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Aggregation of Information and Beliefs in Prediction Markets*

Marco Ottaviani[†] Peter Norman Sørensen[‡]

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Abstract

We analyze a binary prediction market in which traders have heterogeneous prior beliefs and private information. Realistically, we assume that traders are allowed to invest a limited amount of money (or have decreasing absolute risk aversion). We show that the rational expectations equilibrium price underreacts to information. When favorable information to an event is available and is revealed by the market, the price increases and this forces optimists to reduce the number of assets they can (or want to) buy. For the market to equilibrate, the price must increase less than a posterior belief of an outside observer.

Keywords: Prediction markets, private information, heterogeneous prior beliefs, limited budget, underreaction.

JEL Classification: D82 (Asymmetric and Private Information), D83 (Search; Learning; Information and Knowledge), D84 (Expectations; Speculations).

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1 Introduction

This paper studies the theoretical properties of prediction markets as forecasting tools. Prediction markets are incentive-based mechanisms for obtaining forecasts about future events. Often, prediction market forecasts are more accurate than those obtained by more traditional methods, such as expert judgement, or opinion polls.¹ Partly because of the track record of prediction markets as forecasting tools, there is a growing interest in using the information collected in these markets to improve decision making in business and public policy contexts.²

In our model, individual traders have heterogeneous beliefs that originate from two sources. First, individuals have non-common *prior beliefs* (or initial opinions)—these prior beliefs are subjective and thus are uncorrelated with the realization of the outcome. Second, individuals have access to possibly different *information* about the outcome—information has an objective nature because it is correlated with the outcome. This distinction between prior beliefs and information is standard.³

Typically, prediction markets target unique events, such as the outcome of a presidential election, or a merger. Given that traders have limited experience with such situations, it would be unrealistic to assume that they share a common prior belief. Instead, it is natural to allow traders to have heterogeneous prior beliefs. For the purpose of our analysis, the traders' subjective prior beliefs play the role of exogenous parameters, like preferences.

Having different prior beliefs, traders gain from betting actively against one another. While trade in prediction markets may be motivated by the traders' heterogeneous prior beliefs, the designers of these markets typically are interested in extracting the information these traders have.⁴ As we demonstrate in our model, the price that emerges in equilibrium is a generalized average of the participants' posterior beliefs—which combine their prior beliefs with the information that is revealed through the trading process.

We focus on a simple market for a binary event, such as the outcome of a presidential election. Traders can take positions in two Arrow-Debreu contingent assets, each paying

¹See, for instance, Forsythe et al. (1992) and Berg et al. (forthcoming) on the track record of the Iowa Electronic Markets since 1988.

²See Hanson (1999), Wolfers and Zitzewitz (2004), and Hahn and Tetlock (2005).

³An individual updates her belief when learning someone else's information. When exposed to the prior belief of another individual, the same individual would not be led instead to revise her belief.

⁴As Aumann (1976) notes, "reconciling subjective probabilities makes sense if it is a question of implicitly exchanging information, but not if we are talking about 'innate' differences in priors."

one dollar if the corresponding outcome is realized. In our formulation, each trader's initial endowment is constant with respect to the outcome realization. Taking into account a typical institutional feature of prediction markets, we constrain the wealth each trader can invest in the market. The prices of the assets are determined by a competitive trading process. Given that the prices for the two assets sum to one, it is natural to interpret an asset's price as the probability of the corresponding event.

In our analysis, we stack the cards in favor of efficient information aggregation by positing that the market is in a fully-revealing rational expectations equilibrium (REE). In an REE, traders make correct inferences from prices, given common knowledge of the information structure and the prior beliefs. Because the assumption that the market reaches a fully revealing REE is not warranted for some market rules, ours is the most optimistic scenario for information aggregation.⁵

For any given realization of private information, the market price can be interpreted as a posterior probability obtained by updating an implied market prior belief. This implied market prior belief is a generalized average of the traders' prior beliefs. However, we show that this implied market prior belief depends systematically on the particular signal realization. Hence, the incorporation of information into the market price is intertwined with (and cannot easily be separated from) the aggregation of subjective prior beliefs.

Our main result is that the market price underreacts to pre-trade information. When the information is more favorable to an event, resulting in a higher market price for the corresponding asset, the implied market prior probability is lower. Equivalently, more favorable information yields a higher market price, but the price increment is smaller than if the market price adjusted to the information as a Bayesian posterior probability for the event. This result provides a new explanation of the favorite-longshot bias, a regularity that is widely documented in the empirical literature on betting markets.

To understand the logic driving this result, consider a prediction market written on which finalist, Italy or Denmark, will win the European soccer championship. Suppose that the risk-neutral traders are subjectively more optimistic about Italy winning, the further South is their residence. In equilibrium, traders south of a certain threshold latitude spend the maximum amount of money allowed to buy the Italy asset, while, conversely, traders

⁵See Plott and Sunder (1982 and 1988) and Forsythe and Lundholm (1990) for experimental investigations of the conditions leading to REE in settings with differential private information.

north of the threshold latitude bet all they can on Denmark.

Now consider what happens when traders overall possess more favorable information about Italy winning before trade takes place. In the fully revealing REE, this information will be revealed and will cause the price for the Italy asset to be higher, while contemporaneously reducing the price for the Denmark asset. As a result, the southern traders (who are optimistic about Italy) are able to buy fewer Italy assets, which are now more expensive—while the northern traders can afford, and will end up demanding, more Denmark assets (now cheaper). Hence, there would be an excess supply of the Italy asset and excess demand for the Denmark asset. For the market to equilibrate, some northern traders must turn to the Italian side.

In summary, when information more favorable to an outcome is available, the marginal trader who determines the price has a prior belief that is *less* favorable to that outcome. Through this countervailing adjustment, the heterogeneity in priors will dampen the effect of information on price. While we just presented this intuitive explanation under the assumption of risk neutrality, we also show that the result extends to the case of risk-averse agents exhibiting decreasing absolute risk aversion, even when there is no exogenous bound on the amount of money traders can invest.

We contribute to the asset pricing literature by generalizing Grossman's (1976) characterization of REE to allow for wealth effects and heterogeneous priors. Our model's main ingredients (heterogeneous priors, private information, general risk preferences, and REE) then are the same as in Milgrom and Stokey (1982). To their setting, we add the comparative statics of how the equilibrium resulting in the first round of trade must depend on the information that is present in the market. Milgrom and Stokey instead focus on the price adjustment that follows the arrival of new information in a second period, after the first round of trade has occurred. As compared to Varian's (1989) analysis of trade with heterogeneous priors, we introduce here the possibility of wealth effects by imposing a limit on the amount that traders can invest and, more generally, by relaxing the constant absolute risk aversion assumption.

The paper proceeds as follows. Section 2 introduces our baseline model with risk-neutral traders. Section 3 derives the equilibrium and gives a parimutuel re-interpretation. Section 4 demonstrates the general interdependence of information and belief aggregation and the occurrence of the favorite-longshot bias. Section 5 extends the analysis to the case

with risk-averse traders. After a more detailed discussion of the paper’s contribution to the literature in Section 6, we conclude in Section 7. All proofs are in Appendix A.

2 Baseline Model

Our model is inspired by the rules of the Iowa Electronic Markets for a binary prediction market, in which traders can take positions on whether an event, E , is realized (e.g., the Democratic candidate wins the 2008 presidential election) or not. There are two Arrow-Debreu assets corresponding to the two possible realizations: one asset pays out 1 currency unit if event E is realized and 0 otherwise, while the other asset pays out 1 currency unit if the complementary event E^c is realized and 0 otherwise.⁶

Traders enter the market by first obtaining an equal number of both assets. Essentially, the designer of the prediction market initially endows each trader i with w_{i0} units of each of the two assets. One important feature of the market is that there is a limit on how much money each trader can invest.⁷ After entering the market, traders can exchange their assets with other traders. A second key feature of the market is that traders are not allowed to hold a negative quantity of either asset. As explained below in more detail, these two restrictions (on the amount of money invested and on the number of assets a trader can sell) impose a bound on the number of asset units that each individual trader can purchase and eventually hold.⁸

Markets clear when the aggregate demand for asset 1 precisely equals the aggregate demand for asset 2. We normalize the sum of the two asset prices to one, and focus on the price p of the asset paying in event E .

We assume that there is a continuum I of risk-neutral traders who aim to maximize their subjective expected wealth. Trader i maximizes $\pi_i w_i(E) + (1 - \pi_i) w_i(E^c)$, where π_i denotes the trader’s subjective belief. We now turn to the process that determines the trader’s subjective belief, π_i .

⁶The state of the world is given exogenously and cannot be affected by the traders. This assumption is realistic in the case of prediction markets on economic statistics, such as non-farm payroll employment. When applied to corporate decision making, prediction market traders might have incentives to manipulate the outcome. Ottaviani and Sørensen (2006) analyze outcome manipulation, disregarding the wealth effect on which we concentrate in this paper.

⁷For example, in the Iowa Electronic Markets each trader cannot invest more than \$500. Exemption from anti-gambling legislation is granted to small stake markets created for educational purposes.

⁸Our main result (Proposition 2) hinges on the property that this bound is endogenous to the model, because the number of assets each trader eventually holds depends on the market-clearing prices.

Initially, trader i has subjective prior belief q_i . Before trading, trader i privately observes signal s_i . Conditional on state $\omega \in \{E, E^c\}$, we let $f(s|\omega)$ denote the joint probability density of the signal vector $s = \{s_i\}_{i \in I}$.⁹ The likelihood ratio for signal realization s is defined as $L(s) = f(s|E)/f(s|E^c)$. The only constraint imposed on the signal distribution is that there is zero probability of fully revealing signals, so $L(s) \in (0, \infty)$ with probability one. If trader i observes the realized signal vector s , then by Bayes' rule the subjective posterior belief π_i satisfies

$$\frac{\pi_i}{1 - \pi_i} = \frac{q_i}{1 - q_i} L(s). \quad (1)$$

Hence, $L(s)$ is a sufficient statistic for the vector s .

For convenience, we normalize the aggregate endowment of assets to 1. The initial distribution of assets over individuals is described by the cumulative distribution function G . Thus, $G(q) \in [0, 1]$ denotes the share of all assets initially held by individuals with subjective prior belief less than or equal to q . We assume that G is continuous, and that G is strictly increasing on the interval where $G \notin \{0, 1\}$.¹⁰

We assume that the model (i.e., preferences, prior beliefs, and signal distributions) and the rationality of all traders are common knowledge. This means that all traders agree on the conditional distributions $f(s|\omega)$, even though they have heterogeneous prior beliefs. In the terminology of Milgrom and Stokey (1982), traders have concordant beliefs.¹¹

Discussion of Assumptions. Before proceeding, we discuss our main assumptions: heterogeneous priors and rational expectations equilibrium. In economics it is often assumed that individuals have a common prior belief. We have chosen to relax this assumption for the following two reasons:

- First, the common prior assumption is sensible when agents are dealing with commonly experienced events. Prediction markets deal instead with settings in which

⁹We do not assume that the signals are conditionally independent across traders. Indeed, a large number of conditionally independent signals would lead to full information revelation. By allowing for conditional dependence, our model encompasses the realistic scenario in which traders overall do not possess full information about the outcome.

¹⁰This assumption, that the priors are continuously distributed, is made in order to simplify the analysis, but is not essential for our underreaction result. See the discussion in footnote 15.

¹¹Varian (1989), Harris and Raviv (1993), and Kandel and Pearson (1995) relax this assumption by allowing traders to interpret a publicly observed signal differently. We refer to Morris (1994, 1995a) for a characterization of the general conditions for no-trade theorems when the interpretation of new information differs across traders.

traders are unlikely to have experienced similar events in the past. Given that in these settings there is no commonly agreed-upon probability from the outset, it is realistic to assume that the traders have heterogeneous prior beliefs.

- Second, the common prior assumption is not an implication of rational decision making. An alternative approach would have been to relax the rationality assumption by positing, for example, that the link between empirical frequencies and the subjects' common prior is systematically flawed. Instead, we remain within the classic rational framework and are agnostic on the link between the true empirical frequency and the priors. We will show that whatever the true probability is, a systematic bias arises in the market price's response to the information when traders have heterogeneous prior beliefs.

Another question is whether the rational expectations equilibrium is a good analytical tool when the priors are not common. We find it plausible that individuals make some inferences about the state from the realized price in this setting of asymmetric information. Even though these inferences need not be as correct in reality as assumed in the REE, it is a strength of our theory that it works under this narrow, standard assumption.¹²

3 Fully Revealing Rational Expectations Equilibrium

Traders are allowed to exchange assets with other traders in a competitive market. This section characterizes the fully revealing REE in this model.¹³ By normalization, the prices of the two assets sum to one, and we focus on the equilibrium determination of the price p for the asset that pays out in event E .

Solving the choice problem of the risk-neutral traders is straightforward. Suppose trader i has information with likelihood ratio L resulting in a posterior belief equal to π_i , and suppose that the market price is p . The subjective expected return on the asset that pays out in event E is $\pi_i - p$, while the other asset's expected return is $(1 - \pi_1) - (1 - p) =$

¹²Morris (1995b) shows that the REE concept in general relies on strong common knowledge assumptions. In the present setting, we are implicitly assuming that the heterogeneous prior distribution, G , is common knowledge. Such an assumption is no stronger than the usual REE assumption that traders commonly know each others' preferences, for here the subjective prior belief is just a parameter characterizing the individual preferences.

¹³We focus on the necessary properties of this equilibrium, while Radner (1979) discusses the sufficient conditions for its existence.

$p - \pi_i$. With the designer's constraint on asset portfolios, individual demand thus satisfies the following: if $\pi_i > p$, trader i exchanges the entire endowment of the E^c asset into $(1 - p)w_{i0}/p$ units of the E asset. The final portfolio is then w_{i0}/p units of the E asset and 0 of the E^c asset. Conversely, when $\pi_i < p$, the trader's final portfolio is 0 of the E asset and $w_{i0}/(1 - p)$ of the E^c asset. Finally, when $\pi_i = p$, the trader is indifferent between any trade.

The fully revealing REE price is a sufficient statistic for the likelihood ratio $L(s)$. For every L , trader i 's demand solves this trader's maximization problem, given belief $\pi_i(L)$ satisfying (1), and given market price $p(L)$. Market clearing requires that aggregate net trades are zero, or that the aggregate holding of each asset equals aggregate wealth (normalized to 1).

Proposition 1 *The fully revealing rational expectations equilibrium price, p , is the unique solution to the equation*

$$p = 1 - G\left(\frac{p}{(1-p)L + p}\right) \quad (2)$$

and is a strictly increasing function of the information realization L .

Parimutuel Interpretation. Given our focus on REE, the equilibrium is invariant with respect to the specific rules used for market trading. To illustrate this point, we now offer a parimutuel reinterpretation of our equilibrium. Suppose that the amount X_ω was bet on outcome $\omega \in \{E, E^c\}$ in a parimutuel betting market. Every unit bet on outcome E returns $1 + \rho$ units if E is true, where $X_E(1 + \rho) = X_E + X_{E^c} = 1$. The market's implied probability for outcome E is defined as $p = X_E/(X_E + X_{E^c}) = X_E = 1/(1 + \rho)$.

This parimutuel market has an REE which is equal to the rational expectations equilibrium above.¹⁴ Suppose that the implied market probability is p . Again solving the utility maximization problem of trader i , we find the following demand: trader i will bet the amount w_{i0} on event E if $\pi_i > p$, and bet the amount w_{i0} on event E^c if instead $\pi_i < p$. Note that if p is a fully revealing REE price, then market clearing implies that

$$X_E = 1 - G\left(\frac{p}{(1-p)L + p}\right) = p$$

¹⁴As argued by Ottaviani and Sørensen (2005), the assumption that the market reaches an REE might be unrealistic if traders take simultaneous positions in a parimutuel market. The final parimutuel prices are determined on the basis of the positions taken by all traders, who do not observe those prices when they commit to these positions. Nevertheless, we focus here on the benchmark case of REE because this represents the most favorable scenario for information aggregation.

from which it follows that, indeed, $p = X_E$.

4 Underreaction to Information

Using Bayes' rule (1), we can always interpret the price as the posterior belief of a hypothetical market agent with prior belief $p/[(1-p)L+p]$. This implied "market prior" then may be interpreted as an aggregate of the heterogeneous subjective prior beliefs. Our main result is that this aggregate market belief depends in a systematic fashion on the pre-trade information. This means that the aggregation of beliefs cannot be separated from the market's ability to aggregate information.

Proposition 2 *If beliefs are truly heterogeneous, i.e. $q_i \neq q_j$ for some pair of traders, then the market prior*

$$\frac{p}{(1-p)L+p}$$

is strictly decreasing in L .

The arrival of more favorable pre-trade information yields a higher market price that nevertheless underreacts to the information. Consider the inference of the market designer or of an outside observer who desires to extract information from the market price. Given any fixed prior belief q of this observer, the posterior probability $\pi(L)$ for the event E given revealed information L satisfies (1), or

$$\log\left(\frac{\pi(L)}{1-\pi(L)}\right) = \log\left(\frac{q}{1-q}\right) + \log L. \quad (3)$$

The expression on the left hand side is the posterior log-likelihood ratio for event E . It moves one-to-one with changes in $\log L$. Proposition 2 implies that the observer's belief reacts more than the price:

Proposition 3 *If beliefs are truly heterogeneous, then for any two different information realizations $L > L'$,*

$$\log\left(\frac{\pi(L)}{1-\pi(L)}\right) - \log\left(\frac{\pi(L')}{1-\pi(L')}\right) > \log\left(\frac{p(L)}{1-p(L)}\right) - \log\left(\frac{p(L')}{1-p(L')}\right) > 0. \quad (4)$$

To understand the intuition for this underreaction result, consider what happens when the traders have information more favorable to event E (corresponding, say, to the Democratic candidate winning the election), i.e., when L is higher. According to (2), the price

of the E asset, p , is clearly higher when L is higher. Now, this means that traders who are optimistic about a Democratic victory can buy fewer units of asset E , because the bound w_{i0}/p is decreasing in p . On the other hand, traders who are pessimistic about a Democratic victory can buy more units of asset E^c , which they want to buy. If all the traders who were purchasing E before the increase in L were still purchasing E at the higher price that results with higher L , there would be insufficient demand for E . Similarly, there would also be excess demand for E^c . To balance the market it is necessary that some traders who were betting on the Republican candidate before now change sides and put their money on the Democratic candidate. In the new equilibrium, the price must change to move traders from the pessimistic to the optimistic side. Thus, the indifferent trader who determines the equilibrium price is someone with a more pessimistic prior belief about Democratic victory. Hence, although the price p rises with the information L , it rises more slowly than a posterior belief, because of this negative effect on the market prior.¹⁵

The underreaction result derived in this section is driven by the institutional restriction on the amount of money invested (see footnote 7) and, therefore, on the number of assets a trader can sell. In turn, this restriction imposes a bound on the number of assets that each individual can purchase and eventually hold. The result hinges on the fact that this bound (equal to w_{i0}/p) is inversely related to the equilibrium price.¹⁶

Proposition 3 offers a new informational explanation of the favorite-longshot bias, a fact that is widely documented in the empirical literature on betting markets (see Thaler and Ziemba, 1988, and Snowberg and Wolfers, 2005).¹⁷ The bias says that outcomes favored by the market occur more often than if the market price is interpreted as a probability,

¹⁵Our assumption that prior beliefs are distributed continuously in the population is not essential for the result. If the population is finite, or there are gaps in the distribution, then there can be ranges of information, L , over which the equilibrium price is constant. In that case, the rational expectations equilibrium cannot fully reveal L , but can reveal the information needed for the proper allocation of the assets. Underreaction would still hold, but only relative to the limited information revealed by the equilibrium price.

¹⁶Suppose, instead, that the market designer were to impose a direct cap on the number of assets that each trader can buy, rather than on the budget each trader can invest. Then, for a large range of information realizations, the same fixed set of optimists (or pessimists) would buy the full allowance of the E (or E^c) asset. Since the sets of optimists and pessimists are constant, there would be no underreaction. However, typically in prediction markets there is an upper bound on the traders' budget, rather than on the number of nominal positions they can take.

¹⁷In Ottaviani and Sørensen (2005), we explore a fundamentally different explanation of the favorite-longshot bias in parimutuel betting. There, we explicitly assume that simultaneous betting prohibits the adjustment of market odds to the REE. The bias in that paper is generated by the fact that the information revealed in a Bayes-Nash equilibrium is not fully incorporated in the final prices.

and conversely, longshots win less frequently than suggested by the market price. To see how this effect arises in our context, consider again an outside observer of the market with fixed prior belief q .

Proposition 4 *If beliefs are truly heterogeneous, there exists a market price $p^* \in [0, 1]$ with the property that $p(L) > p^*$ implies $\pi(L) > p(L)$, and $p(L) < p^*$ implies $\pi(L) < p(L)$.*

There is a threshold level, p^* , for the realized market price, such that a market price below p^* classifies event E as a longshot and a market price above p^* makes E a favorite. The observer expects longshots to occur less often than indicated by the market price, and favorites to occur more often.

Example. Suppose that the distribution of subjective prior beliefs over the interval $[0, 1]$ is $G(q) = q^\beta / [q^\beta + (1 - q)^\beta]$, where $\beta > 0$ is a parameter that measures the concentration of beliefs. The greater is β , the less spread (according to Rothschild and Stiglitz, 1970) is this symmetric belief distribution around the average belief $q = 1/2$. For $\beta = 1$ beliefs are uniformly distributed, as $\beta \rightarrow \infty$ beliefs become concentrated near $1/2$, and as $\beta \rightarrow 0$ beliefs are maximally dispersed around the extremes of $[0, 1]$. The equilibrium condition (2) becomes

$$\log \left(\frac{p}{1-p} \right) = \log \left(\frac{1 - G \left(\frac{p}{(1-p)L+p} \right)}{G \left(\frac{p}{(1-p)L+p} \right)} \right) = \beta \log \left(\frac{(1-p)L}{p} \right),$$

so that the market price $p(L)$ satisfies the linear relation

$$\log \left(\frac{p(L)}{1-p(L)} \right) = \frac{\beta}{1+\beta} \log L.$$

Hence, $\beta/(1+\beta) \in (0, 1)$ measures the extent to which the price reacts to information. Price underreaction is minimal when β is very large, corresponding to the case with nearly homogeneous beliefs. Conversely, there is an arbitrarily large degree of underreaction when beliefs are maximally heterogeneous (i.e. β is close to zero).

Assume that a market observer's prior is $q = 1/2$ for event E , consistent with a symmetric market price of $p(1) = 1/2$ in the absence of additional information. The posterior belief associated with price p then satisfies

$$\log \left(\frac{\pi(L)}{1-\pi(L)} \right) = \log L = \frac{1+\beta}{\beta} \log \left(\frac{p(L)}{1-p(L)} \right).$$

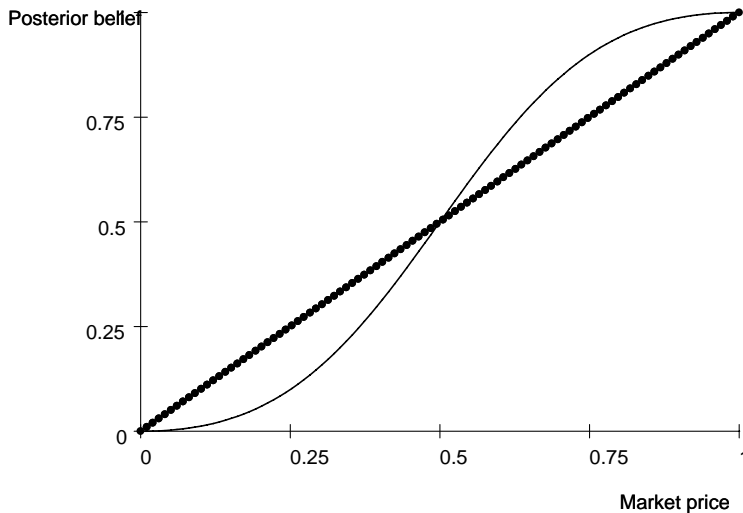


Figure 1: This plot shows the posterior probability for event E as a function of the market price p for the E asset, when the prior beliefs of the risk-neutral traders are uniformly distributed ($\beta = 1$ in the example). The dotted line is the diagonal.

As illustrated in Figure 1 for the case with uniform beliefs ($\beta = 1$), the market price overstates the winning chance of a longshot and understates the winning chance of a favorite by a factor of two.

5 Risk Aversion

So far we have assumed that each individual trader is risk neutral, and thus ends up taking as extreme a position as possible on either side of the market. Now, we show that our result extends nicely to risk averse individuals, under the plausible assumption that traders' absolute risk aversion is decreasing in wealth, even when no exogenous bounds are imposed on the amounts that traders can invest.

Model. Realistically, suppose that each trader is endowed with the same number w_0 of each asset. To properly capture the effect of risk aversion, we suppose that each trader i is also characterized by an initial, state-independent level W_i of additional wealth.¹⁸ Trader i maximizes subjective expected utility of final wealth, $\pi_i u_i(w_i(E)) + (1 - \pi_i) u_i(w_i(E^c))$,

¹⁸See Musto and Yilmaz (2003) for a model in which, instead, traders are subject to wealth risk, because they are differentially affected by the redistribution associated with different electoral outcomes.

where π_i is the trader's subjective belief. We suppose that u_i is twice differentiable with $u'_i > 0$ and $u''_i < 0$, and satisfies the DARA assumption that the Arrow-Pratt coefficient of absolute risk aversion, $-u''_i/u'_i$, is weakly decreasing in wealth, w_i . The cumulative distribution function G again describes the distribution of subjective beliefs, and it is assumed to satisfy the same properties as before. Private information is distributed as before.

Let Δx_i denote the choice variable of trader i , such that $p\Delta x_i$ units of the E^c asset are exchanged for $(1-p)\Delta x_i$ units of the E asset. Note that this is a zero net value trade, since the asset sale generates $(1-p)p|\Delta x_i|$ of cash that is spent on buying the other asset. The final wealth levels in the two states are

$$w_i(E) = W_i + w_0 + (1-p)\Delta x_i \quad (5)$$

and

$$w_i(E^c) = W_i + w_0 - p\Delta x_i. \quad (6)$$

The trading constraints translate into the requirement that $\Delta x_i \in [-w_0/(1-p), w_0/p]$.

Equilibrium. Given price p and posterior belief π_i , trader i chooses Δx_i to maximize posterior expected utility $\pi_i u_i(w_i(E)) + (1-\pi_i)u_i(w_i(E^c))$. If the trading constraints are not binding, then the choice satisfies the necessary first-order condition

$$\frac{\pi_i}{1-\pi_i} \frac{u'_i(w_i(E))}{u'_i(w_i(E^c))} = \frac{p}{1-p}. \quad (7)$$

The endogenously determined net trade Δx_i then satisfies the standard condition that the marginal rate of substitution equals the price ratio. In particular, a trader with $\pi_i = p$ will choose $w_i(E) = w_i(E^c)$, corresponding to $\Delta x_i = 0$. A trader with a posterior belief π_i above the price p will choose $\Delta x_i > 0$. Note that the trading constraint does not bind, unless the trader is nearly risk neutral or the difference between π_i and p is sufficiently large. In analogy with Proposition 1 we have:

Proposition 5 *There exists a unique fully-revealing rational expectations equilibrium. The price, p , is a strictly increasing function of the information realization L .*

Belief Aggregation with CARA. Suppose first that the traders have constant absolute risk aversion (CARA) utility functions, with heterogeneous degrees of risk aversion, such that $u_i(w) = -\exp(-w/t_i)$, where $t_i > 0$ is the constant coefficient of risk tolerance, the inverse of the coefficient of absolute risk aversion. Denoting the relative risk tolerance of trader i in the population by $\tau_i = t_i / \int_0^1 t_j dG(q_j)$, we have:

Proposition 6 *Suppose that the traders have CARA preferences and heterogeneous beliefs. Define an average prior belief q by*

$$\log \frac{q}{1-q} = \int_0^1 \tau_i \log \frac{q_i}{1-q_i} dG(q_i),$$

and for each individual let

$$d_i^* = t_i \log \left(\frac{q_i - qq_i}{q - qq_i} \right). \quad (8)$$

Suppose that $1 + w_0 / \inf_i d_i^* < w_0 / \sup_i d_i^*$. When the realized information L is in the range satisfying

$$1 + \frac{w_0}{\inf_i d_i^*} \leq \frac{qL}{qL + 1 - q} \leq \frac{w_0}{\sup_i d_i^*} \quad (9)$$

then the fully-revealing REE price satisfies Bayes' rule with market prior q , i.e., $p(L) = qL / (qL + 1 - q)$. When L falls outside this range, the price underreacts to changes in L compared to Bayes' rule.

Risk aversion allows for the possibility that no trader meets the constraint. This is more likely to happen when w_0 is large, as also suggested by condition (9). With CARA preferences and when no trader is constrained, trader i chooses net demand d_i^* in equilibrium. The market price thus behaves as a posterior belief and there is no bias.

This result for the case with CARA preferences and unconstrained positions is consistent with Varian's (1989) analysis. As shown by Wilson (1968), under CARA preferences the traders' heterogeneous priors can be aggregated. Once private information is added, the price then behaves as a posterior belief.

Underreaction with DARA. We have seen that CARA preferences lead to an unbiased price reaction to information when the trading constraints are not binding in equilibrium. Now we verify that, for strictly DARA preferences, a bias arises in the price whether traders are constrained or not. When L rises, the rising equilibrium price yields a negative

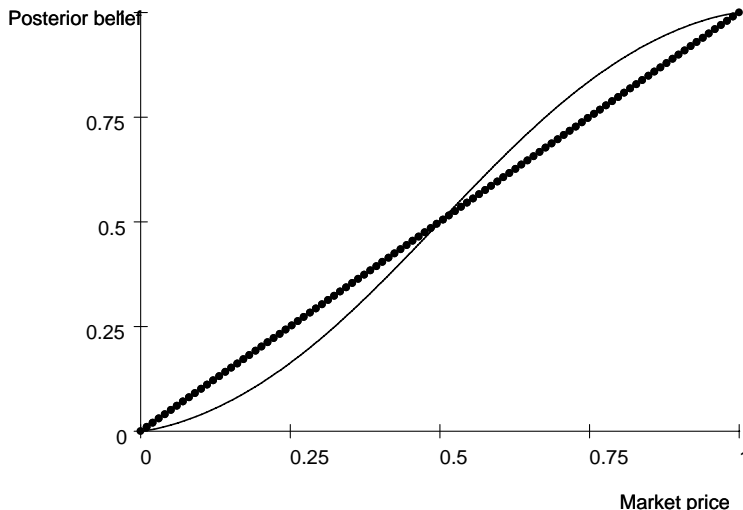


Figure 2: In the example with uniformly distributed risk-averse traders, this plot shows the posterior chance for event E as a function of the E -asset's market price p . The dotted line is the diagonal.

wealth effect on any optimistic individual (with $\pi_i > p$) who is a net demander ($\Delta x_i > 0$). Conversely, pessimistic traders benefit from the price increase. With DARA preferences, the wealth effect implies that optimists become more risk averse while pessimists become less risk averse. An increase in L thus must shift weight from optimists to pessimists when the market price is calculated as a belief average. Hence, although the price rises with L , it rises less fast than a posterior belief, because the weight is shifted more to pessimists when information is more favorable.

Proposition 7 *Suppose that beliefs are truly heterogeneous and that all individuals have strictly decreasing absolute risk aversion. Then the market price underreacts to information, as for any $L \neq L'$,*

$$\left| \log \left(\frac{\pi(L)}{1 - \pi(L)} \right) - \log \left(\frac{\pi(L')}{1 - \pi(L')} \right) \right| > \left| \log \left(\frac{p(L)}{1 - p(L)} \right) - \log \left(\frac{p(L')}{1 - p(L')} \right) \right|.$$

Appendix B provides a graphical illustration of this result for an economy with two types of traders and no exogenous bounds on the wealth traders can invest.

Example. Suppose that prior beliefs are uniformly distributed over the interval $[0, 1]$, with $G(q) = q$, and traders have logarithmic preferences, $u_i(w) = \log w$, satisfying the

DARA assumption. In order to highlight the difference between Propositions 3 and 7, namely the inclusion of individuals with an interior solution to their maximization problem, we again remove completely the exogenous bound on the wealth invested. The individual demand function solving (7) is $\Delta x_i = (W_i + w_0)(\pi_i - p) / [p(1 - p)]$. The equilibrium price is then an average of the posterior beliefs,

$$p(L) = \int_0^1 \pi(L) dq = \int_0^1 \frac{qL}{qL + (1 - q)} dq. \quad (10)$$

For $L \neq 1$, integration by parts of (10) yields $p(L) = L(L - 1 - \log L) / (L - 1)^2$. If we keep fixed $p(1) = \int_0^1 q dq = 1/2$ as the prior belief of the outside observer, Figure 2 illustrates the favorite-longshot bias.

6 Contribution to Literature

This paper bridges the gap between the literature on betting with heterogenous beliefs and the literature on REE. In an influential paper on parimutuel betting, Ali (1977) formulates a static model in which risk-neutral bettors with limited wealth have heterogenous beliefs about a binary outcome. He shows that if the bettor with the median belief thinks that one outcome (defined to be the favorite) is more likely, then the equilibrium fraction of parimutuel bets on this outcome will be lower than the belief of the median bettor. By identifying the belief of the median bettor as the correct benchmark for the empirical probability, Ali concludes that the favorite is underbet as compared to the longshot. Following Ali, the fledgling literature on belief aggregation in prediction markets (Manski, 2006, Gjerstad, 2005, and Wolfers and Zitzewitz, 2005) analyzes the relation between the equilibrium price and the average belief of traders, depending on the traders' preferences for risk. In this literature, traders do not make any inference from market prices.

A key ingredient of Ali's explanation for the favorite-longshot bias is the assumption that the belief of the median bettor corresponds to the empirical probability. But if the traders' beliefs really have information content (about the empirical probability of the state), then their positions should depend on the information about these beliefs that is contained in the market price. This question underlies the rational expectations critique of the Walrasian approach to price formation with heterogeneous beliefs (see e.g. the discussion in Chapter 1 of Grossman, 1989).

The REE literature typically assumes that traders have a common prior belief, so that differences in beliefs across agents only can be attributed to private information. In the context of prediction markets in which there is no liquidity motive for trade, the REE price therefore reveals (a sufficient statistic of) all the information possessed by the traders (Grossman, 1976). After the market opens, the posterior beliefs of all traders would become identical if traders had a common prior—and because the original allocation is Pareto optimal there would be no active trading in equilibrium (Milgrom and Stokey, 1982).

In this paper, we retain the REE assumption of private information and informational inference from prices, but depart from the common prior assumption. Traders have different priors and they trade actively on the basis of those beliefs, as in the literature on prediction markets. But, unlike the prediction market literature, here we do not base our explanation of the favorite-longshot bias on how the equilibrium price relates to a summary statistic of the traders' beliefs. Instead, we conduct our comparative statics exercise with respect to the information that is revealed in the market, holding constant the distribution of the prior beliefs. Our results indicate that the appropriate method for averaging prior beliefs interacts with the realized information.

While the literature on the no-trade theorem focuses on the effect that the arrival of new private information has on an initial equilibrium allocation, we characterize here the REE that results when privately informed traders with heterogeneous priors are asked to trade once. This market-opening scenario is particularly relevant for prediction markets, which are often created with the express purpose of aggregating information and beliefs already held by the traders. Therefore our contribution to the REE literature is the characterization of the interaction of heterogeneous beliefs and information in the *first* round of trade. Our analysis complements Milgrom and Stokey (1982), who focus instead on the second round of trade that follows the arrival of new information.

Following Miller (1977) and Harrison and Kreps (1978), there is an extensive literature in which there is active trade based on heterogeneous prior beliefs.¹⁹ Our model departs from this finance literature in two key ways. First, risk-neutral traders are not wealth-constrained in Harrison and Kreps (1978), but the market is incomplete, so that traders can take positions on only one side of the market. Thus in their model, the entire net

¹⁹See also Morris (1996) and Scheinkman and Xiong's (2004) survey.

supply of the asset is held by the most optimistic trader, whose belief determines the market price. This trader’s belief then can be taken as the market belief, so there is no informational favorite-longshot bias. In our model, markets instead are complete, but the wealth that can be invested is constrained, as is natural in the context of prediction markets. Second, traders do not have private information in most of this literature, with the exception of the models discussed in the next two paragraphs.²⁰

A handful of papers on betting study the interaction of private information with heterogeneous priors. Notably, Shin (1991 and 1992) considers asset pricing by an uninformed monopolist bookmaker in a market in which some traders take positions on the basis of their beliefs while others are perfectly informed about the outcome. Morris (1997) characterizes the equilibrium in a game-theoretic model of bilateral betting with asymmetric information and heterogeneous priors. As in these contributions, in our model heterogeneous beliefs coexist with private information, but we focus here on the competitive equilibrium.

As already discussed in the introduction, we follow Milgrom and Stokey (1982) and Varian (1989) by allowing for the coexistence of private information with heterogeneous priors. We add to Milgrom and Stokey (1982) the characterization of how equilibrium prices react to changes in information in the first round of trade. Our underreaction result crucially depends on wealth effects, which are absent when traders’ positions are not exogenously bounded or when traders have CARA preferences, as in Varian (1989).

7 Conclusion

Prediction markets are special financial markets in which traders’ endowments are constant with respect to different outcome realizations. Our model of these markets has three key ingredients: heterogeneous priors, private information, and limited positions. First, market participants do not share a common prior belief because there is genuine uncertainty about the underlying event. Second, the market designer typically is interested in aggregating the private information of the participants—so it is natural to allow for the presence of this information. Third, prediction market traders are allowed to wager only a limited amount of wealth, so their positions are bounded. The three ingredients of our model

²⁰On the optimal allocation of risk with heterogeneous prior beliefs and risk preferences but without private information, see Wilson’s (1968) classic contribution and the recent developments by Gollier (2006).

are inspired by the special features of prediction markets, but are also relevant for more general financial markets, as Hong and Stein (forthcoming) argue.

In this setting, the REE price reveals all the traders' private information, but underreacts to information. Our result is driven by a wealth effect arising because traders with heterogeneous beliefs take speculative net positions. More generally, according to Proposition 7, underreaction holds under the realistic assumption of decreasing absolute risk aversion preferences, without the need to impose exogenous bounds on positions.

We conclude by discussing three extensions of the model. First, what happens with more than two states? By focusing on markets for a binary event we have managed to characterize the reaction of REE prices to information for *general risk preferences*. In contrast, since Grossman (1976) most of REE literature obtain positive characterizations of equilibrium prices by restricting attention to CARA preferences and normally distributed returns. With more than two states and heterogeneous priors, wealth effects introduce an additional channel through which information affects prices. As a result, information about the likelihood of a state relative to a second state can impact the price of the asset for a third state. The adjustment related by this contagion effect can induce overreaction to information in the relative prices of the assets for the first two states.

Second, our results have interesting implications for price dynamics, in a multi-period trading environment with sequential arrival of information à la Milgrom and Stokey (1982). In current research, we show that the combination of heterogeneous priors with wealth effects yields a simple mechanism for price momentum: price changes are positively correlated with the opening price. Intuitively, the first round's information is swamped by the information revealed in subsequent rounds, and hence over time the price comes to approximate the correctly updated prior belief. Thus, the initial underreaction must be followed by a correcting price momentum.

Finally, we are currently investigating the more general implications of our underreaction result with risk averse traders and unconstrained positions, in the context of broader financial markets in which idiosyncratic uncertainty generates gains from trade. Extending Rubinstein's (1974) results to allow for private information, it can be shown that there is no underreaction bias when traders have common prior but heterogeneous endowments, provided they have hyperbolic absolute risk aversion with common cautiousness parameters. For example, if all traders have logarithmic preferences (our leading DARA example

analyzed in Section 5) the price behaves as a posterior belief when traders have heterogeneous endowments, but common prior. Intuitively, all liquidity-motivated traders take more (or less) extreme positions, as the available information varies. With heterogeneous beliefs, instead, optimists (who buy) buy less on favorable information, while pessimists (who sell) sell more, so that the price must equilibrate in a direction contrary to information. Thus, heterogeneity of priors is important for underreaction and cannot be replaced by heterogeneity in endowments across individuals.

Appendix A: Proofs

Proof of Proposition 1. For a given likelihood ratio L , the prior of an individual with posterior belief π_i is, using (1), $q_i = \pi_i / [(1 - \pi_i)L + \pi_i]$. The E^c asset is demanded in amount $w_{i0} / (1 - p)$ by every individual with $\pi_i < p$, or equivalently $q_i < p / [(1 - p)L + p]$. The aggregate demand for this asset is then $G(p / [(1 - p)L + p]) / (1 - p)$. In equilibrium, this aggregate demand is equalized to the aggregate supply, equal to 1, resulting in equation (2).

We complete the proof by noting that the price defined by (2) reveals L , because it is a strictly increasing function of L . The left-hand side of (2) is a strictly increasing continuous function of p , which is 0 when $p = 0$ and 1 when $p = 1$. For any $L \in (0, \infty)$, the right-hand side is a weakly decreasing continuous function of p , for the cumulative distribution function G is non-decreasing. The right-hand side is equal to 1 at $p = 0$, while it is 0 at $p = 1$. Thus, there exists a unique solution, such that $G \notin \{0, 1\}$. When L rises, the left-hand side is unaffected, while the right-hand side rises for any p , strictly so near the solution to (2) by the assumptions on G . Hence, the solution p must rise with L . \square

Proof of Proposition 2. The market price p is the posterior belief given information L and market prior belief $p / [(1 - p)L + p]$. When L increases, so does p . By equation (2), when p increases, $p / [(1 - p)L + p]$ must fall, because the cumulative distribution function G is non-decreasing. \square

Proof of Proposition 3. By Proposition 1, $p(L) > p(L')$. By (3), (4) is equivalent to

$$\log \left(\frac{p(L)}{1 - p(L)} \right) - \log \left(\frac{p(L')}{1 - p(L')} \right) < \log L - \log L',$$

or

$$\frac{p(L)}{1 - p(L)} \frac{1}{L} < \frac{p(L')}{1 - p(L')} \frac{1}{L'}.$$

Using the strictly increasing transformation $z \rightarrow z / (1 + z)$ on both sides of this inequality, it is equivalent to

$$\frac{p(L)}{[1 - p(L)]L + p(L)} < \frac{p(L')}{[1 - p(L')]L' + p(L')},$$

which is true by Proposition 2. \square

Proof of Proposition 4. By Proposition 3, the function

$$\Psi(L) = \log\left(\frac{\pi(L)}{1 - \pi(L)}\right) - \log\left(\frac{p(L)}{1 - p(L)}\right)$$

is strictly increasing in L . Hence, one of the following three cases will hold. In the first case, there exists an $L^* \in (0, \infty)$ such that $\Psi(L)$ is negative for $L < L^*$ and positive for $L > L^*$ — in this case, the result follows with $p^* = p(L^*)$. In the second case, $\Psi(L)$ is negative for all L , and the result holds for $p^* = 1$. In the third case, $\Psi(L)$ is positive for all L , and the result is true with $p^* = 0$. \square

Proof of Proposition 5. The individual trader solves the problem

$$\max_{\Delta x_i \in [-w_0/(1-p), w_0/p]} \pi_i u_i(w_i(E)) + (1 - \pi_i) u_i(w_i(E^c)).$$

Strict concavity of u_i ensures that the maximand Δx_i is unique. By the Theorem of the Maximum, Δx_i is a continuous function of π_i and p . We first show that the optimizer Δx_i is strictly decreasing in p and weakly increasing in π_i , strictly so when $\Delta x_i \in (-w_0/(1-p), w_0/p)$.

The constraint set $[-w_0/(1-p), w_0/p]$ does not depend on π_i and falls in Veinott's set order when p rises. The trader's objective function $\pi_i u_i(W_i + w_0 + (1-p)\Delta x_i) + (1 - \pi_i) u_i(W_i + w_0 - p\Delta x_i)$ has first derivative

$$\pi_i (1 - p) u_i'(w_i(E)) - (1 - \pi_i) p u_i'(w_i(E^c))$$

with respect to Δx_i . Since $u_i' > 0$, the cross-partial of the objective with respect to the choice variable Δx_i and the exogenous π_i is strictly positive, and hence Δx_i is weakly increasing in π_i , strictly so when the unique Δx_i optimizer satisfies (7). A sufficient condition for a strictly negative cross-partial with respect to Δx_i and p is

$$\Delta x_i [\pi_i (1 - p) u_i''(w_i(E)) - (1 - \pi_i) p u_i''(w_i(E^c))] > 0. \quad (11)$$

Using the first order condition for optimality, the second factor of (11) is positive if and only if

$$-\frac{u_i''(w_i(E^c))}{u_i'(w_i(E^c))} > -\frac{u_i''(w_i(E))}{u_i'(w_i(E))}.$$

By the DARA assumption, this inequality holds if and only if $w_i(E) > w_i(E^c)$, i.e., $\Delta x_i > 0$. Hence the cross-partial is strictly negative, and it follows that Δx_i is strictly decreasing in p .

Equilibrium is characterized by the requirement that the aggregate purchase of asset E must be zero, i.e., $\int_0^1 \Delta x_i(p, q_i, L) dG(q_i) = 0$. When $p = 0$, every trader has $\pi_i > p$ and hence $\Delta x_i > 0$, while the opposite relation holds when $p = 1$. Individual demands are continuous and strictly decreasing in p , so there exists a unique equilibrium price in $(0, 1)$. When L is increased, $\pi_i(L)$ rises, and hence Δx_i rises for every trader. The price must then be strictly increased, in order to restore equilibrium. Finally, since the equilibrating price p is thus a strictly increasing function of L , the equilibrium price schedule is fully revealing. \square

Proof of Proposition 6. Suppose for a moment that no trader is constrained in equilibrium. The necessary and sufficient first order condition (7) for the unconstrained optimum is solved by

$$\Delta x_i = t_i \log \left(\frac{1 - p(L)}{p(L)} \frac{\pi_i(L)}{1 - \pi_i(L)} \right). \quad (12)$$

Market clearing occurs when $\int_0^1 \Delta x_i dG(q_i) = 0$. By (12) and using $\pi_i(L) / (1 - \pi_i(L)) = q_i L / (1 - q_i)$ this is solved by $p(L) = qL / (qL + 1 - q)$. Inserting this market price in the individual demand (12), the resulting equilibrium demand is d_i^* , as given in (8). This analysis describes the equilibrium, provided no individual is constrained. The lower bound constrains no individual when $0 > \inf_i d_i^* \geq -w_0 / (1 - p(L))$, or equivalently $p(L) \geq 1 + w_0 / \inf_i d_i^*$. Likewise, the upper bound is equivalent to $p(L) \leq w_0 / \sup_i d_i^*$.

When a positive mass of traders are constrained, the bias follows from the argument of Proposition 7 reported below. \square

Proof of Proposition 7. The result follows as in the proof of Proposition 3, once we establish that $\log[p(L) / (1 - p(L))] - \log(L)$ is strictly decreasing in L . Suppose, for a contradiction, that $\log[p(L) / (1 - p(L))] - \log(L)$ is non-decreasing near some L . Traders at the boundary $\Delta x_i = -w_0 / (1 - p)$ have their demand decreasing in p , and hence $d\Delta x_i / dL < 0$. Likewise, $d\Delta x_i / dL < 0$ at the other boundary $\Delta x_i = w_0 / p$. We will show that the same effect holds for traders satisfying (7). Since market clearing $\int_0^1 \Delta x_i(p(L), q_i, L) dG(q_i) = 0$ implies $\int_0^1 d\Delta x_i(p, q_i, L) / dL dG(q_i) = 0$, we will then have a contradiction.

Since $\log[\pi_i(L) / (1 - \pi_i(L))] - \log(L)$ is constant, (7) implies that $u'_i(w_i(E)) / u'_i(w_i(E^c))$ is non-decreasing in L . Using the expressions for the final wealth levels (5) and (6), non-

negativity of the derivative of $u'_i(w_i(E))/u'_i(w_i(E^c))$ implies that

$$u''_i(w_i(E)) u'_i(w_i(E^c)) \left[(1-p) \frac{d\Delta x_i}{dL} - \Delta x_i \frac{dp}{dL} \right] \geq -u''_i(w_i(E^c)) u'_i(w_i(E)) \left[p \frac{d\Delta x_i}{dL} + \Delta x_i \frac{dp}{dL} \right].$$

The second derivative of the utility function is negative, so this implies

$$\frac{d\Delta x_i}{dL} \leq \Delta x_i \frac{dp}{dL} \frac{u''_i(w_i(E)) u'_i(w_i(E^c)) - u''_i(w_i(E^c)) u'_i(w_i(E))}{(1-p) u''_i(w_i(E)) u'_i(w_i(E^c)) + p u''_i(w_i(E^c)) u'_i(w_i(E))}. \quad (13)$$

On the right-hand side of (13), $dp/dL > 0$ by Proposition 5, and the denominator is negative. Recall that $\Delta x_i > 0$ if and only if $w_i(E) > w_i(E^c)$. By DARA, this implies that

$$-\frac{u''_i(w_i(E))}{u'_i(w_i(E))} < -\frac{u''_i(w_i(E^c))}{u'_i(w_i(E^c))}$$

or that the numerator is positive. Likewise, when $\Delta x_i < 0$, the numerator is negative. In either case, the right-hand side of (13) is strictly negative. Hence, $d\Delta x_i/dL < 0$ for every trader who satisfies the first-order condition (7). \square

Appendix B: Edgeworth Box Illustration

We now illustrate the logic of Proposition 7 for an economy with two types of traders, with prior beliefs $q_1 < q_2$. As represented in Figure 3, the Edgeworth box is a square because there is no aggregate uncertainty. The initial endowment, e , lies on the diagonal, being the same in the two events: $w_i(E) = W_i + w_0 = w_i(E^c)$.

The strictly risk averse traders have strictly convex indifference curves (not drawn to avoid cluttering the picture). The slope of the indifference curves at any safe allocation is $-\pi_i/(1 - \pi_i) = -q_i L/(1 - q_i)$. Thus, for any allocation along the diagonal, the indifference curve of trader 2 (optimist) is steeper than that of trader 1 (pessimist).

We focus on interior equilibria in which the exogenous trading constraints are not binding. When information L is available and revealed, the marginal rates of substitution of the two traders are equalized at the equilibrium allocation, w^* . Thus, the equilibrium allocation must lie above the diagonal, so that the optimistic trader 2 is a net buyer of the E asset.

How is the equilibrium affected by an exogenous change in information to $L' > L$? As a result of this change in information, indifference curves become steeper by a factor of L'/L . For the sake of argument, suppose that the price were to change without any underreaction from the original $p(L)$ to the Bayes-updated $p' = p(L) L' / (p(L) L' + (1 - p(L)) L)$. At allocation w^* , the marginal rates of substitutions are still equalized, and the straight line dividing the two traders' preferred sets has slope $-p'/(1 - p')$. However, since $p' > p$, this straight line passes through the diagonal below the initial endowment point, proving that w^* cannot be the new equilibrium allocation. Maintaining price p' , the new budget line must pass through the initial endowment. As this new budget line is further up on the diagonal compared to the one passing through w^* , we see illustrated here the positive wealth effect for the pessimistic trader 1 and the negative wealth effect for the optimistic trader 2.

We now turn to the implication of this wealth effect. Given price p' , the choice w_1^* would be optimal for trader 1, if the true budget line were passing through w^* . This point lies above the diagonal in the Edgeworth box. Now, as it is well known since Arrow (1965), DARA implies that the wealth expansion paths diverge from the diagonal. The richer trader 1 demands a riskier bundle (further away from the diagonal) by increasing $w_1(E^c) - w_1(E)$, whereas the poorer trader 2 demands a safer bundle (closer to the

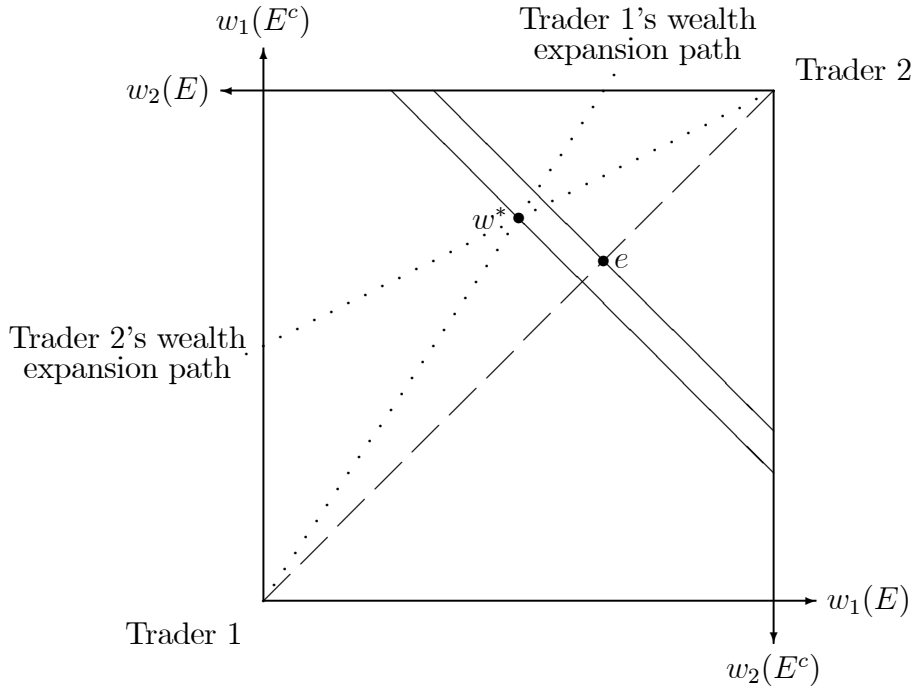


Figure 3: Edgeworth box representation of the underreaction result for the case with DARA preferences and interior solution. The wealth expansion paths are linear for the special case with logarithmic preferences.

diagonal) by decreasing $w_2(E) - w_2(E^c)$.²¹ Therefore p' cannot be an equilibrium after the exogenous information change. To reach an equilibrium both traders must shift their portfolio towards lower $w_i(E^c) - w_i(E)$ so as to eliminate the excess demand for asset E^c as well as the excess supply for asset E . This is achieved by a relative reduction in the price for asset E , so that $p(L') < p'$. Thus, prices must underreact to information.

²¹The trader's choice problem can be reformulated with a safe asset, always paying 1, and a risky asset paying 1 in E^c and -1 in E . The richer trader demands more units of the risky asset, and hence $|w(E^c) - w(E)|$ rises.

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