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The geometry of finite equilibrium datasets*

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Abstract

We investigate the geometry of finite datasets defined by equilibrium prices, income distributions, and total resources. We show that the equilibrium condition imposes no restrictions if total resources are collinear, a property that is robust to small perturbations. We also show that the set of equilibrium datasets is pathconnected when the equilibrium condition does impose restrictions on datasets, as for example when total resources are widely non collinear.

JEL Classification numbers: D31, D51.
Keywords: equilibrium manifold, rationalizability, pathconnectedness.

1. Introduction

The goal of this paper is to improve our understanding of the properties of equilibrium datasets and the geometry of the set they generate through a geometric approach that takes its insight from the global properties of the equilibrium manifold. This approach is totally different from the one in our other paper [2] where we use a linear programming characterization of utilities rationalizing a given dataset.

Our first result is that the equilibrium condition imposes no restrictions on datasets $D = (p^t, (w_i^t), r^t)$ consisting of price, income distribution and total resource vectors if the total resource vectors r^t are collinear. In addition, this property is robust to small perturbations of the total resources. Our second result is that

*This paper is a thoroughly revised and shortened version of a paper circulated in 2002 [1].

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the set of equilibrium datasets is pathconnected when the equilibrium condition does impose restrictions on the datasets.

The results in this paper extend and put in a more systematic setup those of Brown, Matzkin and Snyder about the nature of the restrictions on datasets resulting from the equilibrium condition [3, 5].

The paper is organized as follows. In Section 2, we recall the main assumptions and definitions, and set the notation. Section 3 is devoted to the proof that there are no restrictions imposed by the equilibrium condition on finite datasets when total resources are collinear. Section 4 deals with the pathconnectedness of the set of finite equilibrium datasets. Only the most standard concepts of point-set topology are required for reading this paper. They can be found in, for example, [4].

2. Definitions, assumptions, and notation

Goods and prices

There is a finite number ℓ of goods. Let $p = (p_1, p_2, \dots, p_{\ell-1}, p_\ell) \in \mathbb{R}^\ell_{++}$ be the price vector. We normalize the price vector p by picking the ℓ-th commodity as the numeraire, which is equivalent to setting $p_\ell = 1$. Let S denote the set of strictly positive normalized price vectors.

Individual preferences and demands

A consumer is characterized by a preference preordering represented by a utility function u_i defined on the strictly positive orthant $X = \mathbb{R}^\ell_{++}$ and an endowment vector $\omega_i \in X$. The utility function $u_i : X \to \mathbb{R}$ satisfies the standard assumptions of smooth equilibrium analysis, i.e., is smooth, monotone ($Du_i(x_i) \in X$ for any $x_i \in X$), smoothly strictly quasi-concave (the inequality $y^t D^2 u_i(x_i) y \geq 0$ and equality $y^t Du_i(x_i) = 0$ have a unique solution $y = 0$), and every indifference set $\{x_i \in X \mid u_i(x_i) = u_i\}$ is closed in $\mathbb{R}^\ell$ for any $u_i \in \mathbb{R}$.

Given any price vector $p \in S$ and endowment $\omega_i \in X$, consumer i's demand $f_i(p, p \cdot \omega_i)$ maximizes the utility $u_i(x_i)$ subject to the budget constraint $p \cdot x_i = p \cdot \omega_i$.

Price–income distribution–total resource equilibria

We say that the d-triple $\big(p, (w_i), r\big) \in S \times \mathbb{R}^m_{++} \times X$ is *feasible* if equality

$$\sum_i w_i = p \cdot r \tag{1}$$

is satisfied.

An *equilibrium* d-triple $b = \big(p, (w_i), r\big)$ satisfies equation

$$\sum_i f_i(p, w_i) = r. \tag{2}$$

An equilibrium d-triple is obviously feasible.

We denote by $\mathcal{B}$ the set of feasible d-triples and by $\mathcal{E}$ the subset of $\mathcal{B}$ consisting of equilibrium d-triples.

Let T be a finite number. We define the sets $\mathcal{B}^{(T)}$ and $\mathcal{E}^{(T)}$ as consisting of T pairwise distinct feasible and equilibrium d-triples $D = \big(p^t, (w_i^t), r^t\big)$ respectively.

3. Equilibrium datasets with collinear total resources

Let T be a finite number. We define the sets $\mathcal{B}^{(T)}(r)$ and $\mathcal{E}^{(T)}(r)$ as consisting of T pairwise distinct feasible and equilibrium d-triples $D = \big(p^t, (w_i^t), r^t\big)$ for the *constant* total resources $r \in X$. respectively. Similarly, we denote by $\mathcal{E}^{(T)}[r]$ and $\mathcal{B}^{(T)}[r]$ the sets of equilibrium and feasible datasets with total resources *collinear* with $r \in X$.

3.1. The case of fixed total resources

We first consider the case where total resources are constant and equal to $r \in X$.

Theorem 1. *We have*

$$\mathcal{E}^{(T)}(r) = \mathcal{B}^{(T)}(r)$$

Proof. Let $D = \big(p^t, (w_i^t), r^t\big)$ be a *feasible* dataset. The idea of the proof is show that there exist commodity bundles x_i^t and utility functions u_i for i varying from 1 to m and t from 1 to T such that $x_i^t = f_i(p^t, w_i^t)$ where f_i is the demand function associated with the utility function u_i and $\sum_i x_i^t = r^t$.

Step 1.
We first assume that the following additional property is satisfied for all t, t', and i:

$$\frac{w_i^t}{p^t \cdot r} \neq \frac{w_i^{t'}}{p^{t'} \cdot r} \ . \tag{3}$$

Let

$$x_i^t = \frac{w_i^t}{p^t \cdot r} r.$$

By construction, all the vectors x_i^t for a given i are collinear with the total resource vector $r \in X$. It follows from (3) that all x_i^t for any given i are pairwise distinct.

Step 2.

By construction, the set consisting of the T commodity bundles $\{x_i^1, x_i^2, \dots, x_i^T\}$ is ordered by the partial ordering relation $<$ on X. There is no loss in generality in assuming that the sequence is strictly increasing, i.e., $x_i^1 < x_i^2 < \cdots < x_i^T$.

It follows from this strict ordering relationship that, for $p^t \in S$, we have $p^t \cdot x_i^t < p^t \cdot x_i^{t'}$ whenever $t < t'$. The T price-commodity bundles (p^t, x_i^t) are then well-ranked in the sense of [2], which implies that the T price-commodity bundles (p^t, x_i^t) can be rationalized by some utility function $u_i \in \mathcal{U}$ (Proposition A.2 in [2]).

Step 3.

We can now write $x_i^t = f_i(p^t, w_i^t)$ where f_i is the demand function associated with the utility function u_i. In addition, it follows from the formula defining x_i^t that we have

$$\sum_i x_i^t = \frac{\sum_i w_i^t}{p^t \cdot r} r = r,$$

which implies that the equality

$$\sum_i f_i(p^t, w_i^t) = r = r^t$$

is satisfied for t varying from 1 to T. This proves that the *feasible* dataset $D = \big(p^t, (w_i^t), r^t\big)$ is indeed an *equilibrium* dataset.

Step 4.

The next step is to deal with situations where, for some consumer i at least, there exist t and t' such that the equality

$$\frac{w_i^t}{p^t \cdot r} = \frac{w_i^{t'}}{p^{t'} \cdot r}$$

is satisfied for some $p^t \neq p^{t'}$.

Because of this equality, the commodity bundles $x_i^t = \dfrac{w_i^t}{p^t \cdot r}$ and $x_i^{t'} = \dfrac{w_i^{t'}}{p^{t'} \cdot r}$ are not good candidates for the construction of Step 1 because x_i^t and $x_i^{'t}$ are equal while there are two different candidate "supporting" price vectors p^t and $p^{t'}$. The idea is therefore to perturb x_i^t and $x_i^{t'}$ in such a way that the perturbed sequence $x_i^{'1}, x_i^{'2}, \dots, x_i^{'T}$ remains ordered, and the equality $\sum_i x_i^{'t} = r^t$ satisfied for every t.

In order to do that, consider the line Δ_0 that passes through the origin and that is collinear with the vector $r \in X$. Let Δ be a line parallel to the line Δ_0 and sufficiently close to Δ_0 for the following property to be satisfied. The intersection

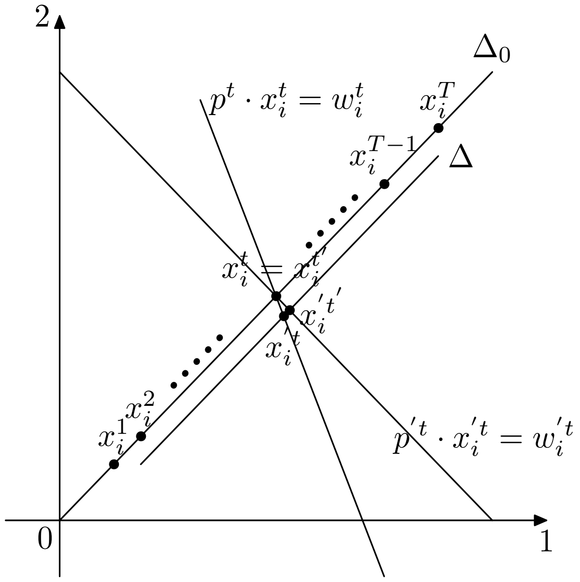

Figure 1: Perturbation of the collection $\{x_i^t\}$ for $1 \leq t \leq T$

points $x_i'^t$ and $x_i'^{t'}$ of Δ with the budget hyperplanes $p^t \cdot x_i^t = w_i^t$ and $p^{t'} \cdot x_i^{t'} = w_i^{t'}$ are distinct and, if we define $x_i'^{t''} = x_i^{t''}$ for $t'' \neq t, t'$, the sequence $x_i'^1, \ldots, x_i'^T$ is still ordered by $<$. This follows from the fact that the sequence x_i^1, x_i^2, $\ldots$, x_i^T is already ordered even if the two elements x_i^t and $x_i^{t'}$ are identical. The new sequence is obtained by just perturbing those two identical elements: the elements $x_i'^t$ and $x_i't'$ can therefore be ordered with all the other elements of the sequence by the relation $<$ provided the perturbation is small enough. In addition, thanks to the choice of the direction of the line Δ, these two elements are themselves ordered.

Once we have perturbed consumer i, it is necessary to perturb accordingly another consumer j so that the total resources remain constant. Therefore, let us pick some arbitrary consumer j and define the new sequence $x_j'^1, x_j'^2, \ldots, x_j'^T$ by

$$x_j'^t = x_j^t + (x_i^t - x_i'^t), x_j'^{t'} = x_j^{t'} + (x_i^{t'} - x_i'^{t'}), x_j'^{t''} = x_j^{t''}$$

with $t'' \neq t, t'$.

With the same line of reasoning as above, we observe that the perturbation that defines the consumption bundles of consumer i can be made small enough for the new sequence to be ordered by relation $<$ and the already distinct elements to remain distinct through the perturbation. In addition, the total resources are now equal, by construction, to the vector r .

Overall, this construction reduces by at least one unit the number of non distinct commodity bundles. It then suffices to iterate this construction for every consumer i and pairs (t, t') such that $x_i^t = x_i^{t'}$. Eventually, one gets for each consumer a collection of ordered sequences of T elements that sum up to the vector of total resources $r \in X$. We can now apply the result established in Step 3.

□

The case of collinear total resources

Theorem 2. *We have*

$$\mathcal{E}^{(T)}[r] = \mathcal{B}^{(T)}[r]$$

Proof. It suffices to observe that the line of proof of Theorem 1 also works for collinear total resources.

□

Remark 1. One checks readily that the proofs of Theorems 1 and 2 extend to the case where the total resources are not collinear, but only sufficiently close to being collinear for the sequence (x_i^t) to remain ordered by the relation $<$ for every i.

Remark 2. The rationalizability of the set of T price-individual consumption vectors (p^t, x_i^t) when the sequence x_i^t consists of pairwise different elements ordered by the relation $<$ used in Step 2 can also be proved directly. It suffices to construct explicitly smooth indifference surfaces through each x_i^t with normal vector at x_i^t the price vector p^t. We leave the details of such construction to the reader.

4. Pathconnectedness of the set $\mathcal{E}^{(T)}$

In general, the set $\mathcal{E}^{(T)}$ of T-tuples of equilibrium triples is a strict subset of the set $\mathcal{B}^{(T)}$. The following theorem states a remarkable global topological property.

Theorem 3. *The set $\mathcal{E}^{(T)}$ is pathconnected.*

Recall that a subset C of a topological space is pathcconnected if any two elements x and y of C can be linked by some continuous path belonging to the set C. This is the same thing as saying that there exists a continuous map $h : [0,1] \to C$ such that $h(0) = x$ and $h(1) = y$.

Proof. The idea of the proof consists in the construction of a continuous path linking two arbitrarily given equilibrium datasets $b = \big(p^t, (w_i^t), r^t\big)$ and $b' = \big(p'^t, (w_i'^t), r'^t\big)$.

We first define the dataset

$$b'' = \big(p'^t, (w_i'^t), \sum_i f_i(p'^t, w_i'^t)\big).$$

These are equilibrium datasets associated with the preferences (or utility functions) corresponding to the demand functions f_i, with i varying from 1 to m.

Note that the components of the dataset b'' are pairwise distinct, so that b'' belongs to the set $\mathcal{E}^{(T)}$. In order to connect b and b', it therefore suffices to connect b to b'' and b'' to b'.

Path from b to b''

Here, the preferences (or utility functions) do not vary. One starts by constructing a continuous path linking $\big(p^t, (w_i^t)\big)_{1\le t\le T}$ to $\big(p'^t, (w_i'^t)\big)_{1\le t\le T}$ in the set $(S \times \mathbb{R}^m_{++})^T$, a path such that all coordinates remain different for the points in the path. (That such a construction is always possible is straightforward. For example, one can start with the line segment linking the two points. If, at some point, some coordinates become equal, it is easy to see that a small perturbation of the path will restore inequality of the coordinates.) Let $\big(p^t(\theta), (w_i^t(\theta))\big)_{1\le t\le T}$, with $\theta \in [0, 1]$, denote the generic point of this path. One then defines

$$r^t(\theta) = \sum_i f_i\big(p^t(\theta), w_i^t(\theta)\big).$$

By definition, the dataset

$$b(\theta) = \big(p^t(\theta), (w_i^t(\theta)), r^t(\theta)\big)$$

is an equilibrium dataset. It follows from the continuity of the individual demand functions that this construction defines a continuous path (for θ varying from 0 to 1) linking the two equilibrium datasets b and b''.

Path from b'' to b'

Here, the T-tuple $\big(p'^t, (w_i'^t)\big)_{1\le t\le T}$ is kept fixed. Consider the points

$$x_i^t = f_i(p'^t, w_i'^t) \quad \text{and} \quad x_i'^t = f_i'(p'^t, w_i'^t)$$

in X. Let K be some convex compact subset of X that contains these $2T$ points in its interior. It follows from the assumptions regarding the preferences associated with the demand functions f_i and f_i' that these preferences can be represented by utility functions u_i and u_i' whose restrictions to the interior of the compact set K are strictly concave. Then, consider for $\theta \in [0, 1]$ the function

$$u_i(\theta) = (1-\theta)u_i + \theta u_i'.$$

The restriction of the function $u_i(\theta)$ to the interior of the compact set K is strictly concave in addition to satisfying the assumptions that we impose on utility functions. The preference preordering defined on the compact K by the utility function $u_i(\theta) \mid K$ can be extended into a preference preordering defined on the strictly positive orthant X. Let $f_i(\theta)$ be the corresponding demand function. One checks readily that the restriction of the demand function $f_i(\theta)$ to the points $(p'^t, w_i'^t)$ varies continuously with $\theta \in [0, 1]$. Define

$$r^t(\theta) = \sum_i f_i\big(\theta)(p'^t, w_i'^t)\big).$$

Then, the equilibrium dataset

$$b(\theta) = \big(p'^t, (w_i'^t), r^t(\theta)\big)_{1 \leq t \leq T}$$

is made of distinct equilibrium d-triples, and varies continuously from b'' to b when θ varies from 0 to 1.

Piecing together the two continuous paths just constructed defines a continuous path linking the two equilibrium datasets b to b'.

□

5. Concluding comments

A better understanding of the global properties of the equilibrium dataset $\mathcal{E}^{(T)}$ would certainly improve the information that can be extracted for real datasets. Pathconnectedness is usually the first global property to be investigated. A positive result regarding pathconnectedness can then be followed by investigations regarding simple connectedness, contractibility, up to the homeomorphism or diffeomorphism type. The approach followed in the current paper to these questions is purely topological. In our other paper [2], we describe an alternative approach that is based on our characterization of utility rankings of equilibrium datasets by the existence of solutions to a linear problem. The two approaches are complementary.

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