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Kings in the Direct Product of Digraphs

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KINGS IN THE DIRECT PRODUCT OF DIGRAPHS

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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Abstract

KINGS IN THE DIRECT PRODUCT OF DIGRAPHS

By Morgan Norge, Master of Science

Virginia Commonwealth University, 2019.

Director: Richard Hammack, Professor, Department of Mathematics and Applied Mathematics.

A ***k*-king** in a digraph D is a vertex that can reach every other vertex in D by a directed path of length at most k. A **king** is a vertex that is a k-king for some k. We will look at kings in the direct product of digraphs and characterize a relationship between kings in the product and kings in the factors. This is a continuation of a project in which a similar characterization is found for the cartesian product of digraphs, the strong product of digraphs, and the lexicographic product of digraphs.

Vita

Morgan Renee Norge was born in Midlothian, VA on May 16, 1996. She graduated from Virginia Commonwealth University with a Bachelor's degree in Applied Mathematics in 2017 and then continued her studies to pursue her Master's in Mathematics from Virginia Commonwealth University.

1 Preliminaries

1.1 Introduction

In this thesis, we consider a problem in graph theory involving kings and product graphs. A graph is simply a structure that is formed with a set of vertices and a set of edges joining vertices that is used to model or represent a relationship. Graph theory can be used in many different fields and is therefore very useful; it is used in computer science, biochemistry, operations research, among many others.

We are going to explore a problem that has to do with a specific type of graph called a digraph. A digraph is also used to model a relationship, but it gives us the ability to model relationships which are directed. For example, a transportation system can be modeled using directed graphs. If we use the example of a train station, and we want to model the route of a train, directed graphs become very useful. In the example of modeling the route of a train, each train station is represented by a vertex, and the route of the train is represented by arcs. For example, given two train stations, Station A and Station B, each train station is represented by a vertex, called v and w. If the train travels from Station B to Stations A, that relationship would be represented by an arc from vertex w to vertex v.

Specifically, we will talk about the direct product of digraphs. The direct product has some very interesting applications, most of which lie in the area of computer science.

In this thesis we will determine the relationship between kings in the direct product of digraphs and its factors. This is a continuation of a project done with Dr. Dewey Taylor and Peter LaBarr, in which we were able to solve the same problem for the cartesian product, the strong product, and the lexicographic product [2].

We will be looking at kings in the direct product of finite digraphs and their factors. Our goal is to come up with necessary and sufficient conditions on two digraphs that ensure that their direct product will have a king, and vice versa. We successfully do so and state our results in Chapter 4.

First, we will review some basic definitions that we will need throughout this paper and we will introduce the direct product. Next, we will discus kings and their background, as well as some interesting applications. We will then move on to state some results by McAndrew that were crucial to us in discovering our results. Finally, we will state and prove our results, as well as explore some future directions.

1.2 Literature Review

We will begin by reviewing some basic Graph Theory definitions that we will need before we formally introduce the problem. First, recall that a graph is an ordered pair, $G(V, E)$, comprised of a set of vertices, V, and a possibly empty set of edges, E consisting of unordered pairs of vertices.

Definition 1. *A **digraph** D is a pair $D = (V(D), A(D))$ where $V(D)$ is a finite set whose elements are called vertices of D and $A(D)$ is a set of ordered pairs of distinct vertices in $V(D)$, called arcs. We view an arc (v_1, v_2) as an arrow pointing from v_1 to v_2.*

Figure 1 shows an example of a digraph D with the vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$ and the arc set $A = \{(v_1, v_2), (v_2, v_3), (v_3, v_1), (v_3, v_5), (v_4, v_2), (v_5, v_4))\}$.

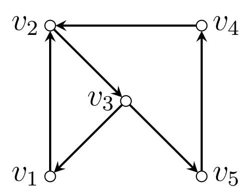

Figure 1: Digraph D

Throughout this thesis, we will primarily be working with digraphs.

Definition 2. *A **walk** in a graph is a sequence of vertices $v_1v_2v_3...v_k$ such that v_iv_{i+1} is an edge for each $1 \leq i \leq k-1$. A **directed walk** in a digraph is a sequence of vertices such that v_iv_{i+1} is an arc for each $1 \leq i \leq k-1$.*

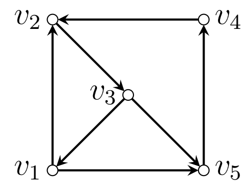

Figure 2: Digraph D

In Figure 2, there are many directed walks; one such example is $\{v_1v_2v_3v_1v_2\}$.

Definition 3. *A **path** in a graph is a walk in which no vertices or edges are repeated. A **directed path** in a digraph is a directed walk in which no vertices or edges are repeated.*

Figure 2 has multiple examples of directed paths; one of which is $\{v_1v_2v_3v_5\}$.

Definition 4. *A **cycle** is a closed path. That is, a path that begins and ends at the same vertex. A **directed cycle** is a closed directed path. That is, a directed path that begins and ends at the same vertex.*

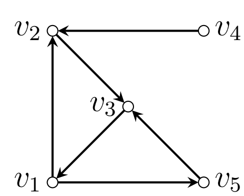

Figure 3: Digraph D

In Figure 3, the sequence of vertices $\{v_1v_2v_3v_1\}$ forms a *directed cycle.* Another example of a directed cycle in digraph D is $\{v_1v_5v_3v_1\}$.

Definition 5. *A vertex v **reaches** another vertex w if there is a directed path from v to w.*

Definition 6. *A **king** in a digraph D is a vertex that can reach every other vertex in D by a directed path. A **k-king** in a digraph D is a vertex that can reach every other vertex in D by a directed path of length at most k.*

The digraph D shown in Figure 4 has a 2-king at the vertex v_1. Figure 5 shows an example of a digraph G with a 3-king at the vertex w_4.

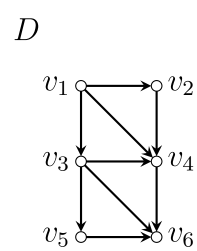

Figure 4: Digraph D, with a 2-king at v_1

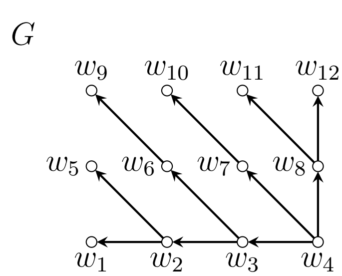

Figure 5: Digraph G, with a 3-king at vertex w_4

Definition 7. *The **direct product** of two digraphs D_1 and D_2 is the digraph denoted by $D_1 \times D_2$ with:*

$$V(D_1 \times D_2) = V(D_1) \times V(D_2)$$

and

$$A(D_1 \times D_2) = (x, x')(y, y') \text{ with } xy \in A(D_1) \text{ and } x'y' \in A(D_2).$$

The example in Figure 6 shows the direct product of two digraphs D_1 and D_2.

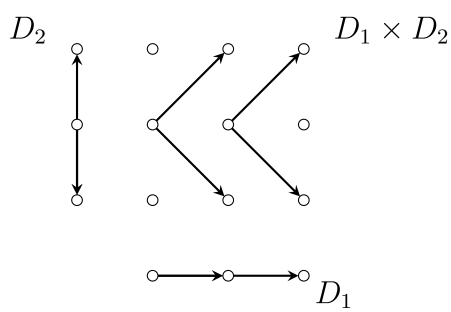

Figure 6: The direct product of D_1 and D_2, $D_1 \times D_2$

Note that D_1 and D_2 are called the **factors** of the product.

Definition 8. *The **underlying graph** G of a digraph D is the graph created using the vertex set of D and replacing the arcs in D with undirected edges.*

Figure 7 shows an example of a digraph D with its underlying graph G.

Figure 7: Digraph D and its underlying graph G

Definition 9. *A digraph D is* ***connected*** *if the underlying graph of D is connected.*

As previously stated, figure 7 shows a digraph D with its underlying graph G. Since G is connected, D is also connected. The digraph $D_1 \times D_2$ in Figure 8 is an example of a digraph that is not connected; we call such a digraph ***disconnected***.

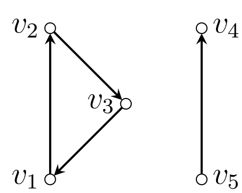

Figure 8: An example of a disconnected digraph

Definition 10. *Given a graph G, a* ***component*** *is a subgraph H of G that is maximally connected, that is, H is connected and any subgraph of G having H as a proper subgraph is disconnected.*

Figure 9 shows an example of a graph G with two components: H and J.

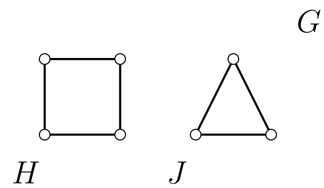

Figure 9: Graph G

Definition 11. *A directed graph is* ***strongly connected*** *if there is a directed path between all pairs of vertices. That is, given any two of its vertices x and y, there is a directed path from x to y and a directed path from y to x.*

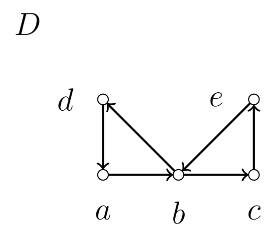

Figure 10: Digraph D

Figure 10 shows an example of a strongly connected digraph D.

Definition 12. *A* ***strong component*** *of a directed graph D is a maximal strongly connected subgraph C of D.*

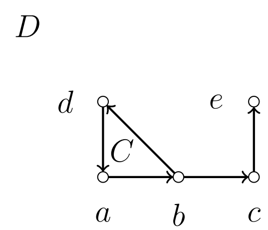

Figure 11: Digraph D

Figure 11 shows a digraph D with a strong component C. The arc set for the strongly connected component C is $\{(a, b), (b, d), (d, a)\}$. The other strong components are $\{c\}$ and $\{e\}$.

2 Kings

Kings originate from tournaments in Graph Theory. Before we can talk about tournaments, we must first review a few definitions.

Definition 13. *A **complete graph** K_n is a simple, undirected graph with n vertices in which every pair of vertices is connected by a unique edge.*

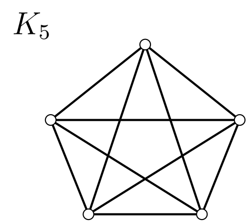

Figure 12: K_5 the complete graph on 5 vertices

Definition 14. *A **tournament** is an orientation of a complete graph.*

Now if we take K_5 from Figure 12 and orient the edges, we get a tournament:

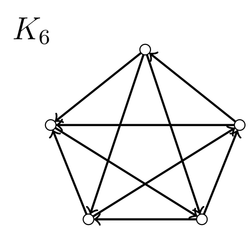

Figure 13: Tournament on K_5

H. G. Landau was among the first mathematicians to study tournaments in depth and study many of the applications of them. Landau himself used tournaments to study and model dominance in flocks of chickens [1]. There are many other very interesting applications of tournaments, such as to the study of voting theory and to the study of social choice theory.

Definition 15. *A* ***king*** *in a tournament is a vertex that can reach every other vertex in the digraph through a directed path of length one or two.*

Hence, every king in a tournament is a 2-king. The use of kings in this thesis, however, differs in that k can take on any positive integer value. It also differs in that we consider arbitrary graphs, not just tournaments.

3 Preliminary Results

In order for a digraph to have a king, it is necessary that the digraph be connected. When working on this problem, we knew that we would need our factors to have some special properties to ensure that their direct product would be connected. In fact, McAndrew [1] has a result that gives us exactly what we need:

Theorem. *If D_1, D_2,..., D_n are strongly connected digraphs, then $D_1 \times D_2 \times ... \times D_n$ has exactly*

$$\frac{d(D_1) \cdot d(D_2) \cdot ... \cdot d(D_n)}{lcm(d(D_1), d(D_2), ..., d(D_n))}$$

strong components.
McAndrew defines $d(D)$ to be the greatest common divisor of the lengths of all directed cycles in D.

Given two integers a and b, it is not hard to verify the simple formula

$$ab = \mathrm{lcm}(a, b) \cdot \gcd(a, b).$$

For example,

$$\begin{aligned}12 \cdot 15 = & \quad \operatorname{lcm}(12, 15) \cdot \gcd(12, 15) \\ = & \quad 60 \cdot 3 \\ = & \quad 180\end{aligned}$$

However, this nice formula only works for two numbers a and b, and no more. For example,

$$\begin{aligned}2 \cdot 4 = & \quad \operatorname{lcm}(2, 4) \cdot \gcd(2, 4) = 4 \cdot 2 \\ 2 \cdot 4 \cdot 8 \neq & \quad \operatorname{lcm}(2, 4, 8) \cdot \gcd(2, 4, 8) = 8 \cdot 2.\end{aligned}$$

However, if we just stick with the two numbers a and b, the formula yields $\frac{a \cdot b}{\operatorname{lcm}(a,b)} = \gcd(a, b)$. Thus, McAndrew's Theorem, adapted with just two factors, becomes the following:

Theorem. *If D_1 and D_2 are strongly connected digraphs, then $D_1 \times D_2$ has $gcd(d(D_1), d(D_2))$ strong components. In particular, $D_1 \times D_2$ is strongly connected if and only if $gcd(d(D_1), d(D_2)) = 1$.*

For example, consider the direct product $C_6 \times C_4$, where C_6 and C_4 are directed cycles on six and four vertices, respectively. The directed cycle C_6 has $d(C_6) = 6$ and the directed cycle C_4 has $d(C_4) = 4$. Then, $\operatorname{lcm}(d(C_6), d(c_4)) = 12$. So, by McAndrew's Theorem, $C_6 \times C_4$ should have exactly $\frac{d(c_6) \cdot d(c_4)}{\operatorname{lcm}(d(c_6), d(C_4))} = \frac{6 \cdot 4}{12} = 2$ strong components. Figure 14 shows

$C_6 \times C_4$, which has two strong components as expected. The red arcs represent one strong component and the blue arcs represent a second strong component.

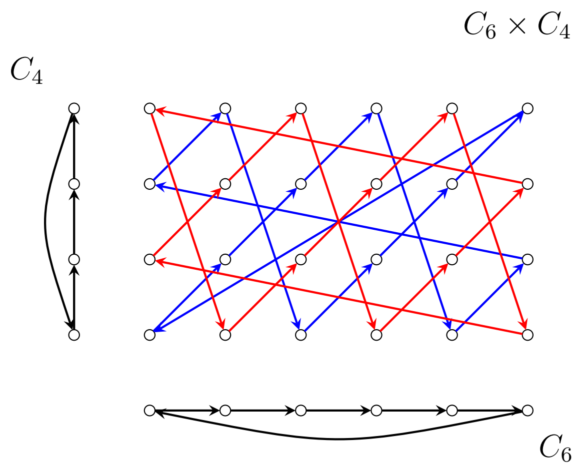

Figure 14: $D_1 \times D_2$

McAndrew's Theorem was extremely useful throughout the process of exploring kings in the direct product of digraphs. The connection between McAndrew's theorem and our results may not be immediately obvious to the reader. McAndrew defines $d(D)$ to be the greatest common divisor of all directed closed walks in D. We will instead define $g_D(v)$, which we will use in place of $d(D)$, and with an appropriately modified hypothesis.

Definition 16. *Given a digraph D and $v, w \in V(D)$, define $g_D(v)$ to be the greatest common divisor of the lengths of all closed directed walks in D containing the vertex v. If there are no directed walks through v, then $g_D(v) = \infty$.*

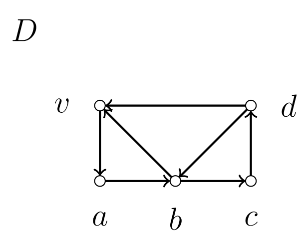

Fig. 9

Figure 15: Digraph D

In figure 15, the digraph D has closed walks $\{(a,b),(b,v),(v,a)\}$, $\{(b,c),(c,d),(d,b)\}$, $\{(a,b),(b,c),(c,d),(d,v),(v,a)\}$, and $\{(a,b),(b,c),(c,d),(d,b),(b,v),(v,a)\}$ with lengths of 3, 3, 5, and 6, respectively. Only three out of these four walks contain v: $\{(a,b),(b,v),(v,a)\}$, $\{(a,b),(b,c),(c,d),(d,v),(v,a)\}$, and $\{(a,b),(b,c),(c,d),(d,b),(b,v),(v,a)\}$, which have lengths 3, 5, and 6, respectively. Therefore, digraph D shown in figure 15 has $g_D(v) = 1$.

We use $g_D(v)$ in place of $d(D)$ because we do not necessarily need the entire direct product to be strongly connected in order to have a king in the product. We will now move into proving some lemmas that will be useful to us as we move towards proving our results.

Lemma 1. Let (v,w) be a vertex of $D_1 \times D_2$. Let C_1 be the strong component of D_1 containing v and let C_2 be the strong component of D_2 containing w. Then $C_1 \times C_2$ is strongly connected if $\gcd(g_{D_1}(v), g_{D_2}(w)) = 1$.

Proof. Suppose that $\gcd(g_{D_1}(v), g_{D_2}(w)) = 1$. From the definitions, we have $d(C_1) \mid g_{D_1}(v)$ and $d(C_2) \mid g_{D_2}(w)$. It follows that if $\gcd(g_{D_1}(v), g_{D_2}(w)) = 1$, then $\gcd(d(D_1), d(D_2)) = 1$. Therefore, $\gcd(d(D_1), d(D_2)) = 1$. Now, from the second form of McAndrew's theorem, $C_1 \times C_2$ is strongly connected. $\square$

Lemma 2. Let C be a strong component in digraph D. If P and Q are two walks in C that begin at a vertex v and end at a vertex v', then

$|P| - |Q| = kg_D(v)$ for some integer k.

Proof. Let P and Q be two walks that begin at a vertex v and end at a vertex v'. Let R be a directed path from v' to v. We know that this directed path exists because v and v' are in the same strong component. The concatination of the walks P and R is a closed walk that contains v, so the length of this concatination must be a multiple of $g_D(v)$. Similarly, the concatination of the walks Q and R is a closed walk that contains v, so the length of this concatination must also be a multiple of $g_D(v)$. Hence, $|P| + |R| = mg_D(v)$ and $|Q| + |R| = ng_D(v)$, where $m, n \in \mathbb{Z}^+$. Thus,

$$\begin{array}{rr} & |P| + |R| = mg_D(v) \\ - & |Q| + |R| = ng_D(v) \\ \hline & |P| - |Q| = (m-n)g_D(v) \end{array}$$

Now, let $m - n = k$, and we have that $|P| - |Q| = kg_D(v)$. □

Lemma 3. A vertex (x, y) is adjacent to a vertex (x', y') in $D_1 \times D_2$ if and only if x is adjacent to x' in D_1 and y is adjacent to y' in D_2.

Proof. Let D_1 and D_2 be digraphs and let $x, x' \in V(D_1)$ and $y, y' \in V(D_2)$. Then let $D_1 \times D_2$ represent the direct product of these two digraphs, in which we have $(x, y), (x', y') \in D_1 \times D_2$. First, suppose that (x, y) is adjacent to (x', y') in $D_1 \times D_2$. We need to show that x is adjacent to x' in D_1 and y is adjacent to y' in D_2. By the definition of the direct product, an arc from (x, y) to (x', y') in $D_1 \times D_2$ only exists if there is an arc from x to x' in D_1 and an arc from y to y' in D_2. Therefore, if (x, y) is adjacent to (x', y') in $D_1 \times D_2$, then x is adjacent to x' in D_1 and y is adjacent to y' in D_2.

Now, suppose that x is adjacent to x' in D_1 and y is adjacent to y' in D_2. We need to show that (x, y) is adjacent to (x', y') in $D_1 \times D_2$. By the definition of the direct product, if we have an arc in D_1 from x to x' and an arc in D_2 from y to y', then we have an arc from (x, y) to (x', y') in $D_1 \times D_2$. □

4 Results

Theorem. *The vertex* (v, w) *is a king in* $D_1 \times D_2$ *if and only if* v *is a king in* D_1 *and* w *is a king in* D_2 *and* $gcd(g_{D_1}(v), g_{D_2}(w)) = 1$.

Proof. First, suppose that (v, w) is a king in $D_1 \times D_2$. We need to show that v is a king in D_1, w is a king in D_2 and $gcd(g_{D_1}(v), g_{D_2}(w)) = 1$.

First, we will show that v is a king in D_1 and w is a king in D_2. In order to do this, we will show that given any $v' \in V(D_1)$ and given any $w' \in V(D_2)$, there exists a $v - v'$ walk in D_1 and a $w - w'$ walk in D_2. Pick the vertex (v', w') in $D_1 \times D_2$. Since (v, w) is a king in $D_1 \times D_2$, there is a directed path from (v, w) to (v', w'). By Lemma 3, P projects onto a walk in each of the factors; this results in a walk from v to v' in D_1 and a walk from w to w' in D_2. Hence, v and w are kings in D_1 and D_2 respectively.

Now, we want to show that $\gcd(g_{D_1}(v), g_{D_2}(w)) = 1$. Suppose that $\gcd(g_{D_1}(v), g_{D_2}(w)) = d$. First, we will show that $d < \infty$. Because (v, w) is a king, there must exist a directed walk from (v, w) to a different vertex (v, w_1). This directed walk then projects onto D_1, as shown in Figure 17. This means that v in D_1 lies on a directed cycle. Thus, $g_{D_1}(v) < \infty$.

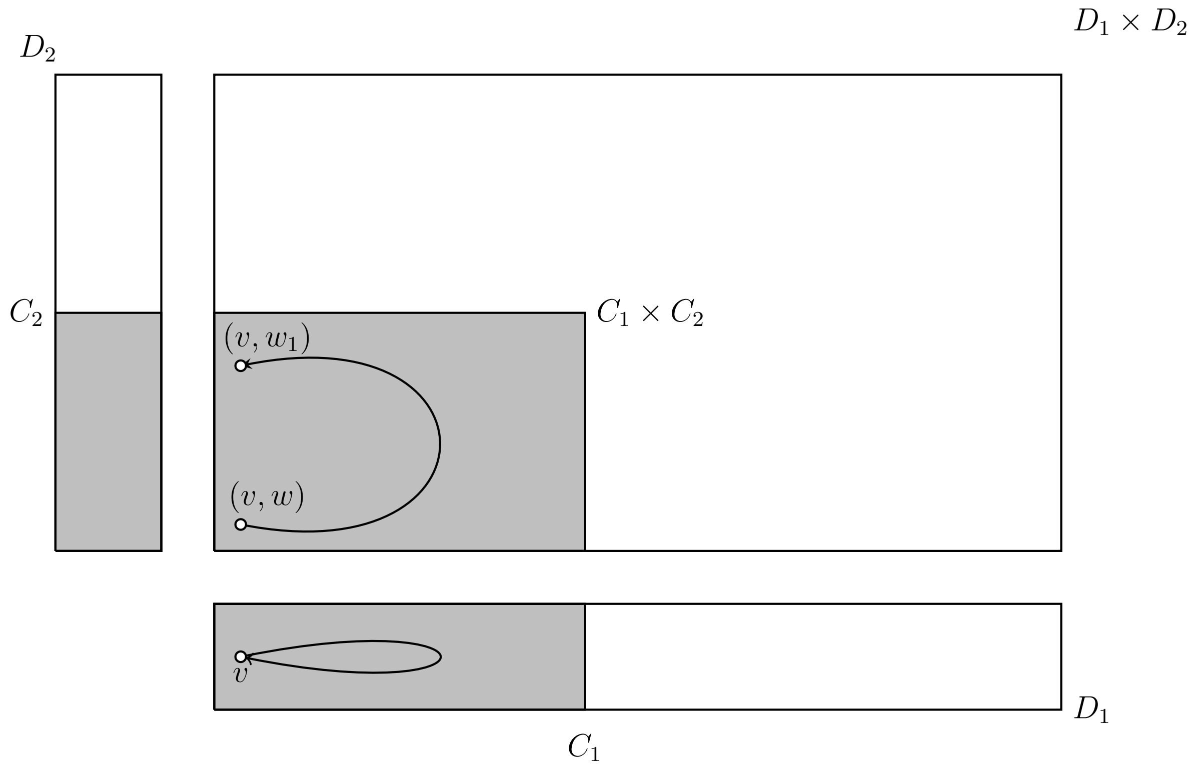

Figure 16: The directed walk in $D_1 \times D_2$ from (v, w) to (v, w_1) projects to a directed walk in D_1 from v to v.

Similarly, Because (v, w) is a king, there must exist a directed walk from (v, w) to a different vertex (v_1, w). This directed walk then projects onto D_2, as shown in Figure 18. Now, w must lie on a directed cycle. Thus, $g_{D_2}(w) < \infty$.

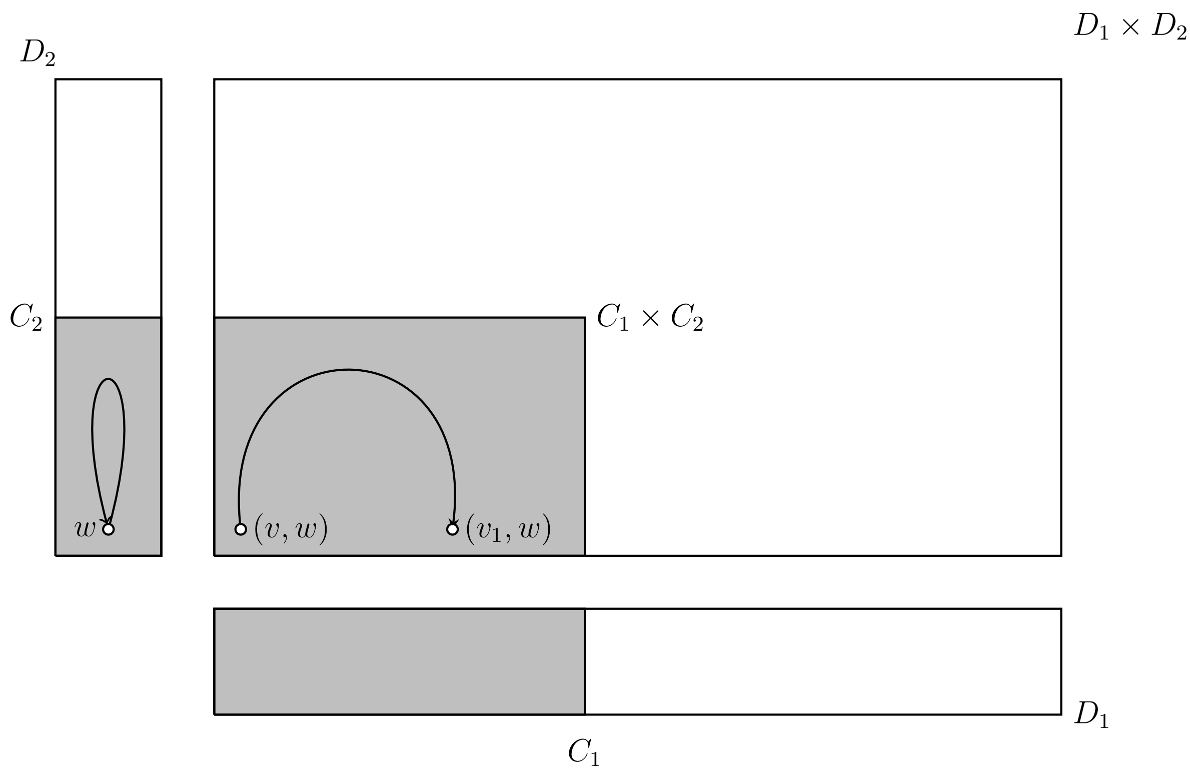

Figure 17: The directed walk in $D_1 \times D_2$ from (v, w) to (v_1, w) projects to a directed walk in D_2 from w to w.

Then, since $g_{D_1}(v) < \infty$ and $g_{D_2}(w) < \infty$, $d = \gcd(g_{D_1}(v), g_{D_2}(w)) < \infty$.

Now that we have shown that $d < \infty$, we will show that $d = 1$. Let C_1 be the strong component in D_1 containing the vertex v and let C_2 be the strong component in D_2 containing the vertex w. Let $x \in V(D_2)$ such that S is an arc from w to x. This arc must exist because C_2 is a strong component in D_2 and as shown previously, $g_{D_2}(w) < \infty$, so there exists a directed walk through w. Let P be a directed walk in $C_1 \times C_2$ from (v, w)

to (v, x), which exists because (v, w) is a king. Let Q be the projection of P onto D_1, which is a closed walk from v to v in C_1 and let R be the projection of P onto D_2, which is a directed path from w to x in C_2.

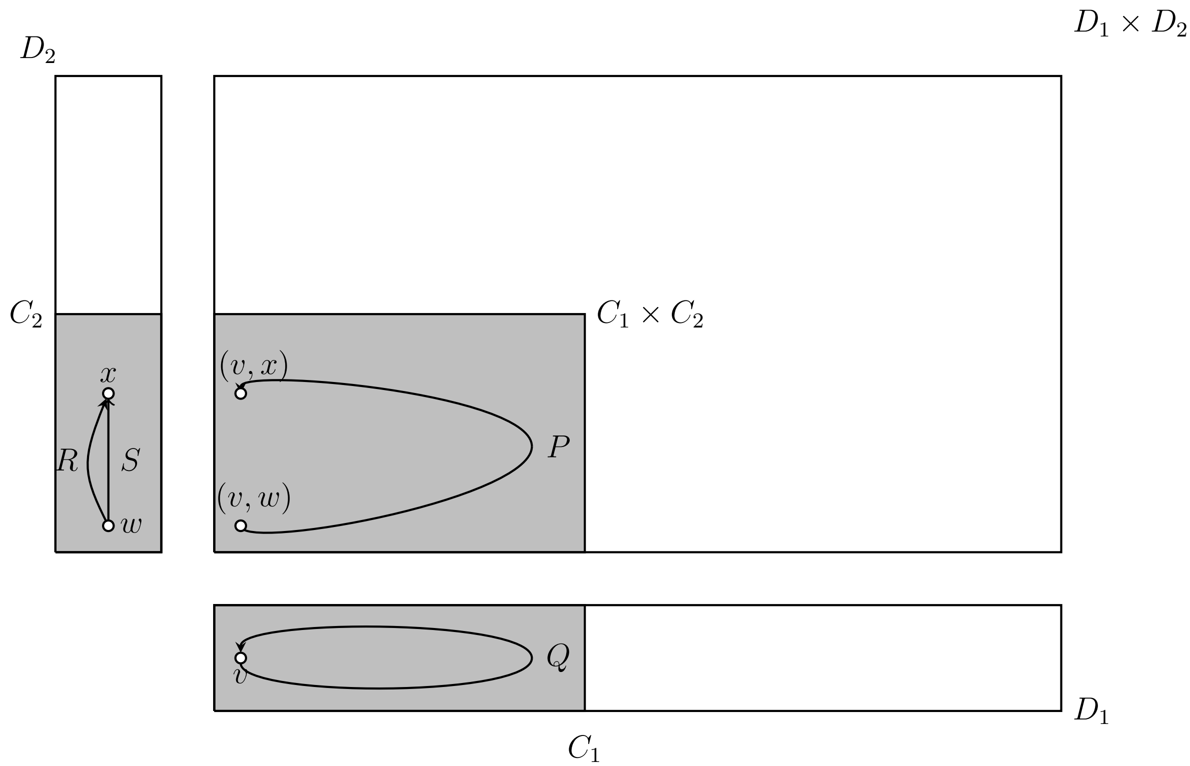

Figure 18: P in $C_1 \times C_2$ projects to Q in C_1, as well as R in C_2. Because C_2 is a strong component, a single arc, S, from w to x exists.

Then we have that $|Q| = md$, where $m \in \mathbb{Z}^+$. This implies that $|P| = md$ and $|R| = md$ as well. By Lemma 2, we have that

$$|S| - |R| = pd$$

$$1 - md = pd$$

$$1 = pd + md$$
$$1 = (p + m)d.$$

Thus, $d = 1$.

Conversely, suppose that v is a king in D_1 and w is a king in D_2 and $\gcd(g_{D_1}(v), g_{D_2}(w)) = 1$. Let C_1 be the strongly connected component in D_1 containing v and let C_2 be the strongly connected component in D_2 containing w. By Lemma 1, we know that we have a strongly connected component $C_1 \times C_2$ in $D_1 \times D_2$ containing (v, w). Pick an arbitrary vertex (v', w') in $D_1 \times D_2$. In order to complete the proof, we must construct a directed path from (v, w) to (v', w'). This will show that (v, w) can reach any vertex in $D_1 \times D_2$ and is a king. Clearly, if (v', w') is in the strong component $C_1 \times C_2$, then we are done because we are guaranteed to have a directed path from (v, w) to (v', w').

Then suppose (v', w') is not contained in $C_1 \times C_2$. Now, consider a walk $Q = vv_1v_2v_3...v_{m-1}v'$ of length m in D_1 and a walk $R = ww_1w_2w_3...w_{n-1}w'$ of length n in D_2. We know that both Q and R must exist because v is a king in D_1 and w is a king in D_2. Without loss of generality, suppose $n \leq m$. Now by the definition of the direct product, if $n = m$ then we must have a walk from (v, w) to (v', w') by the definition of the direct product. Let's call this walk S, where
$S = (v, w), (v_1, w_1), (v_2, w_2), (v_3, w_3)...(v_{m-1}, w_{n-1}), (v', w')$ in $D_1 \times D_2$ and (v, w) can reach (v', w'). Now, if $n < m$, or in other words $|R| < |Q|$, we

want to extend the length of our walk R by $m - n$ to create a new walk R', where $|R'| = |Q|$. This will ensure that $D_1 \times D_2$ will still have a walk from (v, w) to (v', w') in $D_1 \times D_2$. We can create this new walk R' by picking a cycle in the strong component C_2 to walk around and adding exactly $m - n$ arcs to our walk R to get a walk R'. In Figure 20, R is the directed path colored blue and R' is the concatenation of R and the arcs colored red.

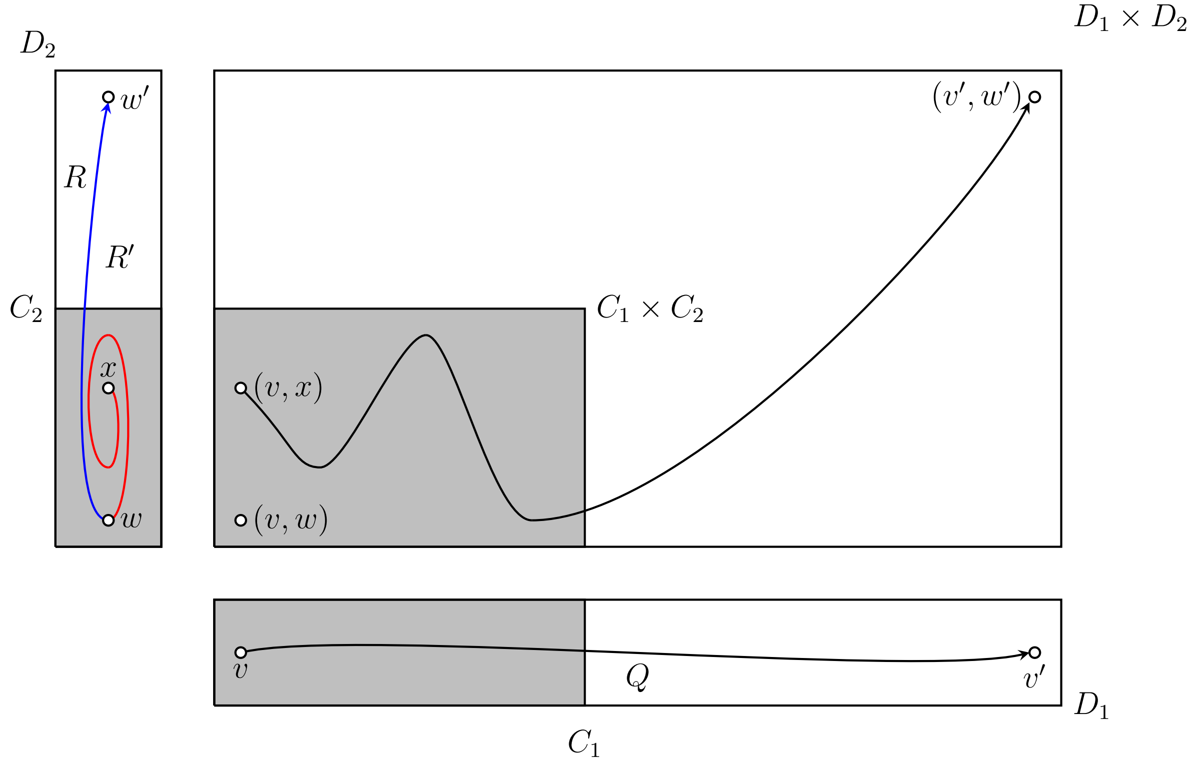

Figure 19: $D_1 \times D_2$

Figure 20 shows one possible example of this. Now, the length of R' is $n + m - n = m$ and we have a walk of length m in D_2 from the vertex w to the vertex x that lies in C_2. Thus, in $D_1 \times D_2$, there is a walk of length m from (v, x) to (v', w'). Since (v, x) is contained in $C_1 \times C_2$, (v, w) can reach (v, x) and, therefore, (v, w) can reach (v', w'). Thus, (v, w) is a king and we have our desired result. $\square$

5 Conclusion

In the future, we hope to be able to specify the value of k associated with the kings. We have not yet explored this extensively, but from the research that we have done, it seems probable that we will need to break this problem down into multiple cases. For example, if we take the direct product of a directed 3-cycle and a directed 4-cycle, the direct product has an 11-king.

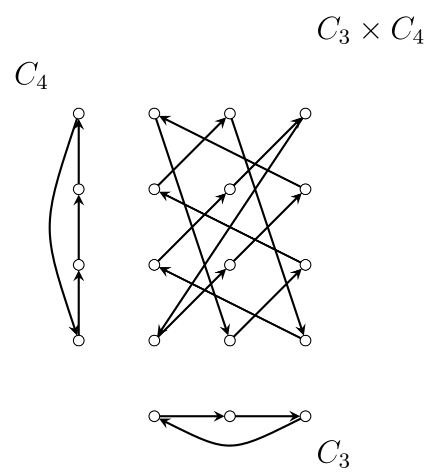

Figure 20: $C_1 \times C_2$, where every vertex is an 11-king

Notice that every vertex in C_3 is a 2-king, and every vertex in C_4 is a 3-king. It is easy to see that if D_1 and D_2 are digraphs, then $\|D_1 \times D_2\| = \|D_1\|\|D_2\|$. From our observations, it seems that because the number of arcs in $D_1 \times D_2$ is the number of arcs in D_1 times the number of arcs in D_2, if $D_1 \times D_2$ has a k-king, $k \leq \|D_1 \times D_2\|$, where $\|D\|$ is the number of arcs in D. This is just one example of a possible result regarding the value of k for kings in the direct product. It seems probable that there are many different cases. We have not yet formalized or verified any results that have to do with the value of k, but it is a question that we hope to explore in the future.

6 References

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