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Publication date:
2005

Document version
Early version, also known as pre-print

Citation for published version (APA):
Harvey, C. M., & Østerdal, L. P. (2005). *Preferences Between Continuous Streams of Events*. Cph.: Department of Economics, University of Copenhagen.

DISCUSSION PAPERS
Department of Economics
University of Copenhagen

05-12

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Preferences Between Continuous Streams of Events*

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August 2005

Abstract

Cost-benefit and health policy studies often model a consequence occurring over time as a continuous stream of events. Such a consequence is measured by the rates at which events occur or by the states that occur, and the value of the consequence is measured by an integral. This paper presents a foundation for such models. It defines conditions on preferences between consequences that are equivalent to an integral value function having a discounting function and an intertemporal equity function.

Key words: discounting, equity, continuous time, value function, evaluation

* This paper is a based on results from unpublished working papers by Harvey (1998a,b). His research was supported in part by U.S. NSF/EPA Grant No. GAD-R825825010.

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1. Introduction

An analyst who is developing an evaluation of alternatives for corporate planning or public policy must judge whether to model consequences as outcomes that occur at discrete times or as outcomes that occur over continuous time. Consequences at discrete times are modeled as sequences defined on a finite or countable set of points and consequences over continuous time are modeled as functions defined on a bounded or unbounded interval. The judgment as to which type of model to use depends on the nature of the data, the nature of the consequences, and the proclivities of the analyst. Each type seems more appropriate under some circumstances.

This paper is concerned with consequences over continuous time. For example, a consequence may be described by the rates of cost and revenue in the development of an oil field, the rates of highway traffic fatalities associated with alcohol consumption, or the levels of a health indicator for a medical patient.

We envision a decision situation in which an attribute is used to measure the outcomes. The attribute may represent a single objective; for example, it may measure rates of monetary events or levels of health or environmental quality. Or the attribute may represent multiple objectives; for example, it may be constructed by assessing tradeoffs between monetary and nonmonetary outcomes and thereby describing the outcomes by monetary amounts. The possible amounts of the attribute may be the rates at which events occur or they may be the states of a system at various times. In either case, we will refer to the attribute amounts as outcomes.

A continuous-time consequence will be a function whose domain is an interval of times t and whose range is within an interval of outcomes x. A model is to contain a family of such functions. In a model, the times will be measured from an initial time $t = 0$ which we will refer to as the present, the time interval on which the consequences are defined will be called the planning period and will be denoted by P, and the outcome interval that contains the range of any consequence will be denoted by I. For continuous-time consequences but not other types of functions, we will use bold type to distinguish between a function and its values. Thus, $x = \mathbf{x}(t)$ denotes the outcome of a continuous-time consequence $\mathbf{x}$ at a time t.

Figure 1 displays a continuous-time consequence defined on the planning period $0 \le t < \infty$. If its outcomes are defined as rates of net gains, then costs occur in the near future and benefits occur in the more distant future.

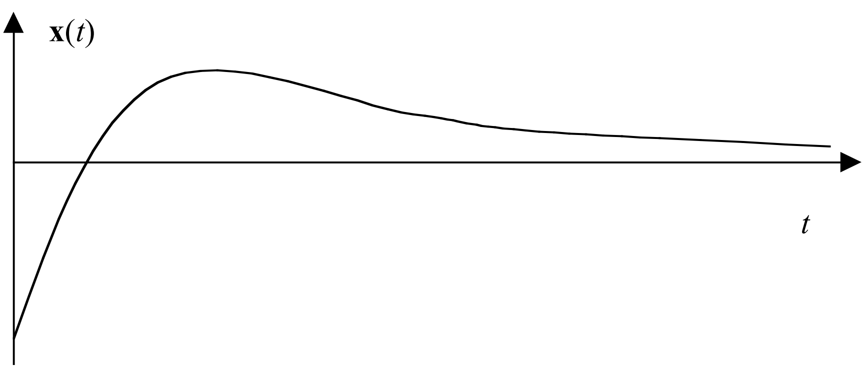

Figure 1: A consequence **x** defined on the planning period $0 \le t < \infty$

This paper develops models of preferences between continuous-time consequences. First, we develop a model for a bounded planning period, and then we extend it to a model for an unbounded planning period. Each model contains continuous-time consequences defined on the planning period and a preference relation defined on some of the pairs of consequences. We define conditions on the preference relation, and we show that the preference relation satisfies the conditions if and only if it is represented by a function that has the integral form:

$$V(\mathbf{x}) = \int_P a(t)\, v(\mathbf{x}(t))\, dt \qquad (1)$$

A function of this form will be called an <u>integral value function</u>, and a model of this type will be called an <u>integral-value model</u>.

A value function (1) represents ordinal preferences. In such a function, the dependence of the importance of an outcome, $x = \mathbf{x}(t)$, on its time and its amount has the product form, $a(t)\,v(x)$. When $v(x) = x$, the function $a(t)$ represents the relative importance assigned to an outcome that occurs at a time t. Although we do not require that $a(t)$ is decreasing to reflect the decreasing importance of future times, we will refer to it as a discounting function.

And when $a(t)$ is constant, the function $v(x)$ represents an attitude toward equity or balance in outcome amounts at different times. Even though we do not require that $v(x)$ is concave to reflect an aversion toward inequity in outcomes, we will refer to it as an equity function.

To develop the integral-value models, we begin with the additive-value model of Debreu (1960) and add a preference condition which implies that the preference relation is represented by an additive value function, $V(x_1,\ldots,x_n) = \sum_{i=1}^{n} a_i\, v(x_i)$, in which the scales $v(x_i)$ are the same. Then, we interpret the vectors $(x_1,\ldots,x_n)$ as continuous-time consequences that are step functions, i.e., functions having constant outcomes x_i on subintervals of the planning period.

We extend in stages the sets of consequences on which the preference relation is defined. For the first model, we extend from the set of step functions defined on a bounded planning period $P = [0, T]$ to the set of Riemann integrable functions defined on $[0, T]$. For the second model, we extend from sets of Riemann integrable functions defined on the bounded planning periods to a set of functions defined on the unbounded planning period $P = [0, \infty)$.

It has been argued that only the exponential discounting functions of constant discounting should be used in a business or public policy study; such arguments are based on the principles of 'temporal consistency' and 'economic efficiency.' Harvey (1994), Ahlbrecht and Weber (1995), and Bleichrodt and Gafni (1996) argue to the contrary that nonconstant discounting models can be reasonable for such a purpose. All three papers discuss the principle of temporal consistency; the first one discusses also the principle of economic efficiency.

The models in this paper have two features that distinguish them from previous models of continuous-time consequences. The first such feature is that the set of consequences over the unbounded planning period $P = [0, \infty)$ that are comparable is defined in terms of the preference relation. Such a dependence is necessary to provide a variety of models with discounting functions that have different behaviours at infinity. This is so since the set of consequences that an integral value function (1) can compare includes only the consequences with a finite value $V(\mathbf{x})$. As we remark in the next section, previous models fix the class of consequences that are comparable, and thus place restrictions on the discounting function.

In particular, the approach taken here provides for discounting functions that decrease more slowly than negative-exponential functions and thus assign more importance to events in the distant future. An analyst can use a model with such a so-called slow discounting function (see, e.g., Harvey, 1995) and compare an evaluation based on that model with an evaluation based on a negative-exponential discounting function.

A slow discounting model can provide insight in a public policy study, e.g., a study on the use a natural resource, infrastructure development, environmental remediation, or public health, in which it is essential to consider the importance of outcomes in the distant future. Such a model also can provide insight for a corporate planning study, e.g., a study on exploration for a deep sea oil deposit, in which substantial returns might be realized in the distant future.

The second feature that distinguishes the models in this paper is that the continuous-time consequences are Riemann integrable on bounded time intervals. Riemann integrable functions are more elementary than the Lebesgue integrable functions considered in previous models. For this reason, we can construct integral value functions by using elementary real analysis—while previous models deduce integral value functions by using existence results from measure theory and functional analysis.

Sets of Riemann integrable functions are sufficiently large to include both step functions and continuous functions. In Harvey (1998a,b), conditions on preferences that imply parametric families of discounting functions and equity functions are defined in terms of indifferences between hypothetical step functions. The actual consequences in a study usually are continuous functions, and thus Riemann integrable functions provide a common ground for examining how preferences between the step functions involved in simple, hypothetical choices can structure preferences between the more complex continuous functions involved in an actual choice.

The results in this paper are ‘if and only if’ results; a preference relation satisfies the conditions in a model if and only if it is represented by an integral function with the properties described in the model. In particular, we do not assume extra ‘technical conditions’ such as differentiability for the functions $a(t)$ and $v(x)$. Proofs of results are provided in an appendix.

2. Previous Research

For two reasons, it seems helpful to devote more attention than usual to previous research. First, it is surprising that the models developed here were not developed long ago—and many readers may assume that they have been. The second reason is that models for continuous-time consequences have been developed, and thus we need to explain how they differ from those in this paper and why the differences are important.

Samuelson (1937) defined a discounting model in which preferences between continuous-time consequences are represented by an integral, $V(\mathbf{x}) = \int_P e^{-rt}\, v(\mathbf{x}(t))\, dt$, where $r > 0$ is a discount rate and $v(x)$ is an intensive-preference utility function. His model soon became the dominant model for studying intertemporal choice.

Samuelson's model is not a measurement theory model. As he was careful to state, he did not deduce his integral function from a list of conditions on preferences. Instead, he assumed that a person acts so as to maximize such an integral. Samuelson's purpose for the model lay elsewhere; he proposed it as a means of inferring a person's intensive-utility function $w(x)$ for consumption rates x from the person's choices of optimal consumption streams given available investment opportunities and given constraints on the person's initial and final wealth.

Koopmans (1960, 1972) developed a model that does include conditions on a preference relation. The consequences are time streams, $x_0, x_1, \ldots$, of events at equally-spaced time points $t = 0, 1, \ldots$. He established that if preferences satisfy certain conditions, then they are represented by an infinite sum, $V(x_0, x_1, \ldots) = \sum_{t=0}^{\infty} (1+r)^{-t}\, v(x_t)$. Koopmans' model is limited to constant discounting, and it assumes that the set of comparable consequences does not depend on the preference relation. A history of discounting models is provided in Loewenstein (1992).

Two types of general discounting models with continuous-time consequences have been developed. First, Grodal (2003, Section 12.3 and Note 12.5.1) presents models with a function of the form $V(\mathbf{x}) = \int_P a(t)\, v(\mathbf{x}(t))\, d\mu(t)$. This work is based on results of Grodal and Mertens that were reported in a 1968 working paper. The models do not define the set of comparable consequences in terms of the preference relation. In particular, constant functions are assumed to be comparable. Thus, the models exclude zero-discounting and certain types of so-called slow discounting (see, e.g., Harvey, 1986, 1995).

The models described by Grodal establish that certain conditions on preferences imply that they are partially represented (see below) by a function $V(\mathbf{x}) = \int_P a(t)v(\mathbf{x}(t))\,d\mu(t)$ but do not establish the converse implication. Of more importance, the models establish only that 'if **x** is preferred to **y** then $V(\mathbf{x}) > V(\mathbf{y})$ for any **x**, **y**.' This result does not imply that 'if $V(\mathbf{x}) > V(\mathbf{y})$ then **x** is preferred to **y**' since it could happen that $V(\mathbf{x}) > V(\mathbf{y})$ while **x** and **y** are indifferent. Thus, the models do not show that $V(\mathbf{x})$ fully represents the preference relation.

As a second type of discounting model with continuous-time consequences, Weibull (1985) develops a model having a preference relation defined on a convex cone C of consequences in a space $L^1(\mu)$ of Lebesgue functions. By means of the Riesz Representation Theorem for linear functionals on $L^1(\mu)$, he shows that the preference relation satisfies certain conditions if and only if it is represented by a function of the form $V(\mathbf{x}) = \int_P a(t)\mathbf{x}(t)\,d\mu(t)$.

Weibull's model differs from those in this paper in three respects. First, a consequence set C does not depend on the preference relation. The set C may be too small for many applications since any consequence in C is in $L^1(\mu)$ and hence must have a finite non-discounted value, $\int_P \mathbf{x}(t)\,d\mu(t)$. In particular, consequences that are constant on an unbounded planning period are excluded. And since C is a cone, the outcome interval I must be unbounded above whenever consequences have positive values and must be unbounded below whenever consequences have negative values. By contrast, the models in this paper can include consequences without finite non-discounted values, and the outcome interval I can be any interval. A bounded or semi-bounded interval I may be needed in an application either because the set of meaningful outcomes is restricted or because the domain of the equity function $v(x)$ is restricted.

The second difference is that $v(x) = x$ in Weibull's model. Thus, the model excludes issues of intertemporal equity that can be represented by a nonlinear equity function $v(x)$.

Third, the set C is defined in terms of Lebesgue integrable functions rather than functions that are Riemann integrable on bounded intervals (and thus are continuous almost everywhere). In an application, it seems unlikely that there would be meaningful consequences that are not continuous almost everywhere. And for that reason, assumptions on preferences are far more difficult to envision for Lebesgue functions than for Riemann functions.

3. Conditions on Preferences

In this section, we define conditions on preferences between continuous-time consequences. These conditions form the basis for the models in the following sections. Similar conditions are well known for multi-variable consequences and for finite probability distributions.

As is conventional, an interval with positive length will be called a nonpoint interval, and an interval with zero length, i.e., a point or the empty set, will be called a point interval. Assume that any outcome interval is a nonpoint interval *I* that contains 0. As a notation for time intervals or outcome intervals, $\langle a,b \rangle$ will designate that there is no requirement as to whether the initial and final values *a* and *b* are in the interval.

First, we need to establish definitions and properties for step consequences. The properties are well-known and are stated here for purposes of readability.

A partition *p* of a time interval $[0,T]$ is a finite set of intervals: $\langle a_0,a_1 \rangle$, … , $\langle a_{m-1},a_m \rangle$ where $0 = a_0 \le a_1 \le \ldots \le a_{m-1} \le a_m = T$ and the intervals are pairwise disjoint and have the union $[0,T]$. The characteristic function $c_i(t)$ for an interval $\langle a_{i-1},a_i \rangle$ is that function which has the value 1 for *t* in $\langle a_{i-1},a_i \rangle$ and has the value 0 otherwise.

A step consequence based on a partition *p* is a function, $\mathbf{x}(t) = \sum_{i=1}^{m} x_i \, c_i(t)$, *t* in [0, *T*], where x_i are amounts in the outcome interval *I* and $c_i(t)$ are the characteristic functions for the intervals $\langle a_{i-1},a_i \rangle$, *i* = 1, … , *m*, in the partition *p*. Thus, a step consequence is a step function defined on the interval $[0,T]$ with values in the interval *I*. Suppose that S_p denotes the set of step consequences based on a partition *p*, and $S_T = \bigcup_p S_p$ denotes the set of step consequences for a time interval $[0,T]$ and an outcome interval *I*.

The integral of a step consequence **x** in a set S_p is the finite sum, $\int \mathbf{x} = \sum_{i=1}^{m} x_i \, (a_i - a_{i-1})$. It can be shown that $\int \mathbf{x}$ depends only on the function **x** and not on the partition p.

Integration is an increasing operation in that for any step consequences **x**, **y**: if $\mathbf{x}(t) \le \mathbf{y}(t)$ for t in $[0,T]$, then $\int \mathbf{x} \le \int \mathbf{y}$. Integration is strictly increasing on any nonpoint interval $\langle a,b \rangle$ in that if $\mathbf{x}(t) \le \mathbf{y}(t)$ for t in $[0,T]$ and $\mathbf{x}(t) < \mathbf{y}(t)$ for t in $\langle a,b \rangle$, then $\int \mathbf{x} < \int \mathbf{y}$; and integration is constant on any nonempty point interval $[a,a]$ in that if $\mathbf{x}(t) = \mathbf{y}(t)$ for $t \ne a$, then $\int \mathbf{x} = \int \mathbf{y}$.

We will model preference relations $\mathbf{x} \succsim \mathbf{y}$ that are defined on pairs **x**, **y** of continuous-time consequences. The statement $\mathbf{x} \succsim \mathbf{y}$ means that '**x** is at least as preferred as **y**.' The preference relations $\sim$ and $\succ$ are assumed to be defined in terms of the relation $\succsim$ by: $\mathbf{x} \sim \mathbf{y}$ provided that $\mathbf{x} \succsim \mathbf{y}$ and $\mathbf{y} \succsim \mathbf{x}$; and $\mathbf{x} \succ \mathbf{y}$ provided that $\mathbf{x} \succsim \mathbf{y}$ and not $\mathbf{y} \succsim \mathbf{x}$. Moreover, $\mathbf{x} \precsim \mathbf{y}$ will denote the statement that $\mathbf{y} \succsim \mathbf{x}$, and $\mathbf{x} \prec \mathbf{y}$ will denote the statement that $\mathbf{y} \succ \mathbf{x}$.

We state below a list of five conditions on a preference relation $\succsim$. In order not to restate the conditions in each result, we define them for a generic set *C* of consequences that are defined on $P = [0,T]$ or on $P = [0,\infty)$ and for a set S_T of step consequences. In the case, $P = [0,T]$, S_T will be a subset of *C*. In the case, $P = [0,\infty)$, each function in S_T will be identified with a function in *C* with the value zero for $t > T$, and thus S_T will be a subset of *C*.

In a rather general topological setting, conditions analogous to (A)-(C) coimply the existence of a continuous value function; see, e.g., Debreu (1954). By contrast, conditions analogous to (A)-(C) plus (D) or (E) coimply the existence of a value function with a special structure, e.g., an additive structure. Here, we do not pursue a result in the spirit of Debreu's; instead, we use the full list of conditions (A)-(E) to coimply the existence of a value function having the form (1).

(A) $\succsim$ is <u>strictly</u> <u>increasing</u> on C: For any **x**, **y** in C,

(a) If $\mathbf{x}(t) \le \mathbf{y}(t)$ a.e. for t in P, then $\mathbf{x} \precsim \mathbf{y}$.

(b) If $\mathbf{x}(t) \le \mathbf{y}(t)$ a.e. for t in P and $\mathbf{x}(t) < \mathbf{y}(t)$ on a nonpoint interval, then $\mathbf{x} \prec \mathbf{y}$.

If outcomes initially are defined as losses, then (A) can be obtained by the usual expedient of measuring the outcomes as reductions in losses. Part (a) implies that $\mathbf{x} \sim \mathbf{y}$ when $\mathbf{x}(t) = \mathbf{y}(t)$ a.e.

(B) $\succsim$ is <u>transitive</u> on C: For any **x**, **y**, **z** in *C*, if $\mathbf{x} \succsim \mathbf{y}$ and $\mathbf{y} \succsim \mathbf{z}$, then $\mathbf{x} \succsim \mathbf{z}$, and $\succsim$ is <u>complete</u> on C: For any **x**, **y** in C, either $\mathbf{x} \succsim \mathbf{y}$ or $\mathbf{y} \succsim \mathbf{x}$.

(C) $\succsim$ is <u>continuous</u> on *C* with respect to S_T: For any **x** in *C* and **w** in S_T, if $\mathbf{w} \prec \mathbf{x}$, then there exists a $\delta > 0$ such that $\int |\mathbf{z} - \mathbf{w}| < \delta$ implies $\mathbf{z} \prec \mathbf{x}$ for any **z** in S_T, and if $\mathbf{w} \succ \mathbf{x}$, then there exists a $\delta > 0$ such that $\int |\mathbf{z} - \mathbf{w}| < \delta$ implies $\mathbf{z} \succ \mathbf{x}$ for any **z** in S_T.

Condition (C) states that $\succsim$ is continuous at each consequence $\mathbf{x}$ in C with regard to step consequences $\mathbf{w}$ in S_T. Suppose, e.g., that $\mathbf{w} \prec \mathbf{x}$ and $\mathbf{w}$ is increased to a step consequence $\mathbf{z}$. Then, the cumulative improvement is, $\int(\mathbf{z}-\mathbf{w}) = \int|\mathbf{z}-\mathbf{w}|$, and whenever this improvement is sufficiently small, then $\mathbf{z} \prec \mathbf{x}$.

In the next two conditions (and the rest of the paper), $\mathbf{x}_Q$ will denote the restriction of a consequence $\mathbf{x}$ to a subinterval Q of P, that is, $\mathbf{x}_Q(t) = \mathbf{x}(t)$ for t in Q and $\mathbf{x}_Q(t) = 0$ otherwise. Moreover, $\mathbf{a}_Q$ will denote the restriction of a constant function with value a to the interval Q.

(D) $\succsim$ is <u>preferentially</u> <u>independent</u> on C: Suppose that $\langle a,b\rangle$ is a bounded subinterval of P and that the following functions are in the set C. Then, $\mathbf{x}_{\langle a,b\rangle} + \mathbf{x}_{P-\langle a,b\rangle} \succsim \mathbf{x}_{\langle a,b\rangle} + \mathbf{y}_{P-\langle a,b\rangle}$ implies that $\mathbf{y}_{\langle a,b\rangle} + \mathbf{x}_{P-\langle a,b\rangle} \succsim \mathbf{y}_{\langle a,b\rangle} + \mathbf{y}_{P-\langle a,b\rangle}$.

Condition (D) states that preferences between two consequence streams in the complement of an interval $\langle a,b\rangle$ are independent of any common consequence stream in $\langle a,b\rangle$. Thus, (D) is analogous to conditions of preferential independence in Debreu (1960) and Gorman (1968).

Two pairs of outcomes $a < b$ and $c < d$ are said to be tradeoffs with respect to a pair of disjoint intervals $\langle\alpha,\beta\rangle$ and $\langle\gamma,\delta\rangle$ provided that $\mathbf{a}_{\langle\alpha,\beta\rangle} + \mathbf{d}_{\langle\gamma,\delta\rangle} \sim \mathbf{b}_{\langle\alpha,\beta\rangle} + \mathbf{c}_{\langle\gamma,\delta\rangle}$. (Intuitively, one is just willing to trade off the improvement from a to b for the worsening from d to c.) For outcomes $\underline{a} < \hat{a} < \overline{a}$ in I, we define $\hat{a}$ to be a <u>tradeoff</u> <u>midvalue</u> of $\underline{a}$, $\overline{a}$ on an interval $\langle\alpha,\beta\rangle$ provided that there exists a disjoint interval $\langle\gamma,\delta\rangle$ and outcomes $c < d$ such that $\underline{a} < \hat{a}$ and $c < d$ are tradeoffs and $\hat{a} < \overline{a}$ and $c < d$ are tradeoffs with respect to $\langle\alpha,\beta\rangle$ and $\langle\gamma,\delta\rangle$.

(E) $\succsim$ is <u>midvalue</u> <u>independent</u> on S_T: For any pair of intervals in $[0,T]$ and any outcomes $\underline{a} < \hat{a} < \overline{a}$ in I, if $\hat{a}$ is a tradeoffs midvalue of $\underline{a}$ and $\overline{a}$ on the first interval, then $\hat{a}$ is a tradeoffs midvalue of $\underline{a}$ and $\overline{a}$ on the second interval.

Condition (E) states that if two outcome pairs, $\underline{a} < \hat{a}$ and $\hat{a} < \overline{a}$, have the same tradeoffs when they occur in one time interval, then they have the same tradeoffs when they occur in any common interval. A variety of analogous conditions for vectors and discrete-time consequences are described in Fishburn (1970), Krantz et al. (1972, page 305), and Harvey (1986, 1995).

4. Consequences on a Bounded Planning Period

In this section, we develop a model for a set of consequences defined on a bounded planning period $P = [0,T]$. Our method of proof is to extend the model of Debreu (1960) for a product set of vector consequences to a model for a set of step consequences and then to extend this model to a model for a set of Riemann integrable functions.

Lemma 4.1 A preference relation $\succsim$ defined on a set S_T of step consequences satisfies the conditions (A)–(E) if and only if $\succsim$ is represented on S_T by a function of the form

$$V(\mathbf{x}) = \sum_{i=1}^{m} \left(A(a_i) - A(a_{i-1}) \right) v(x_i), \ \ \mathbf{x} \ \text{in} \ S_T \tag{2}$$

such that the sum $V(\mathbf{x})$ has the same value for any partition p with $\mathbf{x}$ in S_p and:

(a) The function $A(t)$ is strictly increasing and absolutely continuous on the interval $[0,T]$, and $A(0) = 0$.

(b) The function $v(x)$ is strictly increasing and continuous on the interval I, and $v(0) = 0$.

(c) The function V is continuous at each consequence $\mathbf{w}$ in S_T in that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int |\mathbf{z} - \mathbf{w}| < \delta$ implies $|V(\mathbf{z}) - V(\mathbf{w})| < \varepsilon$ for any $\mathbf{z}$ in S_T.

Moreover, each of the functions $A(t)$ and $v(x)$ is unique up to a positive multiple.

The properties (a)-(c) provide an 'if and only if' result. The properties (a), (b) of $A(t)$ and $v(x)$ do not imply the joint-continuity property (c) of V. For a counterexample, see Harvey (1998b).

Next, we extend the above model from step functions to Riemann integrable functions. We do so by means of two ideas: first, by rewriting the value function (2) as an integral, and second, by approximating a Riemann integrable function by sequences of step functions.

Since a function $A(t)$ is increasing, it is differentiable a.e. with a non-negative derivative. We will define $a(t)$ as the derivative of $A(t)$ where it exists and as 0 otherwise. The function $A(t)$ is an indefinite Lebesgue integral of $a(t)$ since $A(t)$ is absolutely continuous, and thus we can rewrite the value of a step consequence, $\mathbf{x}(t) = \sum_{i=1}^{m} x_i \, c_i(t)$, in the integral form:

$$V(\mathbf{x}) = \sum_{i=1}^{m} \left(\int_{a_{i-1}}^{a_i} a(t)\, dt \right) v(x_i) = \int_0^T a(t)\, v(\mathbf{x}(t))\, dt, \ \ \mathbf{x} \ \text{in} \ S_T \tag{3}$$

Following common usage, we will call $a(t)$ a <u>discounting</u> <u>function</u>. Here, it is not required to be decreasing. Moreover, it is not required to be Riemann integrable. The reason is that if $a(t)$, $0 \le t \le T$, is Riemann integrable, then it is bounded, whereas the absolute continuity of $A(t)$ does not imply that $a(t)$ is bounded. Perhaps, the most important counterexamples are the functions, $A(t) = t^k$, $0 < k < 1$, whose derivatives, $a(t) = k\,t^{k-1}$, are unbounded near the present, $t = 0$. Power discounting functions are often used in models of quality-adjusted life years (QALYs); see, e.g., Pliskin et al. (1980).

Now, consider a function $\mathbf{x}$ that is Riemann integrable on $[0,T]$. Briefly, the definition is that $\mathbf{x}$ is bounded and that any sequence of sums $\sum_{i=1}^{m} x_i (a_i - a_{i-1})$ associated with partitions p of $[0,T]$ converges to the same amount when the maximum lengths of the intervals $\langle a_{i-1}, a_i \rangle$ tend to zero. We will call a function $\mathbf{x}$ defined on $[0,T]$ a <u>Riemann consequence</u> on $[0,T]$ provided that $\mathbf{x}$ is Riemann integrable and also the amounts $\sup\{\mathbf{x}(t) : 0 \le t \le T\}$ and $\inf\{\mathbf{x}(t) : 0 \le t \le T\}$ are in the outcome interval I. The set of Riemann consequences on $[0,T]$ will be denoted by R_T.

A Riemann consequence can be approximated from above by a decreasing sequence of step consequences and from below by an increasing sequence of step consequences. To describe this ‘squeeze property,’ suppose that $\mathbf{x}$ is a Riemann consequence on $[0,T]$ and p is a partition of $[0,T]$. Define $\overline{x}_i = \sup\{\, \mathbf{x}(t) : t \text{ in } \langle a_{i-1}, a_i \rangle\}$ and $\underline{x}_i = \inf\{\, \mathbf{x}(t) : t \text{ in } \langle a_{i-1}, a_i \rangle\}$ for $i = 1, \ldots, m$, and define the step functions $\overline{\mathbf{x}}^p(t) = \sum_{i=1}^{m} \overline{x}_i\, c_i(t)$ and $\underline{\mathbf{x}}^p(t) = \sum_{i=1}^{m} \underline{x}_i\, c_i(t)$. The functions $\overline{\mathbf{x}}^p$, $\underline{\mathbf{x}}^p$ have values in the interval I and thus are in the set S_p associated with the partition p. They will be called <u>upper</u> and <u>lower</u> <u>Darboux</u> <u>step</u> <u>consequences</u> for $\mathbf{x}$.

Lemma A.3 states that for any $\mathbf{x}$ in R_T, there exist sequences $\{\overline{\mathbf{x}}_n\}_{n=1}^{\infty}$, $\{\underline{\mathbf{x}}_n\}_{n=1}^{\infty}$ of upper and lower Darboux step consequences such that $\underline{\mathbf{x}}_1(t) \le \underline{\mathbf{x}}_2(t) \le \ldots \le \mathbf{x}(t) \le \ldots \le \overline{\mathbf{x}}_2(t) \le \overline{\mathbf{x}}_1(t)$ for $0 \le t \le T$, and the sequence of differences $\int \overline{\mathbf{x}}_n - \int \underline{\mathbf{x}}_n$ converges to zero. We use this type of approximation to extend the model in Lemma 4.1 to the following model.

<u>Theorem 4.1</u> For a bounded planning period $P = [0,T]$ and an outcome interval I, consider a preference relation $\succsim$ defined on the set R_T of Riemann consequences on $[0,T]$. The preference relation $\succsim$ satisfies the conditions (A), (B) on R_T, satisfies the condition (C) on the pair of sets R_T, S_T, and satisfies the conditions (D), (E) on the set S_T if and only if $\succsim$ is represented on R_T by a function of the form

$$V(\mathbf{x}) = \int_0^T a(t)\,v(\mathbf{x}(t))\,dt,\ \mathbf{x} \text{ in } R_T \tag{4}$$

such that:

(a) The Lebesgue integral (4) exists for any consequence $\mathbf{x}$ in R_T.

(b) The function $a(t)$ is non-negative and Lebesgue integrable on the interval $[0,T]$.

(c) The functions $A(t) = \int_0^t a(s)\,ds$, $v(x)$, and V satisfy the properties (a)-(c) in Lemma 4.1.

Moreover, each of the functions $A(t)$ and $v(x)$ is unique up to a positive multiple.

5. Consequences on an Unbounded Planning Period

In this section, we extend the model for bounded planning periods $[0,T]$ to a model for the unbounded planning period $P = [0,\infty)$. As a preliminary result, we construct a model for the set of consequences having compact support, i.e., that are zero except on a bounded time interval. Then, we consider a set of consequences having arbitrary behavior at infinity, but we do not require that the preference relation is complete on this set. We define a subset of consequences using the preference relation, and we show that the preference relation restricted to the subset satisfies the conditions (A)-(E) if and only if it is represented by an integral value function.

The model differs from timing models in Koopmans (1960, 1972), Koopmans et al. (1964), Diamond (1965), Burness (1976), Svensson (1980), and Weibull (1985) in that these models assume completeness of a preference relation on a consequence set that is defined independently of the preference relation. Fishburn and Edwards (1997) also define comparability to be independent of preferences: two denumerable time streams of outcomes are comparable if they differ in only a finite number of periods.

By contrast, the models in Wakker (1993) for denumerable probability distributions and those in Harvey (1986, 1995) for denumerable time streams assume completeness of a preference relation on a set that depends on the relation. Wakker argues that comparability dependence is the crucial change in the axioms of Savage (1954) that permits an unbounded utility function, and Harvey argues that it permits an arbitrary sequence of discount weights.

A function defined on the planning period $[0,\infty)$ with values in an outcome interval *I* will be called a Riemann consequence on $[0,\infty)$ provided that its restriction to any subinterval $[0,T]$, $T > 0$, is a Riemann consequence. The set of Riemann consequences on $[0,\infty)$ will be denoted by R_∞. Note that: R_∞ depends on the interval *I*, a function in R_∞ is not required to have a finite integral over $[0,\infty)$, and unless *I* is bounded the function is not required to be bounded.

First, consider a consequence **x** in R_∞ that has compact support, i.e., there is a time $T > 0$ such that **x**(*t*) = 0 for all $t > T$. Such a consequence will be called a finite-horizon consequence, and the set of finite-horizon consequences will be denoted by R_f. We identify a consequence in a set R_T with the corresponding consequence in R_f that has support in $[0,T]$. Then, R_f is the union of the sets R_T for $T > 0$. This identification states in particular that any step consequence is identified with a consequence in R_f, and thus each set S_T is a subset of R_f.

Lemma 5.1 A preference relation $\succsim$ defined on a set R_f of finite-horizon consequences satisfies the conditions (A), (B), and (D) on R_f, satisfies the condition (C) on each pair of sets R_T, S_T, $T > 0$, and satisfies the condition (E) on each set S_T, $T > 0$, if and only if $\succsim$ is represented on R_f by a function of the form

$$V(\mathbf{x}) = \lim_{T\to\infty} \int_0^T a(t)\, v(\mathbf{x}(t))\, dt, \quad \mathbf{x} \text{ in } R_f \tag{5}$$

such that:

(a) The function *a*(*t*) is non-negative on the interval $[0,\infty)$ and is Lebesgue integrable on each subinterval $[0,T]$, $T > 0$, and the function $A(t) = \int_0^t a(s)\, ds$ is strictly increasing on the interval $[0,\infty)$ and is absolutely continuous on each subinterval $[0,T]$, $T > 0$, and *A*(0) = 0.

(b) The function *v*(*x*) is strictly increasing and continuous on the interval *I*, and *v*(0) = 0.

(c) For each $T > 0$, the function *V* is continuous at each consequence **w** in S_T in that for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\int |\mathbf{z} - \mathbf{w}| < \delta$ implies $|V(\mathbf{z}) - V(\mathbf{w})| < \varepsilon$ for any **z** in S_T.

Moreover, each of the functions *A*(*t*) and *v*(*x*) is unique up to a positive multiple.

Next, consider a preference relation $\succsim$ defined on an entire set R_∞. In the model below, we do not require that $\succsim$ is complete on R_∞. Instead, we define a subset $R_\succsim$ of R_∞ in terms of $\succsim$, and we require completeness on $R_{\succsim}$. Roughly speaking, $R_{\succsim}$ contains the consequences in R_∞ that become arbitrarily unimportant in the sufficiently distant future.

To make the idea precise, we use changes that occur in the immediate future period $[0,1)$ to measure the importance of changes that occur in a distant future period $[t,\infty)$. Suppose that $a < b$ are outcomes and $t \geq 1$ is a (distant) time. Then, for a consequence **x** we can compare changes between a and b in the period $[0,1)$ with changes between $\mathbf{x}(t)$ and 0 in the period $[t,\infty)$.

The definition below is chosen to be as simple as possible. See Lemma A.4 for two related conditions. The notation $\mathbf{x}_{[t,\infty)}$, $\mathbf{a}_{[0,1)}$, and so forth is as defined in Section 3.

Definition 5.1 A consequence **x** in a set R_∞ that has a preference relation $\succsim$ will be called comparable provided that for any outcomes $a < b < c$ in I, there exists a time $T \geq 1$ such that

$$\mathbf{a}_{[0,1)} \prec \mathbf{b}_{[0,1)} + \mathbf{x}_{[t,\infty)} \prec \mathbf{c}_{[0,1)} \tag{6}$$

for all $t \geq T$. The set of comparable consequences will be denoted by $R_{\succsim}$.

Lemma 5.2 Suppose that $\succsim$ is a preference relation on R_∞. Then:

(a) For any two consequences **x**, **y** in R_∞, if **x** is in $R_{\succsim}$ and there exists a time $T > 0$ such that $\mathbf{x}(t) = \mathbf{y}(t)$ for all $t \geq T$, then **y** is in $R_{\succsim}$.

(b) If $\succsim$ is strictly increasing on the set of constant consequences on the time interval $[0,1)$, then the set R_f of finite-horizon consequences is a subset of $R_{\succsim}$.

We now present an integral value model for an unbounded planning period. The model in Harvey (1998a) contains a comparability condition similar to that in Definition 5.1. Most likely, other models can be constructed that contain one of the comparability conditions in Lemma A.4.

Theorem 5.1 For the unbounded planning period $P = [0,\infty)$ and an outcome interval I, consider a preference relation $\succsim$ defined on the set R_∞ of Riemann consequences on $[0,\infty)$. Assume that $\succsim$ is strictly increasing on the set of constant consequences on $[0,1)$. Then, the preference relation $\succsim$ satisfies the conditions (A), (B), and (D) on the set $R_{\succsim}$, satisfies the condition (C) on each pair of sets $R_{\succsim}$, S_T, $T > 0$, and satisfies the condition (E) on each set S_T, $T > 0$, if and only if $\succsim$ is represented on $R_{\succsim}$ by a function of the form

$$V(\mathbf{x}) = \lim_{T\to\infty} \int_0^T a(t)v(\mathbf{x}(t))\,dt, \quad \mathbf{x} \text{ in } R_{\succsim} \tag{7}$$

such that:

(a) The limit (7) of Lebesgue integrals exists for any consequence **x** in $R_{\succsim}$.

(b) The functions *a*(*t*), *A*(*t*), *v*(*x*), and *V* satisfy the properties (a)-(c) in Lemma 5.1.

Moreover, each of the functions *A*(*t*) and *v*(*x*) is unique up to a positive multiple.

One may ask whether the subset of consequences in R_∞ such that the limit (7) exists equals the subset $R_{\succsim}$ of R_∞. This equality is not always true, and it is true if and only if the preference relation in Theorem 5.1 satisfies an additional condition. For details, see Harvey (1998b).

6. Prescriptive and Descriptive Uses of the Models

Integral value functions for preferences between continuous-time consequences already play an important role in many areas of economics and decision modeling: e.g., corporate and public policy planning, long-range resource and environmental planning, and medical and public health decision making. The models in this paper provide a foundation for such functions. We view the primary use of the models to be prescriptive, that is, to provide insight into preference conditions and into the implications that the conditions and value assumptions have for policy evaluations.

The models seem overly general for prescriptive use in that the discounting function *a*(*t*) and the equity function *v*(*x*) have general forms. For a tractable model, conditions on preferences should be stated which restrict each function to a single function (up to a positive multiple) or to a parametric family—indeed, we think a one-parameter family should be used when possible.

Typically, *a*(*t*) is restricted to the one-parameter family of negative-exponential functions (which represent a constant, positive discount rate and assign little importance to the far future) or *a*(*t*) is the constant function *a*(*t*) = 1 (which represents a zero discount rate and assigns the same importance to any time whatever). These forms of *a*(*t*) can be viewed as opposite extremes. The two-parameter family of hyperbolic discounting functions, $a(t) = (1 + {}^1\!/_q\, r\,t)^{-q}$, with $q > 0$, $r > 0$ are intermediate forms. Here, *r* is the initial discount rate and the discount rates decrease from *r* toward zero as *t* tends to infinity. One-parameter subfamilies include: $a(t) = (1 + r\,t)^{-1}$

and $a(t) = (1 + {}^{1}/_{2}\, r\, t)^{-2}$. Harvey (1998a, 2003) defines conditions on preferences that imply hyperbolic discounting and these subfamilies in the case of continuous time, and Ahlbrecht and Weber (1995) provide a general discussion of hyperbolic discounting for prescriptive use.

Typically, the equity function $v(x)$ is chosen to be $v(x) = x$. This function represents neutrality toward the variability in a continuous-time consequence. Harvey (1998a) discusses several parametric families of functions $v(x)$ that represent aversion toward intertemporal variability.

By contrast with its prescriptive role, an integral value function seems insufficiently general for descriptive modeling. Behavioral research has found systematic violations of additive value functions $\sum_{t=1}^{\infty} a_t v(x_t)$ and hence by implication of integral value functions (1); see, e.g., Shelley (1993) and Loewenstein and Prelec (1993). We describe here two preference issues that one might wish to include in a descriptive model and that require a more general value function. The development of models for these types of timing preferences are open research problems.

First, the condition (D) of preferential independence excludes the issue of complementarity between the outcomes in different periods. For example, it implies that tradeoffs between the outcomes in any two future periods do not depend on the outcomes in an earlier period. Becker and Murphy (1988) propose an integral value function that depends on both a consequence **x** and a ‘stock function’ related to the cumulative function $X(t) = \int_0^t \mathbf{x}(s)\, ds$. It would be of interest to develop a model in which conditions on preferences imply a value function of the form, $V(\mathbf{x}) = \int_P a(t)\, v(X(t), \mathbf{x}(t))\, dt$, where $v(X(t), \mathbf{x}(t))$ depends on the outcomes $X(t)$ that precede the time t.

Second, the condition (E) of equal tradeoffs midvalues excludes any dependence of tradeoffs between two periods on the predicted circumstances in the periods. However, an individual’s tradeoffs between present and future consumption or a society’s tradeoffs between present and future net benefits may depend on the predicted wealth of the individual or society at the future times. For instance, it has been argued that benefits at a future time should be valued less (and thus discounted more) if the future society has more well-being than the present society; see, e.g., Broome and Ulph (1992) and Broome (1994). It would be of interest to develop a model having a value function of the form, $V(\mathbf{x}) = \int_P a(t)\, v(t, \mathbf{x}(t))\, dt$, where $v(t, \mathbf{x}(t))$ depends on the time t and thus can depend on circumstances that are independent of any specific consequence.

Appendix: Proofs of Results

To begin, we define some notation. Suppose that $p\!:\langle a_{i-1},a_i\rangle$, i = 1, … , m, and $q\!:\langle b_{j-1},b_j\rangle$, j = 1, … , n, denote two partitions of a planning period $P=[0,T]$. Since the mn intersections $\langle a_{i-1},a_i\rangle \cap \langle b_{j-1},b_j\rangle$ are disjoint, they form another partition of $[0,T]$. We will refer to this partition as the conjunction of p and q, and we will denote it by pq. For notational convenience, we define $\langle c_{k-1},c_k\rangle = \langle a_{i-1},a_i\rangle \cap \langle b_{j-1},b_j\rangle$ where the index $k=n(i-1)+j$ ranges from 1 to mn. It follows that, $0=c_0 \le c_1 \le \ldots \le c_{mn}=T$. The sets S_p and S_q are subsets of S_{pq} since, for example, any step consequence, $\mathbf{x}(t)=\sum_{i=1}^m x_i\, c_i(t)$, in S_p can be written as a step consequence, $\mathbf{x}(t)=\sum_{i=1}^m \sum_{j=1}^n x_i\, c_{i,j}(t)$, in S_{pq}.

Lemma A.1 Suppose that a preference relation $\succsim$ defined on a set S_T satisfies the conditions (A)–(E). Then, for any partition p of $[0,T]$, the preference relation $\succsim$ restricted to the subset S_p of S_T is represented by a function of the additive form:

$$V_p(\mathbf{x}) = \sum_{i=1}^m a_{i,p} v_p(x_i), \quad \mathbf{x} \in S_p \qquad \text{(A1)}$$

such that the function $v_p(x)$ is continuous and strictly increasing on the outcome interval I and each coefficient $a_{i,p}$ is positive if $\langle a_{i-1},a_i\rangle$ is a nonpoint interval or zero if $\langle a_{i-1},a_i\rangle$ is a point interval. Moreover, such a function $V_p(\mathbf{x})$ is unique up to a positive linear transformation when the partition p contains at least two nonpoint intervals.

Proof. Assume that a preference relation $\succsim$ satisfies the conditions (A)–(E) on the set S_T. Then, for any partition $p\!:\langle a_{i-1},a_i\rangle$, i = 1, … , m, it satisfies the conditions on the subset S_p. There is a 1–1 correspondence between S_p and the product set I^m of vectors $(x_1,\ldots,x_m)$ with amounts x_i in the interval I, and thus $\succsim$ restricted to S_p induces a preference relation $\succsim_p$ on the set I^m.

Condition (A) implies that the preference relation $\succsim_p$ is strictly increasing on each variable x_i corresponding to a nonpoint interval and is constant on each variable x_i corresponding to a point interval. And conditions (B), (C) imply that $\succsim_p$ is transitive, complete, and continuous.

First, consider any partition p with at least three nonpoint intervals and no point intervals. Suppose that $\langle a_{i-1},a_i\rangle$ is an interval in p. Condition (D) implies that for any amounts x_i, y_i of the i-th variable and any vectors $\tilde{x}, \tilde{y}$ of amounts of the other variables: if $(x_i,\tilde{x}) \succsim_p (x_i,\tilde{y})$, then $(y_i,\tilde{x}) \succsim_p (y_i,\tilde{y})$. Thus, the set of complementary variables is preferentially independent of the i-th variable. Hence, by results of Debreu (1960) and Gorman (1968) the space $(I^m, \succsim_p)$ is represented by an additive value function $V(x_1,\ldots,x_m) = \sum_{i=1}^{m} v_{i,p}(x_i)$ where each function $v_{i,p}(x_i)$ is continuous and strictly increasing on the interval I.

Condition (E) implies that $\succsim_p$ satisfies the condition of 'equal tradeoffs midvalues' defined in Harvey (1986). Hence, by a result there $(I^m, \succsim_p)$ is represented by an additive value function $V(x_1,\ldots,x_m) = \sum_{i=1}^{m} a_{i,p} v_p(x_i)$ where the common function $v_p(x)$ is continuous and strictly increasing on I and the coefficients $a_{i,p}$ are positive. Thus, the corresponding space $(S_p, \succsim)$ is represented by a value function (A1) as described.

As a second case, consider any partition p with at least three nonpoint intervals and at least one point interval. Condition (A) implies that each variable for a point interval is inessential in $(I^m, \succsim_p)$. Hence, by the above argument there exists an additive value function for $(I^m, \succsim_p)$, namely, a sum of terms $a_{i,p} v_p(x_i)$ with $a_{i,p} > 0$, for the essential variables. We can insert terms $a_{i,p} v_p(x_i)$ with $a_{i,p} = 0$ for the inessential variables. Then, $V(x_1,\ldots,x_m) = \sum_{i=1}^{m} a_{i,p} v_p(x_i)$, is a value function for $(I^m, \succsim_p)$, and thus $(S_p, \succsim)$ has a value function (A1) as described.

As the remaining case, consider any partition p with fewer than three nonpoint intervals. Since the period $P = [0,T]$ is a nonpoint interval, there is another partition q such that the conjunction pq has at least three nonpoint intervals. Then, the space $(S_{pq}, \succsim)$ is represented by a function $V_{pq}(\mathbf{x})$ as described in (A1). However, $(S_p, \succsim)$ is a subspace of $(S_{pq}, \succsim)$, and thus by combining terms we can rewrite $V_{pq}(\mathbf{x})$ as a value function for the space $(S_p, \succsim)$.

Finally, if a partition p has at least two non-point intervals, and hence $(I^m, \succsim_p)$ has at least two essential variables, then an additive value function for $(I^m, \succsim_p)$ is unique up to a positive linear transformation, and thus the same is true for a value function (A1) for $(S_p, \succsim)$.

Proof of Lemma 4.1 To show the forward implications, we use the fact that S_T is a union of sets S_p such that the partition p has at least two non-point intervals. Such a partition will be called a proper partition. The conjunction of two proper partitions is a proper partition.

Since the interval I is nonpoint and contains 0, either: (i) there exists an $x^+ > 0$ in I or (ii) there is no $x^+ > 0$ in I but there exists an $x^- < 0$ in I. For any proper partition p, suppose that the coefficients $a_{i,p}$ in (A1) are normalized such that $\sum_{i=1}^{m} a_{i,p} = 1$, and the function $v_p(x)$ in (A1) is normalized such that $v_p(0) = 0$ and $v_p(x^+) = 1$ in case (i) while $v_p(x^-) = -1$ in case (ii). Lemma A.1 implies that for any proper partition *p* such a normalized value function is unique.

Suppose that V_p, V_q and V_{pq} are the normalized value functions for two proper partitions *p*, *q* and their conjunction pq. Then, $V_{pq}(\mathbf{x}) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij,pq} v_{pq}(x_{ij})$ for any $\mathbf{x}$ in S_{pq} where $\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij,pq} = 1$, $v_{pq}(0) = 0$, and either $v_{pq}(x^+) = 1$ or $v_{pq}(x^-) = -1$.

$V_{pq}(\mathbf{x})$ equals $V_p(\mathbf{x})$ for any consequence $\mathbf{x}$ in S_p. For if $\mathbf{x}(t) = \sum_{i=1}^{m} x_i c_i(t)$ is in S_p, then $V_{pq}(\mathbf{x}) = \sum_{i=1}^{m}(\sum_{j=1}^{n} a_{ij,pq}) v_{pq}(x_i)$. Since $\sum_{i=1}^{m}(\sum_{j=1}^{n} a_{ij,pq}) = 1$ and $v_{pq}(x)$ is normalized as above, it follows that the restriction of V_{pq} to S_p equals the normalized value function V_p. Hence, $\sum_{j=1}^{n} a_{ij,pq} = a_{i,p}$ for *i* = 1, … , *m*, and $v_{pq}(x) = v_p(x)$ for *x* in *I*.

Similar reasoning can be used to show that $V_{pq}(\mathbf{x})$ equals $V_q(\mathbf{x})$ for any $\mathbf{x}$ in S_q. Thus, we also have $\sum_{i=1}^{m} a_{ij,pq} = a_{j,q}$ for *j* = 1, … , *n*, and $v_{pq}(x) = v_q(x)$ for *x* in *I*.

It follows that if two step consequences, $\mathbf{x}_p \in S_p$ and $\mathbf{x}_q \in S_q$, are equal, then $V_p(\mathbf{x}_p) = V_q(\mathbf{x}_q)$. For both $\mathbf{x}_p$ and $\mathbf{x}_q$ are in S_{pq}, and thus $V_p(\mathbf{x}_p) = V_{pq}(\mathbf{x}_p) = V_{pq}(\mathbf{x}_q) = V_q(\mathbf{x}_q)$.

A normalized function $v_p(x)$ is independent of *p*. For $v_p(x) = v_{pq}(x) = v_q(x)$, *x* in *I*, for any proper partitions *p* and *q*. We will denote the common function $v_p(x)$ by *v*(*x*).

Next, we show that a normalized coefficient $a_{i,p}$ associated with a proper partition depends only on the endpoints a_{i-1}, a_i of the *i*-th interval $\langle a_{i-1}, a_i \rangle$. For suppose that *p* is a proper partition with an interval $\langle a_{h-1}, a_h \rangle$ and *q* is a proper partition with an interval $\langle b_{k-1}, b_k \rangle$ such that $a_{h-1} = b_{k-1}$ and $a_h = b_k$. The interval $\langle a_{h-1}, a_h \rangle \cap \langle b_{k-1}, b_k \rangle$ in the proper partition *pq* has the endpoints $a_{h-1} = b_{k-1}$ and $a_h = b_k$ Hence, the intervals $\langle a_{i-1}, a_i \rangle \cap \langle b_{k-1}, b_k \rangle$, $i \neq h$, and $\langle a_{h-1}, a_h \rangle \cap \langle b_{j-1}, b_j \rangle$, $j \neq k$, are point intervals, and thus $a_{ik,pq} = 0$ for $i \neq h$ and $a_{hj,pq} = 0$ for $j \neq k$. It follows that $a_{hk,pq} = \sum_{i=1}^{m} a_{ik,pq} = a_{k,q}$ and $a_{hk,pq} = \sum_{j=1}^{n} a_{hj,pq} = a_{h,p}$, and thus $a_{k,q} = a_{h,p}$ as was to be shown. Hence, for any proper partition *p* we can regard a coefficient $a_{i,p}$ as a function of the endpoints a_{i-1}, a_i, that is, $a_{i,p} = f(a_{i-1}, a_i)$.

Supose that *p* is a proper partition with adjacent intervals $\langle a_{h-1}, a_h \rangle$, $\langle a_h, a_{h+1} \rangle$ and *q* is a proper partition with an interval $\langle b_{k-1}, b_k \rangle = \langle a_{h-1}, a_{h+1} \rangle$. Define $h' = h + 1$. Then, $a_{h,p} = \sum_{j=1}^{n} a_{hj,pq} = a_{hk,pq}$, $a_{h',p} = \sum_{j=1}^{n} a_{h'j,pq} = a_{h'k,pq}$, and $a_{k,q} = \sum_{i=1}^{m} a_{ik,pq} = a_{hk,pq} + a_{h'k,pq}$.

Hence, $a_{k,q} = a_{h,p} + a_{h',p}$, and thus $f(a_{h-1},a_{h+1}) = f(a_{h-1},a_h) + f(a_h,a_{h+1})$. It follows that $f(a,c) = f(a,b) + f(b,c)$ for any $a \le b \le c$ in the planning period $[0,T]$.

To solve this functional equation, note that for $a = 0$, it becomes $f(b,c) = f(0,c) - f(0,b)$. We will define $A(t) = f(0,t)$. Thus, $f(b,c) = A(c) - A(b)$. Moreover, $A(0) = f(0,0) = A(0) - A(0) = 0$. See, e.g., Aczél (1966, pp. 223-224) for a general discussion of the functional equation.

The normalized value function $V_p(\mathbf{x}) = \sum_{i=1}^{m} a_{i,p} v_p(x_i)$ for S_p can now be written as:

$$V(\mathbf{x}) = V_p(\mathbf{x}) = \sum_{i=1}^{m} \big(A(a_i) - A(a_{i-1}) \big) v(x_i), \quad \mathbf{x} \in S_p \tag{A2}$$

where the functions $A(t)$ and $v(x)$ are independent of the partition p.

The function V in (A2) is well-defined for any $\mathbf{x} \in S_T$. For if $\mathbf{x} \in S_p$ and $\mathbf{x} \in S_q$ for two different proper partitions p and q, then $V_p(\mathbf{x}) = V_{pq}(\mathbf{x}) = V_q(\mathbf{x})$.

Moreover, the function V represents the space $(S_T, \succsim)$. For consider any $\mathbf{x}$, $\mathbf{y}$ in S_T. Then, $\mathbf{x}$ is in S_p and $\mathbf{y}$ is in S_q for some proper partitions p and q. Therefore, $\mathbf{x}$ and $\mathbf{y}$ are both in the set S_{pq}, and thus $V(\mathbf{x}) = V_{pq}(\mathbf{x})$, $V(\mathbf{y}) = V_{pq}(\mathbf{y})$. Hence, $\mathbf{x} \succsim \mathbf{y}$ if and only if $V_{pq}(\mathbf{x}) \ge V_{pq}(\mathbf{y})$ if and only if $V(\mathbf{x}) \ge V(\mathbf{y})$. It follows that V represents $(S_p, \succsim)$ for any partition p whatever.

Next, we show that the functions $A(t)$, $v(x)$, $V(\mathbf{x})$ have the properties (a)-(c). The function $A(t) = f(0,t)$ is strictly increasing on the interval $[0,T]$ since by Lemma A.1 any coefficient $a_{i,p}$ is positive for a nonpoint i-th interval. Moreover, $A(0) = 0$. And since $A(t)$ is increasing on $[0,T]$, to show that $A(t)$ is absolutely continuous on $[0,T]$ it suffices to show that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any pairwise disjoint open intervals (a_{i-1},a_i), $i = 1,\ldots, n$, in $[0,T]$: $\sum_{i=1}^{n}(a_i - a_{i-1}) < \delta$ implies $\sum_{i=1}^{n}\big(A(a_i) - A(a_{i-1})\big) < \varepsilon$.

As a notational convenience, suppose that $\mathbf{b}$ denotes the step consequence with the value b for t in $[0,T]$. Then, $V(\mathbf{b}) = A(T)v(b) = v(b)$ since $A(T) = f(0,T) = \sum_{i=1}^{m} a_{i,p} = 1$ for any proper partition p. In particular, $V(\mathbf{0}) = v(0) = 0$. By a similar argument, one can show that $V(\mathbf{x}) = v(b)$ for any step consequence $\mathbf{x}$ that equals b except at a finite number of points.

Suppose that case (i) holds, i.e., there is an $x^+ > 0$ in the interval I and the function $v(x)$ is normalized such that $v(x^+) = 1$. The proof in the case (ii) is similar and will be omitted. For any $\varepsilon > 0$, there exists a $b(\varepsilon) > 0$ in I such that $V(\mathbf{b}(\varepsilon)) = v(b(\varepsilon)) < \varepsilon$ since the function $v(x)$ is continuous. Moreover, $\mathbf{0} \prec \mathbf{b}(\varepsilon)$ since by condition (A) the preference relation $\succsim$ is strictly increasing on nonpoint intervals. By condition (C), it follows that there exists a $\delta > 0$ such that $\int |\mathbf{z} - \mathbf{0}| < \delta$ implies $\mathbf{z} \prec \mathbf{b}(\varepsilon)$ for $\mathbf{z}$ in S_T. Thus, $\int |\mathbf{z}| < \delta$ implies $V(\mathbf{z}) < \varepsilon$.

Now consider any pairwise disjoint open intervals (a_{i-1}, a_i), $i = 1, \ldots, n$, in $[0, T]$, and define a step consequence $\mathbf{z}$ by $\mathbf{z}(t) = x^+$ for $a_{i-1} < t < a_i$, $i = 1, \ldots, n$, and $\mathbf{z}(t) = 0$ otherwise. Then, we have $\int |\mathbf{z}| = \sum_{i=1}^n (a_i - a_{i-1}) v(x^+) = \sum_{i=1}^n (a_i - a_{i-1})$ and $V(\mathbf{z}) = \sum_{i=1}^n (A(a_i) - A(a_{i-1})) v(x^+)$ $= \sum_{i=1}^n (A(a_i) - A(a_{i-1}))$. Therefore, $\sum_{i=1}^n (a_i - a_{i-1}) < \delta$ implies $\int |\mathbf{z}| < \delta$ implies $V(\mathbf{x}) < \varepsilon$ implies $\sum_{i=1}^n (A(a_i) - A(a_{i-1})) < \varepsilon$. Thus, $A(t)$ is absolutely continuous on the interval $[0, T]$.

Lemma A.1 establishes that the function $v(x)$ is strictly increasing and continuous on the interval I and that $v(0) = 0$.

It remains to show that the function V is continuous at each consequence $\mathbf{x}$ in S_T, i.e., for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int |\mathbf{z} - \mathbf{x}| < \delta$ implies $|V(\mathbf{z}) - V(\mathbf{x})| < \varepsilon$ for all $\mathbf{z}$ in S_T.

Consider any $\mathbf{x}$ in S_T and any $\varepsilon > 0$. Then $\mathbf{x}$ is in a set S_p. As a first case, assume that the interval I contains an upper endpoint u and that $\mathbf{x}(t) = u$ except perhaps on point intervals in the partition p. Since the function $v(x)$ is continuous on I, there is an outcome $b < u$ in I such that $v(b) > v(u) - \varepsilon$. Then, $V(\mathbf{b}) = v(b) > v(u) - \varepsilon = V(\mathbf{x}) - \varepsilon$. But condition (A) implies $\mathbf{b} \prec \mathbf{x}$, and thus $V(\mathbf{b}) < V(\mathbf{x})$. Hence, $V(\mathbf{x}) - \varepsilon < V(\mathbf{b}) < V(\mathbf{x})$. Condition (A) also implies $\mathbf{z} \precsim \mathbf{x}$, and thus $V(\mathbf{z}) \le V(\mathbf{x})$ for any $\mathbf{z}$ in S_T. By condition (C), there is a $\delta > 0$ such that: $\int |\mathbf{z} - \mathbf{x}| < \delta$ implies $\mathbf{z} \succ \mathbf{b}$ for $\mathbf{z}$ in S_T. Therefore, $\int |\mathbf{z} - \mathbf{x}| < \delta$ implies $V(\mathbf{z}) > V(\mathbf{b})$ implies $V(\mathbf{x}) - \varepsilon < V(\mathbf{z}) \le V(\mathbf{x})$ implies $|V(\mathbf{z}) - V(\mathbf{x})| < \varepsilon$ for any $\mathbf{z}$ in S_T.

The proof is similar and can be omitted if the interval I contains a lower endpoint l and $\mathbf{x}(t) = l$ except perhaps on point intervals in the partition p.

In the remaining case, the partition p contains nonpoint intervals $\langle a_{h-1}, a_h \rangle$, $\langle a_{k-1}, a_k \rangle$ such that x_h is not an upper endpoint of I and x_k is not a lower endpoint of I. For amounts $x'_h > x_h$ and $x'_k < x_k$ to be chosen, define a step consequence $\mathbf{x}_1$ by $\mathbf{x}_1(t) = x'_h$ for t in $\langle a_{h-1}, a_h \rangle$ and $\mathbf{x}_1(t) = \mathbf{x}(t)$ otherwise, and define a step consequence $\mathbf{x}_2$ by $\mathbf{x}_2(t) = x'_k$ for t in $\langle a_{k-1}, a_k \rangle$ and $\mathbf{x}_2(t) = \mathbf{x}(t)$ otherwise. Then, $V(\mathbf{x}_1) - V(\mathbf{x}) = (A(a_h) - A(a_{h-1}))(v(x'_h) - v(x_h)) > 0$, and $V(\mathbf{x}_2) - V(\mathbf{x}) = (A(a_k) - A(a_{k-1}))(v(x'_k) - v(x_k)) < 0$.

Since the function $v(x)$ is continuous, we can choose x'_h, x'_k such that $V(\mathbf{x}_1) - V(\mathbf{x}) < \varepsilon$ and $V(\mathbf{x}) - V(\mathbf{x}_2) < \varepsilon$. Then, by condition (C) there is a $\delta_1 > 0$ such that for $\mathbf{z}$ in $S_T : \int |\mathbf{z} - \mathbf{x}| < \delta_1$

implies $\mathbf{z} \prec \mathbf{x}_1$ implies $V(\mathbf{z}) < V(\mathbf{x}_1) < V(\mathbf{x}) + \varepsilon$, and there is a $\delta_2 > 0$ such that for $\mathbf{z}$ in S_T: $\int |\mathbf{z} - \mathbf{x}| < \delta_2$ implies $\mathbf{z} \succ \mathbf{x}_2$ implies $V(\mathbf{z}) > V(\mathbf{x}_2) > V(\mathbf{x}) - \varepsilon$. Thus, for $\delta = \min\{\delta_1, \delta_2\}$ we have for any $\mathbf{z}$ in S_T: $\int |\mathbf{z} - \mathbf{x}| < \delta$ implies $V(\mathbf{x}) - \varepsilon < V(\mathbf{z}) < V(\mathbf{x}) + \varepsilon$.

For the converse part of the proof, assume that a space $(S_T, \succsim)$ is represented by a function V of the form (A2) where the functions $A(t)$ and $v(x)$ satisfy the properties (a)-(c). Then, it is straightforward to show that $(S_T, \succsim)$ satisfies the conditions (A), (B) and (D), (E).

To show that $(S_T, \succsim)$ satisfies the continuity condition (C), consider any step consequences $\mathbf{x}$, $\mathbf{y}$ with $\mathbf{x} \prec \mathbf{y}$ and thus $V(\mathbf{x}) < V(\mathbf{y})$. Define $\varepsilon = V(\mathbf{y}) - V(\mathbf{x}) > 0$. By property (c), there is a $\delta > 0$ such that for any $\mathbf{z}$ in S_T: $\int |\mathbf{z} - \mathbf{x}| < \delta$ implies $|V(\mathbf{z}) - V(\mathbf{x})| < \varepsilon$. Thus, $\int |\mathbf{z} - \mathbf{x}| < \delta$ implies $V(\mathbf{z}) < V(\mathbf{y})$ which implies $\mathbf{x} \prec \mathbf{y}$. By a similar argument, there is a $\delta > 0$ such that for any $\mathbf{z}$ in S_T: $\int |\mathbf{z} - \mathbf{y}| < \delta$ implies $\mathbf{z} \succ \mathbf{x}$. Hence, condition (C) is satisfied.

To show that each function $A(t)$, $v(x)$ is unique up to a positive multiple, consider two value functions $V(\mathbf{x}) = \sum_{i=1}^{m} (A(a_i) - A(a_{i-1}))v(x_i)$ and $V^*(\mathbf{x}) = \sum_{i=1}^{m} (A^*(a_i) - A^*(a_{i-1}))v^*(x_i)$. For any proper partition p, the uniqueness property of additive value functions implies that there are constants $a_p > 0$, b_p such that $V^*(\mathbf{x}) = a_p V(\mathbf{x}) + b_p$ for any $\mathbf{x}$ in S_p. But $\mathbf{0}$ is in S_p and $V(\mathbf{0}) = V^*(\mathbf{0}) = 0$, and thus $b_p = 0$. Hence, $V^*(\mathbf{x}) = a_p V(\mathbf{x})$.

For any other proper partition q, there is a constant $a_q > 0$ such that $V^*(\mathbf{x}) = a_q V(\mathbf{x})$, and thus $a_p V(\mathbf{x}) = a_q V(\mathbf{x})$ for $\mathbf{x}$ in $S_p \cap S_q$. Since $S_p \cap S_q$ contains the nonzero constant consequences, it follows that $a_p = a_q$. Thus, $V^*(\mathbf{x}) = aV(\mathbf{x})$ for $\mathbf{x}$ in S_T where $a > 0$ denotes the common value.

For a partition p, define a consequence $\mathbf{x}$ in S_p such that $\mathbf{x}(t) = x_1$ for t in the interval $[0, a_1\rangle$ of p and $\mathbf{x}(t) = 0$ otherwise. Then, $A^*(a_1)\, v^*(x_1) = V^*(\mathbf{x}) = aV(\mathbf{x}) = aA(a_1)\, v(x_1)$. However, a_1 is any element of $[0,T]$, and x_1 is any element of I. Thus, $A^*(t)$ is a positive multiple of $A(t)$, and $v^*(x)$ is a positive multiple of $v(x)$.

Next, we establish several real-analysis results for use in Theorems 4.1 and 5.1. For two functions f and g with a common domain D, $f \geq g$ will mean that $f(t) \geq g(t)$ for all t in D.

Lemma A.2 Suppose that $\{f_n(t)\}_{n=1}^{\infty}$ is a sequence of non-negative step functions defined on a bounded interval D such that $f_1 \geq f_2 \geq \ldots$. If $\lim_{m\to\infty} \int f_n = 0$, then $\lim_{m\to\infty} f_n(t) = 0$ a.e. on D.

Proof. For any $t \in D$, the sequence $\{f_n(t)\}_{n=1}^{\infty}$ is non-negative and weakly decreasing, and thus $\lim_{n\to\infty} f_n(t) \geq 0$ exists. Define $E = \{t \in D : \lim_{n\to\infty} f_n(t) > 0\}$. We show that E is a null set, i.e., for any $\varepsilon > 0$ there exists a countable set of open intervals I_j, $j = 1, 2, \ldots$, such that $E \subset \bigcup_{j=1}^{\infty} I_j$ and $\sum_{j=1}^{\infty} |I_j| < \varepsilon$ (where $|I_j|$ denotes the length of an interval I_j).

Consider an $\varepsilon > 0$. For each n = 1, 2, …, there is an $N(n)$ such that $\int f_{N(n)} < {}^1\!/_2\, ({}^1\!/_4)^n \varepsilon$. Define $A_n = \{t \in D : f_{N(n)}(t) > ({}^1\!/_2)^n\}$, n = 1, 2, … . Since $f_{N(n)}$ is a step function, A_n is the union of a finite number of intervals $I_{i,n}$. Moreover, we have the following two properties:

(a) $$\sum_{n=1}^{\infty} \sum_i |I_{i,n}| < {}^{\varepsilon}\!/_2$$

For $({}^1\!/_2)^n \sum_{n=1}^{\infty} \sum_i |I_{i,n}| \leq \int f_{N(n)} < {}^1\!/_2\, ({}^1\!/_4)^n \varepsilon$, and thus $\sum_i |I_{i,n}| < {}^1\!/_2\, ({}^1\!/_2)^n \varepsilon$. By summing over n = 1, 2, … , we deduce the inequality (a).

(b) $$E \subset \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \bigcup_i I_{i,n}$$

For if $t \in E$, then $\lim_{m\to\infty} f_m(t) > ({}^1\!/_2)^n$ for some n, and thus $t \in A_n$.

If some of the intervals $I_{i,n}$ are not open, then we can slightly expand them to open intervals as follows. Relabel the intervals $I_{i,n}$ as J_j, j = 1, 2, … . For each j, choose an open interval I_j such that $J_j \subset I_j$ and $|I_j| < |J_j| + ({}^1\!/_2)^{n+1} \varepsilon$. Then, $E \subset \bigcup_{n=1}^{\infty} \bigcup_i I_{i,n} = \bigcup_{j=1}^{\infty} J_j \subset \bigcup_{j=1}^{\infty} I_j$, and thus $\sum_{j=1}^{\infty} |I_j| < {}^{\varepsilon}\!/_2 + {}^{\varepsilon}\!/_2 = \varepsilon$.

Lemma A.3 Suppose that a function $a(t)$ is non-negative and Lebesgue integrable on an interval $[0,T]$ and a function $v(x)$ is continuous and strictly increasing on an interval I. Then:

(a) For any function $\mathbf{x}$ in the set R_T, the function $a(t)v(\mathbf{x}(t))$ is Lebesgue integrable on $[0,T]$.

(b) For any functions $\mathbf{x}$, $\mathbf{y}$ in R_T such that $\mathbf{x}(t) \leq \mathbf{y}(t)$ a.e. on $[0,T]$, the following relations between $\mathbf{x}$ and $\mathbf{y}$ are equivalent.

(i) There is a subset E of $[0,T]$ having positive measure such that $\mathbf{x}(t) < \mathbf{y}(t)$ for t in E.

(ii) There is a nonpoint subinterval $[\alpha,\beta)$ of $[0,T]$ such that $\sup\{\mathbf{x}(t) : \alpha \leq t < \beta\} < \inf\{\mathbf{y}(t) : \alpha \leq t < \beta\}$.

(iii) $\int_0^T a(t)v(\mathbf{x}(t))dt < \int_0^T a(t)v(\mathbf{y}(t))dt$.

(c) For any function $\mathbf{x}$ in R_T, there exist sequences $\{\bar{\mathbf{x}}_n\}_{n=1}^{\infty}$ and $\{\underline{\mathbf{x}}_n\}_{n=1}^{\infty}$ of functions in S_T such that:

(iv) $\underline{\mathbf{x}}_1 \le \underline{\mathbf{x}}_2 \le \ldots \le \mathbf{x} \le \ldots \le \overline{\mathbf{x}}_2 \le \overline{\mathbf{x}}_1$.

(v) $\int \overline{\mathbf{x}}_n$ and $\int \underline{\mathbf{x}}_n$ converge to $\int \mathbf{x}(t)dt$ as n tends to ∞.

(vi) $\int_0^T a(t)v(\overline{\mathbf{x}}_n(t))dt$ and $\int_0^T a(t)v(\underline{\mathbf{x}}_n(t))dt$ converge to $\int_0^T a(t)v(\mathbf{x}(t))dt$ as n tends to ∞.

Proof. To show property (a), consider a function $\mathbf{x}$ in R_T. Since $\mathbf{x}$ is Riemann integrable, it is bounded and continuous almost everywhere. Moreover, the values $\sup \mathbf{x} = \sup\{\mathbf{x}(t) : 0 \le t \le T\}$ and $\inf \mathbf{x} = \inf\{\mathbf{x}(t) : 0 \le t \le T\}$ are in the interval I. Thus, the function $v(\mathbf{x}((t))$ is bounded by $v(\inf \mathbf{x})$ and $v(\sup \mathbf{x})$ (since the function $v(x)$ is increasing) and is continuous a.e. (since $v(x)$ is continuous). Hence, $v(\mathbf{x}(t))$ is Riemann integrable. But the function $a(t)$ is Lebesgue integrable, and thus the product $a(t)\, v(\mathbf{x}(t))$ is Lebesgue integrable.

To show property (b), we show that (i) implies (ii) implies (iii) implies (i). First, assume that $\mathbf{x}$, $\mathbf{y}$ satisfy (i). Since $\mathbf{x}$ and $\mathbf{y}$ are Riemann integrable on $[0,T]$ (and thus are continuous a.e. on $[0,T]$), there is a time t_0 in E such that $\mathbf{x}$ and $\mathbf{y}$ are continuous at t_0. Define $\varepsilon = \mathbf{y}(t_0) - \mathbf{x}(t_0) > 0$, $a = \mathbf{x}(t_0) + {}^{\varepsilon}/_{3}$, and $b = \mathbf{y}(t_0) - {}^{\varepsilon}/_{3}$. Then, there exists a $\delta > 0$ such that $\mathbf{x}(t) \le a < b \le \mathbf{y}(t)$ for all t such that $|t - t_0| \le \delta$. Thus, $\mathbf{x}$ and $\mathbf{y}$ satisfy (ii).

Second, if $\mathbf{x}$, $\mathbf{y}$ satisfy (ii) then: $\int_0^T a(t)v(\mathbf{y}(t))dt - \int_0^T a(t)v(\mathbf{x}(t))dt \ge \int_\alpha^\beta a(t)(\,v(b) - v(a)\,)\,dt = \int_\alpha^\beta a(t)(\,v(b) - v(a)\,)\,dt = (\,A(\beta) - A(\alpha)\,)(v(b) - v(a)) > 0$.

Third, assume that (i) is false, i.e., $\mathbf{x}(t) = \mathbf{y}(t)$ a.e. on $[0,T]$. Then, $a(t)v(\mathbf{y}(t)) = a(t)v(\mathbf{x}(t))$ a.e. on $[0,T]$, and thus $\int_0^T a(t)v(\mathbf{y}(t))dt = \int_0^T a(t)v(\mathbf{x}(t))dt$.

To show property (c), consider a function $\mathbf{x}$ in R_T. For a partition p, suppose that $\overline{\mathbf{y}}^p$, $\underline{\mathbf{y}}^p$ are the upper and lower Darboux consequences for $\mathbf{x}$ associated with p (as defined in Section 4). For any $\varepsilon > 0$, there exists a partition p such that $\int \overline{\mathbf{y}}^p - \int \underline{\mathbf{y}}^p < \varepsilon$ (see, e.g., Stromberg, 1981, p. 271). Thus, there are sequences of Darboux consequences $\overline{\mathbf{y}}_n$ and $\underline{\mathbf{y}}_n$ such that $\int \overline{\mathbf{y}}_n - \int \underline{\mathbf{y}}_n < {}^1/_n$ for n = 1, 2, … . Define $\overline{\mathbf{x}}_n(t) = \min\{\overline{\mathbf{y}}_i(t) : 1 \le i \le n\}$ and $\underline{\mathbf{x}}_n(t) = \max\{\underline{\mathbf{y}}_i(t) : 1 \le i \le n\}$ for n = 1, 2, … . Then, $\{\overline{\mathbf{x}}_n\}_{n=1}^\infty$ and $\{\underline{\mathbf{x}}_n\}_{n=1}^\infty$ are sequences of functions in S_T that satisfy (iv), (v).

To show (vi), consider the sequence $f_n(t) = \overline{\mathbf{x}}_n(t) - \underline{\mathbf{x}}_n(t)$, n = 1, 2, … , of step functions. By properties (c), (d), the functions f_n are non-negative with $f_1 \ge f_2 \ge \ldots$ and $\lim_{n\to\infty} \int f_n = 0$. Hence, $\lim_{n\to\infty} f_n(t) = 0$ a.e. by Lemma A.2. Thus, for example $\lim_{n\to\infty} \overline{\mathbf{x}}_n(t) = \mathbf{x}(t)$ a.e. (since $\underline{\mathbf{x}}_n \le \mathbf{x} \le \overline{\mathbf{x}}_n$). Hence, $\lim_{n\to\infty} a(t)v(\overline{\mathbf{x}}_n(t)) = a(t)v(\mathbf{x}(t))$ a.e., and since $a(t)v(\overline{\mathbf{x}}_n(t))$, n = 1, 2, … , is a weakly decreasing sequence of Lebesgue integrable functions, the Monotone Convergence

Theorem implies that $\lim_{n\to\infty} \int_0^T a(t)v(\overline{\mathbf{x}}_n(t))\,dt = \int_0^T a(t)v(\mathbf{x}(t))dt$. By a similar argument, the sequence of integrals $\int_0^T a(t)v(\underline{\mathbf{x}}_n(t))dt$ converges to $\int_0^T a(t)v(\mathbf{x}(t))dt$ as n tends to ∞.

Proof of Theorem 4.1. For the forward part of the proof, assume that a preference relation $\succsim$ defined on a set R_T satisfies the stated conditions. Then, $\succsim$ restricted to the subset S_T satisfies the conditions of Lemma 4.1. Hence, there exist functions $A(t)$, $v(x)$, and V with the properties (a)-(c) in Lemma 4.1 such that $V(\mathbf{x}) = \sum_{i=1}^{m}\big(A(a_i) - A(a_{i-1})\big)v(x_i)$ represents $\succsim$ on the set S_T.

As noted in Section 4, the properties of $A(t)$ imply that there exists a non-negative, Lebesgue integrable function $a(t)$ on the interval $[0,T)$ such that $A(t) = \int_0^t a(s)ds$. Thus, for any function $\mathbf{x}$ in S_T, the value $V(\mathbf{x})$ can be written in the integral form, $V(\mathbf{x}) = \int_0^T a(t)v(\mathbf{x}(t))dt$. Moreover, Lemma A.3(a) shows that the Lebesgue integral $V(\mathbf{x}) = \int_0^T a(t)v(\mathbf{x}(t))dt$ exists for any $\mathbf{x}$ in R_T.

Next, we show that V represents the space $(R_T, \succsim)$, i.e., $\mathbf{x} \succsim \mathbf{y}$ if and only if $V(\mathbf{x}) \geq V(\mathbf{y})$ for any $\mathbf{x}$, $\mathbf{y}$ in R_T. To do so, we show the properties below. Properties (ii) and (iii) are sufficient. For $\mathbf{x} \succsim \mathbf{y}$ implies not $\mathbf{x} \prec \mathbf{y}$ which implies $V(\mathbf{x}) \geq V(\mathbf{y})$ by (ii). This states that $V(\mathbf{x}) > V(\mathbf{y})$ or $V(\mathbf{x}) = V(\mathbf{y})$ which implies $\mathbf{x} \succ \mathbf{y}$ or $\mathbf{x} \sim \mathbf{y}$ by (ii) and (iii) which implies $\mathbf{x} \succsim \mathbf{y}$.

(i) For any function $\mathbf{x}$ in R_T and any $\varepsilon > 0$, there exists a function $\mathbf{w}$ in S_T such that $\mathbf{w} \sim \mathbf{x}$ and $|V(\mathbf{w}) - V(\mathbf{x})| < \varepsilon$.

Consider any $\mathbf{x}$ in R_T and any $\varepsilon > 0$. By Lemma A.3(c), there exist $\overline{\mathbf{x}}$, $\underline{\mathbf{x}}$ in S_T such that $\underline{\mathbf{x}} \leq \mathbf{x} \leq \overline{\mathbf{x}}$ and $|V(\overline{\mathbf{x}}) - V(\underline{\mathbf{x}})| < \varepsilon$. Then, $\underline{\mathbf{x}} \precsim \mathbf{x} \precsim \overline{\mathbf{x}}$ by condition (A), and $V(\underline{\mathbf{x}}) \leq V(\mathbf{x}) \leq V(\overline{\mathbf{x}})$ since for any t the integrand $a(t)v(x)$ in V is weakly increasing as a function of x.

If $\overline{\mathbf{x}} \sim \mathbf{x}$ or $\underline{\mathbf{x}} \sim \mathbf{x}$, we are through. Otherwise, $\underline{\mathbf{x}} \prec \mathbf{x} \prec \overline{\mathbf{x}}$. In this case, define the functions, $\mathbf{x}_\lambda = \lambda\overline{\mathbf{x}} + (1-\lambda)\underline{\mathbf{x}}$, $0 \leq \lambda \leq 1$, and the sets $L = \{\lambda \in [0,1] : \mathbf{x}_\lambda \prec \mathbf{x}\}$, $U = \{\lambda \in [0,1] : \mathbf{x}_\lambda \succ \mathbf{x}\}$. Then, L and U are disjoint, 0 is in L, and 1 is in U. Moreover, L and U are open relative to $[0,1]$. To verify this, first note that $\int|\mathbf{x}_\mu - \mathbf{x}_\lambda| \leq T|\mu - \lambda|(\sup\mathbf{x} - \inf\mathbf{x})$ for any λ, μ in $[0,1]$. Define $k = T(\sup\mathbf{x} - \inf\mathbf{x})$. Assume that $k > 0$. Consider, e.g., an amount λ in L, that is, $\mathbf{x}_\lambda \prec \mathbf{x}$. Then, by condition (C) there is a $\delta > 0$ such that $\int|\mathbf{x}_\mu - \mathbf{x}_\lambda| < \delta$ implies $\mathbf{x}_\mu \prec \mathbf{x}$ for any μ in $[0,1]$. Hence, $|\mathbf{x}_\mu - \mathbf{x}_\lambda| \leq \delta k^{-1}$ implies $\mathbf{x}_\mu \prec \mathbf{x}$ which implies μ is in L. Since $[0,1]$ is connected, it follows that there exists a ν in $[0,1]$ that is not in L or U. Hence, $\mathbf{x}_\nu \sim \mathbf{x}$ by the completeness condition (B). Moreover, $\underline{\mathbf{x}} \le \mathbf{x}_\nu \le \overline{\mathbf{x}}$ implies $V(\underline{\mathbf{x}}) \le V(\mathbf{x}_\nu) \le V(\overline{\mathbf{x}})$ implies $|V(\mathbf{x}_\nu) - V(\mathbf{x})| < \varepsilon$. In the case $k = 0$, there is a constant b such that $\mathbf{x}(t) = b$ a.e., and thus $\mathbf{x} \sim \mathbf{b}$ by condition (A).

(ii) $V(\mathbf{x}) < V(\mathbf{y})$ implies $\mathbf{x} \prec \mathbf{y}$ for any functions **x**, **y** in R_T.

Consider any **x**, **y** in R_T with $V(\mathbf{x}) < V(\mathbf{y})$. Define $\varepsilon = {}^1\!/_2\,(V(\mathbf{y}) - V(\mathbf{x}))$. By (i) above, there exist **w**, **z** in S_T such that $\mathbf{w} \sim \mathbf{x}$, $\mathbf{z} \sim \mathbf{y}$, $|V(\mathbf{w}) - V(\mathbf{x})| < \varepsilon$, and $|V(\mathbf{z}) - V(\mathbf{y})| < \varepsilon$. Thus, $V(\mathbf{w}) < V(\mathbf{z})$. Since the function V represents preferences on the set S_T it follows that $\mathbf{w} \prec \mathbf{z}$. Hence, by transitivity we have $\mathbf{x} \prec \mathbf{y}$.

Property (ii) implies the following property:

(ii′) $\mathbf{x} \sim \mathbf{y}$ implies $V(\mathbf{x}) = V(\mathbf{y})$ for any **x**, **y** in R_T.

Consider any **x**, **y** in R_T with $\mathbf{x} \sim \mathbf{y}$. If $V(\mathbf{x}) < V(\mathbf{y})$ or $V(\mathbf{x}) > V(\mathbf{y})$, then $\mathbf{x} \prec \mathbf{y}$ or $\mathbf{x} \succ \mathbf{y}$ by property (ii) which contradicts $\mathbf{x} \sim \mathbf{y}$.

(iii) $V(\mathbf{x}) = V(\mathbf{y})$ implies $\mathbf{x} \sim \mathbf{y}$ for any **x**, **y** in R_T.

Consider any **x**, **y** in R_T with $V(\mathbf{x}) = V(\mathbf{y})$. By property (i) there exist functions **w**, **z** in S_T such that $\mathbf{w} \sim \mathbf{x}$ and $\mathbf{z} \sim \mathbf{y}$. By (ii′), it follows that $V(\mathbf{w}) = V(\mathbf{x})$ and $V(\mathbf{z}) = V(\mathbf{y})$. Hence, $V(\mathbf{w}) = V(\mathbf{z})$. Since V represents preferences on the set S_T, this equality implies that $\mathbf{w} \sim \mathbf{z}$. Therefore, $\mathbf{x} \sim \mathbf{y}$ by transitivity.

For the converse part of the proof, assume that a function V of the form (2) is well-defined on a set R_T, that a preference relation $\succsim$ is represented by V on R_T, and that the functions $a(t)$, $A(t)$, $v(x)$, and V satisfy the properties (b), (c). Then, $\succsim$ satisfies condition (A) on the set R_T by Lemma A.3(b), it satisfies condition (B) on R_T since V is a value function, and it satisfies the conditions (D), (E) on the set S_T by Lemma 4.1.

To show that $\succsim$ satisfies condition (C) on the pair of sets R_T, S_T, consider any functions **x** in R_T and **w** in S_T. If $\mathbf{w} \prec \mathbf{x}$, then $V(\mathbf{w}) < V(\mathbf{x})$. Define $\varepsilon = V(\mathbf{x}) - V(\mathbf{w}) > 0$. By property (c) in Lemma 4.1, there exists a $\delta > 0$ such that: $\int |\mathbf{z} - \mathbf{w}| < \delta$ implies $|V(\mathbf{z}) - V(\mathbf{w})| < \varepsilon$ for any **z** in S_T. But $|V(\mathbf{z}) - V(\mathbf{w})| < \varepsilon$ implies $V(\mathbf{z}) < V(\mathbf{x})$ implies $\mathbf{z} \prec \mathbf{x}$. The argument when $\mathbf{w} \succ \mathbf{x}$ is similar, and thus condition (C) is satisfied.

By Lemma 4.1, each of the functions $A(t)$ and $v(x)$ is unique up to a positive multiple.

Proof of Lemma 5.1 First, we show the forward implications. The assumptions in Lemma 5.1 on a space $(R_T, \succsim)$ imply the assumptions in Theorem 4.1 for each horizon $T > 0$. Thus, for each $T > 0$ there are functions $a_T(t)$, $A_T(t)$, $v_T(t)$, and $V_T(\mathbf{x}) = \int_0^T a_T(t) v_T(\mathbf{x}(t))dt$ with the properties described in Theorem 4.1 such that V_T represents the space $(R_T, \succsim)$.

As noted in Section 5, we identify a function $\mathbf{x}$ in a set R_T with a function in R_f by defining $\mathbf{x}(t) = 0$ for $t > T$. Then, for any two horizons $T < T^*$, S_T is a subset of S_{T^*} and R_T is a subset of R_{T^*}. Moreover, $R_f = \bigcup_{T \geq U} R_T$ for any $U < \infty$. For our purposes, we use: $R_f = \bigcup_{T \geq 1} R_T$.

For each $T \geq 1$, choose functions $A_T(t)$, $v_T(x)$ that are normalized such that $A_T(1) = 1$ and either $v_T(x^+) = 1$ or $v_T(x^-) = -1$. Theorem 4.1 implies that such functions are unique. Thus, for any horizons $T < T^*$ in $[1, \infty)$ we have: $A_{T^*}(t) = A_T(t)$ for $0 \leq t \leq T$, $v_{T^*}(x) = v_T(x)$ for x in I, and $V_{T^*}(\mathbf{x}) = V_T(\mathbf{x})$ for $\mathbf{x}$ in R_T. Hence, the following functions are well-defined: the function $A(t)$, $0 \leq t < \infty$, defined by $A(t) = A_T(t)$ for $0 \leq t \leq T$, the function $v(x)$, x in I, defined by $v(x) = v_T(x)$, and the function $V(\mathbf{x})$, $\mathbf{x}$ in R_f, defined by $V(\mathbf{x}) = V_T(\mathbf{x})$ for $\mathbf{x}$ in R_T. We also define a function $a(t)$, $0 \leq t < \infty$, by $a(t) = A'(t)$ where $A'(t)$ exists and $a(t) = 0$ otherwise. These definitions imply that, $V(\mathbf{x}) = \lim_{T \to \infty} V_T(\mathbf{x}) = \lim_{T \to \infty} \int_0^T a(t) v(\mathbf{x}(t))\, dt$, for any function $\mathbf{x}$ in the set R_f.

By Theorem 4.1, the functions $a(t)$, $A(t)$, $v(x)$, V have the properties (a)-(c) in Lemma 5.1. And since $V(\mathbf{x}) = V_T(\mathbf{x})$ for any $\mathbf{x}$ in R_T, the function V represents each space $(R_T, \succsim)$.

Moreover, V represents the space $(R_f, \succsim)$. For consider any $\mathbf{x}$, $\mathbf{x}^*$ in R_f. Then, $\mathbf{x}$ is in R_T and $\mathbf{x}^*$ is in R_{T^*} for some $T, T^* \geq 1$. Suppose that $T \leq T^*$. Then, both $\mathbf{x}$ and $\mathbf{x}^*$ are in R_{T^*}, and thus they can be compared by the function V_{T^*}. Hence, $\mathbf{x} \succsim \mathbf{x}^*$ if and only if $V_{T^*}(\mathbf{x}) \geq V_{T^*}(\mathbf{x}^*)$ if and only if $V(\mathbf{x}) \geq V(\mathbf{x}^*)$ since $V(\mathbf{x}) = V_{T^*}(\mathbf{x})$ and $V(\mathbf{x}^*) = V_{T^*}(\mathbf{x}^*)$.

To show the converse implications, assume that there are functions $a(t)$, $A(t)$, $v(x)$, and V as described in Lemma 5.1. Then, Theorem 4.1 implies that for each horizon $T > 0$ the preference relation $\succsim$ satisfies the conditions (A), (B) on R_T, satisfies the condition (C) on the pair R_T, S_T, and satisfies the conditions (D), (E) on S_T. Since R_f is the union of the sets R_T, $T > 0$, it follows that $\succsim$ satisfies the conditions (A)–(E) on the sets specified in Lemma 5.1.

To show that each of the functions $A(t)$, $v(x)$ is unique up to a positive multiple, consider two functions $V(\mathbf{x}) = \lim_{T \to \infty} \int_0^T a(t) v(\mathbf{x}(t))\, dt$, $V^*(\mathbf{x}) = \lim_{T \to \infty} \int_0^T a^*(t) v^*(\mathbf{x}(t))\, dt$ with the associated functions $A(t) = \int_0^t a(s)ds$ and $A^*(t) = \int_0^t a^*(s)ds$. For $T > 0$, the functions V, V^* restricted to

R_T reduce to: $V(\mathbf{x}) = \int_0^T a(t)v(\mathbf{x}(t))dt$ and $V^*(\mathbf{x}) = \int_0^T a^*(t)v^*(\mathbf{x}(t))dt$. Thus, by Theorem 4.1 there are constants $c_T > 0$, $d_T > 0$ such that $A(t) = c_T A^*(t)$ for $0 \le t \le T$ and $v(x) = d_T v^*(x)$ for x in I. Hence, for any $T^* \ge T$ we have: $c_T = c_{T^*}$ since $A(t)$, $A^*(t)$ are nonzero for $t > 0$, and $d_{T^*} = d_T$ since I contains nonzero outcomes. Define $c > 0$ as the common value of c_T and define $d > 0$ as the common value of d_T. Then, $A(t) = c\,A^*(t)$ for $0 \le t \le \infty$ and $v(x) = d\,v^*(x)$ for x in I.

Proof of Lemma 5.2 To show (a), consider a function $\mathbf{x}$ in $R_{\succsim}$ and a function $\mathbf{y}$ in R_∞. If there exists a time $T^* > 0$ such that $\mathbf{x}(t) = \mathbf{y}(t)$ for any $t \ge T^*$, then $\mathbf{x}_{[t,\infty)} = \mathbf{y}_{[t,\infty)}$ for any $t \ge T^*$, and thus $\mathbf{y}$ satisfies Definition 5.1 with T replaced by $\max\{T, T^*\}$.

To show (b), consider a function $\mathbf{x}$ in R_f Then, there is a $T \ge 1$ such that $\mathbf{x}_{[t,\infty)} = \mathbf{0}_{[t,\infty)}$ for $t \ge T$. But $a < b < c$ implies that $\mathbf{a}_{[0,1)} \prec \mathbf{b}_{[0,1)} \prec \mathbf{c}_{[0,1)}$ since $\succsim$ is strictly increasing on the set of constant functions on [0,1). Hence, $\mathbf{x}$ satisfies Definition 5.1.

The result below provides alternative definitions of comparability. Part (a) is a special case of part (c), and Definition 5.1 is a special case of part (b).

Lemma A.4 Suppose that a preference relation $\succsim$ on R_∞ satisfies the conditions in Theorem 5.1 and that $\mathbf{x}$ is a function in $R_{\succsim}$. Then:

(a) For any amounts $a < b < c$ in I, there is a horizon $T \ge 1$ such that for any $t \ge T$:

$$\mathbf{a}_{[0,1)} + \mathbf{x}_{[1,t)} \prec \mathbf{b}_{[0,1)} + \mathbf{x}_{[1,\infty)} \prec \mathbf{c}_{[0,1)} + \mathbf{x}_{[1,t)} \qquad \text{(A3)}$$

(b) For any amounts $a < b < c$ in I and any bounded nonpoint interval $[\alpha,\beta)$, there is a $T \ge \beta$ such that for any $t \ge T$:

$$\mathbf{a}_{[\alpha,\beta)} \prec \mathbf{b}_{[\alpha,\beta)} + \mathbf{x}_{[t,\infty)} \prec \mathbf{c}_{[\alpha,\beta)} \qquad \text{(A4)}$$

(c) For any amounts $a < b < c$ in I and any bounded nonpoint interval $[\alpha,\beta)$, there is a $T \ge \beta$ such that for any $t \ge T$:

$$\mathbf{x}_{[0,\alpha)} + \mathbf{a}_{[\alpha,\beta)} + \mathbf{x}_{[\beta,t)} \prec \mathbf{x}_{[0,\alpha)} + \mathbf{b}_{[\alpha,\beta)} + \mathbf{x}_{[\beta,\infty)} \prec \mathbf{x}_{[0,\alpha)} + \mathbf{c}_{[\alpha,\beta)} + \mathbf{x}_{[\beta,t)} \qquad \text{(A5)}$$

Proof. For each part, consider a function **x** in $R_{\succsim}$ and amounts $a < b < c$ in I. We show only the first preferences in (A3)-(A5) since the second preferences can be obtained by similar arguments.

For (A3), note that there is a time $T \geq 1$ such that $\mathbf{a}_{[0,1)} \prec \mathbf{b}_{[0,1)} + \mathbf{x}_{[t,\infty)}$ for any $t \geq T$. Thus, by preferential independence we can switch from $\mathbf{0}_{[1,t)}$ to $\mathbf{x}_{[1,t)}$ in these functions to obtain the preference, $\mathbf{a}_{[0,1)} + \mathbf{x}_{[1,t)} \prec \mathbf{b}_{[0,1)} + \mathbf{x}_{[1,t)} + \mathbf{x}_{[t,\infty)} = \mathbf{b}_{[0,1)} + \mathbf{x}_{[1,\infty)}$, for any $t \geq T$.

For (A4) and (A5), define the sets $E = [0,1] - [\alpha,\beta)$ and $F = [\alpha,\beta) - [0,1]$. Then, E, F, and $[0,1] \cap [\alpha,\beta)$ are disjoint intervals and $E \cup [\alpha,\beta) = [0,1] \cup F$.

For (A4), note that $\mathbf{b}_E + \mathbf{a}_{[\alpha,\beta)} \prec \mathbf{b}_E + \mathbf{b}_{[\alpha,\beta)}$ since preferences are strictly increasing. Thus, there exists an a^+ with $a < a^+ < b$ such that: (i) $\mathbf{b}_E + \mathbf{a}_{[\alpha,\beta)} \prec \mathbf{a}^+_E + \mathbf{b}_{[\alpha,\beta)}$ since preferences are continuous. (This preference can be regarded as a tradeoffs statement.) There exists a $T \geq 1$ such that $\mathbf{a}^+_{[0,1)} \prec \mathbf{b}_{[0,1)} + \mathbf{x}_{[t,\infty)}$ for all $t \geq T$ since **x** is in $R_{\succsim}$. Define $T^* = \max\{T,\beta\}$. Then: (ii) $\mathbf{a}^+_{[0,1)} + \mathbf{b}_F \prec \mathbf{b}_{[0,1)} + \mathbf{b}_F + \mathbf{x}_{[t,\infty)} = \mathbf{b}_E + \mathbf{b}_{[\alpha,\beta)} + \mathbf{x}_{[t,\infty)}$ for $t \geq T^*$ by preferential independence. Hence, (i) and (ii) imply that $\mathbf{b}_E + \mathbf{a}_{[\alpha,\beta)} \prec \mathbf{b}_E + \mathbf{b}_{[\alpha,\beta)} + \mathbf{x}_{[t,\infty)}$ for $t \geq T^*$ by transitivity which implies that $\mathbf{a}_{[\alpha,\beta)} \prec \mathbf{b}_{[\alpha,\beta)} + \mathbf{x}_{[t,\infty)}$ for all $t \geq T^*$ by preferential independence.

Finally, the first preference in (A4) implies the first preference in (A5) by using preferential independence to add the terms $\mathbf{x}_{[0,\alpha)}$ and $\mathbf{x}_{[\beta,t)}$ to both sides.

Definition 5.1 and Theorem 5.1 do not depend on whether the outcome interval I contains an upper endpoint or a lower endpoint. But the proof of Theorem 5.1 does require special arguments for such cases. Here, we make a few definitions. Define $u = \sup I$ and $l = \inf I$. Then, $I = \langle l,u \rangle$ where $l \geq -\infty$ and $u \leq \infty$. If I contains u (and thus u is finite), then the constant function $\mathbf{u}_{[0,\infty)}$ is in R_∞. However, it may or may not be in $R_{\succsim}$. Similar remarks apply if I contains l.

Definition A.1 A function **x** in R_∞ with $\mathbf{x}(t) = u$ a.e. will be called an upper extreme function; a function **x** in R_∞ with $\mathbf{x}(t) = l$ a.e. will be called a lower extreme function; and a function in R_∞ of neither type will be called a non-extreme function.

Proof of Theorem 5.1 To show the forward implications, assume that a preference relation $\succsim$ defined on a set R_∞ satisfies the stated conditions. Then, R_f is a subset of $R_{\succsim}$ by Lemma 5.2. Hence, $\succsim$ restricted to R_f satisfies the conditions of Lemma 5.1, and thus there exist functions

$a(t)$, $A(t)$, $v(x)$, V with the properties (a)-(c) in Lemma 5.1, and $V(\mathbf{x}) = \lim_{T\to\infty} \int_0^T a(t)v(\mathbf{x}(t))dt$ represents $\succsim$ on the set R_f.

To show that $V(\mathbf{x})$ converges for any $\mathbf{x}$ in $R_{\succsim}$, it suffices to show that for any $\varepsilon > 0$ there is a $T > 0$ such that $V(\mathbf{x}_{[0,s)}) - V(\mathbf{x}_{[0,t)}) < \varepsilon$ for any $s, t \geq T$. For it follows that $\{V(\mathbf{x}_{[0,n]})\}_{n=1}^{\infty}$ is a Cauchy sequence and thus has a limit V, and that $\lim_{T\to\infty} V(\mathbf{x}_{[0,t)}) = V$.

For $\mathbf{x}$ in $R_{\succsim}$ and $\varepsilon > 0$, choose $a < b < c$ in I such that $A(1)(v(c) - v(a)) < \varepsilon$. Lemma A.4(a) implies that there is a $T \geq 1$ such that $\mathbf{a}_{[0,1)} + \mathbf{x}_{[1,s)} \prec \mathbf{b}_{[0,1)} + \mathbf{x}_{[1,\infty)} \prec \mathbf{c}_{[0,1)} + \mathbf{x}_{[1,t)}$ for $s, t \geq T$. This preference implies $\mathbf{a}_{[0,1)} + \mathbf{x}_{[1,s)} \prec \mathbf{c}_{[0,1)} + \mathbf{x}_{[1,t)}$ since $\succsim$ is transitive on $R_{\succsim}$ which implies $V(\mathbf{a}_{[0,1)} + \mathbf{x}_{[1,s)}) < V(\mathbf{c}_{[0,1)} + \mathbf{x}_{[1,t)})$ since V represents $\succsim$ on R_f. By adding $V(\mathbf{x}_{[0,1)})$ to both sides and evaluating the integrals, this implies $A(1)v(a) + V(\mathbf{x}_{[0,s)}) < A(1)v(c) + V(\mathbf{x}_{[0,t)})$ which implies $V(\mathbf{x}_{[0,s)}) - V(\mathbf{x}_{[0,t)}) < A(1)\,(v(c) - v(a)) < \varepsilon$.

Next, we show that the function V represents $\succsim$ on the set $R_{\succsim}$. We do so by showing (i)–(iii) below. These properties correspond to the properties (i)–(iii) in the proof of Theorem 4.1.

(i) For any non-extreme function $\mathbf{x}$ in $R_{\succsim}$ and any $\varepsilon > 0$, there exist a horizon $T > 0$ and a step function $\mathbf{w}$ in S_T such that $\mathbf{w} \sim \mathbf{x}$ and $|V(\mathbf{w}) - V(\mathbf{x})| < \varepsilon$.

For a bounded, nonpoint interval $[\alpha,\beta)$, define $\bar{b} = \sup\{\mathbf{x}(t) : \alpha \leq t < \beta\}$ and $\underline{b} = \inf\{\mathbf{x}(t) : \alpha \leq t < \beta\}$. Since $\mathbf{x}$ is a Riemann consequence on $[0,\beta]$, $\bar{b}$ and $\underline{b}$ are finite and in the interval I. There exists an interval $[\alpha,\beta)$ as described such that $\bar{b} < u$ and $V(\bar{\mathbf{b}}_{[\alpha,\beta)}) - V(\mathbf{x}_{[\alpha,\beta)}) < \varepsilon/8$, and there exists an interval $[\alpha,\beta)$ as described (possibly a different interval) such that $\underline{b} > l$ and $V(\mathbf{x}_{[\alpha,\beta)}) - V(\underline{\mathbf{b}}_{[\alpha,\beta)}) < \varepsilon/8$. We prove only the first half of these assertions since the argument for the second half is similar.

If the interval I has the form, $I = \langle l, u)$, choose a point t' at which $\mathbf{x}$ is continuous. Then, there is a interval $[\alpha,\beta)$ containing t' such that $\bar{b} < u$. If I has the form, $I = \langle l, u]$, then $\mathbf{u}_{[0,\infty)}$ is in R_∞ and $\mathbf{x}(t) \leq \mathbf{u}_{[0,\infty)}(t)$ for all t in $[0,\infty)$. Since $\mathbf{x}$ is a non-extreme function, Lemma A.3(b) implies that there is a bounded nonpoint interval $[\alpha,\beta)$ such that $\bar{b} < u$.

We may need to shrink the interval $[\alpha,\beta)$ in order to obtain, $V(\bar{\mathbf{b}}_{[\alpha,\beta)}) - V(\mathbf{x}_{[\alpha,\beta)}) < \varepsilon/8$. For any γ in $\alpha < \gamma \leq \beta$: $V(\bar{\mathbf{b}}_{[\alpha,\gamma)}) - V(\mathbf{x}_{[\alpha,\gamma)}) \leq V(\bar{\mathbf{b}}_{[\alpha,\gamma)}) - V(\underline{\mathbf{b}}_{[\alpha,\gamma)}) = (A(\gamma) - A(\alpha))(\bar{b} - \underline{b})$ where $\bar{b}$, $\underline{b}$ are based on the interval $[\alpha,\gamma)$. Since the difference, $\bar{b} - \underline{b}$, weakly decreases as γ

decreases, we can choose γ sufficiently close to α to obtain the inequality. By an abuse of notation, we will denote the chosen γ by β. Then, $\bar{b} < u$ and $V(\bar{\mathbf{b}}_{[\alpha,\beta)}) - V(\mathbf{x}_{[\alpha,\beta)}) < \varepsilon/8$.

Next, we show that for some $T > 0$ there exist functions $\underline{\mathbf{w}}$, $\bar{\mathbf{w}}$ in S_T such that $\underline{\mathbf{w}} \prec \mathbf{x} \prec \bar{\mathbf{w}}$ and $|V(\bar{\mathbf{w}}) - V(\mathbf{x})| < \varepsilon/2$, $|V(\mathbf{x}) - V(\underline{\mathbf{w}})| < \varepsilon/2$. Again, we show only half of these assertions since the argument for the other half is similar.

The function $\bar{\mathbf{x}} = \mathbf{x}_{[0,\alpha)} + \bar{\mathbf{b}}_{[\alpha,\beta)} + \mathbf{x}_{[\beta,\infty)}$ is in R_∞. And by Lemma 5.2(a) it is in $R_{\succsim}$ since $\bar{\mathbf{x}}(t) = \mathbf{x}(t)$ for $t \geq \beta$. Thus, $\mathbf{x} \precsim \bar{\mathbf{x}}$ since preferences are increasing on $R_{\succsim}$. Below, we choose an amount c such that $\bar{b} < c < u$. Define $\bar{\mathbf{y}}_t = \mathbf{x}_{[0,\alpha)} + \mathbf{c}_{[\alpha,\beta)} + \mathbf{x}_{[\beta,t)}$ for $t \geq \beta$. By Lemma A.4(c), there exists a $T \geq \beta$ such that $\bar{\mathbf{x}} \prec \bar{\mathbf{y}}_t$ for $t \geq T$. Below, we choose a time $t \geq T$. By property (i) in the proof of Theorem 4.1 there exists a function $\bar{\mathbf{w}}$ in S_t such that $\bar{\mathbf{w}} \sim \bar{\mathbf{y}}_t$. Thus, in summary we have $\mathbf{x} \precsim \bar{\mathbf{x}} \prec \bar{\mathbf{y}}_t \sim \bar{\mathbf{w}}$, and by transitivity it follows that $\mathbf{x} \prec \bar{\mathbf{w}}$.

It remains to chose c and t such that $|V(\bar{\mathbf{w}}) - V(\mathbf{x})| < \varepsilon/2$. We will compare $V(\mathbf{x})$ and $V(\bar{\mathbf{w}})$ in four steps. First, we have chosen an interval $[\alpha,\beta)$ with $|V(\bar{\mathbf{b}}_{[\alpha,\beta)}) - V(\mathbf{x}_{[\alpha,\beta)})| < \varepsilon/8$ and thus $|V(\bar{\mathbf{x}}) - V(\mathbf{x})| < \varepsilon/8$. Define $\bar{\mathbf{y}} = \mathbf{x}_{[0,\alpha)} + \mathbf{c}_{[\alpha,\beta)} + \mathbf{x}_{[\beta,\infty)}$. Second, we choose c such that $|V(\bar{\mathbf{c}}_{[\alpha,\beta)}) - V(\bar{\mathbf{b}}_{[\alpha,\beta)})| < \varepsilon/8$ and thus $|V(\bar{\mathbf{y}}) - V(\bar{\mathbf{x}})| < \varepsilon/8$. Third, since $V(\bar{\mathbf{y}}_t)$ converges to $V(\bar{\mathbf{y}})$ as t tends to ∞, we can choose t such that $|V(\bar{\mathbf{y}}_t) - V(\bar{\mathbf{y}})| < \varepsilon/8$. Fourth, by (i) in the proof of Theorem 4.1 there exists a function $\bar{\mathbf{w}}$ in S_t such that $\bar{\mathbf{w}} \sim \bar{\mathbf{y}}_t$ and $|V(\bar{\mathbf{w}}) - V(\bar{\mathbf{y}}_t)| < \varepsilon/8$. By combining the four inequalities, we conclude that $|V(\bar{\mathbf{w}}) - V(\mathbf{x})| < \varepsilon/2$.

Now, define $\mathbf{w}_\lambda = \lambda \bar{\mathbf{w}} + (1-\lambda)\underline{\mathbf{w}}$, $0 \leq \lambda \leq 1$. Then, $|V(\mathbf{w}_\lambda) - V(\mathbf{x})| = |\lambda\big(V(\bar{\mathbf{w}}) - V(\mathbf{x})\big) + (1-\lambda)\big(V(\underline{\mathbf{w}}) - V(\mathbf{x})\big)| < \lambda \cdot \varepsilon/2 + (1-\lambda)\varepsilon/2 = \varepsilon$. We can use the continuity condition (C) as in the proof of Theorem 4.1 to show that there is an amount ν in [0,1] such that $\mathbf{w}_\nu \sim \mathbf{x}$.

(ii) $V(\mathbf{x}) < V(\mathbf{y})$ implies $\mathbf{x} \prec \mathbf{y}$ for any functions **x**, **y** in $R_{\succsim}$.

To show (ii), first consider non-extreme functions **x** and **y**. Define $\varepsilon = 1/2\,(V(\mathbf{y}) - V(\mathbf{x})) > 0$. By (i) above, there exists a horizon $T > 0$ and step functions **w**, **z** in S_T such that $\mathbf{w} \sim \mathbf{x}$, $\mathbf{z} \sim \mathbf{y}$, $|V(\mathbf{w}) - V(\mathbf{x})| < \varepsilon$, and $|V(\mathbf{z}) - V(\mathbf{y})| < \varepsilon$. Thus, $V(\mathbf{w}) < V(\mathbf{z})$, and it follows that $\mathbf{w} \prec \mathbf{z}$ since V represents preferences on the set R_f. Hence, by transitivity we have $\mathbf{x} \prec \mathbf{y}$.

Next, assume that **y** is an upper extreme function. Then, $\mathbf{x}(t) \leq \mathbf{y}(t)$ a.e. on $[0,\infty)$. Moreover, $V(\mathbf{x}) < V(\mathbf{y})$ implies $V(\mathbf{x}_{[0,t)}) < V(\mathbf{y}_{[0,t)})$ for some $t > 0$, and thus by Lemma A.3(b) there is a nonpoint interval $[\alpha,\beta)$ such that **x**(*t*) < **y**(*t*) for *t* in $[\alpha,\beta)$. Hence, $\mathbf{x} \prec \mathbf{y}$ by condition (A). In the case that **x** is a lower extreme function, the arguments are similar.

Finally, if **x** were upper extreme function this would imply $V(\mathbf{x}_{[0,t)}) \geq V(\mathbf{y}_{[0,t)})$ for all $t > 0$ which implies $V(\mathbf{x}) \geq V(\mathbf{y})$. And **y** cannot be a lower extreme function by a similar argument.

(iii) $V(\mathbf{x}) = V(\mathbf{y})$ implies $\mathbf{x} \sim \mathbf{y}$ for any functions **x**, **y** in $R_{\succsim}$.

First, assume that **x** and **y** are non-extreme functions . Then, by (i) there are step functions **w**, **z** such that $\mathbf{w} \sim \mathbf{x}$ and $\mathbf{z} \sim \mathbf{y}$. By arguing as in (ii′) in the proof of Theorem 4.1, it follows that $V(\mathbf{w}) = V(\mathbf{x})$ and $V(\mathbf{z}) = V(\mathbf{y})$. Hence, $V(\mathbf{w}) = V(\mathbf{z})$. This equality implies $\mathbf{w} \sim \mathbf{z}$ since V represents preferences on the set R_f. Therefore, $\mathbf{x} \sim \mathbf{y}$ by transitivity.

Next, assume that **y** is an upper extreme function. Then, $\mathbf{x}(t) \leq \mathbf{y}(t)$ a.e. on $[0,\infty)$. Thus, by Lemma A.3(b) the statement $V(\mathbf{x}) = V(\mathbf{y})$ implies $\mathbf{x}(t) = \mathbf{y}(t)$ a.e. on $[0,\infty)$. Hence, $\mathbf{x} \sim \mathbf{y}$ by condition (A). The arguments are the same in the case that **x** is an upper extreme function, and they are similar in the case that **x** or **y** is a lower extreme function.

To show the converse implications, assume that a function *V* of the form (7) is well-defined on a set $R_{\succsim}$, that a preference relation $\succsim$ is represented by *V*, and that the functions *a*(*t*), *A*(*t*), *v*(*x*), and *V* satisfy the properties in (b). Then, it is straightforward to verify that $\succsim$ satisfies the conditions (A), (B), and (D) on the set $R_{\succsim}$. Lemma 5.1 states that $\succsim$ satisfies condition (E) on each set S_T, $T > 0$, and one can use an argument similar to that in Theorem 4.1 to show that $\succsim$ satisfies condition (C) on each pair of sets $R_{\succsim}$, S_T, $T > 0$.

By Lemma 5.1, each of the functions *A*(*t*) and *v*(*x*) is unique up to a positive multiple.

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