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Interaction Sheaves on Continuous Domains

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Abstract

We introduce a description of the power structure which is inherent in a strategic game form using the concept of an interaction sheaf. The latter assigns to each open set of outcomes a set of interaction arrays, specifying the changes that coalitions can make if outcome belongs to this open set. The interaction sheaf generalizes the notion of effectivity functions which has been widely used in implementation theory, taking into consideration that changes in outcome may be sustained not only by single coalitions but possibly by several coalitions, depending on the underlying strategy choices. Also, it allows us to consider game forms with not necessarily finite sets of outcomes, generalizing the results on solvability of game forms obtained in the finite case in Abdou and Keiding (2003).

Keywords: Nash equilibrium, strong equilibrium, solvability, effectivity, acyclicity.

JEL Classification: C70, D71
AMS Classification: 91A44

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1 Introduction

A game form is strongly solvable if for each assignment of individual preferences over outcomes, the resulting game possesses a strong Nash equilibrium. Several necessary conditions for strong solvability can be found in the literature; Abdou and Keiding (2003) provide conditions which are both necesssary and sufficient, at least for the case where the strategy sets of the game form are all finite. In this paper we present a model of interaction based on power distribution among agents, a model general enough to allow for a representation of classical coalitional models (e.g. effectivity functions) as well as to capture the essential features of strategic ones (e.g. game forms). For this, use an extension of the well-known concept of an effectivity function associated with a game form introduced by Moulin and Peleg (1982) and the property of acyclicity of this extended effectivity function, also known from the implementation literature, cf. e.g. Abdou and Keiding (1991). This extension is done in such a way that the essential interaction inherent in a strategic game form can be represented in the new object. An interaction sheaf is to the notion of equilibrium (e.g. Nash or strong Nash) precisely what the effectivity function is to the core.

In this paper, a game form is said to be $\mathcal{M}$-solvable (where $\mathcal{M}$ is any collection of coalitions) if it has $\mathcal{M}$-equilibria for any assignment of preferences. Nash solvability and strong Nash solvability are special cases of $\mathcal{M}$-solvability. We extend the characterization of $\mathcal{M}$-solvable game forms to the case where the game form may have infinitely many strategies and alternatives, so that strategy sets and outcome space are topological spaces, assumed in general to be compact Hausdorff spaces. In this setup, the notion of an outcome-dependent effectivity function ,which is at the basis of the characterization of solvability, is naturally formalized using the concept of a *sheaf,* which captures the idea of local (outcome-dependent) power of coalitions by specifying the power structure valid at each open set of outcomes. It turns out that the right concept in this setting is the interaction sheaf associated with a game form. Furthermore the notion of an abstract interaction sheaf is introduced and the corresponding notion of settlement set and stability.

An interaction sheaf is an object which is similar to that of an effectivity structure as introduced by Abdou and Keiding (2003), with the difference that it can be used in the context of outcome spaces that are not necessarily finite. The advantages of the present way of formalizing power structures are that (1) it allows for the representation of various equilibrium concepts within the *same* interaction form, whereas the other is specific to one equilibrium concept, and (2) in the current model, it allows for operations like projections that *faithfully* reflect the change in the underlying confederation, and most importantly, (3) since only the interactive form associated to some game form and some equilibrium concept is relevant for stability, it allows for a simple *comparison* between different procedures or mechanisms with respect to stability. An interaction form can thus be viewed as an intrinsic representation of power without a direct reference to strategies or to some equilibrium concept.

The paper is structured as follows: In Section 2, we give the definitions of the basic concepts such as game forms, preferences, equilibria, and in Section 3, we proceed to the concepts which are central for the following, namely interaction sheaves. In this section, we also investigate some general properties of interaction sheaves, and we introduce the notion of a settlement as well as stability of interaction sheaves. The next section is concerned with the characterization of stable interaction sheaves by the property of acyclicity, thus extending a result known from simple games (Nakamura (1979)) and effectivity functions (Keiding (1985) to the present general context of interaction sheaves. Section 5 contains the main result of the paper, showing that solvability of game forms may be characterized in terms of stability or acyclicity of the associated interaction sheaf. The concluding section contains some final comments.

2 Basic definitions and notations

In the present section, we introduce the concepts and the notation which will be needed as we proceed. We use of the following notational conventions: For any set D, we denote by $\mathcal{P}(D)$ the set of all subsets of D and by $\mathcal{P}_0(D) = \mathcal{P}(D)\backslash\{\emptyset\}$ the set of all non-empty subsets of D. Elements of $\mathcal{P}(N)$ are called ***coalitions,*** and a ***confederation*** is a subset $\mathcal{M}$ of $\mathcal{P}_0(N)$.

A ***game form*** is an array $G = (X_1, \cdots, X_n, A, g)$. Here the set $N = \{1, \cdots, n\}$, where $n \geq 2$, is interpreted as the set of ***players***, X_i is the ***strategy set*** of player i, for $i \in N$, A is the set of ***alternatives,*** and $g : \prod_{i \in N} X_i \longrightarrow A$ is the ***outcome function***. For every coalition $S \in \mathcal{P}_0(N)$, the product $\prod_{i \in S} X_i$ is denoted X_S (by convention $X_\emptyset$ is the singleton $\{\emptyset\}$) and $N\backslash S$ is denoted S^c. Similarly if $B \in \mathcal{P}(A)$, $A\backslash B$ is denoted B^c. If $x_N \in X_N$, the notation $g(x_S, X_{S^c})$ stands for $\{g(x_S, y_{S^c}) \mid y_{S^c} \in X_{S^c}\}$ if $S \neq \emptyset$ and for $g(X_N)$ if $S = \emptyset$.

For any set D, $Q(D)$ denotes the set of all quasi-orders on D (that is all binary relations on D which are complete and transitive). We let $\overset{\circ}{R}$ denote the strict preference relation associated with R, that is $a \overset{\circ}{R} b$ if and only if $a\ R\ b$ and not $b\ R\ a$.

We assume that $X_1, \cdots, X_n$ and A are Hausdorff compact topological spaces and that g is continous and onto. We denote by $\mathcal{G}$ the set of open sets of A, by $\mathcal{F}$ that of closed sets. Moreover $\mathcal{G}_0 \equiv \mathcal{G} \setminus \{\emptyset\}$ and $\mathcal{F}_0 \equiv \mathcal{F} \setminus \{\emptyset\}$. A quasi-order R on A is ***continuous*** if for any $a \in A$ the sets $\{b \in A \mid b\ R\ a\}$ and $\{b \in A \mid a\ R\ b\}$ are closed. A continuous real function u induces a continuous quasi-order R by setting $a\ R\ b$ if and only if $u(a) \geq u(b)$.

A ***game in strategic form*** is an array $(X_1, \ldots, X_n; Q_1, \ldots, Q_n)$, where for each

$i \in N =$, X_i is the set of strategies of player i, and Q_i is a quasi-order on $X_N = \prod_{i \in N} X_i$. For $\mathcal{M}$ a confederation , a strategy array $x_N \in X_N$ is an ***$\mathcal{M}$-equilibrium*** of the game $(X_1, \ldots, X_n; Q_1, \ldots, Q_n)$ if there is no coalition $S \in \mathcal{M}$ and $y_S \in X_S$ such that for all $i \in S$:

$$(y_S, x_{S^c}) \overset{\circ}{Q}_i x_N.$$

For $R \in Q(A)$ we put $P(a, R) = \{b \in A \mid b \overset{\circ}{R} a \}$. A preference profile (over A) is a map R_N from N to $Q(A)$, also written as $R_N = (R_1, \ldots, R_N)$, so that a preference profile is an element of $Q(A)^N$. For notational convention, we write $P(a, i, R_N)$ for $P(a, R_i)$, and we use the notations $P(a, S, R_N) = \cap_{i \in S} P(a, i, R_N)$ for $S \in \mathcal{P}_0(N)$. For each preference profile $R_N \in Q(A)$, the game form G induces a game $(X_1, \ldots, X_n; Q_1, \ldots, Q_n)$ with the same strategy spaces as in G and with the Q_i defined by

$$x_N \, Q_i \, y_N \Leftrightarrow g(x_N) \, R_i \, g(y_N)$$

for $x_N, y_N \in X_N$. We denote this game by (G, R_N)

We say that $a \in A$ is an ***$\mathcal{M}$-equilibrium outcome*** of (G, R_N) if there exists an $\mathcal{M}$- equilibrium x_N in (G, R_N) such that $g(x_N) = a$. Let $L \subset Q(A)$ be a subset of preferences. The game form G is said to be ***solvable in $\mathcal{M}$-equilibrium*** or ***$\mathcal{M}$-solvable*** on L if for each preference profile $R_N \in L^N$, the game (G, R_N) has a strong equilibrium. In particular, when $\mathcal{M} = \mathcal{N} = \{\{1\}, \ldots, \{n\}\}$, the set of all singleton coalitions, then an $\mathcal{M}$-equilibrium is simply a Nash equilibrium. Similarly, when $\mathcal{M} = \mathcal{P}_0(N)$, the family of all coalitions, an $\mathcal{M}$-equilibrium is a strong Nash equilibrium.

3 Interaction sheaves

In this paper, we aim at a characterization of $\mathcal{M}$-solvable game forms using a suitable notion of power structure which is inherent in the game form. This approach was initiated by the seminal paper by Moulin and Peleg (1984), where they introduced the effectivity function associated with a game form. For the solvability of game forms, the effectivity function contains too little information, and refined notions of power structures were considered by Abdou and Keiding (2003) in the context of solvability of game forms with finite strategy spaces. Below we introduce a generalization of effectivity functions that will work in the context of solvability of game forms with an infinite number of strategies.

Definition 3.1 *(a) An interaction array on (N, A) is a map $\varphi : \mathcal{P}_0(N) \to \mathcal{P}(A)$ with $\varphi(S) \neq \emptyset$ for some $S \in \mathcal{P}_0(N)$. Let $\mathcal{P}_0(N, A)$ be the set of all interaction arrays. We introduce a partial order $\leq$ on $\mathcal{P}_0(N, A)$ by the formula $\varphi \leq \psi$ if and only if $\varphi(S) \subset \psi(S)$ for all $S \in \mathcal{P}_0(N)$. For $\mathcal{A} \subset \mathcal{P}(A)$. We denote by $\mathcal{A}_0(N, A)$ the set of all interaction arrays with values in $\mathcal{A}$.*

(b) An interaction form with values in $\mathcal{A}$ is a mapping $\mathcal{E} : \mathcal{G}_0 \to \mathcal{A}_0(N, A)$ such that for all $U \in \mathcal{G}_0$, $\varphi, \psi \in \mathcal{A}_0(N, A)$, if $\varphi \in \mathcal{E}[U]$ and $\varphi \leq \psi$ then $\psi \in \mathcal{E}[U]$

(d) The interaction form $\mathcal{E}$ is a an interaction presheaf if $U \subset V \Rightarrow \mathcal{E}[V] \subset \mathcal{E}[U]$ for all $U, V \in \mathcal{G}_0$, and an interaction sheaf if, in addition, for each $U \in \mathcal{G}_0$ and each open covering $(U_i)_{i\in I}$ of U one has $\mathcal{E}[U] = \cap_{i\in I}\mathcal{E}[U_i]$.

Interaction forms are collections of interaction arrays that satisfy natural conditions of non-trivialness and monotonicity, and they can be considered as a formalization of the power structure in society. Part (c) of the definition connects the power structure of the interaction form to the topology of the outcome space.

When $\mathcal{E}$ is a presheaf, we may think of an interaction array in $\mathcal{E}[U]$ as a description of an availlable move of the agents given any state in U. Let $\mathcal{M} \subset \mathcal{P}_0(N)$ be a confederation. In order that a scenario leading to some outcome be viable, it needs to be appoved by all coalitions of $\mathcal{M}$. To interpret the statement $\varphi \in \mathcal{E}[U]$, one may imagine that any outcome in U can occur in different scenarios that are not directly explicited in the model; any scenario leading to some state in U may arouse some coalition $S \in \mathcal{M}$ that objects by threatening to drive the outcome into $\varphi(S)$, in this case a is rejected. The interaction array is the result of a *disjunctive* move of the coalitions, so that the surge of some objecting coalition S is not concomitant to that of another coalition. Within a coalition, action is coordinated, not within a confederation. When a confederation becomes active at a, this activation must be understood as a *collusion* of interests between its components. Indeed the rejection of a is equivalent to the rejection of each scenario leading to a, and each scenario may be opposed by some coalition in $\mathcal{M}$. Our model is universal in the sense that we allow *a priori* all coalitions to react to some state in U. Nevertheless, the fact that $\varphi(S) = \emptyset$ for some S means that coalition S is inhibited or desactivated and therefore that the power represented by φ holds without the participation of S. Therefore the support of φ (i.e. those coalitions S such that $\varphi(S)$ is nonempty) is in fact the *active* confederation behind φ.

Remark 3.2 The discussion of the present section has been confined to situations where the coalition structure is $\mathcal{P}_0(N)$, the set of all nonempty subsets of N. However, restricting to any $\mathcal{M} \subset \mathcal{P}_0(N)$ means simply that we consider only interaction arrays which are projections $\varphi_{|\mathcal{M}}$ to $\mathcal{M}$ of interaction arrays $\varphi \in \mathcal{P}_0(N, A)$, where

$$\varphi_{|\mathcal{M}}(S) = \begin{cases} \varphi(S) & \text{if } S \in \mathcal{M}, \\ \emptyset & \text{otherwise.} \end{cases}$$

$$\mathcal{E}_{|\mathcal{M}}[U] = \{\varphi \in \mathcal{A}_0(N, A) \mid \varphi_{|\mathcal{M}}(S) \in \mathcal{E}[U]\} \tag{1}$$

It turns out (See Proposition 5.2 and the related remark 5.1) that this restriction reflects faithfully what is meant when the coalitions that are allowed to act jointly are those members of the confederation $\mathcal{M}$.

As a first example of an interaction form, we consider the one induced by an effectivity function E, that is a map $E : \mathcal{P}_0(N) \to \mathcal{P}(\mathcal{P}_0(A))$. To define the interaction form $\mathcal{E}^E$ associated with E we let $\mathcal{E}^E[U]$, for $U \in \mathcal{G}_0$, contain all the interaction arrays φ such that $\delta_{(S,B)} \leq \varphi$ where

$$\delta_{(S,B)}(S') = \begin{cases} B & \text{if } S' = S, \\ \emptyset & \text{otherwise.} \end{cases}$$

This is an interaction sheaf, which is constant in the sense that $\mathcal{E}^E[U]$ does not depend on U. Similarly, given an interaction form $\mathcal{E}$ one can extract an effectivity functions $E^{\mathcal{E}}$ with

$$E^{\mathcal{E}}(S) = \{B \in P_0(A) \mid \delta(S, B) \in \mathcal{E}[A]\}.$$

For a more interesting interaction form, consider a game form $G = (X_1, \ldots, X_n, A, g)$ and $\mathcal{M} \subset \mathcal{P}_0(N)$. The β-interaction form associated with $(G, \mathcal{M})$ is defined as

$$\mathcal{E}_\beta^{G,\mathcal{M}}[U] = \{\varphi \in \mathcal{P}_0(N, A) \mid \forall x_N \in g^{-1}(U), \exists S \in \mathcal{M}, \exists y_S \in X_S : g(y_S, x_{S^c}) \in \varphi(S)\} \quad (2)$$

for all $U \in \mathcal{G}_0$. This is actually an interaction sheaf. As is usual, the β-construction shows what coalitions can do by adapting their coordinated strategy choices to the situation, the latter represented by a strategy array with outcome in U. There is a corresponding α-construction: Define the α-interaction form associated with G and $\mathcal{M}$ by:

$$\mathcal{E}_\alpha^{G,\mathcal{M}}[U] = \{\varphi \in \mathcal{P}_0(N, A) \mid \exists x_N \in g^{-1}(U), \forall S \in \mathcal{M}, \forall y_S \in X_S : g(y_S, x_{S^c}) \in \varphi(S)\} \quad (3)$$

The α-interaction form is in general not a sheaf nor even a presheaf. It assigns to any open set the interaction arrays which specify for each coalition S a set of outcomes, namely $\varphi(S)$ that S^c can force, given some fixed strategy array. In Section 5 we study interaction forms associated with continuous game forms. Whether coalitions in the confederation have a real interest to dismantle an outcome a, depends on the actual preferences. This is why we introduce the following:

Definition 3.3 *Let $\mathcal{E}$ be an interaction presheaf, and let $R_N \in Q(A)^N$ be a preference profile. The alternative $a \in A$ is dominated in $\mathcal{E}$ at R_N if there is an open neighbourhood U of a, an interaction array $\varphi \in \mathcal{E}[U]$ such that for all $S \in \mathcal{P}_0(N)$, $\varphi(S) \subset P(a, S, R^N)$.*

A settlement (for $\mathcal{E}$ at R_N) is an alternative which is not dominated at R_N; the set of all settlements is denoted $\mathrm{Stl}(\mathcal{E}, R_N)$. *For L, a set of preferences, the interaction presheaf $\mathcal{E}$ is* ***stable*** *on L if* $\mathrm{Stl}(\mathcal{E}, R_N)$ *is nonempty for all profiles $R_N \in L^N$.*

An outcome a is a settlement if there exists at least one scenario that forces a such that and no coalition has an incentive to disrupt it.

In the next section, we shall consider combinatorial conditions on the interaction presheaf $\mathcal{E}$ which implies that it is stable. In the remainder of this section, we shall have a closer look at the topological properties of $\mathcal{E}$.

First of all we notice that there is a quasi-order $\subset$ defined on interaction presheaves by

$$\mathcal{E} \subset \mathcal{E}' \text{ if and only if } \mathcal{E}[U] \subset \mathcal{E}'[U] \text{ for all } U \in \mathcal{G}_0.$$

Since the intersection of any family of interaction presheaves (sheaves) over A is an interaction presheaf (sheaf), and since the trivial sheaf defined by $\mathcal{E}'[U = \mathcal{A}_0(N, A)$ for all $U \in \mathcal{G}_0$ contains all presheaves, it follows that for any interaction presheaf $\mathcal{E}$, there is a minimal (for $\subset$) interaction sheaf (called the sheaf cover of $\mathcal{E}$ and denoted $\mathcal{E}^+$) containing $\mathcal{E}$, whereby for any $U \in \mathcal{G}$, $\mathcal{E}^+[U]$ is defined as the intersection of all $\mathcal{E}'[U]$ with $\mathcal{E}' \supset \mathcal{E}$.

Proposition 3.4 *Let $\mathcal{E}$ be an interaction presheaf with values in $\mathcal{A}$ and for each open set U, let $\mathcal{R}(U)$ be the set of all open coverings of U. Then*

$$\mathcal{E}^+[U] = \bigcup_{R \in \mathcal{R}(U)} \bigcap_{V \in R} \mathcal{E}[V] \tag{4}$$

and $\mathrm{Stl}(\mathcal{E}, R_N) = \mathrm{Stl}(\mathcal{E}^+, R^N)$ *for any profile R_N.*

PROOF: Let $\mathcal{E}'[U]$ denote the expression on the right hand of (4), then it is staightforward that $\mathcal{E}'$ is a sheaf. Moreover for U open, $\mathcal{E}[U] \subset \mathcal{E}'[U]$ trivially, since U is itself a covering of U, so $\mathcal{E}^+[U] \subset \mathcal{E}'[U]$. It follows that $\mathcal{E}^+ \subset \mathcal{E}'$. Conversely if $\mathcal{E}''$ is a sheaf such that $\mathcal{E}" \supset \mathcal{E}$, then it is easy to see that $\mathcal{E}'' \supset \mathcal{E}'$. It follows that $\mathcal{E}^+ \supset \mathcal{E}'$. The second statement follows from the definition of domination. □

Our definition of an interaction form, designed so as to capture the phenomenon of state-dependent power structure, has taken as primitive notion of local power the interaction arrays corresponding to the open neighbourhoods of the topology, corresponding to the classical notion of a presheaf (cf. eg. Bredon, 1990). Alternatively, we might have considered local power as defined for each point of A (corresponding to studying the sections of a presheaf). Below, we consider such an alternative approach.

Definition 3.5 *An interaction bundle with values in $\mathcal{A}$ is a map $\mathcal{I} : A \to \mathcal{A}_0(N, A)$. The alternative $a \in A$ is dominated in $\mathcal{I}$ at the profile $R_N \in Q(A)^N$ if there exists $\varphi \in \mathcal{I}[a]$ such that for all $S \in \mathcal{P}_0(N)$: $\varphi(S) \subset P(a, S, R_N)$. The set* $\mathrm{Stl}(\mathcal{I}, R_N)$ *of settlements for $\mathcal{I}$ at R_N consists of all the alternatives which are not dominated in $\mathcal{I}$ at R_N.*

It turns out that the β-interaction bundle associated with a game form is related in a straightforward manner to the question of existence of equilibria of that game form (Proposition 5.2). On the other hand the relation between acyclicity and stability is easier to express using interaction presheaves (Theorem 4.4) We now explore the interrelations between the two objects. To every interaction presheaf $\mathcal{E}$ we associate the interaction bundle $\mathcal{E}^\bullet$ defined by

$$\mathcal{E}^\bullet[a] = \bigcup_{U \in \mathcal{U}(a)} \mathcal{E}[U],$$

where $\mathcal{U}(a)$ is the set of all open neighbourhoods of a. The following proposition is obvious.

Proposition 3.6 *Let $\mathcal{E}$ be an interaction presheaf. Then for any preference profile $R_N \in Q(A)^N$, $\mathrm{Stl}(\mathcal{E}, R_N) = \mathrm{Stl}(\mathcal{E}^\bullet, R_N)$.*

Conversely, to any interaction bundle $\mathcal{I}$ we may associate an interaction sheaf $\mathcal{I}^\diamond$ defined by

$$\mathcal{I}^\diamond[U] = \bigcap_{a \in U} \mathcal{I}[a]$$

for any $U \in \mathcal{G}$. The question whether any sheaf can be obtained in this way is answered in the following:

Proposition 3.7 *For any interaction presheaf $\mathcal{E}$ one has $\mathcal{E} \subset \mathcal{E}^{\bullet\diamond}$ and $\mathcal{E}^+ = \mathcal{E}^{\bullet\diamond}$. Moreover $\mathcal{E}$ is a sheaf if and only if $\mathcal{E} = \mathcal{E}^{\bullet\diamond}$. It follows that any sheaf $\mathcal{E}$ can be obtained as $\mathcal{I}^\diamond$ where $\mathcal{I}$ is the interaction bundle $\mathcal{E}^\bullet$.*

PROOF: The interaction form $\mathcal{E}^{\bullet\diamond} = (\mathcal{E}^\bullet)^\diamond$ is a sheaf by its construction, and clearly $\mathcal{E} \subset \mathcal{E}^{\bullet\diamond}$, so that $\mathcal{E}^+ \subset \mathcal{E}^{\bullet\diamond}$. Let $U \in \mathcal{G}_0$, let $\mathcal{E}' \supset \mathcal{E}$ be a sheaf, and for any $a \in A$ let $\mathcal{U}(a)$ be the set of all open neighbourhoods of a. Writing out the definitions and using inclusion, we have that

$$\mathcal{E}^{\bullet\diamond}[U] = \bigcap_{a \in U} \bigcup_{V \in \mathcal{U}(a)} \mathcal{E}[V] \subset \bigcap_{a \in U} \bigcup_{V \in \mathcal{U}(a)} \mathcal{E}'[V].$$

Now we have (set theoretic equality):

$$\bigcap_{a \in U} \bigcup_{V \in \mathcal{U}(a)} \mathcal{E}'[V] = \bigcup_{(V_b) \in \Pi_{b \in U} \mathcal{U}(b)} \bigcap_{b \in U} \mathcal{E}'[V_b],$$

where the union is over all possible collections $V = (V_b)_{b \in A}$, with $V_b \in \mathcal{U}(b)$ for all $b \in A$. Since $\mathcal{E}'$ is a sheaf, we have that $\cap_{b \in U} \mathcal{E}'[V_b] = \mathcal{E}'[\cup_{b \in U} V_b] \subset \mathcal{E}'[U]$. It follows that $\mathcal{E}^{\bullet\diamond}[U] \subset \mathcal{E}'[U]$. Since $\mathcal{E}^{\bullet\diamond} \subset \mathcal{E}'$ for all $\mathcal{E}' \supset \mathcal{E}$, we have $\mathcal{E}^{\bullet\diamond} \subset \mathcal{E}^+$. □

Proposition 3.8 *(i) For any interaction presheaf $\mathcal{E}$: $\mathcal{E}^\bullet = (\mathcal{E}^+)^\bullet$. (ii) For presheaves $\mathcal{E}$ and $\mathcal{F}$, $\mathcal{E}^\bullet = \mathcal{F}^\bullet$ if and only if $\mathcal{E}^+ = \mathcal{F}^+$.*

Let $a \in A$. Since $\mathcal{E} \subset \mathcal{E}^+$ one has $\mathcal{E}^\bullet(a) \subset (\mathcal{E}^+)^\bullet(a)$. Let $\varphi \in (\mathcal{E}^+)^\bullet(a)$. then for some open neigborhood U of a, $\varphi \in \mathcal{E}^+(U)$ and by Proposition 3.8 , there exists some open cover $(U_i)_{i\in I}$ of U such that $\varphi \in \mathcal{E}(U_i)$ for all $i \in I$. Since there exists $i_0 \in I$ such that $a \in U_{i_0}$ we have that $\varphi \in \mathcal{E}^\bullet[a]$. It follows that the equality $\mathcal{E}^+ = \mathcal{F}^+$ implies $\mathcal{E}^\bullet = \mathcal{F}^\bullet$. Conversely if $\mathcal{E}^\bullet = \mathcal{F}^\bullet$, then by Proposition 3.7 $\mathcal{E}^+ = \mathcal{E}^{\bullet\diamond} = \mathcal{F}^{\bullet\diamond} = \mathcal{F}^+$ □

We remark that an interaction bundle $\mathcal{I}$ may be not equal to $\mathcal{I}^{\diamond\bullet}$, that is for some $a \in A$, $\mathcal{I}(a) \neq \cup_{U\in\mathcal{U}(a)}\mathcal{I}^\diamond[U]$.

Example 3.9 Let $f : A \to A$ be any map and for any $a \in A$, let $\mathcal{I}(a) \equiv \mathcal{U}(a)$ be the set of all neighborhoods of a. $\mathcal{I}$ can be viewed as an interaction bundle with $N = 1$. In this case one has $\mathcal{I}(a) = \cup_{U\in\mathcal{U}(a)}\mathcal{I}^\diamond[U]$ if and only if f is continuous at a.

This justifies the following:

Definition 3.10 *An interaction bundle $\mathcal{I}$ is said to be regular if $\mathcal{I} = \mathcal{I}^{\diamond\bullet}$.*

If $\mathcal{I}$ is regular then or any preference profile $R_N \in Q(A)^N$, $\mathrm{Stl}(\mathcal{I}, R_N) = \mathrm{Stl}(\mathcal{I}^\diamond, R_N)$; this is a consequence of Proposition 3.6.

Proposition 3.11 *Let $\mathcal{I}$ be an interaction bundle. In order that $\mathcal{I}$ be regular it is necessary and sufficient that $\mathcal{I} = \mathcal{E}^\bullet$ for some presheaf $\mathcal{E}$; moreover in this case there exists a unique sheaf $\mathcal{E}$ such that $\mathcal{I} = \mathcal{E}^\bullet$, namely $\mathcal{E} = \mathcal{I}^\diamond$.*

Proof. The result follows immediately from Propositions 3.7 and 3.8. □

One can summarrize the situation as follows: The operation $\bullet$ takes an interaction presheaf to some regular interaction bundle and its restriction to the set of sheaves is injective, its inverse being the operation $\diamond$.

4 Stability of interaction presheaves

In this section, we introduce a combinatorial property of interaction presheaves which is shown to be equivalent to stability. This extends the results of Abdou and Keiding (2003) to the topological setup, given that the interaction presheaf satisfies a mild continuity assumption and preferences are representable by continuous real functions.

We need some more notation, extending the notion of range of an interaction array as presented in Definition3.1: For $\varphi \in \mathcal{P}_0(N,A)$ and $i \in N$ the i-range of φ is the set

$$R^i(\varphi) = \bigcup_{S:i\in S} \varphi(S).$$

Definition 4.1 *Let $\mathcal{E}$ be an interaction presheaf. A* **cycle** *in $\mathcal{E}$ is family $(U^k,\varphi^k)_{k=1}^r$, where $U^k \in \mathcal{G}$, $\varphi^k \in \mathcal{E}[U^k]$, $k = 1,\dots,r$ with the properties:*

(i) $\cup_{k=1}^r U^k = A$,

(ii) if $i \in N$ and $\emptyset \neq J \subset \{1,\dots,r\}$ then there exists $k \in J$ such that $U^k \cap R^i(\varphi^j) = \emptyset$ for all $j \in J$.

If $\mathcal{E}$ has no cycles, then $\mathcal{E}$ is said to be acyclic.

For proving equivalence of stability and acyclicity, we need the following lemma which is a slight extension of a classical topological result about normal spaces. Here and in the sequel, $\overline{W}$ denotes the closure of the set W.

Lemma 4.2 *Let $\{U_i \mid i = 1,\dots,p\}$ be a finite collection of open sets in a normal topological space E. Then there are open sets W_i with the properties*

(i) $\overline{W}_i \subset U_i$ for $i = 1,\dots,p$,

(ii) $\cup_{i\in J} U_i = E \Rightarrow \cup_{i\in J}\overline{W}_i = E$ for all subsets J of $\{1,\dots,r\}$.

PROOF: Let $\mathcal{J} \subset \mathcal{P}(\{1,\dots,p\})$ be a collection of subsets of indices such that for all $J \in \mathcal{J}$, $\cup_{i\in J} U_i = E$. Choose any $h \in \{1,\dots,p\}$ such that $h \in I$ for some $J \in \mathcal{J}$, and let $\mathcal{J}_h = \{J \in \mathcal{J} \mid h \in J\}$. Then the sets U_h^c and $\left(\cap_{j\in J, j\neq h} U_j^c\right)$ are disjoint for each $J \in \mathcal{J}_h$, so that

$$U_h^c \cap \left[\cup_{J\in\mathcal{J}_h} \cap_{j\in J, j\neq h} U_j^c\right] = \emptyset.$$

By normality of E, there are disjoint open sets V and W_h such that $U_h^c \subset V$ and $\left[\cup_{J\in\mathcal{J}_h} \cap_{j\in J,j\neq h} U_j^c\right] \subset W_h$, $\cap_{i=2}^p U_i^c \subset W_1$. Clearly $\overline{W}_h \subset V^c \subset U_h$, and for each $J \in \mathcal{J}_h$, W_h contains $\cup_{j\in J,j\neq h} U_j^c$, so that $(W_h, (U_j)_{j\in J,j\neq h})$ is a covering of E.

Replacing the family $(U_1,\dots,U_p)$ by $(W_h,(U_j)_{j\neq h})$ and repeating the procedure, we eventually get a family $(W_1,\dots,W_p)$ with the desired properties. □

A family $(W_i)_{i=1}^p$ with the properties stated in Lemma 4.2 is called a ***refinement*** of $(U_i)_{i=1}^p$.

We shall also need an alternative formulation of the condition defining a cycle.

Lemma 4.3 *Let $\mathcal{E}$ be an interaction sheaf. Then the following are equivalent:*

(1) $\mathcal{E}$ has a cycle,

(2) there is a family $(W^k,\varphi^k)_{k=1}^r$ with $W^k \in \mathcal{G}$ and $\varphi^k \in \mathcal{E}[W^k]$, each k, such that

(i') $\cup_{k=1}^n W^k = A$,

(ii') for each $i \in N$ there is a permutation $(k_1, \dots, k_r)$ of $(1, \dots, r)$ such that for any $j \in \{1, \dots, r\}$,

$$\left[\overline{W}^{k_1} \cup \cdots \cup \overline{W}^{k_j}\right] \cap \left[R^i(\varphi^{k_j}) \cup \cdots \cup R^i(\varphi^{k_r})\right] = \emptyset.$$

Proof: (1)⇒(2): Let $(U^k, \varphi^k)_{k=1}^r$ be a cycle in $\mathcal{E}$, and let $(W^k)_{k=1}^r$ be a refinement of $(U^k)_{k=1}^r$ (Lemma 4.2). Let $i \in N$ be arbitrary. By property (ii) in Definition 4.1 applied to $J = \{1, \dots, r\}$, we get the existence of $k_1 \in \{1, \dots, r\}$ such that $\overline{W}^{k_1}$ and $\cup_{k=1}^r R^i(\varphi^k)$ have empty intersection. Now, let $2 \leq j \leq r$ and assume that indices $k_1, \dots, k_{j-1}$ have been constructed such that

$$\left[\overline{W}^{k_1} \cup \cdots \cup \overline{W}^{k_{j-1}}\right] \cap \left[\cup_{k \in \{1,\dots,r\} \setminus \{k_1,\dots,k_{j-2}\}} R^i(\varphi^k)\right] = \emptyset;$$

applying now property (ii) of Definition 4.1 with $J = \{1, \dots, r\} \setminus \{k_1, \dots, k_{j-1}\}$ we get $k_j \in J$ such that

$$\left[\overline{W}^{k_1} \cup \cdots \cup \overline{W}^{k_j}\right] \cap \left[\cup_{k \in \{1,\dots,r\} \setminus \{k_1,\dots,k_{j-1}\}} R^i(\varphi^k)\right] = \emptyset;$$

Repeating the procedure r times yields a permutation $(k_1, \dots, k_r)$ with the desired properties.

(2)⇒(1): We check that (W^k, φ^k) is a cycle in $\mathcal{E}$, and it satisfies to show that (ii) is fulfilled. Thus, let $i \in N$ and let J be a nonempty subset of $\{1, \dots, r\}$. Let j^0 be such that $J \subseteq \{k_{j^0}, \dots, k_r\}$ where $(k_1, \dots, k_r)$ is the permutation defined in (ii'). Then each set $R^i(\varphi^j) \cap W^{k_{j^0}}$ for $j \geq j^0$, so that $W^{k_{j^0}}$ does not intersect any of the sets $R^i(\varphi^j)$, for $j \in J$, and we have shown that (ii) is satisfied. □

Theorem 4.4 *A closed valued interaction presheaf $\mathcal{E}$ is stable if and only if it is acyclic.*

Proof. Assume that $\mathcal{E}$ is not stable. Then $\mathrm{Stl}(\mathcal{E}, u)$ is empty for some continuous profile $u = (u_1, \dots, u_n)$, that is for any $a \in A$ there is $U^a \in \mathcal{G}$ and $\varphi^a \in \mathcal{E}[U^a]$ such that such that $\varphi^a(S) \subseteq P(a, S, u_i)$ for all $S \in \mathcal{P}_0(N)$, or expressed otherwise, $u_i(a) < \min\{u_i(b) \mid b \in R^i(\varphi^a)\}$ for all $i \in N$ such that $R^i(\varphi^a) \neq \emptyset$. Since the u_i are continuous, there exists an open neighbourhood W^a of a such that $\sup\{u_i(c) \mid c \in W^a\} < \min\{u_i(b) \mid b \in R^i(\varphi^a)\}$ for all $i \in N$ such that $R^i(\varphi^a) \neq \emptyset$. Moreover, by the presheaf property, $\varphi^a \in \mathcal{E}[U^a \cap W^a]$.

Since A is compact, there exist $a_1, \dots, a_r$ such that the family $(U^{a_1} \cap W^{a_1}, \dots, U^{a_r} \cap W^{a_r})$ is a covering of A. Put $V^k = U^{a_k} \cap W^{a_k}$, $\varphi^k = \varphi^{a_k}$, $k = 1, \dots, r$. We show that $(V^k, \varphi^k)_{k=1}^r$ is a cycle.

Clearly, $\cup_{k=1}^r V^k = A$; to check (ii) in Definition 4.1, let $i \in N$ and $\emptyset \neq J \subset \{1, \dots, r\}$. Let $k \in J$ such that $\sup\{u_i(c) \mid c \in V^k\} = \min_{j \in J} \sup\{u_i(c) \mid c \in V^j\}$.

We claim that $V^k \cap R^i(\varphi^j) = \emptyset$ for all $j \in J$. Indeed, this is trivially true if $R^i(\varphi^j) = \emptyset$. If $R^i(\varphi^j) \neq \emptyset$, let $a \in V^k$ and $b \in R^i(\varphi^j)$. We then have

$$u_i(a) \leq \sup_{c \in V^k} u_i(c) \leq \sup_{c \in V^j} u_i(c) < u_i(b),$$

which proves the claim and shows that $(V^k, \varphi^k)_{k=1}^r$ is indeed a cycle.

Conversely let $(U^k, \varphi^k)_{k=1}^r$ be a cycle in $\mathcal{E}$. We construct a profile $(u_1, \ldots, u_n)$ such that $\mathrm{Stl}(\mathcal{E}, u)$ is empty. To begin, using Lemma 4.2 we choose an open covering $(W^1, \ldots, W^r)$ of A such that $\overline{W}^k \subset U^k$ for $k = 1, \ldots, n$.

Now, by Lemma 4.3 there is a permutation $(k_1, \ldots, k_r)$ of $(1, \ldots, r)$ such that for any $j \in \{1, \ldots, r\}$,

$$\left[\overline{W}^{k_1} \cup \cdots \cup \overline{W}^{k_j}\right] \cap \left[R^i(\varphi^{k_j}) \cup \cdots \cup R^i(\varphi^{k_r})\right] = \emptyset.$$

We construct for each $i \in N$ continuous function u_i on A such that

$$\begin{array}{lcll} u_i(c) & \leq & h-1 & \text{for } h \in \{1, \ldots, r\} \text{ and } c \in \overline{W}^{k_h}, \\ u_i(b) & \geq & h & \text{for } b \in R^i(\varphi^{k_h}) \end{array}$$

This may be done as follows: Since A is a normal topological space, for any $h \in \{1, \ldots, r\}$ there is a continuous function $v^h : A \to [0,1]$ such that

$$v^h(a) = \begin{cases} 0 & \text{if } a \in \overline{W}^{k_1} \cup \cdots \cup \overline{W}^{k_h}, \\ 1 & \text{if } a \in R^i(\varphi^{k_h}) \cup \cdots \cup R^i(\varphi^{k_r}). \end{cases}$$

The function $u_i = \sum_{h=1}^r v^h$ has the desired properties.

Consider now the profile $(u_1, \ldots, u_n)$. If $a \in \overline{W}^j$ and $i \in N$, then there exists an index h (depending on i) such that $k_h = j$. It follows that $u_i(a) \leq h$ and for any $b \in R^i(\varphi^j)$, $u_i(b) \geq h+1$, therefore $R^i(\varphi^j) \subset P(a, u_i)$ for each i, and by the presheaf property we have $P(W^k, \cdot, u) \in \mathcal{E}[W^k]$, so that every $a \in W^k$ is dominated.

Since $(W_1, \ldots, W_n)$ is a covering of A, the set of undominated alternatives at the profile $(u_1, \ldots, u_n)$ is empty. □

Remark 4.5 Theorem 4.4 has been proved for the class of continuous preferences preferences representable by continuous functions. It is easy to see, by a slight modidication of the first part of the proof, that the same result holds for the larger class of continuous preferences.

The characterization of stable interaction presheaves given in Theorem 4.4 gives a purely combinatorial property of the power structure, which in principle may be verified without recourse to preference profiles and notions of domination. In order to exploit this fact in characterizing solvable game forms, we need to investigate the relation between equilibria of the game form and the settlements of its associated interaction forms and bundles. This is done in the following section.

5 Game form solvability and stability of interaction forms

We now return to the discussion of game forms; let $G = (X_1, \cdots, X_n, A, g)$ be a game form such that the strategy spaces X_i for $i = 1, \ldots, n$ as well as the outcome space A are compact Hausdorff spaces, and where g is continuous and onto. In Section 3, we introduced the associated β-interaction presheaf $\mathcal{E}^G_\beta$. We assume that a confederation $\mathcal{M} \subset \mathcal{P}_0(N)$ is given. We define the associated β-interaction bundle $\mathcal{I} \equiv \mathcal{I}^{G,\mathcal{M}}_\beta$ by

$$\begin{aligned}\mathcal{I}_\beta[a] = \{\varphi \in \mathcal{P}_0(N, A) \quad | \quad & \forall x_N \in g^{-1}(a), \\ & \exists S \in \mathcal{M},\, y_S \in X_S : g(y_S, x_{S^c}) \in \varphi(S)\} \qquad (5)\end{aligned}$$

and the corresponding interaction sheaf (2). Similarly we define the α-interaction bundle $\mathcal{I}_\alpha \equiv \mathcal{I}^{G,\mathcal{M}}_\alpha$ and the corresponding interaction form $\mathcal{E}_\alpha \equiv \mathcal{E}^{G,\mathcal{M}}_\alpha$ given by (3). One has $\varphi \in \mathcal{E}_\beta[U]$ if and only if $\varphi^c \notin \mathcal{E}_\alpha[U]$, where by definition $\varphi^c(S) = \varphi(S)^c$ for all $S \in P_0(N)$. It is also clear that $\mathcal{E}_\beta = (\mathcal{I}_\beta)^\diamond$.

Remark 5.1 One advantage of our present model compared to that of Abdou and Keiding (2003) is that restrictions on confederations as given in (1) reflect *faithfully* the shift of power from one confederation to another in the game form G. This is because we have:

$$\mathcal{E}^{G,\mathcal{M}}_\beta = (\mathcal{E}^{G,\mathcal{P}_0(M)}_\beta)_{|\mathcal{M}}$$

where the second member is the projection of $\mathcal{E}^{G,\mathcal{P}_0(M)}_\beta$ on $\mathcal{M}$. It follows that one needs only to know $\mathcal{E}^{G,\mathcal{P}_0(M)}_\beta$ in order to deduce $\mathcal{E}^{G,\mathcal{M}}_\beta$ for all confederations $\mathcal{M}$.

The following result is straightforward but central for our characterization of solvable game forms. It shows that the concept of β-interaction bundle is to the $\mathcal{M}$- equilibrium of the game (G, R_N) what the β effectivity function is to the β-core of that game :

Proposition 5.2 *For any R_N the set of $\mathcal{M}$-equilibrium outcomes of (G, R_N) is equal to* $\mathrm{Stl}(\mathcal{I}_\beta, R_N)$.

Proof: Let $a \in A$ be an $\mathcal{M}$-equilibrium outcome of (G, R_N). There exists an $\mathcal{M}$-equilibrium of (G, R^N) $x_N \in X$ such that $g(x_N) = a$ and for all $S \in \mathcal{M}$ and $y_S \in X_S$, $g(y_S, x_{S^c}) \notin P(a, S, R_N)$, and consequently, the interaction array $P(a, \cdot, R_N)$ does not belong to $\mathcal{I}_\beta[a]$. If follows that a is not dominated in $\mathcal{I}_\beta[a]$ at R_N, or equivalently $a \in \mathrm{Stl}(\mathcal{E}_\beta, R_N)$.

Conversely, if $a \in \mathrm{Stl}(\mathcal{I}_\beta, R_N)$ then the interaction array $P(a, \cdot, R_N)$ is not in $\mathcal{I}_\beta[a]$. But then there must be some strategy array $x_N \in X$ with $g(x_N) = a$ such that $g(y_S, x_{S^c}) \notin P(a, S, R_N)$ for all $S \in \mathcal{M}$ and all $y_S \in X_S$, showing that x_N is an $\mathcal{M}$-equilibrium. □

In order to apply the main result of Section 4, we need to work with interaction presheafs or interaction bundles which are either closed- or open-valued. Since however the relevant presheaf has a particular structure, we need a closer look at the β-interaction presheaf as well as other, related, constructions. We recall that the set of closed (resp. open) valued interaction arrays is denoted $\mathcal{F}_0(N, A)$ (resp. $\mathcal{G}_0(N, A)$). We define the presheaves $\overline{\mathcal{E}}_\beta$ ($\mathring{\mathcal{E}}_\beta$), $\overline{\mathcal{E}}_\alpha$ ($\mathring{\mathcal{E}}_\alpha$) by restricting for each $U \in \mathcal{G}_0$ to the interaction arrays which take only closed (open) sets as values. Similarly, we introduce the interaction bundles $\overline{\mathcal{I}}_\beta$, $\mathring{\mathcal{I}}_\beta$, $\overline{\mathcal{I}}_\alpha$ ($\mathring{\mathcal{I}}_\alpha$)

Proposition 5.3 *For any continuous R_N, the set of $\mathcal{M}$-equilibrium outcomes of (G, R_N) is equal to* $\mathrm{Stl}(\mathring{\mathcal{I}}_\beta, R_N)$. □

We shall make use of a topology on the set of interaction arrays: For any $\psi \in \mathcal{P}_0(N, A)$, define the lower interval I_ψ and the upper interval J_ψ by :

$$I_\psi = \{\varphi \in \mathcal{F}_0(N, A) \mid \varphi \leq \psi\}$$

$$J_\psi = \{\varphi \in \mathcal{G}_0(N, A) \mid \psi \leq \varphi\}$$

The collection $\{I_\psi \mid \psi \in \mathcal{G}_0(N, A)\}$ is a basis for a topology on $\mathcal{P}_0(N, A)$ which is called the ***upper topology***. $\mathcal{F}_0(N, A)$ will be endowed with its topology as a subspace of $\mathcal{P}_0(N, A)$.

Lemma 5.4 *For any $a \in A$ and any $U \in \mathcal{G}_0$ we have:*
(i) $\mathcal{I}_\alpha(a) = \{\varphi \in \mathcal{P}_0(N, A) \mid I_\varphi \cap \overline{\mathcal{I}}_\alpha(a) \neq \emptyset\}$
(ii) $\mathcal{E}_\alpha[U] = \{\varphi \in \mathcal{P}_0(N, A) \mid I_\varphi \cap \overline{\mathcal{E}}_\alpha[U] \neq \emptyset\}$
(iii) $\mathcal{I}_\beta(a) = \{\varphi \in \mathcal{P}_0(N, A) \mid J_\varphi \subset \mathring{\mathcal{I}}_\beta(a)\}$,
(iv) $\mathcal{E}_\beta[U] = \{\varphi \in \mathcal{P}_0(N, A) \mid J_\varphi \subset \mathring{\mathcal{E}}_\beta[U]\}$

Proof: If $I_\varphi \cap \overline{\mathcal{I}}_\alpha(a) \neq \emptyset$ then clearly $\varphi \in \mathcal{I}_\alpha(a)$. Conversely, assume that $\varphi \in \mathcal{I}_\alpha(a)$, then there exists $x_N \in X_N$ such that $g(x_N) = a$ and for all $S \in \mathcal{M}$, $g(x_{S^c}, X_S) \subset \varphi(S)$. Since the sets $g(x_{S^c}, X_S)$ for $S \in \mathcal{P}_0(N)$ are closed, the interaction array ψ defined by $\psi(S) := g(x_{S^c}, X_S)$, $(S \in \mathcal{P}_0(N))$ belongs to $\overline{\mathcal{I}}_\alpha(a)$ and $\psi \leq \varphi$. This proves assertion (i).The verification of the other assertions is left to the reader. □

Lemma 5.5 *The correspondence $\overline{\mathcal{I}}_\alpha$ from A to $\mathcal{F}_0(N, A)$ has closed graph in $A \times \mathcal{F}_0(N, A)$.*

Proof: Let $(a, \varphi) \in A \times \mathcal{F}_0(N, A)$, let $\mathcal{V}_a$ be the set of open neighbourhoods of a, let $\mathcal{V}_\varphi$ be the family $\mathcal{V}_\varphi := (V : V \in \mathcal{G}_0(N, A), \varphi \leq V)$ and let $\mathcal{V} = \mathcal{V}_a \times \mathcal{V}_\varphi$. Assume that (a, φ) belongs to the closure of the graph of $\overline{\mathcal{I}}_\alpha$, that is for any $V \equiv (V_1, V_2) \in \mathcal{V}$ there exist $b^V \in V_1$ and $\psi^V \in I_{V_2} \cap \overline{\mathcal{I}}_\alpha(b^V)$. By the definition of $\overline{\mathcal{I}}_\alpha$, there exists $x_N^V \in X_N$ such that $g(x_N^V) = b^V$ and for all $S \in \mathcal{M}$, $g(x_{S^c}^V, X_S) \subset \psi^V(S) \subset V_2(S)$.

The set $\mathcal{V}$ ordered by componentwise inclusion is a directed set. Since X is compact, the net $(x_N^V)_{V \in \mathcal{V}}$ admits a convergent subnet, say $(x_N^{V_t})_{t \in T}$, where $(T, \geq)$ is a directed set. Let x_N be its limit; by continuity of g, $g(x_N) = a$. We claim that for all $S \in \mathcal{M}$ $g(x_{S^c}, X_S) \subset \varphi(S)$. Indeed, for any $t \in T$, continuity of g implies that $g(x_{S^c}, X_S) \subset \overline{V_{2,t}(S)}$. Let $V \in \mathcal{V}_\varphi$. Since A is normal and due to the subnet property, there exists some $t \in T$ such that $\varphi(S) \subset \overline{V_{2,t}(S)} \subset V_2(S)$ for all $S \in \mathcal{S}$. Therefore $g(x_{S^c}, X_S) \subset V_2(S)$. Since the last inclusion is true for all $V_2 \in \mathcal{V}_\varphi$, again by normality it follows that $g(x_{S^c}, X_S) \subset \varphi(S)$. We conclude that $\varphi \in \overline{\mathcal{I}}_\alpha(a)$. □

Lemma 5.6 *$\mathring{\mathcal{I}}_\beta$ is regular: $(\mathring{\mathcal{E}}_\beta)^\bullet = \mathring{\mathcal{I}}_\beta$.*

Proof: Let $a \in A$. For any open neighbourhood U of a, $(\mathring{\mathcal{E}})_\beta[U] \subset (\mathring{\mathcal{I}})_\beta(a)$, so that $(\mathring{\mathcal{E}}_\beta)^\bullet \subset \mathring{\mathcal{I}}_\beta$. Conversely, if $\varphi \in (\mathring{\mathcal{I}})_\beta(a)$ then $\varphi^c \notin \overline{\mathcal{I}}_\alpha(a)$, where φ^c is the interaction array defined by $\varphi^c(S) = \varphi(S)^c$, all S. It follows from Lemma 5.5 that there exist $U \in \mathcal{G}_0$, $W \in \mathcal{G}_0(N, A)$ such that $a \in U$, $\varphi^c \in I_W$ and for all $b \in U$, $\overline{\mathcal{I}}_\alpha(b) \cap I_W = \emptyset$; in view of Lemma 5.4(i), $W \notin \mathring{\mathcal{I}}_\alpha(b)$ or equivalently $W^c \in \overline{\mathcal{I}}_\beta(b)$. Since $W^c(S) \subset \varphi(S)$ for all $S \in \mathcal{P}_0(N)$, we have $\varphi \in (\mathring{\mathcal{I}})_\beta^\diamond[U]$. □

Example 5.7 Let $\pi : X_1 \times X_2 \to X_1 \times X_2$ the identity, $A := X_1 \times X_2$, $\pi_i (i = 1, 2)$ the projections. Let $\mathcal{M} = \{\{1\}\}$. For any $(x_1, x_2) \in A$, $U \in \mathcal{G}_0$, let

$$I_\beta(x_1, x_2) = \{B \subset A \mid x_2 \in \pi_2(B)\},$$

$$E_\beta[U] = \{B \subset A \mid \pi_2(U) \subset \pi_2(B)\},$$

then we have :

$$\mathcal{I}_\beta(x_1, x_2) = \{\varphi \mid \varphi(1) \in I_\beta(x_1, x_2)\},$$
$$\mathcal{E}_\beta(U) = \{\varphi \mid \varphi(1) \in E_\beta[U]\}.$$

We remark that the set $\{(y_1, x_2)\} \in I_\beta(x_1, x_2)$ but unless x_2 is isolated there is no $U \in \mathcal{U}_{(x_1,x_2))}$ such that $\pi_2(U) = \{x_2\}$. Therefore unless X_2 is finite, $\mathcal{E}_\beta^\bullet \neq \mathcal{I}_\beta$. It follows that unless X_2 is finite $\mathcal{I}_\beta$ and $\overline{\mathcal{I}}_\beta$ are not regular. Moreover let $R : X_2 \to X_1$ a map, then the graph of R, Graph(R) is an element of $E_\beta[A]$ but unless R continuous, there is no a closed $B \in E_\beta[A]$ such that $B \subset$ Graph(R). It follows that there is no analog of Lemma 5.4(i) for $\mathcal{I}_\beta$.

Lemma 5.8 *For any $a \in A$ and any $U \in \mathcal{G}_0$ we have:*
(i) $\mathring{\mathcal{I}}_\beta(a) = \{V \in \mathcal{G}_0(N,A) \mid I_V \cap \overline{\mathcal{I}}_\beta(a) \neq \emptyset\}$.
(ii) $\mathring{\mathcal{E}}_\beta[U] = \{V \in \mathcal{G}_0(N,A) \mid I_V \cap \overline{\mathcal{E}}_\beta[U] \neq \emptyset\}$,

Proof: By the definitions, if $V \in \mathcal{G}_0(N,A)$ and $I_V \cap \overline{\mathcal{I}}_\beta(a) \neq \emptyset$, then $V \in \mathring{\mathcal{E}}_\beta(a)$. Conversely, if $V \in \mathring{\mathcal{I}}_\beta(a)$ then $V^c \notin \overline{\mathcal{I}}_\alpha(a)$ so that by Lemma 5.5 there exists $W \in \mathcal{G}_0(N,A)$ such that $V^c \in I_W$ and $I_W \cap \overline{\mathcal{I}}_\alpha(a) = \emptyset$. In view of Lemma 5.4(i), $W \notin \mathring{\mathcal{I}}_\alpha(a)$ or equivalently $W^c \in \overline{\mathcal{I}}_\beta(a)$. Since $W^c \in I_V$ we have $W^c \in I_W \cap \overline{\mathcal{I}}_\beta(a)$. This proves (i). The proof of the other assertion is left to the reader. □

Lemma 5.9 *For each continuous profile and let R_N we have*

$$\mathrm{Stl}(\mathcal{I}_\beta, R_N) = \mathrm{Stl}(\mathring{\mathcal{I}}_\beta, R_N) = \mathrm{Stl}(\mathring{\mathcal{E}}_\beta, R_N) = \mathrm{Stl}(\overline{\mathcal{I}}_\beta, R_N) = \mathrm{Stl}(\overline{\mathcal{E}}_\beta, R_N)$$

and this set equals the set of equilibrium outcomes of G at R_N.

Proof: Let R_N be a continuous profile. Put $V(S) = P(a,S,R_N)$ $(S \in P_0(N))$. Clearly V has open values. The first equality follows by Proposition 5.3. The second equality follows from regularity of $\mathring{\mathcal{I}}_\beta$ (Lemma 5.6). In view of Lemma 5.8 (i), $\mathrm{Stl}(\mathring{\mathcal{I}}_\beta, R_N) = \mathrm{Stl}(\overline{\mathcal{I}}_\beta, R_N)$. In view of Lemma 5.8 (ii) $\mathrm{Stl}(\mathring{\mathcal{E}}_\beta, R_N) = \mathrm{Stl}(\overline{\mathcal{E}}_\beta, R_N)$. □

Theorem 5.10 *G is $\mathcal{M}$-solvable if and only if $\overline{\mathcal{E}}_\beta$ ($\mathring{\mathcal{E}}_\beta$) is acyclic.*

Proof. Applying Theorem 4.4 to $\overline{\mathcal{E}}_\beta$, one has that $\mathcal{M}$-solvabilty of G is equivalent to acyclicity of the interaction sheaf $\overline{\mathcal{E}}_\beta$. Moreover by Lemma 5.8(ii) any cycle of $\mathring{\mathcal{E}}_\beta$ gives rise to a cycle of $\overline{\mathcal{E}}_\beta$.

Conversely given a cycle $(C^k, \varphi^k)_{k=1}^r$ of $\overline{\mathcal{E}}_\beta$, applying Lemma 4.2 one may replace the open cover $(C^k)_{k=1}^r$ of A by an open cover $(U^k)_{k=1}^r$ such that $\overline{U}^k \subset C^k$ for $k = 1, \ldots, r$. The array $(\overline{U}^k, \varphi^k)_{k=1}^r$ still verifies the combinatorial properties of cycles. By a further application of Lemma 4.2 to the family $((\overline{U}^k)^c, (\varphi^k(S))^c, k = 1, \cdots, r, S \in P_0(N))$ of open sets, one can replace $(\overline{U}^k, \varphi^k)$ by (W^k, ψ^k) taking open values such that $\overline{U}^k \subset W^k$ and $\varphi^k(S) \subset \psi^k(S)$ for all $S \in \mathcal{P}_0(N)$, $k = 1 \cdots, r$, thus getting a family $(U^k, \psi^k)_{k=1}^r$ which satifies conditions (i) and (ii) of cycles. Since $U^k \subset C^k$ one has $\psi^k \in \mathring{\mathcal{E}}_\beta[C^k] \subset \mathring{\mathcal{E}}_\beta[U^k]$. Thus $(U^k, \psi^k)_{k=1}^r$ is a cycle in $\mathring{\mathcal{E}}_\beta$. □

6 Interaction sheaves over convex domains

In the present section, we consider a special case which however turns up in many applications, namely that where the domain A is a convex and compact subset of some Euclidean space $\mathbb{R}^d$. Let $\mathcal{C}$ denotes the set of all convex and closed subsets of A and $\mathcal{C}_0(A,N)$ the set of all interaction arrays φ such that $\varphi(S) \in \mathcal{C}$ for all $S \in \mathcal{P}_0(N)$. Working with convex domains, it seems natural to restrict preferences to all $R \in Q(A)$ which are ***convex*** in the sense that for each $a \in A$, the set $P(a,R)$ is convex. Denoting this subset of $Q(A)$ by $Q_{\mathcal{C}}(A)$, we say that an interaction presheaf is ***c-stable*** if $\mathrm{Stl}(\mathcal{E}, R_N) \neq \emptyset$ for each $R_N \in Q_{\mathcal{C}}(A)^N$. Adding a linear structure of the outcome space means that in some cases, the properties of acyclicity and consequently of $\mathcal{S}$-solvability may take another form due to the restriction on the set of admissible preferences.

We then have to revise the results in Section 4 so as to take the convexity of domain and preferences into consideration. For this, we must modify the definition of a cycle given in Definition 4.1. We use the notation $\mathrm{co}(B)$ for the convex hull of $B \subset A$.

Definition 6.1 *Let $\mathcal{E}$ be an interaction sheaf on (N,A), A* ***convex cycle*** *in $\mathcal{E}$ is a family $(U^k,\varphi^k)_{k=1}^r$ with $U^k \in \mathcal{G}_0$ and $\varphi^k \in \mathcal{E}[U^k]$, each k, such that*

(i) $\cup_{k=1}^n U^k = A$,

(ii) if $i \in N$ and $\emptyset \neq J \subset \{1,\dots,r\}$ then there exists $k \in J$ such that $U^k \cap \mathrm{co}(\cup_{j\in J} R^i(\varphi^j)) = \emptyset$.

Lemma 4.3 in this context is still valid provided condition (ii') takes a new form, precisely

Lemma 6.2 *Let $\mathcal{E}$ be an interaction sheaf. Then the following are equivalent:*

(1) $\mathcal{E}$ has a cycle,

(2) there is a family $(W^k,\varphi^k)_{k=1}^r$ with $W^k \in \mathcal{G}_0$ and $\varphi^k \in \mathcal{E}[W^k]$, each k, such that

(i') $\cup_{k=1}^n W^k = A$,

(ii') for each $i \in N$ there is a permutation $(k_1,\dots,k_r)$ of $(1,\dots,r)$ such that for any $j \in \{1,\dots,r\}$,

$$\left[\overline{W}^{k_1} \cup \cdots \cup \overline{W}^{k_j}\right] \cap \mathrm{co}\left(\left[R^i(\varphi^{k_j}) \cup \cdots \cup R^i(\varphi^{k_r})\right]\right) = \emptyset.$$

Let $\mathcal{C}_0(A,N)$ be the set of all interaction arrays that are closed-and-convex valued. We define $\mathcal{E}_{\mathcal{C}}[U] = \mathcal{E}[U] \cap \mathcal{C}_0(A,N)$.

Lemma 6.3 *$\mathcal{E}$ is acyclic if and only if $\mathcal{E}_{\mathcal{C}}$ is. $\mathcal{E}$ is stable on $Q_{\mathcal{C}}$ if and only if $\mathcal{E}_{\mathcal{C}}$ is.*

We have the following counterpart of Theorem 4.4 in the context of convex cycles.

Lemma 6.4 *Let $C_1 \subset \cdots, \subset C_p$ be an increasing sequence of compact and convex sets of $\mathbb{R}^d$ such that $0 \in \overset{\circ}{C}_1$ and $d(C_k, C_{k+1}^c) > 0$ for $k = 1, \ldots, p-1$. Then there exists a continuous quasiconvex function v such that:*

$$v(x) \leq k \iff x \in C_k, \ k = 1, \ldots, p.$$

Proof. For any convex set containing 0, denote by $J_C(x) = \inf\{\lambda > 0 \mid x \in \lambda C\}$ ($= +\infty$ if the latter set is empty) and let ∂C denote the boundary of C. Define v as follows:

$$\begin{aligned} v(x) &= J_{C_1}(x) && \text{if } x \in C_1, \\ v(x) &= k + \frac{J_{C_k}(x) - 1}{J_{C_k}(x) - J_{C_{k+1}}(x)} && \text{if } x \in C_{k+1} - C_k, k = 1, \ldots, p-1, \\ v(x) &= p - 1 + J_{C_p}(x) && \text{if } x \in \mathbb{R}^d \setminus C_p \end{aligned}$$

Then v is continuous on any point $x \in \mathbb{R}^d \setminus \cup_{k=1}^p \partial C_k$ since the functions J_{C_k} are continuous and $J_{C_k}(x) - J_{C_{k+1}}(x) > 0$ for $x \neq 0$. Moreover, if $x \in \partial C_k$, then $v(x) = k$, and the continuity of v at x is easily verified.

We check the quasiconvexity of v: let $x, y \in \mathbb{R}^d$, $\lambda \in [0,1]$, and let $z = (1-\lambda)x + \lambda y$. Assume $v(x) \leq v(y)$; we distinguish 3 cases:

Case 1: $y \in C_1$. We have $x \in C_1$ so that by convexity of J_{C_1}, $v(z) \leq \max\{v(x), v(y)\}$.

Case 2: $y \in C_{k+1} \setminus C_k$ where $1 \leq k \leq p-1$. Let $\alpha = v(y) - k$. If $z \in C_k$ then $v(z) \leq v(y)$. If $z \in C_{k+1} \setminus C_k$ then

$$\begin{aligned} &(1-\alpha)\, J_{C_k}(z) + \alpha J_{C_{k+1}}(z) \\ &\quad \leq (1-\alpha)[(1-\lambda)J_{C_k}(x) + \lambda J_{C_k}(y)] + \alpha[(1-\lambda)J_{C_{k+1}}(x) + \lambda J_{C_{k+1}}(y)] \\ &\quad = (1-\lambda)[(1-\alpha)J_{C_k}(x) + \alpha J_{C_{k+1}}(x)] + \lambda[(1-\alpha)J_{C_k}(y) + \alpha J_{C_{k+1}}(y)] \\ &\quad \leq (1-\lambda).1 + \lambda.1 \\ &\quad = 1. \end{aligned}$$

Here the first inequality follows from convexity of J_{C_k} and $J_{C_{k+1}}$ and the fact that $0 \leq \alpha \leq 1$. When $x \in C_k$ $J_{C_k}(x) \leq 2$ and $J_{C_{k+1}}(x) \leq 1$, so that $(1-\alpha)J_{C_k}(x) + \alpha J_{C_{k+1}}(x) \leq 1$. When $x \in \mathcal{C}_{k+1} \setminus C_k$ then $v(x) - k \leq \alpha$ so that again $(1-\alpha)J_{C_k}(x) + \alpha J_{C_{k+1}}(x) \leq 1$. In both cases $v(y) - k = \alpha$ so that $(1-\alpha)J_{C_k}(y) + \alpha J_{C_{k+1}}(y) = 1$. This justifies the second inequality. We conclude that $v(z) - k \leq \alpha$ or equivalently $v(z) \leq v(y)$.

Case 3: $y \in \mathbb{R}^d - C_p$. If $z \in C_p$ then $v(z) \leq p \leq v(y)$. If $z \in \mathbb{R}^d - C_p$ then either $x \in \mathbb{R}^d - C_p$ and by convexity of J_{C_p} we have $v(z) \leq v(y)$ or $x \in C_p$ then $J_{C_p}(x) \leq 1$ than again by convexity of J_{C_p} we have $J_{C_p}(z) \leq \max 1, J_{C_p}(y) = J_{C_p}(y)$ so that $v(z) \leq v(y)$. □

Now we have the ingredients for proving a counterpart of Theorem 4.4. Since the method of proof is the same, once we have established the necessary ingredients in the form of Lemma 6.2-6.4 above, we shall the details and present only an outline of the proof.

Theorem 6.5 *Let $\mathcal{E}$ be an interaction presheaf. Then $\mathcal{E}$ is stable on $Q_{\mathcal{C}}(A)$ if and only if $\mathcal{E}$ has no convex cycles.*

PROOF: (Outline) If $\mathcal{E}$ is not stable on $Q_{\mathcal{C}}(A)$ then existence of a convex cycle follows the same steps of the general case. The only precision to add is that empty intersections of condition 2(ii) of Lemma 4.3 extend to the convex hull of the sets $R^i(\varphi^{k_j}) \cup \cdots \cup R^i(\varphi^{k_r}))$, so that the condition 2(ii) of Lemma 6.2 is satisfied.

Conversely, if a convex cycle exists, then for each player i, let $(k_1, \ldots, k_r)$ be the permutation given in (2) of Lemma 6.2, and let $B^i_j = \text{co}(R^i(\varphi^{k_j}) \cup \cdots \cup R^i(\varphi^{k_r}))$.

Remove empty sets B^i_k from the list if there are any, and let $B^i_1, \ldots, B^i_{p^i}$ be the remaining decreasing family of sets. Without loss of generality we may assume that B^i_p has non-empty interior and $d(B^i_k, (B^i_{k-1})^c) > 0$ for $k = 2, \ldots, p$. Then apply Lemma 6.4 to obtain a continuous quasiconvex function v^i such that $v^i(x) \leq p^i+1-k$ if and only if $x \in B^i_k$. The profile $(u_1, \cdots, u_n)$, where $u^i = -v^i(x)$, $i = 1, \ldots, n$, has an empty settlement set. $\square$

7 Concluding remarks

In the previous sections, we have introduced the concept of an interaction sheaf and used it for the characterization of solvable game forms. This was done in a topological framework. Equivalence between acyclicity and stability is proved for the class of continuous preferences. In fact the Hausdorff assumption on the compact set A provides a continuous class preferences rich enough to separate closed sets by respecting some combinatorial property. The results are thus similar to those of the discrete framework (e.g. Abdou and Keiding (2003)). If the context requires restricted domains of preferences, the notion of acyclicity has to be modified in accordance to that domain (see Kolpin (1991) for the effectivity function case). As an interesting framework for this restriction we considered the case of convex domains and convex continuous preferences.

The interaction sheaf represents conflicts in an intrinsic way since strategy sets are not explicitely described. The interpretation of an interactive sheaf adopted thoughout this paper is of the β-type. The power described is the upsetting power, the dual of which would be the stabilizing, or forcing power. Consistently with this interpretation the interaction sheaf of a game form as presented here contains exactly the information needed to decide upon the question of solvability, and it cannot be excluded that future problems may need a further development of the concepts used, so that we might not yet have reached the final form of describing the power structure in a game form. However, the interaction sheaf seems to be suitable for quite many problems, of which we have only touched upon a few. Also, it should be observed that the construction may be applied not only to strategic game forms but also to conflict situations which are presented in a less simple form (indexed families of game forms, generalized game forms), pointing to a more basic role of the interaction sheaf for analyzing conflict situations. They may be either very simple, if they reflect the power the effectivity power where each coalition acts separately, or more complex, where individuals act jointly (Nash) and even more complex when all coalitions act jointly (strong Nash). But since all those specific forms can be extracted by projection from a unique form, the model allows for comparision of different contextual interactions and the study of the degree of unstability when stability is not achieved. A closer study of these possibilities will however be a matter of future research.

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