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ON THE CLASSIFICATION OF NONSIMPLE GRAPH C^* -ALGEBRAS

SØREN EILERS AND MARK TOMFORDE

ABSTRACT. We prove that a graph C^* -algebra with exactly one proper nontrivial ideal is classified up to stable isomorphism by its associated six-term exact sequence in K -theory. We prove that a similar classification also holds for a graph C^* -algebra with a largest proper ideal that is an AF-algebra. Our results are based on a general method developed by the first named author with Restorff and Ruiz. As a key step in the argument, we show how to produce stability for certain full hereditary subalgebras associated to such graph C^* -algebras. We further prove that, except under trivial circumstances, a unique proper nontrivial ideal in a graph C^* -algebra is stable.

1. INTRODUCTION

The classification program for C^* -algebras has for the most part progressed independently for the classes of infinite and finite C^* -algebras. Great strides have been made in this program for each of these classes. In the finite case, Elliott's Theorem classifies all AF-algebras up to stable isomorphism by the ordered K_0 -group. In the infinite case, there are a number of results for purely infinite C^* -algebras. The Kirchberg-Phillips Theorem classifies certain simple purely infinite C^* -algebras up to stable isomorphism by the K_0 -group together with the K_1 -group. For nonsimple purely infinite C^* -algebras many partial results have been obtained: Rørdam has shown that certain purely infinite C^* -algebras containing exactly one proper nontrivial ideal are classified up to stable isomorphism by the associated six-term exact sequence of K -groups [Rør97], Restorff has shown that nonsimple Cuntz-Krieger algebras satisfying Condition (II) are classified up to stable isomorphism by their filtrated K -theory [Res06, Theorem 4.2], and Meyer and Nest have shown that certain purely infinite C^* -algebras with a linear ideal lattice are classified up to stable isomorphism by their filtrated K -theory [MN, Theorem 4.14]. However, in all of these situations the nonsimple C^* -algebras that are classified have the property that they are either AF-algebras or purely infinite, and consequently all of their ideals and quotients are of the same type.

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Recently, the first named author with Restorff and Ruiz have provided a framework for classifying nonsimple C^* -algebras that are not necessarily AF-algebras or purely infinite C^* -algebras. In particular, these authors have shown in [ERR] that certain extensions of classifiable C^* -algebras may be classified up to stable isomorphism by their associated six-term exact sequence in K -theory. This has allowed for the classification of certain nonsimple C^* -algebras in which there are ideals and quotients of mixed type (some finite and some infinite).

In this paper we consider the classification of nonsimple graph C^* -algebras. Simple graph C^* -algebras are known to be either AF-algebras or purely infinite algebras, and thus are classified by their K -groups according to either Elliott's Theorem or the Kirchberg-Phillips Theorem. Therefore, we begin by considering nonsimple graph C^* -algebras with exactly one proper nontrivial ideal. These C^* -algebras will be extensions of simple C^* -algebras that are AF or purely infinite by other simple C^* -algebras that are AF or purely infinite — with mixing of the types allowed. These nonsimple graph C^* -algebras are similar to the extensions considered in [ERR], however, the results of [ERR] do not apply directly. Instead, we must do a fair bit of work, using the techniques from the theory of graph C^* -algebras, to show that the machinery of [ERR] can be used to classify these extensions; it is verifying the requirement of **fullness** that is most difficult in this context. Ultimately, however, we are able to show that a graph C^* -algebra with exactly one proper nontrivial ideal is classified up to stable isomorphism by the six-term exact sequence in K -theory of the corresponding extension. Additionally, we are able to show that a graph C^* -algebra with a largest proper ideal that is an AF-algebra is also classified up to stable isomorphism by the six-term exact sequence in K -theory of the corresponding extension.

It is also worthwhile to note that the extensions of graph C^* -algebras classified in this paper constitute a very large class. Every AF-algebra is stably isomorphic to a graph C^* -algebra, and every Kirchberg algebra with free K_1 -group is stably isomorphic to a graph C^* -algebra. Thus the extensions we consider comprise a wide variety of extensions of AF-algebras (respectively, purely infinite algebras) by purely infinite algebras (respectively, AF-algebras).

While there is little hope to generalize the methods in [ERR] to general (even finite) ideal lattices in a context covering all graph C^* -algebras, the classifications we obtain in this paper suggest that a complete classification of graph C^* -algebras generalizing the Cuntz-Krieger case solved in [Res06] may be possible by other methods. Such a result may involve generalizing the work of Boyle and Huang to graph C^* -algebras and mimicking the approach used by Restorff; or perhaps it may be accomplished by generalizing Kirchberg's isomorphism theorem to allow for subquotients which are AF-algebras, and then in this special case (probably using the global vanishing of one connecting map of K -theory) overcoming the difficulties of projective dimension of the invariants exposed by Meyer and Nest. Neither of these

approaches seem within immediate reach, but both appear plausible to be successful at some future stage. It is also an open problem, and possibly much less difficult, to establish isomorphism directly between unital graph C^* -algebras by keeping track of the class of the unit in the K_0 -groups. We mention that the methods of [RR07] do not seem to generalize to this setting.

This paper is organized as follows. In §2 we establish notation and conventions for graph C^* -algebras and extensions. In §3 we derive a number of preliminary results for graph C^* -algebras with the goal of applying the methods of [ERR]. In §4 we use our results from §3 and the results of [ERR] to prove our two main theorems: In Theorem 4.5 we show that if A is a graph C^* -algebra with exactly one proper nontrivial ideal I , then A is classified up to stable isomorphism by the six-term exact sequence in K -theory coming from the extension $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$. In Theorem 4.7 we show that if A is a graph C^* -algebra with a largest proper ideal I that is an AF-algebra, then A is classified up to stable isomorphism by the six-term exact sequence in K -theory coming from the extension $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$. In §5 we consider a variety of examples, and also use our results to classify the stable isomorphism classes of the C^* -algebras of all graphs having exactly two vertices and satisfying Condition (K). (Be aware that although these graphs have only two vertices, the graphs are allowed to contain a finite or countably infinite number of edges.) We find that even for this small collection of graphs, the associated C^* -algebras fall into a variety of stable isomorphism classes, and there are quite a few cases to consider. We conclude in §6 by proving that if A is a graph C^* -algebra that is not a nonunital AF-algebra, and if A contains a unique proper nontrivial ideal I , then I is stable.

2. NOTATION AND CONVENTIONS

We establish some basic facts and notation for graph C^* -algebras and extensions.

2.1. Notation and conventions for graph C^* -algebra. A (directed) graph $E = (E^0, E^1, r, s)$ consists of a countable set E^0 of vertices, a countable set E^1 of edges, and maps $r, s : E^1 \rightarrow E^0$ identifying the range and source of each edge. A vertex $v \in E^0$ is called a *sink* if $|s^{-1}(v)| = 0$, and v is called an *infinite emitter* if $|s^{-1}(v)| = \infty$. A graph E is said to be *row-finite* if it has no infinite emitters. If v is either a sink or an infinite emitter, then we call v a *singular vertex*. We write E_{sing}^0 for the set of singular vertices. Vertices that are not singular vertices are called *regular vertices* and we write E_{reg}^0 for the set of regular vertices. For any graph E , the *vertex matrix* is the $E^0 \times E^0$ matrix A_E with $A_e(v, w) := |\{e \in E^1 : s(e) = v \text{ and } r(e) = w\}|$. Note that the entries of A_E are elements of $\{0, 1, 2, \dots\} \cup \{\infty\}$.

If E is a graph, a *Cuntz-Krieger E -family* is a set of mutually orthogonal projections $\{p_v : v \in E^0\}$ and a set of partial isometries $\{s_e : e \in E^1\}$ with orthogonal ranges which satisfy the *Cuntz-Krieger relations*:

$$(1) \quad s_e^* s_e = p_{r(e)} \text{ for every } e \in E^1;$$

- (2) $s_e s_e^* \leq p_{s(e)}$ for every $e \in E^1$;
(3) $p_v = \sum_{s(e)=v} s_e s_e^*$ for every $v \in E^0$ that is not a singular vertex.

The *graph algebra* $C^*(E)$ is defined to be the C^* -algebra generated by a universal Cuntz-Krieger E -family.

A *path* in E is a sequence of edges $\alpha = \alpha_1 \alpha_2 \dots \alpha_n$ with $r(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i < n$, and we say that α has length $|\alpha| = n$. We let E^n denote the set of all paths of length n , and we let $E^* := \bigcup_{n=0}^{\infty} E^n$ denote the set of finite paths in G . Note that vertices are considered paths of length zero. The maps r, s extend to E^* , and for $v, w \in E^0$ we write $v \geq w$ if there exists a path $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = w$. Also for a path $\alpha := \alpha_1 \dots \alpha_n$ we define $s_\alpha := s_{\alpha_1} \dots s_{\alpha_n}$, and for a vertex $v \in E^0$ we let $s_v := p_v$. It is a consequence of the Cuntz-Krieger relations that $C^*(E) = \overline{\text{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta)\}$.

We say that a path $\alpha := \alpha_1 \dots \alpha_n$ of length 1 or greater is a *cycle* if $r(\alpha) = s(\alpha)$, and we call the vertex $s(\alpha) = r(\alpha)$ the *base point* of the cycle. A cycle is said to be *simple* if $s(\alpha_i) \neq s(\alpha_1)$ for all $1 < i \leq n$. The following is an important condition in the theory of graph C^* -algebras.

Condition (K): No vertex in E is the base point of exactly one simple cycle; that is, every vertex is either the base point of no cycles or at least two simple cycles.

For any graph E a subset $H \subseteq E^0$ is *hereditary* if whenever $v, w \in E^0$ with $v \in H$ and $v \geq w$, then $w \in H$. A hereditary subset H is *saturated* if whenever $v \in E_{\text{reg}}^0$ with $r(s^{-1}(v)) \subseteq H$, then $v \in H$. For any saturated hereditary subset H , the *breaking vertices* corresponding to H are the elements of the set

$$B_H := \{v \in E^0 : |s^{-1}(v)| = \infty \text{ and } 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty\}.$$

An *admissible pair* (H, S) consists of a saturated hereditary subset H and a subset $S \subseteq B_H$. For a fixed graph E we order the collection of admissible pairs for E by defining $(H, S) \leq (H', S')$ if and only if $H \subseteq H'$ and $S \subseteq H' \cup S'$. For any admissible pair (H, S) we define

$$I_{(H,S)} := \text{the ideal in } C^*(E) \text{ generated by } \{p_v : v \in H\} \cup \{p_{v_0}^H : v_0 \in S\},$$

where $p_{v_0}^H$ is the *gap projection* defined by

$$p_{v_0}^H := p_{v_0} - \sum_{\substack{s(e)=v_0 \\ r(e) \notin H}} s_e s_e^*.$$

Note that the definition of B_H ensures that the sum on the right is finite.

For any graph E there is a canonical gauge action $\gamma : \mathbb{T} \rightarrow \text{Aut } C^*(E)$ with the property that for any $z \in \mathbb{T}$ we have $\gamma_z(p_v) = p_v$ for all $v \in E^0$ and $\gamma_z(s_e) = z s_e$ for all $e \in E^1$. We say that an ideal $I \triangleleft C^*(E)$ is *gauge invariant* if $\gamma_z(I) \subseteq I$ for all $z \in \mathbb{T}$.

There is a bijective correspondence between the lattice of admissible pairs of E and the lattice of gauge-invariant ideals of $C^*(E)$ given by $(H, S) \mapsto I_{(H,S)}$ [BHRS, Theorem 3.6]. When E satisfies Condition (K), all ideals of $C^*(E)$ are gauge invariant [Tom06, Theorem 2.24] and the map $(H, S) \mapsto I_{(H,S)}$ is onto the lattice of ideals of $C^*(E)$. When $B_H = \emptyset$, we write I_H in place of $I_{(H,\emptyset)}$ and observe that I_H equals the ideal generated by $\{p_v : v \in H\}$. Note that if E is row-finite, then B_H is empty for every saturated hereditary subset H .

2.2. Notation and conventions for extensions. All ideals in C^* -algebras will be considered to be closed two-sided ideals. An element a of a C^* -algebra A (respectively, a subset $S \subseteq A$) is said to be *full* if a (respectively, S) is not contained in any proper ideal of A . A map $\phi : A \rightarrow B$ is *full* if $\text{im } \phi$ is full in B .

If A and B are C^* -algebras, an *extension* of A by B consists of a C^* -algebra E and a short exact sequence

$$\epsilon : \quad 0 \longrightarrow B \xrightarrow{\alpha} E \xrightarrow{\beta} A \longrightarrow 0.$$

We say that the extension ϵ is *essential* if $\alpha(B)$ is an essential ideal of E , and we say that the extension ϵ is *unital* if E is a unital C^* -algebra. For any extension there exist unique $*$ -homomorphisms $\eta_\epsilon : E \rightarrow \mathcal{M}(B)$ and $\tau_\epsilon : A \rightarrow \mathcal{Q}(B) := \mathcal{M}(B)/B$ which make the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & A & \longrightarrow & 0 \\ & & \parallel & & \eta_\epsilon \downarrow & & \tau_\epsilon \downarrow & & \\ 0 & \longrightarrow & B & \xrightarrow{i} & \mathcal{M}(B) & \xrightarrow{\pi} & \mathcal{Q}(B) & \longrightarrow & 0 \end{array}$$

commute. The $*$ -homomorphism τ_ϵ is called the *Busby invariant* of the extension, and the extension is essential if and only if τ_ϵ is injective. An extension ϵ is *full* if the associated Busby invariant τ_ϵ has the property that $\tau_\epsilon(a)$ is full in $\mathcal{Q}(A)$ for every $a \in A \setminus \{0\}$.

For an extension ϵ , we let $K_{\text{six}}(\epsilon)$ denote the cyclic six-term exact sequence of K -groups

$$\begin{array}{ccccc} K_0(B) & \longrightarrow & K_0(E) & \longrightarrow & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \longleftarrow & K_1(E) & \longleftarrow & K_1(B) \end{array}$$

where $K_0(B)$, $K_0(E)$, and $K_0(A)$ are viewed as (pre-)ordered groups. Given two extensions

$$\epsilon_1 : \quad 0 \longrightarrow B_1 \xrightarrow{\alpha_1} E_1 \xrightarrow{\beta_1} A_1 \longrightarrow 0$$

$$\epsilon_2 : \quad 0 \longrightarrow B_2 \xrightarrow{\alpha_2} E_2 \xrightarrow{\beta_2} A_2 \longrightarrow 0$$

we say $K_{\text{six}}(\mathfrak{e}_1)$ is isomorphic to $K_{\text{six}}(\mathfrak{e}_2)$, written $K_{\text{six}}(\mathfrak{e}_1) \cong K_{\text{six}}(\mathfrak{e}_2)$, if there exist isomorphisms $\alpha, \beta, \gamma, \delta, \epsilon$, and ζ making the following diagram commute

$$\begin{array}{ccccc}
K_0(B_1) & \longrightarrow & K_0(E_1) & \longrightarrow & K_0(A_1) \\
& \searrow \alpha & \downarrow \beta & & \swarrow \gamma \\
& & K_0(B_2) & \longrightarrow & K_0(E_2) & \longrightarrow & K_0(A_2) \\
& & \uparrow & & \downarrow & & \\
& & K_1(A_2) & \longleftarrow & K_1(E_2) & \longleftarrow & K_1(B_2) \\
& \swarrow \zeta & & & \uparrow \epsilon & & \swarrow \delta \\
K_1(A_1) & \longleftarrow & K_1(E_1) & \longleftarrow & K_1(B_1)
\end{array}$$

and where α, β , and γ are isomorphisms of (pre-)ordered groups.

3. PRELIMINARY GRAPH C^* -ALGEBRA RESULTS

In this section we develop a few results for graph C^* -algebras in order to apply the methods of [ERR] in §4. However, several of these results are interesting in their own right.

Lemma 3.1. *If E is a graph such that $C^*(E)$ contains a unique proper nontrivial ideal I , then the following six conditions are satisfied:*

- (1) E satisfies Condition (K),
- (2) E contains exactly three saturated hereditary subsets $\{\emptyset, H, E^0\}$,
- (3) E contains no breaking vertices; i.e., $B_H = \emptyset$,
- (4) I is a gauge invariant ideal and $I_H = I$,
- (5) If X is a nonempty hereditary subset of E , then $X \cap H \neq \emptyset$, and
- (6) E has at most one sink, and if v is a sink of E then $v \in H$.

Proof. Suppose that E does not satisfy Condition (K). Then by [Tom06, Proposition 1.17] there exists a saturated hereditary subset $H \subseteq E^0$ such that $E \setminus H$ contains a cycle $\alpha = e_1 \dots e_n$ with no exits. The set $X = \{s(e_i)\}_{i=1}^n$ is a hereditary subset of $E \setminus H$, and I_X is an ideal in $C^*(E \setminus H)$ Morita equivalent to $M_n(C(\mathbb{T}))$ (see [BHRS, Proposition 3.4] and [Rae05, Example 2.14] for details). Thus I_X , and hence $C^*(E \setminus H)$, contains a countably infinite number of ideals. Since $C^*(E \setminus H) \cong C^*(E)/I_{(H, B_H)}$ [BHRS, Proposition 3.4], this implies that $C^*(E)$ has a countably infinite number of ideals. Hence if $C^*(E)$ has a finite number of ideals, E satisfies Condition (K).

Because E satisfies Condition (K), it follows from [DT02, Theorem 3.5] that the ideals of $C^*(E)$ are in one-to-one correspondence with the pairs (H, S) where H is saturated hereditary, and $S \subseteq B_H$ is a subset of the breaking vertices of H . Since E contains a unique proper nontrivial ideal, it follows that E contains a unique saturated hereditary subset H not equal to

E^0 or \emptyset , and that there are no breaking vertices; i.e., $B_H = \emptyset$. It must also be the case that $I = I_H$. Moreover, since E satisfies Condition (K), [BHRS, Corollary 3.8] shows that all ideals of $C^*(E)$ are gauge-invariant.

In addition, suppose X is a hereditary subset with $X \cap H = \emptyset$. Since H is hereditary, none of the vertices in H can reach X , and thus the saturation \overline{X} contains no vertices of H , and $\overline{X} \cap H = \emptyset$. But then \overline{X} is a saturated hereditary subset of E that does not contain the vertices of H , and hence must be equal to \emptyset . Thus if X is a nonempty hereditary subset of E , then $X \cap H \neq \emptyset$.

Finally, suppose v is a sink of E . Consider the hereditary subset $X := \{v\}$. From the previous paragraph it follows that $X \cap H \neq \emptyset$ and hence $v \in H$. In addition, there cannot be a second sink in E , for if v' is a sink, then $X := \{v\}$ and $Y := \{v'\}$ are distinct hereditary sets. Since v cannot reach v' , we see that v is not in the saturation \overline{Y} . Similarly, since v' cannot reach v , we have that v' is not in the saturation \overline{X} . Thus \overline{X} and \overline{Y} are distinct saturated hereditary subsets of E that are proper and nontrivial, which is a contradiction. It follows that there is at most one sink in E . \square

Remark 3.2. According to [DHS03, Lemma 1.3] and [BHRS, Corollary 3.5] the C^* -algebras I and A/I are in this case also graph algebras. As they are necessarily simple, they must be either Kirchberg algebras or AF-algebras. We will denote the four cases thus occurring as follows

Case	I	A/I
[11]	AF	AF
[1 ∞]	AF	Kirchberg
[∞ 1]	Kirchberg	AF
[$\infty\infty$]	Kirchberg	Kirchberg

Definition 3.3. Let A be a C^* -algebra. A proper ideal $I \triangleleft A$ is a *largest proper ideal* of A if whenever $J \triangleleft A$, then either $J \subseteq I$ or $J = A$.

Observe that a largest proper ideal is always an essential ideal. Also note that if A is a C^* -algebra with a unique proper nontrivial ideal I , then I is a largest proper ideal; and if A is a simple C^* -algebra then $\{0\}$ is a largest proper ideal.

Lemma 3.4. *Let E be a graph, and suppose that I is a largest proper ideal of $C^*(E)$. Then I is gauge invariant and $I = I_{(H, B_H)}$ for some saturated hereditary subset H of E^0 . Furthermore, if K is any saturated hereditary subset of E , then either $K \subseteq H$ or $K = E^0$.*

Proof. Let γ denote the canonical gauge action of \mathbb{T} on $C^*(E)$. For any $z \in \mathbb{T}$ we have that $\gamma_z(I)$ is a proper ideal of $C^*(E)$. Since I is a largest proper ideal of $C^*(E)$, it follows that $\gamma_z(I) \subseteq I$. A similar argument shows that $\gamma_{z^{-1}}(I) \subseteq I$. Thus $\gamma_z(I) = I$ and I is gauge invariant. It follows from

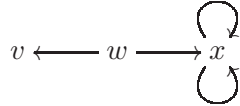
[BHRS, Theorem 3.6] that $I = I_{(H,S)}$ for some saturated hereditary subset H of E^0 and some subset $S \subseteq B_H$. Because I is a largest proper ideal, it follows that $S = B_H$, and hence $I = I_{(H,B_H)}$. Furthermore, if K is a saturated hereditary subset, then either $I_{(K,B_K)} \subseteq I_{(H,B_H)}$ or $I_{(K,B_K)} = C^*(E)$. Hence either $K \subseteq H$ or $K = E^0$. \square

Lemma 3.5. *Let E be a graph and suppose that I is a largest proper ideal of $C^*(E)$ with the property that $C^*(E)/I$ is purely infinite. Then $I = I_{(H,B_H)}$ for some saturated hereditary subset H of E^0 , and there exists a cycle γ in $E \setminus H$ and an edge $f \in E^1$ with $s(f) = s(\gamma)$ and $r(f) \in H$. Furthermore, if $x \in E^0$, then $x \geq s(\gamma)$ if and only if $x \in E^0 \setminus H$.*

Proof. Lemma 3.4 shows that $I = I_{(H,B_H)}$ for some saturated hereditary subset H of E^0 . It follows from [BHRS, Corollary 3.5] that $C^*(E)/I_{(H,B_H)} \cong C^*(E \setminus H)$, where $E \setminus H$ is the subgraph of E with $(E \setminus H)^0 := E^0 \setminus H$ and $(E \setminus H)^1 := E^1 \setminus r^{-1}(H)$. Since $C^*(E \setminus H)$ is purely infinite, it follows from [DT05, Corollary 2.14] that $E \setminus H$ contains a cycle α . Define $K := \{x \in E^0 : x \not\geq s(\alpha)\}$. Then K is saturated hereditary, $H \subseteq K$, and $K \neq E^0$. Hence $I_{(H,B_H)} \subseteq I_{(K,B_K)} \neq C^*(E)$, and the fact that $I_{(H,B_H)}$ is a largest proper ideal implies that $I_{(H,B_H)} = I_{(K,B_K)}$ so that $H = K$. Hence for $x \in E^0$ we have $x \geq s(\alpha)$ if and only if $x \in E^0 \setminus H$.

Consider the set $J := \{x \in E^0 : s(\alpha) \geq x\}$. Then J is a hereditary subset and we let \bar{J} denote its saturation. Since $I_{(H,B_H)}$ is a largest proper ideal, it follows that either $\bar{J} \subseteq H$ or $\bar{J} = E^0$. Since $s(\alpha) \in \bar{J} \setminus H$, we must have $\bar{J} = E^0$. Choose any element $w \in H$. Since $w \in \bar{J}$ it follows that there exists $v \in J$ with $w \geq v$. But since $w \geq v$ and H is hereditary, it follows that $v \in H$. Hence $v \in J \cap H$, and there is a path from $s(\alpha)$ to a vertex in H . Choose a path $\mu = \mu_1 \mu_2 \dots \mu_n$ with $s(\mu) = s(\alpha)$, $r(\mu_{n-1}) \notin H$, and $r(\mu_n) \in H$. Since $r(\mu_{n-1}) \notin H$ the previous paragraph shows that there exists a path ν with $s(\nu) = r(\mu_{n-1})$ and $r(\nu) = s(\alpha)$. Let $\gamma := \nu \mu_1 \dots \mu_{n-1}$ and let $f := \mu_n$. Then γ is a cycle in $E \setminus H$ and f is an edge with $s(f) = s(\gamma)$ and $r(f) \in H$. Furthermore, since $s(\alpha)$ is a vertex on the cycle γ , we see that for any $x \in E^0$ we have $x \geq s(\gamma)$ if and only if $x \geq s(\alpha)$. It follows from the previous paragraph that if $x \in E^0$, then $x \geq s(\gamma)$ if and only if $x \in E^0 \setminus H$. \square

Remark 3.6. Note that the conclusion of the above lemma does not hold if I is a maximal proper ideal that is not a largest proper ideal. For example, if E is the graph



and $H = \{v\}$ then $I := I_H$ is a maximal ideal that is AF, and $C^*(E)/I_H \cong M_2(\mathcal{O}_2)$ is purely infinite. However, there is no edge from the base point of a cycle to $H = \{v\}$.

Definition 3.7. We say that two projections $p, q \in A$ are *equivalent*, written $p \sim q$, if there exists an element $v \in A$ with $p = vv^*$ and $q = v^*v$. We write $p \lesssim q$ to mean that p is equivalent to a subprojection of q ; that is, there exists $v \in A$ such that $p = vv^*$ and $v^*v \leq q$. Note that $p \lesssim q$ and $q \lesssim p$ does not imply that $p \sim q$ (unless A is finite).

If $e \in G^1$ then we see that $p_{r(e)} = s_e^*s_e$ and $s_e s_e^* \leq p_{s(e)}$. Therefore $p_{r(e)} \lesssim p_{s(e)}$. More generally we see that $v \geq w$ implies $p_w \lesssim p_v$.

Lemma 3.8. *Let A be a C^* -algebra with an increasing countable approximate unit $\{p_n\}_{n=1}^\infty$ consisting of projections. Then the following are equivalent.*

- (i) A is stable.
- (ii) For every projection $p \in A$ there exists a projection $q \in A$ such that $p \sim q$ and $p \perp q$.
- (iii) For all $n \in \mathbb{N}$ there exists $m > n$ such that $p_n \lesssim p_m - p_n$.

Proof. The equivalence of (i) and (ii) is shown in [Hr98, Theorem 3.3]. The equivalence of (ii) and (iii) is shown in [Hje01, Lemma 2.1]. \square

Lemma 3.9. *Let A be a C^* -algebra. Suppose p_1, p_2, \dots, p_n are mutually orthogonal projections in A , and q_1, q_2, \dots, q_n are mutually orthogonal projections in A with $p_i \sim q_i$ for $1 \leq i \leq n$. Then $\sum_{i=1}^n p_i \sim \sum_{i=1}^n q_i$.*

Proof. Since $p_i \sim q_i$ there exists $v_i \in A$ such that $v_i^*v_i = p_i$ and $v_i v_i^* = q_i$. Thus for $i \neq j$ we have $v_j^*v_i = v_j^*v_j v_j^* v_i v_i^* v_i = v_j^* q_j q_i v_i = 0$ and $v_i v_j^* = v_i v_i^* v_i v_j^* v_j v_j^* = v_i p_i p_j v_j^* = 0$. Hence $(\sum_{i=1}^n v_i)^* \sum_{i=1}^n v_i = \sum_{i=1}^n v_i^* v_i = \sum_{i=1}^n p_i$ and $\sum_{i=1}^n v_i (\sum_{i=1}^n v_i)^* = \sum_{i=1}^n v_i v_i^* = \sum_{i=1}^n q_i$. Thus $\sum_{i=1}^n p_i \sim \sum_{i=1}^n q_i$. \square

Proposition 3.10. *Let E be a graph with no breaking vertices, and suppose that I is a largest proper ideal of $C^*(E)$ and such that $C^*(E)/I$ is purely infinite and I is AF. Then there exists a projection $p \in C^*(E)$ such that $pC^*(E)p$ is a full corner of $C^*(E)$ and pIp is stable.*

Proof. Lemma 3.5 implies that $I = I_{(H, B_H)}$ for some saturated hereditary subset H of E^0 , and there exists a cycle γ in $E \setminus H$ and an edge $f \in E^1$ with $s(f) = s(\gamma)$ and $r(f) \in H$; and furthermore, if $x \in E^0$, then $x \geq s(\gamma)$ if and only if $x \in E^0 \setminus H$. Since E has no breaking vertices, we have that $B_H = \emptyset$ so that $I_{(H, B_H)}$ is the ideal generated by $\{p_v : v \in H\}$ and we may write $I_{(H, B_H)}$ as I_H .

Let $v = s(f) = s(\gamma)$ and let $w = r(f)$. Define $p := p_v + p_w$. Suppose $J \triangleleft C^*(E)$ and $pC^*(E)p \subseteq J$. Since $v \notin H$ we see that $p_v \notin I$ and hence $p_v \in pC^*(E)p \setminus I \subseteq J \setminus I$. Thus $J \not\subseteq I$ and the fact that I is a largest proper ideal implies that $J = C^*(E)$. Hence $pC^*(E)p$ is a full corner of $C^*(E)$.

In addition, since there are no breaking vertices

$$pIp = pI_H p$$

$$\begin{aligned}
&= p \left(\overline{\text{span}}\{s_\alpha s_\beta^* : r(\alpha) = r(\beta) \in H\} \right) p \\
&= \overline{\text{span}}\{p s_\alpha s_\beta^* p : r(\alpha) = r(\beta) \in H\} \\
&= \overline{\text{span}}\{s_\alpha s_\beta^* : r(\alpha) = r(\beta) \in H \text{ and } s(\alpha), s(\beta) \in \{v, w\}\}.
\end{aligned}$$

Let $S := \{\alpha \in E^* : s(\alpha) = v \text{ and } r(\alpha) = w\}$. Since S is a countable set we may list the elements of S and write $S = \{\alpha_1, \alpha_2, \alpha_3, \dots\}$. Define $p_0 := p_w$ and $p_n := p_w + \sum_{k=1}^n s_{\alpha_k} s_{\alpha_k}^*$ for $n \in \mathbb{N}$.

We will show that for $\mu, \nu \in S$ we have

$$(3.1) \quad s_\mu^* s_\nu := \begin{cases} p_{r(\mu)} & \text{if } \mu = \nu \\ 0 & \text{otherwise.} \end{cases}$$

First suppose that $s_\mu^* s_\nu \neq 0$. Then one of μ and ν must extend the other. Suppose μ extends ν . Then $\mu = \nu\lambda$ for some $\lambda \in E^*$. Thus $s(\lambda) = r(\nu) = w$ and $r(\lambda) = r(\mu) = w$. However, I_H is an AF-algebra, and $C^*(E_H)$ is strongly Morita equivalent to I_H [BHRS, Proposition 3.4], so $C^*(E_H)$ is an AF-algebra. Thus E_H contains no cycles. Since λ is a path in E_H with $s(\lambda) = r(\lambda) = w$, and since E_H contains no cycles, we may conclude that $\lambda = w$. Thus $\mu = \nu$. A similar argument works when ν extends μ . Hence the equation in (3.1) holds. It follows that the elements of the set $\{s_\alpha s_\alpha^* : \alpha \in S\} \cup \{p_w\}$ are mutually orthogonal projections, and hence $\{p_n\}_{n=0}^\infty$ is an sequence of increasing projections.

Next we shall show that $\{p_n\}_{n=0}^\infty$ is an approximate unit for pIp . Given $s_\alpha s_\beta^*$ with $r(\alpha) = r(\beta) \in H$ and $s(\alpha), s(\beta) \in \{v, w\}$, we consider two cases.

CASE I: $s(\alpha) = w$. Then for any $\alpha_k \in S$ we see that $(s_{\alpha_k} s_{\alpha_k}^*) s_\alpha s_\beta^* = s_{\alpha_k} s_{\alpha_k}^* p_w s_\alpha s_\beta^* = 0$. In addition, $p_w (s_\alpha s_\beta^*) = s_\alpha s_\beta^*$. Thus $\lim_{n \rightarrow \infty} p_n s_\alpha s_\beta^* = s_\alpha s_\beta^*$.

CASE II: $s(\alpha) = v$. Then $\alpha = \alpha_j \lambda$ for some $\alpha_j \in S$ and some $\lambda \in E_H^*$ with $s(\lambda) = w$. We have $p_w (s_\alpha s_\beta^*) = 0$, and also (3.1) implies that

$$(s_{\alpha_k} s_{\alpha_k}^*) s_\alpha s_\beta^* = s_{\alpha_k} s_{\alpha_k}^* s_{\alpha_j} s_\lambda s_\beta^* = \begin{cases} s_\alpha s_\beta^* & k = j \\ 0 & k \neq j. \end{cases}$$

Thus $\lim_{n \rightarrow \infty} p_n s_\alpha s_\beta^* = s_\alpha s_\beta^*$.

The above two cases imply that $\lim_{n \rightarrow \infty} p_n x = x$ for any $x \in \text{span}\{s_\alpha s_\beta^* : r(\alpha) = r(\beta) \in H \text{ and } s(\alpha), s(\beta) \in \{v, w\}\}$. Furthermore, an $\epsilon/3$ -argument shows that $\lim_{n \rightarrow \infty} p_n x = x$ for any $x \in pI_H p = \overline{\text{span}}\{s_\alpha s_\beta^* : r(\alpha) = r(\beta) \in H \text{ and } s(\alpha), s(\beta) \in \{v, w\}\}$. A similar argument shows that $\lim_{n \rightarrow \infty} x p_n = x$ for any $x \in pI_H p$. Thus $\{p_n\}_{n=1}^\infty$ is an approximate unit for $pI_H p$.

We shall now show that $pI_H p$ is stable. For each $n \in \mathbb{N}$ define

$$\lambda^n := \underbrace{\gamma \gamma \dots \gamma}_n f.$$

For any $k, n \in \mathbb{N}$ we have

$$s_{\lambda^n} s_{\lambda^n}^* \sim s_{\lambda^n}^* s_{\lambda^n} = p_{r(\lambda^n)} = p_w = s_{\alpha_k}^* s_{\alpha_k} \sim s_{\alpha_k} s_{\alpha_k}^*.$$

For any $n \in \mathbb{N}$ choose q large enough that $|\lambda^q| \geq |\alpha_k|$ for all $1 \leq k \leq n$. Then for all $j \in \mathbb{N}$ we see that $\lambda^{q+j} \in S$ and $\lambda^{q+j} \neq \alpha_k$ for all $1 \leq k \leq n$. Thus for any $1 \leq k \leq n$ we have

$$s_{\alpha_k} s_{\alpha_k}^* \sim s_{\alpha_k}^* s_{\alpha_k} = p_{r(\alpha_k)} = p_w = p_{r(\lambda^{q+k})} = s_{\lambda^{q+k}}^* s_{\lambda^{q+k}} \sim s_{\lambda^{q+k}} s_{\lambda^{q+k}}^*$$

and

$$p_w = s_{\lambda^q}^* s_{\lambda^q} \sim s_{\lambda^q} s_{\lambda^q}^*.$$

It follows from Lemma 3.9 that

$$p_n = p_w + \sum_{k=1}^n s_{\alpha_k} s_{\alpha_k}^* \lesssim \sum_{k=0}^n s_{\lambda^{q+k}} s_{\lambda^{q+k}}^* \lesssim p_m - p_n$$

where m is chosen large enough that $\lambda^{q+k} \in \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ for all $0 \leq k \leq n$. Lemma 3.8 shows that pI_{HP} is stable. \square

4. CLASSIFICATION

In this section we state and prove our main results. We apply the methods of [ERR] to classify certain extensions of graph C^* -algebras in terms of their six-term exact sequences of K -groups. To do this we will need to discuss classes of C^* -algebras satisfying various properties. We give definitions of these properties here, and obtain a lemma that is a consequence of [ERR, Theorem 3.10].

Definition 4.1 (see Definition 3.2 of [ERR]). We will be interested in classes \mathcal{C} of separable nuclear unital simple C^* -algebras in the bootstrap category \mathcal{N} satisfying the following properties:

- (I) Any element of \mathcal{C} is either purely infinite or stably finite.
- (II) \mathcal{C} is closed under tensoring with M_n , where M_n is the C^* -algebra of n by n matrices over \mathcal{C} .
- (III) If A is in \mathcal{C} , then any unital hereditary C^* -subalgebra of A is in \mathcal{C} .
- (IV) For all A and B in \mathcal{C} and for all x in $KK(A, B)$ which induce an isomorphism from $(K_*^+(A), [1_A])$ to $(K_*^+(B), [1_B])$, there exists a $*$ -isomorphism $\alpha : A \rightarrow B$ such that $KK(\alpha) = x$.

Definition 4.2. If B is a separable stable C^* -algebra, then we say that B has the *corona factorization property* if every full projection in $\mathcal{M}(B)$ is Murray-von Neumann equivalent to $1_{\mathcal{M}(B)}$.

Lemma 4.3 (Cf. Theorem 3.10 of [ERR]). *Let \mathcal{C}_I and \mathcal{C}_Q be classes of unital nuclear separable simple C^* -algebras in the bootstrap category \mathcal{N} satisfying the properties of Definition 4.1. Let A_1 and A_2 be in \mathcal{C}_Q and let B_1 and B_2 be in \mathcal{C}_I with $B_1 \otimes \mathbb{K}$ and $B_2 \otimes \mathbb{K}$ satisfying the corona factorization property. Let*

$$\begin{aligned} \mathfrak{e}_1 : \quad & 0 \longrightarrow B_1 \otimes \mathbb{K} \longrightarrow E_1 \longrightarrow A_1 \longrightarrow 0 \\ \mathfrak{e}_2 : \quad & 0 \longrightarrow B_2 \otimes \mathbb{K} \longrightarrow E_2 \longrightarrow A_2 \longrightarrow 0 \end{aligned}$$

be essential and unital extensions. If $K_{\text{six}}(\mathfrak{e}_1) \cong K_{\text{six}}(\mathfrak{e}_2)$, then $E_1 \otimes \mathbb{K} \cong E_2 \otimes \mathbb{K}$.

Proof. Tensoring the extension ϵ_1 by \mathbb{K} we obtain a short exact sequence ϵ'_1 and vertical maps

$$\begin{array}{ccccccccc} \epsilon_1 : & 0 & \longrightarrow & B_1 \otimes \mathbb{K} & \longrightarrow & E_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow & & \downarrow & & \\ \epsilon'_1 : & 0 & \longrightarrow & (B_1 \otimes \mathbb{K}) \otimes \mathbb{K} & \longrightarrow & E_1 \otimes \mathbb{K} & \longrightarrow & A_1 \otimes \mathbb{K} & \longrightarrow & 0 \end{array}$$

from ϵ_1 into ϵ'_1 that are full inclusions. These full inclusions induce isomorphisms of K -groups and hence we have that $K_{\text{six}}(\epsilon_1) \cong K_{\text{six}}(\epsilon'_1)$. In addition, since ϵ_1 is essential, $B_1 \otimes \mathbb{K}$ is an essential ideal in E_1 , and the Rieffel correspondence between the strongly Morita equivalent C^* -algebras E_1 and $E_1 \otimes \mathbb{K}$ implies that $(B_1 \otimes \mathbb{K}) \otimes \mathbb{K}$ is an essential ideal in $E_1 \otimes \mathbb{K}$, so that ϵ'_1 is an essential extension. Furthermore, since $B_1 \otimes \mathbb{K}$ is stable and ϵ_1 is essential and full [ERR, Proposition 1.5], it follows from [ERR, Proposition 1.6] that ϵ'_1 is full. Moreover, since $\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$, we may rewrite ϵ'_1 as

$$\epsilon'_1 : \quad 0 \longrightarrow B_1 \otimes \mathbb{K} \longrightarrow E_1 \otimes \mathbb{K} \longrightarrow A_1 \otimes \mathbb{K} \longrightarrow 0.$$

By a similar argument, there is an essential and full extension

$$\epsilon'_2 : \quad 0 \longrightarrow B_2 \otimes \mathbb{K} \longrightarrow E_2 \otimes \mathbb{K} \longrightarrow A_2 \otimes \mathbb{K} \longrightarrow 0$$

such that $K_{\text{six}}(\epsilon'_2) \cong K_{\text{six}}(\epsilon_2)$. Thus $K_{\text{six}}(\epsilon'_1) \cong K_{\text{six}}(\epsilon'_2)$, and [ERR, Theorem 3.10] implies that $E_1 \otimes \mathbb{K} \cong E_2 \otimes \mathbb{K}$. \square

Lemma 4.4. *Let A be a C^* -algebra and let I be a largest proper ideal of A . If $p \in A$ is a full projection, then the inclusion map $pIp \hookrightarrow I$ and the inclusion map $pAp/pIp \hookrightarrow A/I$ are both full inclusions.*

Proof. Since p is a full projection, we see that A is Morita equivalent to pAp and the Rieffel correspondence between ideals takes the form $J \mapsto pJp$. If J is an ideal of I with $pIp \subseteq J$, then by compressing by p we obtain $pIp \subseteq pJp$. Since the Rieffel correspondence is a bijection, this implies that $I \subseteq J$, and because J is an ideal contained in I , we get that $I = J$. Hence $pIp \hookrightarrow I$ is a full inclusion. Furthermore, because I is a largest proper ideal of A , we know that A/I is simple and thus $pAp/pIp \hookrightarrow A/I$ is a full inclusion. \square

Theorem 4.5. *If A is a graph C^* -algebra with exactly one proper nontrivial ideal I , then A classified up to stable isomorphism by the six-term exact sequence*

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \\ \uparrow & & & & \downarrow \\ K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I) \end{array}$$

with all K_0 -groups considered as ordered groups. In other words, if A is a graph C^* -algebra with precisely one proper nontrivial ideal I , if A' is a

graph C^* -algebras with precisely one proper nontrivial ideal I' , and if

$$\mathfrak{e}_1 : \quad 0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

$$\mathfrak{e}_2 : \quad 0 \longrightarrow I' \longrightarrow A' \longrightarrow A'/I' \longrightarrow 0$$

are the associated extensions, then $A \otimes \mathbb{K} \cong A' \otimes \mathbb{K}$ if and only if $K_{\mathbf{six}}(\mathfrak{e}_1) \cong K_{\mathbf{six}}(\mathfrak{e}_2)$.

Moreover, in cases $[\mathbf{1}\infty]$, $[\infty\mathbf{1}]$, and $[\infty\infty]$, the order structure on $K_0(A)$ may be removed from the invariant leaving it still complete. And in case $[\mathbf{11}]$, the ordered group $K_0(A)$ is a complete invariant in its own right.

Proof. It is straightforward to show that $A \otimes \mathbb{K} \cong A' \otimes \mathbb{K}$ implies that $K_{\mathbf{six}}(\mathfrak{e}_1) \cong K_{\mathbf{six}}(\mathfrak{e}_2)$. Thus we need only establish the converse. To do this, we begin by assuming that $K_{\mathbf{six}}(\mathfrak{e}_1) \cong K_{\mathbf{six}}(\mathfrak{e}_2)$.

We define \mathcal{C}_{API} as the union of the class of unital simple and separable AF-algebras and the class of simple, nuclear, unital, and separable purely infinite C^* -algebras in the bootstrap category. The category \mathcal{C}_{API} meets all the requirements in the list in Definition 4.1: We clearly have that each algebra in \mathcal{C}_{API} is either purely infinite or stably finite, and that \mathcal{C}_{API} is closed under passing to matrices and unital hereditary subalgebras. We also need to prove that the Elliott invariant is complete for \mathcal{C}_{API} , and this follows by the classification results of Elliott [Ell76, Theorem 4.3] and Kirchberg-Phillips (see [Kir, Theorem C] and [Phi00, §4.2]) after noting that the classes are obviously distinguishable by the nature of the positive cone in K_0 . Finally, as recorded in [ERR, Theorem 3.9], the stabilizations of the C^* -algebras in \mathcal{C}_{API} all have the corona factorization property according to [KN06, Theorem 5.2] and [Ng, Proposition 2.1].

It follows from [DHS03, Lemma 1.3] and [BHRS, Corollary 3.5] that I and A/I are simple graph C^* -algebras and thus each of I and A/I is either an AF-algebra or a purely infinite algebra [DT05, Remark 2.16]. Similarly for I' and A'/I' . Since $K_{\mathbf{six}}(\mathfrak{e}_1) \cong K_{\mathbf{six}}(\mathfrak{e}_2)$, we see that $K_0(I) \cong K_0(I')$ and $K_0(A/I) \cong K_0(A'/I')$ as ordered groups. By considering the positive cone in these groups, we may conclude that I and I' are either both purely infinite or both AF-algebras, and also A/I and A'/I' are either both purely infinite or both AF-algebras. Thus A and A' both fall into one of the four cases described in Remark 3.2.

Cases $[\infty\infty]$, $[\infty\mathbf{1}]$

Write $A = C^*(E)$ for some graph E . Since I is a largest proper ideal, Lemma 3.4 implies that $I = I_{(H, B_H)}$ for some saturated hereditary subset $H \subsetneq E^0$. If we let $v \in E^0 \setminus H$, and define $p := p_v$, then $p \notin I$. Since I is a largest proper ideal in A , this implies that the projection p is full. Thus we obtain a full hereditary subalgebra pAp , and as noted in Lemma 4.4 we

have that all vertical maps in

$$\begin{array}{ccccccccc} \epsilon'_1 : & 0 & \longrightarrow & pIp & \longrightarrow & pAp & \longrightarrow & pAp/pIp & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow & & \downarrow & & \\ \epsilon_1 : & 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \end{array}$$

are full inclusions. It follows that all of the above maps induce isomorphisms on the K -groups and $K_{\text{six}}(\epsilon_1) \cong K_{\text{six}}(\epsilon'_1)$.

In addition, since pIp is nonunital and purely infinite, the ideal pIp is stable (by Zhang's dichotomy) and we may write $pIp \cong B_1 \otimes \mathbb{K}$ for a suitably chosen $B_1 \in \mathcal{C}_{API}$. We now let $E_1 := pAp$ and $A_1 := pAp/pIp$. With this notation, ϵ'_1 takes the form

$$\epsilon'_1 : \quad 0 \longrightarrow B_1 \otimes \mathbb{K} \longrightarrow E_1 \longrightarrow A_1 \longrightarrow 0$$

with B_1 and A_1 unital C^* -algebras in \mathcal{C}_{API} . Furthermore, ϵ'_1 is an essential extension because the ideal I is a largest proper ideal in A , and thus also the ideal $pIp \cong B_1 \otimes \mathbb{K}$ is a largest proper ideal in $A_1 \otimes \mathbb{K}$, which implies that $pIp \cong B_1 \otimes \mathbb{K}$ is an essential ideal.

By a similar argument, we may find an extension

$$\epsilon'_2 : \quad 0 \longrightarrow B_2 \otimes \mathbb{K} \longrightarrow E_2 \longrightarrow A_2 \longrightarrow 0$$

with $E_2 := qA'q$ for a full projection $q \in A'$, the C^* -algebras B_2 and A_2 in \mathcal{C}_{API} with $B_2 \otimes \mathbb{K}$ satisfying the corona factorization property, and ϵ'_2 an essential and full extension with $K_{\text{six}}(\epsilon'_2) \cong K_{\text{six}}(\epsilon_2)$. It follows from Lemma 4.3 that $E_1 \otimes \mathbb{K} \cong E_2 \otimes \mathbb{K}$, or equivalently, that $pAp \otimes \mathbb{K} \cong qA'q \otimes \mathbb{K}$. Furthermore, because pAp is a full corner of A , and qAq is a full corner of A' , we obtain that $pAp \otimes \mathbb{K} \cong A \otimes \mathbb{K}$ and $qA'q \otimes \mathbb{K} \cong A' \otimes \mathbb{K}$. It follows that $A \otimes \mathbb{K} \cong A' \otimes \mathbb{K}$. We also observe that in this case the order structure on $K_0(A)$ is a redundant part of the invariant.

Case [1 ∞]

As seen in Lemma 3.1, we may write $A = C^*(E)$ where E has no breaking vertices. By Proposition 3.10 there exists a projection $p \in A$ such that pAp is a full corner inside A , and pIp is stable. Moreover, since I is an AF-algebra by hypothesis and pIp is a hereditary subalgebra of I , it follows from [Ell76b, Theorem 3.1] that pIp is an AF-algebra. Hence we may choose a unital AF-algebra B_1 with $pIp \cong B_1 \otimes \mathbb{K}$. The extension

$$\epsilon'_1 : \quad 0 \longrightarrow B_2 \otimes \mathbb{K} \longrightarrow pAp \longrightarrow pAp/pIp \longrightarrow 0$$

is essential. In addition, an argument as in Cases [$\infty\infty$], [$\infty\mathbf{1}$] shows that $K_{\text{six}}(\epsilon'_1) \cong K_{\text{six}}(\epsilon_1)$. We may perform a similar argument for A' , and arguing as in Cases [$\infty\infty$], [$\infty\mathbf{1}$] and applying Lemma 4.3 we obtain that $A \otimes \mathbb{K} \cong A' \otimes \mathbb{K}$. Again, the order structure on $K_0(A)$ is a redundant part of the invariant.

Case [11]

Since I and A/I are AF-algebras, it follows from a result of Brown that A is an AF-algebra [Bro81] (or see [Eff81, §9.9] for a detailed proof). Similarly, A' is an AF-algebra. It follows from Elliott's Theorem that $K_0(A)$ order isomorphic to $K_0(A')$ implies that $A \otimes \mathbb{K} \cong A' \otimes \mathbb{K}$. Moreover, in this case the ordered group $K_0(A)$ is a complete invariant. \square

Remark 4.6. Joint work in progress by Ruiz and the first named author provides information about the necessity of using order on the K_0 -groups of $K_{\text{six}}(-)$. There are examples of pairs of non-isomorphic stable AF-algebras A and A' with exactly one ideal I and I' such that $K_{\text{six}}(\mathfrak{e}) \simeq K_{\text{six}}(\mathfrak{e}')$ with group isomorphisms which are positive at $K_0(I)$ and $K_0(A/I)$ but not at $K_0(A)$. By [KST, Corollary 4.8], A and A' may be realized as graph C^* -algebras. In the other cases, one may prove that any isomorphism between $K_{\text{six}}(\mathfrak{e})$ and $K_{\text{six}}(\mathfrak{e}')$ will automatically be positive at $K_0(A)$ if it is positive at $K_0(I)$ and $K_0(A/I)$. Thus it is possible that any isomorphism of our reduced invariant lifts to a $*$ -isomorphism in the $[\infty\infty]$, $[\infty\mathbf{1}]$, and $[\mathbf{1}\infty]$ cases, but this has only been established in the $[\infty\infty]$ case, cf. [ER].

Our proof of Theorem 4.5 can be modified slightly to give us an additional result.

Theorem 4.7. *If A is the C^* -algebra of a graph satisfying Condition (K), and if A has a largest proper ideal I such that I is an AF-algebra, then A is classified up to stable isomorphism by the six-term exact sequence*

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \\ & & \uparrow & & \downarrow \\ K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I) \end{array}$$

with $K_0(I)$ considered as an ordered group.

In other words, if A is the C^ -algebra of a graph satisfying Condition (K) with a largest proper ideal I that is an AF-algebra, if A' is the C^* -algebra of a graph satisfying Condition (K) with a largest proper ideal I' that is an AF-algebra, and if*

$$\begin{array}{l} \mathfrak{e}_1 : \quad 0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0 \\ \mathfrak{e}_2 : \quad 0 \longrightarrow I' \longrightarrow A' \longrightarrow A'/I' \longrightarrow 0 \end{array}$$

are the associated extensions, then $A \otimes \mathbb{K} \cong A' \otimes \mathbb{K}$ if and only if $K_{\text{six}}(\mathfrak{e}_1) \cong K_{\text{six}}(\mathfrak{e}_2)$.

Proof. To begin, using the desingularization of [DT05] we may find a row-finite graph F such that $C^*(F)$ is stably isomorphic to A . Since $C^*(F)$ is Morita equivalent to A , the C^* -algebra $C^*(F)$ has a largest proper ideal that is an AF-algebra, and the associated six-term exact sequence of K -groups is isomorphic to $K_{\text{six}}(\mathfrak{e}_1)$. Hence we may replace A by $C^*(F)$ for

the purposes of the proof. Likewise for A' . Thus we may, without loss of generality, assume that A and A' are C^* -algebras of row-finite graphs, and in particular that A and A' are C^* -algebras of graphs with no breaking vertices. To obtain the result, we simply argue as in Case [1 ∞] of the proof of Theorem 4.5, using [ERR, Theorem 3.13] in place of [ERR, Theorem 3.10], and noting that Proposition 3.10 applies since the graphs have no breaking vertices. \square

5. EXAMPLES

To illustrate our methods we give a complete classification, up to stable isomorphism, of all C^* -algebras of graphs with two vertices that have precisely one proper nontrivial ideal. Combined with other results, this allows us to give a complete classification of all C^* -algebras of graphs satisfying Condition (K) with exactly two vertices.

If E is a graph with two vertices, and if $C^*(E)$ has exactly one proper ideal, then E must have exactly one proper nonempty saturated hereditary subset with no breaking vertices. This occurs precisely when the vertex matrix of E has the form

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

where $a, d \in \{0, 2, 3, \dots, \infty\}$ and $b \in \{1, 2, 3, \dots, \infty\}$ with the extra conditions

$$a = 0 \implies b = \infty \quad \text{and} \quad b = \infty \implies (a = 0 \text{ or } a = \infty),$$

Computing K -groups using [DT02], we see that in all of these cases the K_1 -groups of $C^*(E)$, the unique proper nontrivial ideal I , and the quotient $C^*(E)/I$ all vanish. Thus the six-term exact sequence becomes $0 \rightarrow K_0(I) \rightarrow K_0(C^*(E)) \rightarrow K_0(C^*(E)/I) \rightarrow 0$, and using [DT02] to compute the K_0 -groups and the induced maps we obtain the following cases.

a	d	b	$K_0(I) \rightarrow K_0(C^*(E)) \rightarrow K_0(C^*(E)/I)$	Case
0	0	∞	$\mathbb{Z}_{++} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z}_{++}$	[11]
0	n	∞	$\mathbb{Z}_{d-1} \rightarrow \mathbb{Z}_{d-1} \oplus \mathbb{Z} \rightarrow \mathbb{Z}_{++}$	$[\infty \mathbf{1}]$
0	∞	∞	$\mathbb{Z}_{\pm} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z}_{++}$	$[\infty \mathbf{1}]$
n	0	$1, n$	$\mathbb{Z}_{++} \rightarrow \text{coker}\left(\begin{bmatrix} b \\ a-1 \end{bmatrix}\right) \rightarrow \mathbb{Z}_{a-1}$	$[\mathbf{1}\infty]$
n	n	$1, n$	$\mathbb{Z}_{d-1} \rightarrow \text{coker}\left(\begin{bmatrix} d-1 & b \\ 0 & a-1 \end{bmatrix}\right) \rightarrow \mathbb{Z}_{a-1}$	$[\infty\infty]$
n	∞	$1, n$	$\mathbb{Z}_{\pm} \rightarrow \text{coker}\left(\begin{bmatrix} b \\ a-1 \end{bmatrix}\right) \rightarrow \mathbb{Z}_{a-1}$	$[\infty\infty]$
∞	0	$1, n, \infty$	$\mathbb{Z}_{++} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z}_{\pm}$	$[\mathbf{1}\infty]$
∞	n	$1, n, \infty$	$\mathbb{Z}_{d-1} \rightarrow \mathbb{Z}_{d-1} \oplus \mathbb{Z} \rightarrow \mathbb{Z}_{\pm}$	$[\infty\infty]$
∞	∞	$1, n, \infty$	$\mathbb{Z}_{\pm} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z}_{\pm}$	$[\infty\infty]$

where “ n ” indicates an integer ≥ 2 , “ \mathbb{Z}_{++} ” indicates a copy of \mathbb{Z} ordered with $\mathbb{Z}_+ = \mathbb{N}$ and “ \mathbb{Z}_{\pm} ” indicates a copy of \mathbb{Z} ordered with $\mathbb{Z}_+ = \mathbb{Z}$. In addition, in all cases we have written the middle group in such a way that the map from $K_0(I)$ to $K_0(C^*(E))$ is $[x] \mapsto [(x, 0)]$, and the map from $K_0(C^*(E))$ to $K_0(C^*(E)/I)$ is $[(x, y)] \mapsto [y]$. Note that in all but the first case, the order structure of the middle K_0 -groups is irrelevant and need not be computed.

Theorem 5.1. *Let E and E' be graphs each with two vertices such that $C^*(E)$ and $C^*(E')$ each have exactly one proper nontrivial ideal, and write the vertex matrix of E as $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ and the vertex matrix of E' as $\begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix}$. Then*

$$C^*(E) \otimes \mathbb{K} \cong C^*(E') \otimes \mathbb{K}$$

if and only if the following three conditions hold:

- (1) $a = a'$
- (2) $d = d'$
- (3) If $a \in \{2, \dots\}$ then
 - (a) If $d \in \{0, \infty\}$ then $[b] = [z][b']$ in \mathbb{Z}_{a-1} for a unit $[z] \in \mathbb{Z}_{a-1}$
 - (b) If $d \in \{2, \dots\}$ then $[z_1][b] = [z_2][b']$ in $\mathbb{Z}_{\gcd(a-1, d-1)}$ for a unit $[z_1] \in \mathbb{Z}_{d-1}$ and a unit $[z_2] \in \mathbb{Z}_{a-1}$.

Proof. Suppose $C^*(E) \otimes \mathbb{K} \cong C^*(E') \otimes \mathbb{K}$. Then $K_0(I) \cong K_0(I')$ as ordered groups and $K_0(C^*(E)/I) \cong K_0(C^*(E')/I')$ as ordered groups. From a consideration of the invariants in the above table, this implies that $a = a'$, $d = d'$, and the invariants for $C^*(E)$ and $C^*(E')$ both fall into the same case (i.e. the same row) of the table. Thus we need only consider the two cases described in (3)(a) and (3)(b).

CASE I: $a \in \{2, \dots\}$ and $d \in \{0, \infty\}$.

In this case there are isomorphisms α , β , and γ such that

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \text{coker}\left(\begin{bmatrix} b & \\ & a-1 \end{bmatrix}\right) & \longrightarrow & \mathbb{Z}_{a-1} \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \text{coker}\left(\begin{bmatrix} b' & \\ & a-1 \end{bmatrix}\right) & \longrightarrow & \mathbb{Z}_{a-1} \longrightarrow 0 \end{array}$$

commutes. Since the only automorphisms on \mathbb{Z} are $\pm \text{Id}$, we have that $\alpha(x) = \pm x$. Also, since the only automorphisms on \mathbb{Z}_{a-1} are multiplication by a unit, $\gamma([x]) = [z][x]$ for some unit $[z] \in \mathbb{Z}_{a-1}$. By the commutativity of the left square $\beta([1, 0]) = [(\pm 1, 0)]$. Also, by the commutativity of the right square, $\beta([0, 1]) = ([y, z])$ for some $y \in \mathbb{Z}$. It follows from the \mathbb{Z} -linearity of β that $\beta[(r, s)] = [(\pm r + sy, sz)]$, so β is equal to left multiplication by the matrix $\begin{bmatrix} \pm 1 & y \\ 0 & z \end{bmatrix}$. We must have $\beta[(b, a-1)] = [(0, 0)]$, and thus $[(\pm b + (a-1)y, (a-1)z)] = [(0, 0)]$ in $\text{coker}\left(\begin{bmatrix} b' & \\ & a-1 \end{bmatrix}\right)$. Hence $\pm b + (a-1)y = b't$ and $(a-1)z = (a-1)t$ for some $t \in \mathbb{Z}$. It follows that $z = t$ and $\pm b + (a-1)y = b'z$, so $b \equiv \pm z \pmod{a-1}$. Since $[\pm z]$ is a unit for \mathbb{Z}_{a-1} it follows that $[b] = [z][b']$ in \mathbb{Z}_{a-1} for a unit $[z] \in \mathbb{Z}_{a-1}$. Thus the condition in (a) holds.

CASE II: $a \in \{2, \dots\}$ and $d \in \{2, \dots\}$. In this case there are isomorphisms α , β , and γ such that

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}_{d-1} & \longrightarrow & \text{coker}\left(\begin{bmatrix} d-1 & b \\ 0 & a-1 \end{bmatrix}\right) & \longrightarrow & \mathbb{Z}_{a-1} \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & \mathbb{Z}_{d-1} & \longrightarrow & \text{coker}\left(\begin{bmatrix} d-1 & b' \\ 0 & a-1 \end{bmatrix}\right) & \longrightarrow & \mathbb{Z}_{a-1} \longrightarrow 0 \end{array}$$

commutes. Since the only automorphisms on \mathbb{Z}_{d-1} are multiplication by a unit, we have that $\alpha([x]) = [z_1][x]$ for some unit $[z_1] \in \mathbb{Z}_{d-1}$. Likewise, $\gamma([x]) = [z_2][x]$ for some unit $[z_2] \in \mathbb{Z}_{a-1}$. By the commutativity of the left square $\beta([1, 0]) = [(z_1, 0)]$. Also, by the commutativity of the right square, $\beta([0, 1]) = ([y, z_2])$ for some $y \in \mathbb{Z}$. It follows from the \mathbb{Z} -linearity of β that $\beta[(r, s)] = [(z_1 r + ys, z_2 s)]$, so β is equal to left multiplication by the matrix $\begin{bmatrix} z_1 & y \\ 0 & z_2 \end{bmatrix}$. Since $\begin{bmatrix} d-1 & b \\ 0 & a-1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ a-1 \end{bmatrix}$, we must have $\beta[(b, a-1)] = [(0, 0)]$, and thus $[(z_1 b + y(a-1), z_2(a-1))] = [(0, 0)]$ in $\text{coker}\left(\begin{bmatrix} d-1 & b' \\ 0 & a-1 \end{bmatrix}\right)$. Hence $z_1 b + y(a-1) = (d-1)s + b't$ and $z_2(a-1) = (a-1)t$ for some $s, t \in \mathbb{Z}$. It follows that $z_2 = t$ and $z_1 b + y(a-1) = (d-1)s + b't$. Writing $(d-1)s - y(a-1) = k \gcd(a-1, d-1)$ we obtain $z_1 b - z_2 b' = k \gcd(a-1, d-1)$ so that $z_1 b \equiv z_2 b' \pmod{\gcd(a-1, d-1)}$ and $[z_1][b] = [z_2][b']$ in $\mathbb{Z}_{\gcd(a-1, d-1)}$. Thus the condition in (b) holds.

For the converse, we assume that the conditions in (1)–(3) hold. Consider the following three cases.

CASE I: $a = 0$ or $a = \infty$. In this case, by considering the invariants listed in the above table, we see that we may use the identity maps for the three vertical isomorphisms to obtain a commutative diagram. Thus the six-term exact sequences are isomorphic, and it follows from Theorem 4.5 that $C^*(E) \otimes \mathbb{K} \cong C^*(E') \otimes \mathbb{K}$.

CASE II: $a \in \{2, \dots\}$ and $[b] = [z][b']$ in \mathbb{Z}_{a-1} for a unit $[z] \in \mathbb{Z}_{a-1}$. Then $b \cong zb' \pmod{a-1}$. Hence $zb' - b = (a-1)y$ for some $y \in \mathbb{Z}$. Consider $\begin{bmatrix} 1 & y \\ 0 & z \end{bmatrix} : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$. It is straightforward to check that this matrix takes $\text{im} \begin{bmatrix} b \\ a-1 \end{bmatrix}$ into $\text{im} \begin{bmatrix} b' \\ a-1 \end{bmatrix}$. Thus multiplication by this matrix induces a map $\beta : \text{coker}(\begin{bmatrix} b \\ a-1 \end{bmatrix}) \rightarrow \text{coker}(\begin{bmatrix} b' \\ a-1 \end{bmatrix})$. In addition, if we let $\alpha = \text{Id}$ and let γ be multiplication by $[z]$, then it is straightforward to verify that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \text{coker}(\begin{bmatrix} b \\ a-1 \end{bmatrix}) & \longrightarrow & \mathbb{Z}_{a-1} \longrightarrow 0 \\ & & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \text{coker}(\begin{bmatrix} b' \\ a-1 \end{bmatrix}) & \longrightarrow & \mathbb{Z}_{a-1} \longrightarrow 0 \end{array}$$

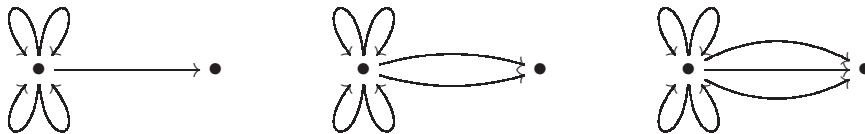
commutes. Since α and γ are isomorphisms, an application of the five lemma implies that β is an isomorphism. It follows from Theorem 4.5 that $C^*(E) \otimes \mathbb{K} \cong C^*(E') \otimes \mathbb{K}$.

CASE III: Suppose that $[z_1][b] = [z_2][b']$ in $\mathbb{Z}_{\text{gcd}(a-1, d-1)}$ for a unit $[z_1] \in \mathbb{Z}_{d-1}$ and a unit $[z_2] \in \mathbb{Z}_{a-1}$. Then $z_1b - z_2b' = k \text{gcd}(a-1, d-1)$ for some $k \in \mathbb{Z}$. Furthermore, we may write $k \text{gcd}(a-1, d-1) = s(d-1) - y(a-1)$ for some $s, y \in \mathbb{Z}$. Consider $\begin{bmatrix} z_1 & y \\ 0 & z_2 \end{bmatrix} : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$. It is straightforward to check that this matrix takes $\text{im} \begin{bmatrix} d-1 & b \\ 0 & a-1 \end{bmatrix}$ into $\text{im} \begin{bmatrix} d-1 & b' \\ 0 & a-1 \end{bmatrix}$. Thus multiplication by this matrix induces a map $\beta : \text{coker}(\begin{bmatrix} d-1 & b \\ 0 & a-1 \end{bmatrix}) \rightarrow \text{coker}(\begin{bmatrix} d-1 & b' \\ 0 & a-1 \end{bmatrix})$. In addition, if we let α be multiplication by $[z_1]$ and let γ be multiplication by $[z_2]$, then it is straightforward to verify that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \text{coker}(\begin{bmatrix} d-1 & b \\ 0 & a-1 \end{bmatrix}) & \longrightarrow & \mathbb{Z}_{a-1} \longrightarrow 0 \\ & & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \text{coker}(\begin{bmatrix} d-1 & b' \\ 0 & a-1 \end{bmatrix}) & \longrightarrow & \mathbb{Z}_{a-1} \longrightarrow 0 \end{array}$$

commutes. Since α and γ are isomorphisms, an application of the five lemma implies that β is an isomorphism. It follows from Theorem 4.5 that $C^*(E) \otimes \mathbb{K} \cong C^*(E') \otimes \mathbb{K}$. \square

Example 5.2. Consider the three graphs



which all have graph C^* -algebras with precisely one proper nontrivial ideal. By Theorem 5.1 the C^* -algebras of the two first graphs are stably isomorphic to each other, but not to the C^* -algebra of the third graph.

Remark 5.3. We mention that with existing technology it would be very difficult to see directly that the C^* -algebras of the two first graphs in Example 5.2 are stably isomorphic. One approach would be to form the *stabilized graphs* (see [Tom04, §4]) and then attempt to transform one graph to the other through operations that preserve the stable isomorphism class of the associated C^* -algebra (e.g., in/splittings, delays). However, even in this concrete example it is unclear what sequence of operations would accomplish this and we speculate that it would not be possible using the types of operations mentioned above and their inverses. In addition, the second author's Ph.D. thesis (see [Tom02] and the resulting papers [RTW], [Tom01], and [Tom03]) deals with extensions of graph C^* -algebras and shows that under certain circumstances two essential one-sink extensions of a fixed graph G have stably isomorphic C^* -algebras if they determine the same class in $\text{Ext } C^*(G)$ [Tom01, Theorem 4.1]. In Example 5.2, the three displayed graphs are all essential one-sink extensions of the graph with one vertex and four edges, whose C^* -algebra is \mathcal{O}_4 . We also have that $\text{Ext } \mathcal{O}_4 \cong \mathbb{Z}_3$, and the first two graphs in Example 5.2 determine the classes [1] and [2] in \mathbb{Z}_3 , respectively. Consequently, we cannot apply [Tom01, Theorem 4.1], and we see that the methods of this paper have applications to situations not covered by [Tom01, Theorem 4.1]. (As an aside, we mention that the second author has conjectured that if G is a finite graph with no sinks or sources, if $C^*(G)$ is simple, and if E_1 and E_2 are one-sink extensions of G , then $C^*(E_1)$ is stably isomorphic to $C^*(E_2)$ if and only if there exists an automorphism on $\text{Ext } C^*(G)$ taking the class of the extension determined by E_1 to the class of the extension determined by E_2 . We see that Example 5.2 is consistent with this conjecture since there is an automorphism of \mathbb{Z}_3 taking [1] to [2].)

Using the Kirchberg-Phillips Classification Theorem and our results in Theorem 5.1 we are able to give a complete classification of the stable isomorphism classes of C^* -algebras of graphs satisfying Condition (K) with exactly two vertices. We state this result in the following theorem. As one can see, there are a variety of cases and possible ideal structures for these stable isomorphism classes.

Theorem 5.4. *Let E and E' be graphs satisfying Condition (K) that each have exactly two vertices. Let A_E and $A_{E'}$ be the vertex matrices of E and E' , respectively, and order the vertices of each so that $c \leq b$ and $c' \leq b'$. Then $C^*(E) \otimes \mathbb{K} \cong C^*(E') \otimes \mathbb{K}$ if and only if one of the following five cases occurs.*

(i) $A_E = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A_{E'} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ with

$$(b \neq 0 \text{ and } c \neq 0) \quad \text{or} \quad (a = 0, 0 < b < \infty, c = 0, \text{ and } d \geq 2)$$

and

$$(b' \neq 0 \text{ and } c' \neq 0) \quad \text{or} \quad (a' = 0, 0 < b' < \infty, c' = 0, \text{ and } d' \geq 2)$$

and if B_E is the $E^0 \times E_{\text{reg}}^0$ submatrix of $A_E^t - I$ and $B_{E'}$ is the $(E')^0 \times (E')_{\text{reg}}^0$ submatrix of $A_{E'}^t - I$, then

$$\text{coker}(B_E : \mathbb{Z}^{E_{\text{reg}}^0} \rightarrow \mathbb{Z}^{E^0}) \cong \text{coker}(B_{E'} : \mathbb{Z}^{(E')_{\text{reg}}^0} \rightarrow \mathbb{Z}^{(E')^0})$$

and

$$\ker(B_E : \mathbb{Z}^{E_{\text{reg}}^0} \rightarrow \mathbb{Z}^{E^0}) \cong \ker(B_{E'} : \mathbb{Z}^{(E')_{\text{reg}}^0} \rightarrow \mathbb{Z}^{(E')^0}).$$

In this case $C^*(E)$ and $C^*(E')$ are purely infinite and simple.

(ii) $A_E = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ and $A_{E'} = \begin{bmatrix} 0 & b' \\ 0 & 0 \end{bmatrix}$ with $0 < b < \infty$ and $0 < b' < \infty$. In this case $C^*(E) \cong M_{b+1}(\mathbb{C})$ and $C^*(E') \cong M_{b'+1}(\mathbb{C})$, so that both C^* -algebras are simple and finite-dimensional.

(iii) $A_E = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ and $A_{E'} = \begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix}$ with $b \neq 0$ and $b' \neq 0$,

$$a = 0 \implies b = \infty \quad \text{and} \quad b = \infty \implies (a = 0 \text{ or } a = \infty),$$

and

$$a' = 0 \implies b' = \infty \quad \text{and} \quad b' = \infty \implies (a' = 0 \text{ or } a' = \infty),$$

and the conditions (1)–(3) of Theorem 5.1 hold. In this case $C^*(E)$ and $C^*(E')$ each have exactly one proper nontrivial ideal and have ideal structure of the form

$$\begin{array}{c} A \\ | \\ I \\ | \\ \{0\}. \end{array}$$

(iv) $A_E = \begin{bmatrix} a & \infty \\ 0 & d \end{bmatrix}$ and $A_{E'} = \begin{bmatrix} a' & \infty \\ 0 & d' \end{bmatrix}$ with $a \in \{2, 3, \dots\}$ and $a' \in \{2, 3, \dots\}$, and with $a = a'$ and $d = d'$. In this case $C^*(E)$ and $C^*(E')$ each have exactly two proper nontrivial ideals and have ideal structure of

the form

$$\begin{array}{c} A \\ \downarrow \\ I \\ \downarrow \\ J \\ \downarrow \\ \{0\}. \end{array}$$

(v) $A_E = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ and $A_{E'} = \begin{bmatrix} a' & 0 \\ 0 & d' \end{bmatrix}$ with
 $(a = a' \text{ and } d = d') \quad \text{or} \quad (a = d' \text{ and } d = a')$.

In this case $C^*(E) \cong C^*(E') \cong I \oplus J$, where $I := \begin{cases} \mathcal{O}_a & \text{if } a \geq 2 \\ \mathbb{C} & \text{if } a = 0 \end{cases}$

and $J := \begin{cases} \mathcal{O}_d & \text{if } d \geq 2 \\ \mathbb{C} & \text{if } d = 0 \end{cases}$, and each C^* -algebra has exactly two proper nontrivial ideals and ideal structure of the form

$$\begin{array}{ccc} & A & \\ & \swarrow \quad \searrow & \\ I & & J \\ & \swarrow \quad \searrow & \\ & \{0\}. & \end{array}$$

Remark 5.5. We are not able to classify C^* -algebras of graphs with exactly two vertices that do not satisfy Condition (K). For example if E and E' are graphs with vertex matrices $A_E = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ and $A_{E'} = \begin{bmatrix} 1 & b' \\ 0 & 1 \end{bmatrix}$, then $C^*(E)$ and $C^*(E')$ each have uncountably many ideals, and are extensions of $C(\mathbb{T})$ by $C(\mathbb{T} \otimes \mathbb{K})$. Using existing techniques, it is unclear when $C^*(E)$ and $C^*(E')$ will be stably isomorphic.

We conclude this section with an example showing an application of Theorem 4.7 to C^* -algebras with multiple proper ideals.

Example 5.6. Consider the two graphs



The ideal $I := I_{\{v,w\}}$ in $C^*(E)$ is a largest proper ideal that is an AF-algebra, and the six-term exact sequence corresponding to

$$0 \rightarrow I \rightarrow C^*(E) \rightarrow C^*(E)/I \rightarrow 0$$

is

$$0 \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \operatorname{coker}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \rightarrow 0$$

where the middle map is $[(x, y)] \mapsto [(x, y, 0)]$. Likewise, the ideal $I' := I_{\{v',w'\}}$ in $C^*(E')$ is a largest proper ideal that is an AF-algebra, and the six-term exact sequence corresponding to

$$0 \rightarrow I' \rightarrow C^*(E') \rightarrow C^*(E')/I' \rightarrow 0$$

is

$$0 \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \operatorname{coker}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) \rightarrow 0$$

where the middle map is $[(x, y)] \mapsto [(x, y, 0)]$. If we define $\beta : \operatorname{coker}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \rightarrow \operatorname{coker}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$ by $\beta[(x, y, z)] = [(x + z, y + z, z)]$, then we see that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \operatorname{coker}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) & \longrightarrow & 0 \\ & & \downarrow \operatorname{Id} & & \downarrow \beta & & \\ 0 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \operatorname{coker}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) & \longrightarrow & 0 \end{array}$$

commutes. An application of the five lemma shows that β is an isomorphism. It follows from Theorem 4.7 that $C^*(E) \otimes \mathbb{K} \cong C^*(E') \otimes \mathbb{K}$.

In the examples above, both connecting maps in the six-term exact sequences vanish. Since all C^* -algebras considered (and, more generally, all graph C^* -algebras satisfying Condition (K)) have real rank zero, the exponential map $\partial : K_0(A/I) \rightarrow K_1(I)$ is always zero. However, the index map $\partial : K_1(A/I) \rightarrow K_0(I)$ does not necessarily vanish and may carry important information. In forthcoming work, the authors and Carlsen explain how to compute this map for graph C^* -algebras.

6. STABILITY OF IDEALS

In this section we prove that if A is a graph C^* -algebra that is not an AF-algebra, and if A contains a unique proper nontrivial ideal I , then I is stable.

Definition 6.1. If v is a vertex in a graph E we define

$$L(v) := \{w \in E^0 : \text{there is a path from } w \text{ to } v\}.$$

We say that v is *left infinite* if $L(v)$ contains infinitely many elements.

Definition 6.2. If $E = (E^0, E^1, r, s)$ is a graph, then a *graph trace* on E is a function $g : E^0 \rightarrow [0, \infty)$ with the following two properties:

- (1) For any $v \in G^0$ with $0 < |s^{-1}(v)| < \infty$ we have $g(v) = \sum_{s(e)=v} g(r(e))$.
- (2) For any infinite emitter $v \in G^0$ and any finite set of edges $e_1, \dots, e_n \in s^{-1}(v)$ we have $g(v) \geq \sum_{i=1}^n g(r(e_i))$.

We define the *norm* of a graph trace g to be the (possibly infinite) quantity $\|g\| := \sum_{v \in E^0} g(v)$, and we say a graph trace g is *bounded* if $\|g\| < \infty$.

Lemma 6.3. *Let E be a graph such that $C^*(E)$ is simple. If there exists $v \in E^0$ such that v is left infinite, then $C^*(E)$ is stable.*

Proof. Since $C^*(E)$ is simple, it follows from [DT05, Corollary 2.15] that E is cofinal. Therefore, the vertex v can reach every cycle in E , and any vertex that is on a cycle in E is left infinite. In addition, if $g : E^0 \rightarrow [0, \infty)$ is a bounded graph trace on E , then since v is left infinite, it follows that $g(v) = 0$. Furthermore, it follows from [Tom06, Lemma 3.7] that

$$H := \{w \in E^0 : g(w) = 0\}$$

is a saturated hereditary subset of vertices. Since $C^*(E)$ is simple, it follows from [DT05, Theorem 3.5] that the only saturated hereditary subsets of E are E^0 and \emptyset . Because $v \in H$, we have that $H \neq \emptyset$ and hence $H = E^0$, which implies that $g \equiv 0$. Since we have shown that every vertex on a cycle in E is left infinite, and that there are no nonzero bounded graph traces on E , it follows from [Tom06, Theorem 3.2(d)] that $C^*(E)$ is stable. \square

Proposition 6.4. *Let E be a graph such that $C^*(E)$ contains a unique proper nontrivial ideal I , and let $\{E^0, H, \emptyset\}$ be the saturated hereditary subsets of E . Then there are two possibilities:*

- (1) *The ideal I is stable; or*
- (2) *The graph C^* -algebra $C^*(E)$ is a nonunital AF-algebra, and H is infinite.*

Proof. By Lemma 3.1, we see that E contains a unique saturated hereditary subset H not equal to either E^0 or \emptyset , and also $I = I_H$. In addition, it follows from [DHS03, Lemma 1.6] that I_H is isomorphic to the graph C^* -algebra $C^*({}_H E_\emptyset)$, where ${}_H E_\emptyset$ is the graph described in [DHS03, Definition 1.4]. In particular, if we let

$$F_H := \{\alpha \in E^* : s(\alpha) \notin H, r(\alpha) \in H, \text{ and } r(\alpha_i) \notin H \text{ for } i < |\alpha|\}$$

then

$${}_H E_\emptyset^0 := H \cup F_H \quad \text{and} \quad {}_H E_\emptyset^1 := \{e \in E^1 : s(e) \in H\} \cup \{\bar{\alpha} : \alpha \in F_H\}$$

where $s(\bar{\alpha}) = \alpha$, $r(\bar{\alpha}) = r(\alpha)$, and the range and source of the other edges is the same as in E . Note that since I is the unique proper nontrivial ideal in $C^*(E)$, we have that $I \cong C^*({}_H E_\emptyset)$ is simple.

Consider three cases.

CASE I: H is finite.

Choose a vertex $v \in E^0 \setminus H$. By Lemma 3.1 v is not a sink in E , and thus there exists an edges $e_1 \in E^1$ with $s(e_1) = v$ and $r(e_1) \notin H$. Continuing

inductively, we may produce an infinite path $e_1e_2e_3\dots$ with $r(e_i) \notin H$ for all i . (Note that the vertices of this infinite path need not be distinct.) We shall show that for each i there is a path from $r(e_i)$ to a vertex in H . Fix i , and let

$$X := \{w \in E^0 : \text{there is a path from } r(e_i) \text{ to } w\}.$$

Then X is a nonempty hereditary subset, and by Lemma 3.1 it follows that $X \cap H \neq \emptyset$. Thus there is a path from $r(e_i)$ to a vertex in H . Since this is true for all i , it must be the case that F_H is infinite. In the graph ${}_H E_\emptyset$ there is an edge from each element of F_H to an element in H . Since H is finite, this implies that there is a vertex in $H \subseteq {}_H E_\emptyset^0$ that is reached by infinitely many vertices, and hence is left infinite. It follows from Lemma 6.3 that $I \cong C^*({}_H E_\emptyset)$ is stable. Thus we are in the situation described in (1).

CASE II: H is infinite, and E contains a cycle.

Let $\alpha = \alpha_1 \dots \alpha_n$ be a cycle in E . Since H is hereditary, the vertices of α must either all lie outside of H or all lie inside of H . If the vertices all lie in H , then the graph ${}_H E_\emptyset$ contains a cycle, and since $C^*({}_H E_\emptyset)$ is simple, the dichotomy for simple graph C^* -algebras [DT05, Remark 2.16] implies that $C^*({}_H E_\emptyset)$ is purely infinite. Since H is infinite, it follows that ${}_H E_\emptyset^0$ is infinite and $C^*({}_H E_\emptyset)$ is nonunital. Because $C^*({}_H E_\emptyset)$ is a simple, separable, purely infinite, and nonunital C^* -algebra, Zhang's Theorem [Zha90] implies that $I \cong C^*({}_H E_\emptyset)$ is stable. Thus we are in the situation described in (1).

If the vertices of α all lie outside H , then the set

$$X := \{w \in E^0 : \text{there is a path from } r(\alpha_n) \text{ to } w\}$$

is a nonempty hereditary set. It follows from Lemma 3.1 that $X \cap H \neq \emptyset$. Thus there exists a vertex $v \in H$ and a path β from $r(\alpha_n)$ to v with $r(\beta_i) \notin H$ for $i < |\beta|$. Consequently there are infinitely many paths in F_H that end at v (viz. $\beta, \alpha\beta, \alpha\alpha\beta, \alpha\alpha\alpha\beta, \dots$). Hence there are infinitely many vertices in ${}_H E_\emptyset$ that can reach v , and v is a left infinite vertex in ${}_H E_\emptyset$. It follows from Lemma 6.3 that $I \cong C^*({}_H E_\emptyset)$ is stable. Thus we are in the situation described in (1).

CASE III: H is infinite, and E does not contain a cycle.

Since E does not contain a cycle, it follows from [DT05, Corollary 2.13] that $C^*(E)$ is an AF-algebra. In addition, since H is infinite it follows that E^0 is infinite and $C^*(E)$ is nonunital. Thus we are in the situation described in (2). \square

Corollary 6.5. *If E is a graph with a finite number of vertices and such that $C^*(E)$ contains a unique proper nontrivial ideal I , then I is stable. Furthermore, if $\{E^0, H, \emptyset\}$ are the saturated hereditary subsets of E , then $C^*(E_H)$ is a unital C^* -algebra and $I \cong C^*(E_H) \otimes \mathcal{K}$.*

Proof. Since E^0 is finite it is the case that $C^*(E)$ is unital, and it follows from Proposition 6.4 that I is stable. Furthermore, since $I = I_H$ it follows from [BPRS00, Theorem 4.1] and [BPRS00, Proposition 3.4] that I is Morita equivalent to $C^*(E_H)$. Since I and $C^*(E_H)$ are separable, it follows that I

and $C^*(E_H)$ are stably isomorphic. Thus $I \cong I \otimes \mathcal{K} \cong C^*(E_H) \otimes \mathcal{K}$. Finally, since $E_H^0 = H \subseteq E^0$ is finite, $C^*(E_H)$ is unital. \square

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