UNIVERSITY OF COPENHAGEN

Individual Preference Rankings Compatible with Prices, Income Distributions and Total Resources

Balasko, Yves; Tvede, Mich

Publication date:
2009

Document version
Publisher's PDF, also known as Version of record

Citation for published version (APA):
Balasko, Y., & Tvede, M. (2009). *Individual Preference Rankings Compatible with Prices, Income Distributions and Total Resources*. Department of Economics, University of Copenhagen.

Discussion Papers
Department of Economics
University of Copenhagen

No. 09-09

Individual Preference Rankings Compatible with Prices, Income Distributions and Total Resources

Yves Balasko, Mich Tvede

Studiestræde 6, DK-1455 Copenhagen K., Denmark
Tel.: +45 35 32 30 82 – Fax: +45 35 32 30 00
http://www.econ.ku.dk

ISSN: 1601-2461 (online)

Individual preference rankings compatible with prices, income distributions and total resources*

Yves Balasko † Mich Tvede‡

November 2008

Abstract

We consider the problem of determining the individual preference rankings that are necessarily implied by a dataset consisting of prices, income distributions and total resources.

We show the equivalence between the compatibility with individual preference rankings and the existence of a solution to a set of linear equalities and inequalities. Using this characterization, we give new proofs of the rationalizability of finite data sets where total resources are close to being collinear and the contractibility and pathconnectedness of the set that consists of rationalizable finite datasets.

Keywords: Equilibrium manifold, rationalizability, testability, pathconnectedness.
JEL-classification: D1, D5.

1. Introduction

We define the d-triple $d = \big(p, (w_i), r\big)$ as a vector whose three components are: a price vector p, an income distribution vector (w_i); a total resource vector r.

In an exchange economy made of m consumers, where consumer i is characterized by the utility function u_i and endowment vector ω_i, any arbitrarily given price vector p defines the d-triple $\big(p, (w_i), r\big)$ where $w_i = p \cdot \omega_i$ and $r = \sum_i \omega_i$. The (indirect) utility of the d-triple $d = \big(p, (w_i), r\big)$ for consumer i is given by the expression $\hat{v}_i(d) = \hat{u}_i(p, w_i)$ where $\hat{u}_i$ is the indirect utility function associated with the direct utility function u_i.

A dataset D consists of a finite number T of d-triples $d^t = \big(p^t, (w_i^t), r^t\big)$, with $t \in \mathbb{T} = \{1, \dots, T\}$. The ranking

$$\hat{v}_i(d^{t_1}) \leq \hat{v}_i(d^{t_2}) \leq \cdots \leq \hat{v}_i(d^{t_T})$$

of the indirect utility levels $\hat{v}_i(d^t)$ reflects the preference ranking of the dataset D by consumer i, with i varying from 1 to m.

*This paper is an expanded version of a paper circulated in 2003 that contains the first reference to and proof of the linear programming characterization of ranking compatibility [2]. We are grateful to a referee of this Journal for valuable comments and suggestions.

†Department of Economics and Related Studies, University of York, U.K., email: yb501@york.ac.uk
‡Department of Economics, University of Copenhagen, email: Mich.Tvede@econ.ku.dk

We consider in this paper the inverse problem of determining the individual preference rankings that are necessarily implied by a given dataset D despite the fact that individual utility functions are unknown. This problem requires the determination of the utility functions that rationalize the dataset D and may have many as well as zero solutions. Different rankings can be compatible with the dataset D and the issue in this paper is to determine them.

It is known that the equilibrium manifold determines the consumers' preferences [1, 5]. But the minimum this requires are the points of an open subset of the equilibrium manifold, i.e., an infinite number of points. If the dataset T is finite, the only results available so far are those about the "testability" of the equilibrium manifold [4]. There is a big gap between the determination of consumers' preferences and the testability issue. That gap is partially filled in by the results of the current paper on the determination of the individual rankings compatible with a finite dataset D

In applications, prices, individual incomes and aggregate demand are observable. Utility functions or preferences are not. The determination of the individual preference rankings compatible with finite datasets is therefore a precious information that is of particular interest in cost-benefit analyses and public finance for example.

The main result of this paper is the equivalence between rankings of finite datasets and existence of solutions to a set of linear equalities and inequalities. This problem has a remarkable structure from which follows that its computational complexity is relatively modest. We use this characterization to improve our understanding of the set of finite datasets compatible with given individual rankings. We show that this set is open, which implies that the compatibility of a dataset with given individual rankings is robust to perturbations. We also prove that this set is contractible, which implies in particular that it is pathconnected. The economic interest of global topological properties like contractibility, pathconnectedness and more generally, the nature of the various homology , cohomology and homotopy groups that make up the core of algebraic topology may seem remote to many economists. Nevertheless, once an economically relevant property has been identified and, here, this property is the compatibility of finite datasets with given individual rankings, the property can be identified with the set of suitably defined elements that satisfy that property. At an abstract level, this set is usually a subset of some bigger set and the issue is then to get the best "understanding" possible of that subset. For example, if equations and inequations are often used to characterize subsets of a given set, it is generally not an easy task to determine from their equations and inequations the "picture" of the subsets. Global topological properties are the tools that mathematicians have invented to bypass our visual limitations in dimensions higher than three. From that perspective, contractibility of the set of finite datasets compatible with given individual rankings is quite remarkable. Though not convex, this set has already many of the nice properties of convex sets. We also use our characterization to give an alternative proofs of the rationalizability of finite data sets when total resources are close to being collinear [3].

The paper is organized as follows. In Section 2, we recall the main assumptions and definitions, and set the notation. Section 3 addresses the equivalence between the compatibility of finite datasets with specified individual rankings and the existence of solutions to some set of linear equalities and inequalities. Section 4 is devoted to a proof that finite datasets where total resources are collinear or almost collinear are rationalizable. Section

5 deals with the contractibility and pathconnectedness of the set of datasets compatible with a given individual rankings, a property that is applied to show the pathconnectedness of the set of rationalizable datasets. Section 6 concludes the paper with a few comments. The concepts of well ranked and strongly ranked datasets are defined in the Appendix. Their properties are a crucial ingredient of our characterization of dataset rankings.

2. Definitions, assumptions and notation

Goods and prices

There is a finite number ℓ of goods. Let $p = (p_1, p_2, \dots, p_{\ell-1}, p_\ell) \in \mathbb{R}^\ell_{++}$ be the price vector. We normalize the price vector p by picking the ℓ-th commodity as the numeraire: $p_\ell = 1$. Let S denote the set of strictly positive normalized price vectors.

Consumers and their utility functions

There is a finite number $m \geq 2$ of consumers. A consumer is characterized by the consumption set $X = \mathbb{R}^\ell_{++}$, an endowment vector $\omega_i \in X$ and a utility function $u_i : X \to \mathbb{R}$.

We assume that consumer i's utility function u_i belongs to the class $\mathcal{U}$ of smooth maps from X into $\mathbb{R}$ that satisfy the following properties whose mathematical and economic interpretations are standard: 1) $Du_i(x_i) \in X$ for any $x_i \in X$ (smooth monotonicity); 2) The combined inequality $y^T D^2 u_i(x_i)\, y \geq 0$ and equality $y^T Du_i(x_i) = 0$, where $x_i \in X$, have a unique solution $y = 0 \in \mathbb{R}^\ell$ (smooth strict quasi-concavity); 3) The indifference surfaces $u_i^{-1}(a)$ are closed in $\mathbb{R}^\ell$ for all $a \in \mathbb{R}$. (The latter condition does not follow from the continuity of u_i because the consumption space X is open in $\mathbb{R}^\ell$, not closed; economically, this property means that every commodity is strictly necessary.)

We denote by $\Omega = X^m$ the set of endowments of all consumers.

Individual demand functions

Given the utility function $u_i \in \mathcal{U}$, the demand of consumer i for the price vector $p \in S$ and income $w_i > 0$ is $f_i(p, w_i) = \arg\max u_i(x_i)$ subject to the budget constraint $p \cdot x_i \leq w_i$. Walras law is the identity $p \cdot f_i(p, w_i) = w_i$.

Let $\omega = (\omega_i) \in \Omega$ denote the vector of individual endowments. The equilibrium manifold is the subset of $S \times \Omega$ that consists of the elements (p, ω) that satisfy equation

$$\sum_i f_i(p, p \cdot \omega_i) = \sum_i \omega_i. \tag{1}$$

Feasible and equilibrium d-triples

Let $(w_i) \in \mathbb{R}^m_{++}$ denote the income distribution between the m consumers making up the economy.

The d-triple $(p, (w_i), r) \in S \times \mathbb{R}^m_{++} \times X$ is *feasible* if the equality

$$w_1 + w_2 + \cdots + w_m = p \cdot r \tag{2}$$

is satisfied. Note that the definition of feasibility does not require utility functions.

The d-triple $(p, (w_i), r) \in S \times \mathbb{R}^m_{++} \times X$ is an *equilibrium d-triple* for the utility profile $u = (u_i) \in \mathcal{U}^m$ if the equality

$$\sum_i f_i(p, w_i) = r \tag{3}$$

is satisfied. It follows from Walras law satisfied by the individual demand functions that an equilibrium d-triple is feasible.

Datasets

From now on, we consider datasets $D = (d^t)_{1 \leq t \leq T}$ consisting of T pairwise distinct feasible d-triples $d^t = (p^t, (w_i)^t, r^t) \in S \times \mathbb{R}^m_{++} \times X$. The set of these datasets is denoted by $\mathcal{D}$ while $\mathcal{E}(u)^{[T]}$ represents the subset of elements of $\mathcal{D}$ whose components are equilibrium d-triples for the utility profile $u = (u_1, \ldots, u_m) \in \mathcal{U}^m$. The utility profile $u = (u_i) \in \mathcal{U}^m$ then *rationalizes* the dataset D. We also introduce the set $\mathcal{E}^{[T]}$ of rationalizable datasets $D \in \mathcal{D}$. We have $\mathcal{E}^{[T]} = \cup_{u \in \mathcal{U}^m} \mathcal{E}(u)^{[T]}$.

Datasets and compatible preference (pre)orderings

Let $D \in \mathcal{E}(u)^{[T]}$. The ranking

$$\hat{v}_i(d^{t_1}) \leq \hat{v}_i(d^{t_2}) \leq \cdots \leq \hat{v}_i(d^{t_T})$$

of the indirect utility levels $\hat{v}_i(d^t)$ defines a preordering $\preceq_i$ on the set $\mathbb{T}$. The consumers' preference preorderings of the dataset D (given the utility profile $u = (u_1, \ldots, u_m) \in \mathcal{U}^m$) is the m-tuple $\preceq = (\preceq_i)$ defined by the m consumers' preorderings of the set $\mathbb{T}$.

An ordering is a preordering that is antisymmetric (i.e., $t \preceq_i t'$ and $t' \preceq_i t$ imply $t = t'$). The ordering $\prec_i$ is a *refinement* of the preordering $\preceq_i$ if $t \prec_i t'$ implies $t \preceq_i t'$. Note that any preordering can be refined into an ordering.

It will be important in the sequel to differentiate orderings from preorderings. We will use the notation $\prec = (\prec_i)$ for orderings of the dataset D.

3. Compatibility of datasets and preference (pre)orderings

The main result of this section is the equivalence between the compatibility of a given dataset D with the preference **ordering** $\prec = (\prec_i)$ and the solution of a set of linear equalities and inequalities.

3.1. A set of linear equalities and inequalities

Theorem 1. *Let $\prec = (\prec_i)$ be some preference ordering of the set $\mathbb{T}$. Let $D = (d^t) \in \mathcal{D}$ be a dataset made of T feasible d-triples.*

There exists a utility profile $u = (u_1, u_2, \ldots, u_m) \in \mathcal{U}^m$ inducing the preference ordering $\prec = (\prec_i)$ if and only if the system of linear equalities and inequalities

$$LP(D, \prec): \quad \begin{cases} p^{t'} \cdot x_i^t > w_i^{t'} & \text{whenever } t' \prec_i t; & (L1) \\ p^t \cdot x_i^t = w_i^t; & & (L2) \\ \sum_i x_i^t = r^t. & & (L3) \end{cases}$$

has a solution $\{(x_i^t)\}$*, with* $1 \le i \le m$ *and* $1 \le t \le T$*.*

Proof. The condition is necessary. Let $x_i^t = f_i(p^t, w_i^t)$. Equality *(L3)* follows from the equilibrium condition. Equality *(L2)* follows from Walras law. Inequality *(L1)* then follows from inequality $u_i(x_i^{t'}) < u_i(x_i^t)$.

The condition is sufficient. Pick some arbitrary consumer i. We use equality *(L2)* to substitute $p^t \cdot x_i^t$ to w_i^t in inequality *(L1)*. The collection of inequalities *(L1)* implies that the data (p^t, x_i^t) are well-ranked for the ordered index set $(\mathbb{T}, \prec_i) = \{t_{i_1} \prec_i t_{i_2} \prec_i \cdots \prec_i t_{i_T}\}$ in the sense of Definition A.1 of the Appendix. It then suffices to apply Proposition A.2 of the Appendix to rationalize these data by some utility function $u_i \in \mathfrak{U}$ with the property that the strict inequality $u_i(x_i^{t'}) < u_i(x_i^t)$ is equivalent to $t' \prec_i t$. It then suffices to do this for every consumer. Equality *(L3)* implies that the equilibrium condition is satisfied for every t varying from 1 to T.

□

3.2. Extension to preference preorderings

The following Proposition enables us to use Theorem 1 in the case of preorderings instead of orderings.

Proposition 1. *Let* $\preceq = (\preceq_i)$ *be some preference preordering of the set* $\mathbb{T}$ *associated with the utility profile* $u = (u_i) \in \mathfrak{U}^m$*. Then, for any ordering* $\prec = (\prec_i)$ *that is a refinement of the preordering* $\preceq = (\preceq_i)$*, there exists a utility profile* $u' = (u'_i) \in \mathfrak{U}^m$ *such that the dataset* D *belongs to* $\mathcal{E}^{[T]}(u')$ *and the induced preference ranking coincides with the ordering* $\prec = (\prec_i)$*.*

Proof. This is essentially Proposition A.8 of the Appendix. □

The following corollary is just a reformulation of Proposition 1.

Corollary 1. *Any dataset* D *that is rationalized by the utility profile* $u = (u_i)$ *that induces a preference preordering can be rationalized by another utility profile* $u' = (u'_i)$ *such that the induced preference preordering of the datasets is actually an ordering.*

Proposition 1 enables us to check the compatibility of the dataset D with the preference preordering $\preceq = (\preceq_i)$ by reducing this question to the compatibility of the dataset D with any preference ordering $(\prec) = (\prec_i)$ that is a refinement of $\preceq = (\preceq_i)$. We solve the latter question by applying Theorem 1.

3.3. Structure and size of the linear programming problem

Checking whether the linear system $LP(D, \prec)$ has a solution is equivalent to solving a linear programming problem. This problem has $m\ell T$ real unknowns and $m\ell T + \ell T + mT(T+1)/2$ constraints (including the sign constraints). For a given economy, m and ℓ are constant and the only variable parameter is the number T of equilibrium data. The numbers of unknowns and constraints are linear and quadratic respectively in T. This situation is similar to the one observed by Varian [7] for Afriat's inequalities in the case of one consumer. But, at variance with Afriat's set of inequalities that are practically impossible to solve for large T because of the size of the problem, the set of inequalities and equalities in Theorem (1) decomposes into T smaller linear subproblems.

The number of unknowns of each subproblem is equal to ℓm while the number of constraints varies from $m\ell + \ell + m$ to $m\ell + \ell + mT$ depending on the value of t and of the ranking profile $\prec$. The average value of the number of constraints is therefore equal to $m\ell + \ell + mT/2$. Both average and maximal values are linear in T. This makes each one of the linear subproblems far more tractable than the general problem, an advantage that more than compensates for the fact that there exist T such problems.

4. The set of rationalizable datasets and related sets

Given the preference ordering $\prec = (\prec_i)$, we denote by $\left(\mathcal{E}^{[T]}\middle|\prec\right)$ the set of T pairwise distinct datasets $D \in \mathcal{D}$ that are compatible with the preference ordering $\prec$. It would be most interesting to have a precise description of the set $\left(\mathcal{E}^{[T]}\middle|\prec\right)$ as as subset of $\mathcal{D}$. Such description is impossible. The best we can hope for at the moment are some global topological properties of the set $\left(\mathcal{E}^{[T]}\middle|\prec\right)$ like pathconnectedness or contractibility. We investigate these properties using the characterization provided by Theorem 1.

Recall that $\mathcal{E}^{[T]}$ denotes the set of T pairwise distinct rationalizable datasets $D \in \mathcal{D}$. We then have:

Proposition 2.

$$\mathcal{E}^{[T]} = \bigcup_{\prec} (\mathcal{E}^{[T]}|\prec).$$

Proof. Obvious given Proposition 1 and the discussion that followed.

□

Proposition 2 enables us to derive properties of the set of rationalizable data $\mathcal{E}^{[T]}$ from those of the sets $(\mathcal{E}^{[T]}\middle|\prec)$, sets for which we can use Theorem 1.

4.1. Openness of the set of datasets compatible with a given preference preordering

Theorem 1 tells us that the set $\left(\mathcal{E}^{[T]}\middle|\prec\right)$ consists of the datasets D such that the set defined by the linear system $LP(D, \prec)$ is non empty. We are going to apply this characterization to prove the following proposition:

Proposition 3. *The set $\left(\mathcal{E}^{[T]}\middle|\prec\right)$ is open.*

Proof. Let us show that if $D' = (p'^t, (w_i'^t), r'^t) \in \mathcal{D}$ is a dataset close enough to D, then the set defined by $LP(D', \prec)$ is also non empty. The idea is to perturb the solution x of $LP(D, \prec)$ into a solution x' of $LP(D', \prec)$. Define $\bar{x}_i^t = (x_i^{1t}, x_i^{2t}, \ldots, x_i^{\ell-1\,t})$ consisting of the first $\ell - 1$ coordinates of x_i^t. For i varying from 1 to $m-1$, let $x_i'^{\ell t} = w_i'^t - \bar{p}'^t \cdot \bar{x}_i^t$ where $\bar{p}'^t = (p_1'^t, \ldots, p_{\ell-1}'^t)$ and $x_i'^t = (\bar{x}_i^t, x_i'^{\ell t})$. Then, define $x_m'^t$ by

$$x_m'^t = r'^t - \sum_{i=1}^{m-1} x_i'^t.$$

Equalities $L2$ and $L3$ are satisfied by construction and $L1$ is satisfied because these inequalities are strict for D' sufficiently close to D. □

An obvious consequence of this proposition is:

Corollary 2. *The set of rationalizable datasets $\mathcal{E}^{[T]}$ is open in $\mathcal{D}$.*

4.2. Rationalizability of datasets with collinear total resources

Theorem 2. *Let $D = (d^t)_{1\leq t\leq T}$ be a dataset such that the total resources r^t are all collinear. The dataset D is then rationalizable.*

The vectors of total resources r^t are all collinear with some vector $r \in X$. Let us define $x_i^t \in X$ and $\lambda_i^t > 0$ by

$$x_i^t = \frac{w_i^t}{p^t \cdot r^t} r^t = \lambda_i^t r.$$

The following lemma is obvious:

Lemma 1. *Generically on D, the λ_i^t are pairwise distinct.*

Proof of Theorem 2. It follows from Lemma 1 combined with Proposition 3 that we can assume that the dataset D is such that the λ_i^t's are pairwise distinct. They then define for each consumer i the ordering $\prec_i$ of the set $\mathbb{T}$:

$$\lambda_i^{t_1} < \lambda_i^{t_2} < \cdots < \lambda_i^{t_T}.$$

It then suffices to check that (x_i^t) for $1 \leq i \leq m$ and $1 \leq t \leq T$ is a solution of the linear system $LP(D, \prec)$, which is obvious. $\square$

5. Topological properties of sets of compatible and rationalizable datasets

We now apply Theorem 1 to get a simple proof of the contractibility of $\left(\mathcal{E}^{[T]}\middle|\prec\right)$. In a second stage, this property is used to prove the pathconnectedness of the set of rationalizable datasets $\mathcal{E}^{[T]}$.

5.1. Contractibility of $\left(\mathcal{E}^{[T]}\middle|\prec\right)$

Proposition 4. *The set $\left(\mathcal{E}^{[T]}\middle|\prec\right)$ is contractible.*

The topological space Z is said to be *contractible* if there exists a continuous map $h : Z \times [0, 1] \to Z$ that satisfies the following properties: 1) $h(., 1)$ is the identity map of Z, i.e., $h(., 1) = \mathrm{id}_Z$; 2) $h(z, 0) = z_0 \in Z$ for $z \in Z$ and some $z_0 \in Z$. The intuition behind this definition is that a contractible space can be continuously deformed into a point, here the point z_0.

A related idea is the one of a *deformation retract*. By definition, the topological subspace Z_0 of Z is a deformation retract if there exists a continuous map $h : Z \times [0, 1] \to Z$ that satisfies the following properties: 1) $h(., 1)$ is the identity map of Z, i.e., $h(., 1) = \mathrm{id}_Z$; 2) $h(z, 0) \in Z_0$ for $z \in Z$; 3) $h(z, 0) = z$ for all $z \in Z_0$. If the subspace Z_0 is a deformation retract of Z and is contractible, then Z is also contractible. (First, contract Z to Z_0, and then Z_0 to a point.)

It is intuitively clear and almost obvious that a contractible space is pathconnected.

Lemma 2. *Let* $D = \big((p^t, (w_i^t), r^t)\big) \in \big(\mathcal{E}^{[T]}\big|\prec\big)$ *and* $D^* = \big((p^t, (w_i^{*t}), r^{*t})\big) \in \big(\mathcal{E}^{[T]}\big|\prec\big)$. *Then, for any real number* $\lambda \in [0, 1]$, *the dataset* $D(\lambda) = \lambda D + (1 - \lambda)D^*$ *belongs to* $\big(\mathcal{E}^{[T]}\big|\prec\big)$.

Proof. It suffices to observe that the linear system $LP(D, \prec)$ in Theorem 1 is linear with respect to x_i^t and w_i^t. □

We now define special datasets that belong to the set $\big(\mathcal{E}^{[T]}\big|\prec\big)$ for a given ordering $\prec = (\prec_i)$. Let $t_{i_1} \prec_i t_{i_2} \prec_i \cdots \prec_i t_{i_T}$ represent the ordering $\prec_i$. Let $(\mu_{t_{i_1}}, \mu_{t_{i_2}}, \ldots, \mu_{t_{i_T}})$ be a strictly increasing sequence of strictly positive (real) numbers:

$$0 < \mu_{t_{i_1}} < \mu_{t_{i_2}} < \cdots < \mu_{t_{i_T}}.$$

Let $\tau_i \in X$ be some strictly positive vector. Define the vector $x_i^{*t} = \mu_t \tau_i \in X$. The sequence x_i^{*t} satisfies the strict (vector) inequalities

$$x_i^{*t_{i_1}} < x_i^{*t_{i_2}} < \cdots < x_i^{*t_{i_T}}.$$

Let $p = (p^t)$, with $p^t \in S$ for $t = 1, 2, \ldots, T$. Define $w_i^{*t} = p^t \cdot x_i^{*t}$ for $1 \leq i \leq m$ and $r^{*t} = \sum_i x_i^{*t}$. We denote by $D^*(p)$ the dataset $(p^t, (w_i^{*t}), r^{*t})$.

Lemma 3. *The dataset* $D^*(p) = (p^t, w^*, r^*)$ *belongs to* $\big(\mathcal{E}^{[T]}\big|\prec\big)$ *for any* $p = (p^t) \in S^T$.

Proof. This is essentially a rehash of Theorem 2.

□

Proof of the contractibility property

Proof of Proposition 4. Let $\mathcal{X}_0$ be the subset of $\big(\mathcal{E}^{[T]}\big|\prec\big)$ consisting of the datasets $D^*(p)$ where the price sequence $p = (p^1, p^2, \ldots, p^T)$ is varied in S^T. The map $p \to D^*(p)$ is continuous. The inverse map is the projection $(p^t, (w_i^t), r^t) \to p = (p^t)$. These two maps are continuous, which proves that $\mathcal{X}_0$ is homeomorphic to S^m and is therefore contractible as a Cartesian product of contractible spaces.

We now define the map $h : \big(\mathcal{E}^{[T]}\big|\prec\big) \times [0, 1] \to \big(\mathcal{E}^{[T]}\big|\prec\big)$ by

$$h(D, \lambda) = \lambda D + (1 - \lambda)D^*(p)$$

where $p = (p^t)$ is fixed. This map is clearly continuous. In addition, $h(D, 1) = D$, $h(D, 0) = D^*(p) \in \mathcal{X}_0$, and for $D = D^*(p)$, it comes $h(D^*(p), \lambda) = D^*(p)$. The set $\mathcal{X}_0$ is therefore a deformation retract of $\big(\mathcal{E}^{[T]}\big|\prec\big)$ by the map h. The set $\mathcal{X}_0$ being contractible, the set $\big(\mathcal{E}^{[T]}\big|\prec\big)$ itself is contractible. □

5.2. Pathconnectedness of the set of rationalizable datasets

Lemma 4. *The intersection*

$$\bigcap_{\prec} \big(\mathcal{E}^{[T]}\big|\prec\big)$$

is non empty.

Proof. The idea of the proof is to identify a dataset $\tilde{D} \in \mathcal{E}^{[T]}$ that is compatible with all possible preference orderings $\prec$.

Step 1. Let $u \in \mathcal{U}$ be some arbitrary utility function. Let x^1, x^2, ..., x^t, ..., x^T be T distinct consumption bundles in X yielding the same utility level: $u(x^1) = u(x^2) = \cdots = u(x^t) = \cdots = u(x^T))$. Let p^t be the supporting price vector of x^t for $t = 1, 2, \ldots, T$.

The strict inequality $p^t \cdot x^t < p^t \cdot x^{t'}$ for $t \neq t'$ follows from the strict quasi-concavity of the utility function u combined with $x^t \neq x^{t'}$ (and $u(x^t) = u(x^{t'})$).

Step 2. These inequalities imply as a special case the inequalities

$$p^t \cdot x^t < p^t \cdot x^{t'} \quad \text{for } 1 \leq t \prec_i t' \leq T$$

for any ordering $\prec_i$ of $\mathbb{T} = \{1, 2, \ldots, T\}$.

Step 3. Let $\tilde{D}$ be the dataset $(\tilde{p}^t, (\tilde{w}_i^t), \tilde{r}^t)$ where $\tilde{p}^t = p^t$, $\tilde{w}_i^t = \tilde{p}^t \cdot x^t$, and $\tilde{r}^t = mx^t$ for t varying from 1 to T. Let also $u = (u, u, \ldots, u)$ denote the utility profile associated with each consumer having $u \in \mathcal{U}$ as utility function.

Let $x_i^t = x^t$ for $i = 1, 2, \ldots, m$ and $t = 1, 2, \ldots, T$. Then, it is obvious that (x_i^t) with $1 \leq i \leq m$ and $1 \leq t \leq T$ is a solution of the linear system $LP(\tilde{D}, \prec)$ for any preference ordering $\prec$. We conclude by applying Theorem 1.

□

We can now prove:

Proposition 5. *The set of rationalizable datasets $\mathcal{E}^{[T]}$ is pathconnected.*

Proof. The set $\mathcal{E}^{[T]}$ is the union of the path-connected sets $\left(\mathcal{E}^{[T]} \middle| \prec\right)$ for all preference orderings $\prec = (\prec_i)$. These sets have a non empty intersection by Lemma 4. Let $\tilde{D}$ be some element of the intersection $\cap_{\prec}\left(\mathcal{E}^{[T]} \middle| \prec\right)$. It then suffices to join the datasets $D \in \left(\mathcal{E}^{[T]} \middle| \prec\right)$ and $D' \in \left(\mathcal{E}^{[T]} \middle| \prec'\right)$ to the element $\tilde{D}$ by two continuous paths contained in $\left(\mathcal{E}^{T]} \middle| \prec\right)$ and $\left(\mathcal{E}^{T]} \middle| \prec'\right)$ respectively to define a continuous path in $\mathcal{E}^{[T]}$ linking the two datasets D and D'.

□

6. Concluding comments

The analysis developed in this paper has been limited to the pure exchange model. Obviously, serious attempts at applying our analysis to real world data should consider extending our theoretical results to economies that are more complex than the pure exchange ones. Production should explicitely be taken into account. More generally, theoretical developments should exploit the structures induced by time and uncertainty up to the operation of asset markets. This paper should be seen as a feasibility study for such an approach.

References

[1] Y. Balasko. The equilibrium manifold keeps the memory of individual demand functions. *Economic Theory*, 24:493–501, 2004.

[2] Y. Balasko and M. Tvede. Individual preferences compatible with a finite number of equilibrium data: A linear programming characterization. Preprint, 2003.

[3] Y. Balasko and M. Tvede. The geometry of equilibrium data sets. Preprint, 2008.

[4] D.J. Brown and R.L. Matzkin. Testable restrictions on the equilibrium manifold. *Econometrica*, 64:1249–1262, 1996.

[5] P-A. Chiappori, I. Ekeland, H. Kübler, and H. Polemarchakis. Testable implications of general equilibrium theory: A differentiable approach. Working paper, University of Chicago, Revised February 2002.

[6] P.A. Chiappori and J-C. Rochet. Revealed preferences and differentiable demand. *Econometrica*, 55:687–691, 1987.

[7] H.R. Varian. The nonparametric approach to demand analysis. *Econometrica*, 50:945–973, 1982.

A. Rationalizability of well-ranked individual price-consumption data

We consider a set of individual price-consumption data (p^t, x_i^t) indexed by a set I. The index set I is equipped with an ordering denoted by $\prec$.

The utility function $u_i : X \to \mathbb{R}$ *rationalizes* the set of individual price-consumption data $\{(p^t, x_i^t)$ with $t \in I$ if

$$u_i(x_i^t) = \max u_i(x_i) \qquad \text{subject to} \quad p^t \cdot x_i \leq p^t \cdot x_i^t$$

for $t \in I$.

A.1. Well-ranked individual data for a specified ordering

Definition A. 1. *The individual price-consumption data (p^t, x_i^t) with $t \in I$ is well-ranked for the ordering $\prec$ of the index set I if the strict inequality*

$$p^t \cdot x_i^t < p^t \cdot x_i^{t'} \tag{4}$$

is satisfied whenever $t \prec t'$.

Note that the property of being well-ranked depends on the ordering $\prec$ on I.

Proposition A. 2. *Any set of well-ranked individual price-consumption data for the ordered index set $(I, \prec)$ can be rationalized by some utility function $u_i \in \mathcal{U}$.*

Proof. There is no loss of generality if we identify the index set I to the set of integers $\mathbb{T} = \{1, 2, \dots, T\}$ and the order $\prec$ to the ordinary $1 < 2 < \cdots < T$.

It follows from [6] that it suffices that the data can be rationalized by a piecewise-linear utility function. (The domain of the utility function in [6] is a compact subset of the strictly positive orthant $X = \mathbb{R}^\ell_{++}$; this domain has to be extended here to X, which is straightforward.)

The idea is therefore to find a function of the form

$$u_i(x_i) = \inf_{1 \leq t \leq T} a_t + b_t p^t \cdot x_i$$

where a_t and b_t are > 0 and such that the strict inequality

$$a_t + b_t p^t \cdot x_i^t < a_{t'} + b_{t'} p^{t'} \cdot x_i^t \tag{5}$$

is satisfied for every t and t' where $1 \leq t \neq t' \leq T$.

It follows readily from the $(T-1)^2$ inequalities (5) that x_i^t is supported for the utility function u_i by the price vector p^t, with $1 \leq t \leq T$. This implies that the price-consumption data are rationalized by the utility function $u_i \in \mathcal{U}^{\mathrm{pl}}$.

The proof consists therefore in finding these coefficients a_t and b_t with $1 \leq t \leq T$. It starts with a_1 and b_1 arbitrary > 0, the computation of the other coefficients proceeding by induction on T.

Case $T = 2$. Let $a_2 = a_1 + b_1 p^1 \cdot x_i^1$. The goal is to find b_2 such that the two inequalities

$$a_1 + b_1 p^1 \cdot x_i^1 < a_2 + b_2 p^2 \cdot x_i^1 \tag{6}$$

$$a_2 + b_2 p^2 \cdot x_i^2 < a_1 + b_1 p^1 \cdot x_i^2 \tag{7}$$

are satisfied.

Inequality (6) is satisfied for any $b_2 > 0$ since $a_2 = a_1 + b_1 p^1 \cdot x_i^1$ and $p^1 \cdot x_i^1 > 0$ from $x_i^1 \neq 0$. Inequality $a_2 < a_1 + b_1 p^1 \cdot x_i^2$ is equivalent to $a_1 + b_1 p^1 \cdot x_i^1 < a_1 + b_1 p^1 \cdot x_i^2$, itself equivalent to $p^1 \cdot x_i^1 < p^1 \cdot x_i^2$, inequality that follows from the fact that the dataset D is strongly well-ranked. It then suffices to pick $b_2 > 0$ small enough for the strict inequality

$$b_2 p^2 \cdot x_i^2 < b_1 p^1 \cdot (x_i^2 - x_i^1)$$

to be satisfied.

T arbitrary. We now use the induction assumption for $T-1$. With $a_1 > 0$ and $b_1 > 0$ arbitrary, there exists $a_2, \ldots, a_{T-1}$ and $b_2, \ldots, b_{T-1}$, all > 0, with $a_t = a_{t-1} + b_{t-1} p^{t-1} \cdot x_i^{t-1}$ for $2 \leq t \leq T-1$, such that inequality (5) is satisfied for $0 \leq t \neq t' \leq T-1$.

Let $a_T = a_{T-1} + b_{T-1} p^{T-1} \cdot x_i^{T-1}$. We have $a_0 < a_1 < \cdots < a_{T-1} < a_T$ since each $b_t p^t \cdot x_i^t$ is > 0.

Inequality $a_t + b_t p^t \cdot x_i^t \leq a_T + b_T p^T \cdot x_i^t$, $t \leq T-1$.

It follows from the monotonicity of the sequence a_t that inequality

$$a_t + b_t p^t \cdot x_i^t = a_{t+1} \leq a_T + b_T p^T \cdot x_i^t$$

is satisfied for any $t \leq T-1$ and any $b_T > 0$.

Inequality $a_T < a_t + b_t p^t \cdot x_i^T$, $t \leq T-1$.

By the induction assumption, inequality

$$a_T < a_t + b_t p^t \cdot x_i^{T-1} \tag{8}$$

is satisfied for $t < T-1$. The price-consumption data are well-ranked, which implies the strict inequality $p^t \cdot x_i^{T-1} < p^t \cdot x_i^T$ from which follows the strict inequality $a_t + b_t p^t \cdot x_i^{T-1} < a_t + b_t p^t \cdot x_i^T$ which, combined with inequality (8), yields

$$a_T = a_{T-1} + b_{T-1} p^{T-1} \cdot x_i^{T-1} < a_t + b_t p^t \cdot x_i^T \tag{9}$$

for $t < T-1$.

If follows from inequality (9) that it suffices to pick $b_T > 0$ such that

$$b_T p^T \cdot x_i^T < \inf_{1 \leq t \leq T-1} (a_t + b_t p^t \cdot x_i^T - a_T)$$

to have the inequalities

$$a_T + b_T p^T \cdot x_i^T < a_t + b_t p^t \cdot x_i^T \tag{10}$$

satisfied for $1 \leq t \leq T-1$.

□

Proposition A. 3. *Let (p^t, x_i^t) with $t = 1, 2, \ldots, T$ be a set of pairwise distinct price-consumption data (i.e., $(p^t, x_i^t) \neq (p^{t'}, x_i^{t'})$ for $t \neq t'$) rationalized by a utility function $u_i \in \mathcal{U}$ and such that the inequalities*

$$u_i(x_i^1) \leq u_i(x_i^2) \leq \cdots \leq u_i(x_i^T)$$

are satisfied. Then these data are well-ranked for the ordered index set $(\mathbb{T}, <) = \{1 < 2 < \cdots < T\}$.

Proof. Let t be arbitrary between 1 and T. For any $t' > t$, we have $u_i(x_i^t) \leq u_i(x_i^{t'})$. This implies the inequality $p^t \cdot x_i^t \leq p^t \cdot x_i^{t'}$. This inequality is strict for $t \neq t'$ because the utility function u_i is strictly quasi-concave and $x_i^t \neq x_i^{t'}$. □

A.2. Strongly ranked individual data for a given ordering

The utility function whose existence is established in Proposition A.2 does not guarantee us that the utility levels $u_i(x_i^t)$ are strictly increasing

$$u_i(x_i^1) \leq u_i(x_i^2) \leq \cdots \leq u_i(x_i^T) \quad (11)$$

when the individual price-consumption data are well-ranked for the index set $(\mathbb{T}, <)$.

This leads us to strengthen the concept of well-ranked data set with respect to an ordered index set as follows:

Definition A. 4. *The individual price-consumption data (p^t, x_i^t) indexed by the ordered set $(\mathbb{T}, <)$ are* strongly ranked *if, in addition to being well-ranked, they satisfy the inequality*

$$p^{t+1} \cdot x_i^t \leq p^{t+1} \cdot x_i^{t+1}$$

for every $t \neq T$.

A set of indexed data can be well-ranked without being strongly ranked. The following proposition reflects the additional information associated with strongly ranked data.

Proposition A. 5. *Let (p^t, x_i^t) be strongly ranked data for the ordered index set $(\mathbb{T}, <) = \{1 < 2 < \cdots < T\}$. Then, the strict inequalities*

$$u_i(x_i^1) < u_i(x_i^2) < \cdots < u_i(x_i^T) \quad (12)$$

are satisfied for any utility function $u_i \in \mathcal{U}$ that rationalizes these data.

Proof. It follows from the definition of strongly ranked data sets that the inequality

$$p^t \cdot x_i^{t-1} \leq p^t \cdot x_i^t$$

is satisfied for $t \neq 1$. It then follows from $x_i^{t-1} \neq x_i^t$ that we have

$$u_i(x_i^{t-1}) < u_i(x_i^t)$$

for $t \neq 1$. □

By Proposition A.2, any well-ranked data set is rationalizable by some utility function $u_i \in \mathcal{U}$.

It follows from Proposition 5 that the utility rankings of the commodity bundles x_i^t are independent of the utility function $u_i \in \mathcal{U}$ that rationalizes the strongly ranked price-consumption data (p^t, x_i^t).

In the next proposition, we see that, if well-ranked data are not always strongly ranked, it is nevertheless possible to embed these well-ranked data into a larger set of strongly ranked data.

Proposition A. 6. *Any set of well-ranked data for the ordered index set $(\mathbb{T}, <) = \{1, 2, \ldots, T\}$ can be embedded into some larger set of strongly ranked data for some ordered index set $(J, \prec)$ (with $\mathbb{T} \subset J$), with* 1 *being the smallest element of $(J, \prec)$.*

Proof. The proof works by induction on T, the number of well-ranked data. For $T = 1$, there is nothing to prove because any data set of one element is well-ranked and strongly ranked.

Induction argument for T arbitrary. The induction hypothesis can be stated as follows: any set of T well-ranked data (p^t, x_i^t) for the ordered index set $(\mathbb{T}, <) = \{1 < 2 < \cdots < T\}$ can be embedded into a larger set of strongly ranked data for the ordered index set $(J, \prec)$ whose smallest element is the element 1, the smallest element of the set I. This property is assumed to be satisfied for $T - 1$, and we establish that it is then true for T.

Let therefore T well-ranked data (p^t, x_i^t) for the ordered set $(\mathbb{T}, <) = \{1 < 2 < \cdots < T\}$. The $T - 1$ data (p^t, x_i^t) indexed by the ordered subset $(I', <) = \{2 < 3 < \cdots < T\}$ are also well-ranked. Therefore, by the induction assumption, we can embed these $T - 1$ data into a larger set of strongly ordered data with an ordered index set $(J', \prec)$ whose smallest element is the element $2 \in (I', <)$.

One checks readily that these strongly ranked data and the pair (p^1, x_i^1) of the original data set define a set of data that are well-ranked for the ordered set $(J'', \prec) = \{1\} \cup (J', \prec)$ with $1 \prec t$ for $t \in J'$.

Let us show that these data are either strongly ranked for the ordered index set $(J'', \prec)$ or that we can find an additional pair $(p^{A_{12}}, x_i^{A_{12}})$ such that the data set consisting of (p^1, x_i^1), $(p^{A_{12}}, x_i^{A_{12}})$, and the subsequent (p^t, x_i^t) for $t \in J'$ are strongly ranked for the ordered index set $(J, \prec) = \{1 \prec A_{12} \prec 2\} \cup (J', \prec)$.

If the inequality $p^2 \cdot x_i^1 \leq p^2 \cdot x_i^2$ is satisfied, then the full data set is clearly strongly ranked for the ordered index set $(J'', \prec)$ and there is nothing more to prove. (It suffices to take $(J, \prec) = (J'', \prec)$, and to observe that 1 is the smallest element of both index set $(\mathbb{T}, <)$ and $(J, \prec)$.)

Assume now that the inequality $p^2 \cdot x_i^1 > p^2 \cdot x_i^2$ is satisfied. Define $\epsilon = \inf_{2 \leq t \leq T} p^1 \cdot (x_i^t - x_i^1)$. Let us show that there exists some $x_i^{A_{12}} \in \mathbb{R}^\ell_{++}$ that satisfies the following equalities:

$$p^1 \cdot x_i^{A_{12}} = p^1 \cdot x_i^1 + \epsilon/2 \quad , \quad p^2 \cdot x_i^{A_{12}} \leq p^2 \cdot x_i^2 \tag{13}$$

The set

$$\{x_i \in X \mid p^2 \cdot x_i \leq p^2 \cdot x_i^2\}$$

is convex and bounded from below by 0, contains elements arbitrarily close to 0, and also contains the point x_i^2. The image of this set by the linear map $x_i \to p^1 \cdot x_i$ is therefore an interval of the form $(0, A]$. Now $p^1 \cdot x_i^2$ belongs to this interval as the image of the point x_i^2. This implies that $p^1 \cdot x_i^1 + \epsilon/2$ also belongs to this interval by the definition of ϵ, which proves the existence of some point $x_i^{A_{12}}$ satisfying the above equalities and inequalities.

Set $p^{A_{12}} = p^1$. Let us now show that the set consisting of (p^1, x_i^1), $(p^{A_{12}}, x_i^{A_{12}})$, and of the data (p^t, x_i^t) for $t \in J'$ is strongly ranked with the respect to the ordered index set $(J, \prec) = \{1 < A_{12} < 2\} \cup (J', \prec)$. By construction, we have $p^1 \cdot x_i^1 \leq p^1 \cdot x_i^{A_{12}} = p^1 \cdot x_i^1 + \epsilon/2$ and $p^{A_{12}} \cdot x_i^{A_{12}} = p^1 \cdot x_i^{A_{12}} = p_1 \cdot x_i^1 + \epsilon/2 < p^{A_{12}} \cdot x_i^t$ for $t \geq 2$, which proves that these data are indeed well-ranked. The inequality

$$p^{A_{12}} \cdot x_i^1 = p^1 \cdot x_i^1 \leq p^1 \cdot x_i^1 + \epsilon/2 = p^1 \cdot x_i^{A_{12}} = p^{A_{12}} \cdot x_i^{A_{12}}$$

and the inequality

$$p^2 \cdot x_i^{A_{12}} \leq p^2 \cdot x_i^2$$

are satisfied by construction. These inequalities combined with the strong ranking of the data for $t \geq 2$ imply that these data are strongly ranked for the ordered index set $(J, \prec) = \{1 < A_{12} < 2 < \cdots < T\}$. Note that the element 1 is the smallest element of the ordered set $(J, \prec)$. □

Proposition A. 7. *Any set of well-ranked data (p^t, x_i^t) for the ordered index set $(\mathbb{T}, <)$ is rationalizable by a utility function $u_i \in \mathfrak{U}$ for which the strict inequalities*

$$u_i(x_i^1) < u_i(x_i^2) < \cdots < u_i(x_i^T)$$

are satisfied.

Proof. These data being well-ranked for the ordered index set $(\mathbb{T}, <) = \{1 < 2 < \cdots < T\}$, it follows from Proposition A.6 that they can be embedded into a larger set of strongly ranked data indexed by some ordered indexed set $(J, <)$, with $(\mathbb{T}, <) \subset (J, \prec)$. The strict inequalities of the Proposition then follow from the fact that the order of $\mathbb{T}$ is the restriction of the order of J. □

Proposition A. 8. *Let (p^t, x_i^t) with $t = 1, 2, \ldots, T$ be a set of pairwise distinct data (i.e., $(p^t, x_i^t) \neq (p^{t'}, x_i^{t'})$ for $t \neq t'$) rationalized by some utility function $u_i \in \mathfrak{U}$ and such that the weak inequalities*

$$u_i(x_i^1) \leq u_i(x_i^2) \leq \cdots \leq u_i(x_i^T)$$

are satisfied. Then these data can be rationalized by a utility function $\tilde{u}_i \in \mathfrak{U}$ such that the strict inequalities

$$\tilde{u}_i(x_i^1) < \tilde{u}_i(x_i^2) < \cdots < \tilde{u}_i(x_i^T)$$

are satisfied.

Proof. By Proposition A.3, the data are well-ranked for the ordered index set $(\mathbb{T}, <) = \{1 < 2 < \cdots < T\}$. It then follows from Proposition A.7 that there exists a utility function $\tilde{u}_i \in \mathfrak{U}$ that rationalizes the T pairs (p^t, x_i^t) (for $t = 1, \ldots, T$) and such that the strict inequalities

$$\tilde{u}_i(x_i^1) < \tilde{u}_i(x_i^2) < \cdots < \tilde{u}_i(x_i^T)$$

are satisfied. □