Growth rate of binary words avoiding xxx^R

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Abstract

Consider the set of those binary words with no non-empty factors of the form xxx^R. Du, Mousavi, Schaeffer, and Shallit asked whether this set of words grows polynomially or exponentially with length. In this paper, we demonstrate the existence of upper and lower bounds of the form $n^{\lg n+o(\lg n)}$ on the number of such words of length n, where $\lg n$ denotes the base-2 logarithm of n.

Key words: pattern with reversal, word avoiding xxx^R, growth rate of a language

1. Introduction

In this paper we study the binary words avoiding the pattern xxx^R. Here the notation x^R denotes the "reversal" or "mirror image" of x. For example, the word 011 011 110 is an instance of xxx^R, with $x = 011$. The avoidability of patterns with reversals has been studied before, for instance by Rampersad and Shallit [10] and by Bischoff, Currie, and Nowotka [2, 3, 6].

The question of whether a given pattern with reversal is avoidable may initially seem somewhat trivial. For instance, the pattern xx^R is avoided by the periodic word $(012)^\omega$ and xxx^R, the pattern studied in this paper, is avoided by the periodic word $(01)^\omega$. However, looking at the entire class of binary words that avoid xxx^R reveals that these words have a remarkable structure.

Du, Mousavi, Schaeffer, and Shallit [7] looked at binary words avoiding xxx^R. They noted that there are various periodic words that avoid this pattern and also proved that a certain aperiodic word studied by Rote [12] and related to the Fibonacci word also avoids the pattern xxx^R. They posed a variety of conjectures and open problems concerning binary words avoiding xxx^R, notably:

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Does the number of such words of length n grow polynomially or exponentially with n?

The growth rate of words avoiding a given pattern over a certain alphabet is a fundamental problem in combinatorics on words (see the survey by Shur [13]). Typically, for families of words defined in terms of the avoidability of a pattern, this growth is either polynomial or exponential. For instance, there are exponentially many ternary words of length n that avoid the pattern xx and exponentially many binary words of length n that avoid the pattern xxx [4]. Similarly, there are exponentially many words over a 4-letter alphabet that avoid the pattern xx in the abelian sense [5]. Indeed, the vast majority of avoidable patterns lead to exponential growth. Polynomial growth is rather rare: The two known examples are binary words avoiding overlaps [11] and words over a 4-letter alphabet avoiding the pattern $abwbcxaybazac$ [1]. It was therefore quite natural for Du et al. to suppose that the growth of binary words avoiding xxx^R was either polynomial or exponential. However, we will show that in this case the growth is intermediate between these two possibilities. To our knowledge, this is the first time such a growth rate has been shown in the context of pattern avoidance.

Our main result is a "structure theorem" analogous to the well-known result of Restivo and Salemi [11] concerning binary overlap-free words. The existence of such a structure theorem was conjectured by Shallit (personal communication) but he could not precisely formulate it. The result of Restivo and Salemi implies the polynomial growth of binary overlap-free words. In our case, the structure theorem we obtain leads to an upper bound of the form $n^{\lg n+o(\lg n)}$ for binary words avoiding xxx^R (here $\lg n$ denotes the base-2 logarithm of n). We also are able to establish a lower bound of the same type. In Table 1 we give an exact enumeration for small values of n.

The sequence $(a_n)_{n\geq 1}$ is sequence A241903 of the On-Line Encyclopedia of Integer Sequences [9].

2. Blocks L and S

Define

$$\mathcal{K} = \{z \in 0\{0,1\}^*1 : z \text{ avoids } xxx^R\}.$$

Let the transduction $h : \{S,L\}^* \to \{0,1\}^*$ be defined for a sequence $u = \prod_{i=0}^{n} u_i$, $u_i \in \{S,L\}$ by

$$h(u_i) = \begin{cases} 00100 & u_i = S \text{ and } i \text{ even} \\ 11011 & u_i = S \text{ and } i \text{ odd} \\ 00100100 & u_i = L \text{ and } i \text{ even} \\ 11011011 & u_i = L \text{ and } i \text{ odd.} \end{cases}$$

Then define

$$\mathcal{M} = \{u \in \{S,L\}^* : h(u) \text{ avoids } xxx^R\}.$$

n	a_n	n	a_n	n	a_n	n	a_n
1	2	17	282	33	2018	49	8598
2	4	18	324	34	2244	50	9266
3	6	19	372	35	2490	51	9964
4	10	20	426	36	2756	52	10708
5	16	21	488	37	3044	53	11484
6	24	22	556	38	3354	54	12300
7	34	23	630	39	3690	55	13166
8	48	24	712	40	4050	56	14062
9	62	25	804	41	4438	57	15000
10	80	26	908	42	4856	58	15974
11	100	27	1024	43	5300	59	16994
12	124	28	1152	44	5772	60	18076
13	148	29	1296	45	6272	61	19206
14	178	30	1454	46	6800	62	20388
15	210	31	1626	47	7370	63	21632
16	244	32	1814	48	7966	64	22924

Table 1: Number of binary words a_n of length n avoiding xxx^R

Theorem 1. *Let $z \in \mathcal{K}$. Then there exists a constant C such that z can be written*

$$z = ph(u)st$$

where $|p|, |s| \le C$, $u \in \mathcal{M}$, and $t \in (\epsilon + 1)(01)^*(\epsilon + 1)$.

Proof. Word z cannot contain 000 or 111 as a factor, so write $z = f(v)$ where $v \in \{ab, ad, cb, cd\}^*$, and

$$f : a \mapsto 0, b \mapsto 1, c \mapsto 00, d \mapsto 11.$$

Write $v = prs$ where r is a maximal string of alternating a's and b's in v; thus r lies in $(\epsilon + b)(ab)^*(\epsilon + a)$. If $|s| \ge 2$, then we claim that $|r| = 1$ or $|pr| < 3$. For suppose that $|r| \ge 2$, $|pr| \ge 3$ and $|s| \ge 2$. Let s_1, s_2 be the first two letters of s. Then s_1 must be c or d; otherwise, rs_1 is an alternating string of a's and b's that is longer than r. Suppose $s_1 = c$. (The other case is similar.) Since $|r| \ge 2$ and $|pr| \ge 3$, we conclude that prs_1s_2 has $yabcs_2$ as a suffix, some $y \in \{b, d\}$. But then z contains a factor $f(yabcs_2)$, which has a factor $1f(abc)1 = 101001 = xxx^R$, where $x = 10$. This is impossible.

If ab or ba is a factor of v, we can write $v = prs$ as above, with $|r| \ge 2$. This implies that $|s| \le 1$ or $|pr| \le 2$. If $|pr| \le 2$, then $p = \epsilon$, $|r| = 2$, since $|r| \ge 2$; in this case $pr = ab$. If $|s| \le 1$, then, since z ends in 1, either $s = \epsilon$ or $s = d$. In the first case, ab is a suffix of v; in the second ad is a suffix. It follows that every instance of ab or ba in v either occurs in a prefix of length 2, or in a suffix of the form $(\epsilon + b)(ab)^*(\epsilon + ad)$. The given suffix maps under f to a suffix $t \in (\epsilon + 1)(01)^*(\epsilon + 1)$ of z. We therefore can write $z = p_1z_1t$ such that $|p_1| \le 2$, and $z_1 = f(v_1)$, for some $v_1 \in \{ad, cb, cd\}^*$ where ba is not a factor of v_1.

Write $v_1 = prs$ where r is a maximal string of alternating c's and d's in v_1. First of all, note that $|r| < 7$; we check that $f(cdcdcdc)$ contains xxx^R with $x = 0f(d)0$, and, symmetrically, $f(dcdcdcd)$ contains xxx^R with $x = 1f(c)1$. We claim that $|r| < 3$ or $|pr| < 7$. For otherwise, suppose that $|r| \geq 3$, and $|p'r| = 7$, where p' is a suffix of p. Assume that the first letter of r is c. (The other case is similar.) Since $|r| < 7$, $p' \neq \epsilon$. Since r is maximal, the last letter of p' is a b. If $|p'| = 1$, then $f(p'r) = f(bcdcdcd)$, which contains xxx^R with $x = 1f(c)1$; this is impossible. If $|p'| \geq 2$, then cb is a suffix of p' (since ab is not a factor of v_1). However, then $p'r$ contains the factor $cbcdc$, and $f(cbcdc) = 001001100 = xxx^R$, where $x = 001$, so this is also impossible. It follows that every instance of cdc or dcd in v_1 occurs in a prefix of v_1 of length 6. Removing a prefix p' of length at most 7 from v_1 then gives a suffix v_2, such that the first letter of v_2 is a or c, and neither of cdc and dcd is a factor of v_2. We can thus write $z = p_2 z_2 t$ where $z_2 = f(v_2)$, $v_2 \in \{ad, cb, cd\}^*$, words ba, cdc, dcd are not factors of v_2, and $|p_2| \leq |p_1| + |f(p')| \leq 2 + 2(7) - 1 = 15$. (Here, at most 6 letters of p' can be c or d, since $cdcdcdc$ and $dcdcdcd$ lead to instances of xxx^R.)

Suppose that v' is any factor of v_2 of length 8. We claim that v' contains one of cd or dc as a factor. Since $v' \notin \{a,b\}^*$, one of c and d is a factor of v'. Suppose then that c is a factor of v'. (The other case is similar.) Suppose that neither of cd nor dc is a factor of v'. It follows that v' is $bcbcbcbc$ or $cbcbcbcb$; each of these contains $cbcbcbc$, and $f(cbcbcbc)$ contains $010010010 = xxx^R$ where $x = 010$.

We may thus write $v_2 = p' \left(\prod_{i=0}^{n} a_i\right) s'$, with $n \geq -1$, $|p'|, |s'| \leq 7$, such that each a_i begins and ends with c or d, and neither of cd or dc is a factor of any a_i. By $n = -1$ we allow the possibility that the product term is empty. As a convention, we write the product as empty if $|v_2|_{cd} + |v_2|_{dc} \leq 1$; for $i \geq 0$, then the last letter of p' and the first letter of s' are in $\{c, d\}$. Suppose $n \geq 0$. Consider a_i, $i \geq 0$. Without loss of generality, let a_i begin with c. The letter preceding a_i is either the last letter of a_{i-1}, or the last letter of p', and must be a d. We cannot have $|a_i| = 1$, which would force $a_i = c$; word a_i is then followed by the first letter of a_{i+1} or of s', which must be d. Then dcd is a factor of v_2, which is impossible. Thus $|a_i| \geq 2$. Since cd is not a factor of a_i, a_i begins with cb. Since a_i ends with c or d (not in b), $a_i \neq cb$, so that $|a_i| \geq 3$. Since ba is not a factor of v_2, a_i therefore begins with cbc. If $a_i \neq cbc$, then, since cd is not a factor of a_i, word a_i begins with $cbcb$, and arguing as previously, with $cbcbc$. If $cbcbc$ is a proper prefix of a_i, then a_i begins with $cbcbcb$. However, $f(cb)^3 0$ contains an instance of xxx^R, so this is impossible: If a_i begins with c, then $a_i \in \{cbc, cbcbc\}$. By the same reasoning, if a_i begins with d, then $a_i \in \{dad, dadad\}$.

Let $v_3 = (p')^{-1} v_2 (s')^{-1} = \prod_{i=0}^{n} a_i$. Deleting up to the first 5 letters, if necessary, we assume that $a_0 \in \{cbc, cbcbc\}$ (i.e., if a_0 begins with dad or $dadad$, then delete these letters). Then $z = p_3 z_3 s_3 t$ where $z_3 = f(v_3)$, $|p_3| \leq |f(p')| + |p_2| + 5 \leq 2(4) + 3 + 15 + 5 = 31$, $|s_3| = |f(s')| \leq 2(4) + 3 = 11$. Here we use the fact that at most 4 of the letters of p' or s' can be in $\{c, d\}$; otherwise the pigeonhole principle would force an occurrence of cd or dc in one of these.

We can write v_3 in the form $g(u)$ where $u \in \{S, L\}^*$. Here write $u = \prod_{i=0}^{m} u_i$, each $u_i \in \{S, L\}$, and let g be the transducer

$$g(u_i) = \begin{cases} cbc & u_i = S \text{ and } i \text{ even} \\ dad & u_i = S \text{ and } i \text{ odd} \\ cbcbc & u_i = L \text{ and } i \text{ even} \\ dadad & u_i = L \text{ and } i \text{ odd.} \end{cases}$$

Thus z_3 has the form $h(u)$ where h is the transducer

$$h(u_i) = \begin{cases} 00100 & u_i = S \text{ and } i \text{ even} \\ 11011 & u_i = S \text{ and } i \text{ odd} \\ 00100100 & u_i = L \text{ and } i \text{ even} \\ 11011011 & u_i = L \text{ and } i \text{ odd.} \end{cases}$$

We have thus proved the theorem with $C = \max(31, 11) = 31$. □

To study the growth rate of $\mathcal{K}$, it thus suffices to study the growth rate of $\mathcal{M}$.

The transducer h is sensitive to the index of a word modulo 2; thus, suppose r, $s \in \{S, L\}^*$ and r is a suffix of s. If $|r|$ and $|s|$ have the same parity, then $h(r)$ is a suffix of $h(s)$. However, if $|r|$ and $|s|$ have opposite parity, then $\overline{h(r)}$ is a suffix of $h(s)$. (Here the overline indicates binary complementation.)

3. Suitable pairs of words

Let $\mathcal{S}, \mathcal{L} \in \{S, L\}^*$. Say that the pair $\langle \mathcal{S}, \mathcal{L} \rangle$ is **suitable** if

1. $|\mathcal{S}|$, $|\mathcal{L}|$ are odd.
2. There exist non-empty $\ell, \mu, p \in \{0, 1\}^*$ such that
   (a) $h(\mathcal{L}) = \ell \ell^R$
   (b) $h(\mathcal{S}) = \ell\mu = \mu^R \ell^R$
   (c) $h(\mathcal{L}) = \ell\mu\overline{\mu^R}p$

We see that $\langle S, L \rangle$ is suitable; specifically, we could choose $\mu = 0$, $\ell = 0010$, $p = 00$.

Since $|\mathcal{S}|$, $|\mathcal{L}|$ are odd, the transducer h is sensitive to the index of a word modulo 2, where lengths (and indices) are measured in terms of $\mathcal{S}$ and $\mathcal{L}$; i.e., if we use length function $||w|| = |w|_{\mathcal{S}} + |w|_{\mathcal{L}}$; thus, suppose r, $s \in \{\mathcal{S}, \mathcal{L}\}^*$ and r is a suffix of s. If $||r||$ and $||s||$ have the same parity, then $h(r)$ is a suffix of $h(s)$. However, if $||r||$ and $||s||$ have opposite parity, then $\overline{h(r)}$ is a suffix of $h(s)$.

Lemma 2. *Let $\mathcal{S}, \mathcal{L} \in \{S, L\}^*$. Suppose that $\langle \mathcal{S}, \mathcal{L} \rangle$ is suitable.*

1. *Word $h(\mathcal{L})p^{-1}$ is a prefix of $h(\mathcal{S}\mathcal{S})$.*
2. *Word $h(\mathcal{S})$ is both a prefix and suffix of $h(\mathcal{L})$.*

Proof. The first of these properties is immediate from property 2(c) of the definition of suitability. For the second, we see that $h(\mathcal{L}) = \ell\mu\overline{\mu^R}p = \mu^R\ell^R\overline{\mu^R}p = p^R\overline{\mu}\ell\mu$ (in the last step we use the fact that $h(\mathcal{L}) = h(\mathcal{L})^R$). $\square$

Now suppose that $\mathcal{S}$ and $\mathcal{L}$ are fixed and $\langle\mathcal{S},\mathcal{L}\rangle$ is suitable. Define morphism $\Phi : \{\mathcal{S},\mathcal{L}\}^* \to \{\mathcal{S},\mathcal{L}\}^*$ by $\Phi(\mathcal{S}) = \mathcal{SL}$, $\Phi(\mathcal{L}) = \mathcal{SLL}$.

Morphism Φ is conjugate to the square of the Fibonacci morphism D, where $D(\mathcal{L}) = \mathcal{LS}$, $D(\mathcal{S}) = \mathcal{L}$; namely, $\Phi = \mathcal{L}^{-1}D^2\mathcal{L}$. This implies that for $k \geq 1$ $||\Phi^k(\mathcal{S})|| = F_{2k+1}$, $||\Phi^k(\mathcal{L})|| = F_{2k+2}$, where F_k is the kth Fibonacci number, counting from $F_1 = F_2 = 1$.

Lemma 3. *Let $\beta \in \{\mathcal{S},\mathcal{L}\}^*$. Then*

1. *$h(\Phi(\mathcal{S}\beta))$ is a prefix of $h(\Phi(\mathcal{L}\beta))$ and $h(\Phi^2(\mathcal{S}\beta))$ is a prefix of $h(\Phi^2(\mathcal{L}\beta))$.*
2. *$\overline{h(\Phi(\mathcal{S}\beta))}$ is a suffix of $h(\Phi(\mathcal{L}\beta))$.*
3. *$\overline{h(\Phi^2(\mathcal{S}\beta))}$ is a suffix of $h(\Phi^2(\mathcal{L}\beta))$.*
4. *$h(\Phi(\mathcal{L}))p^{-1}$ is a prefix of $h(\Phi(\mathcal{SS}))$.*
5. *$h(\Phi^2(\mathcal{L}))(\overline{p})^{-1}$ is a prefix of $h(\Phi^2(\mathcal{SS}))$.*

Proof. Since $\Phi(\mathcal{S})$ is a prefix of $\Phi(\mathcal{L})$, $\Phi(\mathcal{S}\beta)$ is a prefix of $\Phi(\mathcal{L}\beta)$, so that $h(\Phi(\mathcal{S}\beta))$ is a prefix of $h(\Phi(\mathcal{L}\beta))$. Similarly, $h(\Phi^2(\mathcal{S}\beta))$ is a prefix of $h(\Phi^2(\mathcal{L}\beta))$, establishing (1).

Since $\mathcal{S}$ is a suffix of $\mathcal{L}$, we see that $\Phi(\mathcal{S})$ is a suffix of $\Phi(\mathcal{L})$. Because $|\Phi(\mathcal{L})|$ is odd, while $|\Phi(\mathcal{S})|$ is even, it follows that $\overline{h(\Phi(\mathcal{S}))}$ is a suffix of $h(\Phi(\mathcal{L}))$. More generally, if $\beta \in \{\mathcal{S},\mathcal{L}\}^*$, $\overline{h(\Phi(\mathcal{S}\beta))}$ is a suffix of $h(\Phi(\mathcal{L}\beta))$, establishing (2). The proof of (3) is similar.

For (4), $h(\Phi(\mathcal{L}))p^{-1} = h(\mathcal{SLL})p^{-1} = h(\mathcal{SL})h(\mathcal{L})p^{-1}$, which is a prefix of $h(\mathcal{SL})h(\mathcal{SS})$, which is in turn a prefix of $h(\mathcal{SL})h(\mathcal{SL}) = h(\Phi(\mathcal{SS}))$.

For (5), $h(\Phi^2(\mathcal{L}))(\overline{p})^{-1} = h(\Phi(\mathcal{SL})\Phi(\mathcal{L}))(\overline{p})^{-1} = h(\Phi(\mathcal{SL}))\overline{h(\Phi(\mathcal{L}))p^{-1}}$ (since $|\Phi(\mathcal{SL})|$ is odd), which is a prefix of $h(\Phi(\mathcal{SL}))\overline{h(\Phi(\mathcal{SS}))} = h(\Phi(\mathcal{SL})\Phi(\mathcal{SS}))$, which is in turn a prefix of $h(\Phi(\mathcal{SLSSL})) = h(\Phi^2(\mathcal{SS}))$. $\square$

In order to count the words in which we are interested, we prove a sequence of lemmas, ending with the structure result, Lemma 8. This lemma, and the lemmas leading up to it are very technical; roughly speaking, they consider the structure of sets which are 'almost' $\Phi^{-1}(\mathcal{M})$, $\Phi^{-2}(\mathcal{M})$ and $\Phi^{-3}(\mathcal{M})$. We exclude certain words b from these sets, on the basis that (variously) $h(\Phi(b))$, $h(\Phi^2(b))$, $h(\Phi^3(b))$ contain instances of xxx^R.

Define the set $\mathcal{B} \subseteq \{\mathcal{S},\mathcal{L}\}^*$:

$$\begin{aligned}
\mathcal{B} \quad = \quad & (\mathcal{S}+\mathcal{L})\mathcal{SSSL}(\mathcal{L}+\mathcal{SS}+\mathcal{SL}) \cup \mathcal{LSSL}(\mathcal{L}+\mathcal{SS}+\mathcal{SL}) \cup (\mathcal{S}+\mathcal{L})\mathcal{LLLLL}(\mathcal{S}+\mathcal{L}) \\
& \cup(\mathcal{S}+\mathcal{L})\mathcal{LSLLL}(\mathcal{S}+\mathcal{L}) \\
& \cup\Phi((\mathcal{S}+\mathcal{L})\mathcal{SS}(\mathcal{S}+\mathcal{L})) \cup \Phi((\mathcal{S}+\mathcal{L})\mathcal{LLLSL}(\mathcal{L}+\mathcal{SS}+\mathcal{SL})) \\
& \cup\Phi^2(\mathcal{LLL}(\mathcal{S}+\mathcal{L})) \cup \Phi^2((\mathcal{S}+\mathcal{L})\mathcal{LSS}(\mathcal{S}+\mathcal{L})) \cup \Phi^2((\mathcal{S}+\mathcal{L})\mathcal{SSSSS}(\mathcal{S}+\mathcal{L}))
\end{aligned}$$

Lemma 4. *Let $u \in \mathcal{M}$. Then no word of $\mathcal{B}$ is a factor of u.*

Proof. It suffices to show that for each word $b \in \mathcal{B}$, $h(b)$ contains a non-empty factor xxx^R. $\mathcal{B}$ is written as a union, and we make cases based on which piece of the union b belongs:

$b \in (\mathcal{S}+\mathcal{L})\mathcal{SSSL}(\mathcal{L}+\mathcal{SS}+\mathcal{SL})$: In this case, it suffices to show that $h(\mathcal{SSSSLL})(\bar{p})^{-1}$ contains a non-empty factor xxx^R, because of the results of Lemma 2. In particular, $h(\mathcal{SSSSLL})(\bar{p})^{-1}$ is a suffix of $h(\mathcal{LSSSLL})(\bar{p})^{-1}$, which is a prefix of $h(\mathcal{LSSSLSS})$, which is a prefix of $h(\mathcal{LSSSLSL})$. Again, $h(\mathcal{SSSSLL})(\bar{p})^{-1}$ is a prefix of $h(\mathcal{SSSSLSS})$, which is a prefix of $h(\mathcal{SSSSLSL})$. Now

$$\begin{aligned}
& h(\mathcal{SSSSLL})(\bar{p})^{-1} \\
= \;& (\ell\mu)(\overline{\mu^R\ell^R})(\ell\mu)(\overline{\mu^R\ell^R})(\ell\ell^R)(\overline{\ell\mu\overline{\mu^R}}) \\
= \;& \ell\mu\overline{\mu^R\ell^R}\ell\mu\overline{\mu^R\ell^R}\ell\ell^R\overline{\ell\mu\overline{\mu^R}} \\
= \;& \ell \;\; \mu\overline{\mu^R\ell^R}\ell \;\; \mu\overline{\mu^R\ell^R}\ell \;\; \ell^R\overline{\ell\mu}\mu^R
\end{aligned}$$

which contains an instance of xxx^R with $x = \mu\overline{\mu^R\ell^R}\ell$.

$b \in \mathcal{LSSL}(\mathcal{L}+\mathcal{SS}+\mathcal{SL})$: In this case, it suffices to show that $h(\mathcal{LSSLL})p^{-1}$ contains a non-empty factor xxx^R, because of the results of Lemma 2. But

$$\begin{aligned}
& h(\mathcal{LSSLL})p^{-1} \\
= \;& (p\overline{\mu}\mu^R\ell^R)(\overline{\ell\mu})(\mu^R\ell^R)(\overline{\ell\ell^R})(\ell\mu\overline{\mu^R}) \\
= \;& p\overline{\mu}\mu^R\ell^R\overline{\ell\mu}\mu^R\ell^R\overline{\ell\ell^R}\ell\mu\overline{\mu^R} \\
= \;& p \;\; \overline{\mu}\mu^R\ell^R\overline{\ell} \;\; \overline{\mu}\mu^R\ell^R\overline{\ell} \;\; \overline{\ell^R}\ell\mu\overline{\mu^R}
\end{aligned}$$

which contains the instance xxx^R with $x = \overline{\mu}\mu^R\ell^R\overline{\ell}$.

$b \in (\mathcal{S}+\mathcal{L}))\mathcal{L}^5(\mathcal{S}+\mathcal{L})$: In this case, it suffices to show that $h(\mathcal{SL}^5\mathcal{S})$ contains a non-empty factor xxx^R, because of the results of Lemma 2. But

$$\begin{aligned}
& h(\mathcal{SL}^5\mathcal{S}) \\
= \;& (\mu^R\ell^R)(\overline{\ell\ell^R})(\ell\ell^R)(\overline{\ell\ell^R})(\ell\ell^R)(\overline{\ell\ell^R})(\ell\mu) \\
= \;& \mu^R\ell^R\overline{\ell\ell^R}\ell\ell^R\overline{\ell\ell^R}\ell\ell^R\overline{\ell\ell^R}\ell\mu \\
= \;& \mu^R \;\; \ell^R\overline{\ell\ell^R}\ell \;\; \ell^R\overline{\ell\ell^R}\ell \;\; \ell^R\overline{\ell\ell^R}\ell \;\; \mu
\end{aligned}$$

which contains the instance xxx^R with $x = \ell^R\overline{\ell\ell^R}\ell$.

$b \in (\mathcal{S}+\mathcal{L})\mathcal{LSLLL}(\mathcal{S}+\mathcal{L})$: In this case, it suffices to show that $h(\mathcal{SLSLLLS})$ contains a non-empty factor xxx^R, because of the results of Lemma 2. Here

$$\begin{aligned}
& h(\mathcal{SLSLLLS}) \\
= \;& (\ell\mu)(\overline{\ell\ell^R})(\ell\mu)(\overline{\ell\ell^R})(\ell\ell^R)(\overline{\ell\ell^R})(\mu^R\ell^R) \\
= \;& \ell\mu\overline{\ell\ell^R}\ell\mu\overline{\ell\ell^R}\ell\ell^R\overline{\ell\ell^R}\mu^R\ell^R \\
= \;& \ell \;\; \mu\overline{\ell\ell^R}\ell \;\; \mu\overline{\ell\ell^R}\ell \;\; \ell^R\overline{\ell\ell^R}\mu^R \;\; \ell^R
\end{aligned}$$

which contains the instance xxx^R with $x = \mu\overline{\ell\ell^R}\ell$.

$b \in \Phi((\mathcal{S}+\mathcal{L})\mathcal{S}\mathcal{S}(\mathcal{S}+\mathcal{L}))$: In this case, it suffices to show that $h(\Phi(\mathcal{S}\mathcal{S}\mathcal{S}\mathcal{S}))$ contains a non-empty factor xxx^R, because of the results of Lemma 3. In particular, $h(\Phi(\mathcal{S}\mathcal{S}\mathcal{S}\mathcal{S}))$ is a prefix of $h(\Phi(\mathcal{S}\mathcal{S}\mathcal{S}\mathcal{L}))$, $\overline{h(\Phi(\mathcal{S}\mathcal{S}\mathcal{S}\mathcal{S}))}$ is a suffix of $h(\Phi(\mathcal{L}\mathcal{S}\mathcal{S}\mathcal{S}))$, and $\overline{h(\Phi(\mathcal{S}\mathcal{S}\mathcal{S}\mathcal{L}))}$ is a suffix of $h(\Phi(\mathcal{L}\mathcal{S}\mathcal{S}\mathcal{L}))$. However,

$$\begin{aligned}
& h(\Phi(\mathcal{S}\mathcal{S}\mathcal{S}\mathcal{S})) \\
= \; & (\mu^R\ell^R\overline{\ell\ell^R})(\mu^R\ell^R\overline{\ell\ell^R})(\mu^R\ell^R\overline{\ell\ell^R}(\mu^R\ell^R\overline{\ell\ell^R}) \\
= \; & \mu^R\ell^R\overline{\ell} \;\; \overline{\ell^R}\mu^R\ell^R\overline{\ell} \;\; \overline{\ell^R}\mu^R\ell^R\overline{\ell} \;\; \overline{\ell^R}\mu^R\ell^R\overline{\ell} \;\; \overline{\ell^R}
\end{aligned}$$

containing an instance of xxx^R, with $x = \overline{\ell^R}\mu^R\ell^R\overline{\ell}$.

$b \in \Phi((\mathcal{S}+\mathcal{L})\mathcal{L}\mathcal{L}\mathcal{L}\mathcal{S}\mathcal{L}(\mathcal{L}+\mathcal{S}\mathcal{S}+\mathcal{S}\mathcal{L}))$: In this case, it suffices to show that $h(\Phi(\mathcal{S}\mathcal{L}\mathcal{L}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{L}))p^{-1}$ contains a non-empty factor xxx^R, because of the results of Lemma 3. But

$$\begin{aligned}
& h(\Phi(\mathcal{S}\mathcal{L}\mathcal{L}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{L}))p^{-1} \\
= \; & (\mu^R\ell^R\overline{\ell\ell^R})(\mu^R\ell^R\overline{\ell\ell^R}\ell\ell^R)(\overline{\mu^R\ell^R}\ell\ell^R\overline{\ell\ell^R})(\mu^R\ell^R\overline{\ell\ell^R}\ell\ell^R)(\overline{\mu^R\ell^R}\ell\ell^R)(\overline{\ell\mu}\ell\ell^R\overline{\ell\ell^R})(\ell\mu\overline{\ell\ell^R}\ell\mu\overline{\mu^R}) \\
= \; & \mu^R \;\; \ell^R\overline{\ell\ell^R}\mu^R\ell^R\overline{\ell\ell^R}\ell\ell^R\overline{\mu^R\ell^R}\ell \;\; \ell^R\overline{\ell\ell^R}\mu^R\ell^R\overline{\ell\ell^R}\ell\ell^R\overline{\mu^R\ell^R}\ell \;\; \ell^R\overline{\ell\mu}\ell\ell^R\overline{\ell\ell^R}\ell\mu\overline{\ell\ell^R}\ell \;\; \mu\overline{\mu^R}
\end{aligned}$$

an instance of xxx^R with $x = \ell^R\overline{\ell\ell^R}\mu^R\ell^R\overline{\ell\ell^R}\ell\ell^R\overline{\mu^R\ell^R}\ell$.

$b \in \Phi^2(\mathcal{L}\mathcal{L}\mathcal{L}(\mathcal{S}+\mathcal{L}))$: In this case, it suffices to show that $h(\Phi^2(\mathcal{L}\mathcal{L}\mathcal{L}\mathcal{S}))$ contains a non-empty factor xxx^R, because of the results of Lemma 3. But

$$\begin{aligned}
& h(\Phi^2(\mathcal{L}\mathcal{L}\mathcal{L}\mathcal{S})) \\
= \; & (\ell\mu\overline{\ell\ell^R}\ell\mu\overline{\ell\ell^R}\ell\ell^R\overline{\ell\mu}\ell\ell^R\overline{\ell\ell^R})(\ell\mu\overline{\ell\ell^R}\ell\mu\overline{\ell\ell^R}\ell\ell^R\overline{\ell\mu}\ell\ell^R\overline{\ell\ell^R})(\ell\mu\overline{\ell\ell^R}\mu^R\ell^R\overline{\ell\ell^R}\ell\ell^R\overline{\mu^R\ell^R}\ell\ell^R\overline{\ell\ell^R}) \\
& (\mu^R\ell^R\overline{\ell\ell^R}\ell\mu\overline{\ell\ell^R}\ell\ell^R) \\
= \; & \ell\mu\overline{\ell} \cdot \overline{\ell^R}\ell\mu\overline{\ell\ell^R}\ell\ell^R\overline{\ell\mu}\ell\ell^R\overline{\ell\ell^R}\ell\mu\overline{\ell} \cdot \overline{\ell^R}\ell\mu\overline{\ell\ell^R}\ell\ell^R\overline{\ell\mu}\ell\ell^R\overline{\ell\ell^R}\ell\mu\overline{\ell} \cdot \overline{\ell^R}\mu^R\ell^R\overline{\ell\ell^R}\ell\ell^R\overline{\mu^R\ell^R}\ell\ell^R\overline{\ell\ell^R}\mu^R\ell^R\overline{\ell} \\
& \cdot\overline{\ell^R}\ell\mu\overline{\ell\ell^R}\ell\ell^R
\end{aligned}$$

containing an instance of xxx^R, with $x = \overline{\ell^R}\ell\mu\overline{\ell\ell^R}\ell\ell^R\overline{\ell\mu}\ell\ell^R\overline{\ell\ell^R}\ell\mu\overline{\ell}$.

$b \in \Phi^2((\mathcal{S}+\mathcal{L})\mathcal{L}\mathcal{S}\mathcal{S}(\mathcal{S}+\mathcal{L}))$: In this case, it suffices to show that $h(\Phi^2(\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{S}\mathcal{S}))$ contains a non-empty factor xxx^R, because of the results of Lemma 3. Now

$$\begin{aligned}
& h(\Phi^2(\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{S}\mathcal{S})) \\
= \; & (\ell\mu\overline{\ell\ell^R}\ell\mu\overline{\ell}\,\overline{\ell^R}\ell\ell^R)(\overline{\ell\mu}\ell\ell^R\overline{\ell\mu}\ell\ell^R\overline{\ell\ell^R}\ell\mu\overline{\ell}\,\overline{\ell^R}\ell\ell^R)(\overline{\ell\mu}\ell\ell^R\overline{\ell\mu}\ell\ell^R\overline{\ell\ell^R})(\ell\mu\overline{\ell}\,\overline{\ell^R}\mu^R\ell^R\overline{\ell\ell^R}\ell\ell^R) \\
& (\overline{\mu^R\ell^R}\ell\ell^R\overline{\mu^R\ell^R}\ell\ell^R\overline{\ell}\,\overline{\ell^R}) \\
= \; & \ell\mu\overline{\ell\ell^R}\ell\mu\overline{\ell} \cdot \overline{\ell^R}\ell\ell^R\overline{\ell\mu}\ell\ell^R\overline{\ell\mu}\ell\ell^R\overline{\ell\ell^R}\ell\mu\overline{\ell} \cdot \overline{\ell^R}\ell\ell^R\overline{\ell\mu}\ell\ell^R\overline{\ell\mu}\ell\ell^R\overline{\ell\ell^R}\ell\mu\overline{\ell} \\
& \cdot\overline{\ell^R}\mu^R\ell^R\overline{\ell\ell^R}\ell\ell^R\overline{\mu^R\ell^R}\ell\ell^R\overline{\mu^R\ell^R}\ell\ell^R\overline{\ell} \cdot \overline{\ell^R}
\end{aligned}$$

containing an instance of xxx^R, with $x = \overline{\ell^R}\ell\ell^R\overline{\ell\mu}\ell\ell^R\overline{\ell\mu}\ell\ell^R\overline{\ell\ell^R}\ell\mu\overline{\ell}$.

$b \in \Phi^2((\mathcal{S}+\mathcal{L})\mathcal{SSSSS}(\mathcal{S}+\mathcal{L}))$: In this case, it suffices to show that $h(\Phi^2(\mathcal{S}^7))$ contains a non-empty factor xxx^R, because of the results of Lemma 3. Finally,

$$\begin{aligned}
& h(\Phi^2(\mathcal{S}^7)) \\
= \; & (\ell\mu\overline{\ell\ell^R}\ell\mu\overline{\ell\ell^R}\ell\ell^R)(\overline{\ell\mu}\ell\ell^R\overline{\mu^R\ell^R}\ell\ell^R\overline{\ell\ell^R})(\ell\mu\overline{\ell\ell^R}\ell\mu\overline{\ell\ell^R}\ell\ell^R)(\overline{\ell\mu}\ell\ell^R\overline{\mu^R\ell^R}\ell\ell^R\overline{\ell\ell^R}) \\
& (\ell\mu\overline{\ell\ell^R}\ell\mu\overline{\ell\ell^R}\ell\ell^R)(\overline{\ell\mu}\ell\ell^R\overline{\mu^R\ell^R}\ell\ell^R\overline{\ell\ell^R})(\ell\mu\overline{\ell\ell^R}\ell\mu\overline{\ell\ell^R}\ell\ell^R) \\
= \; & \ell\mu\overline{\ell} \cdot \overline{\ell^R}\ell\mu\overline{\ell\ell^R}\ell\ell^R\overline{\ell\mu}\ell\ell^R\overline{\mu^R\ell^R}\ell\ell^R\overline{\ell\ell^R}\mu^R\ell^R\overline{\ell} \cdot \overline{\ell^R}\ell\mu\overline{\ell\ell^R}\ell\ell^R\overline{\ell\mu}\ell\ell^R\overline{\mu^R\ell^R}\ell\ell^R\overline{\ell\ell^R}\mu^R\ell^R\overline{\ell} \\
& \cdot \overline{\ell^R}\ell\mu\overline{\ell\ell^R}\ell\ell^R\overline{\ell\mu}\ell\ell^R\overline{\mu^R\ell^R}\ell\ell^R\overline{\ell\ell^R}\mu^R\ell^R\overline{\ell} \cdot \overline{\ell^R}\mu^R\ell^R\overline{\ell\ell^R}\ell\ell^R
\end{aligned}$$

containing an instance of xxx^R, with $x = \overline{\ell^R}\ell\mu\overline{\ell\ell^R}\ell\ell^R\overline{\ell\mu}\ell\ell^R\overline{\mu^R\ell^R}\ell\ell^R\overline{\ell\ell^R}\mu^R\ell^R\overline{\ell}$. $\square$

4. Parsing words of $\mathcal{M}$ using Φ

Lemma 5. *Let $y \in \{\mathcal{S},\mathcal{L}\}^* \cap \mathcal{M}$. Then y can be written*

$$y = p_1\Phi(y_1)s_1t_1,$$

where $|p_1|, |s_1| \le 9$, $y_1 \in \{\mathcal{S},\mathcal{L}\}^*$, and $t_1 \in (\epsilon+\mathcal{S}+\mathcal{S}^2+\mathcal{S}^3)\mathcal{LS}^* + \mathcal{S}^*(\epsilon+\mathcal{L}+\mathcal{LS})$. (Here all lengths are as words of $\{\mathcal{S},\mathcal{L}\}^*$; thus, for example $|p_1| = |p_1|_{\mathcal{L}} + |p_1|_{\mathcal{S}}$.)

Proof. Suppose that $|y|_{\mathcal{L}} = n$. If $n = 0$, the lemma is true, letting $t_1 = y$. If $n = 1$, write $y = \mathcal{S}^k\mathcal{L}\mathcal{S}^j$. Since by Lemma 4, $\mathcal{SSSSLSS}$ cannot be a factor of $y \in \mathcal{M}$, we have $k \le 3$ or $j \le 1$; thus we can again let $t_1 = y$, and we are again done.

Suppose from now on, that $n \ge 2$, and write $y = (\prod_{i=1}^n \mathcal{S}^{m_i}\mathcal{L})\mathcal{S}^{m_{n+1}}$, where each $m_i \ge 0$. For $1 \le i \le n-1$, word $\mathcal{LS}^{m_{i+1}}\mathcal{L}$ has one of $\mathcal{LL}$, $\mathcal{LSL}$ or $\mathcal{LSS}$ as a prefix, depending on whether $m_{i+1} = 0, 1$ or $m_{i+1} \ge 2$, respectively. This implies that for $1 \le i \le n-1$, we have $m_i \le 3$, since by Lemma 4, no word of $\mathcal{S}^4(\mathcal{LL}+\mathcal{LSL}+\mathcal{LSS})$ can be a factor of $y \in \mathcal{M}$. For $2 \le i \le n-1$, we have $m_i \le 1$, since no word of $\mathcal{L}(\mathcal{S}^2+\mathcal{S}^3)(\mathcal{LL}+\mathcal{LSL}+\mathcal{LSS})$, can appear in y. Since $\mathcal{S}^4\mathcal{LS}^2$ cannot be a factor of $y \in \mathcal{M}$, if $m_{n+1} \ge 2$, then $m_n \le 3$. We have thus established that

$$y \in (\epsilon+\mathcal{S}+\mathcal{S}^2+\mathcal{S}^3)\mathcal{L}\left((\epsilon+\mathcal{S})\mathcal{L}\right)^* \left((\epsilon+\mathcal{S}+\mathcal{S}^2+\mathcal{S}^3)\mathcal{LSSS}^* + \mathcal{S}^*\mathcal{L}(\epsilon+\mathcal{S})\right)$$

Write $y = p'y't_1$, where

$$p' \in (\epsilon+\mathcal{S}+\mathcal{S}^2+\mathcal{S}^3), y' \in \mathcal{L}\left((\epsilon+\mathcal{S})\mathcal{L}\right)^*,$$

$$t_1 \in (\epsilon+\mathcal{S}+\mathcal{S}^2+\mathcal{S}^3)\mathcal{LSSS}^* + \mathcal{S}^*\mathcal{L}(\epsilon+\mathcal{S}).$$

In particular, $\mathcal{SS}$ is not a factor of y'.

Without loss of generality, suppose $|y| \geq 7$ and $|y'| \geq 6$. (If $|y| \leq 6$ or $|y'| \leq 5$, let $p_1 = p'y'$, $y_1 = s_1 = \epsilon$, and the lemma holds. Write $y' = p''y''s_1$, where $|p''| = 4$, $|s_1| = 2$. We next consider the placement in y, y', y'' of hypothetical factors $\mathcal{L}^k$, $k \geq 3$:

- $\mathcal{L}^k$, $k \geq 6$, cannot be a factor of y: If $\mathcal{L}^6$ is a factor of y, so is one of $\mathcal{S}\mathcal{L}^6$, $\mathcal{L}^6\mathcal{S}$ or $\mathcal{L}^7$, since $|y| \geq 7$; this is impossible.

- $\mathcal{L}^5$ can only appear in y as a prefix or suffix: Otherwise, y contains some two-sided extension of $\mathcal{L}^5$. As $\mathcal{L}^6$ is not a factor of y, this must be $\mathcal{S}\mathcal{L}^5\mathcal{S}$. This is impossible by Lemma 4.

- $\mathcal{L}^4$ is not a factor of $\rho y''\sigma$, where ρ is the last letter of p'' and σ is the first letter of s_1: The length 5 left extension of an occurrence of $\mathcal{L}^4$ in $\rho y''\sigma$ cannot be $\mathcal{L}^5$ because of the previous paragraph; it must be $\mathcal{S}\mathcal{L}^4$. Since $\mathcal{S}\mathcal{S}$ is not a factor of y', the further left extension $\mathcal{L}\mathcal{S}\mathcal{L}^4$ must thus also be a factor of y'. However, this forces y' to contain one of the further left extensions $\mathcal{L}\mathcal{L}\mathcal{S}\mathcal{L}^4$ and $\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L}^4$, which is impossible.

- $\mathcal{L}^3$ is not a factor of y'': Suppose that $\mathcal{L}^3$ is a factor of y''. By the previous paragraph, its extension $\mathcal{S}\mathcal{L}^3\mathcal{S}$ is a factor of $\rho y''\sigma$. Since $\mathcal{S}\mathcal{S}$ is not a factor of y', the extension of $\mathcal{S}\mathcal{L}^3\mathcal{S}$ to $\mathcal{L}\mathcal{S}\mathcal{L}^3\mathcal{S}$ must be a factor of y'. One of the further left extensions $\mathcal{L}\mathcal{L}\mathcal{S}\mathcal{L}^3\mathcal{S}$ and $\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L}^3\mathcal{S}$ must thus occur in y', but these are impossible by Lemma 4.

We have now shown that neither of $\mathcal{S}^2$ and $\mathcal{L}^3$ can be a factor of y''. Thus

$$y'' \in (\mathcal{L} + \mathcal{L}\mathcal{L})(\mathcal{S}\mathcal{L} + \mathcal{S}\mathcal{L}\mathcal{L})^*.$$

Let p''' be the longest prefix of y'' of the form $\mathcal{L}^k$, and write $y'' = p'''y_1$. Letting $p_1 = p'p''p'''$, we have $|p_1| \leq 3 + 4 + 2$, so the lemma holds. $\square$

5. Parsing words of $\mathcal{M}$ using Φ^2

Lemma 6. *Let $y_1 \in \{\mathcal{S}, \mathcal{L}\}^*$, such that $\Phi(y_1) \in \mathcal{M}$. Then y_1 can be written*

$$y_1 = p_2\Phi(y_2)s_2t_2,$$

where $|p_2|, |s_2| \leq 4, y_2 \in \{\mathcal{L}, \mathcal{S}\}^*$ *and*

$$t_2 \in \left((\epsilon + \mathcal{L} + \mathcal{L}^2 + \mathcal{L}^3)\mathcal{S}\mathcal{L}^* + \mathcal{L}^*(\epsilon + \mathcal{S} + \mathcal{S}\mathcal{L})\right)(\epsilon + \mathcal{S} + \mathcal{L}).$$

Proof. From Lemma 4, no word of

$$\begin{aligned}&(\mathcal{S} + \mathcal{L})\mathcal{S}\mathcal{S}(\mathcal{S} + \mathcal{L}) \cup (\mathcal{S} + \mathcal{L})\mathcal{L}\mathcal{L}\mathcal{L}\mathcal{S}\mathcal{L}(\mathcal{L} + \mathcal{S}\mathcal{S} + \mathcal{S}\mathcal{L})\\ &\cup\Phi(\mathcal{L}\mathcal{L}\mathcal{L}(\mathcal{S} + \mathcal{L})) \cup \Phi((\mathcal{S} + \mathcal{L})\mathcal{L}\mathcal{S}\mathcal{S}(\mathcal{S} + \mathcal{L})) \cup \Phi((\mathcal{S} + \mathcal{L})\mathcal{S}\mathcal{S}\mathcal{S}\mathcal{S}\mathcal{S}(\mathcal{S} + \mathcal{L}))\end{aligned}$$

can appear in y_1. This includes all length 4 two-sided extensions of $\mathcal{S}\mathcal{S}$; it follows that $\mathcal{S}\mathcal{S}$ can only appear in y_1 as a prefix or suffix.

If $|y_1| \le 1$, we are done. In this case, let $p_2 = y_1$, $y_2 = s_2 = t_2 = \epsilon$. Therefore, we will assume that $|y_1| \ge 2$, and write $y_1 = p'y's'$, $|p'| = |s'| = 1$. Then $\mathcal{SS}$ is not a factor of y'.

Suppose that $|y'|_{\mathcal{S}} = n$. If $n = 0$, the lemma is true, letting $p_2 = p'$, $y_2 = s_2 = \epsilon$, $t_2 = y's'$. If $n = 1$, write $y' = \mathcal{L}^k\mathcal{SL}^j$. Since $\mathcal{L}^4\mathcal{SL}^2$ is not a factor of y_1, $k \le 3$ or $j \le 1$; thus we can let $p_2 = p'$, $t_2 = y's'$, and we are again done.

Suppose from now on, that $n \ge 2$, and write $y' = (\prod_{i=1}^{n} \mathcal{L}^{m_i}\mathcal{S})\mathcal{L}^{m_{n+1}}$, where each $m_i \ge 0$. For $1 \le i \le n-1$, $m_{i+1} \le 1$, since $\mathcal{SS}$ is not a factor of y'. It follows that for $1 \le i \le n-2$ $\mathcal{SL}^{m_{i+1}}\mathcal{SL}^{m_{i+2}}$ has one of $\mathcal{SLSL}$ or $\mathcal{SLL}$ as a prefix. This implies that for $1 \le i \le n-2$, we have $m_i \le 3$, since $\mathcal{L}^4\mathcal{SLSL}$ and $\mathcal{L}^4\mathcal{SLL}$ are not factors of y_1. In fact, for $2 \le i \le n-2$, we have $m_i \le 2$, since $\mathcal{SL}^3\mathcal{SLSL}$ and $\mathcal{SL}^3\mathcal{SLL}$ are not factors of y_1. We have thus established that

$$y' \in (\epsilon + \mathcal{L} + \mathcal{L}^2 + \mathcal{L}^3)\,(\mathcal{SL} + \mathcal{SLL})^*\,\mathcal{SL}^j\mathcal{SL}^k$$

Since $\mathcal{L}^4\mathcal{SL}^2$ is not a factor of y_1, we require $k \le 3$ or $j \le 1$. Write $y' = p''y_2\mathcal{S}t''$ where $p'' \in (\epsilon + \mathcal{L} + \mathcal{L}^2 + \mathcal{L}^3)$, $y_2 \in (\mathcal{SL} + \mathcal{SLL})^*$, $t'' \in \mathcal{SL}^k\mathcal{SL}^j$, $k \le 3$ or $j \le 1$. Let $p_2 = p'p''$, $s_2 = \mathcal{S}$, $t_2 = t''s'$. The lemma is established. □

6. Parsing words of $\mathcal{M}$ using Φ^3

Lemma 7. *Let $y_2 \in \{\mathcal{S}, \mathcal{L}\}^*$ such that $\Phi^2(y_2) \in \mathcal{M}$. Then y_2 can be written*

$$y_2 = p_3\Phi(y_3)s_3,$$

where $|p_3|, |s_3| \le 6, y_3 \in \{\mathcal{L}, \mathcal{S}\}^*$.

Proof. From Lemma 4, no word of

$$\mathcal{LLL}(\mathcal{S} + \mathcal{L}) \cup (\mathcal{S} + \mathcal{L})\mathcal{LSS}(\mathcal{S} + \mathcal{L}) \cup (\mathcal{S} + \mathcal{L})\mathcal{SSSSS}(\mathcal{S} + \mathcal{L})$$

can appear in y_2. These include both of the length 4 right extensions of $\mathcal{LLL}$; it follows that $\mathcal{LLL}$ can only appear in y_2 as a suffix. They also include all of the length 5 two-sided extensions of $\mathcal{LSS}$; Thus $\mathcal{LSS}$ can appear in y_2 only as a prefix or suffix. Finally, they include all length 7 two-sided extensions of $\mathcal{S}^5$. Thus, $\mathcal{S}^5$ can only appear in y_2 as a suffix or prefix. If $|y_2| \le 4$, we are done. Assume that $|y_2| \ge 5$, and write $y_2 = p'y's'$, $|p'| = 4$, $|s'| = 1$. Then $\mathcal{LLL}$ is not a factor of y'. We also claim that $\mathcal{SS}$ is not a factor of y'. Otherwise, y_2 has a factor $\rho\mathcal{SS}$ which is not a suffix, with $|\rho| = 4$. However, the length 5 suffix of $\rho\mathcal{SS}$ is not a prefix or suffix of y_2, and contains either $\mathcal{S}^5$ or $\mathcal{LSS}$ as a factor; this is impossible.

Since neither of $\mathcal{L}^3$ or $\mathcal{S}^2$ is a factor of y_2, we have $y' \in (\epsilon + \mathcal{L} + \mathcal{L}^2)(\mathcal{SL} + \mathcal{SLL})^*(\epsilon + \mathcal{S})$, and can write $y' = \mathcal{L}^k\Phi(y_3)\mathcal{S}^j$ where $k \le 2$, $s \le 1$. The lemma therefore holds. □

7. A hierarchy of S's and L's

Combining Lemmas 5 through 7 gives the following:

Lemma 8. *Let $y \in \{\mathcal{S}, \mathcal{L}\}^* \cap \mathcal{M}$. Then y can be written*

$$y = p_1 \Phi(p_2 \Phi(p_3 \Phi(y_3) s_3) s_2 t_2) s_1 t_1,$$

where $|p_1|, |s_1| \leq 9, |p_2|, |s_2| \leq 4, |p_3|, |s_3| \leq 6$, and

$$t_1 \in (\epsilon + \mathcal{S} + \mathcal{S}^2 + \mathcal{S}^3)\mathcal{L}\mathcal{S}^* + \mathcal{S}^*(\epsilon + \mathcal{L} + \mathcal{L}\mathcal{S}),$$

$$t_2 \in \big((\epsilon + \mathcal{L} + \mathcal{L}^2 + \mathcal{L}^3)\mathcal{S}\mathcal{L}^* + \mathcal{L}^*(\epsilon + \mathcal{S} + \mathcal{S}\mathcal{L})\big)\,(\epsilon + \mathcal{S} + \mathcal{L}).$$

Corollary 9. *Let $y \in \{\mathcal{S}, \mathcal{L}\}^* \cap \mathcal{M}$. Then there is a constant κ such that y can be written*

$$y = \pi \Phi^3(y_3)\sigma,$$

where σ can be written $\sigma_1 \Phi(\mathcal{L}^j)\sigma_2 \mathcal{S}^k \sigma_3$, with $|\pi\sigma_1\sigma_2\sigma_3| \leq \kappa$.

Lemma 10. *Suppose that $\langle \mathcal{S}, \mathcal{L}\rangle$ is suitable, and $|h(\mathcal{S})|$ is odd, $|h(\mathcal{L})|$ even. Let*

$$\Sigma = (\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L})^{-1}\Phi^3(\mathcal{S})\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L},\ \Lambda = (\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L})^{-1}\Phi^3(\mathcal{L})\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L}.$$

Then $\langle \Sigma, \Lambda\rangle$ is suitable, and $|h(\Sigma)|$ is odd, $|h(\Lambda)|$ even.

Proof. Each of $|\Sigma|$, $|\Lambda|$ is odd. Let

$$\hat{\ell} = h(\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{L}\mathcal{S})\ell, \quad \hat{\mu} = \ell^R\overline{h(\mathcal{S}\mathcal{L})}, \quad \hat{p} = \overline{\hat{\ell}^R}h(\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L})$$

$$\begin{aligned}
h(\Sigma) &= h((\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L})^{-1}\Phi^3(\mathcal{S})\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L}) \\
&= h((\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L})^{-1}\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L}) \\
&= h(\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L}) \\
&= h(\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{L}\mathcal{S})\ell\ell^R\overline{h(\mathcal{S}\mathcal{L})} \\
&= \hat{\ell}\hat{\mu}
\end{aligned}$$

For a word $z \in \{\mathcal{S}, \mathcal{L}\}^*$ with $|z|$ even, we observe that $\overline{h(z^R)} = (h(z))^R$. Therefore, we also have

$$\begin{aligned}
\Sigma &= h(\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L}) \\
&= h(\mathcal{L}\mathcal{S})h(\mathcal{L})\overline{h(\mathcal{S}\mathcal{L}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L})} \\
&= (\overline{h(\mathcal{S}\mathcal{L})})^R\ell\ell^R\left(h(\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{L}\mathcal{S})\right)^R \\
&= \hat{\mu}^R\hat{\ell}^R
\end{aligned}$$

Further,

$$\begin{aligned}
h(\Lambda) &= h((\mathcal{SLSL})^{-1}\Phi^3(\mathcal{L})\mathcal{SLSL}) \\
&= h((\mathcal{SLSL})^{-1}\mathcal{SLSLLSLSLLSLLSLSLLSLLSLSL}) \\
&= h(\mathcal{LSLSLLSLLSLSLLSLLSLSL}) \\
&= h(\mathcal{LSLSLLSLLS})h(\mathcal{L})\overline{h(\mathcal{SLLSLLSLSL})} \\
&= h(\mathcal{LSLSLLSLLS})\ell \;\; \ell^R \left(h(\mathcal{LSLSLLSLLS})\right)^R \\
&= \hat{\ell}\hat{\ell}^R
\end{aligned}$$

Finally,

$$\begin{aligned}
h(\Lambda) &= h(\mathcal{LSLSLLSLLS})\ell \;\; \ell^R\overline{h(\mathcal{SLLSLLSLSL})} \\
&= h(\mathcal{LSLSLLSLLS})\ell \;\; \ell^R\overline{h(\mathcal{SL})}\;\overline{h(\mathcal{LS})}\;\overline{h(\mathcal{L})}h(\mathcal{LSLSL}) \\
&= h(\mathcal{LSLSLLSLLS})\ell \;\; \ell^R\overline{h(\mathcal{SL})}\;\overline{h(\mathcal{LS})}\;\overline{\hat{\ell}}\;\overline{\hat{\ell}^R}h(\mathcal{LSLSL}) \\
&= \hat{\ell}\hat{\mu}\overline{\hat{\mu}^R}\hat{p}.
\end{aligned}$$

$\square$

This result combines with Corollary 9 to allow us to parse words of $\mathcal{M}$. Let $L_0 = L$, $S_0 = S$. Supposing that $\langle S_i, L_i\rangle$ is suitable, let $\mathcal{L} = L_i$, $\mathcal{S} = S_i$, and

$$L_{i+1} = (S_iL_iS_iL_i)^{-1}\Phi^3(L_i)S_iL_iS_iL_i, \quad S_{i+1} = (S_iL_iS_iL_i)^{-1}\Phi^3(L_i)S_iL_iS_iL_i.$$

Since $\langle S, L\rangle$ is suitable, all of the pairs $\langle S_i, L_i\rangle$ will be suitable by Lemma 10. Suppose $y \in \{S, L\}^* \cap \mathcal{M}$. By repeatedly applying Corollary 9, we write $y = \hat{\pi}\upsilon\hat{\sigma}$ where $\upsilon \in \{S_i, L_i\}^*$.

8. Upper bound on growth rate

If H is a language, denote by $\#(H, n)$ the number of words of H of length n. If N_0 is a positive integer, the statements

$$\text{For } n > 1,\ \#(\mathcal{H}, n) \le n^{\lg n + o(\lg n)}. \tag{1}$$

and

$$\text{For } n > N_0,\ \#(\mathcal{H}, n) \le n^{\lg n + o(\lg n)}. \tag{2}$$

are equivalent; let $K = 1 + \max_{n \le N_0} \#(\mathcal{H}, n)$. Suppose that for $n > N_0$, $\#(\mathcal{H}, n) \le n^{\lg n + o(\lg n)}$. Then for $n > 1$,

$$\begin{aligned}
\#(\mathcal{H}, n) &\le Kn^{\lg n + o(\lg n)} \\
&= n^{\lg n + o(\lg n) + \log_n K} \\
&\le n^{\lg n + o(\lg n)}.
\end{aligned}$$

Define

$$\mathcal{N} = \{z \in \{0,1\}^* : z \text{ avoids } xxx^R\}.$$

Theorem 11. *For $n > 1$, $\#(\mathcal{N}, n) \le n^{\lg n + o(\lg n)}$.*

Recall that

$$\mathcal{K} = \{z \in 0\{0,1\}^*1 : z \text{ avoids } xxx^R\}.$$

Suppose that for $n > 1$, we have $\#(\mathcal{K}, n) \le n^{\lg n + o(\lg n)}$. Then, since neither of 000 and 111 can be a factor of a word of $\mathcal{N}$,

$$\mathcal{N} \subseteq (\epsilon, 1, 11)\mathcal{K}(\epsilon, 0, 00)$$

so that for $n > 5$,

$$\begin{aligned} \#(\mathcal{N}, n) &\le \#(\mathcal{K}, n) + 2\#(\mathcal{K}, n-1) + 3\#(\mathcal{K}, n-2) + 2\#(\mathcal{K}, n-3) + \#(\mathcal{K}, n-4) \\ &\le 9n^{\lg n + o(\lg n)} \\ &\le n^{\lg n + o(\lg n)} \end{aligned}$$

where the last inequality uses $9 \le n^{\log_n 9} = n^{o(\lg n)}$ to absorb the 9 into the $n^{o(\lg n)}$ term. Replacing $N_0 = 5$ by 1, it follows that for $n > 1$, $\#(\mathcal{N}, n) \le n^{\lg n + o(\lg n)}$.

It remains to show that for $n > 1$, $\#(\mathcal{K}, n) \le n^{\lg n + o(\lg n)}$.

Let $\mathcal{T} = (\epsilon + 1)(01)^*(\epsilon + 1)$. Thus

$$\#(T, n) = \begin{cases} 1, & n \le 1 \\ 2, & n \ge 2. \end{cases}$$

Let C be the constant from Theorem 1, and let $\mathcal{P}$ be the set of binary words of length C or less. From Theorem 1, each word k of $\mathcal{K}$ has the form $ph(w)st$ where $p, s \in \mathcal{P}$, $w \in \mathcal{M}$ and $t \in \mathcal{T}$.

Suppose that we can show that for $n > 1$, $\#(\mathcal{M}, n) \le n^{\lg n + o(\lg n)}$. Since $8|w| \ge |h(w)| \ge 5|w|$, if $|k| = n$, then $n/8 \le |w| \le n/5$. Thus, if $n > 8$, then $|w| \ge n/8 > 1$ and $\#(\mathcal{M}, |w|) \le |w|^{\lg |w| + o(\lg |w|)}$, so that there are at most $(n/5)^{\lg(n/5) + o(\lg(n/5))}$ choices for w. There are at most 2^{C+1} choices for each of p and s, and once p, s and w are chosen, there are at most 2 choices for t. In total,

$$\begin{aligned} \#(\mathcal{K}, n) &\le 2^{C+1}(n/5)^{\lg(n/5) + o(\lg(n/5))}2^{C+1}2 \\ &\le 2^{2C+3}n^{\lg n + o(\lg n)} \\ &\le n^{\lg n + o(\lg n)} \end{aligned}$$

where the last inequality uses $2^{2C+3} \le n^{2C+3} = n^{o(\lg n)}$ to absorb 2^{2C+3} into the $n^{o(\lg n)}$ term. Replacing $N_0 = 8$ by 1, it follows that for $n > 1$, $\#(\mathcal{K}, n) \le n^{\lg n + o(\lg n)}$.

We have thus reduced the proof of Theorem 11 to the following:

Theorem 12. *For $n > 1$, $\#(\mathcal{M}, n) \le n^{\lg n + o(\lg n)}$.*

Proof. Let $y \in \mathcal{M}$ have length n. Choose $\langle \mathcal{S}, \mathcal{L} \rangle = \langle S, L \rangle$. Then iteration of Corollary 9 gives

$$y = p_1 \Phi^3(p_2 \Phi^3(p_3 \cdots p_m \Phi^3(\epsilon) s_m \cdots s_3) s_2) s_1,$$

where $m \le (\lg n)/3$. For $i \in \{1, \cdots, m\}$ we have

$$s_i = \sigma_{1,i} \Phi(\mathcal{L}^{j_i}) \sigma_{2,i} \mathcal{S}^{k_i} \sigma_{3,i}$$

Since $|p_i \sigma_{3,i} \sigma_{2,i} \sigma_{1,i}| \le \kappa$, there is a constant α such that there are at most α choices for $(p_i, \sigma_{i,3}, \sigma_{i,2}, \sigma_{i,1})$. This gives a number of choices for $\{(p_i, \sigma_{i,3}, \sigma_{i,2}, \sigma_{i,1})\}_{i=1}^m$ which is polynomial in n.

This leaves the problem of bounding the number of choices of the j_i and k_i.

We have

$$\begin{aligned} n &\ge |\Phi^3(\Phi^3(\cdots \Phi^3(\epsilon) \Phi(\mathcal{L}^{j_m}) \mathcal{S}^{k_m} \cdots \Phi(\mathcal{L}^{j_3}) \mathcal{S}^{k_3}) \Phi(\mathcal{L}^{j_2}) \mathcal{S}^{k_2}) \Phi(\mathcal{L}^{j_1}) \mathcal{S}^{k_1}| \\ &= \sum_{i=1}^{m} \big(j_i |\Phi^{3i-2}(\mathcal{L})| + k_i |\Phi^{3i-3}(\mathcal{S})| \big) \\ &= \sum_{i=1}^{m} (j_i F_{6i-2} + k_i F_{6i-5}) \end{aligned}$$

It follows that the number of choices for the j_i, k_i is less than or equal to the number of partitions (with repetition) of n with parts chosen from $\{F_{3i+1}\}_{i=0}^{\infty}$. Since $F_{3i+1} \ge 2^i$, this is less than or equal to the number of partitions of n into powers of 2. Mahler [8] showed that the number $p(n, r)$ of partitions of n into powers of r satisfies

$$\lg p(n, r) \sim \frac{\lg^2 n}{\lg^2 r};$$

thus, $p(n, 2) \sim n^{\lg n + o(\lg n)}$. The result follows. $\square$

9. Lower bound on growth

Let $\psi : \{S, L\}^* \to \{S, L\}^*$ be given by

$$\psi(S) = LSL, \psi(L) = LSLSL.$$

Since $\psi(S)$, $\psi(L)$ are palindromes, we have

$$\psi(u^R) = (\psi(u))^R, u \in \{S, L\}^*.$$

Letting $\langle \mathcal{S}, \mathcal{L} \rangle = \langle S, L \rangle$, we find that $\psi = (\mathcal{L}\mathcal{S}\mathcal{L})^{-1} D^3 \mathcal{L}\mathcal{S}\mathcal{L}$. It follows that $|\psi^k(S)| = F_{3k+1}$, $|\psi^k(L)| = F_{3k+2}$.

Define languages $\mathscr{L}_i$ by

$$\mathscr{L}_0 = LS^*, \mathscr{L}_{i+1} = \psi(\mathscr{L}_i) LS^*.$$

Let $\mathscr{L} = \cup_{i=0}^{\infty}\mathscr{L}_i$.

A word $w \in \mathscr{L}$ has the form

$$w = \psi(\psi(\psi(\cdots\psi(\psi(LS^{k_m})LS^{k_{m-1}})\cdots)LS^{k_2})LS^{k_1})LS^{k_0}$$

so that the number of words of $\mathscr{L}$ of length n is the number of partitions of n of the form

$$n = \sum_{i=0}^{m} \left(F_{3i+2} + k_i F_{3i+1}\right).$$

Since $F_{i+1} \leq 2^i$, this is greater than or equal to the number of partitions of n of the form

$$n = \sum_{i=0}^{m} \left(2^{3i+1} + k_i 2^{3i}\right),$$

which is greater than or equal to the number of partitions of n of the form

$$n = \sum_{i=0}^{m} (k_i + 1)2^{3i+1}.$$

This, in turn, is at least half of the number of partitions of n of the form

$$n = \sum_{i=0}^{m} k_i 2^{3i+1},$$

which is the number of partitions of $n/2$ of the form

$$n/2 = \sum_{i=0}^{m} k_i 8^i.$$

Following Mahler [8], this is $p(n/2, 8) \sim n^{\lg n + o(\lg n)}$. We will show that no word of $h(\mathscr{L})$ has a non-empty factor xxx^R, so that this gives a lower bound on $\mathcal{N}$.

One checks the following:

Lemma 13. *No word of $\mathscr{L}$ has any of the following factors:*

$$L^3, SSL, SLSLS, LSLSLLSLSLLSLSL = \psi(L^3), LLSLLSLLSLSL$$

$$LSLLSLSLLSLLSLSLLSL = \psi(SLSLS).$$

Theorem 14. *No word of $h(\mathscr{L})$ contains a non-empty word of the form xxx^R.*

Proof. Suppose $w \in \mathscr{L}$, and xxx^R is a non-empty factor of $h(w)$. Let

$$W = \left((h(S) + h(L))(\overline{h(S)} + \overline{h(L)})\right)^* = \left((00100 + 00100100)(11011 + 11011011)\right)^*.$$

Thus $h(w)$ is a factor of a word of W. Note that none of 000, 111, 0101, 1010, 001011, 110011, 010010010, is a factor of any word of W, nor thus, of w. Also, $\ell = 0010$ is always followed by 01 in any word of W, while $\overline{\ell^R} = 1011$ is always preceded by 01.

If $|x| \leq 2$, then $h(w)$ contains a factor 000, 111, 010110 or 101011. The last two contain 0101, so this is impossible. Assume therefore that $|x| \geq 3$ and write $x = x'\alpha\beta\gamma$, where α, β, $\gamma \in \{0, 1\}$. Then $\alpha\beta\gamma\gamma\beta\alpha$ is a factor of xxx^R. Suppose that $\gamma = 0$. (The other case is similar.) Since 000 is not factor of w, we can assume that $\beta = 1$. Since 110011 is not a factor of w, $\alpha\beta\gamma = 010$. If $|x| = 3$, then xxx^R is 010010010, which is not a factor of w. We conclude that $|x| \geq 4$. Since 1010 is not a factor of w, $\ell = 0010$ is a suffix of x. Write $x = x''\ell$, so that

$$xxx^R = x''\ell x''\ell\ell^R(x'')^R = x''\ell x'' h(L)(x'')^R.$$

Since $x''\ell x''$ precedes $h(L)$ in a word of W, the length 4 suffix of $x''\ell x''$ must be 1011; since x'' follows ℓ in $h(w)$, it follows that x'' begins with 0. Therefore, $|x''| \geq 5$. It follows that x'' must end with 11011, so that, in fact, $|x''| \geq 6$, and 011011 is a suffix of x''. If $|x''| = 6$, then

$$xxx^R = 0110110010\ 0110110010\ 0100110110 = 011011h(SSL)110110.$$

This forces SSL to be a factor of w, which is impossible, since $w \in \mathscr{L}$. Thus $|x''| \geq 7$.

Since 0101 is not a factor of w, if suffix 011011 of x'' is preceded by 1, it is preceded by 11, and $\overline{h(L)}\ell$ is a suffix of x. This forces xx^R to have

$$\overline{h(L)}\ell\ell^R\overline{h(L)^R} = \overline{h(L)}h(L)\overline{h(L)}$$

as a factor, forcing LLL to be a factor of w, which is impossible. We conclude that 0011011 is a suffix of x''. Since x'' follows ℓ in w, 01 must be a prefix of x''. Suppose 011 is a prefix of x''. Since 0011011 is a suffix, then $x''\ell x''$ has factor

$$0011011\ell 011 = 00\overline{h(S)}h(S)11$$

and w has a factor $SSuL$ for some u; this is impossible. We conclude that 010 is a prefix of x''; since 0101 is not a factor of w, in fact, $0100 = \ell^R$ is a prefix of x''. In total,

$$xxx^R = \ell^R\hat{x}\ell\ \ell^R\hat{x}\ell\ \ell^R\hat{x}^R\ell$$

The 'bracketing' by ℓ and ℓ^R forces w to contain a factor $uLuLu^R$, where $|u|$ is odd.

Consider the shortest factor $uLuLu^R$ or w, where $|u|$ is odd.

If the last letter of u is L, then LLL is a central factor of uLu. This is impossible. Thus S is a suffix of u. If $u = S$, then $uLuLu^R = SLSLS$, which is not a factor of any word of $\mathscr{L}$. We conclude that $|u| > 1$, so that $|u| \geq 3$, since $|u|$ is odd.

Since SSL is not a factor of w, the length 3 suffix of uL is LSL. This makes $LSLSL$ a central factor of uLu^R. Since $SLSLS$ is not a factor of w, the length

3 suffix of u is LLS. If $u = LLS$, then Lu has prefix LLL, which is not a factor of w. We conclude that $|u| \geq 5$.

Since neither of LLL and SS is a factor of w, we conclude that $LSLLS$ is the length 5 suffix of u. If $u = LSLLS$, then $uLuLu^R = LSLLSLLSLLSLSLLSL$, with illegal factor $LLSLLSLLSLSL$. Thus $|u| \geq 7$.

If the length 7 suffix of u is $LSLSLLS$, then a central factor uLu^R is $LSLSLLSLSLLSLSL$, which is not a factor of w. We conclude that the length 7 suffix is $SLLSLLS$.

Write $w = \psi(v)LS^k$ for some $v \in \mathscr{L}$, some $k \geq 0$. Since $|w|_L > 1$, $v \neq \epsilon$. Then w has suffix LLS^k, and prefix $uLuLSL$ of $uLuLu^R$ must be a factor of $\psi(v)$. Let $L(LSL)^mL$ be a factor of uLu where m is as large as possible. Since $uLuLSL$ has suffix $LSLSL$, and $uLuLSL$ is a factor of $\psi(v)$, word $L(LSL)^mLSLSL$ must be a factor of $uLuLSL$. If $m \geq 2$, then $uLuLu^R$ has illegal factor $LLSLLSLLSLSL$. We conclude that $m = 1$, so that $LLSLLSLL$ is not a factor of uLu

In the context of uLu, word u follows the suffix $LLSLLSL$ of uL. Therefore, u cannot have L as a prefix or uLu contains the factor $LLSLLSLL$. It follows that SL is a prefix of u. However, a prefix of u cannot be SLS; otherwise uLu would have factor $uLSLS$ which has illegal suffix $SLSLS$. It follows that the length 3 prefix of u is SLL.

Write

$$u = SL : Lu'SL : LSL : LS$$

The colons indicate boundaries in u between instances of $\psi(S)$ and $\psi(L)$. Thus, we may write $u = SL\psi(u'')LS$, for some word u'' in $\mathscr{L}$. Since $|\psi(S)| \equiv |\psi(L)| \equiv 1$ (modulo 2), we have

$$|u| \equiv |\psi(u'')| \equiv |u''| \text{ (modulo 2).}$$

Then

$$\begin{aligned} uLuLu^R &= SL\psi(u'')LSLSL\psi(u'')LSLSL(\psi(u''))^R LS \\ &= SL\psi(u''Lu''L(u'')^R)LS. \end{aligned}$$

Recall that $w = \psi(v)LS^k$. Although the suffix LS of $uLuLu^R$ may occur here as a prefix of LS^k, certainly $uLuLu^R(LS)^{-1}$ is in $\psi(v)$. We conclude that $u''Lu''L(u'')^R$ is a factor of $\mathscr{L}$, where u'' has odd length shorter than u. This is a contradiction. $\square$

References

[1] K. Baker, G. McNulty, W. Taylor, "Growth problems for avoidable words", Theoret. Comput. Sci. 69 (1989), 319–345.

[2] B. Bischoff, J. D. Currie, and D. Nowotka, "Unary patterns with involution", Internat. J. Found. Comp. Sci. 23 (2012), 1641–1652.

[3] B. Bischoff and D. Nowotka, “Pattern avoidability with involution”. In WORDS 2011, pp. 65-70, 2011. Available at `http://rvg.web.cse.unsw.edu.au/eptcs/content.cgi?WORDS2011` .

[4] F.-J. Brandenburg, “Uniformly growing k-th power-free homomorphisms”, Theoret. Comput. Sci. 23 (1983), 69–82.

[5] A. Carpi, “On the number of Abelian square-free words on four letters”, Discrete Appl. Math. 81 (1998), 155–167.

[6] J. D. Currie, “Pattern avoidance with involution”. Available at `http://arxiv.org/abs/1105.2849`, 2011.

[7] C. F. Du, H. Mousavi, L. Schaeffer, J. Shallit, “Decision algorithms for Fibonacci-automatic words with applications to pattern avoidance”. Available at `http://arxiv.org/abs/1406.0670`, 2014.

[8] Kurt Mahler, “On a special functional equation”, J. London Math. Soc. 1.2 (1940), 115–123.

[9] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, `http://oeis.org` .

[10] N. Rampersad and J. Shallit, “Words avoiding reversed subwords”, J. Combin. Math. Combin. Comput. 54 (2005), 157-164.

[11] A. Restivo, S. Salemi, “Overlap free words on two symbols”. In M. Nivat, D. Perrin, eds., *Automata on Infinite Words*, Vol. 192 of *Lecture Notes in Computer Science*, pp. 198–206, Springer-Verlag, 1984.

[12] G. Rote, “Sequences with subword complexity $2n$”, J. Number Theory 46 (1994), 196-213.

[13] A. M. Shur, “Growth properties of power-free languages”. In *Developments in Language Theory*, Vol. 6795 of *Lecture Notes in Computer Science*, pp. 28–43, Springer-Verlag 2011.