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*Publication date:*
1998

*Document version*
Early version, also known as pre-print

*Citation for published version (APA):*
McIntosh, J. P. (1998). *Duality Theory and the Consistent Estimation of Technological Parameters: Why Cost Function Estimation Can Be Wrong*. Department of Economics, University of Copenhagen.

# Discussion Papers

Department of Economics

University of Copenhagen



No. 98-01


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http://www.econ.ku.dk



ISSN: 1601-2461 (online) ISSN 0902-6452 (print)

# Duality Theory And The Consistent Estimation Of Technological Parameters: Why Cost Function Estimation Can Be Wrong

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December 15, 1997

## Abstract

In this article we show that technological parameter estimates obtained by estimating a cost function that is derivable as the dual of a production function can be biased and inconsistent if the stochastic structure of the model arises from certain types of behavioural assumptions made about rational agents. We consider a speci¯c example in which ¯rms are uncertain about prices. We show that when actual prices di®er from expected prices and ¯rms have to make decisions on the basis of their expectations, the inherited stochastic speci¯cation of the dual system is highly non-linear in the disturbance terms making consistent parameter estimation impossible by conventional methods. This is demonstrated by a Monte Carlo simulation study of two text-book examples using synthetic data. It is also shown that this type of result can arise when the researcher derives the error structure from the assumption that agents make optimization errors.

[1] From August 1, 1997 to December 31, 1997

# 1 Introduction[2]

It has been a tradition in the design of econometric models to use a two stage procedure. The ¯rst stage speci¯es a set of relationships, often a set of ¯rst-order conditions derived from an optimization problem. The second stage completes the model by adding a set of random disturbance terms to each equation. In many cases the two stages are independent and the error structure has no behavioral content. In such cases it is legitimate to ask why these models contain random components.[3] Recently, economists have realized the importance of making stochastic speci¯cations an integral part of economic models. Following Fuss et al (1978, section 7), McElroy (1987, page 738) proposes a `general error model' in which the \... estimated behavioral equations (e.g. cost and derived demand equations) inherit their stochastic speci¯cation from the underlying maximization problem." While her application of this idea to the theory of producer behavior is somewhat limited, since it focuses exclusively on the asymmetries between the researcher and the agent with respect to the observability of information, the idea is extremely important.

Our research extends this approach by also considering the uncertainty that agents face and uses the fact that when expectations are not ful¯lled exactly, errors are produced. These errors arise in a natural way and satisfy the econometrician's need for a stochastic element in the model; however, their presence has dramatic implications for the way the model should be estimated. In what follows we consider both primal and dual (i.e. cost function) approaches to estimating technological parameters and the implications of this \derived" stochastic speci¯cation on the consistency of these estimates.

The approach taken here is an experimental one. While it is possible to deal with these issues theoretically by deriving the properties of the various estimators by analytical methods, this is by no means a trivial task. As an alternative, we construct a number of \laboratory experiments" in which it is possible to generate data sets which satisfy a particular hypothesis. Various estimation procedures are then applied to the models using this data to see if the procedure works. This is the only alternative to theoretical derivations. In particular, it should be emphasized that using data generated by \real" agents like ¯rms or individual plants can never be used to resolve this problem because the researcher can never be sure of either the technology being used or the objectives of the agents.

The data used in this experiment are generated to satisfy various criteria. In particular, having speci¯ed the technology and the expectations mechanism, we generate factor and commodity prices both expected and realized and from these, assuming expected pro¯t maximization in a perfectly competitive environment, we compute the inputs and outputs. Parameter estimates are then obtained for both the primal and dual versions of the model and their

[2] Earlier versions of this paper were presented at the European Meetings of the Econometric Society in Istanbul and the Institute of Economics, University of Copenhagen. We are grateful to the seminar participants for their comments as well as those of Zvi Griliches, Ian Irvine, and Yair Mundlak.

[3] A good example of this methodology in the area of producer behaviour may be found in the survey article by Jorgenson (1987, page 1861).

statistical properties evaluated.

The main ¯nding is that, while there are no problems in estimating the primal model: the production function and a set of marginal productivity conditions, the estimation of the dual: the cost function and a set of conditional input demand functions, is generally impossible to estimate consistently. The reason for this lies in the nature of individual agent's decision making process. Decisions are made on the basis of expected prices, hence cost depends on both expected factor prices and actual factor prices. This causes the disturbance terms to enter the cost function and conditional input demand functions non-linearly and, consequently, none of the estimation methods available produces consistent estimates.

Over the last two decades the profession has produced a very large literature on the estimation of technology parameters. Much of it attempts to do this with cost functions.[4] Based on the results presented below we have serious reservations about this approach. The methodological perspective mentioned in the ¯rst paragraph requires, as a minimum, consistency between the theoretical model of the agent's behavior and the econometric model estimated by the researcher. This means that the disturbance terms must have their counterpart in the theoretical model otherwise the \true" model is not being estimated. However, when we attempt to ¯nd economically plausible or reasonable scenarios which generate a stochastic structure for an econometric model the resulting disturbance terms often turn out to be very \unreasonable". These lead to serious estimation problems, many of which, arise in systems of equations involving the cost function and the conditional input demand functions. Consequently, it is our view that traditional results on economies of scale, economies of scope, or the complementarity or substitutability of inputs should be viewed with considerable caution if they have been obtained using this dual approach.[5]

The literature in this area suggests that researchers have a free choice over the representation of technology; it is largely a matter of convenience whether one estimates the production side or the cost side. One of the consequences of embedding the stochastic structure in the economic core of the model is that this statement is no longer generally true. A number of cases emerge where it does matter. In the examples considered below the maintained hypothesis can only be formulated in the context of the primal problem and attempting to employ a °exible functional form for the cost function and the conditional input demand functions or factor shares simply fails to represent the hypothesis.

The paper has the following format. In the next section a very simple model of producer behavior under uncertainty is outlined. In Section 3 both primal and dual representations of two self-dual production technologies, the Cobb-Douglas and the CES, are estimated by a number of di®erent procedures. This section also shows that it is important that output be speci¯ed as an endogenous variable if the theory suggests that it should be.[6] The last section contains

[4] A good summary of this literature can be found in Berndt (1991, Ch. 9).

[5] We are not the ¯rst to note di±culties with cost function estimation. Mundlak (1996) shows that cost function estimation can be ine±cient.

[6] This may appear to some readers to be obvious and in any case not worth discussing. However, it should be pointed out that a very high proportion of studies which estimate cost functions treat output as being exogenous.

a discussion of the implications of these results.

## 2 A Static Stochastic Model of Producer Behavior

Consider the following static optimization problem. The agent is a firm whose technology is represented by a production function $f$ and which maximizes expected profit,

$$\pi(v) = E[\hat{p}\theta f(v;\alpha) - \hat{w} \cdot v] \tag{1}$$

by the appropriate choice of $v$, where $v$ is a $k$-dimensional input vector and the $\alpha$ vector is a set of technological parameters. $f$ and $\alpha$ are the same for all firms. $\theta$ is a firm-specific productivity variable and $(\hat{p}, \hat{w})$ is the price vector of the firm's output and inputs, respectively. These are also specific to each firm, although this is not indicated in the notation. In equation (1) expectations are taken with respect to $(\hat{p}, \hat{w})$, which is not known to the firm when it makes its input-output decision. We follow Fuss et al (1978) and McElroy (1987) by assuming that $\theta$ is known by the firm's management but is not observable to the researcher.

The solution to this problem is given by a set of first-order conditions

$$\bar{p}\theta \frac{\partial f}{\partial v_i} = \bar{w}_i \quad i = 1 \ldots k. \tag{2}$$

where $(\bar{p}, \bar{w})$ is the expected value of $(\hat{p}, \hat{w})$. These equations can be solved to get

$$v_i = v_i\left(\frac{\bar{w}}{\bar{p}\theta}\right) \quad i = 1 \ldots k \tag{3}$$

and

$$y(\cdot) = \theta f(v(\cdot);\alpha) \tag{4}$$

which are the unconditional input demand functions and the firm's output supply function, respectively.

For more on this point see McIntosh (1982).

As we mentioned in the introduction this is a laboratory exercise so that the data sets that satisfy the above equations are generated artificially. This is done by specifying a distribution for $(\hat{p}; \hat{w})$ and then drawing an independent sample from it. $\theta$ is generated in a similar fashion. The procedure is straightforward and all that is required is that $(\hat{p}; \hat{w})$ and $\theta$ are positive and that $\theta$ is independent of $(\hat{p}; \hat{w})$. In order to proceed with the experiments we have to specify what the researcher knows about the problem and the mechanism whereby stochastic elements enter the structural equations. First, since $\theta$ is unobservable to the researcher it is treated as an unobservable univariate random effect on productivity. Expected prices are not observed by the researcher but the realization of the process, $(p; w)$, is observed, and furthermore, the researcher knows how expectations are formed. Specifically, it is known that there is a set of relationships between $(p; w)$ and their expected values, $(\hat{p}; \hat{w})$ of the form: $\hat{p} = \pi p$ and $\hat{w}_i = \omega_i w_i$ where $\pi$ and $\omega_i$ are independent random variables whose means are all equal to one.[7] Once a functional form for $f$ is assumed and parameter values assigned, equations (3) and (4) can be used to get the inputs and the output. Like $(p; w)$, $(\pi; \omega)$ and $(\hat{p}; \hat{w})$ are specific to each firm.

The data available to the researcher for each firm are the quantities $(y; v)$ and the price vector, $(p; w)$, and the researcher knows the functional form of $f$, knows that firms maximize expected profits and are price takers in all markets so that $(p; w)$ is exogenous, knows how expectations are formed and knows that there is an unobservable random effect. Given this information the standard procedure for obtaining estimates of $\alpha$ is to represent the model by a set of non-linear structural equations to which conventional estimation methods are applied. For the primal problem these are the expected marginal productivity conditions together with the production function. As we shall see in the next section, this system of equations can be estimated consistently by conventional estimation procedures.

However, researchers often prefer to use the firm's cost function and a set of conditional factor demand functions (or their shares in total cost) rather than the production function and a set of expected marginal productivity conditions. Given $(y; \theta)$ conditional input demand functions are well defined and solve

$$C(\hat{w}; y/\theta) \quad = \quad Min E[\hat{w}.x : f(x; \alpha) \geq y/\theta] \qquad (5)$$

This gives

$$x_i \quad = \quad x_i(\hat{w}; y/\theta) \qquad i = 1 \ldots k \qquad (6)$$

as the conditional input demand functions. Now economists traditionally write the cost function as $C(\hat{w}; y/\theta) = \hat{w}.x(\hat{w}; y/\theta)$; but this does not represent the firm's true cost in this case.

[7] This makes expected prices and factor prices equal to their actual values on average so this convention could be considered as a multiplicative version of rational expectations. In the final section the case of additive errors is discussed.

The actual cost that the firm has to pay is equal to $w.x(\tilde{w}; y/\theta)$ because firms have to make input decisions on the basis of expected factor prices before they observe actual factor prices. Consequently the true cost function must be defined as[8]

$$C_T(w; \tilde{w}; y/\theta) = \sum_{i=1}^{k} w_i x_i(\tilde{w}; y/\theta) \tag{7}$$

and depends on actual factor prices as well as expected factor prices.[9] As we shall see in the next section consistent parameter estimates can be obtained only when the following relationship exists between the two cost functions

$$C_T(w; \tilde{w}; y/\theta) = h(\omega) C(w; y/\theta) \tag{8}$$

# 3 Monte Carlo Simulations

In this section we consider two self-dual technologies, the first of which, is a very simple two factor Cobb-Douglas production function

$$y = \theta v_1^{\alpha_1} v_2^{\alpha_2} \tag{9}$$

The generated data has the following properties. $\ln(\theta)$ is a zero mean normal random variable with standard deviation $\sigma_\theta$. $(\ln(p); \ln(w_1); \ln(w_2))$ has a trivariate normal distribution with zero mean. $(\ln(\pi); \ln(\omega_1); \ln(\omega_2))$ are independent unit variance normal random variables normalized so that the sample average of $(\pi; \omega)$ is equal to the unit vector. Given $\theta$ and $\tilde{p} = \pi p$ and $\tilde{w}_i = \omega_i w_i$, $v$ and $y$ can be computed from equations (3) and (4) once values for $(\alpha_1; \alpha_2)$ are specified. In this example these are chosen to be $(0.3; 0.6)$.

The primal problem structural equations for this model are obtained by substituting $(\pi p; \omega_1 w_1; \ldots \omega_k w_k)$ for $(\tilde{p}; \tilde{w})$ in equation (2), taking logs and adding the log of the equation $y = \theta f(v; \alpha)$. The resulting system has an error structure of the form $u_i = \ln(\omega_i) - \ln(\pi) - \ln(\theta)$ for $i = 1; \ldots k$ and $u_y = \ln(\theta)$ which arises in a natural way from the agent's expectations and the random effects in the production function and does not depend on any ex-post ad hoc error specification assumptions. These equations together with those for the dual are displayed in the Appendix.

[8] As notation we use the vector, v, to represent the unconditional factor demand function and the vector, x, for the conditional factor demand function.

[9] Notice that the dual profit function has similar properties.

The error terms in this system are distributed multivariate normal with zero means and the maximization of the likelihood function for this system yields consistent estimates of $(\alpha_1, \alpha_2)$. The first row of Table 1 gives the averages of the parameter estimates and their standard errors for 50 replications, each with 1000 observations. It is clear from these results that, as expected, the application of the maximum likelihood estimation procedure generates consistent parameter estimates.

Turning to the dual problem for the Cobb-Douglas example the true cost function is given by

$$
\begin{aligned}
C_T(w, \hat{w}, y/\theta) &= h_{CD}(\omega)(k_1 + k_2) w_1^{\frac{\alpha_1}{\alpha_1+\alpha_2}} w_2^{\frac{\alpha_2}{\alpha_1+\alpha_2}} (y/\theta)^{\frac{1}{\alpha_1+\alpha_2}} && (10) \\
&= h_{CD}(\omega) C_{CD}(w, y/\theta) && (11)
\end{aligned}
$$

where

$$
h_{CD}(\omega) = \frac{k_1 \omega_2 + k_2 \omega_1}{\omega_1 \omega_2 (k_1 + k_2)} \omega_1^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \omega_2^{\frac{\alpha_2}{\alpha_1+\alpha_2}} \qquad (12)
$$

and

$$
\begin{aligned}
k_1 &= (\alpha_1/\alpha_2)^{\frac{\alpha_2}{\alpha_1+\alpha_2}} && (13) \\
k_2 &= (\alpha_1/\alpha_2)^{\frac{-\alpha_1}{\alpha_1+\alpha_2}} && (14)
\end{aligned}
$$

$C_{CD}(w, y/\theta)$ is the standard cost function arising from the Cobb-Douglas technology. Its formula may be found in Varian (1992, p. 55).

The dual system consists of the logarithm of the cost function defined by equation (10), and the logarithms of the two conditional input demand functions where, as in the primal problem, $(\hat{p}, \hat{w})$ is replaced by its equivalent $(\pi p, \omega_1 w_1, \omega_2 w_2)$. Here the error structure is more complicated. Given the form of $h_{CD}(\omega)$ the presence of $\ln(h_{CD}(\omega))$ in the cost function error term destroys the normality of the error structure. The mean of this term is also positive which requires an adjustment of the constant term in this equation if consistent parameter estimates are to be obtained.

The dual is estimated, first by the Full Information Maximum Likelihood (FIML) procedure, in spite of the fact that this estimation method is inappropriate, because this is what is usual in the literature referred to in the introduction. In addition to the non-normality of the error structure and the need for a constant term adjustment, this estimation procedure falsely specifies $y$ as an exogenous variable when it is clearly endogenous. These estimates appear in the second row of Table 1. These values bear little relation to the true parameter values

TABLE 1

Average Parameter Estimates For The Cobb-Douglas Example; $\alpha_1 = 0.3$, $\alpha_2 = 0.6$
(Standard Errors)

| Model | Estimation Method | $\hat{\alpha}_1$ | $\hat{\alpha}_2$ |
|---|---|---|---|
| Primal | FIML | 0.3003<br>(0.0046) | 0.5998<br>(0.0048) |
| Dual | FIML | 0.2050<br>(0.0034) | 0.7316<br>(0.0039) |
| Dual | NL3SLS | 0.2527<br>(0.0064) | 0.6549<br>(0.0072) |
| Dual<br>Sub-system | NL3SLS | 0.3003<br>(0.0065) | 0.6000<br>(0.0073) |

so the conventional approach of applying maximum likelihood to the cost function and a set of conditional factor demand functions appears to be rather unsatisfactory. The estimates in row 3 of Table 1 are the result of applying Non-Linear Three Stage Least Squares (NL3SLS) to the system. There is a considerable improvement in the quality of the estimates but they are still many standard errors away from the true parameter values. This improvement is due to two factors. First, we are able to recognize the endogeneity of ln(y) by not including it on the instrument list. Secondly, the NL3SLS estimator does not depend on any distributional assumption for its consistency[10] so that the non-normality of the error structure does not bias the estimates by mispecifying the objective function which, in this case, is a weighted sum of squares rather than a likelihood function. However, consistent estimates for the dual can be obtained when the logarithm of the cost function has its constant term adjusted (for example, by adding a nuisance parameter) so that the mean of the error is equal to zero or when the cost function is dropped from the system. The results of doing the latter using just the sub-system of conditional input demand functions are shown in the fourth row of Table 1.

In the second example the production function is specified as CES and the data on prices, random effects, and expectation errors are the same as in the first example. The production function is of the form[11]

[10] See Amemiya (1985, page 256) for the consistency requirements of this estimator.
[11] This particular specification is due to Jehle (1991, p. 245).

TABLE 2

Average Parameter Estimates For The CES
Example: $®_1$ = 0:5, $®_2$ = 0:75
(Standard Errors)

| Model | Estimation Method | $\hat{®}_1$ | $\hat{®}_2$ |
|---|---|---|---|
| Primal | FIML | 0.5001<br>(0.0083) | 0.7497<br>(0.0038) |
| Dual | FIML | 0.7087<br>(0.0101) | 0.8605<br>(0.0037) |
| Dual | NL3SLS | 0.7543<br>(0.0221) | 0.8310<br>(0.0155) |
| Dual<br>Sub-system | NL3SLS | 0.3925<br>0.0161 | 0.7612<br>0.0174 |

$$y = µ(v_1^{®_1} + v_2^{®_1})^{®_2=®_1} \tag{15}$$

with $®_2 < 1$.

The results for the CES case which are shown in Table 2 are, with one exception, similar to those for the Cobb-Douglas. Primal estimation produces consistent estimates which are very close to the true values of $(®_1; ®_2)$. These were chosen to be (0:5; 0:75) in this example. Likewise, dual estimation leads to inconsistent estimates and these estimates are somewhat worse relative to the true values than was the case for the Cobb-Douglas. Unlike the Cobb-Douglas example, however, estimating the dual sub-system of conditional input demand functions without the cost function does not produce consistent parameter estimates.

# 4 Discussion

The examples employed here are self-dual and explicitly exclude the translog and more general °exible functional forms. There is a very good reason for this. As shown earlier, when ¯rms have to make decisions on their expectations before they observe actual factor and output prices, the resulting cost function depends on both expected and realized prices. The form of this derived cost function can only be obtained by knowing the algebraic representation of the production function, but, as is clear from the foregoing, even where such knowledge is available this will not generally result in consistent estimates of the technological parameters. In particular, it would be incorrect to use a translog cost function which has as its only arguments observed factor prices because that would \assume away" the maintained hypothesis that ¯rms act before they can observe factor and output prices. Since the translog functional form is not self-dual there is no obvious way of deriving a translog cost function which includes both expected and actual factor prices in a way which correctly represents the maintained hypothesis.[12]

One concludes from examples of this type that the estimation of non self-dual °exible functional forms can be carried out successfully only on the production side and attempts to estimate the dual system will generate misleading results.

In Section 2 the expectation formation mechanism was assumed to be multiplicative in nature. This is unusual and most formulations of the rational expectations hypothesis have been of the additive variety. The ¯rst point to note is that none of the results is altered by changing to an additive error model. In the present context the additive formulation has two undesirable characteristics. When we write $\hat{w}_i = w_i + !_i$ the condition that $\hat{w}_i$ ¸ 0 makes the independence of $!_i$ from $w_i$ problematic. There are ways of getting around this like assuming that the variance of $!_i$ is su±ciently small relative to, the mean of $w_i$ or the smallest value of $w_i$, whichever is the more appropriate. But these assumptions are arbitrary and they generate relatively \small" errors which is not always what is required. Secondly, as a practical matter, it does not seem reasonable to us that forecast errors should be independent of the size of the variable being predicted. Errors which are proportional to the value of the variable are much more attractive. As a result we concentrated on the multiplicative model.

The conclusion to be drawn from these two examples is that while it may be desirable to make the stochastic speci¯cation of the model responsive to the demands of economic theory, this may impose constraints on the way the model can be estimated. Furthermore, this problem is much more general than one might have expected from the examples in the previous section. It can arise in a wide variety of circumstances and researchers should be aware of such possibilities. Consider an alternative justi¯cation for the presence of random components in a system of structural equations: optimization errors. It is sometimes suggested that stochastic elements occur in regression equations because agents make optimization errors. These lead to discrepancies between marginal products and relative prices. While the conceptual basis may be di®erent from the model based on expectation errors, the logic and the algebra are the same. Consequently, the same problems will arise in connection with the estimation of the cost function.

We might also considered the case of measurement errors. Our approach di®ers from that of Fuss et al (1978) in the sense that the measurement error mechanism produces true and

[12] McElroy (1987) is able to estimate the dual system and her application is, in fact, a translog cost function example. As readers will recall, however, her errors arise because the econometrician is unable to observe what the agent observes. When, in addition, there are prediction errors or optimization errors (see below) and these feed back into the system through measured variables like cost, the resulting model is much more complicated.

observed variables with the same mean so that there is no systematic bias in the error. Like Fuss et al the results depend on whether the variable is a price or quantity. In the general case measurement errors involving quantities prevent the consistent estimation of both the primal and dual systems; whereas measurement errors in prices only prevent dual system estimation. The situation for the two special cases examined here is somewhat better; both primal and dual estimation is possible for both types of error for the Cobb-Douglas and the dual of the CES can be estimated in the presence of quantity measurement errors.

# 5 Appendix

The structural equations for the Cobb-Douglas primal model are

$$\begin{aligned} \ln(\alpha_1) + (\alpha_1 - 1)\ln(v_1) + \alpha_2\ln(v_2) - \ln(w_1/p) &= u_1 & (16)\\ \ln(\alpha_2) + \alpha_1\ln(v_1) + (\alpha_2 - 1)\ln(v_2) - \ln(w_2/p) &= u_2 & (17)\\ \ln(y) - \alpha_1\ln(v_1) - \alpha_2\ln(v_2) &= u_y & (18) \end{aligned}$$

and for the dual they are

$$\begin{aligned} \ln(C_T) - [\alpha_1\ln(w_1) + \alpha_2\ln(w_2) + \ln(y)]/(\alpha_1 + \alpha_2) - \ln(k_1 + k_2) &= u_c^* & (19)\\ \ln(x_1) - [-\alpha_2\ln(w_1) + \alpha_2\ln(w_2) + \ln(y)]/(\alpha_1 + \alpha_2) - \ln(k_1) &= u_1^* & (20)\\ \ln(x_2) - [\alpha_1\ln(w_1) - \alpha_1\ln(w_2) + \ln(y)]/(\alpha_1 + \alpha_2) - \ln(k_2) &= u_2^* & (21) \end{aligned}$$

# References


[1] Amemiya, T. (1985). Advanced Econometrics, Harvard University Press, Cambridge Mass.

[2] Berndt, E. R. (1991). The Practice of Econometrics: Classic and Contemporary, Addison-Wesley, Reading Mass. U.S.A.

[3] Fuss, M., D. McFadden and Y. Mundlak (1978). "A Survey Of Functional Forms In The Economic Analysis Of Production", Ch. II in Production Economics: A Dual Approach To Theory And Applications, edited by M. Fuss and D. McFadden, North-Holland Press, Amsterdam.

[4] Jehle, G.A. (1991). Advanced Microeconomic Theory, Prentice Hall, Englewood Clifts, New Jersey.

[5] Jorgenson, D. W. (1987). "Econometric Methods For Modelling Producer Behavior" ch. 11 in Handbook of Econometrics, edited by Zvi Griliches and Michael Intriligator, North-Holland Press, Amsterdam

[6] McElroy, M. B. (1987). "Additive General Error Models For Production, Cost, and Derived Demand or Share Systems", Journal of Political Economy 95, 738-57.

[7] McIntosh, James (1982). \Dynamic Interrelated Factor Systems: The United Kingdom 1950-1978" Economic Journal, 92 Supp., 79-86.

[8] Mundlak, Y. (1996). \Production Function Estimation: Reviving The primal", Econometrica 64, 431-38.

[9] Varian, H. (1992). Microeconomic Analysis, 3rd Ed., W. W. Norton & Co., New York.