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NEWTON SLOPES FOR ARTIN-SCHREIER-WITT TOWERS

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Abstract. We fix a monic polynomial $f(x) \in \mathbb{F}_q[x]$ over a finite field and consider the Artin-Schreier-Witt tower defined by $f(x)$; this is a tower of curves $\cdots \to C_m \to C_{m-1} \to \cdots \to C_0 = \mathbb{A}^1$, with total Galois group $\mathbb{Z}_p$. We study the Newton slopes of zeta functions of this tower of curves. This reduces to the study of the Newton slopes of L-functions associated to characters of the Galois group of this tower. We prove that, when the conductor of the character is large enough, the Newton slopes of the L-function form arithmetic progressions which are independent of the conductor of the character. As a corollary, we obtain a result on the behavior of the slopes of the eigencurve associated to the Artin-Schreier-Witt tower, analogous to the result of Buzzard and Kilford.

Contents

1. Introduction

We fix a prime number p. Let $\mathbb{F}_q$ be a finite extension of $\mathbb{F}_p$ of degree a so that $q = p^a$. For an element $b \in \overline{\mathbb{F}}_p$, let $\hat{b}$ denote its Teichmüller lift in $\mathbb{Z}_p^{\mathrm{ur}}$. We fix a monic polynomial $f(x) = x^d + b_{d-1}x^{d-1} + \cdots + b_0 \in \mathbb{F}_q[x]$ whose degree d is not divisible by p. Set $b_d := 1$. Let $\hat{f}(x)$ denote the polynomial $x^d + \hat{b}_{d-1}x^{d-1} + \cdots + \hat{b}_0 \in \mathbb{Z}_q[x]$. The *Artin-Schreier-Witt* tower associated to $f(x)$ is the sequence of curves C_m over $\mathbb{F}_q$ defined by the following equations:

$$C_m : \quad \underline{y}_m^F - \underline{y}_m = \sum_{i=0}^{d} (b_i x^i, 0, 0, \dots)$$

where $\underline{y}_m = (y_m^{(1)}, y_m^{(2)}, \dots)$ are viewed as Witt vectors of length m, and $\bullet^F$ means raising each Witt coordinate to pth power. In explicit terms, this means that C_1 is the usual Artin-Schreier curve given by $y^p - y = f(x)$, and C_2 is the curve above C_1 given by an additional equation (over $\mathbb{F}_q$)

$$z^p - z + \frac{y^{p^2} - y^p - (y^p - y)^p}{p} = \frac{\hat{f}^\sigma(x^p) - \hat{f}(x)^p}{p},$$

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where $\hat{f}^{\sigma}(x) := x^d + \hat{b}_{d-1}^p x^{d-1} + \cdots + \hat{b}_0^p$.

It is clear that the Artin-Schreier-Witt tower is a tower of smooth affine curves $\cdots \to C_m \to C_{m-1} \to \cdots \to C_0 := \mathbb{A}^1_{\mathbb{F}_q}$, forming a tower of Galois covers of $\mathbb{A}^1$ with total Galois group $\mathbb{Z}_p$. This tower is totally ramified at ∞. Thus, each curve C_m has only one point at ∞, which is $\mathbb{F}_q$-rational and smooth. It is well known that the zeta function of the affine curve C_m is

$$Z(C_m, s) = \exp\left(\sum_{k\geq 1} \frac{s^k}{k} \cdot \#C_m(\mathbb{F}_{q^k})\right) = \frac{P(C_m, s)}{1-qs},$$

where $P(C_m, s) \in 1 + s\mathbb{Z}[s]$ is a polynomial of degree $2g(C_m)$, pure of q-weight 1, and $g(C_m)$ denotes the genus of C_m.

A natural interesting problem, in the spirit of Iwasawa theory, is to understand the q-adic Newton slopes of this sequence $P(C_m, s)$ of polynomials, especially their stable properties as $m \to \infty$. This seems to be a difficult problem for a general tower of curves, and in fact it is not clear if one should expect any stable property for the q-adic Newton slopes. For the Artin-Schreier-Witt tower of curves considered in this paper, we discover a surprisingly strong stability property for the q-adic Newton slopes.

Our problem for the zeta functions easily reduces to the corresponding problem for the L-functions attached to the tower of curves. In this paper, all characters are assumed to be continuous. For a finite character $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$, we put $\pi_\chi = \chi(1) - 1$. Let m_χ be the nonnegative integer so that the image of χ has cardinality p^{m_χ}; we call p^{m_χ} the *conductor* of χ. Then, when χ is nontrivial, $\mathbb{Q}_p(\pi_\chi)$ is a finite totally ramified (cyclotomic) extension of $\mathbb{Q}_p$ of degree $(p-1)p^{m_\chi - 1}$, and π_χ is a uniformizer. Such a character χ defines an L-function $L(\chi, s)$ over $\mathbb{A}^1_{\mathbb{F}_q}$ given by

$$L(\chi, s) = \prod_{x\in|\mathbb{A}^1|} \frac{1}{1 - \chi\left(\mathrm{Tr}_{\mathbb{Q}_{q^{\deg(x)}}/\mathbb{Q}_p}\left(\hat{f}(\hat{x})\right)\right) s^{\deg(x)}} \in 1 + s\mathbb{Z}_p[\pi_\chi][\![s]\!],$$

where $|\mathbb{A}^1|$ denotes the set of closed points of $\mathbb{A}^1_{\mathbb{F}_q}$ and $\hat{x}$ denotes the Teichmüller lift of any of the conjugate geometric points in the closed point x. The L-function $L(\chi, s)$ is known to be a polynomial of degree $p^{m_\chi - 1}d - 1$ if χ is non-trivial. The L-function of the trivial character is given by $L(1, s) = 1/(1-qs)$, which is just the zeta function of $\mathbb{A}^1_{\mathbb{F}_q}$. The zeta functions of the curves in the Galois tower admit the following decompositions:

$$Z(C_m, s) = \prod_{\chi, 0\leq m_\chi \leq m} L(\chi, s), \qquad P(C_m, s) = \prod_{\chi, 1\leq m_\chi \leq m} L(\chi, s).$$

Hence the study of the polynomial $P(C_m, s)$ reduces to the study of $L(\chi, s)$ for various nontrivial finite characters χ.

In this paper we study certain periodicity behavior of the Newton polygon of the L-function $L(\chi, s)$. We first explain our conventions on Newton polygons.

Notation 1.1. Let R be a ring with valuation and ϖ an element with positive valuation. Let $v_\varpi(\cdot)$ denote the valuation on R normalized so that $v_\varpi(\varpi) = 1$. Then the *ϖ-adic Newton polygon* of a polynomial or a power series $1 + a_1 s + a_2 s^2 + \cdots$ with coefficients in R, is the lower convex hull of the set of points $(i, v_\varpi(a_i))$ for $i = 0, 1, \ldots$ (put $a_0 = 1$). The *slopes* of such a polygon are the slopes of each of its width 1 segments, counted with multiplicity and put in increasing order. Clearly, changing the choice of ϖ in R results in rescaling the slopes.

If χ_1 are χ_2 are two characters with the same conductor $m_{\chi_1} = m_{\chi_2} = m > 0$, then their L-functions $L(\chi_1, s)$ and $L(\chi_2, s)$ are Galois conjugate polynomials over $\mathbb{Q}(\zeta_{p^m})$ and hence have the same p-adic Newton polygon. Our main result is the following

Theorem 1.2. *Let $m_0 = 1 + \lceil \log_p(\frac{a(d-1)^2}{8d}) \rceil$ and let $0 < \alpha_1, \ldots, \alpha_{dp^{m_0-1}-1} < 1$ denote the slopes of the q-adic Newton polygon of $L(\chi_0, s)$ for a finite character $\chi_0 : \mathbb{Z}_p \to \mathbb{C}_p^\times$ with $m_{\chi_0} = m_0$. Then, for every finite character $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$ with $m_\chi \geq m_0$, the q-adic Newton polygon of $L(\chi, s)$ has slopes*

$$\bigcup_{i=0}^{p^{m_\chi - m_0}-1} \{\frac{i}{p^{m_\chi-m_0}}, \frac{\alpha_1 + i}{p^{m_\chi-m_0}}, \ldots, \frac{\alpha_{dp^{m_0-1}-1} + i}{p^{m_\chi-m_0}}\} - \{0\},$$

In other words, the q-adic Newton slopes of $L(\chi, s)$ form a union of dp^{m_0-1} arithmetic progressions, with increment $p^{m_0-m_\chi}$.

In short, this theorem says that the Newton slopes of the L-function for an Artin-Schreier-Witt tower enjoy a certain periodicity property. From this, one can easily deduce a nice description for the Newton slopes of the zeta function $P(C_m, s)$, as $m \to \infty$. For an integer $m \geq 1$, write

$$P(m, s) = \prod_{m_\chi = m} L(\chi, s) \in 1 + s\mathbb{Z}[s],$$

which is a polynomial of degree $(p-1)p^{m-1}(p^{m-1}d - 1)$. Then,

$$P(C_m, s) = \prod_{k=1}^{m} P(k, s) \in 1 + s\mathbb{Z}[s]$$

is a polynomial of degree

$$2g(C_m) = \sum_{k=1}^{m} (p-1)p^{k-1}(p^{k-1}d - 1) = (p-1)\left(d\frac{p^{2m}-1}{p^2-1} - \frac{p^m-1}{p-1}\right).$$

With the notations of the previous theorem, we deduce

Corollary 1.3. *For each integer $m \geq m_0$, the q-adic Newton polygon of $P(m, s)$ has slopes*

$$\bigcup_{i=0}^{p^{m-m_0}-1} \{\frac{i}{p^{m-m_0}}, \frac{\alpha_1 + i}{p^{m-m_0}}, \ldots, \frac{\alpha_{dp^{m_0-1}-1} + i}{p^{m-m_0}}\} - \{0\},$$

each counted with multiplicity $(p-1)p^{m-1}$.

Example 1.4. We say that $L(\chi, s)$ is *ordinary* if its Newton polygon is equal to its Hodge polygon; see for example [W4, §1.2] or [LW, §3]. The Newton slopes are easily determined in this case. For instance, when $p \equiv 1 \pmod{d}$, the example after Corollary 3.4 in [W4] says that for any character χ of order p, the L-function $L(\chi, s)$ is ordinary. Theorem 2.9 in [LW] then implies that $L(\chi, s)$ is ordinary for all non-trivial χ. One deduces that for each integer $m \geq 1$, the q-adic Newton polygon of $P(m, s)$ has slopes

$$\bigcup_{i=0}^{p^{m-1}-1} \{\frac{i}{p^{m-1}}, \frac{\frac{1}{d}+i}{p^{m-1}}, \ldots, \frac{\frac{d-1}{d}+i}{p^{m-1}}\} - \{0\} = \{\frac{1}{dp^{m-1}}, \frac{2}{dp^{m-1}}, \ldots, \frac{dp^{m-1}-1}{dp^{m-1}}\},$$

each counted with multiplicity $(p-1)p^{m-1}$. Corollary 1.3 is an analogue of this result which does not require $L(\chi, s)$ to be ordinary.

Both the proof of Theorem 1.2 and the proof of [LW, Theorem 2.9] referenced in Example 1.4, rely on the study of a T-adic version of the characteristic function, which is an entire version of the L-function. This T-adic characteristic function interpolates the characteristic functions associated to all finite characters. The key observation we make in this paper is that the obvious lower bound of the $T^{a(p-1)}$-adic Newton polygon of the T-adic characteristic function agrees, at all points in an arithmetic progression, with the $\pi_\chi^{a(p-1)}$-adic polygon for the characteristic function for the specialization at a finite character χ. Therefore, the two Newton polygons agree at these points. As a byproduct, we prove that a strong form of the T-adic Riemann hypothesis holds for the T-adic L-function in [LW], namely, the splitting field of the meromorphic T-adic L-function is a uniformly finite extension.

The theorem is largely motivated by the conjectural behavior of the Coleman-Mazur eigencurve near the boundary of weight space [BK], which concerns a different tower of curves. Similar to that situation, we can also define an eigencurve parametrizing the zeros of various $L(\chi, s)$, if properly normalized. In our setup, we can prove the following

Theorem 1.5. *The eigencurve constructed for the Artin-Schreier-Witt tower, when restricted to the rim of the weight space, is an infinite union of subspaces which are finite and flat over the weight annulus, with slopes given by arithmetic progressions governed by the weight parameter.*

We refer to Section 4 for a precise statement. We feel that this theorem provides evidence for the case of Coleman-Mazur eigencurves; however, we do point out that the two cases are essentially different in several ways.

We end the introduction with a few questions.

Question 1.6. We restricted ourselves to the case when all coefficients of $\hat{f}(x)$ are Teichmüller lifts. It would be interesting to know to what extent one can loosen this condition.

Question 1.7. We restricted ourselves to the case of $\mathbb{Z}_p$-extensions. One can certainly slightly modify the defining equation for C_m by changing $y_{\underline{m}}^F$ to $y_{\underline{m}}^{F^r}$ to consider a $\mathbb{Z}_{p^r}$-extension for some r. It would be interesting to know if the analogous statements of our main theorems continue to hold in this case.

Question 1.8. It is also natural to consider an Artin-Schreier-Witt extension of higher dimensional tori, as in [LW]. In this case, the expected slopes in Theorem 1.2 will be of a different form.

Question 1.9. We consider only Artin-Schreier-Witt towers in this paper. It would be interesting to know if the Newton slopes of the L-functions for other $\mathbb{Z}_p$-towers of curves in the literature enjoy a similar periodicity property.

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2. T-adic exponential sums

In this section, we recall properties of the L-function associated to a T-adic exponential sum as considered by C. Liu and the second author in [LW]; its specializations to appropriate values of T interpolate the L-functions considered above.

Definition 2.1. We use T as an indeterminate and let f and $\hat{f}$ be as in the introduction. For a positive integer k, the *T-adic exponential sum* of f over $\mathbb{F}_{q^k}^\times$ is the sum:

$$S^*(k,T) := \sum_{x \in \mathbb{F}_{q^k}^\times} (1+T)^{\mathrm{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(\hat{f}(\hat{x}))} \in \mathbb{Z}_p[\![T]\!].[1]$$

Note that the sum is taken over $\mathbb{F}_{q^k}^\times$ as opposed to $\mathbb{F}_{q^k}$. The $*$-notation simply reminds us that we are working over the torus $\mathbb{G}_m$. The associated *T-adic L-function* of f over $\mathbb{G}_{m,\mathbb{F}_q}$ is the generating function

(2.1.1)

$$L^*(T,s) = \exp\big(\sum_{k=1}^\infty S^*(k,T)\frac{s^k}{k}\big) = \prod_{x \in |\mathbb{G}_m|} \frac{1}{1-(1+T)^{\mathrm{Tr}_{\mathbb{Q}_{q^{\deg(x)}}/\mathbb{Q}_p}(\hat{f}(\hat{x}))} s^{\deg(x)}} \in 1 + s\mathbb{Z}_p[\![T]\!][\![s]\!].$$

We put

$$L(T,s) = \frac{L^*(T,s)}{\big(1-(1+T)^{\mathrm{Tr}_{\mathbb{Q}_q/\mathbb{Q}_p}(\hat{f}(0))}s\big)} = \prod_{x \in |\mathbb{A}^1|} \frac{1}{1-(1+T)^{\mathrm{Tr}_{\mathbb{Q}_{q^{\deg(x)}}/\mathbb{Q}_p}(\hat{f}(\hat{x}))} s^{\deg(x)}} \in 1+s\mathbb{Z}_p[\![T]\!][\![s]\!].[2]$$

Note that $L(T,s)$ (resp. $L^*(T,s)$) is the L-function over $\mathbb{A}^1$ (resp. $\mathbb{G}_m$) associated to the character $\mathrm{Gal}(C_\infty/\mathbb{A}^1) \cong \mathbb{Z}_p \to \mathbb{Z}_p[\![T]\!]^\times$ sending 1 to $1+T$. It is clear that for a finite character $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$, we have $L(T,s)|_{T=\pi_\chi} = L(\chi,s)$ for $\pi_\chi = \chi(1)-1$.

The T-adic L-function is a T-adic meromorphic function in s. It is often useful to consider a related holomorphic function in s defined as follows:

Definition 2.2. The *T-adic characteristic function* of f over $\mathbb{G}_{m,\mathbb{F}_q}$ is the generating function

$$C^*(T,s) = \exp\big(\sum_{k=1}^\infty \frac{1}{1-q^k} S^*(k,T)\frac{s^k}{k}\big).[3]$$

Clearly, we have

$$C^*(T,s) = L^*(T,s)L^*(T,qs)L^*(T,q^2s)\cdots, \quad \text{and} \quad L^*(T,s) = \frac{C^*(T,s)}{C^*(T,qs)}.$$

In particular, $C^*(T,s) \in 1 + s\mathbb{Z}_p[\![T]\!][\![s]\!]$.

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[1] This sum agrees with $S_f(k,T)$ in [LW] (for the one-dimensional case).

[2] Our $L^*(T,s)$ (but not $L(T,s)$) agrees with the $L_f(T,s)$ in [LW] (for the one-dimensional case).

[3] Our $C^*(T,s)$ agrees with the $C_f(T,s)$ in [LW] (for the one-dimensional case); we will not introduce a version $C(T,s)$ without the star since it will not be used in our proof.

Similarly, for a nontrivial finite character $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$, we put

$$L^*(\chi, s) = \big(1 - \chi(\mathrm{Tr}_{\mathbb{Q}_q/\mathbb{Q}_p}(\hat{f}(0)))s\big)L(\chi, s).$$

This is just the L-function of the character χ over $\mathbb{G}_{m,\mathbb{F}_q}$. The *characteristic function* for χ is defined to be

$$C^*(\chi, s) := L^*(\chi, s)L^*(\chi, qs)L^*(\chi, q^2s)\cdots.$$

It follows that

$$C^*(\chi, s) = C^*(T, s)|_{T=\pi_\chi} \quad \text{and} \quad L^*(\chi, s) = \frac{C^*(\chi, s)}{C^*(\chi, qs)}.$$

Theorem 2.3. *The T-adic characteristic function $C^*(T, s)$ is T-adically entire in s. Moreover, the $T^{a(p-1)}$-adic Newton polygon of $C^*(T, s)$ lies above the polygon whose slopes are $0, \frac{1}{d}, \frac{2}{d}, \ldots$, i.e., it lies above the polygon with vertices $(k, \frac{k(k-1)}{2d})$.*

Proof. The first statement is [LW, Theorem 4.8], and the second one follows from the Hodge bound [LW, Theorem 5.2]. Note that the polyhedron Δ in *loc. cit.* is nothing but the interval $[0, d]$, and the function $W(k)$ in *loc. cit.* is the constant function 1. So $\mathrm{HP}_q(\Delta)$ in *loc. cit.* is nothing but the polygon written in the statement of our theorem, after being renormalized from a T-adic Newton polygon to a $T^{a(p-1)}$-adic Newton polygon, namely, after scaling vertically by $1/a(p-1)$.

For the convenience of the reader, we sketch the proof in the simpler case when $a = 1$ (that is, $q = p$). We first recall that the Artin-Hasse exponential series is defined as

$$E(\pi) = \exp\big(\sum_{i=0}^{\infty} \frac{\pi^{p^i}}{p^i}\big) = \prod_{p\nmid i, i\geq 1} \big(1 - \pi^i\big)^{-\mu(i)/i} \in 1 + \pi + \pi^2\mathbb{Z}_p[\![\pi]\!],$$

where $\mu(\cdot)$ is the Möbius function. We take π to be the uniformizer in $\mathbb{Q}_p[\![T]\!]$ such that $T = E(\pi) - 1 = \pi+$ higher degree terms. Simple iteration calculation shows that $\pi \in T + T^2\mathbb{Z}_p[\![T]\!]$.

For our given polynomial $\hat{f}(x) = \sum_{i=0}^d \hat{b}_i x^i \in \mathbb{Z}_p[x]$, we put

$$E_f(x) = \prod_{i=0}^{d} E(\hat{b}_i \pi x^i) \in \mathbb{Z}_p[\![\pi]\!][\![x]\!].$$

Dwork's splitting lemma [LW, Lemma 4.3] (which can be checked by hand) says that for $x_0 \in \mathbb{F}_{p^k}$, we have

$$(1 + T)^{\mathrm{Tr}_{\mathbb{Z}_{p^k}/\mathbb{Z}_p}(\hat{f}(\hat{x}_0))} = E(\pi)^{\mathrm{Tr}_{\mathbb{Z}_{p^k}/\mathbb{Z}_p}(\hat{f}(\hat{x}_0))} = \prod_{j=0}^{k-1} E_f(\hat{x}_0^{p^j}).$$

Let B denote the Banach module over $\mathbb{Z}_p[\![\pi^{1/d}]\!]$ with the formal basis $\Gamma = \{1, \pi^{1/d}x, \pi^{2/d}x^2, \ldots\}$, that is,

$$B = \left\{ \sum_{i=0}^{\infty} c_i \pi^{i/d} x^i \mid c_i \in \mathbb{Z}_p[\![\pi^{1/d}]\!] \right\}.$$

It is clear that we can write

$$E_f(x) = \sum_{j=0}^{\infty} u_j \pi^{j/d} x^j \in B, \quad \text{for } u_j \in \mathbb{Z}_p.$$

Let ψ_p denote the operator on B defined by

$$\psi_p(\sum_{i\geq 0}^{\infty} c_i x^i) = \sum_{i\geq 0}^{\infty} c_{pi} x^i.$$

Let M be the matrix of the composite linear operator $\psi_p \circ E_f : B \to B$ with respect to the basis Γ, where E_f is just multiplication by the power series $E_f(x)$. One checks that

$$(\psi_p \circ E_f)\big(\pi^{i/d} x^i\big) = \sum_{j=0}^{\infty} u_{pj-i}\pi^{(p-1)j/d} \cdot (\pi^{j/d} x^j)$$

Therefore, the infinite matrix $M = (u_{pj-i}\pi^{(p-1)j/d})_{0\leq i,j<\infty}$ has the shape

$$M = \begin{pmatrix} * & * & * & \cdots \\ \pi^{\frac{p-1}{d}} * & \pi^{\frac{p-1}{d}} * & \pi^{\frac{p-1}{d}} * & \cdots \\ \pi^{\frac{2(p-1)}{d}} * & \pi^{\frac{2(p-1)}{d}} * & \pi^{\frac{2(p-1)}{d}} * & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $*$ denotes an element in $\mathbb{Z}_p[\![\pi^{1/d}]\!]$. It follows that the matrix M is nuclear.

Now, one computes that

$$(p-1)\mathrm{Tr}(M) = (p-1)\sum_{i=0}^{\infty} u_{(p-1)i}\pi^{(p-1)i/d} = \sum_{x\in\mathbb{F}_p^\times}(\sum_{j=0}^{\infty} u_j \pi^{j/d}\hat{x}^j) = \sum_{x\in\mathbb{F}_p^\times} E_f(\hat{x}) = \sum_{x\in\mathbb{F}_p^\times}(1+T)^{f(\hat{x})}.$$

Similarly, using the obvious property $G(x)\circ\psi_p = \psi_p \circ G(x^p)$ which holds for any power series $G(x)$, one checks that for every positive integer k, $(p^k-1)\mathrm{Tr}(M^k)$ is given by

$$(p^k-1)\mathrm{Tr}((\psi_p\circ E_f)^k) = (p^k-1)\mathrm{Tr}(\psi_p^k \circ \prod_{i=0}^{k-1} E_f(x^{p^i})) = \sum_{x\in\mathbb{F}_{p^k}^\times}\prod_{i=0}^{k-1} E_f(\hat{x}^{p^i}) = S^*(k,T).$$

This is the additive form of the T-adic Dwork's trace formula. Its equivalent multiplicative form is

$$C^*(T,s) = \exp\big(\sum_{k=1}^{\infty}\frac{1}{1-p^k}S^*(k,T)\frac{s^k}{k}\big) = \exp\big(\sum_{k=1}^{\infty} -\mathrm{Tr}(M^k)\frac{s^k}{k}\big) = \det\big(I - Ms).$$

From the nuclear shape of M, it is clear that the $T^{(p-1)}$-adic (i.e., $\pi^{(p-1)}$-adic) Newton polygon of the characteristic power series of M lies above the Hodge polygon of M, that is the polygon with slopes $0, \frac{1}{d}, \frac{2}{d}, \ldots$. □

3. Periodicity of Newton polygons

Using the interplay of the $T^{a(p-1)}$-adic Newton polygon of $C^*(T,s)$ and the $\pi_\chi^{a(p-1)}$-adic Newton polygon of $L^*(\chi,s)$, one can deduce strong periodicity results. From this, we recover results for $L(T,s)$ easily.

Notation 3.1. Let $C^*(T,s) = 1 + a_1(T)s + a_2(T)s^2 + \cdots \in 1 + s\mathbb{Z}_p[\![T]\!][\![s]\!]$ be the power series expansion in s. Put $a_0 = 1$. Then Theorem 2.3 says that

$$v_{T^{a(p-1)}}(a_n(T)) \geq \frac{n(n-1)}{2d}.$$

In other words, each $a_n(T)$ can be written as a power series in T:

$$a_n(T) = a_{n,\lambda_n}T^{\lambda_n} + a_{n,\lambda_n+1}T^{\lambda_n+1} + a_{n,\lambda_n+2}T^{\lambda_n+2} + \cdots,$$

with $a_{n,i} \in \mathbb{Z}_p$, $a_{n,\lambda_n} \neq 0$, and

$$\lambda_n \geq \frac{n(n-1)a(p-1)}{2d}.$$

We call $a_{n,\lambda_n}T^{\lambda_n}$ the *leading term* of $a_n(T)$, and a_{n,λ_n} the *leading coefficient* of $a_n(T)$.

Proposition 3.2. (1) *For every nontrivial finite character $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$, the $\pi_\chi^{a(p-1)}$-adic Newton polygon of $C^*(\chi, s)$ lies above the polygon whose slopes are $0, \frac{1}{d}, \frac{2}{d}, \ldots$, that is the polygon with vertices $(n, \frac{n(n-1)}{2d})$ for $n \in \mathbb{Z}_{\geq 0}$.*

(2) *Let χ_1 denote a nontrivial character of $\mathbb{Z}_p$ factoring through the quotient $\mathbb{Z}_p/p\mathbb{Z}_p$. The q-adic Newton polygon of $C^*(\chi_1, s)$ lies below the polygon starting at $(0,0)$, and then with a segment of slope 0, then $d-1$ segments of slope $\frac{1}{2}$, then a segment of slope 1, then $d-1$ segments of slope $1+\frac{1}{2}$, and so on.*

(3) *The q-adic Newton polygon of $C^*(\chi_1, s)$ contains the line segment connecting $\big(nd, \frac{n(nd-1)}{2}\big)$ and $\big(nd+1, \frac{n(nd-1)}{2} + n\big)$ for all $n \in \mathbb{Z}_{\geq 0}$.*

Proof. Part (1) follows from Theorem 2.3 and the simple fact that the $T^{a(p-1)}$-adic Newton polygon of $C^*(T, s)$ lies below the $\pi_\chi^{a(p-1)}$-adic Newton polygon of $C^*(T, s)|_{T=\pi_\chi} = C^*(\chi, s)$.

For (2), we first recall that

$$C^*(\chi_1, s) = L^*(\chi_1, s)L^*(\chi_1, qs)L^*(\chi_1, q^2 s)\cdots.$$

It suffices to show that the q-adic Newton polygon of $L^*(\chi_1, s)$ lies below the polygon starting at $(0,0)$ and with a segment of slope 0, and then $d-1$ segments of slope $\frac{1}{2}$. But this is clear, as $L^*(\chi_1, s)$ is the product of the factor $1-\chi(\mathrm{Tr}_{\mathbb{Q}_q/\mathbb{Q}_p}(\hat{f}(0)))s$, which has slope zero, with the L-function $L(\chi_1, s)$, whose q-adic Newton polygon starts at $(0,0)$ and ends at $(d-1, \frac{d-1}{2})$.

Part (3) follows from the combination of (1) and (2) because the upper bound and lower bound of the q-adic Newton polygon of $C^*(\chi_1, s)$ agree on the line segments connecting $\big(nd, \frac{n(nd-1)}{2}\big)$ and $\big(nd+1, \frac{n(nd-1)}{2}+n\big)$ for all $n \in \mathbb{Z}_{\geq 0}$. Indeed, the upper bound y-coordinate corresponding to $x = nd$ in (2) is $n\frac{d-1}{2} + \frac{n(n-1)}{2}d = \frac{n(nd-1)}{2}$. □

Remark 3.3. Proposition 3.2(3) above, concerning the agreement of the Newton polygon upper and lower bounds, is the key point of this paper. See Figure 1 for an illustration of the upper and lower bounds in the special case $d = 4$.

Proposition 3.4. *For an integer k congruent to 0 or 1 modulo d, the leading term $a_{k,\lambda_k}T^{\lambda_k}$ of $a_k(T)$ satisfies $\lambda_k = \frac{k(k-1)a(p-1)}{2d}$ and $a_{k,\lambda_k} \in \mathbb{Z}_p^\times$.*

Proof. From Proposition 3.2(3), we know the q-adic Newton polygon for $C^*(\chi_1, s)$ passes through the points $\Big(nd, \frac{n(nd-1)}{2}\Big)$ and $\Big(nd+1, \frac{n(nd-1)}{2} + n\Big)$. By considering the lower bound for the Newton polygon given by Proposition 3.2(1), the slope in the preceding segment must be less than $\frac{d-1}{d} + (n-1)$, and the slope in the following segment must be greater than $n + \frac{1}{d}$. This shows that these lattice points $\Big(nd, \frac{n(nd-1)}{2}\Big)$ and $\Big(nd+1, \frac{n(nd-1)}{2} + n\Big)$ must be vertices

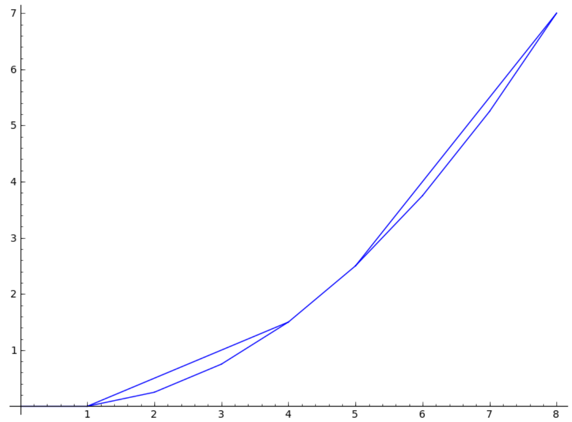

FIGURE 1. The upper and lower bounds for the Newton polygon of the characteristic function over the interval $[0, 8]$ for the special case $d = 4$.

for both the q-adic Newton polygon of $C^*(\pi_{\chi_1}, s)$ and the $T^{a(p-1)}$-adic Newton polygon of $C^*(T, s)$. Hence we must have

$$v_{\pi_{\chi_1}}(a_{nd}(\pi_{\chi_1})) = a(p-1)\frac{n(nd-1)}{2} = v_T(a_{nd}(T))$$

$$v_{\pi_{\chi_1}}(a_{nd+1}(\pi_{\chi_1})) = a(p-1)\left(\frac{n(nd-1)}{2} + n\right) = v_T(a_{nd+1}(T)).$$

This shows that both λ_{nd} and λ_{nd+1} have the stated values and also shows that the leading coefficients of $a_{nd}(T)$ and $a_{nd+1}(T)$ are indeed p-adic units. $\square$

Corollary 3.5. *For every nontrivial finite character $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$, the $\pi_\chi^{a(p-1)}$-adic Newton polygon of $C^*(\chi, s)$ contains the line segment connecting $\left(nd, \frac{n(nd-1)}{2}\right)$ and $\left(nd+1, \frac{n(nd-1)}{2}+n\right)$ for all $n \in \mathbb{Z}_{\geq 0}$. Therefore, the $\pi_\chi^{a(p-1)}$-adic Newton polygon of $C^*(\chi, s)$ has the same upper bound polygon as described in Proposition 3.2(2).*

Proof. By Proposition 3.4, for $k = nd$ and $nd + 1$, we have

$$a_k(T) = a_{k,\lambda_k} T^{\lambda_k} + \text{ terms with higher power in } T,$$

where $\lambda_k = \frac{k(k-1)a(p-1)}{2d}$ and $a_{k,\lambda_k} \in \mathbb{Z}_p^\times$. Thus, we have

$$v_{\pi_\chi^{a(p-1)}}\big(a_k(\pi_\chi)\big) = \frac{k(k-1)a(p-1)}{2d}/(a(p-1)) = \frac{k(k-1)}{2d}.$$

Note that this agrees with the lower bound of the $\pi_\chi^{a(p-1)}$-adic Newton polygon of $C^*(\chi, s)$ given by Proposition 3.2(1). So it forces the $\pi_\chi^{a(p-1)}$-adic Newton polygon of $C^*(\chi, s)$ to contain the segment connecting $\left(nd, \frac{n(nd-1)}{2}\right)$ and $\left(nd+1, \frac{n(nd-1)}{2} + n\right)$. The last statement follows from the convexity of Newton polygons. $\square$

Corollary 3.6. *The $T^{a(p-1)}$-adic Newton polygon of $C^*(T, s)$ contains a line segment connecting $\left(nd, \frac{n(nd-1)}{2}\right)$ and $\left(nd+1, \frac{n(nd-1)}{2} + n\right)$ for all $n \in \mathbb{Z}_{\geq 0}$.*

Proof. Note that the lower bound of the $T^{a(p-1)}$-adic Newton polygon of $C^*(T,s)$ given by Theorem 2.3 is achieved at the points $x = nd$ or $nd+1$, as shown in Proposition 3.4. The corollary follows by convexity. □

Lemma 3.7. *For every nontrivial finite character $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$, the maximum gap between the lower bound and the upper bound for the $\pi_\chi^{a(p-1)}$-adic Newton polygon of $C^*(\chi,s)$, given in Proposition 3.2(1) and Corollary 3.5, is $\frac{(d-1)^2}{8d}$.*

Proof. It suffices to consider the block above the x-axis interval $[1,d]$, because the upper and lower bounds in later intervals are obtained from the bounds in this interval by adding the same constant to both. At the positive integer $x = i$ in the interval $[1,d]$, the gap is $g(i) = \frac{i-1}{2} - \frac{i(i-1)}{2d} = \frac{i(d+1)-d-i^2}{2d}$. The maximum of this function is $g\left(\frac{d+1}{2}\right) = \frac{(d-1)^2}{8d}$. □

Theorem 3.8. *For a finite character $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$ with $m_\chi \geq 1 + \log_p(\frac{a(d-1)^2}{8d})$, the $\pi_\chi^{a(p-1)}$-adic Newton polygon of $C^*(\chi,s)$ is independent of the character χ.*

Proof. By definition, the $\pi_\chi^{a(p-1)}$-adic Newton polygon of $C^*(\chi,s)$ is the convex hull of points $\big(i, v_{\pi_\chi^{a(p-1)}}(a_i(\pi_\chi))\big)$ for all $i \geq 0$. Since we have already given a (very strong) upper bound for this polygon in Corollary 3.5, we need only to consider those points which lies below this upper bound, which we will prove to be independent of the choice of χ, provided that $m_\chi \geq 1 + \log_p(\frac{a(d-1)^2}{8d})$. We assume this inequality for the rest of the proof.

For each integer $i \geq 0$, write $a_i(T) = a_{i,\lambda_i}T^{\lambda_i} + a_{i,\lambda_i+1}T^{\lambda_i+1} + \cdots$ for $a_{i,j} \in \mathbb{Z}_p$ and $a_{i,\lambda_i} \neq 0$. Then $\lambda_i \geq \frac{i(i-1)a(p-1)}{2d}$ by Theorem 2.3. Let λ_i' denote the minimal integer such that $a_{i,\lambda_i'}$ is a p-adic unit. We claim the following:

- If the point $\big(i, v_{\pi_\chi^{a(p-1)}}(a_i(\pi_\chi))\big)$ lies below the upper bound polygon in Corollary 3.5, then $v_{\pi_\chi^{a(p-1)}}(a_i(\pi_\chi)) = \lambda_i'/a(p-1)$.

Note that the number λ_i' is independent of χ. So the theorem follows from this claim.

We now prove the claim by studying $a_i(\pi_\chi) = a_{i,\lambda_i}\pi_\chi^{\lambda_i} + a_{i,\lambda_i+1}\pi_\chi^{\lambda_i+1} + \cdots$. For $j \in [\lambda_i, \lambda_i')$, the coefficient $a_{i,j}$ belongs to $\mathbb{Z}_p$ and is divisible by p; so the valuation of the term is

$$v_{\pi_\chi^{a(p-1)}}\left(a_{i,j}\pi_\chi^j\right) \geq v_{\pi_\chi^{a(p-1)}}(p) + \frac{\lambda_i}{a(p-1)} = \frac{p^{m_\chi-1}}{a} + \frac{\lambda_i}{a(p-1)} \geq \frac{(d-1)^2}{8d} + \frac{\lambda_i}{a(p-1)},$$

which corresponds to a point lying on or above the upper bound polygon by Lemma 3.7. Therefore, if the point $\big(i, v_{\pi_\chi^{a(p-1)}}(a_i(\pi_\chi))\big)$ lies below the upper bound polygon, the valuation must come from the valuation of the term $a_{i,\lambda_i'}\pi_\chi^{\lambda_i'}$, which has $\pi_\chi^{a(p-1)}$-valuation $\lambda_i'/a(p-1)$. This proves the claim and hence the theorem. □

Remark 3.9. In fact, the same proof shows that, for any (not necessarily finite) character $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$ with $|\pi_\chi| = |\chi(1) - 1| \geq \frac{8d}{a(p-1)(d-1)^2}$, the $\pi_\chi^{a(p-1)}$-adic Newton polygon of $C^*(\chi,s)$ is independent of χ.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Recall our setup: put $m_0 = 1 + \lceil \log_p(\frac{a(d-1)^2}{8d}) \rceil$ and $\chi_0 : \mathbb{Z}_p \to \mathbb{C}_p^\times$, a finite character with $m_{\chi_0} = m_0$. Let $0 < \alpha_1, \ldots, \alpha_{dp^{m_0-1}-1} < 1$ denote the slopes of the

q-adic Newton polygon of $L(\chi_0, s)$. Then $0, \alpha_1, \ldots, \alpha_{dp^{m_0-1}-1}$ are the slopes of the q-adic Newton polygon of $L^*(\chi_0, s)$, and hence

$$\bigcup_{i\geq 0}\{i, \alpha_1 + i, \ldots, \alpha_{dp^{m_0-1}-1} + i\} \tag{3.9.1}$$

are the slopes of the q-adic Newton polygon of $C^*(\chi_0, s)$. Since $v(q) = a(p-1)p^{m_0-1}v(\pi_{\chi_0})$, the slopes of the $\pi_{\chi_0}^{a(p-1)}$-adic Newton polygon of $C^*(\chi_0, s)$ are rescaled to

$$\bigcup_{i\geq 0}\{p^{m_0-1}i, p^{m_0-1}(\alpha_1 + i), \ldots, p^{m_0-1}(\alpha_{dp^{m_0-1}-1} + i)\}. \tag{3.9.2}$$

Now given a finite character $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$ with $m_\chi \geq m_0$, Theorem 3.8 says that the slopes of the $\pi_\chi^{a(p-1)}$-adic Newton polygon of $C^*(\chi, s)$ are also given by (3.9.2). Now $v(q) = a(p-1)p^{m-1}v(\pi_\chi)$, the slopes of the q-adic Newton polygon of $C^*(\chi, s)$ are rescaled to

$$\bigcup_{i\geq 0}\{p^{m_0-m}i, p^{m_0-m}(\alpha_1 + i), \ldots, p^{m_0-m}(\alpha_{dp^{m_0-1}-1} + i)\}. \tag{3.9.3}$$

Using the relation

$$L(\chi, s) = \frac{1}{1 - \chi(\mathrm{Tr}_{\mathbb{Q}_q/\mathbb{Q}_p}(\hat{f}(0)))s} \cdot \frac{C^*(\chi, s)}{C^*(\chi, qs)},$$

it is clear that the slopes of the q-adic Newton polygon of $L(\chi, s)$ are as described in Theorem 1.2. $\square$

Remark 3.10. It would be interesting to know whether the slopes of the $T^{a(p-1)}$-adic Newton polygon of $C^*(T, s)$ satisfy a similar periodicity property, i.e., whether the slopes form a disjoint union of finite number of arithmetic progressions. This is known to be true if $p \equiv 1 \bmod d$ (see the references given in Example 1.4), but open in general.

We give another application. Recall that the T-adic L-function $L(T, s)$ is a T-adic meromorphic (but in general not rational) function in s over the field $\mathbb{Q}_p((T))$. Thus, adjoining all the infinitely many zeros and poles of $L(T, s)$ to the field $\mathbb{Q}_p((T))$ would generally give an infinite extension. The following result shows this is not the case.

Theorem 3.11. *The splitting field $K_{p,d}$ over $\mathbb{Q}_p((T))$ of all T-adic L-functions $L(T, s)$ for* all *monic polynomials $f \in \overline{\mathbb{F}}_p[x]$ of degree d is a finite extension of $\mathbb{Q}_p((T))$.*

Proof. It suffices to prove that the splitting field $K'_{p,d}$ over $\mathbb{Q}_p((T))$ of all T-adic characteristic functions $C^*(T, s)$ for all monic polynomials $f \in \overline{\mathbb{F}}_p[x]$ of degree d is a finite extension of $\mathbb{Q}_p((T))$. By Corollary 3.6, the power series $C^*(T, s)$ factors as an infinite product of polynomials (in the variable s) of degree $\leq d$. So the splitting field $K'_{p,d}$ is contained in the extension $\widehat{K}_{p,d}$ of $\mathbb{Q}_p((T))$ given by adjoining zeros of all irreducible polynomials of degree $\leq d$. But this is a finite extension of $\mathbb{Q}_p((T))$ as proved as follows.

First, since $\mathbb{Q}_p$ has characteristic zero, all extensions of $\mathbb{Q}_p((T))$ are tamely ramified. If we only adjoin irreducible polynomials of degree $\leq d$, then $\widehat{K}_{p,d}$ is an extension of $\mathbb{Q}_p((T))$ with T-ramification degree $\leq d!$. It then suffices to bound the residual extension; that is to prove that the composite of all finite Galois extension of $\mathbb{Q}_p$ of degree $\leq d!$ is still a finite extension. In fact, we will show that there are only finitely many Galois extensions of $\mathbb{Q}_p$ with degree $\leq d!$. For this, we first notice that there are only finitely many choices of Galois groups with order $\leq d!$, which are all solvable. So we just need to prove that there are only finitely many *abelian* extensions at each step. But this is clear from local class field theory. □

Remark 3.12. One may be able to give a more precise bound on the extension degree of $K_{p,d}$ over $\mathbb{Q}_p((T))$. We leave this to the interested reader.

Remark 3.13. The above result proves a strong form of the T-adic Riemann hypothesis for the T-adic L-function in [LW] in the sense of Goss [Go], see [W1] and [Sh] for evidence for Goss's original conjecture for his characteristic p zeta functions. For Dwork's unit root zeta function which is known to be a p-adic meromorphic function, the corresponding p-adic Riemann hypothesis, even the weaker version about the finiteness of the ramification of the splitting field over $\mathbb{Q}_p$, is essentially completely open; see Conjecture 1.3 in [W3]. For the characteristic power series of the U_p-operator acting on the p-adic Banach space of overconvergent p-adic modular forms of a given level and weight, the finiteness of the ramification of the splitting field over $\mathbb{Q}_p$ is also unknown; see Conjecture 6.1 in [W2]. An example of the p-adic Riemann hypothesis for zeta functions of divisors is given in [WH].

Remark 3.14. Another natural problem is to study the possible simplicity of the zeros of the T-adic characteristic series $C^*(T,s)$. The simplicity is known in the case $p \equiv 1 \pmod d$. It would be interesting to know if simplicity remains true in general.

A similar proof gives the following more classical application.

Theorem 3.15. *Let $E_{p,d}(m)$ be the splitting field over $\mathbb{Q}_p$ of all zeta functions $Z(C_m,s)$ for* all *monic polynomials $f \in \bar{\mathbb{F}}_p[x]$ of degree d. Then there is an explicit constant B_d depending only on d such that for all $m \geq 1$, we have*

$$[E_{p,d}(m) : \mathbb{Q}_p] \leq B_d p^{m-1}.$$

Furthermore, the inequality is an equality with $B_d = d$ if $p \equiv 1 \pmod d$.

4. Eigencurves for Artin-Schreier-Witt towers

For the Igusa tower coming from modular curves, Coleman and Mazur also study a certain analogous T-adic characteristic function; from this, they define an eigencurve parametrizing the zeros of the characteristic function. This eigencurve has many applications in number theory.

One of the striking results about the Coleman-Mazur eigencurve is its nice behavior near the boundary of the weight space, as shown by Buzzard and Kilford in [BK] when $p = 2$. Unfortunately, such a result is only known for very small prime numbers p, or for small discs near the boundary of the weight space [XZ].

In this section, we study the analogous construction for the Artin-Schreier-Witt tower of curves, and we prove strong geometric properties of the analogous eigencurve near the boundary of the weight space.

Definition 4.1. Let $\mathcal{W}$ denote the the rigid analytic open unit disc associated to $\mathbb{Z}_p[\![T]\!]$. The *eigencurve* $\mathcal{C}_f$ associated to the Artin-Schreier-Witt tower for $f(x)$ is defined to be the zero locus of $C^*(T,s)$, viewed as a rigid analytic subspace of $\mathcal{W} \times \mathbb{G}_{m,\mathrm{rig}}$, where s is the coordinate of the second factor. Denote the natural projection to the first factor by $\mathrm{wt} : \mathcal{C}_f \to \mathcal{W}$; and denote the *inverse* of the natural projection to the second factor by

$$\boldsymbol{\alpha} : \mathcal{C}_f \xrightarrow{\mathrm{pr}_2} \mathbb{G}_{m,\mathrm{rig}} \xrightarrow{x \mapsto x^{-1}} \mathbb{G}_{m,\mathrm{rig}}.$$

For a closed point w on $\mathcal{W}$, we use $v_{\mathcal{W}}(w)$ to denote the p-adic valuation of the T-coordinate of w. Similarly, for a closed point $z \in \mathbb{G}_{m,\mathrm{rig}}$, we use $v_{\mathbb{G}_m}(z)$ to denote the p-adic valuation of the s-coordinate of z.

Theorem 4.2. *The following properties hold for the eigencurve $\mathcal{C}_f$:*

(1) *The formal power series $C^*(T,s)$ can be written as an infinite product $\prod_{i=0}^{\infty} P_i(s)$, where each polynomial $P_i(s) = 1 + b_{i,1}(T)s + \cdots + b_{i,d}(T)s^d$ belongs to $\mathbb{Z}_p[\![T]\!][s]$, whose $T^{a(p-1)}$-adic Newton polygon accounts for the segment between $x \in [(i-1)d, id-1]$ of the $T^{a(p-1)}$-adic Newton polygon of $C^*(T,s)$, and the leading term of $b_{i,d}(T)$ has coefficients in $\mathbb{Z}_p^\times$.*

(2) *The eigencurve $\mathcal{C}_f$ is an infinite disjoint union $\coprod_{i\geq 0} \mathcal{C}_{f,i}$, where each $\mathcal{C}_{f,i}$ is the zero locus of the polynomial $P_i(s)$ and it is a finite and flat cover of $\mathcal{W}$ of degree d.*

(3) *Put $r = \frac{8d}{a(p-1)(d-1)^2}$. Let $\mathcal{W}^{\geq r}$ denote the annulus inside $\mathcal{W}$ where $|T| \geq r$. Then there exist an integer $l \in \mathbb{N}$ and (distinct) rational numbers $\beta_1, \dots, \beta_l \in [0,1)$ such that each $\mathcal{C}_{f,i} \times_{\mathcal{W}} \mathcal{W}^{\geq r}$ is a disjoint union $\coprod_{j=1}^{l} \mathcal{C}_{f,i}^{(j)}$ of closed subspaces of $\mathcal{C}_{f,i}$, each being finite and flat over $\mathcal{W}^{\geq r}$, and is characterized by the following property:*

$$\forall z \in \mathcal{C}_{f,i}^{(j)}, \quad v_{\mathbb{G}_m}\big(\boldsymbol{\alpha}(z)\big) = ap^{m_0-1}(p-1)(\beta_j + i)v_{\mathcal{W}}\big(\mathrm{wt}(z)\big)$$

Proof. The decomposition in (1) follows from the basic fact on the relation between Newton polygons and factorizations, in light of Corollary 3.6. Moreover, by Proposition 3.4, all $b_{i,j}(T)$ has coefficients in $\mathbb{Z}_p$ and the leading term of $b_{i,d}(T)$ is a p-adic unit.

Having the factorization at hand, it is clear that $\mathcal{C}_f$ is the union of the zero loci of the $P_i(s)$'s, which are closed analytic subspaces $\mathcal{C}_{f,i}$ of $\mathcal{W} \times \mathbb{G}_{m,\mathrm{rig}}$. Moreover, since for any character $\chi : \mathbb{Z}_p \to \mathbb{C}_p^\times$ with $\pi_\chi := \chi(1) - 1$, the slopes of the $\pi_\chi^{a(p-1)}$-adic Newton polygons of $P_i(s)|_{T=\pi_\chi}$ sits in $[i-1, i)$. So the zeros of $P_i(s)|_{T=\pi_\chi}$ are distinct for different i. This implies that all subspaces $\mathcal{C}_{f,i}$ are disjoint and concludes the proof of (2).

For (3), let $\chi_0 : \mathbb{Z}_p \to \mathbb{C}_p^\times$ be a finite character with $m_{\chi_0} = 1 + \lceil \log_p \frac{a(d-1)^2}{8d} \rceil$. Let $0, \alpha_1, \dots, \alpha_{dp^{m_0-1}-1}$ be the q-adic slopes in $L^*(\chi_0, s)$ (counting multiplicities). These are rational numbers in the interval $[0,1)$. Each point $w \in \mathcal{W}$ gives rise to a (not necessarily finite) character $\chi_w : \mathbb{Z}_p \to \mathbb{C}_p^\times$; put $\pi_{\chi_w} = \chi_w(1) - 1$. By Remark 3.9, the slopes of the $\pi_{\chi_w}^{a(p-1)}$-adic Newton polygon of $C^*(T,s)|_{T=\pi_{\chi_w}}$ are exactly

$$\bigcup_{i \in \mathbb{Z}_{\geq 0}} \big\{p^{m_0-1}i, p^{m_0-1}(\alpha_1 + i), \dots, p^{m_0-1}(\alpha_{dp^{m_0-1}-1} + i)\big\}. \tag{4.2.1}$$

Then the slopes of the p-adic Newton polygon of $C^*(T,s)|_{T=\pi_{\chi_w}}$ should be given by the numbers in (4.2.1) times the normalizing factor $a(p-1)v_p(\pi_{\chi_w}) = a(p-1)v_{\mathcal{W}}(w)$.

Let $0 = \beta_1 < \cdots < \beta_l < 1$ be the slopes $0, \alpha_1, \dots, \alpha_{dp^{m_0-1}-1}$ with repeated numbers removed. Then the slope information above implies that the intersections $\mathcal{C}_{f,i}^{(j)}$ of $\mathcal{C}_{f,i}$ with the subdomain

$$\big\{z \in \mathcal{W}^{\geq r} \times \mathbb{G}_{m,\mathrm{rig}} \;\big|\; v_{\mathbb{G}_m}\big(\boldsymbol{\alpha}(z)\big) = ap^{m_0-1}(p-1)(\beta_j + i)v_{\mathcal{W}}\big(\mathrm{wt}(z)\big)\big\}$$

form a finite cover of $\mathcal{C}_{f,i} \times_{\mathcal{W}} \mathcal{W}^{\geq r}$ by affinoid subdomains. The union

$$\mathcal{C}_{f,i} \times_{\mathcal{W}} \mathcal{W}^{\geq r} = \coprod_{j=1}^{l} \mathcal{C}_{f,i}^{(j)}$$

is a disjoint union. Each $\mathcal{C}_{f,i}^{(j)}$ is finite and flat over $\mathcal{W}^{\geq r}$ because its fiber over every point of $\mathcal{W}^{\geq r}$ is exactly the multiplicity of β_j in the collection of α's above. The assertions in (3) are now proved. $\square$

Remark 4.3. The analogous statement of Theorem 4.2(2) for Coleman-Mazur eigencurve is probably too strong to be true. But it is generally believed that, at least under certain conditions, the analogous statement of Theorem 4.2(3) for Coleman-Mazur eigencurve holds.

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