledge that can be used to observe human mathematical activities (Chevallard, 1992). The object of knowledge that will be learnt by a human related to mathematics can be identified into two aspects, a practical block and a knowledge block, which are main components of praxeological reference models. The practical block is formed by a type of task (T) and a technique (τ). A type of task (T) is a specific class of problems such as finding a number between two rational numbers. The students need a technique (τ) to solve this problem such as finding the middle numbers by adding two rational numbers and then dividing by two. Then, the knowledge block consists of a technology (θ) used to explain the practical block and a theory (Θ) to justify and reason about the technology (θ). The technology (θ) for the case is that between two different rational numbers, someone can find at least a number. While the theory of the ordered field of rational numbers, especially denseness-in-itself of rational numbers is uses to justify the technology. Those four elements (T, τ, θ, Θ) are interdependent.

When I look at back to the process of transposition of mathematical knowledge, I consider that knowledge is collectively produced in communities. In the case of learning mathematics by PLSTs in a teacher college, some mathematical knowledge does not only transpose from scholars to PLSTs, but is also shared among them. They tend to share information and experiences within the group and learning from the activity itself (Engeström, 1987).

Lave & Wenger (2012) introduce the concept of *communities of practice* (CoP) to describe a group of people sharing an interest, a craft or a profession. There are three essential conditions of CoP (Lave & Wenger, 2012): 1) *mutual engagement* (members establishing norms and building collaborative relationships), 2) *joint enterprise* (members creating a shared understanding of what are the common objectives), and 3) *shared repertoire* (members producing resources – material or symbolic – which are recognized as their own by the group and its members). I interpret the *mutual engagement* based on the theory of social norms (Yackel and Cobb, 1996) as an interaction among PLSTs during a discussion such as questioning each other's thinking, explaining their ways of thinking, working together to solve problems, solving problems using a variety of approaches, and so on. Then, the *joint enterprise* that I also interpret based on the theory of sociomathematical norm (Yackel and Cobb, 1996) is an acceptable mathematical explanation and justification by PLSTs during the discussion. Finally, the *share repertoire* is interpreted as results, mathematical and didactical techniques, for the tasks argued by PLSTs during their discussion.

The concept of density of rational numbers

The concept of density of rational numbers is closely related to the concept of infinity (Vamvakoussi, & Vosniadou, 2004). More specifically, between any two different rational numbers there are infinitely many rational numbers. The numbers that students know as concrete objects change into continuities, and some numbers are really difficult to explain to pupils.

Some pupils probably get difficulty to deal with the concept of density. It is because they just look at a finite number of different numbers between two given rational numbers. They come to the idea of discreteness as a fundamental presupposition, which constrains pupils' understanding of density (Vamvakoussi, & Vosniadou, 2004). Specifically, pupils in the discreteness–density knowledge give seemingly inconsistent answers when a teacher asks how many numbers between two rational numbers when those numbers representing in two different forms, for instance, decimals and fractions.

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Arithmetic mean is one way prompting pupils to find the concept of infinity. For example, between two rational numbers a and b, there is m_0 as the arithmetic mean, repeat the procedure for a and m_0 to find m_1 and so on. It may lead the pupils to infer that there are infinitely many numbers between a and b. This is a way of approaching the notion of actual infinity in a potential manner (Vamvakoussi, & Vosniadou, 2004). Then, Brousseau (1997, p.166) explained the properties of rational numbers in order to make measurements are mostly topological properties that are related to the idea of arithmetic mean. He said between two rational numbers, we can always put a number in between, and we can measure all the intervals so obtains. Moreover, when pupils work with the operations of rational numbers tends to choose decimals instead of fractions because they allow pupils rapid calculations and a convenient representation of rational measurement. This situation probably leads pupils to the inconsistent answers that I explain in the previous paragraph.

Design of hypothetical teacher tasks

The present study is part of a pilot study, in which five HTTs about rational numbers have been tested. In this paper, I just focus on the third HTT that is about density of rational numbers. The task was chosen based on pupils' difficulties to figure out how many numbers between two rational numbers (Vamvakoussi et al, 2011). The task given to the students is originally written in Danish that was translated from English as follows:

> You ask fifth grade students how many numbers there are between $\frac{2}{5}$ and $\frac{4}{5}$, and how many numbers between 0.4 and 0.8.
>
> *Your students say that there is only one number between* $\frac{2}{5}$ *and* $\frac{4}{5}$ *namely* $\frac{3}{5}$ *; they also say 3 numbers between 0.4 and 0.8.*
>
> How do you interpret this claims? (*solve individually in 4 minutes*)
>
> Explain your ideas to teach this students? (*discuss in pairs in 5 minutes, use the space below if necessary, and write your ideas to support the discussion*)

In this study, 11 first year PLSTs from Metropolitan University College (MUC), Denmark, volunteered to work in a group of two, but a group consists of 3 students. Each group worked and discussed for 9 minutes in different schedules. All students wrote their answer on the paper for the individual task, but only few students wrote their answers for the discussion task. I also recorded their activities using video recording for the discussion task. The data was collected at Wednesday, January 6th, 2016.

A-priori analysis

The task given to PLSTs can be described into praxeological reference models. There are three possible tasks can be interpreted as follows:

T₁ = given two different rational numbers, $\frac{a}{b} = m, c_1 c_2 \cdots$ and $\frac{c}{d} = n, d_1 d_2 \cdots$, find how many numbers between $\frac{a}{b}$ and $\frac{c}{d}$, and $m, c_1 c_2 \cdots$ and $n, d_1 d_2 \cdots$.

T₂ = given two different student answers about denseness of rational numbers between $\frac{a}{b} = m, c_1 c_2 \cdots$ and $\frac{c}{d} = n, d_1 d_2 \cdots$, interpret these answers.

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T* = given problems and student responses to the type of task T, determine what ideas as a teacher to teach students.

The first two type of tasks, $\mathbf{T_1}$ and T_2, are used to assess pre-service lower secondary teachers' mathematical knowledge about density of rational numbers. Meanwhile, the last type of task, **T***, is used to evaluate their didactical knowledge related to teach density of rational numbers.

I consider that the type of task $\mathbf{T_2}$ is interrelated to the type of task $\mathbf{T_1}$ because when someone interprets that students' claim is true, one probably will show a mathematical technique to solve the type of task $\mathbf{T_1}$. Instead of describe each mathematical technique for both of them, I rather concern to describe the mathematical techniques to the type of task $\mathbf{T_1}$ as follows:

$\boldsymbol{\tau_{11}}$ = change fractions into decimals or vice versa. e.g. 2/5 = 0.4 and 4/5 = 0.8, so there are same numbers between two decimals and two fractions.

$\boldsymbol{\tau_{12}}$ = first show that there is one number between x and y (x, y representing the general terms for rational numbers such as $\frac{a}{b}$, and $m, c_1c_2 \cdots$). There exists z, so $x < z < y$, then use this to find z_1 so that $x < z_1 < z$, continue to z_2 so that $x < z_2 < z_1$, and etc.

There are some possible techniques to find z such as:

$\boldsymbol{\tau_{12a}}$ = find z between $\frac{a}{b}$ and $\frac{c}{d}$ using a formula: $\frac{a+c}{b+d}$.

$\boldsymbol{\tau_{12b}}$ = find z between $\frac{a}{b}$ and $\frac{c}{d}$ using a formula: $\frac{ad+bc}{2bd}$.

$\boldsymbol{\tau_{12c}}$ = if $b = d$, take a number m between a and c, then $\frac{a}{b} < \frac{m}{d} < \frac{c}{b}$.

$\boldsymbol{\tau_{12d}}$ = find z between two decimals $m, c_1c_2 \cdots$ and $n, d_1d_2 \cdots$ by considering a number between two numbers after comma. e.g. between 0.4 and 0.8, there exist for instance 0.6.

$\boldsymbol{\tau_{12e}}$ = find z between $m, c_1c_2 \cdots$ and $n, d_1d_2 \cdots$ using a formula: $\frac{n, c_1c_2\cdots + n, d_1d_2\cdots}{2}$.

$\boldsymbol{\tau_{13}}$ = represent those numbers in a number line, find other numbers between two numbers using one mathematical technique from $\boldsymbol{\tau_{12}}$.

There is also a specific mathematical technique that can only be applied by decimals ($\boldsymbol{\tau_{14}}$) or by fractions ($\boldsymbol{\tau_{15}}$).

$\boldsymbol{\tau_{14}}$ = put 0s after decimals and show that the numbers between two decimals can be written as many as by adding 0s.

$\boldsymbol{\tau_{15}}$ = find equal/equivalent fractions for $\frac{a}{b}$ and $\frac{c}{b}$, and show that the bigger denominators, the more fractions with the same denominator to be found.

There are also possibilities that someone gives correct mathematical techniques for fractions but not for decimals or vice versa. The incorrect mathematical technique for fractions as follows:

$\boldsymbol{\tau_{16}^{-}}$ = change both fractions into the same denominator (in case they have different denominators), find some natural numbers between two numerators. e.g. between $\frac{2}{5}$ and $\frac{4}{5}$ is $\frac{3}{5}$, because 3 is a number between 2 and 4.

Meanwhile, the incorrect mathematical technique for decimals can be written as:

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τ_{17}^{-} = consider the numbers as natural numbers by omitting commas, and find natural numbers between them. e.g. there are 3 numbers between 0.4 and 0.8 because there are 3 numbers between 4 and 8.

Actually, those mathematical techniques described above are just partial techniques that can be more varied when I implement my research into a big scale research. Meanwhile, the possible technology (**θ**) to justify is that between two rational numbers, there exists at least a rational number, and I consider that it is based on the theory (**Θ**) of the order field of rational numbers.

Theorem. Whenever $q < s$ are rational numbers, there is a rational number r such that $q < r < s$.

Proof. $r = \frac{1}{2}(q + s)$ is rational and satisfies the inequality.

Corollary. We can construct a sequence of rational numbers $r_1, r_2, \ldots$ such that $q < r_k < s$ for all k (and, in fact, $r_1 < r_2 < \ldots$).

Those mathematical techniques lead me to describe some of didactical techniques that could be applied to solve the type of task **T***. To make it simple to recognize those tasks, I put * on the type of didactical techniques correspond to mathematical techniques such as τ_{11}*. This didactical technique means that PLSTs explain to pupils using the mathematical technique of τ_{11}. From those, I get 7 different didactical techniques, but I also consider other didactical techniques that can be coded as τ_{18}* and so on.

τ_{18}* = shows to pupils that 0.*a* and/or 0.*ab* lie in between, and ask them to consider about other numbers in between two rational numbers.

τ_{19}* = explain to pupils to change decimals into percentage.

τ_{20}* = ask pupils to compare decimals/percentages and fractions through pizza experiment.

τ_{21}* = ask students to reduce fractions. e.g. $\frac{8}{10}$ can be reduced into $\frac{4}{5}$.

τ_{22}* = explain to pupils through a simple example such as how many numbers between 0 and 1.

τ_{23}* = use visual representations such as ruler and relate to measurement.

τ_{24}* = show pupils a contextual activity through dividing pizza into more slices.

τ_{25}* = introduce other contextual activities related to everyday life.

Result

A-posteriori analysis

The mathematical techniques described by PLSTs were not only taken from their answers on the worksheets but also elaborated from their discussion. The reason to do this because of the mathematical task given on the worksheet was not explicitly stated (question a), so not all of them wrote their answers to the type of task $\mathbf{T}_1$. Instead of describing the mathematical and didactical techniques separately, I consider to describe them together and show links between the mathematical and didactical techniques.

Starting from group 1 consisting two female students, student A and student B, graduating from B level. None of them wrote a mathematical technique in their papers explicitly, but student B shared a wrong mathematical technique, τ_{16}^{-}, during the discussion. She said "*it is true that $\frac{3}{5}$ is the only number that is not present. It is the number that we find in the middle when we say 2, 3, 4"*. In the discussion, they also tried to link between teaching fractions and decimals altogether. Student A, for instance, said *"and we could take the*

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decimals in percentage and get 40% and 80%. Then we could ask them to remove 40 % of the pizza and $\frac{2}{5}$ *of the pizza"*. Here, I categorized those didactical techniques into $\boldsymbol{\tau_{19}}$* and $\boldsymbol{\tau_{20}}$*. Student A also wrote in her worksheet and suggested during the discussion to teach pupils how to rewrite fractions into decimals and vice versa ($\boldsymbol{\tau_{11}}$*). Both of them also argued that pupils have to find reduce fraction such as asking pupils whether they could reduced $\frac{8}{10}$ (categorize as $\boldsymbol{\tau_{21}}$*). However, even they knew that both decimals and fractions were same value and only different representations, but they still shared an agreement that there was only a number between two fractions in the end of the discussion.

Group 2 consists of 2 male students, student C and student D, who graduated from A level. There was only student D wrote explicitly two mathematical techniques to solve the type of task $\mathbf{T_1}$ on his worksheet. The first mathematical technique is clearly about finding equal/equivalent fractions ($\boldsymbol{\tau_{15}}$), and the second one I interpret as $\boldsymbol{\tau_{12}}$ because he wrote $\frac{2.5}{5}$ between $\frac{2}{5}$ and $\frac{3}{5}$. Then, the first didactical technique to solve the task was $\boldsymbol{\tau_{11}}$* suggested by student C. This technique was supported by student D that he said *"We can make them try to write the other numbers as fractions, such that 0.5, 0.6, 0.7 are written as* $\frac{5}{10}, \frac{6}{10}$, *and* $\frac{7}{10}$, *and they also get* $\frac{4}{10}$ *and* $\frac{8}{10}$. *We get* $\frac{2}{5}$ *and* $\frac{4}{5}$ *when we reduced those and in the middle we have that 0.6 become* $\frac{4}{5}$. *We could also write (writes down on paper:)* $\frac{2.5}{5}$ *between* $\frac{2}{5}$ *and* $\frac{3}{5}$*"*. The other didactical techniques could be drawn from this argumentation were $\boldsymbol{\tau_{12}}$* (related to $\boldsymbol{\tau_{12c}}$*), $\boldsymbol{\tau_{15}}$* and $\boldsymbol{\tau_{21}}$*. During the discussion, student D stated precisely that there ware infinitely many numbers, and the numbers, for instance, complex numbers.

Group 3 also consist of two male students, student E graduated from B level and student F from A level. They also did not write mathematical techniques in their worksheets, but I can imply what techniques did they use during the discussion. Student F started the discussion with the didactical technique of $\boldsymbol{\tau_{18}}$* that was showing to pupils 0.7 and 0.79 lie between 0.4 and 0.8. Then, student E suggested to give a simple example and student F supported by suggesting to show pupils numbers between 0 and 1. He said *"this is a* $\frac{1}{2}$, *they know that, and we can write it here. We can continue and write an interval, 0.25, they can do that. Then we can introduce those numbers now (points on the numbers 0.4 and 0.8). Then they can see that there must be numbers in between them, the same way that they saw the numbers in between 1 and 2"*. From this answers, I interpret that student F had the mathematical idea of $\boldsymbol{\tau_{12}}$ that probably links to the didactical idea of $\boldsymbol{\tau_{12}}$* and also $\boldsymbol{\tau_{22}}$*. On the other hands, this group got difficulties for the first time to realize that there are many numbers between two fractions. They realized after student F suggested pupils to rewrite fractions into decimals. Here, they applied the didactical technique of $\boldsymbol{\tau_{11}}$*, and finally realized that both questions are same, so there were also many numbers in between. Then, they suggested to used the didactical technique of $\boldsymbol{\tau_{15}}$* to teach fractions as well.

Group 4 consists of 3 male students, student G, student H, and student I, who graduated from A level. Student G and student I justified pupils' mistake based on the mathematical techniques of $\boldsymbol{\tau_{16}}^-$ and $\boldsymbol{\tau_{17}}^-$. Student G also wrote in his worksheet that $\frac{4}{6}$ lies between two fractions and 0.41 lies between two decimals. I interpret that he used the mathematical technique of $\boldsymbol{\tau_{12}}$ as well. Then, the discussion started with student G argumentation about 0.41 lies between 0.4 and 0.8. To teach pupils about denseness of decimals, student H suggested to use visual representation through ruler, and this idea was supported by others. I categorize this didactical technique as $\boldsymbol{\tau_{24}}$*. Then, student G said *"we need to be concrete about the pizza. We should divide in* $\frac{2}{5}$ *and then* $\frac{4}{5}$ *and also* $\frac{3}{5}$. *Then we can show them that there also a slice in between these divisions of the pizza."* This didactical technique ($\boldsymbol{\tau_{24}}$*) was supported by other students. The last didactical technique was

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$\boldsymbol{\tau}_{12}$* that I found from student H argumentation *"they have to understand the relation between decimals and fractions such that they can transform the numbers. If they struggle with one of the notations, then they can transform it to the other notation"*. However, I could not see explicitly whether all of them agree that there are infinity many numbers in between two fractions and two decimals.

The last group was group 5 consisted a male student, student J, graduated from A level with 3 years teaching experiences, and a female student, student K, graduated from B level with 1 year teaching experience. During the discussion, they tended to use various contextual problems to teach pupils. The first one was to use pizza and division to give pupils an idea about numbers in between. It seems for me this didactical technique in line with $\boldsymbol{\tau}_{24}$*. Other ideas were pouring milks from a jar to cups, and sharing chocolate bars, but they did not give clear explanation how to use it in teaching denseness of rational numbers. All of them I categorize as $\boldsymbol{\tau}_{25}$* as part of mathematics found in everyday life. It was also stated by student J in his worksheet that students should be introduced to everyday mathematics. The other didactical idea was $\boldsymbol{\tau}_{12}$* about change fractions into decimals or vice versa. Even this group also did not speak about infinity many numbers during the discussion, student J stated on his worksheet that between two numbers there are many numbers, and student K wrote those numbers, fractions and decimals, were just written in different ways.

Analysis for the collective aspects

By this short case study in which only 9 minutes for each group to discuss the HTT about denseness of rational numbers. I realize that it is not so easy to give a deep analysis for the collective aspects emerging during PLSTs' discussion. Based on what Lave & Wenger (2012) introduce the concept of *communities of practice* (CoP) to describe a group of people sharing an interest, a craft or a profession, I interpret this notion as how PLSTs share their mathematical and didactical techniques to their colleagues.

Without any doubt, PLSTs worked together to solve especially the type of task **T*** because I stated clearly on their worksheet to discuss in pairs in 5 minutes. One interesting part is that the way they start the discussion for sharing their thinking. I found that only a group, group 2, started the discussion by asking a question to the other student what he thinks on the case. Other groups directly started by giving his/her mathematical thinking about the task. Both shows the way they build *mutual engagement* for the discussion. Since there were only 2 students for each group except for the group 4, one student shared his/her thinking and the other gave an agreement by saying "yes" or "no" sometimes adding some argumentations or posing a questions. As an example when student F said *" Yes, we could also write 0.75 up here, and 0.75 could be placed between 0.4 and 0.8"*. Then student E said *"Yes, what about the fractions?"*. It is also one of norms that common appears during the discussion and part of *mutual engagement.* Meanwhile, there are various approaches they used to solve the type of task **T*** such as group 5 suggested to relate the problem to the real word mathematics.

During the discussion, PLSTs shared their didactical techniques to solve the tasks. Some of their techniques supported by mathematical explanation, for instance, when student C from group 2 purposed the didactical technique $\boldsymbol{\tau}_{11}$*, student D gave a mathematical explanation related to the rewrite numbers from decimals into fractions or vice versa. Sometimes, they give justification in the the technology (**θ**) and theory level (**Θ**). It can be seen from the statement of student D that is still related to justify $\boldsymbol{\tau}_{11}$*. Student D said *"Then they can see that the numbers are the same - they are only written in different notations. This way we see that there are many other numbers in between. There are also decimal numbers, there are infinitely many numbers"*. I categorize this process as positive joint enterprise because they come to the correct mathematical technique

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with a better mathematical explanation and justification. However, I also found that some students shared a didactical technique without a mathematical explanation. Group 5, for instance, gave a lot of didactical techniques by using various contextual problems such as student J said *"We could also use a plate of chocolate".* She did not give any argumentations how to use it in teaching pupils about density of rational numbers, and also her colleague did not ask for clarification and justification about it.

At the part of a-posteriori analysis, I described some mathematical and didactical techniques to solve the tasks. Those are results from the *shared repertoire* of PLSTs to show their mathematical and didactical knowledge on density of rational numbers. Actually, some mathematical techniques (table 1) were not exactly stated by PLSTs on their worksheet or during the discussion, but I draw based on the idea that when the group purposed the didactical techniques. Then, I found two common mathematical and didactical techniques that are $\boldsymbol{\tau_{11}}$ related to $\boldsymbol{\tau_{11}*}$ (change fractions into decimals or vice versa) and $\boldsymbol{\tau_{12}}$ related to $\boldsymbol{\tau_{12}*}$ (showing for two rational numbers, it can be showed at least a number in between).

Table 1. Mathematical and didactical techniques

Group	Mathematical techniques	Didactical techniques
1	τ_{11}, τ_{16}^{-}	$\tau_{11}*, \tau_{19}*, \tau_{20}*, \tau_{21}*$
2	$\tau_{11}, \tau_{12}, \tau_{15}$	$\tau_{11}*, \tau_{12}*(\tau_{12c}*), \tau_{15}*, \tau_{21}*$
3	$\tau_{11}, \tau_{12}, \tau_{15}$	$\tau_{11}*, \tau_{12}*, \tau_{15}*, \tau_{18}*, \tau_{22}*,$
4	$\tau_{12}, \tau_{16}^{-}, \tau_{17}^{-}$	$\tau_{12}*, \tau_{24}*$
5	τ_{12}	$\tau_{12}*, \tau_{24}*, \tau_{25}*$

Concluding remarks

A-priori and a-posteriori analysis for the of PLSTs' knowledge on rational numbers have not been finished yet. I still realize that mathematical and didactical techniques described based on the praxeological reference models on the a-priori analysis still need to be developed and well organized. Some didactical techniques are probably similar, so it makes quite difficult to distinguee among them especially the didactical techniques from $\boldsymbol{\tau_{18}*}$ to $\boldsymbol{\tau_{25}*}$. Far from this, I can see that PLSTs showed some mathematical and didactical techniques during the discussion even a-posteriori analysis I did were not really valid and reliable yet. I leave this condition as a challenge for developing a better praxeological reference models to analyse the result.

Meanwhile, the process PLSTs sharing their knowledge about density of rational numbers appears in the sense of collective aspects. I can not make a general conclusion for the process of *mutual engagement, joint enterprise*, and *shared repertoire* because one group has different ways to share their ideas, and of course this happens in a setting of research. Further remarks for this research is that there is a challenge to look at in deep for the collective aspects of PLSTs sharing their knowledge by design a better framework and look at the challenge from the perspective of study and research path (SRP) (Barbequero et.al, 2015)

Acknowledgment: We would like to thank to the Ministry of Research, Technology, and Higher Education of the Republic of Indonesia for funding this PhD research.

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