Existence and uniqueness of certain automorphisms on K3 surfaces

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Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit der Existenz und gegebenenfalls Eindeutigkeit von Automorphismen von K3 Flächen sowohl über $\mathbb{C}$ als auch in positiver Charakteristik.

Wir beweisen, dass eine K3 Fläche mit vielen Geradenbündeln und einem, nicht symplektischen Automorphismus endlicher Ordnung, welcher durch eine n-te Einheitswurzel (wobei n die Ordnung teilt) auf den globalen 2-Formen operiert, eindeutig durch diese n-te Einheitswurzel ζ_n und ein Ideal in $\mathbb{Z}[\zeta_n]$ bestimmt ist. Wir geben zwei Anwendungen.

Zunächst verallgemeinern wir Vorontsovs Theorem zur Klassifizierung von komplexen K3 Flächen mit einem nicht-symplektischen Automorphismus, welcher mit kleiner Ordnung (anstelle von Trivial) auf der Néron-Severi Gruppe operiert. Dieses wird zu einer Klassifikation von rein nicht-symplektischen Automorphismen hoher Ordnung ausgebaut. Projektive Modelle werden angegeben.

In der zweiten Anwendung zeigen wir auf der einen Seite die Existenz und Eindeutigkeit einer K3 Fläche mit treuer $(\mathbb{Z}/5\mathbb{Z})^2$-Gruppenwirkung. Lässt man nun andererseits die Endlichkeit der Gruppe fallen, so existieren unendlich viele K3 Flächen mit einem symplektischen und einem nicht-symplektischen Automorphismus der Ordnung 5.

Im weiteren Verlauf wenden wir uns Automorphismen unendlicher Ordnung zu. Diese fallen in zwei Klassen, je nachdem ob sie mit einer Faserung kompatibel sind oder nicht. Im ersten Fall verschwindet die algebraische (oder topologische) Entropie, während sie im zweiten Fall der Logarithmus einer Salem Zahl ist. Dies ist eine ganzalgebraische Zahl λ, welche konjugiert zu $1/\lambda$ ist und deren weitere Konjugierten auf dem Einheitskreis liegen. Der Grad dieser Salem Zahl ist der Salem Grad des Automorphismus. Im komplex projektiven Fall ist er höchstens 20, während auf supersingulären K3 Flächen in positiver Charakteristik Salem Grad 22 möglich ist. Wir beginnen mit der expliziten Konstruktion solcher Automorphismen auf den supersingulären K3 Flächen mit Artin Invariante $\sigma = 1$ in Charakteristik $p \equiv 3 \mod 4$. Weiter beweisen wir, dass *jede* supersinguläre K3 Fläche einen solchen Automorphismus besitzt. Im letzten Teil der Arbeit wird eine Strategie entwickelt, um zu entscheiden, ob eine gegebene Salem Zahl auf einer supersingulären K3 Fläche realisiert ist. Im Fall der Charakteristik $p = 5$ zeigen wir, dass die minimale Salem Zahl λ_d im Grad d genau dann realisiert ist, wenn $18 \neq d \leq 22$ gerade ist. Nebenbei entscheiden wir den von McMullen in **[64]** offen gelassenen Fall der Realisierung von λ_{12} auf einer komplexen projektiven K3 Fläche positiv.

Die benutzten (und weiterentwickelten) Methoden sind Gittertheorie, Kleben, lineare und quadratische Programmierung, die Torelli Sätze und Computer Algebra. Wir sehen sie als ersten Schritt hin zu einer vollständigen Klassifizierung der endlichen, treuen Gruppenwirkungen auf K3 Flächen.

Schlüsselwörter: supersinguläre K3-Fläche, Automorphismus, Salem Zahl.

Abstract

A projective K3 surface over an algebraically closed field $k = \overline{k}$, is a smooth, projective surface with $h^1(X, \mathcal{O}_X) = 0$ and $\omega_X \cong \mathcal{O}_X$. In this thesis we study the existence and uniqueness of K3 surfaces with certain automorphisms, both over $\mathbb{C}$ and in positive characteristic.

We prove that a K3 surface with many line bundles and a finite order non-symplectic automorphism, acting by an n-th root of unity on the global 2-forms, is determined by this n-th root of unity ζ_n and an ideal in the cyclotomic integers $\mathbb{Z}[\zeta_n]$. Two applications are given.

First, we generalize Vorontsov's theorem to a classification of K3 surfaces admitting a purely non-symplectic automorphism which acts with small order (instead of trivially) on the Néron-Severi group. Further this is extended to a classification of purely non-symplectic automorphisms of high order. Projective models are given. Then, we prove that there is a unique pair (X, G) up to isomorphism where X is a K3 surface and $(\mathbb{Z}/5\mathbb{Z})^2 \cong G$ a subgroup of the automorphism group of X. However, dropping the commutativity or rather finiteness of the group, we prove the existence of an infinite sequence of K3 surfaces admitting both a symplectic and a non-symplectic automorphism of order 5.

Then, we turn to automorphisms of infinite order. They fall into two classes depending on whether they are compatible with an elliptic fibration or not. In the first case the algebraic (or topological) entropy of the automorphism is zero, while in the second case it is the logarithm of a Salem number. That is an algebraic integer $\lambda > 1$ which is conjugate to $1/\lambda$ and all whose other conjugates lie on the unit circle. The degree of the Salem number is called the Salem degree of the automorphism. It is at most 20 over $\mathbb{C}$, but in positive characteristic on supersingular K3 surfaces Salem degree 22 is possible. We start by giving an explicit construction of such automorphisms on the supersingular K3 surface of Artin invariant one in characteristic $p \equiv 3 \mod 4$. Then, we prove that every supersingular K3 surface admits such an automorphism. In the last part of the thesis we give a strategy to prove or disprove the existence of an automorphism on a supersingular K3 surface realizing a given Salem number. We apply the strategy in characteristic 5, and prove that the minimal Salem number λ_d of degree d is realized if and only if $d \leq 22$ is even and $d \neq 18$. As a by product, we close a case left open by McMullen in **[64]** - the existence of a complex projective K3 surface automorphism realizing λ_{12}.

The main tools applied (and developed) are lattice theory, gluing, linear and quadratic programming as well as the Torelli theorems and computer algebra. The tools developed in this thesis can be seen as first step towards the full classification of all finite groups acting faithfully on some K3 surface.

Keywords: supersingular K3 surface, automorphism, Salem number.

Contents

CHAPTER 1

Introduction

Most results of this thesis are published in the papers [**19, 20, 18, 38**]. Chapter 7 is joint work with Víctor González-Alonso.

We assume the reader is familiar with the general theory of algebraic geometry, say, at the level of Hartshorne's book [**42**]. Further, we assume the general theory of algebraic surfaces as in [**10**]. Nevertheless, we will remind the reader of some facts needed.

1. Complex K3 surfaces and the Torelli theorem

In this section we review some standard facts about complex K3 surfaces and their moduli needed in the sequel. Later we will meet K3 surfaces in positive characteristic as well, especially supersingular ones. See Chapter 5. Our main reference is [**10**].

Definition 1.1. A complex K3 surface is a smooth, 2-dimensional complex manifold X with vanishing irregularity $h^1(X, \mathcal{O}_X) = 0$ and trivial canonical bundle $\omega_X \cong \mathcal{O}_X$.

An algebraic K3 surface over a field k is defined in just the same way by replacing 'complex manifold' with 'projective variety over k'.

Common examples are

- double covers of $\mathbb{P}^2$ branched over a smooth sextic,
- smooth quartics in $\mathbb{P}^3$,
- smooth intersections of a quadric and a cubic in $\mathbb{P}^4$,
- a smooth complete intersection of 3 quadrics in $\mathbb{P}_5$,
- Kummer surfaces, that is the minimal resolution $\widetilde{A/\langle\pm 1\rangle}$ of the quotient of an abelian surface by the involution $x \mapsto -x$.

Instead of requiring smoothness we may allow ADE-singularities. They admit a crepant resolution, and then the minimal model of the surface is still a K3 surface.

It is a non-trivial fact that all complex K3 surfaces are simply connected. To see this, one first shows that any two K3 surfaces are diffeomorphic. In particular their topology agrees. Then, one proves that smooth quartics in $\mathbb{P}^3_{\mathbb{C}}$ are simply connected.

Let X be a complex K3 surface. Its second singular cohomology equipped with the cup product is an even unimodular lattice

$$H^2(X, \mathbb{Z}) \cong 3U \oplus 2E_8 =: L_{K3}$$

of signature $(3, 19)$. Such a lattice is unique up to isometry. Since complex K3 surfaces are Kähler, there is a Hodge decomposition:

$$H^2(X, \mathbb{Z}) \otimes \mathbb{C} \cong H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

where $H^{i,j}(X) \cong H^j(X, \Omega^i_X)$, $H^{i,j}(X) = \overline{H^{j,i}(X)}$ and $H^{1,1}(X) = (H^{2,0}(X) \oplus H^{0,2}(X))^{\perp}$ has signature $(1, 19)$. By Lefschetz' Theorem on $(1, 1)$ classes we can recover the *Néron-Severi group* from the Hodge structure as

$$\mathrm{NS}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z}).$$

Its rank ρ is called the Picard number of X. A sublattice $N \subseteq L$ is called *primitive* if $N = N \otimes \mathbb{Q} \cap L$. The *transcendental lattice* $T(X)$ is defined as the smallest primitive sublattice $T \subseteq H^2(X, \mathbb{Z})$ whose complexification contains $H^{2,0}(X) \subseteq T \otimes \mathbb{C}$. The surface X is projective if and only if NS has signature $(1, \rho - 1)$. In this case $T(X) = \mathrm{NS}(X)^{\perp}$.

The set

$$\{x \in H^{1,1}(X, \mathbb{R}) \mid x^2 > 0\}$$

has two connected components $\mathcal{C}_X$ and $\mathcal{C}'_X$. One of them, say $\mathcal{C}_X$, contains the cone of Kähler classes. It is called the *positive cone*. A class $d \in H^2(X, \mathbb{Z})$ is called effective (resp. ample) if $\delta = c_1(\mathcal{O}_X(D))$ for D an effective (resp. ample) divisor where $c_1 : \operatorname{Pic} X \to H^2(X, \mathbb{Z})$ denotes the first Chern class. By Riemann-Roch if $\delta \in H^2(X, \mathbb{Z})$ with $\delta^2 = -2$, then δ or $-\delta$ is effective.

$$\Delta_X = \{\delta \in \mathrm{NS}(X) \mid \delta^2 = -2, \delta \text{ effective}\}$$

Then the Kähler cone is

$$\mathcal{C}_X^+ = \{x \in \mathcal{C}_X \mid x.\delta > 0 : \forall \delta \in \Delta_X\},$$

and (if X is projective,) the ample cone

$$\alpha_X = \mathrm{NS}(X) \otimes \mathbb{R} \cap C_X^+ = \{x \in \mathrm{NS}(X) \otimes \mathbb{R} \mid x.\delta > 0 : \forall \delta \in \Delta_X\}.$$

We see that the integral points of the Kähler cone are the ample classes.

Let X, Y be complex K3 surfaces and

$$f : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$$

an isometry of lattices whose $\mathbb{C}$-linear extension maps $H^{2,0}(X)$ to $H^{2,0}(Y)$. We call f a *Hodge isometry*. If moreover $f(\mathcal{C}_X) = \mathcal{C}_Y$, and f maps effective classes on X to effective classes on Y, then f is called *effective*.

The Hodge structure of a K3 surface determines it up to isomorphism as is reflected by the (weak) Torelli theorems.

Theorem 1.2 (Weak Torelli). [**10**, VIII 11.2] *Let X and Y be two complex K3 surfaces and suppose there is a Hodge isometry $H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$. Then X and Y are isomorphic.*

Theorem 1.3 (Strong Torelli). [**10**, VIII 11.1] *Let X, Y be complex K3 surfaces and*

$$f : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$$

an effective Hodge isometry, then $f = F^*$ for a unique isomorphism $F : Y \to X$.

If X is projective, we may instead assume that f maps ample classes to ample classes on Y.

The Hodge decomposition is determined by the period $H^{2,0}(X) = \mathbb{C}\omega$ and the space of all such Hodge structures on L_{K3},

$$\Omega(L_{K3}) = \{\mathbb{C}\omega \in \mathbb{P}(L_{K3} \otimes \mathbb{C}) \mid \omega.\omega = 0,\ \omega.\overline{\omega} > 0\},$$

is called the period domain.

A marked K3 surface is a complex K3 surface X together with an isometry $\phi : H^2(X, \mathbb{Z}) \to L_{K3}$ called marking. Two marked K3 surfaces (X, ϕ_X) and (Y, ϕ_Y)

are equivalent if there is an isomorphism $f : X \to Y$ with $\phi_Y = \phi_X \circ f^*$. The *period point* of a marked K3 surface (X, ϕ_X) is $\phi_{\mathbb{C}}(H^{2,0}) \in \Omega_{L_{K3}}$. We can make sense of this notion in families as follows. Let $f : \mathcal{X} \to U$ be a flat family of K3 surfaces over a contractible open set U. Let X be a fiber and $\phi : H^2(X, \mathbb{Z}) \to L_{K3}$ a marking. Then the marking of X extends to the family via $\phi_U : R^2 f_* \mathbb{Z} \to (L_{k3})_U$ where $(L_{k3})_U$ is a constant sheaf on U with fiber L_{K3}.

In particular, we have a period map for marked families. Now, the local Torelli theorem states that the period map for the versal deformation family is an isomorphism.

THEOREM 1.4 (Surjectivity of the period map). [**10**, VIII 14.2] *Every point of* $\Omega(L_{K3})$ *is the period point of a marked* $K3$ *surface.*

REMARK 1.5. It is possible to construct a coarse moduli space of (unmarked) K3 surfaces by taking the quotient of the period domain by $O(L_{K3})$. However, this quotient is badly behaved. For example it is non-Hausdorff. This can be solved by considering (pseudo-)*polarized* K3 surfaces instead.

Recall that a divisor H is called pseudo ample if the rational map induced by the linear system $|nH|$ is birational for large $n \in \mathbb{N}$. Let H be an effective divisor on a K3 surface X and $h = c_1(H)$. Then H is (pseudo)-ample if and only if $h^2 > 0$ and $h.\delta > 0 (\geq 0)$ for all effective classes δ with $\delta^2 = -2$. If H is pseudo ample, then $|nH|$ contracts all (-2) curves orthogonal to H and the resulting surface is embedded (cf. [**60**][**69**, p.322]). This results in ADE singularities cf. [**10**, III §2 (2.1,3.4)].

DEFINITION 1.6. Let L be a lattice. A (pseudo-)ample L-polarized K3 surface is a pair (X, ι) where X is a K3 surface and $\iota : L \hookrightarrow \mathrm{NS}(X)$ a primitive embedding such that $\iota(L)$ contains a (pseudo-)ample class.

Two L-polarized K3 surfaces $(X, \iota), (Y, j)$ are called equivalent if there is an isomorphism $f : X \to Y$ with $j = i \circ f^*$. As it turns out, lattice polarizations behave well in families and one can obtain a well behaved moduli theory. The period domain

$$\Omega(L) = \{[\omega] \in \Omega(L_{K3}) \mid \omega.l = 0 \ \ \forall l \in L\}$$

of (pseudo-)ample L polarized $K3$ surfaces is of dimension $20 - \operatorname{rk} L$.

EXAMPLE 1.7. If $i : X \to \mathbb{P}^2$ is a K3 surface branched over a smooth sextic, then $h = c_1(i^* \mathcal{O}_{\mathbb{P}^2}(1))$ has $h^2 = 2$. We see that X is (2) polarized. If $i : X \hookrightarrow \mathbb{P}^3$ is a smooth quartic, then $h = c_1(i^* \mathcal{O}_{\mathbb{P}^3}(1))$ has self intersection 4. Since $h^2 = 4$, it is indivisible, and hence X is (4)-polarized.

EXAMPLE 1.8. [**41**, p.24] Let X be a complex K3 surface and $\pi : X \to \mathbb{P}^1$ be an elliptic fibration (see. Chap 6 § 2) with section $O : \mathbb{P}^1 \to X$. We can obtain X as the minimal model of the hypersurface in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-4) \oplus \mathcal{O}_{\mathbb{P}^1}(-6))$ given by the Weierstraß equation

$$ZY^2 = X^3 + \alpha(s,t) X Z^2 + \beta(s,t) Z^3$$

where (s,t) are coordinates on $\mathbb{P}^1$ and $\alpha(s,t)$, $\beta(s,t)$ are homogeneous of degrees 8 and 12 respectively. Denote by $f = [\pi^{-1}(t)]$ the class of a fiber and by $o = [\mathcal{O}]$ the class of the section. Then

$$\mathbb{Z}o + \mathbb{Z}f \cong \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \cong U$$

is unimodular, hence primitive in NS. We see that elliptic fibrations are U-polarized K3 surfaces.

Since our primary interest lies more in (isolated) examples than in families, we refer to **[29, 69, 41]** for further details.

If we consider just a single K3 surface X, we will usually omit the (X) from notation and just write $H^{i,j}, \mathrm{NS}, T$, etc. From now on all K3 surfaces are assumed to be projective.

CHAPTER 2

Lattices

Lattices lie at the heart of the theory of K3 surfaces. In this chapter we fix notation and develop the tools needed to study automorphisms of K3 surfaces.

1. Basic definitions and notation

A *lattice* is a finitely generated free abelian group L together with a non-degenerate symmetric bilinear form

$$\langle -, - \rangle : L \times L \longrightarrow \mathbb{Z}.$$

If confusion is unlikely, we will write $x.y$ and x^2 instead of $\langle x, y\rangle$ and $\langle x, x\rangle$ where $x, y \in L$. The *signature* of L is the pair (n_+, n_-), where n_+ (resp. n_-) is the number of positive (resp. negative) eigenvalues of the $\mathbb{R}$-bilinear extension of $\langle -, -\rangle$. A lattice is called *even* if $\langle x, x\rangle \in 2\mathbb{Z}$ for any $x \in L$, otherwise it is called *odd*. The *orthogonal group* of L is the group of *isometries* of L, that is,

$$O(L) = \{f : L \to L \mid \langle f(x), f(y)\rangle = \langle x, y\rangle \ \forall x, y \in L\} \subseteq GL(L).$$

As a matter of notation, if L_1 and L_2 are two lattices, the direct sum $L_1 \oplus L_2$ is meant to be the *orthogonal* direct sum, unless any other bilinear form is specified.

The *dual lattice* of L is defined as

$$L^\vee = \{y \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle x, y\rangle \in \mathbb{Z} \quad \forall\, x \in L\}\,.$$

The non-degeneracy of the bilinear form implies that the natural map $L^\vee \to \mathrm{Hom}(L, \mathbb{Z})$ defined by $x \mapsto \langle x, -\rangle$ is an isomorphism. Given an isometry $f \in O(L)$ its $\mathbb{Q}$-linear extension satisfies $f_{\mathbb{Q}}(L^\vee) = L^\vee$. In this way $L \mapsto L^\vee$ is a *covariant* functor.

The *discriminant group* of L is defined as $D_L = L^\vee/L$, and naturally inherits a symmetric bilinear form

$$b_L : D_L \times D_L \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

In case L is even, there is a natural quadratic form (the *discriminant form*):

$$q_L : D_L \to \mathbb{Q}/2\mathbb{Z}.$$

We say that a bilinear or quadratic form is totally isotropic on some subspace if it vanishes identically on this subspace. The discriminant form of a lattice L determines its overlattices as follows:

$$\left\{ \begin{matrix} \text{Lattices M} \\ L \subseteq M \subseteq L^\vee \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \text{Totally isotropic} \\ \text{subspaces } \overline{M} \subseteq D_L \end{matrix} \right\}$$

If L is even, then q_L is zero on $\overline{M} = M/L$ if and only if M is even.

The *determinant* of L, denoted $\det(L)$, is the determinant of the Gram matrix of $\langle -,-\rangle$ with respect to any basis of L, and coincides up to sign with the order of the discriminant group D_L. More precisely

$$\det(L) = (-1)^{n_-}\,|D_L|\,.$$

A lattice is called *unimodular* if $|\det L| = 1$. Given a finite abelian group D, its *length* $l(D)$ is its minimum number of generators. In particular $l(D_L) \leq \operatorname{rk} L^\vee = \operatorname{rk} L$ for a lattice L. The discriminant group (and form) decomposes as an orthogonal direct sum of p-groups

$$(D_L, q_L) \cong \bigoplus_p \left(q|(D_L)_p : (D_L)_p \to \mathbb{Q}_p/2\mathbb{Z}_p\right).$$

By polarization, $b|(D_L)_p^2$ and $q|(D_L)_p$ carry the same information for odd primes $p \neq 2$. If $(D_L)_p$ is an $\mathbb{F}_p$-vector space, then $q|(D_L)_p$ takes values in $\frac{2}{p}\mathbb{Z}/2\mathbb{Z}$. In this case, if $p \neq 2$, we denote by q_p the quadratic form $q_p(x) \equiv px^2 \mod p$ on $(D_L)_p$ with values in $\mathbb{F}_p$. Such a form is determined up to isometry by its rank and determinant $\det(q_p) \in \mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2$ (cf. [**63**, §3]).

We say that two lattices M and N are in the same *genus* if $N \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong M \otimes_{\mathbb{Z}} \mathbb{Z}_p$ are isometric over the p-adic integers for all primes p and $N \otimes_{\mathbb{Z}} \mathbb{R} \cong M \otimes_{\mathbb{Z}} \mathbb{R}$ over the real numbers.

Theorem 2.1. [**75**, 1.9.4] *The signature (n_+, n_-) and discriminant form q determine the genus of an even lattice and vice versa.*

Locally this means that $N \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $q|(D_L)_p$ carry the same information. The local data at different primes is connected by the oddity formula

$$\operatorname{sig}(L) + \sum_{p\geq 3} p\text{-excess}(L) \equiv \operatorname{oddity}(L) \mod 8$$

where the $\operatorname{sig}(L)$ is defined as $n_+ - n_-$. The p-excess is an invariant of $q|(D_L)_p$ for $p \geq 3$ while the oddity is an invariant of the even part $q|(D_L)_2$.

We need only the following simple cases:

- If L is an even lattice and $2 \nmid \det L$, then the oddity (L) is zero.
- If $(D_L)_p$ is an $\mathbb{F}_p$-vector space, $p \neq 2$, then

  $$p\text{-excess}(L) \equiv \dim(D_L)_p(p-1) + 4k_p \mod 8,$$

  where $k_p = 1$ if the Legendre symbol $\left(\frac{\det q_p}{p}\right) = -1$, and zero else.

For precise definitions and a more detailed discussion of the classification of quadratic forms, we refer to [**26**, Chapter 15].

A lattice L such that D_L is annihilated by n, is called *n-elementary*. Indefinite p-elementary lattices ($p \neq 2$, a prime number) of rank at least 3 are determined up to isometry by their signature pair and determinant. To get uniqueness for $p = 2$ one needs to introduce an extra invariant, namely the parity of q_L [**85**, Sec. 1].

Definition 2.2. A *supersingular K3 lattice* is an even lattice N of rank 22, signature $(1,21)$ such that the discriminant group $D_N \cong \mathbb{F}_p^{2\sigma}$, p prime, $\sigma \in \{1,\dots,10\}$. For $p = 2$, the extra condition $q_N(x) \equiv 0 \mod \mathbb{Z}$ for all $x \in D_N$ is imposed.

Notation. The symbols A_n, D_n, E_n denote the negative definite root lattices of the respective type. U denotes the even indefinite unimodular lattice of rank 2 and H_5 is the even lattice of determinant -5 and signature $(1,1)$.

2. Gluing lattices

We call an embedding of lattices $M \hookrightarrow L$ *primitive* if L/M is torsion free. Primitive sublattices arise as kernels of endomorphisms and also in geometry, such as $\mathrm{NS}(X)$ or the transcendental lattice $T(X)$ in $H^2(X,\mathbb{Z})$ of a complex $K3$ surface.

Two primitive embeddings $i,j : M \hookrightarrow L$ are called isomorphic if there is a commutative diagram with $f \in O(L)$.

$$\begin{tikzcd} M \arrow[r, hook, "i"] \arrow[dr, hook, "j"'] & L \arrow[d, "f"] \\ & L \end{tikzcd}$$

We say that S embeds uniquely into L if all primitive embeddings are isomorphic. A (weakened) criterion for this to happen is given in the next theorem.

THEOREM 2.3. [**75**, Theorem 1.14.4][**68**, 2.8] *Let M be an even lattice of signature (m_+, m_-) and L an even unimodular lattice of signature (l_+, l_-). Then there is a unique primitive embedding of M into L if*

(1) $l(D_M) + 2 \leq \operatorname{rk} L - \operatorname{rk} M$;
(2) $l_+ > m_+$ and $l_- > m_-$.

We mention the related

THEOREM 2.4. [**75**, Theorem 1.14.2] *Let M be an even, indefinite lattice such that $\operatorname{rk} M \geq 2 + l(D_M)$, then the genus of M contains only one class, and the homomorphism $O(M) \to O(q_M)$ is surjective.*

DEFINITION 2.5. Let M and N be two lattices. A *primitive extension* of M and N is an overlattice $M \oplus N \hookrightarrow L$ (of the same rank), such that M and N are primitive sublattices of L.

A *glue map* is a map ϕ defined on certain subgroups

$$D_M \supseteq G_M \xrightarrow[\phi]{\sim} G_N \subseteq D_N,$$

with the extra condition that $q_M = -q_N \circ \phi$. Given such ϕ, we define *the glue*

$$G_\phi := \{x + \phi(x) | x \in G_M\} \subseteq D_M \oplus D_N$$

as the graph of ϕ. By construction, G_ϕ is a totally isotropic subspace of $D_M \oplus D_N$. Hence, we can define an integral lattice $L = M \oplus_\phi N$ via

$$L/(M \oplus N) = G_\phi.$$

The reader may check that $M \oplus N \hookrightarrow L$ is indeed a primitive extension.

Conversely, given a primitive extension as above, we get that the isotropic subspace $L/(M \oplus N) =: G_\phi$ defines a glue map ϕ via

$$G_M \cong L/(M \oplus N) = G_\phi \cong G_N. \tag{1}$$

It is defined on the spaces $G_M := p_M(L)/M$ and $G_N := p_N(L)/N$ where

$$p_M : M^\vee \oplus N^\vee \to M^\vee \text{ and } p_N : M^\vee \oplus N^\vee \to N^\vee$$

are the orthogonal projections.

THEOREM 2.6. [**75**, Prop 1.15.1] *There is a one to one to one correspondence*

$$\left\{ \begin{matrix} \textit{Primitive extensions} \\ M \oplus N \hookrightarrow L \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ \begin{matrix} \textit{Glue maps} \\ D_M \supseteq G_M \xrightarrow[\phi]{\sim} G_N \subseteq D_N \end{matrix} \right\}.$$

With the glue map ϕ understood, we will often drop it from notation and simply denote by $G = L/(M \oplus N)$ the glue of a primitive extension. It is not hard to see that $D_L = G^\perp/G$ where $\perp$ denotes the orthogonal subspace with respect to $b_{M\oplus N}$.

There is the following constraint on the size of the glue:

LEMMA 2.7.

$$|D_N/G_N| \cdot |D_M/G_M| = \det L$$

PROOF. Divide the standard formula

$$\det M \det N = [L : M \oplus N]^2 \det L$$

by $[L : M \oplus N]^2$ and use the isomorphisms (1). □

If L is unimodular, this recovers the well known fact that

$$D_M = G_M \xrightarrow[\phi]{\cong} G_N = D_N.$$

For example, $D_{T(X)} \cong D_{\mathrm{NS}(X)}$ for a K3 surface X over $\mathbb{C}$.

EXAMPLE 2.8. We shall compute a simple example which we will meet again later. Consider two rank one lattices B and C generated by $b \in B, c \in C$ with $b^2 = 2$, $c^2 = -18$. Then $B^\vee$ is generated by $b/2$ and $C^\vee$ is generated by $c/18$. The 2-torsion part of their discriminant groups is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ generated by $\overline{b/2} = b/2 + \mathbb{Z}b$ and $\overline{c/2} = c/2 + \mathbb{Z}c$. There is a unique isomorphism $\overline{b/2} \mapsto \overline{c/2}$. Note that it is actually a glue map since

$$q_B(\overline{c/2}) + q_C(\overline{c/2}) = 1/2 + 2\mathbb{Z} - 9/2 + 2\mathbb{Z} = 2\mathbb{Z}.$$

Hence, its graph $G_\phi = \{\overline{0}, \overline{b/2} + \overline{c/2}\} \subseteq D_B \oplus D_C$ is isotropic and defines an (even) overlattice of $B \oplus C$ generated by $\{b, c, b/2 + c/2\}$. In the basis $(b, b/2 + c/2)$ its Gram matrix is $\begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix}$.

We now prove some technical results that will be needed in the sequel.

LEMMA 2.9. *Let $N \hookrightarrow L$ be a primitive embedding. Then there is a surjection $D_L \twoheadrightarrow D_N/G_N$.*

PROOF. We have the following induced diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, two heads] & L^\vee \arrow[r] \arrow[d, two heads] & L^\vee/L = D_L \arrow[r] \arrow[d, dashed, two heads] & 0 \\ 0 \arrow[r] & p_N(L) \arrow[r] & N^\vee \arrow[r] & N^\vee/p_N(L) = D_N/G_N \arrow[r] & 0 \end{tikzcd}$$

where the primitivity of $N \hookrightarrow L$ gives the surjectivity of the central vertical arrow. The snake lemma then implies the desired surjection. □

COROLLARY 2.10. *Let $M \oplus N \hookrightarrow L$ be a primitive extension and p a prime number. If L is p-elementary, then the quotient D_N/G_N is an $\mathbb{F}_p$-vector space.*

3. Extending isometries

Clearly, an isometry $f = f_M \oplus f_N$ defined on $M \oplus N$ extends to a primitive extension L if and only if $\overline{f}(L/(M \oplus N)) = L/(M \oplus N)$, i.e. $\overline{f}(G_\phi) = G_\phi$. In other words $\overline{f_M} : D_M \to D_M$ preserves G_M, f_N preserves G_N and $\phi \circ \overline{f_M} = \overline{f_N} \circ \phi$.

$$\left\{\begin{matrix}\text{Primitive extensions } f \in O(L) \\ \text{of } f_M \oplus f_N \in O(M \oplus N)\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{Glue maps } \phi : G_M \xrightarrow{\sim} G_N \\ \text{satisfying } \phi \circ f_M = f_N \circ \phi\end{matrix}\right\}$$

This imposes compatibility conditions on the minimal polynomials of the two actions.

The following theorem is striking in its simplicity and its consequences. It is probably known to the experts, though the author does not know a reference.

Theorem 2.11. *Let $M \oplus N \hookrightarrow L$ be a primitive extension and f_M, f_N be isometries of M and N with minimal polynomials $m(x)$ and $n(x)$. Suppose that $f = f_M \oplus f_N$ extends to L. Then*

$$dL \subseteq M \oplus N$$

where $d\mathbb{Z} = (m(x)\mathbb{Z}[x] + n(x)\mathbb{Z}[x]) \cap \mathbb{Z}$.

Proof. By definition of d we can find $u, v \in \mathbb{Z}[x]$ such that

$$d = u(x)n(x) + v(x)m(x).$$

Then $d \cdot \mathrm{id} = u(f)n(f) + v(f)m(f)$ and further

$$\begin{aligned} dL &= (u(f)n(f) + v(f)m(f))L \\ &\subseteq u(f)n(f)L + v(f)m(f)L \\ &\subseteq \ker m(f) \oplus \ker n(f) \\ &= M \oplus N. \end{aligned}$$

In the last step we used the primitivity of $M \oplus N \hookrightarrow L$. □

Corollary 2.12. *If L is r-elementary, i.e. $rD_L = 0$, then M is dr-elementary, i.e.,*

$$drD_M = 0.$$

In particular,

$$\det M \mid (dr)^{\operatorname{rk} M}.$$

Proof. We take the chain of inclusions

$$M \oplus N \subseteq L \subseteq L^\vee \subseteq M^\vee \oplus N^\vee$$

and project everything orthogonally to $M^\vee$ as in eqn. 1:

$$M \subseteq p_M(L) \subseteq M^\vee \subseteq M^\vee.$$

From Theorem 2.11 we get that $dL \subseteq M \oplus M$, and projecting this gives

$$d\, p_M(L) = p_M(dL) \subseteq M.$$

Now, use Lemma 2.9 to see that $rD_M \subseteq G_M$ and conclude. □

Note that d divides the resultant $\operatorname{res}(m(x), n(x))$ and both have the same prime factors. For a case where $d < res(m(x), n(x))$ consider $x^2 + 1, x^2 - 4$. We deduce the following corollary. It was originally stated in [**64**, Theorem 4.3] for *unimodular* primitive extensions.

Corollary 2.13. *Let M, N, L be lattices and*

$$M \oplus N \hookrightarrow L$$

a primitive extension with glue $G_M \cong G \cong G_N$. Let f_M, f_N be isometries of M and N with characteristic polynomials χ_M and χ_N. If $f_M \oplus f_N$ extends to L, then any prime dividing $|G|$ also divides the resultant $\operatorname{res}(\chi_M, \chi_N)$.

When the discriminant groups consist of $\mathbb{F}_p$-vector spaces the situation is especially simple.

Theorem 2.14. [**63**, Theorem 3.1] *Let L_1, L_2 be two lattices with discriminant groups $D_i = D_{L_i}$, and let $f_i \in O(L_i)$, $i = 1, 2$ be isometries. Suppose $p \in \mathbb{Z}$ is a prime number such that*

- *the p-primary parts $(D_i)_p$ are vector spaces over $\mathbb{F}_p$,*
- *the maps $\overline{f}_i$ on $(D_i)_p$ have the same characteristic polynomial $S(x)$, and*
- *$S(x) \in \mathbb{F}_p[x]$ is a separable polynomial with $S(1)S(-1) \neq 0$.*

Then there is a gluing map $\phi_p : (D_1)_p \xrightarrow{\cong} (D_2)_p$ such that $f_1 \oplus f_2$ extends to the overlattice $L_1 \oplus_{\phi_p} L_2$.

Note that we can piece together gluing maps ϕ_p for different primes p to get a simultaneous glue map $\phi = \oplus_p \phi_p$.

In good situations we can arrange the conditions of the preceding theorem by "twisting". See 2.21.

For later use, we mention the following

Proposition 2.15. *Let L be a p-elementary lattice and $f \in O(L)$ an isometry. Then the characteristic polynomial $\chi_{\overline{f}|D_L}(x) \in \mathbb{F}_p[x]$ divides the reduction of $\chi_{f|L}(x)$ modulo p.*

Proof. Consider the following exact sequence of $\mathbb{F}_p$-vector spaces.

$$0 \longrightarrow L/pL^\vee \longrightarrow L^\vee/pL^\vee \longrightarrow L^\vee/L \longrightarrow 0$$

It is compatible with the action of f on each part. Thus the splitting of this sequence is compatible with f. To conclude the proof recall that $\chi_{f|L^\vee} = \chi_{f|L}$ and notice that $\chi_{\overline{f}|(L^\vee/pL^\vee)} \equiv \chi_{f|L^\vee} \mod p$. □

4. Real orthogonal transformations and the sign invariant

In this section we review the sign invariant of a real orthogonal transformation. Proofs and details can be found in [**40**].

For $p, q \in \mathbb{N}$ we denote by $\mathbb{R}^{p,q}$ the vector space $\mathbb{R}^{p+q}$ equipped with the quadratic form

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2.$$

Let $SO_{p,q}(\mathbb{R}) = SO(\mathbb{R}^{p,q})$ be the Lie group of real orthogonal transformations of determinant one, preserving the quadratic form. If the characteristic polynomial $s(x)$ of $F \in SO_{p,q}(\mathbb{R})$ is of even degree $2n = p+q$ and separable, then it is reciprocal, i.e., $x^{2n}s(x) = s(x^{-1})$. It has a trace polynomial $r(x)$ defined by

$$s(x) = x^n r(x + x^{-1}).$$

Its roots are real of the form $\lambda + \lambda^{-1}$ where λ is a root of $s(x)$. Call $\mathcal{T}$ the set of roots of $r(x)$ in the interval $(-2, 2)$. They correspond to conjugate pairs of roots $\lambda + \overline{\lambda}$ of $s(x)$ on the unit circle. We have an orthogonal direct sum decomposition

$$\mathbb{R}^{p,q} = \bigoplus_{\tau \in \mathbb{R}} E_\tau, \quad E_\tau := \ker(F + F^{-1} - \tau I).$$

On $E_\tau, \tau \in \mathcal{T}$, F acts by rotation by angle $\theta = \arccos(\tau/2)$. Hence, E_τ is either positive or negative definite. For $\tau \in \mathcal{T}$, this is encoded in the *sign invariant*.

$$\epsilon_F(\tau) = \begin{cases} +1 & \text{if } E_\tau \text{ has signature } (2,0), \\ -1 & \text{if } E_\tau \text{ has signature } (0,2). \end{cases}$$

Denote by $2t$ the number of roots of $s(x)$ outside the unit circle. We can recover the signature via

$$(p,q) = (t,t) + \sum_{\tau \in \mathcal{T}} \begin{cases} (2,0) & \text{if } \epsilon_F(\tau) = +1 \\ (0,2) & \text{if } \epsilon_F(\tau) = -1 \end{cases}$$

Two isometries $F, G \in SO_{p,q}(\mathbb{R})$ with characteristic polynomial $s(x)$ are conjugate in $O_{p,q}(\mathbb{R})$ iff $\epsilon_F = \epsilon_G$.

5. Lattices in number fields

In this section we review the theory of lattice isometries associated to certain reciprocal polynomials as exploited in **[64]**. For further reading, consider **[11, 12, 13]**.

A pair (L, f) where L is a lattice and $f \in O(L)$ an isometry with minimal polynomial $p(x)$, is called a $p(x)$-*lattice*. We call two $p(x)$-lattices (L, f) and (N, g) isomorphic if there is an isometry $\alpha : L \to N$ with $\alpha \circ f = g \circ \alpha$. Notice that this definition differs from that of McMullen in **[64]** where $p(x)$ is the characteristic polynomial instead.

Example 2.16. If X is a complex K3 surface and f an automorphism of X acting by multiplication with $\zeta_n = \exp\left(\frac{2\pi i}{n}\right)$ on $H^0(X, \Omega_X^2)$, then $(T(X), f)$ is a $c_n(x)$-lattice, where $c_n(x)$ denotes the n-th cyclotomic polynomial. To see this note that

$$H^{2,0} \subseteq (\ker c_n(f^*|T)) \otimes_{\mathbb{Z}} \mathbb{C} \subseteq T \otimes \mathbb{C}.$$

Since the kernel is defined over $\mathbb{Z}$, the equality $T = \ker c_n(f|T)$ follows from the minimality of T. As $\omega.\overline{\omega} > 0$, the sign invariant is given by

$$\epsilon_{f|T}(\zeta_n^k + \zeta_n^{-k}) = \begin{cases} +1 & \text{if k = 1} \\ -1 & \text{else} \end{cases}$$

where $k \in (\mathbb{Z}/n\mathbb{Z})^\times$.

Given an element $a \in \mathbb{Z}[f + f^{-1}] \subseteq \mathrm{End}(L)$ one can define a new inner product

$$\langle g_1, g_2 \rangle_a := \langle a g_1, g_2 \rangle$$

on L. We denote the resulting lattice by $L(a)$, and call this operation a *twist*. The pair $(L(a), f)$ is called a twisted $p(x)$-lattice. If L is even, then so is $L(a)$.

Conversely, if we start with an irreducible, reciprocal polynomial $p(x) \in \mathbb{Z}[x]$ of degree $d = 2e$, we can associate a $p(x)$-lattice to it as follows. Recall that $r(y)$ denotes the associated trace polynomial defined by $p(x) = x^e r(x + x^{-1})$. Then

$$K := \mathbb{Q}[f] = \mathbb{Q}[x]/p(x)$$

is an extension of degree 2 of

$$k := \mathbb{Q}[f + f^{-1}] = \mathbb{Q}[y]/r(y)$$

with Galois involution σ defined by $f^\sigma = f^{-1}$.

Now we can define the *principal* $p(x)$-lattice (L_0, f_0) by

$$L_0 := \mathbb{Z}[x]/p(x)$$

with isometry f_0 given by multiplication with x. As inner product we take

$$\langle g_1, g_2\rangle_0 := \sum_i \frac{g_1(x_i)g_2(x_i^{-1})}{r'(x_i + x_i^{-1})} = \mathrm{Tr}_{\mathbb{Q}}^K\left(\frac{g_1 g_2^\sigma}{r'(x+x^{-1})}\right)$$

where the sum is taken over the roots x_i of $p(x)$ and $r'(y)$ is the formal derivative of $r(y)$. This form is even with $|\det L_0| = |p(1)p(-1)|$.

EXAMPLE 2.17. For $p(x) = c_5(x)$ the trace polynomial is $r(y) = y^2 + y - 1$. We set $\zeta = \zeta_5$ and compute $L_0(\zeta + \zeta^4)$. Its bilinear form is given by

$$(x, y) \mapsto Tr_{\mathbb{Q}}^K\left(\frac{(\zeta+\zeta^4)xy^\sigma}{r'(\zeta+\zeta^4)}\right) = \frac{1}{5}Tr_{\mathbb{Q}}^K((\zeta^3+\zeta^2+3)xy^\sigma).$$

In the basis $(1, \zeta, \zeta^2, \zeta^3)$, we compute the gram matrix and the matrix representation of f:

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

we see that $L_0(\zeta + \zeta^4) \cong A_4(-1)$.

The situation is particularly nice if K has class number one, $\mathbb{Z}[x]/p(x)$ is the full ring of integers $\mathcal{O}_K$ of K and $|p(1)p(-1)|$ is square-free. In this case $p(x)$ is called a *simple* reciprocal polynomial and we get the following theorem.

THEOREM 2.18. [**64**, 5.2] *Let $p(x)$ be a simple reciprocal polynomial, then every $p(x)$-lattice of rank $\deg p(x)$ is isomorphic to a twist $L_0(a)$ of the principal $p(x)$-lattice.*

REMARK 2.19. If we drop the condition that $|p(1)p(-1)|$ is square-free, we have to allow twists in $r'(x + x^{-1})\mathcal{D}_K^{-1} \cap \mathcal{O}_k = 1/(x - x^{-1})\mathcal{O}_K \cap \mathcal{O}_k$, where $\mathcal{D}_K = (p'(x))\mathcal{O}_K$ is the different of K. If K/k ramifies over 2, these need not be even in general [**11**, §2.6]. Dropping the condition on the class number leads to so called ideal lattices surveyed in [**13**].

If $\mathbb{Z}[f] \cong \mathbb{Z}[x]/p(x)$ is the full ring of integers $\mathcal{O}_K$, then all the usual objects such as discriminant group, glue, (dual) lattice, etc. will be $\mathcal{O}_K$-modules.

LEMMA 2.20. *Let $p(x)$ be a simple reciprocal polynomial. Then there is an element $b \in \mathcal{O}_K$ of absolute norm $|p(1)p(-1)|$ such that $L_0^\vee = \frac{1}{b}\mathcal{O}_K$. If $a \in \mathcal{O}_k$ is a twist, then*

$$L_0(a)^\vee/L_0(a) \cong \mathcal{O}_K/ab\mathcal{O}_K$$

as $\mathcal{O}_K$-modules.

PROOF. Since $L_0^\vee \subseteq K$ is a finitely generated $\mathcal{O}_K$-module, it is a fractional ideal. By simplicity of $p(x)$, $\mathcal{O}_K$ is a PID and fractional ideals are of the form $\frac{1}{b}\mathcal{O}_K$, for some $b \in \mathcal{O}_K$. Then $L_0(a)^\vee = \frac{1}{a}L_0^\vee = \frac{1}{ab}L_0$ and $D_{L_0(a)} \cong \mathcal{O}_K/ab\mathcal{O}_K$. □

Given a unit $u \in \mathcal{O}_K^\times$ and $a \in \mathcal{O}_K \setminus \{0\}$ the twist $L_0(uu^\sigma a)$ is isomorphic to $L_0(a)$ via $x \mapsto ux$ as $p(x)$-lattice. Conversely, if $v \in \mathcal{O}_k$ and $L_0(va) \cong L_0(a)$ as $p(x)$-lattices, then, by non-degeneracy of the trace map, we can find $u \in \mathcal{O}_K$ with $v = uu^\sigma$. Since the cokernel of the norm map $N : \mathcal{O}_K^\times \to \mathcal{O}_k^\times$ is finite, the associates of $a \in \mathcal{O}_k$ give only finitely many non-isomorphic twists.

By Lemma 2.20 the prime decomposition of $a \in \mathcal{O}_k$ in $\mathcal{O}_K$ determines the $\mathcal{O}_K$-module structure of the discriminant, while twisting by a unit may change the signature and discriminant form as follows.

Let $\mathcal{T}$ denote the set of real places of $k = \mathbb{Q}[y]/r(y)$ that become complex in K. They correspond to the real roots τ of $r(y)$ in the interval $(-2, 2)$. Each such real place is an embedding of $\nu_\tau : k \hookrightarrow \mathbb{R}$ given by $b(y) \mapsto b(\tau)$. Its sign is recorded by $\operatorname{sign}_\tau(b(x)) = \operatorname{sign}(\nu_\tau(b(x)))$. We call the resulting quantity the sign invariant. If $(L_0(b), f_0)$ is a twist of the principal lattice, then (cf. [**40**, 4.2])

$$\epsilon_{f_0}(\tau) = \operatorname{sign}_\tau(b/r'(y)).$$

Recall that for $n \in \mathbb{N}$ a lattice is called n-elementary if $nD_L = 0$. Let $\mathfrak{I} \subseteq \mathcal{O}_K$ be an ideal. We call a $p(x)$-lattice $\mathfrak{I}$-elementary if $\mathfrak{I}D_L = 0$.

We remark that if $(N, f_N) \hookrightarrow (L, f_L)$ is a primitive embedding of $p(x)$-lattices, then the maps appearing in the previous section are $\mathbb{Z}[f] \cong \mathbb{Z}[x]/p(x)$-module homomorphisms. For example, if L is $\mathfrak{p}$-elementary, then D_N/G_N is annihilated by $\mathfrak{p}$ as well, or equivalently it is an $\mathcal{O}_K/\mathfrak{p}$-vector space.

Theorem 2.21. [**63**, Theorem 4.3] *Suppose that $\mathcal{O}_k = \mathbb{Z}[f+f^{-1}]$ is a Dedekind domain of class number one, $p \in \mathbb{Z}$ a prime number such that*

$$\overline{S}(x) = \det(xI - f) \mod p$$

is separable, $\overline{S}(1)\overline{S}(-1) \neq 0$ and $\gcd(p, \det L) = 1$. Let $\overline{S}_1(x)$ be a reciprocal factor of $\overline{S}(x)$. Then there exists a twist $M = L(a)$, with $a \in \mathbb{Z}[f + f^{-1}]$ dividing p, such that

$$\overline{S}_1(x) = \det(xI - \overline{f}|(D_M)_p).$$

CHAPTER 3

How to determine a K3 surface from a finite automorphism

In this section we pursue the question when an automorphism determines a (complex) K3 surface up to isomorphism. We prove that if the automorphism is finite non-symplectic and the transcendental lattice small, then the isomorphism class of the K3 surface is determined by an n-th root of unity and an ideal in $\mathbb{Z}[\zeta_n]$. As application, we give a generalization of Vorontsov's theorem and prove that there is a unique pair (X, G) up to isomorphism where X is a complex K3 surface with a faithful action of $G \cong (\mathbb{Z}/5\mathbb{Z})^2$. We give a description of its Néron-Severi lattice. Equations of its generators are given in the authors Masters thesis [**17**].

An automorphism f of a K3 surface is called *symplectic* if it acts trivially on the global holomorphic 2-forms, $f^*|H^0(X, \Omega_X^2) = \mathrm{id}$, and non-symplectic otherwise. Furthermore, we call f *purely* non-symplectic if all non-trivial powers are non-symplectic. Note that K3 surfaces admitting a non-symplectic automorphism of finite order are always algebraic [**74**, 3.1]. The minimal resolution of the quotient of a K3 surface by a finite symplecitc automorphism is still a K3 surface. If the automorphism is non-symplectic, then it is either an Enriques surface or rational. Being symplectic or not governs the deformation behavior of the automorphism. A symplectic automorphism deforms (at least) in $\operatorname{rk} T(X) - 2$ dimensions, while a non-symplectic automorphism acting by order n on the holomorphic 2-form deforms in $\operatorname{rk} T(X)/\varphi(n) - 1$ dimensions where φ is the Euler totient function. In order to determine a K3 surface by some fixed data d, the pair (X, d) should not deform. In the symplectic case this means that $\operatorname{rk} T(X) = 2$. There one can reconstruct the K3 surface up to isomorphism from the (oriented) transcendental lattice by means of a Shioda-Inose structure [**93**]. This section is concerned with the non-symplectic case, i.e. $\operatorname{rk} T(X) = \varphi(n)$.

We prove that if X is a complex K3 surface and $f \in \operatorname{Aut}(X)$ of finite order with

$$\operatorname{rk} T(X) = \varphi(\operatorname{order}(f|H^{2,0}(X))),$$

then the action of f on $H^0(X, \Omega_X^2)$ and $\operatorname{NS}^\vee(X)/\operatorname{NS}(X)$ determine the isomorphism class of X. This is encoded in a root of unity and an ideal in $\mathbb{Z}[\zeta_n]$. If f is of infinite order, we need the additional data of a primitive embedding $T(X) \hookrightarrow L_{K3}$. In many cases it is unique.

We shall give two applications of this theorem. The first is a generalization of Vorontsov's Theorem 3.12 and the classification of purely non-symplectic automorphisms of high order. The second application is the uniqueness of a K3 surface with a (faithful) $(\mathbb{Z}/5\mathbb{Z})^2$ action in characteristic different from 5 (Chap. 4).

Finite abelian groups of symplectic automorphisms on complex K3 surfaces were studied by Nikulin in [**74**]. A full classification of symplectic groups acting on K3 surfaces and an intriguing connection to the Mathieu group M_{23} is given by Mukai in [**70**]. Later Kondo, in [**51**], found a beautiful connection to Niemeyer lattices. The possible symplectic group actions were then classified in [**104, 43**]. The classification is extended to positive characteristic in [**30**]. Non-symplectic automorphisms of prime order and their fixed points were classified in [**5**].

1. Small cyclotomic fields

Motivated by the action of a non-symplectic automorphism on the transcendental lattice of a K3 surface, we study $c_n(x)$-lattices more closely. In order to do this we review some of the general theory on cyclotomic fields. Our main reference is [**103**].

For $n \in \mathbb{N}$, we denote by $K = \mathbb{Q}(\zeta_n)$ the n-th cyclotomic field and by $c_n(x)$ the n-th cyclotomic polynomial. The Euler totient function $\varphi(n)$ records the degree of $c_n(x)$. The maximal real subfield of K is $k = \mathbb{Q}[\zeta_n + \overline{\zeta}_n]$. The rings of integers of these two fields are

$$\mathcal{O}_K = \mathbb{Z}[\zeta_n] \quad \text{and} \quad \mathcal{O}_k = \mathbb{Z}[\zeta_n + \overline{\zeta}_n].$$

LEMMA 3.1. *The cyclotomic polynomials $c_n(x)$ are simple reciprocal polynomials for $2 \leq \varphi(n) \leq 21$, $n \neq 2^d$.*

PROOF. The only non-trivial part is that the class numbers are one. This is stated in [**58**]. □

Note that even though $|c_{2^d}(1)c_{2^d}(-1)| = 4$ is not square-free, every *even* c_{2^d}-lattice $(2 \leq d \leq 5)$ is a twist of the principal c_{2^d}-lattice (cf. Remark 2.19).

LEMMA 3.2. [**103**, Prop. 2.8] *If $n \in \mathbb{N}$, has two distinct prime factors, then $(1-\zeta_n)$ is a unit in $\mathcal{O}_K$.*

The kernel $\mathcal{O}_k^{\times +}$ of the map

$$\operatorname{sign} : \mathcal{O}_k^\times \to \{\pm 1\}^{\frac{\varphi(n)}{2}}$$

is the set of *totally positive units* of $\mathcal{O}_k$.

PROPOSITION 3.3. [**92**, A.2] *If the relative class number $h^-(K) = h(K)/h(k)$ is odd, then $\mathcal{O}_k^{\times +} = N_k^K(\mathcal{O}_K^\times)$.*

COROLLARY 3.4. *Let $n \in \mathbb{N}$ with $\varphi(n) \leq 20$. Set $K := \mathbb{Q}(\zeta_n)$ the n-th cyclotomic field. Then the group homomorphism*

$$\operatorname{sign} : \mathcal{O}_k^\times / N(\mathcal{O}_K) \to \{\pm 1\}^{\frac{\varphi(n)}{2}}$$

is injective.

PROOF. As $\mathbb{Q}[\zeta_n]$ is a PID for $\phi(n) \leq 20$, the relative class number is one and we may apply Proposition 3.3. □

The first cases where the relative class number is even is for $n = 39, 56, 29$. There $h^-(\mathbb{Q}[\zeta_n]) = 2, 2, 2^3$ (cf. [**103**, §3]), and the sign map has a kernel of order $2, 2, 2^3$ as well. We refer the interested reader to [**92, 47**] for more on the relation of class numbers and totally positve units.

PROPOSITION 3.5. *The isomorphism class of a $c_n(x)$-lattice (T, f), with $2 \leq \operatorname{rk} T = \varphi(n) \leq 20$ is given by the kernel of*

$$\mathbb{Z}[x]/c_n(x) \to O(T^\vee/T), \quad x \mapsto \overline{f}$$

and the sign invariant of f.

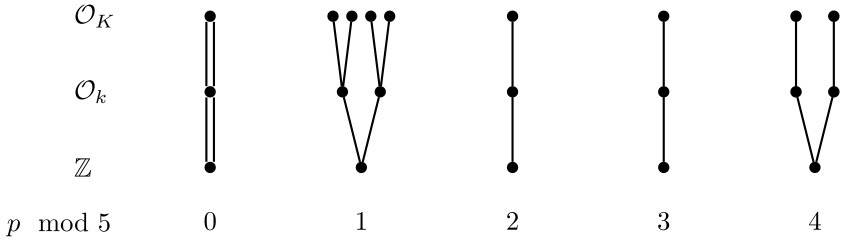


FIGURE 1. Prime decompositions in $\mathbb{Q}[\zeta_5]$

PROOF. By Theorem 2.18, $(T, f) \cong (L_0(ub), f_0)$. Lemma 2.20 shows that the $\mathcal{O}_K$-module structure of the discriminant determines the prime decomposition of $b \in \mathcal{O}_k$. By Corollary 3.4 the different isomorphism classes of $(L_0(ub), f_0)$ for fixed b and some $u \in \mathcal{O}_k^\times$ are determined by their sign invariant. $\square$

REMARK 3.6. Note that we did not exclude 2-powers from this proposition.

EXAMPLE 3.7. Recall that we denote by $\zeta_5 = \exp(\frac{2\pi i}{5})$ a fifth root of unity and by $c_5(x) = x^4 + x^3 + x^2 + x + 1$ its minimal polynomial. Set $K = \mathbb{Q}[\zeta_5]$ and $k = \mathbb{Q}[\zeta_5 + \zeta_5^{-1}] = \mathbb{Q}[\sqrt{5}]$. Denote their rings of integers by $\mathcal{O}_K = \mathbb{Z}[\zeta_5] = \mathbb{Z}[x]/c_5(x)$ and $\mathcal{O}_k = \mathbb{Z}[\zeta_5 + \zeta_5^{-1}]$. Both are principal ideal domains, and the primes factor as given in Figure 1. Since $c_5(x)$ is a simple reciprocal polynomial, every rank 4 $c_5(x)$-lattice is isomorphic to a twist $L_0(t), t \in \mathcal{O}_k$ of the principal $c_5(x)$-lattice by Theorem 2.18. The principal $c_5(x)$-lattice L_0 has determinant $5 = |c_5(1)c_5(-1)|$. The only prime above 5 is $(\zeta_5 - 1)$. Hence, after a twist $t \in \mathcal{O}_k$

$$D_{L_0(t)} = \tfrac{1}{(\zeta_5-1)t}\mathcal{O}_K/\mathcal{O}_K \cong \mathcal{O}_K/(\zeta_5 - 1)t$$

as $\mathcal{O}_K$-modules by Lemma 2.20. The $\mathcal{O}_K$-module structure of the discriminant group determines the prime decomposition of t up to units. The cokernel of the norm map $\mathcal{O}_K^\times \to \mathcal{O}_k^\times$ consists of 4 elements. Thus we get 4 inequivalent twists $\pm t, \pm(\zeta_5 + \zeta_5^{-1})t$ two of which have signature $(2, 2)$ but different sign invariants and the other ones signatures $(4, 0)$ and $(0, 4)$.

For later use, we note two theorems on cyclotomic polynomials.

THEOREM 3.8. **[2, 31]** *The resultant of two cyclotomic polynomials c_m, c_n of degrees $0 < m < n$ is given by*

$$\operatorname{res}(c_n, c_m) = \begin{cases} p^{\varphi(m)} & \text{if } n/m = p^e \text{ is a prime power,} \\ 1 & \text{otherwise.} \end{cases}$$

THEOREM 3.9. **[35, 31]**

$$(\mathbb{Z}[x]c_n + \mathbb{Z}[x]c_m) \cap \mathbb{Z} = \begin{cases} p & \text{if } n/m = p^e \text{ is a prime power,} \\ 1 & \text{otherwise.} \end{cases}$$

2. Uniqueness Theorem

PROPOSITION 3.10. *Let $X/\mathbb{C}$ be a K3 surface and $f \in \mathrm{Aut}(X)$ an automorphism with $f^*(\omega) = \zeta_n^k \omega$ on $0 \neq \omega \in H^0(X, \Omega_X^2)$ where $\zeta_n = e^{2\pi i/n}$, $k \in (\mathbb{Z}/n\mathbb{Z})^\times$. Suppose that*

(1) $\operatorname{rk} T_X = \varphi(n)$,
(2) $T_X \hookrightarrow L_{K3}$ *uniquely.*

Then the isomorphism class of X is determined by (I, ζ_n^k) where I is the kernel of

$$\mathbb{Z}[x]/c_n(x) \to O(T^\vee/T), \qquad x \mapsto \overline{f^*}.$$

Conversely, if (Y, f_Y) is another pair satisfying (1), (2) and $\zeta_X = \zeta_Y$, but $I_X \neq I_Y$, then $X \ncong Y$.

PROOF. Let X, Y be K3 surfaces and f_X, f_Y be automorphisms as in the theorem. Set $\tau = \zeta_n^k + \zeta_n^{-k}$ and $E_\tau = \ker(f|_T + f|_T^{-1} - \tau id_T)$. Looking at $\omega, \overline{\omega} \in E_\tau \otimes \mathbb{C}$ with $\omega.\overline{\omega} > 0$, we see that E_τ has signature $(2,0)$. Since the signature of T is $(2, \varphi(n)-2)$, this determines the sign invariants of (T_X, f_X) and (T_Y, f_Y). This is recorded by the complex n-th root of unity ζ_n^k. By assumption their discriminants have the same $\mathcal{O}_K$-module structure and Proposition 3.5 implies that $(T_X, f_X) \cong (T_Y, f_Y)$ as $c_n(x)$-lattices.

Hence, we can find an isometry $\psi_T : T_X \to T_Y$ such that $f_Y \circ \psi_T = \psi_T \circ f_X$. The latter condition assures that ψ_T is compatible with the eigenspaces of f_X, f_Y. Since $\operatorname{rk} T_X = \varphi(n)$, the eigenspaces for ζ_n^k are $H^{2,0}(X)$ and $H^{2,0}(Y)$. In particular,

$$\psi_T(H_X^{2,0}) = H_Y^{2,0}.$$

Now, choose markings ϕ_X and ϕ_Y on X and Y. They provide us with two embeddings ϕ_X and $\phi_Y \circ \psi$ of T_X into L_{K3}. By assumption (2) any two embeddings are isomorphic. That is, we can find $\psi \in O(L_{K3})$ such that the following diagram commutes.

$$\begin{tikzcd}
T_X \arrow[r, "f_X"] \arrow[d, "\psi_T"] & T_X \subseteq H^2(X,\mathbb{Z}) \arrow[r, "\phi_X"] \arrow[d, "\psi_T"] & L_{K3} \arrow[d, "\exists\psi"] \\
T_Y \arrow[r, "f_Y"] & T_Y \subseteq H^2(Y,\mathbb{Z}) \arrow[r, "\phi_Y"] & L_{K3}
\end{tikzcd}$$

By construction $\phi_Y^{-1} \circ \psi \circ \phi_X$ is a Hodge isometry. By the weak Torelli Theorem X and Y are isomorphic. Conversely, let $f_1, f_2 \in \mathrm{Aut}(X)$ with $f_1^*\omega = f_2^*\omega = \zeta_n\omega$. Note that $f_1 \circ f_2^{-1}$ is symplectic. Then $(f_1 \circ f_2^{-1})|T$,i.e., $f_1|T = f_2|T$ and in particular $I_1 = I_2$. □

REMARK 3.11. Replacing f by a power f^k with k coprime to n, we can fix the action on the 2-forms. This corresponds to the Galois action $\zeta_n \mapsto \zeta_n^k$ on $\mathbb{Q}(\zeta_n)$. In case the embedding is not unique, one can fix the isometry class of NS. Then isomorphism classes of primitive embeddings with $T^\perp = \mathrm{NS}$ are given by glue maps $\phi : T^\vee/T \to \mathrm{NS}^\vee/\mathrm{NS}$ with $-q_T = q_{\mathrm{NS}} \circ \phi$ modulo the action of $O(\mathrm{NS})$ on the right. We can also allow for an action of the centralizer of $f|T$ in $O(T)$ on the left. It should be noted that the proposition can be applied to automorphisms of infinite order too.

3. Vorontsov's Theorem

In this section we give a generalization as well as a (new) uniform proof of Vorontsov's Theorem and related results using the uniqueness theorem of the preceeding section. On the way we can correct results in [**97, 98, 45, ?**].

Let X be a complex K3 surface and

$$\rho : \mathrm{Aut}(X) \to O(\mathrm{NS}(X))$$

be the representation of the automorphism group of X on the Néron-Severi group. Set $H(X) := \ker \rho$ and $h(X) := |H(X)|$ its order. Nikulin [**74**] showed that $H(X)$ is a finite cyclic group and $\varphi(h(X)) \mid \mathrm{rk}\, T(X)$. Vorontsov's Theorem looks at the extremal case $\varphi(h(X)) = \mathrm{rk}\, T(X)$. That is, X has an automorphism of (maximal) order $h(X)$ acting trivially on $\mathrm{NS}(X)$.

Theorem 3.12. [**102, 50, 56, 81**] *Set* $\Sigma := \{66, 44, 42, 36, 28, 12\}$ *and* $\Omega := \{3^k\ (1 \leq k \leq 3),\ 5^l\ (l = 1, 2),\ 7, 11, 13, 17, 19\}$.

(1) Let $X/\mathbb{C}$ be a K3 surface with $\varphi(h(X)) = \mathrm{rk}\, T(X)$. Then $h(X) \in \Sigma \cup \Omega$. The transcendental lattice $T(X)$ is unimodular iff $h(x) \in \Sigma$.

(2) Conversely, for each $N \in \Sigma \cup \Omega$, there exists, modulo isomorphisms, a unique K3 surface $X/\mathbb{C}$ such that $h(X) = N$ and $\varphi(h(X)) = \mathrm{rk}\, T(X)$. Moreover, $T(X)$ is unimodular iff $N \in \Sigma$.

For our alternative proof, we first show that h and the conditions given determine the transcendental lattice (T, f) as a $c_h(x)$-lattice (up to powers of f) and that T embeds uniquely into the K3-lattice. Then Theorem 3.17 provides the uniqueness. Next we show that for each $h \in \Sigma \cup \Omega$ we can find (T, f) such that f is of order h and acts trivially on $T^\vee/T$. This f can be glued to the identity, which trivially preserves an ample cone. Then the strong Torelli theorem provides the existence. Alternatively, the equations of X and f are well known and found in Tables 2, 3.

Proposition 3.13. *Let f be a non-symplectic automorphism acting with order n on the global 2-forms of a complex K3 surface X with $\varphi(n) = \mathrm{rk}\, T(X)$. Then*

$$\det T(X) \mid \mathrm{res}(c_n, \mu)$$

where $\mu(x)$ is the minimal polynomial of $f|\,\mathrm{NS}(X)$. If $\mathrm{res}(c_n, \mu) \neq 0$ and f is purely non-symplectic, then T is n-elementary, i.e. $nD_T = 0$.

Proof. If c_n and μ have a common factor, then $\mathrm{res}(c_n, \mu) = 0$, and the statement is certainly true. We may assume that $\gcd(c_n, \mu) = 1$. Then we know that

$$T = T(X) = \ker c_n(f|H^2(X, \mathbb{Z}))$$

and we can view (T, f) as a $c_n(x)$-lattice. Then $D_T \cong \mathcal{O}_K/I$, $K = \mathbb{Q}(\zeta_n)$, for some ideal $I < \mathcal{O}_K$. The isomorphisms $D_{\mathrm{NS}} \cong D_T \cong \mathcal{O}_K/I$ are compatible with f. In particular $\mu(f|\,\mathrm{NS}) = 0$ implies that $\mu(\overline{f}|D_T) = 0$, i.e., $\mu(\zeta_n) \in I$. By definition of norm and resultant

$$|\det T(X)| = |\mathcal{O}_K/I| = N(I) \mid N(\mu(\zeta_n)) = \mathrm{res}(c_n, \mu).$$

It remains to prove that $nD_T = 0$. By Corollary 2.12 it is enough to show that

$$n\mathbb{Z} \subseteq (\mathbb{Z}[x]c_n + \mathbb{Z}[x]\mu) \cap \mathbb{Z}. \tag{2}$$

Since f is assumed to be purely non-symplectic, we may replace μ by $\prod_{d|n} c_d$. Then (2) follows from Theorem 3.9. □

Corollary 3.14. *Suppose that* $\operatorname{rk} T(X) = \varphi(n)$ *and* f *is purely non-symplectic with* $\gcd\left(\mu_{f|\,\mathrm{NS}}, c_n\right) = 1$. *Then we have the following restrictions on* $T(X)$

(0) *T has signature* $(2, \varphi(n) - 2)$;

(1) $2 \leq \varphi(n) \leq 21$;

(2)
$$(x-1) \mid \mu \quad \text{and} \quad \mu \mid \prod_{k<n} c_k$$
where $\deg \mu \leq 22 - \varphi(n)$ *and* $\det T \mid \operatorname{res}(c_n, \mu)$;

(3) $\exists b \in \mathcal{O}_k$ *such that* $T \cong L_0(b)$ *is a twist of the principal* $c_n(x)$*-lattice.*

The resulting determinants are listed in Table 1.

Proof. (0) and (1) are clear. (2) Since f is of finite order, $\mu = \mu(f|\,\mathrm{NS})$ is separable and we apply Proposition 3.13. (3) By assumption T is a $c_n(x)-$lattice of rank $\deg c_n$. For $2 \leq \varphi(n) \leq 21$ all cyclotomic polynomials $c_n(x)$ are simple and Theorem 2.18 provides the claim.

It remains to compute the values of Table 1. We shall do the computation for $n = 28$. The other cases are similar. By Theorem 3.8 a factor $c_k(x)$ of $\mu(x)$ will contribute to the resultant if and only if n/k is a prime power. Hence, the only possibilities are $c_4, c_7, c_{14}, c_4c_7, c_4c_{14}$ which are of degree $2, 6, 6, 8, 8$ and give resultants $7^2, 2^6, 2^6, 2^67^2, 2^67^2$. The principal $c_{28}(x)$-lattice is unimodular and up to units there is only a single twist above 2 of norm 2^6 and a single twist above 7 of norm 7^2. This results in the 4 possible determinants $1, 2^6, 7^2, 2^67^2$. We can exclude 7^2 and 2^67^2 since there is no twist of the right signature $(2, 10)$. This leaves us with determinants 1 and 2^6. □

Table 1. Possible determinants of the transcendental lattice

n	$\varphi(n)$	$\det T$	n	$\varphi(n)$	$\det T$
$3, 6$	2	3	20	8	$2^4, 2^45^2$
4	2	2^2	$21, 42$	12	$1, 7^2$
$5, 10$	4	5	24	8	$2^2, 2^6, 2^23^4, 2^63^4$
$7, 14$	6	7	$25, 50$	20	5
8	4	$2^2, 2^4$	$27, 54$	18	$3, 3^3$
$9, 18$	6	$3, 3^3$	28	12	$1, 2^6$
$11, 22$	10	11	32	16	$2^2, 2^4, 2^6$
12	4	$1, 2^23^4, 2^4$	$33, 66$	20	1
$13, 26$	12	13	36	12	$1, 3^4, 2^63^2$
$15, 30$	8	$5^2, 3^4$	40	16	2^4
16	8	$2^2, 2^4, 2^6, 2^8$	44	20	1
$17, 34$	16	17	48	16	2^2
$19, 38$	18	19	60	16	–

Lemma 3.15. *Let* $2 \leq \varphi(n) \leq 20$. *For each* $p \mid n$, *there is a unique prime ideal* $\mathcal{O}_k$ *above* p. *In particular the* $c_n(x)$*-lattices* T *of Table 1 are determined up to isomorphism by their determinants and sign invariant. They admit a unique primitive embedding into* L_{k3}, *except* $(n, \det T) = (32, 2^6)$ *which does not embed in* L_{K3}.

PROOF. The c_n-lattices are twists of the principal c_n-lattice. Twists correspond to ideals in $\mathcal{O}_k$ which are well known from the theory of cyclotomic fields.

Since $\varphi(n)+l(T^\vee/T) \leq 20$ for all pairs (n,d) except $(25,5)$, $(27,3^3)$ and $(32,2^6)$ Theorem 3.17 provides uniqueness (and existence) of a primitive embedding outside those cases.

We have to check in case $n = 25$ that T embeds uniquely into the K3-lattice. It has rank 20 and determinant 5. Its orthogonal complement NS is an indefinite lattice of determinant 5. It is unique in its genus and the canonical map $O(\mathrm{NS}) \to O(\mathrm{NS}^\vee/\mathrm{NS})$ is surjective since both groups are generated by $-id$. By [**75**, 1.14.1] the embedding of T into L_{k3} is unique.

For the case $(27,3^3)$, we need more theory not explained here, see e.g. [**67**]. By [**67**, VIII 7.6] NS is 3-semiregular and p-regular for $p \neq 3$. Now [**67**, VIII 7.5] provides surjectivity of $O(\mathrm{NS}) \to O(\mathrm{NS}^\vee/\mathrm{NS})$ and uniqueness in its genus (alternatively cf. [**65, 66**]). Uniqueness of the embedding follows again with [**75**, 1.14.1].

It remains to check that $(32,2^6)$ does not embed into the K3-lattice. Suppose that it does. Then its orthogonal complement is isomorphic to $A_1(-1)\oplus 5A_1$ which is the only lattice of signature $(1,5)$ and discriminant group $\mathbb{F}_2^6$ (cf. [**26**, Tbl. 15.5]). Its discriminant form takes half integral values. Up to sign it is isomorphic to the discriminant form of its orthogonal complement $T \cong U(2)\oplus U(2)\oplus D_4\oplus E_8$ which takes integral values, contradicting the existence of a primitive embedding. □

PROPOSITION 3.16. *Let X be a complex K3 surface and $f \in \mathrm{Aut}(X)$ an automorphism of finite order with $f^*(\omega) = \zeta_n\omega$ on $0 \neq \omega \in H^0(X,\Omega_X^2)$. Suppose that* $\mathrm{rk}\, T_X = \varphi(n)$. *Then there is a unique primitive embedding*

$$T_X \hookrightarrow L_{K3}.$$

PROOF. If $\varphi(n) \leq 10$, then $\mathrm{rk}\, T_X + l(D_{T(X)}) \leq 2\,\mathrm{rk}\, T_X = 2\varphi(n) = 20$ and Theorem 2.3 provides uniqueness of the embedding. If $\varphi(n) > 10$, then $\varphi(n) \geq 12$ and ζ_n is not an eigenvalue of $f|\,\mathrm{NS}$. By Corollary 3.14 there are only finitely many possibilities of T_X up to isometry. Uniqueness of the embedding is checked individually in Lemma 3.15. □

THEOREM 3.17. *Let $X_i, i = 1,2$ be complex K3 surfaces and $f_i \in \mathrm{Aut}(X_i)$ automorphisms of finite order with $f_i^*(\omega_i) = \zeta_n\omega_i$ on $0 \neq \omega_i \in H^0(X_i,\Omega_{X_i}^2)$ such that* $\mathrm{rk}\, T(X_i) = \varphi(n)$. *Then $X_1 \cong X_2$ if and only if $I_1 = I_2$ where I_i is the kernel of*

$$\mathbb{Z}[x]/c_n(x) \to O(\mathrm{NS}(X_i)^\vee/\mathrm{NS}(X_i)), \qquad x \mapsto \overline{f_i^*}.$$

PROOF. Combine Propositions 3.16 and 3.10. □

Let X/k be an algebraic variety and $f : X/k \to X/k$ an automorphism. We call f tame, if its order is coprime to the base characteristic $\mathrm{char}\, k$.

THEOREM 3.18. *Let $X_i/k, i = 1,2$ be K3 surfaces over an algebraically closed field of characteristic not 2 or 3. Let f_i be tame automorphisms with $f_i^*\omega_i = \zeta_n\omega_i \in H^0(X_i,\Omega_{X_i}^2)$, and $\varphi(n) = 22-\rho$. If $I_1 \cong I_2$, then $X_1 \cong X_2$ where I_i is the kernel of*

$$\mathbb{Z}[x]/c_n(x) \to O(\mathrm{NS}(X_i)^\vee/\mathrm{NS}(X_i)), \quad x \mapsto \overline{f_i^*}.$$

If X is supersingular of Artin invariant σ, $n > 2$ and $N(I) = p^{2\sigma}$, then X is determined up to isomorphism by I alone.

PROOF. In the characteristic 0 case we can work over $\mathbb{C}$. In the tame case we can lift (X,NS,f) [**46**, Thm. 3.2]. This preserves I, and since f is tame, there is a unique lift of ζ_n to an n-th root as well. We can apply the previous Theorem 3.17. For the supersingular case see Theorem 5.11. □

Proof of Theorem 3.12. Let X be as in the theorem and f be a generator of $H(X)$, that is, $f|\operatorname{NS} = id$ and (T, f) is a simple $c_h(x)$-lattice. As usual the discriminant group is a finite $\mathcal{O}_K$-module, and we can find an ideal $I < \mathcal{O}_K$ such that

$$\operatorname{NS}^\vee/\operatorname{NS} \cong T^\vee/T \cong \mathcal{O}_K/I.$$

The isometry f acts via multiplication by x on the right hand side. The condition that it acts trivially on NS translates to

$$(x-1) \in I.$$

(1) If n has distinct prime factors, then, by Lemma 3.2, $(x-1)$ is a unit. Hence, $I = \mathcal{O}_K$ and T is unimodular.
(2) If $n = p^k$, then $\mathcal{O}_K$ is totally ramified over p and $(x-1)$ is prime of norm p. In particular, either $I = (x-1)\mathcal{O}_K$, or T is unimodular.

Collecting the entries (n, d) with $d = 1$ and n even from Table 1 leads to Σ, while (2) leads to (p^k, p), i.e., Ω. Now Lemma 3.15 and Theorem 3.17 provide uniqueness of the K3 surface up to isomorphism. Note that the 2-power entries in Table 1 do not satisfy (2). Instead of isolated examples there are only families with trivial action (see **[88]**).

For the existence part, note that for each $n \in \Sigma\cup\Omega$ there is a $c_n(x)$-lattice (T, f) of signature $(2, \varphi(n) - 2)$ with trivial action on the discriminant group. It embeds primitively into the K3-lattice and we can glue f to the identity on the orthogonal complement. Then the strong Torelli theorem and the surjectivity of the period map provide the existence of the desired K3 surface and its automorphism.

□

Lemma 3.19. *The pair $(54, 3^3)$ is not realized by a K3 surface.*

Proof. Suppose that (X, f) realizes $(54, 3^3)$. Then by Corollary 3.14 the characteristic polynomial of $f|H^2(X,\mathbb{Z})$ is $c_{54}c_6c_2c_1$. The resultants look as follows: $\operatorname{res}(c_{54}, c_6) = 3^2, \operatorname{res}(c_6, c_2) = 3, \operatorname{res}(c_2, c_1) = 2$. A similar reasoning as in Proposition 3.13 for each factor c_i yields a unique gluing diagram for $Ci = \ker c_i(f|H^2(X,\mathbb{Z}))$. Here an edge symbolizes a glue map. Associated to an edge is the glue G.

$$\begin{tikzcd} C54 \arrow[r, dash, "\mathbb{F}_3^2"] \arrow[dr, dash, "\mathbb{F}_3"] & C6 \arrow[d, dash, "\mathbb{F}_3"'] \\ C1 \arrow[r, dash, "\mathbb{F}_2"'] & C2 \end{tikzcd}$$

In particular this determines the lattices $C2$ and $C1$ with Gram matrices (-18) and (2). There is a unique gluing of these lattices computed in Example 2.8. It results in the lattice

$$C1C2 = \ker c_1c_2(f|H^2(X,\mathbb{Z})) \cong \begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix}.$$

We know the determinant of $C6$ is 3^3. Twisting the principal c_6-lattice results in $C6 = A_2(3)$. The gluing of $C1C2$ and $C6$ equals $\operatorname{NS}(X)$. It has discriminant group $\mathbb{F}_3^3$. The lattices $C1C2$ and $C6$ are glued over $\mathbb{F}_3$. Their discriminant groups are $D_1 = \mathbb{Z}/9\mathbb{Z}$ and $D_2 = \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{F}_3$. The gluing must result in a 3-elementary lattice. We can apply Lemma 2.9 to get that the glue $3D_1 = G_1 \cong G_2 = 3D_2$. This gluing is uniquely determined (up to ± 1) and a quick calculation shows that the resulting lattice is not 3-elementary. $\square$

TABLE 2. Realized determinants in ascending order of $\varphi(n) \leq 10$

n	$\det T$	X	f	
$3, 6$	3	$y^2 = x^3 - t^5(t-1)^5(t+1)^2$	$(\zeta_3 x, \pm y, t)$	[**50**]
4	2^2	$y^2 = x^3 + 3t^4 x + t^5(t^2 - 1)$	$(-x, \zeta_4 y, -t)$	
$5, 10$	5	$y^2 = x^3 + t^3 x + t^7$	$(\zeta_5^3 x, \pm\zeta_5^2 y, \zeta_5^2 t)$	[**50**]
8	2^2	$y^2 = x^3 + tx^2 + t^7$	$(\zeta_8^6 x, \zeta_8 y, \zeta_8^6)$	[**88**]
	2^4	$t^4 = (x_0^2 - x_1^2)(x_0^2 + x_1^2 + x_2^2)$	$(\zeta_8 t; x_1 : x_0 : x_2)$	
12	1	$y^2 = x^3 + t^5(t^2 - 1)$	$(-\zeta_3 x, \zeta_4 y, -t)$	[**50**]
	$2^2 3^2$	$y^2 = x^3 + t^5(t^2 - 1)^2$	$(-\zeta_3 x, \zeta_4 y, t)$	
	2^4	$y^2 = x^3 + t^5(t^2 - 1)^3$	$(-\zeta_3 x, \zeta_4 y, -t)$	
$7, 14$	7	$y^2 = x^3 + t^3 x + t^8$	$(\zeta_7^3 x, \zeta_7 y, \zeta_7^2 t)$	[**50**]
$9, 18$	3	$y^2 = x^3 + t^5(t^3 - 1)$	$(\zeta_9^2 x, \pm\zeta_9^3 y, \zeta_9^3 t)$	[**50**]
	3^3	$y^2 = x^3 + t^5(t^3 - 1)^2$	$(\zeta_9^2 x, \pm y, \zeta_9^3 t)$	
16	2^2	$y^2 = x^3 + t^2 x + t^7$	$(\zeta_{16}^2 x, \zeta_{16}^{11} y, \zeta_{16}^{10} t)$	[**87**, 4.2]
	2^4	$y^2 = x^3 + t^3(t^4 - 1)x$	$(\zeta_{16}^6 x, \zeta_{16}^9 y, \zeta_{16}^4 t)$	[**28**, 4.1]
	2^6	$y^2 = x^3 + x + t^8$	$(-x, iy, \zeta_{16} t)$	[**87**, 2.2]
20	2^4	$y^2 = x^3 + (t^5 - 1)x$	$(-x, \zeta_4 y, \zeta_5 t)$	
	$2^4 5^2$	$y^2 = x^3 + 4t^2(t^5 + 1)x$	$(-x, \zeta_4 y, \zeta_5 t)$	
24	2^2	$y^2 = x^3 + t^5(t^4 + 1)$	$(\zeta_3 \zeta_8^6 x, \zeta_8 y, \zeta_8^2 t)$	
	2^6	$y^2 = x^3 + (t^8 + 1)$	$(\zeta_3 x, y, \zeta_8 t)$	
	$2^2 3^4$	$y^2 = x^3 + t^3(t^4 + 1)^2$	$(\zeta_3 \zeta_8^6 x, \zeta_8 y, \zeta_8^6 t)$	
	$2^6 3^4$	$y^2 = x^3 + x + t^{12}$	$(-x, \zeta_{24}^6 y, \zeta_{24} t)$	
$15, 30$	5^2	$y^2 = x^3 + 4t^5(t^5 + 1)$	$(\zeta_3 x, \pm y, \zeta_5 t)$	
	3^4	$y^2 = x^3 + t^5 x + 1$	$(\zeta_{15}^{10} x, \pm y, \zeta_{15} t)$	
$11, 22$	11	$y^2 = x^3 + t^5 x + t^2$	$(\zeta_{11}^5 x, \pm\zeta_{11}^2 y, \zeta_{11}^2 t)$	[**50**]

THEOREM 3.20. *Let X be a K3 surface and f a purely non-symplectic automorphism of order n such that $\operatorname{rk} T = \varphi(n)$ and ζ_n is not an eigenvalue of $f|\operatorname{NS} \otimes \mathbb{C}$.*

Set $d = |\det \operatorname{NS}|$, then X is determined up to isomorphism by the pair (n, d). Conversely, all possible pairs (n, d) and equations for X and f are given in Tables 2, 3.

PROOF. Comparing Tables 1 and 2, 3 we have to exclude the pairs $(16, 2^8)$ and $(54, 3^3)$. For $(16, 2^8)$, this is done in [**87**, 4.1]. The pair $(54, 3^3)$ is ruled out in Lemma 3.19.

By Lemma 3.15 the transcendental lattice is uniquely determined by (n, d) and embeds uniquely into L_{K3}. By Theorem 3.17 X is determined up to isomorphism by (ζ_n, I), where I is the kernel of

$$\mathbb{Z}[x]/c_n(x) \to O(T^\vee/T), \qquad x \mapsto \overline{f}$$

and $f^*\omega_X = \zeta_n \omega_X$ for $0 \neq \omega_X \in H^0(X, \Omega_X^2)$. By Lemma 3.15, I is determined uniquely by (n, d). Replacing f with f^k, $(n, k) = 1$, does not affect (n, d), hence I.

TABLE 3. Purely non-symplectic automorphisms with $\varphi(n) \geq 12$

n	$\det T$	X	f	
$13, 26$	13	$y^2 = x^3 + t^5x + t$	$(\zeta_{13}^5 x, \pm\zeta_{13} y, \zeta_{13}^2 t)$	**[50]**
$21, 42$	1	$y^2 = x^3 + t^5(t^7 - 1)$	$(\zeta_{42}^2 x, \zeta_{42}^3 y, \zeta_{42}^{18} t)$	**[50]**
$21, 42$	7^2	$y^2 = x^3 + 4t^4(t^7 - 1)$	$(\zeta_3\zeta_7^6 x, \pm\zeta_7^2 y, \zeta_7 t)$	
$21, 42$	7^2	$y^2 = x^3 + t^3(t^7 + 1)$	$(\zeta_3\zeta_7^3 x, \pm\zeta_7 y, \zeta_7^3 t)$	
21	7^2	$x_0^3x_1 + x_1^3x_2 + x_0x_2^3 - x_0x_3^3$	$(\zeta_7 x_0 : \zeta_7 x_1 : x_2 : \zeta_3 x_3)$	
28	1	$y^2 = x^3 + x + t^7$	$(-x, \zeta_4 y, -\zeta_7 t)$	**[50]**
	2^6	$y^2 = x^3 + (t^7 + 1)x$	$(-x, \zeta_4 y, \zeta_7 t)$	
	2^6	$y^2 = x^3 + (t^7 + 1)x$	$\left(x - (y/x)^2, \zeta_4\left(y - (y/x)^3\right), \zeta_7 t\right)$	
$17, 34$	17	$y^2 = x^3 + t^7x + t^2$	$(\zeta_{17}^7 x, \zeta_{17} y, \zeta_{17}^2 t)$	**[50]**
34	17	$x_0x_1^5 + x_0^5x_2 + x_1^2x_2^4 = y^2$	$(-y; x_0 : \zeta_{17}x_1, \zeta_{17}^5 x_2)$	
32	2^2	$y^2 = x^3 + t^2x + t^{11}$	$(\zeta_{32}^{18} x, \zeta_{32}^{11} y, \zeta_{32}^2 t)$	**[80]**
	2^4	$y^2 = x_0(x_1^5 + x_0^4x_2 + x_1x_2^4)$	$(\zeta_{32} y; \zeta_{32}^2 x_0 : x_1 : \zeta_{32}^{24} x_2)$	
36	1	$y^2 = x^3 - t^5(t^6 - 1)$	$(\zeta_{36}^2 x, \zeta_{36}^3 y, \zeta_{36}^{30} t)$	**[50]**
	3^4	$y^2 = x^3 + x + t^9$	$(-x, \zeta_4 y, -\zeta_9 t)$	
	2^63^2	$x_0x_3^3 + x_0^3x_1 + x_1^4 + x_2^4$	$(x_0 : \zeta_9^3 x_1 : \zeta_4\zeta_9^3 x_2 : \zeta_9 x_3)$	
40	2^4	$z^2 = x_0(x_0^4x_2 + x_1^5 - x_2^5)$	$(x_0 : \zeta_{20}x_1 : \zeta_4 x_2; \zeta_8 z)$	**[56]**
48	2^2	$y^2 = x^3 + t(t^8 - 1)$	$(\zeta_{48}^2 x, \zeta_{48}^3 y, \zeta_{48}^6 t)$	**[56]**
$19, 38$	19	$y^2 = x^3 + t^7x + t$	$(\zeta_{19}^7 t, \zeta_{19} y, \zeta_{19}^2 t)$	**[50]**
$27, 54$	3	$y^2 = x^3 + t(t^9 - 1)$	$(\zeta_{27}^2 x, \zeta_{27}^3 y, \zeta_{27}^6 t)$	**[50]**
27	3^3	$x_0x_3^3 + x_0^3x_1 + x_2(x_1^3 - x_2^3)$	$(x_0 : \zeta_{27}^3 x_1 : \zeta_{27}^{21} x_2 : \zeta_{27} x_3)$	
$25, 50$	5	$z^2 = (x_0^6 + x_0x_1^5 + x_1x_2^5)$	$(z; x_0 : \zeta_{25}^5 x_1 : \zeta_{25}^4 x_2)$	**[50]**
$33, 66$	1	$y^2 = x^3 + t(t^{11} - 1)$	$(\zeta_{66}^2 x, \zeta_{66}^3 y, \zeta_{66}^6 t)$	**[50]**
44	1	$y^2 = x^3 + x + t^{11}$	$(-x, \zeta_4 y, \zeta_{11}, t)$	**[50]**

However, in this way we can fix a primitive n-th root of unity ζ_n.

It remains to compute the Néron-Severi group of the examples in Tables 2, 3 not found in the literature. In most cases this can be done by collecting singular fibers of an elliptic fibration or determining the *fixed lattice* $S(f^k) = H^2(X, \mathbb{Z})^f$ of a suitable power of the automorphism f through its fixed points. The corresponding tables of fixed lattices are collected in [**5**].

$(4, 2^2)$ We see two fibers of type II^* over $t = 0, \infty$ and two fibers of type I_2 over $t = \pm 1$. Then $\mathrm{NS} \cong U \oplus 2E_8 \oplus 2A_1$ as expected. The two form is given in local coordinates by $dx \wedge dt/2y$, and $f^*(dx \wedge dt/2y) = -dx \wedge -dt/(2y\zeta_4) = \zeta_4^3 dx \wedge dt/2y$. Hence the action is non-symplectic. The fixed lattice is $U \oplus 2E_8$ while the I_2 fibers are exchanged. Giving that $f|\,\mathrm{NS}$ has order two.

$(8, 2^4)$ The fourfold cover of $\mathbb{P}^2$ is a special member of a family in [**4**, Ex. 5.3]. It has five A_3 singularities. The fixed locus of the non-symplectic involution f^4 consists of 8 rational curves, where each A_3 configuration contains 1 fixed curve. Hence, its fixed lattice is of rank 18 and determinant -2^4. It equals NS.

$(12,2^4)$ We get fibers of type $1 \times II, 1 \times II^*, 2 \times I_0^*$ which results in the lattice $\mathrm{NS} = U \oplus E_8 \oplus 2D_4$.

$(12,2^2 3^2)$ This time zero section and fibers span the lattice $\mathrm{NS} = U \oplus E_8 \oplus 2A_2 \oplus D_4$.

$(15,5^2)$ This elliptic K3 surface arises as a degree 5 base change from the rational surface $Y : y^2 = x^3 + 4t(t+1)$. We see the section $(x,y) = (\zeta_3^k, 1+2t)$ of Y and then $(x,y) = (\zeta_3^k, 1+2t^5)$ generating the Mordell-Weil group of $X_{(15,5^2)}$. Alternatively one can compute that NS is the fixed lattice of f^3.

$(15,3^4)$ The 5-th power f^5 of f is a non-symplectic automorphism of order 3 acting trivially on NS. It has 2 fixed curves of genus 0 lying in the E_7 fiber and 6 isolated fixed points over $t = 0$ and $t = \infty$. The classification of the fixed lattices of non-symplectic automorphisms of order 3 provides the fixed lattice of f^5 which equals NS. In order to get explicit generators of the Mordell-Weil group we can base change with $t \mapsto t^3$ from the rational surface $y^2 = 2x^3 + tx + t^4$ with sections $(x,y) = (t, t^2+t), (0, t^2)$.

$(34,17)$ In the first case the fixed locus of g_1^{17} consists of a curve $(y = 0)$ of genus 8 and a rational curve - the zero section. This leads to the fixed lattice $U \oplus 2(-2)$.

Since the fixed locus of g_2^{17} is a curve of genus 8, $S(g^{17}) \cong (2) \oplus 2(-2)$. Note that there is an A_4 singularity at zero. Since the fixed lattices of the two automorphisms are different, the actions are distinct as well.

$(20,2^4)$ The elliptic fibration has 5 fibers of type III and a single fiber of type III^*. This results in the lattice $U \oplus E_7 \oplus 5A_1$ spanned by fiber components and the zero section. It has determinant 2^6. Since there is also a 2-torsion section, $\det \mathrm{NS} = 2^4$.

$(20,2^4 5^2)$ In this case $X_{(20,2^4 5^2)}$ has a single fiber of type I_0^* and 6 fibers of type III. This results in the lattice $U \oplus D_4 \oplus 6A_1$ of rank 12 and determinant 2^8. Again there is 2-torsion. We reach a lattice of determinant 2^6. We get X by a degree 5 base change from $y^2 = x^3 + 4t^2(t+1)x$ with section $(x,y) = (t^2, t^3 + 2t^2)$. We get the sections $(x,y) = (t^6, t^9 + 2t^4)$ and $(-x(t), iy(t))$ generating the Mordell-Weil lattice $2A_1^\vee(5)$ of X.

$(21,7^2)$ We can base change this elliptic fibration from $y^2 = x^3 + 4t^4(t-1)$ to get the sections. $(x,y) = (\zeta_3^k t^6, 2t^2 + t^9)$ generating the Mordell-Weil lattice of X. To double check note that f^3 is an order 7 non-symplectic automorphism acting trivially on NS and not fixing a curve of genus 0 point-wise. There is only a single possible fixed lattice of rank 10, namely $U(7) \oplus E_8$. For the other possible action see Lemma 3.44.

$(24,2^2)$ The fibration has 4 type II fibers, one type I_0^* and an II^* fiber. We get $\mathrm{NS} = U \oplus D_4 \oplus E_8$.

$(24,2^6)$ In this case the fixed locus of f^{12} consists of 4 rational curves and a curve of genus 1. This leads to a fixed lattice of rank 14 and determinant 2^6 as expected.

$(24, 2^2 3^4)$ The trivial lattice is $U \oplus D_4 \oplus 4A_2$. It equals NS for absence of torsion sections.

$(24, 2^6 3^4)$ Since X has a purely non-symplectic automorphism of order 24, the rank of NS is either 6 or 14. $\mathrm{Fix}(f^{12})$ consists of 2 smooth curves of genus 0. Its fixed lattice $S(f^{12})$ is 2-elementary of rank 10 and determinant -2^8. We see that $\operatorname{rk}\mathrm{NS}(X) = 14$. Since the orthogonal complement of $S = S(f^{12})$ is of rank 4, the glue G_S is an at most 4 dimensional subspace. Then $2^4 \leq |D_S/G_S| \leq |D_{\mathrm{NS}}|$ by Lemma 2.7. Hence $2^4 \mid \det \mathrm{NS}$. Note that $S(f^{12}) = \ker f^{12} - 1$ and then $\ker c_{24}c_8(f) = S(f^{12})^\perp$ is of rank 12. This shows that the characteristic polynomial of $f|\,\mathrm{NS}$ is divided by c_8 but not by c_{24}. We are in the situation of the theorem. As $2^4 \mid \det \mathrm{NS}$, it is either -2^6 or $-2^6 3^4$. We show that $3 \mid \det \mathrm{NS}(X)$. A computation reveals that $\mathrm{Fix}(f^8)$ consists of a smooth curve of genus 1 and 3 isolated fixed points. This leads to a fixed lattice

$$S(f^8) = \ker(c_8c_4c_2c_1)(f) \cong U(3) \oplus 3A_2$$

of rank 8 and determinant -3^5. Now we view $S(f^8)$ as a primitive extension of $\ker c_8(f) \oplus \ker c_4c_2c_1(f)$. The rank of both summands is 4, while the length of the discriminant group of $S(f^8)$ is 5. Then each summand must contribute to the discriminant group. We see that $3 \mid \det \ker c_8(f)$. However,

$$3 \nmid \mathrm{res}(c_8, c_{12}c_6c_4c_3c_2c_1) = 2^4.$$

In particular the 3 part of $D_{\ker c_8(f)}$ cannot be glued inside NS. Then $3 \mid \det \mathrm{NS}$.

$(27, 3^3)$ The action of f^9 has an isolated fixed point and a fixed curve of genus 3. We see that the fixed lattice of f^9 is $U(3) \oplus A_2 = \mathrm{NS}$. It is spanned by the 4 lines at $x_3 = 0$. Note that f^3 acts trivially on NS while f does not.

$(28, 2^6)$ This fibration has 8 fibers of type III and a 2-torsion section. Together they generate the Néron-Severi group.

$(32, 2^2)$ The elliptic fibration has a singular fiber of type I_0^*, of type II and 16 of type I_1. Thus $\mathrm{NS} =\cong U \oplus D_4$. Here f has 6 isolated fixed points.

$(32, 2^4)$ The fixed locus of $S(f^{16})$ is the strict transform of $y = 0$ which is the disjoint union of a rational curve and a curve of genus 5. Thus $\det NS = 2^4$. Note that f has 4 isolated fixed points.

$(36, 3^4)$ The fixed curves of f^{12} are a smooth of genus 0 over $t = 0$ and the central rational curve in the D_4 fiber. This leads to the fixed lattice $U \oplus 4A_2 = \mathrm{NS}$.

$(36, 2^6 3^2)$ If we can show that $2 \mid \det \mathrm{NS}$, then the only possibility is $\det \mathrm{NS} = -2^6 3^2$. The action of f^{18} fixes a smooth curve of genus 3 and nothing else. Hence, its fixed lattice S is 2-elementary of rank 8 and determinant 2^8. Denote by $C = S^\perp \subseteq \mathrm{NS}$ the orthogonal complement of S inside NS. It has rank 2. Assume that $2 \nmid \det \mathrm{NS}$. Then $(D_S)_2 \cong (D_C)_2$ which is impossible, since $(D_S)_2$ has dimension 8, while $(D_C)_2$ is generated by at most 2 elements. □

Remark 3.21. The pair $(21, 7^2)$ contradicts the main result of J. Jang in [**45**, 2.1]. There it is claimed that a purely non-symplectic automorphism of order 21 acts trivially on NS. As a consequence it is claimed that there is only a single K3 surface of order 21. However there are two. The pair $(28, 2^6)$ and its uniqueness are probably known to J. Jang independently.

The pair $(32, 2^4)$ contradicts the main result of S. Taki in [**98**]. There the uniqueness of $(X, \langle g\rangle)$ where g is a purely non-symplectic automorphism of order 32 is claimed.

In [**97**, 1.8, 4.8] S. Taki classifies non symplectic automorphisms of 3-power order acting trivially on NS. The author is missing a case. It is claimed that if $\mathrm{NS}(X) = U(3) \oplus A_2$ then there is no purely non-symplectic automorphism of order 9 acting trivially on NS. The pair $(27, 3^3)$ contradicts this result - there the automorphism acts with order 3 on NS. It is a special member of the family

$$x_0x_3^3 + x_0^3x_1 + x_2(x_1 - x_2)(x_1 - ax_2)(x_1 - bx_2)$$

with automorphism given by $(x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : \zeta_9^3 x_1 : \zeta_9^3 x_2 : \zeta_9 x_3)$ and generically trivial action on NS as a fixed point argument shows. It was found by first computing the action of f on cohomology through gluing, thus proving its existence. Then one specializes a family with automorphism of order 3 given in [**3**, 4.9].

4. Classification of non-symplectic automorphisms of high order

Let X_i, $i = 1, 2$, be K3 surfaces and $G_i \subset \mathrm{Aut}(X_i)$ subsets of their automorphism groups. We say that the pairs $(X_1, G_1) \cong (X_2, G_2)$ are isomorphic if there is an isomorphism $\phi : X_1 \to X_2$ with $\phi \circ G_1 = G_2 \circ \phi$.

Theorem 3.22. *Let X be a K3 surface and $\mathbb{Z}/n\mathbb{Z} \cong G \subseteq \mathrm{Aut}(X)$ a purely non-symplectic subgroup with $\varphi(n) \geq 12$. All possible pairs are found in Table 3.*

Theorem 3.23. *Let $X_{(n,d)}$ be as in Tables 2,3.*

(1) For $(n, d) = (66, 1), (44, 1), (50, 5), (42, 1), (28, 1), (36, 1), (32, 2^2), (32, 2^4), (40, 2^4), (54, 3), (27, 3^3), (24, 2^2), (16, 2^2)$, we have

$$Aut(X_{(n,d)}) = \langle g_{(n,d)}\rangle \cong \mathbb{Z}/n\mathbb{Z}.$$

(2) For $(n, d) = (28, 2^6), (12, 1), (16, 2^4), (20, 2^4)$ we have

$$\mathrm{Aut}(X) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}.$$

Remark 3.24. In [**56**] (1) is proven for $n = 66, 44, 50$ and 40 by a different method. By the classification in [**76**], all other entries of Tables 2 and 3 have infinite automorphism group.

Before proving the theorems we refine our terminology.
A g-lattice is a pair (A, a), where A is a lattice and $a \in O(A)$ an isometry. A morphism $\phi : (A, a) \to (B, a)$ of g-lattices is an isometry $\phi : A \to B$ such that $\phi \circ a = b \circ \phi$.

Definition 3.25. We call two primitive extensions of g-lattices

$$(A_i, a_i) \oplus (B_i, b_i) \hookrightarrow (C_i, c_i), \quad i = 1, 2$$

isomorphic if there is a commutative diagram

$$\begin{array}{ccccc} (A_1, a_1) & \oplus & (B_1, b_1) & \longrightarrow & (C_1, c_1) \\ \downarrow & & \downarrow & & \downarrow \\ (A_2, a_2) & \oplus & (B_2, b_2) & \longrightarrow & (C_2, c_2) \end{array}$$

of g-lattices.

We leave the proof of the following proposition to the reader.

PROPOSITION 3.26. *There is a one to one correspondence between isomorphism classes of primitive extensions and the double coset*

$$\mathrm{Aut}(A,a) \setminus \left\{ \begin{array}{c} \textit{Glue maps } \phi : G_A \xrightarrow{\sim} G_B \\ \textit{satisfying } \phi \circ a = b \circ \phi \end{array} \right\} / \, \mathrm{Aut}(B,b)$$

where $g.\phi.h = g \circ \phi \circ h$ *for* $g \in Aut(A,a)$ *and* $h \in \mathrm{Aut}(B,b)$.

DEFINITION 3.27. A g-lattice (A,a) has *few extensions* if

$$Aut(A,a) \to \mathrm{Aut}(q_A, \overline{a}) = \{g \in O(q_A) \mid g \circ a = a \circ g\}$$

is surjective.

EXAMPLE 3.28. Let (A,a) be a g-lattice such that $D_A \cong \mathbb{F}_p$. Then $\mathrm{Aut}(D_A, \overline{a}) = \{\pm id_{D_A}\}$, and we see that (A,a) has few extensions.

LEMMA 3.29. *Let* (A,a) *be a simple* $c_n(x)$*-lattice. Then* $\mathrm{Aut}(A,a) = \langle \pm a \rangle$.

PROOF. Let $h \in \mathrm{Aut}(A,a)$. Then h is a $\mathbb{Z}[a]$-module homorphism, i.e. $h \in \mathbb{Z}[a]^\times \subseteq K^\times$. Since h is an isometry and (A,a) is simple, we get that

$$Tr^K_{\mathbb{Q}}(hh^\sigma x) = Tr^K_{\mathbb{Q}}(x) \quad \forall x \in K.$$

By non-degeneracy of the trace form, we get $hh^\sigma = 1$, i.e. $|h| = 1$. By Kronecker's theorem, h is a root of unity. $\square$

PROPOSITION 3.30. *Let* $(L_0(a), f)$ *be a twist of the principal, simple* $p(x)$*-lattice and* $I < \mathcal{O}_K$ *such that* $D_{L_0(a)} \cong \mathcal{O}_K/I$. *Then*

$$\mathrm{Aut}(q_{L_0(a)}, \overline{f}) = \{[u] \in (\mathcal{O}_K/I)^\times \mid uu^\sigma \equiv 1 \mod I\}.$$

PROOF. Let $g \in \mathrm{Aut}(q_{L_0(a)}, \overline{f})$. Under the usual identifications $g = [u] \in (\mathcal{O}_K/I)^\times$. Note that $(f - f^{-1})\mathcal{O}_K = \mathcal{D}^K_k$ is the relative different of K/k, and set $d = f - f^{-1}$. Then $L_0(a)^\vee = \frac{1}{ad}\mathcal{O}_K$ and $I = ad\mathcal{O}_K$ (Lemma 2.20). Since $[u]$ preserves the discriminant form, we get that $b(x,y) = b([u]x,[u]y)$ for all $x, y \in \frac{1}{ad}\mathcal{O}_K$, i.e.

$$Tr^K_{\mathbb{Q}}\left(\frac{1 - uu^\sigma}{r'(f + f^{-1})ad^2}\mathcal{O}_K\right) \subseteq \mathbb{Z}.$$

By definition of the different, this is equivalent to

$$\frac{1 - uu^\sigma}{r'(f + f^{-1})ad^2} \in \mathcal{D}_K^{-1},$$

and consequently

$$1 - uu^\sigma \in ad^2 r'(f + f^{-1})\mathcal{D}_K^{-1}.$$

Now, the different $\mathcal{D}_K = p'(f)\mathcal{O}_K = (f - f^{-1})r'(f + f^{-1})\mathcal{O}_K$. Hence

$$1 - uu^\sigma \in ad\mathcal{O}_K = I$$

as claimed. Conversely, let $u \equiv 1 \mod I$. A similar computation shows that the discriminant quadratic form $q_{L_0(a)}$ is preserved if and only if

$$1 - uu^\sigma \in add^\sigma\mathcal{O}_k = a(\mathcal{D}^K_k)^2 \cap k.$$

However, we already know $1 - uu^\sigma \in a\mathcal{D}^K_k \cap k$. By simplicity, we know that the norm $N(\mathcal{D}^K_k) = |p(1)p(-1)|$ is squarefree, and hence

$$\mathcal{D}^K_k \cap k = (\mathcal{D}^K_k)^2 \cap k.$$

$\square$

REMARK 3.31. Instead of $|p(-1)p(1)|$ being squarefree, one may assume K/k to be tamely ramified and the proof works. However, $\mathbb{Q}[\zeta_{2^k}]$ is ramified at two and then $\mathcal{D}_k^K \cap k$ is the prime ideal of $\mathcal{O}_k$ above 2. In this case we need the condition $uu^\sigma \equiv 1 \mod I\mathcal{D}_k^K \cap k$.

LEMMA 3.32. *All entries in Table 1 except* $(24, 2^6 3^4)$ *have simple glue.*

PROOF. We do the calculation for $(27, 3^3)$. The other cases are similar. Set $\zeta = \zeta_{27}$. Then $I = (1-\zeta)^3$, and $\mathcal{O}_K/I = \mathbb{Z}[\zeta]/(1-\zeta)^3$ has 18 units. They are given by

$$u_\epsilon = \epsilon_0 + \epsilon_1(1-\zeta) + \epsilon_2(1-\zeta)^2, \quad (\epsilon_0 \in \{1,2\},\ \epsilon_1, \epsilon_2 \in \{0,1,2\})$$

We compute that

$$0 \overset{!}{\equiv} (1 - u_\epsilon u_\epsilon^\sigma) \equiv \epsilon_0(\epsilon_1 + \epsilon_2)(2 - \zeta - \zeta^{-1}) \mod (1-\zeta)^3.$$

We get 6 distinct solutions. However $\pm\zeta^k$ for $k \in 1,2,3$ are all distinct modulo $(1-\zeta)^3$. The claim follows. □

For $n \in \mathbb{N}$ we denote by $\mathcal{S}_n$ the symmetric group of n elements and by $\mathcal{D}_n$ the dihedral group - the symmetry group of a regular polygon with n sides.

LEMMA 3.33. *Let L be a hyperbolic lattice. Fix a chamber of the positive cone and denote by $O^+(L)/W(L) \cong \Gamma(L) \subseteq O^+(L)$ the subgroup generated by the isometries preserving the chamber. Set*

$$\phi : \Gamma(L) \to O(q_L)$$

then for $L \neq U(3) \oplus A_2$ in Table 4 ϕ is surjective. For $L = U(3) \oplus A_2$ the cokernel of ϕ is generated by $-id$. It is injective as well for all L in the table except $U(2) \oplus 2D_4$ and $U(2) \oplus D_4 \oplus E_8$ where its kernel is of order 2.

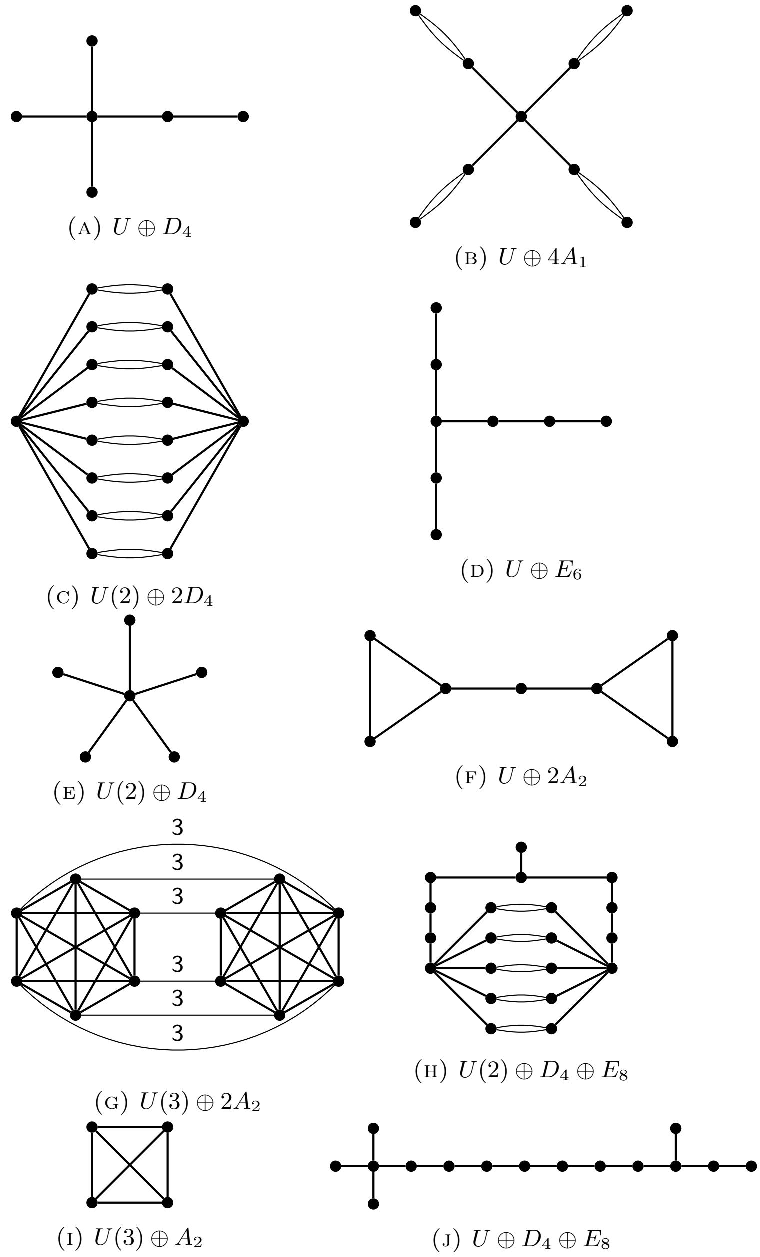
TABLE 4. Symmetry groups of a chamber

L	$\Gamma(L)$	L	$\Gamma(L)$
$U \oplus A_2$	$\mathcal{S}_2$	$U \oplus 2A_2$	$\mathcal{D}_4$
$U(3) \oplus A_2$	$\mathcal{D}_4$	$U(3) \oplus 2A_2$	$\mathcal{S}_6 \times \mathcal{S}_2$
$U \oplus 4A_1$	$\mathcal{S}_4$	$U \oplus D_4$	$\mathcal{S}_3$
$U(2) \oplus D_4$	$\mathcal{S}_5$	$U \oplus E_6$	$\mathcal{S}_2$
$U \oplus E_8$	$\mathcal{S}_1$	$U(2) \oplus 2D_4$	$\mathcal{S}_8 \times \mathcal{S}_2$
$U \oplus D_4 \oplus E_8$	$\mathcal{S}_3$	$U(2) \oplus D_4 \oplus E_8$	$S_5 \times S_2$

PROOF. In all cases we can compute a fundamental root system using Vinberg's algorithm [**101**, §3]. An isometry preserves the chamber corresponding to the fundamental root system if and only if it preserves the fundamental root system. We get a sequence

$$0 \to O^+(L)/W(L) \cong \Gamma(L) \to Sym(\Gamma) \to 0$$

where $Sym(\Gamma)$ denotes the symmetry group of the dual graph of a fundamental root system. Since the fundamental roots form a basis of $L \otimes \mathbb{Q}$, the sequence is exact. The calculation of $\ker\phi$ is done by computer. For $L = U(2) \oplus 2D_4$ see also [**52**, 2.6]. □



(A) $U \oplus D_4$

(B) $U \oplus 4A_1$

(C) $U(2) \oplus 2D_4$

(D) $U \oplus E_6$

(E) $U(2) \oplus D_4$

(F) $U \oplus 2A_2$

(G) $U(3) \oplus 2A_2$

(H) $U(2) \oplus D_4 \oplus E_8$

(I) $U(3) \oplus A_2$

(J) $U \oplus D_4 \oplus E_8$

FIGURE 2. Dynkin diagrams of the fundamental root systems

REMARK 3.34. The observation that for $L = U(3) \oplus A_2$, $-id|D_L$ generates the cokernel of ϕ gives another proof that $(54, 3^3)$ is not realized.

PROOF OF THEOREM 3.22. Fix some pair (X, G) with (n, d) and write $G = \langle g \rangle$ for $g \in G$ such that $g^*\omega = \zeta_n \omega$. In order to prove the theorem, we have to show that $(H^2(X, \mathbb{Z}), g)$ is unique up to isomorphism as a g-lattice. We have seen that (n, d) determines $(T, g|_T)$ (and $X/ \cong$ by Thm. 3.17). By Lemma 3.32, $(T, g|_T)$ has simple glue. Hence the isomorphism class of $(H^2(X, \mathbb{Z}), g)$ is determined by the isomorphism class of $(\mathrm{NS}, g|_{\mathrm{NS}})$. What remains is to determine all possible isomorphism classes for $(\mathrm{NS}, g|_{\mathrm{NS}})$ and (n, d) fixed. This is done in the following lemmas. $\square$

LEMMA 3.35. *For* (p,p)*,* $p = 13, 17, 19$*,* $g|_{\mathrm{NS}} = id$.

PROOF. Since the order of g on NS is strictly smaller than $n = p$ in these cases, it can only be one. $\square$

PROOF OF THEOREM 3.23. (1) By Lemma 3.33, we have for these lattices that $O^+(\mathrm{NS})/W(\mathrm{NS}) \cong O(q_{\mathrm{NS}})$. Consequently, automorphisms are determined by their action on the transcendental lattice and this group is generated by $g_{(n,d)}$. (2) In this case $\phi : \Gamma(\mathrm{NS}) \to O(q|_{\mathrm{NS}})$ has a kernel of order two and there are exactly two possibilities for $g|\,\mathrm{NS}$. They differ by an element of the kernel corresponding to a symplectic automorphism of order two. $\square$

We note the following theorem for later use.

THEOREM 3.36. [**72**, 3.3.14 (ii)] *Let* (L, f) *be a* c_n*-lattice of rank* $m\phi(n)$*.*

$$\det L \in \begin{cases} p^m \cdot \left(\mathbb{Q}^\times\right)^2, & \text{for } n = p^k, p \neq 2, \\ \left(\mathbb{Q}^\times\right)^2, & \text{else.} \end{cases}$$

Recall the notation $Ci = \ker c_i(g|H^2(X,\mathbb{Z}))$, $CiCj = \ker c_i c_j(g|H^2(X,\mathbb{Z}))$, and note that $g|\,\mathrm{NS}$ preserves a chamber of the positive cone if and only if

$$\ker\left(g^{n-1} + g^{n-2} \cdots + 1\right)|_{\mathrm{NS}}$$

is root free (see Chap. 7 §1). In this case we call g unobstructed and obstructed else.

LEMMA 3.37. *For* **(38, 19)** *set*

$$R = \begin{pmatrix} -2 & -1 \\ -1 & -10 \end{pmatrix}.$$

Then $\mathrm{NS} \cong U \oplus R$ *and* $g|_{\mathrm{NS}}$ *is given by the gluing of*

$$C1 \cong U \oplus (-2) \text{ and } C2 \cong (-38)$$

along 2.

PROOF. There are 3 cases

$$\chi_{g|\,\mathrm{NS}} = (x-1)^r (x+1)^{4-r}, \quad (r = 1, 2, 3).$$

Note that $C1$ is 2-elementary and $\det C1 \mid 2^m$ where $m = min\{r, 4-r\}$ (Thm 2.11).

- $r = 1$: Here $C1 = (2)$ and $C2 = (-2) \oplus R$ which is the unique even, negative definite lattice of determinant -38.
- $r = 2$: We see $\det C2 \mid 2^2 19$ and there are two such lattices - R and $R(2)$. However, R has roots, and $R(2)$ has wrong 19 glue, since the Legendre symbol $\left(\frac{2}{19}\right) = -1$.
- $r = 3$: We have the single choice $C2 = (-38)$ and $C1 = U \oplus (-2)$. Indeed the gluing exists and is unique.

$\square$

LEMMA 3.38. **(34, 17)** *There are two pairs* (X, g_1) *and* (X, g_2) *for* $(34, 17)$*. The action of* $g_1|_{\mathrm{NS}}$ *is given by the gluing of*

$$C1 = U \oplus (-2) \oplus (-2) \quad \text{and} \quad C2 = -\begin{pmatrix} 6 & 2 \\ 2 & 12 \end{pmatrix}.$$

along 2.

The action of $g_2|_{\mathrm{NS}}$ *is given by the gluing of*

$$C1 = (2) \oplus (-2) \oplus (-2) \quad \text{and} \quad C2 = -2\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

along 2.

PROOF. Here we need a little more work. Note that $\mathrm{NS} \in II_{(1,5)}(17^{-1})$. There are the 5 cases

$$\chi_{g|\,\mathrm{NS}} = (x-1)^r(x+1)^{6-r}, \qquad r \in \{1, \dots 5\}.$$

In any case $17 \mid \det C2 \mid 2^m 17$ where $m = \min\{r, 6-r\}$. The 17 part of the genus symbol of $C2$ is 17^{-1}, and moreover $2(D_{C2})_2 = 0$. Then the genus symbol looks as follows $II_{(0,6-r)}(2^v_* 17^{-1})$.

$r = 1$: Then $C1 = (2)$ and $\det C2 = -34$. Hence in order to glue above 2, $C2$ must be an element of $II_{(0,5)}(2_7 17^{-1})$ or $II_{(0,5)}(2_3^{-1} 17^{-1})$, but both genera are empty as they contradict the oddity formula.

$r = 2$: Here $C2$ is even of signature $(0,4)$ and determinant $d = -17, -2{\cdot}17, -4{\cdot}17$. Looking at the tables in **[77]**, we see that there are $1, 0, 7$ such forms, and all of them contain roots.

$r = 3$: From the tables in **[21]**, we extract the following. If $v = 0, \pm 2$ the respective genera are empty. If $v = 1$, there is a single genus, namely $II_{(0,3)}(2^1_1 17^{-1})$ containing two classes - both have maximum -2. However, for $v = 3$ there are 9 negative definite ternary forms of determiant $2^3 17$. Only a single one of them has the right 2-genus symbol and no roots. It is given by

$$C2 = -2\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}.$$

Indeed, here $C1 = (2) \oplus (-2) \oplus (-2)$ works just fine, and as $|O(q_{C1})| = 2$ it is evident that the gluing is unique as well.

$r = 4$: Here $\det C2 \mid 2^2 17$, and we get the possibilities

$$-\begin{pmatrix} 2 & 0 \\ 0 & 34 \end{pmatrix}, -\begin{pmatrix} 4 & 2 \\ 2 & 18 \end{pmatrix}, -\begin{pmatrix} 6 & 2 \\ 2 & 12 \end{pmatrix}.$$

The first two have wrong 17 glue. We are left with the third one. It has

$$(q_R)_2 \cong (1/2) \oplus (1/2).$$

Then there is the single possibility $C1 \cong U \oplus (-2) \oplus (-2)$. Surjectivity of $O(C1) \to O(q_{C1})$, hence uniqueness of the extension is provided by Theorem 2.4.

$r = 5$: Here $C2 = (-34) \in II_{(0,1)}(2_7 17)$ has wrong 17-glue.

□

For the next lemma we use the holomorphic (see **[8**, p.542] and **[9**, p.567]) and topological Lefschetz formula. We give a short account. See **[97]** for a similar application.

Recall that g is a purely non-symplectic automorphism of the K3 surface X with $g^*\omega = \zeta_n \omega$, where $0 \neq \omega \in H^0(X, \Omega^2_X)$. Let x be a fixed point of g. Then the local action of g at x can be linearized and diagonalized (in the holomorphic category). We call it of type (i,j) if it is of the following form

$$\begin{pmatrix} \zeta_n^i & 0 \\ 0 & \zeta_n^j \end{pmatrix}.$$

This implies that the fixed point set X^g is the disjoint union of isolated fixed points and smooth curves $C_1, \dots, C_N$. Set

$$a_{ij} = \frac{1}{(1-\zeta_n^i)(1-\zeta_n^j)} \quad \text{and} \quad b(g) = \frac{(1+\zeta_n)(1-g)}{(1-\zeta_n)^2}.$$

Denote by $m_{i,j}$ the number of isolated fixed points of type (i,j), and set $g_l = g(C_l)$ the genus of the fixed curve C_l. The topological Lefschetz formula is

$$e(X^f) = \sum_{i=0}^{4} (-1)^i Tr(g^* | H^i(X, \mathbb{Z}))$$

which in our setting amounts to

$$M + \sum_{l=1}^{N} (2 - 2g(C_l)) = 2 + Tr(g^*|T) + Tr(g^*|\,\mathrm{NS})$$

where $M = \sum_{\substack{i+j=n+1 \\ 1<i\leq j<n}} m_{i,j}$ is the number of isolated fixed points. The holomorphic Lefschetz formula is

$$1 + \overline{\zeta_n} = \sum_{i=0}^{2} (-1)^i Tr(g^*|H^i(X, \mathcal{O}_X)) = \sum_{\substack{i+j=n+1 \\ 1<i\leq j<n}} a_{ij} m_{ij} + \sum_{l=1}^{N} b(g_l).$$

LEMMA 3.39. *For* $\mathbf{(26, 13)}$ *the action of* $g|_{\mathrm{NS}}$ *is unique and given by the gluing of*

$$C1 = U \oplus D_4 \oplus A_1 \quad \textit{and} \quad C2 = -\begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 10 \end{pmatrix}$$

along 2^3.

PROOF. We already know the uniqueness of (X, g^2). One can check that g^2 has 9 isolated fixed points and a (pointwise) fixed curve of genus 0. The local types are given by

$$m_{2,12} = 3, m_{3,11} = 3, m_{4,10} = 2, m_{5,9} = 1.$$

Since $X^g \subseteq X^{g^2}$, either g fixes a curve of genus 0 and at most 9 isolated points, or g does not fix a curve and at most 11 points.

A calculation of the holomorphic Lefschetz formula yields the following possibilities: X^g fixes a curve of genus zero and 7 or 9 points. The possible local actions are

$$m_{2,25} = 4, m_{5,22} = 1, m_{11,16} = 1, m_{12,15} = 1;$$
$$m_{2,25} = 4, m_{9,18} = 1, m_{10,17} = 2, m_{11,16} = 1, m_{12,15} = 1;$$
$$m_{2,25} = 3, m_{3,24} = 2, m_{4,23} = 2, m_{5,22} = 1, m_{11,16} = 1.$$

X^g fixes $4, 5, 6$ or 7 points with local contributions

$$m_{5,22} = 1, m_{7,20} = 1, m_{11,16} = 1, m_{12,15} = 1;$$
$$m_{5,22} = 1, m_{11,16} = 1, m_{12,15} = 1, m_{13,14} = 2;$$
$$m_{7,20} = 1, m_{9,18} = 1, m_{10,17} = 2, m_{11,16} = 1, m_{12,15} = 1;$$
$$m_{9,18} = 1, m_{10,17} = 2, m_{11,16} = 1, m_{12,15} = 1, m_{13,14} = 2.$$

In any case the fixed locus has Euler characteristic $e(X^g) \in \{4, 5, 6, 7\}$. Write

$$\chi_{g|\,\mathrm{NS}} = (x-1)^r (x+1)^{10-r}$$

for the characteristic polynomial of the action of g on NS. Then the topological Lefschetz formula reads

$$e(X^g) = 2 + Tr(g^*|T) + Tr(g^*|\,\mathrm{NS}) = 2 + 1 + r - (10 - r) = 2r - 7,$$

and consequently $2r \in \{11, 12, 13, 14\}$, i.e., r is either 6 or 7.

We view NS as a primitive extension of $C1 \oplus C2$. Since $res(c_1, c_{26}) = 1$, we see that $13 \mid \det C2$. Further, $C1$ is 2-elementary. We conclude that $|\det C2| = 2^k 13$ where $k \leq \min\{r, 10 - r\}$.

$r = 6$: Looking at the tables in [**77**], we see that for $k = 0, 1, 2, 3$ all even forms of signature $(0, 4)$ and determinant $-2^k 13$ have roots. For $k = 4$ there are three forms without roots. However, none of them has 2-discriminant $(\mathbb{Z}/2\mathbb{Z})^4$.

$r = 7$: Here we use the tables of [**21**] to list even forms of signature $(0, 3)$ and determinant $2^k 13$.

- For $k = 0$ there is no lattice of this determinant.
- For $k = 1$ there is a single class, but it is obstructed.
- For $k = 2$ there are two genera of this determinant, but their 2 discriminant group is isomorphic to $\mathbb{Z}/4\mathbb{Z}$.
- For $k = 3$ there is a single genus with right 2 discriminant and 13 glue. It is

$$II_{(0,3)}(2_1^{-3} 13^{-1})$$

and consists of the two classes

$$-\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 26 \end{pmatrix}, \quad \text{and} \quad -\begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 10 \end{pmatrix} \cong C2$$

one of which, $C2$, has no roots. Then

$$C1 \cong U \oplus D_4 \oplus A_1.$$

We have to check uniqueness of the gluing. This is provided by the surjectivity of

$$O(C1) \to O(q_{C1})$$

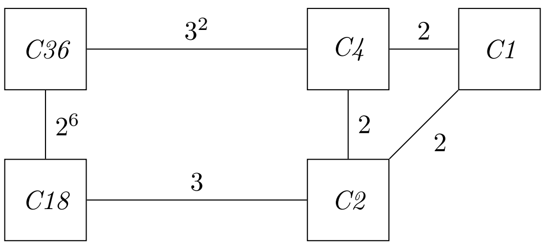
which follows from [**75**, 1.14.2].

□

Lemma 3.40. *For* $(\mathbf{36, 2^6 3^2})$ *, the characteristic polynomial is*

$$\chi_g = c_{36} c_{18} c_4 c_2 c_1$$

and the gluings are given by



This determines the g-lattice $(\mathrm{NS}, g|\,\mathrm{NS})$ *uniquely up to isomorphism.*

Proof. The possible contributors to the resultant are c_9, c_{18}, c_4 and c_{12}. First the 2^6 contribution is coming from either c_9 or c_{18} dividing $\chi(g|\,\mathrm{NS})$. Then there is no room for c_{12} left. Thus the 3^2 contribution is coming from c_4. This leaves us with

$$\chi_g = c_{36} c_{18} c_4 (x \pm 1)(x - 1) \quad \text{or} \quad c_{36} c_9 c_4 (x \pm 1)(x - 1).$$

Since the principal $c_4(x)$-lattice has determinant 2^2, we have to glue it over 2^2. This determines the characteristic polynomial to be $c_{36} c_{18} c_4 c_2 c_1$ or $c_{36} c_9 c_4 c_2 c_1$. At this point we know $C36, C4 \cong (-6) \oplus (-6), C18/C9 \cong E_6(2)$ and their gluings which exist by Theorem 2.14. Then

$$(q_{C1C2})_3 \cong (q_{E6(2)})_3(-1) \cong (2/3).$$

The case $C1C2 = C1 \oplus C2$ leads to $C2 = (-2)$ which is obstructed or $C2 = (-6)$ which has the wrong 3-glue. Thus we have to glue. Then $C1 \cong (4)$ and $C2 \cong (-12)$ as $C1 \cong (12)$ has wrong 3-glue. This gluing is unique since $(\mathbb{Z}/4\mathbb{Z})^\times = \{\pm 1\}$. Since $(D_{C2})_3$ can be glued to $C18$ but not to $C9$, we have

$$\chi_g = c_{36}c_{18}c_4c_2c_1.$$

The only step at which we have non-trivial freedom in the choice of glue is between $C1C2$ and $C4$. This freedom is due to the action of $\overline{g|_{C4}}$. Thus is does not affect the isomorphism class of $(C1C2C4, g_1 \oplus g_2 \oplus g_4)$ and uniqueness of (NS, g) up to isomorphism follows. □

LEMMA 3.41. *For* $\mathbf{(36, 3^4)}$ *the action of* $g|_{\mathrm{NS}}$ *is uniquely determined by the following gluing diagram.*

$$\begin{tikzcd} \boxed{C36} \arrow[r, dash, "3^4"] & \boxed{C12} \arrow[r, dash, "2^2"] & \boxed{C3} \arrow[r, dash, "3"] & \boxed{C1 \cong U \oplus A_2} \end{tikzcd},$$

PROOF.

- **Claim:** $\operatorname{rk} C12 = 4$
  The $\operatorname{res}(c_{36}, c_4) = 3^2$ is too small. Thus $c_{12} \mid \chi_g$. Suppose $c_{12}^2 \mid \chi_g$. Then
  $$\chi_g = c_{12}^2(x \pm 1)(x - 1).$$
  Hence $C12$ is even of signature $(0, 8)$ and $D_{C12} \cong \mathbb{F}_3^4$. According to Magma there are two classes in this genus (one of them is $4A_2$). Both have roots. At this point we know that $C12$ is a twist of the principal c_{12} lattice.
- **Claim:** $c_3c_6, c_3^2, c_6^2 \nmid \chi_g$
  In this case there is no room for c_4 and $3^4 \mid \det C12$. Now, $\operatorname{res}(c_{12}, c_{36}c_6c_3) = 2^4 3^4$ and $\det C12 \in 3^4, 2^2 3^4, 2^4 3^4$. However, only for $\det C12 = 2^2 3^4$ there is a twist of the right signature, and we compute
  $$(D_{C12})_2 \cong \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$$
  $$\begin{tikzcd} \boxed{\text{C36}} \arrow[r, dash, "3^4"] & \boxed{\text{C12}} \arrow[r, dash, "2^2"] & \boxed{\text{C3C6}} \arrow[r, dash, "3^*"] & \boxed{\text{C1C2}} \end{tikzcd}$$
  Looking at the resultant $\operatorname{res}(c_3c_6, c_{12}c_2c_1) = 3^2 2^2$ and the fact that $\det C6C3$ is a square (Thm 3.36), we see that there are the two possibilities
  $$\det C3C6 \in \{2^2, 2^2 3^2\}.$$
  Either $C3C6 \cong D_4$ which is unique in its genus, or $C3C6 \in II_{(0,4)}(2^{-2}3^2)$. According to Magma this genus consists of two classes containing roots. One of them is $A_2(2) \oplus A_2$.
- **Claim:** $\det C12 = 3^4 2^2$
  The resultant $\operatorname{res}(c_{12}, c_{36}c_{3,6}c_4c_2c_1) = 2^2 3^6$, and the possible determinants of $C12$ are $3^2, 3^4, 3^6, 2^2 3^2, 2^2 3^4, 2^2 3^6$. But 3^2 has roots and for $3^4, 2^2 3^2, 2^2 3^6$ there is no twist of the right signature. This leaves us with $2^2 3^4$ or 3^6. We show 3^6 is not possible. In this case the gluings look as follows.
  $$\begin{tikzcd} \boxed{\text{C36}} \arrow[r, dash, "3^4"] & \boxed{\text{C12}} \arrow[r, dash, "3^2"] & \boxed{\text{C4}} \arrow[r, dash, "2^*"] & \boxed{\text{Rest}} \end{tikzcd}$$

In particular $(D_{C4})_3 \cong \mathbb{F}_3^2$ with a non-degenerate form must be glued to $(g^2+1)D_{C12}$ which is totally isotropic. This is impossible.

- **Claim:** $c_4 \nmid \chi_g$
  Since $\det C12 = 3^4 2^2$, c_3 or c_6 must divide χ_g. This leaves us with an undetermined factor of χ_g of degree 3. Suppose c_4 divides it. Then $C4$ is principal, and by counting resultants we obtain
  $$\det C4 \in \{2^2, 2^2 3^2, 2^2 3^4\}.$$
  But 2^2 leads to $C4 = (-2) \oplus (-2)$ which is obstructed. If $\det C4 = 2^2 3^2$, then it must be glued over 3 to either $C36$ or $C12$. But for both the only possible glue is totally isotropic. We are left with $\det C4 = 2^2 3^4$ and then $C4$ must be glued over 3^2 with both $C36$ and $C12$. The gluing with $C12$ leads to roots, i.e. $C12C4$ is obstructed. We conclude that c_4 cannot divide χ_g.
- **Claim:**
  $$\boxed{\text{C36}} \overset{3^4}{\text{———}} \boxed{\text{C12}} \overset{2^2}{\text{———}} \boxed{\text{C3 or C6}} \overset{3}{\text{———}} \boxed{U \oplus A_2}$$
  Counting resultants yields that the determinant of $C3$ (resp. $C6$) is at most $2^2 3$, while it is at least $2^2 3$ as $C12$ needs a gluing partner. Note that $U \oplus A_2$ is the only lattice of determinant 3 and signature $(1,3)$. A calculation shows that all gluings exist.
- **Claim:** $g|U \oplus A_2 = id$
  From Lemma 3.33, we know that there are only two possibilities for $g|U \oplus A_2$. For the non-identity possibility one computes $C2 = (-6)$. However, the gluing of $C6 \cong A_2(2)$ and (-6) along 3 results in a lattice containing a root. Hence this case is obstructed.

The only case with non-trivial freedom is the gluing of $C12$ and $C3$ along 2^2. However $\mathrm{Aut}(D12, g|_{D12}) \to \mathrm{Aut}((q_{D12})_2, \overline{g})$ is surjective, and hence the gluing is unique.

□

The following are the most complicated cases. The proofs are computer aided.

LEMMA 3.42. *For* $\mathbf{(21, 7^2)}$ *there are* 3 *cases distinguished by their invariant lattice:*

(1) $U \oplus E_6$,
(2) $U \oplus 2A_2$,
(3) $U(3) \oplus A_2$.

PROOF. Since $7^2 = \mathrm{res}(c_{21}, c_3)$, we get that $c_3 \mid \chi_g$.
Claim: $c_7 \nmid \chi_g$.
Suppose it does. Then $\chi_g = c_{21}c_7c_3c_1^2$. The resultant $\mathrm{res}(c_7, c_{21}c_3c_1) = 3^6 7$. But the 3^6 contribution is coming from $C21$. Hence $\det C7 = 7$ and $C7 \cong A_6$ which is a root lattice.
We distinguish cases by $\mathrm{rk}\, C3$. Note that
$$\mathrm{NS} \cong U(7) \oplus E_8 \in II_{(1,9)}(7^{-2}).$$

- $\mathbf{rk}\, \boldsymbol{C3} = \mathbf{2}$:
  Clearly $C3 = A_2(7) \in II_{(0,2)}(3^{-1}7^{-2})$ is root free, and we can take $C1 = U \oplus E_6$ which has simple glue by Theorem 2.4.
  $$\boxed{C21} \overset{7^2}{\text{———}} \boxed{C3} \overset{3}{\text{—}} \boxed{C1 \cong U \oplus E_6}$$

- **rk $C3 = 4$**:
$C3 \in II_{(0,4)}(3^2 7^{-2})$ There are 3 classes in this genus:

$$A_2 \oplus A_2(7), \quad \begin{pmatrix} -6 & 3 \\ 3 & -12 \end{pmatrix} \oplus \begin{pmatrix} -2 & 1 \\ 1 & -4 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 & 1 & 1 \\ 2 & 4 & -1 & 2 \\ 1 & -1 & 8 & 3 \\ 1 & 2 & 3 & 8 \end{pmatrix}$$

Obviously the first two contain roots. The third one does not, and there is an isometry

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

as well. A computation shows that this isometry represents the only conjugacy class of elements with characteristic polynomial c_3^2. Take $C1 = U \oplus A_2 \oplus A_2$ which has simple glue by Theorem 2.4 as well.

- **rk $C3 = 6$**:
  (1) $C3 \in II_{(0,6)}(3^{-1} 7^{-2})$ This genus contains 4 classes all of which contain roots.
  (2) $C3 \in II_{(0,6)}(3^3 7^{-2})$ There are 9 classes in this genus. Two of them without roots. Of these only a single one has an isometry of characteristic polynomial c_3^3. We note that there is only a single conjugacy class.

$$C3 = \begin{pmatrix} 4 & -2 & 1 & 2 & 1 & 0 \\ -2 & 4 & -2 & -1 & -2 & 0 \\ 1 & -2 & 4 & 2 & 2 & -1 \\ 2 & -1 & 2 & 4 & 1 & -2 \\ 1 & -2 & 2 & 1 & 6 & 2 \\ 0 & 0 & -1 & -2 & 2 & 6 \end{pmatrix}, \quad g|C3 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

  We can take $C1 = U(3) \oplus A_2$. We have seen the surjectivity of $O(C1) \to O(q_{C1})$ already in the proof of Lemma 3.15. Hence $C1$ has simple glue and the construction is unique.

- **rk $C3 = 8$**:
In this case $\det C3$ is a square and dividing $3^2 7^2$. Hence there are two possibilities for the genus of $C3$.
  (1) $C3 \in II_{(0,8)}(7^{-2})$ There are three classes in this genus. All of them contain roots.
  (2) $C3 \in II_{(0,8)}(3^{-2} 7^{-2})$ contains the single class $A_2(7) \oplus E_6$ which is obstructed.

□

LEMMA 3.43. *For* $(\mathbf{42}, \mathbf{7^2})$ *there are exactly two actions of* $g|\operatorname{NS}$ *distinguished by*

(1) $C1C2 \cong U \oplus E_6$, $\quad \chi(g|\operatorname{NS}) = c_6 c_2^2 c_1^6$.
(2) $C1C2 \cong U \oplus 2A_2$, $\quad \chi(g|\operatorname{NS}) = c_6 c_3 c_2 c_1^5$.

PROOF. We distinguish along the three cases of $(21, 7^2)$. Recall that $C3C6$ is glued along 7^2 with $C42$. Hence

$$\chi(g|(D_{C3C6})_7) = \chi(g|D_{C42}) = x^2 - x + 1.$$

- **rk $C3C6 = 2$**:
In this case we have the following glue diagram,

$$\boxed{C42} \overset{7^2}{\text{———}} \boxed{C6} \overset{3}{\text{—————}} \boxed{C1C2 \cong U \oplus E_6}.$$

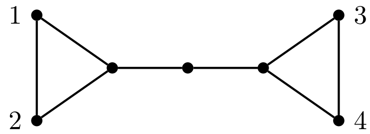
It remains to determine $g|U \oplus E_6$. This is the content of Lemma 3.33. Uniqueness of the glue is evident.

- **rk $C3C6 = 4$**:
There are two conjugacy classes $[f_1]$,$[f_2]$ of $O(C3C6)$ with the right characteristic polynomials:

(1) $f_1 = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $\chi(f_1) = c_3 c_6$;

(2) $f_2 = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$, $\chi(f_2) = c_6^2$.

On the discrimiant form we get
(1) $(q, \overline{f}_1) \cong [(2/3) \oplus (2/3), id \oplus -id]$,
(2) $(q, \overline{f}_2) \cong [(2/3) \oplus (2/3), -id \oplus -id]$.
Here $C1C2 \cong U \oplus 2A_2$ and Lemma 3.33 yields the fundamental root system:



Its symmetry group $O^+(C1C2)/W(C1C2) \cong \mathcal{D}_4 \cong \langle (12), (13)(24) \rangle$ is of order 8. It has 5 conjugacy classes. One of them is of order 4 which is too high. The other ones and their action on q_{C1C2} are represented by
(1) $()$, $[(1/3) \oplus (1/3), id \oplus id]$;
(2) (34), $[(1/3) \oplus (1/3), id \oplus -id]$;
(3) $(12)(34)$, $[(1/3) \oplus (1/3), -id \oplus -id]$;
(4) $(13)(24)$, $[(2/3) \oplus (2/3), id \oplus -id]$.
Comparing the actions, there are two combinations for a gluing.
  - f_2 and $(12)(34)$ is obstructed since new roots appear.
  - f_1 and (34) works.

Since $Aut(q_{C1C2}, \overline{g}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ is generated by the images of $-id|C1C2$ and (34), it has simple glue.

- **rk $C3C6 = 6$**:
In this case there are two possible isometries with the right characteristic polynomials. Namely,
(1) with characteristic polynomial c_6^3, In this case we have the following gluings

$$\boxed{C42} \overset{7^2}{\text{———}} \boxed{C6} \overset{3^3}{\text{———}} \boxed{C2} \overset{2}{\text{———}} \boxed{C1},$$

and there $C2 \in II_{(0,3)}(2_*^{\pm} 3^{-3})$. By the tables in [**21**] there is no such lattice. (Alternatively check [**26**, Chap. 15, (31-35)].)

(2) With characteristic polynomial $c_3 c_6^2$.
   (a) rk $C2 = 2$ Then $\det C2 \mid 3^2 2^2$, and the only possibility is $C2 = (-6) \oplus (-6)$. Then the gluings look as follows

$$\begin{tikzcd}
\boxed{C42} \arrow[r, dash, "7^2"] & \boxed{C6} \arrow[r, dash, "3^2"] \arrow[d, dash, "2^2"] & \boxed{C2} \arrow[d, dash, "2^2"] \\
& \boxed{C3} \arrow[r, dash, "3"] & \boxed{C1}
\end{tikzcd}.$$

Then $C1 = (6) \oplus (-2)$ or $(2) \oplus (-6)$, but only the first one glues along 2 with $C2$ as well. Now, $C3 \cong A_2(2)$ must be glued to (6) along 3. This is impossible.

(b) $\operatorname{rk} C2 = 3$ and $C1 = (6)$. Hence $\det C2 = -2 \cdot 3^2$, and consequently $C2 = A_2 \oplus (-2)$ is obstructed.

□

LEMMA 3.44. *Affine Weierstraß models for $X_{(21,7^2)}$ and the automorphisms of order* 21 *and* 42 *corresponding to the cases (1),(2) in Lemmas 3.42 and 3.42 are given below. For case (3) there is a singular projective model.*

(1) $y^2 = x^3 + t^4(t^7+1)$, $\quad (x,y,t) \mapsto (\zeta_3\zeta_7^6 x, \pm\zeta_7^2 y, \zeta_7 t)$;
(2) $y^2 = x^3 + t^3(t^7+1)$, $\quad (x,y,t) \mapsto (\zeta_3\zeta_7^3 x, \pm\zeta_7 y, \zeta_7 t)$;
(3) $x_0^3x_1 + x_1^3x_2 + x_0x_2^3 - x_0x_3^3$, $\quad (x_0,x_1,x_2,x_3) \mapsto (\zeta_7 x_0, \zeta_7 x_1, x_2, \zeta_3 x_3)$.

PROOF. We identify the three cases by computing the fixed lattice of f^{14}.

(1) There is an E_6 fiber at $t = 0$.
(2) There is a fiber of type I_0^* at $t = 0$ and a fiber of type IV at $t = \infty$. Now g^{14} fixes exactly one isolated point in each fiber and a curve of genus 3. This leads to the fixed lattice $U \oplus A_2 \oplus A_2$ of g^{14}. Here U consists of the zero section and the class of a fiber, one A_2 is in the IV fiber and the other one in the I_0^* fiber, namely the component of multiplicity 2 and the one meeting the zero section.
(3) There are 3 singularities of type A_2 located at the points $(0:0:1:\zeta_3^k)$. The fixed points of g^{14} are a smooth curve of genus 3 at $x_3 = 0$ and the isolated point $(0:0:0:1)$. Hence g^{14} has invariant lattice $U(3) \oplus A_2$. We see that $c_3^3 \mid \chi_{g|\mathrm{NS}}$.

□

REMARK 3.45. In general it is a hard problem to find equations for a K3 surface with a given Hodge structure. In practice equations are found by a mixture of theoretical knowledge, computer algebra, heuristics and intuition. In other words by an 'educated guess'. Below we give some heuristics.

Let (X, g) be a pair of a complex K3 surface and a finite automorphism. We want to find a (possibly singular) birational model of (X, g). By definition $\mathrm{NS}(X)^g$ is primitive. Since g preserves an ample class, this defines a (pseudo-)ample $C1$-polarization on X. Let $D \in \mathrm{NS}(X)^g$ be a (nef) divisor. Then g acts linearly on $H^0(X, D)$ and since it is of finite order (in characteristic 0), we can diagonalize this action. A relatively simple case is when we can choose some $U \subseteq C1 \subseteq L \cong \mathrm{NS}$. This induces an equivariant elliptic fibration with reducible singular fibers given by the roots of $U^\perp \subseteq \mathrm{NS}$. As a next step one can determine the action on the base. Since Weierstraß equations are quite accessible, it is often possible to write down the equation at this point.

If there is no $U \subseteq \mathrm{NS}^g$, we instead choose $D \in \mathrm{NS}$ with $0 < D^2$ small. We may assume D effective, hence pseudo-ample. Since $D^\perp \subseteq \mathrm{NS}$ is negative definite it is easy to compute its roots (and the action of g on them) which correspond to

ADE singularities. At this point further considerations depend on the geometric situation. For example one can compute the fixed locus of g^k and try to specialize known families.

CHAPTER 4

(Non-)symplectic automorphisms of order 5

In [**5**] the authors proved that the moduli space $\mathcal{M}^5_{K3}$ of K3 surfaces admitting a non-symplectic automorphism of order 5 has two irreducible components distinguished by whether the automorphism fixes a curve pointwise or not. In [**36**] A. Garbagnati and A. Sarti showed that the moduli space of complex K3 surfaces admitting both a symplectic and a non-symplectic automorphism of order 5 is zero dimensional. So at most countably many such surfaces may exist. The authors then gave a single example lying in the intersection of the two irreducible components of $\mathcal{M}^5_{K3}$. It is given as the minimal resolution S of the double cover of $\mathbb{P}^2$ branched over the sextic $x_0(x_0^5 + x_1^5 + x_2^5)$. The (non)-symplectic automorphisms are induced by multiplying coordinates with $5th$ roots of unity. In particular the automorphisms commute. One can ask if this example is unique. We give two different answers to this question:

- No, there exist infinitely many non-isomorphic complex K3 surfaces with both a symplectic and non-symplectic automorphism of order 5. The automorphisms generate an infinite subgroup.
- Yes, the pair (X, G) is unique up to isomorphism, where X is a K3 surface and $(\mathbb{Z}/5\mathbb{Z})^2 \cong G \subseteq \operatorname{Aut}(X)$.

The proofs are carried out by reformulating all statements in terms of Hodge structures and lattices.

1. Preliminaries

We want to consider K3 surfaces admitting a symplectic as well as a non-symplectic automorphism of order five. In this section we collect the necessary background material.

Theorem 4.1. [**74**] *Let G be a finite abelian group acting symplecticly on a complex K3 surface X. Then the action of G on the K3-lattice is unique up to isometries of $H^2(X,\mathbb{Z})$. Hence, the isometry class of $\Omega_G := (H^2(X,\mathbb{Z})^G)^\perp$ is determined by G. Conversely Ω_G is primitively embedded in $\operatorname{NS}(X)$ if and only if G acts as a group of symplectic automorphisms on X.*

We need only the following case for our purposes.

Proposition 4.2. [**74**, Prop. 10.1] *If X is a K3 surface with a symplectic automorphism σ of order 5, then the invariant lattice $H^2(X,\mathbb{Z})^\sigma$ is isomorphic to $U \oplus 2U(5)$.*

Theorem 4.3. [**5**] *Let X be a K3 surface with a non-symplectic automorphism τ of order 5 such that τ fixes a curve of genus g and additional k curves of genus 0. Then this data is as in Table 1 and all cases occur. The number of isolated fixed points and their local type is given by n_1, n_2.*

Let $\tau \in O(L_{K3})$ be an isometry of prime order p with hyperbolic invariant lattice $S(\tau) = \ker(\tau - id)$ and $[\tau]$ its conjugacy class. A $[\tau]$-polarized K3 surface is

TABLE 1. Non-symplectic automorphisms of order 5

n_1	n_2	g	k	$(H^2(X,\mathbb{Z})^\sigma)^\perp$	$H^2(X,\mathbb{Z})^\sigma$
1	0	2	0	$H_5 \oplus U \oplus E_8 \oplus E_8$	H_5
3	1	1	0	$H_5 \oplus U \oplus E_8 \oplus A_4$	$H_5 \oplus A_4$
3	1	–	–	$H_5 \oplus U(5) \oplus E_8 \oplus A_4$	$H_5 \oplus A_4^*(5)$
5	2	1	1	$U \oplus H_5 \oplus E_8$	$H_5 \oplus E_8$
5	2	0	0	$U \oplus H_5 \oplus A_4^2$	$H_5 \oplus A_4^2$
7	3	0	1	$U \oplus H_5 \oplus A_4$	$H_5 \oplus A_4 \oplus E_8$
9	4	0	2	$U \oplus H_5$	$H_5 \oplus E_8 \oplus E_8$

a pair (X,ρ) consisting of a K3 surface X and a non-symplectic automorphism ρ such that

$$\rho^*(\omega_X) = \zeta_p \omega_X \quad \rho^* = \phi \circ \tau \circ \phi^{-1}$$

for some marking $\phi : L_{K3} \to H^2(X,\mathbb{Z})$. We say two $[\tau]$-polarized K3 surfaces $(X,\rho),(X',\rho')$ are isomorphic if there is an isomorphism $f : X \to X'$ with $f^{-1} \circ \rho' \circ f = \rho$. As usual set $T(\tau) = S(\tau)^\perp$ and let

$$V^\tau := \{x \in L_{K3} \otimes \mathbb{C} \mid \tau(x) = \zeta_p x\} \subseteq T(\tau) \otimes \mathbb{C}$$

be a complex eigenspace of τ. We set

$$D^\tau = \{\omega \in \mathbb{P}(V^\tau) \mid (\omega,\omega) = 0, (\omega,\overline{\omega}) > 0\}$$

and

$$\Delta^\tau = \bigcup_{d \in T(\tau), d^2=-2} D^\tau \cap d^\perp.$$

With $\Gamma^\tau = \{\gamma \in O(L_{K3}) \mid \gamma \circ \tau = \tau \circ \gamma\}$ we get the following theorem.

THEOREM 4.4. **[5]** *The orbit space $\mathcal{M}^\tau := \Gamma^\tau \setminus (D^\tau \setminus \Delta^\tau)$ parametrizes isomorphism classes of $[\tau]$-polarized K3 surfaces. Two pairs $(X,\rho),(X',\rho')$ of K3 surfaces with non-symplectic automorphism of prime order are polarized by the same $\rho \in O(L_{K3})$ if and only if $S(\rho) \cong S(\rho')$.*

2. Simultaneous symplectic and non-symplectic actions of order 5

We saw that K3 surfaces with a non-symplectic automorphism τ and transcendental lattice T of rank 4 are determined by the $c_5(x)$-isometry class of (T,τ). In this section we ask which $c_5(x)$-lattices arise in this fashion from K3 surfaces and which of them admit a symplectic automorphism of order 5. Similar methods have recently been applied in **[15]**. There the authors prove the existence of a non-symplectic automorphism of order 23 on a holomorphic symplectic manifold deformation equivalent to the Hilbert scheme of two points on a K3 surface.

LEMMA 4.5. *A complex K3 surface X admits a symplectic automorphism of order 5 if and only if the transcendental lattice embeds primitively into*

$$T \hookrightarrow U \oplus 2U(5).$$

PROOF. Proposition 4.2 provides the only if part. Now assume that

$$T \hookrightarrow U \oplus 2U(5).$$

Since T has signature $(2, 22-\rho)$, this implies $l(D_T) \leq \operatorname{rk} T \leq 5$. By Theorem 2.3 $T \hookrightarrow L_{K3}$ is unique and so is the orthogonal complement NS of T. It contains $\Omega_{\mathbb{Z}/5\mathbb{Z}}$. Now apply Theorem 4.1. □

3. The commutative case

PROPOSITION 4.6. *The pair (S, G) where S is a complex K3 surface and $G = (\mathbb{Z}/5\mathbb{Z})^2$ is unique up to isomorphism. It is given as the minimal resolution of the double cover of the projective plane branched over the sextic curve given by*

$$x_0(x_0^5 + x_1^5 + x_2^5)$$

with diagonal G action. The transcendental and Néron-Severi lattice of S are isometric to

$$\mathrm{NS} \cong H_5 \oplus 2A_4 \oplus E_8, \quad T \cong H_5 \oplus U(5).$$

PROOF. Let X be a (complex) K3 surface with a faithful $G = (\mathbb{Z}/5\mathbb{Z})^2$ action. Then G/G_s is cyclic [**74**, Thm 3.1] where $G_s = \ker G \to O(H^{2,0}(X))$. Since G is not symplectic by the classification of (abelian) symplectic actions in [**74**, Thm. 4.5], $G_s \neq G$. This leaves us with $G_s \cong \mathbb{Z}/5\mathbb{Z}$. Then $G = \langle \sigma, \tau \rangle$ with a symplectic automorphism σ and τ a non-symplectic automorphism such that $\sigma \circ \tau = \tau \circ \sigma$. First we show that T has rank 4 and the action of τ on $T^\vee/T$ is of the form $\mathcal{O}_K/(x-1)^3$. Then Theorem 3.17 provides the uniqueness of X.

The transcendental lattice T embeds in the invariant lattice of σ,

$$i : T \hookrightarrow U \oplus 2U(5) \cong H^2(X, \mathbb{Z})^\sigma.$$

In particular $\operatorname{rk} T \leq 5$. As X admits the non-symplectic automorphism τ of order 5. Thus $\operatorname{rk} T \equiv 0 \mod 4$. This leaves us with $\operatorname{rk} T = 4$, i.e. the Picard number of X is 18. Since τ and σ commute, τ acts on both sides of the embedding i. This observation is our starting point. Denote by $R := i(T)^\perp$ then

$$T \oplus R \hookrightarrow U \oplus 2U(5)$$

is a primitive extension compatible with the action of τ. It corresponds to a glue map ϕ

$$D_T \supseteq G_T \xrightarrow[\phi]{\sim} G_R \subseteq D_R$$

with $\phi \circ \tau = \tau \circ \phi$ such that $q_T(x) = -q_R(\phi(x))$. By definition τ acts with order 5 on T. The pair $(T, \tau|_T)$ is a $c_5(x)$-lattice. Since $c_5(x)$ is a simple reciprocal polynomial, this pair is isomorphic to a twist $L_0(a)$ of the principal lattice

$$(T, \tau) \cong (L_0(a), f).$$

Our next goal is to determine the prime factorization of a. The rank of R is too small for an action of order 5, so there τ restricts to the identity. The resultant equals $\operatorname{res}(x^4 + x^3 + x^2 + x + 1, x - 1) = c_5(1) = 5$. By Theorem 2.13 this forces $G_T \cong G_R$ to be 5-groups, i.e., gluing can occur only over 5. In view of $U \oplus 2U(5)$ being 5-elementary we know that D_T and D_R are 5-groups as well. Since D_T is a 5-group, it suffices to consider twists above 5. There is only a single prime ideal $(x-1)\mathcal{O}_K$ over 5. Hence, up to units we may only twist by associates t of $(x-1)^2$. Recall from Lemma 2.20 that

$$D_{L_0(t^k)} \cong \mathcal{O}_K/(x-1)^{2k+1}.$$

Since the action of τ preserves G_T, the glue G_T is actually isomorphic to an ideal in $\mathcal{O}_K/(x-1)^{2k+1}\mathcal{O}_K$. These are of the form $(x-1)^h\mathcal{O}_K/(x-1)^{2k+1}$.

$$G_T \cong \mathcal{O}_K/(x-1)^{2k+1-h}\mathcal{O}_K \tag{3}$$

$$D_T/G_T \cong \mathcal{O}_K/(x-1)^h\mathcal{O}_K \tag{4}$$

The action of τ on $G_T \cong G_R$ is the identity. It is given by multiplying by x. This means that

$$x \equiv 1 \mod (x-1)^{2k+1-h}$$

which is the case for $2k+1-h \in \{0,1\}$. Since the gluing results in a 5-elementary lattice, D_T/G_T (and D_R/G_R) are $\mathbb{F}_5$-vector spaces. We get $h \leq 4$ and then $k \in \{0,1,2\}$. We want to show $k=1$.
First suppose $k=0$, then D_T has length 1 and D_R at most length 2. However, $D_{U\oplus 2U(5)} \cong \mathbb{F}_5^4$ has length 4 and is a sub-quotient of $D_T \oplus D_R$. This is impossible. Now suppose $k=2$, then $D_{L_0(t^2)} \cong \mathbb{F}_5^3 \oplus \mathbb{Z}/25\mathbb{Z}$ which is not a vector space. Hence, we have to glue, that is, $h=4$ and $G_T = 5D_T \cong \mathbb{F}_5$. Solving

$$|D_T/G_T||D_R/G_R| = |D_{U\oplus 2U(5)}| = 5^4 \tag{5}$$

we get $|D_R| = 5$. Thus we arrive at a glue map $5D_T = G_T \cong G_R = D_R$, but the discriminant form on G_T is 0 while on D_R it is non-degenerate - a contradiction. We are left with $k=1$ and $D_T \cong \mathcal{O}_K/(x-1)^3 \cong \mathbb{F}_5^3$.

The uniqueness of a symplectic action is well known, and we have computed the conjugacy class of $\tau|(H^2(X,\mathbb{Z})^\sigma$. Set $\Omega = \left(H^2(X,\mathbb{Z})^\sigma\right)^\perp$ and recall $\Omega \cong \Omega_{\mathbb{Z}/5\mathbb{Z}}$. From the gluing we know (the conjugacy class of) $\overline{\tau} \in O(q_\Omega)$. Hence the action of $\tau|\Omega$ is unique up to the kernel of $O(\Omega) \to O(q_\Omega)$ which is generated by σ. Finally, the few extension property of $(T, \tau|T)$ provides us with the uniqueness of τ (up to multiplication by σ) and hence G. □

In order to extend this result to positive characteristic we recall some results and definitions concerning the lifting of an automorphism. For details, we refer to [**46**]. Let k be an algebraically closed field of positive characteristic p and X/k a K3 surface. We get the canonical surjection

$$\pi : H^2_{cris}(X/W) \to H^2_{cris}(X/W)/pH^2_{cris}(X/W) \cong H^2_{dR}(X/k).$$

Attached to $H^2_{dR}(X/k)$ is the Hodge filtration $F^iH^2_{dR}(X/k), 0 \leq i \leq 3$ where $F^2H^2_{dR}(X/k) \cong H^0(X,\Omega^2_X)$. Any isotropic line $M \subseteq H^2_{cris}(X/W)$ with $\pi(M) = F^2H^2_{dR}(X/k)$ corresponds to a formal lift $\mathcal{X}$ of X/k. It is algebraic if and only if there is an ample line bundle L with $c_1(L) \in M^\perp$. An automorphism of X/k lifts to $\mathcal{X}$ (and its algebraization) iff it preserves M.

LEMMA 4.7. *Let k be an algebraically closed field of positive characteristic $p \neq 2$. Let X/k be a K3 surface, σ a symplectic and τ a non-symplectic automorphism of order five. If σ and τ commute, then the triple (X,σ,τ) lifts to characteristic zero.*

PROOF. Since there is no non-symplectic automorphism of order 5 in characteristic 5, we can assume $p \neq 5$. By the discussion above we have to find an isotropic rank one submodule $M \subseteq H^2_{cris}(X/W)$ and an ample line bundle L with $c_1(L) \in M^\perp$ such that $\pi(M) = F^2H^2_{dR}(X,k)$ and $\sigma(M) = M, \tau(M) = M$. We imitate the reasoning of [**46**, 3.7].

Assume that X is of finite height. As in (the proof of) [**46**, 3.7] we can lift (X,τ) with M inside $H^2(X/W)_{[1+1/h]} \subseteq T_{cris} = \mathrm{NS}^\perp \subseteq H^2_{cris}(X/W)$. Since σ is symplectic, it acts as identity on T_{cris} (cf. [**46**, 3.5]). Hence, it trivially preserves M and σ lifts together with (X,τ) (to an algebraic K3 surface).

Now assume that X is supersingular. Set $H := H^2_{cris}(X/W)$ and let $\zeta \in W$ be a $5th$ root of unity such that τ acts on $F^2H^2_{dR}(X,k) \cong H^0(X,\Omega^2_X)$ by $\overline{\zeta} = \zeta + pW$.

Recall that $p \neq 5$ and hence, by Hensel's lemma, $t^5 - 1 \in W[t]$ splits. In particular the action of σ and τ on H is semisimple. The simultaneous eigenspaces

$$E_{k,l} = \ker(\sigma - \zeta^k I) \cap \ker(\tau - \zeta^l I), \quad 0 \leq k, l \leq 4$$

induce decompositions

$$H \cong \bigoplus_{k,l=1}^{22} E_{k,l} \quad \text{and} \quad H/pH \cong \bigoplus_{k,l=1}^{22} E_{k,l}/pE_{k,l}.$$

Note that $E_{k,l}/pE_{k,l}$ is the simultaneous eigenspace corresponding to $\overline{\zeta}^k$ and $\overline{\zeta}^l$. Take any $0 \neq v \in F^2 H^2_{dR}(X,k) \subseteq E_{0,1}/pE_{0,1}$ and lift it to some $m \in E_{0,1}$. Set $M = Wm$. By construction $\pi(M) = F^2 H^2_{dR}(X,k)$ and M is preserved by both σ and τ. Any eigenvector of τ with eigenvalue ζ is isotropic ($\zeta^2 \neq 1$). We can find a τ^* invariant ample line bundle. It is orthogonal to M. This completes the proof by showing that the lift induced by M is algebraic. $\square$

By lifting the triple (X, σ, τ) we can reduce to the complex case and get the following proposition.

Proposition 4.8. *Let k be an algebraically closed field of odd or zero characteristic not 5. There is a unique K3 surface S/k admitting a commuting pair of a symplectic and a non-symplectic automorphism of order 5. In characteristic five no such surface exists.*

Example 4.9. [**30**, §6] The general unitary group $GU(3, \mathbb{F}_{5^2})$ acts on the Hermitian form $x_0^6 + x_1^6 + x_2^6$ over $\mathbb{F}_{5^2}$. Then $\mathrm{PSU}(3, \mathbb{F}_{5^2})$ acts symplectically on the double cover of $\mathbb{P}^2$ branched over $x_0^6 + x_1^6 + x_2^6 = 0$. It is a supersingular K3 surface of Artin invariant $\sigma = 1$. We note that $(\mathbb{Z}/5\mathbb{Z})^2 \hookrightarrow \mathrm{PSU}(3, \mathbb{F}_{5^2})$.

4. The non-commutative case

In this section we will prove the existence of an infinite number of K3 surfaces with a non-symplectic and a symplectic automorphism of order 5. We present them as a sequence of $[\rho]$-polarized K3 surfaces. So essentially we are constructing their Hodge structures. Then surjectivity of the period map and the Torelli-Theorem provide their existence.

Proposition 4.10. *Let $X/\mathbb{C}$ be a K3 surface with a non-symplectic automorphism τ and a symplectic automorphism σ both of order 5. Then*

(1) there is a primitive embedding $i : T \hookrightarrow U \oplus 2U(5)$,

(2) $(T, \tau) \hookrightarrow (N, f)$ where N is as in Table 1, f is of order 5, its characteristic polynomial is a perfect power of $c_5(x)$ and f acts as identity on $N^\vee/N$,

(3) the orthogonal complement C of T in N does not contain any roots.

Conversely these conditions are sufficient for a $c_5(x)$-lattice to arise as (T, τ) from a K3 surface.

Proof. The first condition is Lemma 4.5. To see the necessity of the second condition note that $N = (H^2(X,\mathbb{Z})^\tau)^\perp =: T(\tau)$ and $S(\tau) := H^2(X,\mathbb{Z})^\tau$. Since the K3-lattice is unimodular, we get an isomorphism $T(\tau)^\vee/T(\tau) \cong S(\tau)^\vee/S(\tau)$ compatible with the action of τ on both sides. Since τ is the identity on the right side, it is the identity on the left side as well. For the third condition, note that the orthogonal complement of T in $T(\tau)$ lies in NS. It can be shown with Riemann-Roch, that if $x \in \mathrm{NS}$ is a root, then x or $-x$ is effective. Suppose x is. Let h be an ample class. If $x \in N \cap \mathrm{NS}$, then $0 < h.(x + \tau(x) + \tau^2(x) + \tau^3(x) + \tau^4(x)) = h.0 = 0$. Thus these roots are an obstruction for τ to preserve the effective cone in NS.

Let us turn to the sufficiency: By (2) we can extend τ to an isometry f of $N =: T(f)$ which we can then glue to the identity on the matching $S(f)$ to obtain an isometry f on the K3-lattice. We realize X as an $[f]$-polarized K3 surface. After replacing f by f^n and τ by τ^n, we can assume that $(\omega, \overline{\omega}) > 0$ for any non-zero $\omega \in \eta := \ker(\tau - \zeta_5 \operatorname{id}) \subseteq T \otimes \mathbb{C}$. This eigenspace η is our candidate period in N^f. Once we show that $\eta \notin \Delta^f$ (as defined in Sect. 2, we can apply Theorem 4.4. Assume $\exists d \in T(f)$ with $d^2 = -2$ and $(d, \omega) = 0$. Then $d \in \eta^\perp \cap T(f) = C$, but C has no roots. Hence, such d do not exist. We get the existence of an $[f]$-polarized K3 surface X with period η.

In particular X has a non-symplectic automorphism of order 5 and transcendental lattice isometric to T. Then (1) and Lemma 4.5 imply that X has a symplectic automorphism of order 5 as well.

□

We start our search for different K3 surfaces with both a symplectic and a non-symplectic automorphism of order 5 by analyzing (1). Set $C = i(T)^\perp$. Then $l((D_T)_p) = l((D_C)_p) \leq 2$ for $p \neq 5$. Hence, we cannot twist by inert primes - these result in length 4. We can only twist by the prime above 5 or by primes above $p \equiv 1, 4 \mod 5$.

Lemma 4.11. *Let $r_1, ..., r_n$ be primes in $\mathcal{O}_k$ above the distinct primes $p_1, ..., p_n \equiv 1 \mod 5$ and s be the prime over 5. Then for $r = \prod_i r_i$*

$$L_0(sr) \hookrightarrow U \oplus 2U(5)$$

primitively given that $L_0(sr)$ has signature $(2, 2)$.

Proof. A different perspective to primitive embeddings is a primitive extension. We glue $L_0(sr)$ and $H_5(\prod_i p_i)$ to obtain $U \oplus 2U(5)$. Since $p_i \equiv 1 \mod 5$, the prime $r_i \in \mathcal{O}_k$ splits in $\mathcal{O}_K$ as $r_i = r_{i1} r_{i2}$.

$$(D_{L_0(sr)})_{p_i} \cong \mathcal{O}_K / r_i \cong \mathcal{O}_K / r_{i1} \times \mathcal{O}_K / r_{i2} \cong \mathbb{F}_{p_i}^2$$

And the form on $(D_{L_0(sr)})_{p_i}$ in a basis of eigenvectors can be normalized to

$$q_{p_i} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In particular q_{p_i} has determinant $-1 \in \mathbb{F}_{p_i}^\times / \mathbb{F}_{p_i}^{\times 2}$. Since the dimension is even, $q_{p_i} \cong q_{p_i}(-1)$. So for a glue map to exist it is enough to show that the discriminant form on $(D_{H_5(p)})_{p_i}$ is isomorphic to q_{p_i}. It can be computed directly:

$$\det 5 \prod_i p_i \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = -5^3 \prod_i p_i^2.$$

Its square class is given by the Legendre symbol

$$\left(\frac{-5^3}{p_i}\right) = \left(\frac{-1}{p_i}\right) \left(\frac{p_i}{5}\right) = \left(\frac{-1}{p_i}\right)$$

as desired. □

So for condition (1) we have a nice list of examples. It remains to check conditions (2) and (3). Set $T := L_0(sr)$. We are searching for a gluing of $c_5(x)$-lattices

$$(T, f_T) \oplus (C, f_C) \hookrightarrow (N, f)$$

where N is a $(x-1)$-elementary c_5-lattice as in Table 1. As a first try we can take $N \cong U \oplus H_5$. Then $C = 0$ and $T \cong N$ which is not the case. As a second try take $N \cong U \oplus H_5 \oplus A_4$. We will see that it does not work and develop along the way the methods to handle the third try successfully.

Now C is of rank 4. So by Theorem 2.18 it is a twist of the principal $c_5(x)$-lattice as well. Since N is 5-elementary, the $p \neq 5$-parts of the discriminant groups of T and C are isomorphic. For the 5-glue, we use that N is $(x-1)$ elementary. In particular $(D_T/G_T)_5 \cong \mathcal{O}_K/(x-1)$ which implies that $(D_C/G_C)_5 \cong \mathcal{O}_K/(x-1)$ as well. We end up with a primitive extension

$$L_0(sr) \oplus L_0(\epsilon sr) \hookrightarrow U \oplus H_5 \oplus A_4$$

where $\epsilon \in \mathcal{O}_k^\times$ is chosen such that $C := L_0(\epsilon sr)$ has signature $(0,4)$. Since we have to glue over 5, we need some more knowledge of how to glue $c_5(x)$-lattices.

Lemma 4.12. *Let q be a non-degenerate quadratic form on $\mathbb{F}_5[X]/(X-1)^3$ where multiplication by X is an isometry. Then q can be normalized as follows:*

$$\begin{pmatrix} 0 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 2 \\ 1 & -2 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

In the first case $\det q$ *is a square and the second case not. In any case $a \in \mathbb{F}_5$ and the basis is given by $u \cdot 1, u \cdot (X-1), u \cdot (X-1)^2$ for some unit $u \in \mathbb{F}_5^\times$.*

Proof. We start in the basis $1, X, X^2$ of $\mathbb{F}_5[X]/(X-1)^3$. By invariance under multiplication by X the Gram-matrix of q is of the form

$$\begin{pmatrix} a & b & 4b-3a \\ b & a & b \\ 4b-3a & b & a \end{pmatrix}.$$

It has determinant $8(b-a)^3$. We can change the basis to $1, X-1, (X-1)^2$. In this basis the Gram-matrix is given by

$$\begin{pmatrix} a & b-a & 2(b-a) \\ b-a & -2(b-a) & 0 \\ 2(b-a) & 0 & 0 \end{pmatrix}$$

After multiplying the basis by an element $u \in \mathbb{F}_5^\times$ we can assume that $(b-a) \in \{1,2\}$. Finally, by replacing 1 by $1 + u(x-1)^2$ for some $u \in \mathbb{F}_5$, we get $a = 0$. □

Lemma 4.13. *Let q_1 and q_2 be isomorphic quadratic forms over $\mathcal{O}_K/(x-1)^3$ invariant under multiplication by x. Let $G_1 = G_2 = (x-1)\mathcal{O}_K/(x-1)^3$. Then we can find an $\mathcal{O}_K$-module isomorphism $\phi : G_1 \to G_2$ with $q_1(x) = -q_2(\phi(x))$ and graph Γ such that $\Gamma^\perp/\Gamma \cong (\mathcal{O}_K/(x-1))^2$.*
For $\tilde{G}_i = (x-1)^2\mathcal{O}_K/(x-1)^3$, we can find a glue map with

$$\Gamma^\perp/\Gamma \cong \mathcal{O}_K/(x-1)^3 \oplus \mathcal{O}_K/(x-1).$$

This sum can be chosen orthogonal. The square class of the $\mathcal{O}_K/(x-1)^3$-part is independent of choices and different from that of q_1.

Proof. Assume $\det q_1$ a square mod 5. First we normalize the forms as in Lemma 4.12 (recall that -1 is a square mod 5). That is, we can find
$v \in \mathbb{F}_5[X]/(x-1)^3$ such that the q_i are given by the following matrices

$$q_1 = \begin{pmatrix} 0 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad q_2 = \begin{pmatrix} 0 & -2 & 1 \\ -2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

in the bases $e_i = (x-1)^i v$ for q_1 and $b_i = 2(x-1)^i v$, for q_2, $i \in \{0,1,2\}$.

Then we define ϕ by $\phi(e_1) = b_1$ and $\phi(e_2) = b_2$. By construction this reverses the signs and is an $\mathcal{O}_K$-module isomorphism. It remains to compute $\Gamma^\perp/\Gamma$ where

$$\Gamma = \langle e_1 + b_1, e_2 + b_2 \rangle.$$

Thus

$$\Gamma^{\perp} = \langle e_2, b_2, e_0 + b_0, e_1 + b_1 \rangle.$$

By definition multiplication by $(x-1)$ raises the index of the e_i, b_i by one. Hence, we get the desired module structure of $\Gamma^{\perp}/\Gamma$. For $\tilde{G}_i$, take $\phi(e_2) = b_2$ and do the computation. The proofs are the same for $\det q_i$ a non-square. □

Let us return to the hoped for primitive extension

$$L_0(sr) \oplus L_0(\epsilon sr) \hookrightarrow U \oplus H_5 \oplus A_4.$$

In order to glue, the discriminant forms of $L_0(sr)$ and $L_0(\epsilon sr)$ must be isomorphic. By the oddity formula

$$\text{signature } (L) + \sum_{p \geq 3} p\text{-excess } (L) \equiv \text{ oddity } (L) \mod 8.$$

In the proof of Lemma 4.11, we saw that the $p \neq 5$ parts of both discriminant forms are equal. So if we subtract the oddities of both forms we end up with

$$4 + 5\text{-excess}(L_0(sr)) - 5\text{-excess}(L_0(\epsilon sr)) \equiv 0 \mod 8.$$

In particular the discriminant forms cannot be isomorphic as their 5-excess differs.

Our next try, compatible with the oddity formula, is

$$L_0(sr) \oplus C \hookrightarrow U \oplus H_5 \oplus A_4 \oplus A_4.$$

and indeed this turns out to work.

Lemma 4.14. *There is a negative definite, root-free c_5-lattice C with discriminant group isomorphic to $\mathcal{O}_K/r \times \mathcal{O}_K/(x-1)^3 \times \mathcal{O}_K/(x-1)$. Such that the determinant of the discriminant form on $\mathcal{O}_K/(x-1)^3$ is a non-square.*

Proof. We take C as a primitive extension of $L_0(sr) \oplus L_0(s)$ where the glue over the 5−part is isomorphic to $\mathcal{O}_K/(x-1)$. For this, we have to check that both sides have isomorphic forms on the 5 part. To do this we use the oddity formula. Recall that

$$p_i\text{-excess}(L_0(sr)) = 2(p_i - 1) + 4k_{p_i}$$

In the proof of Lemma 4.11 we have seen that $\det q_{p_i} = -1$. Write $p_i = 4k + r$, $0 \leq r < 4$. If $p_i \equiv 1 \mod 4$, the determinant -1 of q_{p_i} is a square mod p_i. Thus k_{p_i} vanishes and

$$p_i\text{-excess}(L_0(sr)) \equiv 2(4k + 0) \cdot 0 \equiv 0 \mod 8.$$

For $p_i \equiv 3 \mod 4$, we get

$$p_i\text{-excess}(L_0(sr)) \equiv 2(4k + 2) + 4 \cdot 1 \equiv 0 \mod 8.$$

Both lattices are negative definite and the $p \neq 5$-excess and oddity vanish for both forms. From the oddity formula

$$0 \equiv 4 + 5\text{-excess} \equiv 4 + 3 \cdot (5-1) + 4k_5 \mod 8$$

we can see $k_5 = 0$ for both forms. We conclude that the determinant of each discriminant form over 5 is a square. This can be confirmed by a direct computation for $L_0(s)$. Now we may apply Lemma 4.13 to get the gluing and the condition on the determinants right.

It remains to check that C is root-free. First we remark that there are embeddings

$$L_0(sr) \hookrightarrow L_0(s) \hookrightarrow L_0 = A_4.$$

Suppose the sublattice $L_0(s)$ contains a root x. Then $(x, f(x), f^2(x), f^3(x))$ is a basis of $L_0(sr)$ consisting of roots, and hence $L_0(sr) = A_4$. This is impossible for determinant reasons. Secondly, notice that we glue over an isotropic subspace. This implies that $C \hookrightarrow H_1 \oplus H_2$ for some even lattices H_i. Then any point of $h \in C$ which is not in $L_0(sr) \oplus L_0(s)$ can be written as $h = h_1 + h_2$ with $0 \neq h_i \in H_i$. In particular $h^2 = h_1^2 + h_2^2 \leq -2 - 2$. □

THEOREM 4.15. *There exists an infinite series of K3 surfaces admitting a symplectic and a non-symplectic automorphism of order* 5. *Their transcendental lattices are given as follows:*

let $r_1, ..., r_n$ *be primes in* $\mathcal{O}_k$ *over the distinct primes* $p_1, ...p_n \equiv 1 \mod 5$. *Let* $s \in \mathcal{O}_k$ *be the prime over* 5. *Then for* $r = \prod_i r_i$,

$$T = L_0(sr).$$

PROOF. We have to check the conditions of Proposition 4.10. (1) is Lemma 4.11. (2) We claim that there is a primitive extension of c_5-lattices

$$L_0(sr) \oplus C \hookrightarrow U \oplus H_5 \oplus A_4 \oplus A_4 = T(\tau)$$

such that $T(\tau)^\vee / T(\tau) \cong (\mathcal{O}_K/(x-1))^3$. For this, take the C from the previous Lemma 4.14 which satisfies (3). The $p \neq 5$ part glues automatically by Theorem 2.14. It remains to check the 5-part of the construction. This is provided by Lemmas 4.13 and 4.14. □

REMARK 4.16. A similar construction with slightly different gluings also works for $p \equiv 4 \mod 5$.

CHAPTER 5

Supersingular K3 surfaces

In this section we recall the basic facts about supersingular K3 surfaces that will be used all along.

A *singular* K3 surface X over $\mathbb{C}$ is one whose Picard number

$$\rho(X) = 20 = h^{1,1}(X),$$

which is the maximal possible. Here 'singular' is meant in the sense of exceptional rather than non-smooth. Now, let X be a K3 surface defined over an algebraically closed field κ of characteristic p. Its Picard number is bounded only by the second Betti number $b_2(X) = 22$.

K3 surfaces reaching the maximum possible Picard number

$$\rho(X) = \operatorname{rk} \mathrm{NS}(X) = 22 = b_2(X)$$

are called *(Shioda) supersingular.*

Remark 5.1. Artin introduced in [**6**] a different notion of supersingularity. Namely, a K3 surface X is (Artin) supersingular if its Brauer group has infinite height, or equivalently, if the second crystalline cohomology is purely of slope 1. Due to the Igusa-Artin-Mazur inequality for varieties of finite height [**7**], any Shioda supersingular K3 is also Artin supersingular. The converse follows from the Tate conjecture (even if the surface is not defined over a finite field, see for example [**55**, Theorem 4.8]). The Tate conjecture is known for K3 surfaces defined over finite fields of odd characteristic [**78, 79, 25, 57, 59**] and has recently been announced also for $p = 2$ [**48**]. Therefore, both definitions of supersingularity are equivalent, and from now on we will thus refer to any such K3 surface simply as "supersingular".

Supersingular K3 surfaces are classified according to their *Artin invariant* σ defined by $\det \mathrm{NS}(X) = -p^{2\sigma}, \sigma \in \{1, ..., 10\}$ [**6**]. By work of Ogus [**83**], there is a unique K3 surface of Artin invariant $\sigma = 1$, over $k = \overline{k}$ of characteristic p. We shall denote it by $X(p)$.

Supersingular K3 surfaces arise from singular K3 surfaces as follows:

Proposition 5.2. [**89**, 4.1] *Let X be a singular K3 surface defined over a number field L and $d = \det \mathrm{NS}(X)$. If $\mathfrak{p}$ is a prime of good reduction above $p \in \mathbb{N}$, then $X_p := X \times \operatorname{Spec} \overline{\mathbb{F}}_p$ is supersingular if p is inert in $\mathbb{Q}(\sqrt{d})$.*

As noticed by Shimada [**90**], if $\det \mathrm{NS}(X)$ is coprime to p, then the Artin invariant is $\sigma = 1$. The reason for this is that $NS(X) \hookrightarrow \mathrm{NS}(X_p)$ implies $\sigma = 1$.

1. Torelli theorems for supersingular K3 surfaces

In this section we introduce some versions of the Torelli theorems proved by Ogus in [**83**]. Though crystalline cohomology plays a central role in the development and proof of these results (and even in some statements), we avoid it in order to lighten the exposition, using only the Néron-Severi lattice. The interested reader is referred to [**55, 82, 83**] for the details.

Recall that a *supersingular K3 lattice* is an even lattice N of rank 22, signature $(1,21)$ such that the discriminant group $D_N \cong \mathbb{F}_p^{2\sigma}$, $p>2$, $\sigma \in \{1,\dots,10\}$. For $p=2$, we impose the extra condition $q_N(x) \equiv 0 \mod \mathbb{Z}$ for all $x \in D_N$. These conditions determine N up to isometry.

The Néron-Severi lattice $\mathrm{NS}(X)$ of a supersingular K3 surface X is a supersingular K3 lattice for $p = \operatorname{char} k$ and $1 \le \sigma \le 10$ (cf. [**85**, sect. 8]). We call σ the *Artin invariant* of X.

Recall that, the induced bilinear form on $D_{\mathrm{NS}(X)}$ takes values in

$$\left(\frac{1}{p}\mathbb{Z}\right)/\mathbb{Z} \cong \mathbb{F}_p$$

and is *non-neutral* that is, there is no totally isotropic subspace $P \subset D_{\mathrm{NS}(X)}$ of dimension $\sigma = \frac{1}{2}\dim_{\mathbb{F}_p} D_{\mathrm{NS}(X)}$. To see this, note that neutrality would imply the existence of an even, unimodular overlattice of signature $(1,21)$. It is well known that such a lattice does not exist.

A positive characteristic analogue of a Hodge structure is a *crystal*, associated to the crystalline cohomology groups. On a supersingular K3 surface X such a crystal is determined by the kernel P'_X of the first deRham-Chern class map

$$c^1_{dR} \otimes \kappa : \mathrm{NS}(X) \otimes \kappa \to H^2_{dR}(X,\kappa).$$

Since κ has characteristic p, we have

$$\mathrm{NS}(X) \otimes \kappa \cong (\mathrm{NS}(X)/p\,\mathrm{NS}(X)) \otimes \kappa,$$

and indeed P'_X is contained in the subspace $(p\,\mathrm{NS}(X)^\vee/p\,\mathrm{NS}(X))\otimes\kappa$ which is clearly isomorphic to

$$(\mathrm{NS}(X)^\vee/\,\mathrm{NS}(X)) \otimes \kappa = D_{\mathrm{NS}(X)} \otimes \kappa.$$

Furthermore $P'_X \subseteq D_{\mathrm{NS}(X)} \otimes \kappa$ is a "strictly characteristic subspace", which in general is defined as follows:

Definition 5.3. [**82**, Definition 3.19] Let D be a 2σ dimensional $\mathbb{F}_p$-vector space equipped with a non-degenerate, non-neutral, symmetric bilinear form. Let $\mathrm{Fr}_\kappa : \kappa \to \kappa$ be the Frobenius automorphism of κ, and set

$$\psi = \mathrm{id}_D \otimes \mathrm{Fr}_\kappa : D \otimes \kappa \longrightarrow D \otimes \kappa.$$

The subspace $P \subseteq D \otimes \kappa$ is called *characteristic* if

(1) $\dim_\kappa P = \sigma$;
(2) $\dim_\kappa (P + \psi(P)) = \sigma + 1$;
(3) P is totally isotropic.

If moreover

$$\sum_{i \ge 0} \psi^i(P) = D \otimes \kappa,$$

then P is called *strictly* characteristic.

Note that P is (strictly) characteristic iff $\psi(P)$ is. We call $P_X = \psi^{-1}(P'_X)$ the *period* of X. The next Theorem shows that every strictly characteristic subspace occurs as the period of some K3 surface.

Remark 5.4. In the theorem below, $\bar\iota : D_N \otimes \kappa \cong D_{\mathrm{NS}(X)} \otimes \kappa$ denotes the isomorphism induced by ι. Since this situation will appear repeatedly, we fix the following notation: if $f : N \to M$ is an isometry of lattices, we denote by $\overline{f} : D_N \to D_M$ the induced group isomorphism (or its κ-linear extension).

THEOREM 5.5 (Surjectivity of the period map [**83**]). *Given any supersingular K3 lattice N and a strictly characteristic subspace $P \subset D_N \otimes \kappa$, then there is a K3 surface X and an isometry $N \overset{\iota}{\cong} \mathrm{NS}(X)$, such that $\bar{\iota}(P) = \psi{-}1(\ker c^1_{dR})$.*

In order to formulate a strong Torelli theorem, we need to consider the chamber structure of the positive cone in $\mathrm{NS}(X) \otimes \mathbb{R}$, which is analogous to that in characteristic 0. Let L be an even lattice of signature $(1, n)$, denote by

$$\Delta_L = \left\{\delta \in L \,|\, \delta^2 = \langle \delta, \delta \rangle = -2\right\}$$

the set of *roots* of L, and by

$$V_L = \left\{x \in L \otimes \mathbb{R} \,|\, x^2 > 0 \text{ and } (\delta, x) \neq 0 \quad \forall \delta \in \Delta_L\right\}.$$

According to [**83**, Proposition 1.10], the set V_L is open and each of its connected components meets $L \subset L \otimes \mathbb{R}$. These connected components of V_L are called *chambers* of V_L.

If $L = \mathrm{NS}(X)$ for a supersingular K3 surface X, then there is exactly one chamber α_X, the ample cone, such that a line bundle H is ample if and only if $[H] \in \alpha_X$. It turns out that, together with a strictly characteristic subspace P, the choice of a chamber α in V_L determines a marked K3 surface with ample cone α *up to unique isomorphism*. Indeed, this is a consequence of the following

THEOREM 5.6. [**83**, Theorem II' and Theorem II''] *Let $\kappa = \overline{\kappa}$ be a field of characteristic $p > 3$ and X, Y supersingular K3 surfaces over κ. If $f : \mathrm{NS}(X) \to \mathrm{NS}(Y)$ is an isometry, then there is a unique isomorphism $F : Y \to X$ with $f = F^*$ provided that*

(1) $f(\alpha_X) = \alpha_Y$ and
(2) $\overline{f}(P_X) = P_Y$.

REMARK 5.7. The original statements of Ogus involve N-marked K3 surfaces, that is, pairs (X, η) where $\eta : N \hookrightarrow \mathrm{NS}(X)$ is a finite index injection of a supersingular K3 lattice. This allows to consider families of surfaces with varying Artin invariant σ, which can very well happen. Indeed, it is a crucial property used in the proofs. All the definitions we introduced above (characteristic subspaces, ample chambers, ...) carry over to this context with mild modifications. However, since we do not need this approach here, we avoid it for the sake of simplicity.

Our main application of these results is the following immediate Corollary:

COROLLARY 5.8. *Let $\kappa = \overline{\kappa}$, $\mathrm{char}\,\kappa > 3$, N a supersingular K3 lattice and $P \subset D_N \otimes \kappa$ a strictly characteristic subspace. If $f \in O(N)$ preserves a connected component of V_N and $\overline{f}(P) = P$, then there is a supersingular K3 surface X and an automorphism $F : X \to X$ such that $N \overset{\iota}{\cong} \mathrm{NS}(X)$, $\bar{\iota}(P) = P_X$ and $f = \iota^{-1} \circ F^* \circ \iota$.*

2. The classification of characteristic subspaces by Ogus

The following lemma follows easily from Definition 5.3.

LEMMA 5.9. *If $P \subset D \otimes \kappa$ is a strictly characteristic subspace and $\dim_{\mathbb{F}_p} D = 2\sigma$, then*

$$l = P \cap \psi(P) \cap \cdots \cap \psi^{\sigma-1}(P)$$

is a line. Furthermore, P can be recovered as $P = l + \psi^{-1}(l) + \cdots + \psi^{-(\sigma-1)}l)$ and hence $l + \psi(l) + \cdots + \psi^{2\sigma-1}(l) = D \otimes \kappa$.

It follows from [**82**, (3.20.1)] that $\psi^{1-\sigma}(l_X) = F^2_{Hodge}(H^2_{dR}(X,\kappa)) \cong H^0(X,\Omega^2_X)$. We pick a basis e_0 of l and set $e_i = \psi^i(l)$. Then $e_0, ..., e_{\sigma-1}$ form a basis of the strictly characteristic subspace $\psi^{\sigma-1}(P)$. Define

$$a_i(e, D, P) := \langle e_0, e_{\sigma+i}\rangle. \tag{6}$$

Since the scalar product is non-degenerate, $a_0 \neq 0$, and after changing e_0 by a scalar, unique up to a $p^\sigma+1$ root of unity, we can assume $a_0 = 1$. If we replace e_0 by ζe with ζ a $p^\sigma+1$ root of unity, then a_i is replaced by $\zeta^{p^{\sigma+i}+1}a_i = \zeta^{1-p^i}a_i$. It is a theorem of Ogus that these coordinates are in bijection with isomorphism classes of characteristic subspaces on $D\otimes\kappa$. The following lemma shows how to construct a characteristic subspace out of some given coordinates a_i.

LEMMA 5.10. [**82**] *In the basis $(e_1, ..., e_{2\sigma})$, the intersection matrix $\langle e_i, e_j\rangle_{D\otimes\kappa}$ has the form:*

$$\begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix}$$

where A is the $\sigma\times\sigma$ matrix:

$$\begin{pmatrix} 1 & a_1 & a_2 & a_3 & \dots & a_{\sigma-1} \\ 0 & 1 & \mathrm{Fr}(a_1) & \mathrm{Fr}(a_2) & \dots & \mathrm{Fr}(a_{\sigma-2}) \\ 0 & 0 & 1 & \mathrm{Fr}^2(a_1) & \dots & \mathrm{Fr}^2(a_{\sigma-3}) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

The Frobenius-linear endomorphism ψ of $D\otimes\kappa$ is given by $\psi(e_i) = e_{i+1}$ for $i = 1, \dots, 2\sigma-2$ and $\psi(e_{2\sigma-1}) = \lambda_0 e_0 + \dots \lambda_{\sigma-1}e_{\sigma-1} + \mu_0 e_\sigma + \dots + \mu_{\sigma-1}e_{2\sigma-1}$. Both λ and μ are determined by $\mu_0 = 0$, and

$$\lambda A = (1, 0, \dots, 0) \quad \mu A^t = (*, \mathrm{Fr}(a_{\sigma-1}), \dots, \mathrm{Fr}^{\sigma-1}(a_1)).$$

We can use this lemma to define an $\mathbb{F}_p$ vector space, a bilinear form and a characteristic subspace as follows: Since $\lambda_0 = 1$, ψ, as defined above, is bijective and equips $\kappa^{2\sigma}$ with an $\mathbb{F}_p$ structure D. Now, $\mathrm{Fr}(\langle x, y\rangle) = \langle\psi(x), \psi(y)\rangle$ for $x, y \in k^{2\sigma} = D\otimes\kappa$ assures that the bilinear form descends to an $\mathbb{F}_p$-valued form on D. To conclude set P' as the span of the first σ standard basis vectors of $\kappa^{2\sigma}$. By construction, P' is strictly characteristic and $P\colon = \psi^{1-\sigma}(P')$ has the desired coordinates $a_i(D, P) = a_i$.

3. Isometries vs strictly characteristic subspaces

The following theorem answers the question which lattice isometries come from some supersingular K3 surface.

THEOREM 5.11. *Let $\kappa = \overline{\kappa}$ be an algebraically closed field of* $\operatorname{char}\kappa > 3$, *$N_{p,\sigma}$ be a supersingular K3 lattice and $f \in O(N_{p,\sigma})$. There exists a supersingular K3 surface X/κ of Artin invariant σ, an automorphism $F \in \mathrm{Aut}(X)$ and an isometry $\phi : N_{p,\sigma} \xrightarrow{\cong} \mathrm{NS}(X)$ such that*

$$f = \phi^{-1}F^*\circ\phi$$

if and only if

- *f is positive;*
- *the minimal polynomial $\mu = \mu(\overline{f}|D_{N_{p,\sigma}}) \in \mathbb{F}_p[x]$ is irreducible and*
- *either $\mu = (x\pm 1)$ or $\deg\mu = 2n$ is even and σ/n odd.*

In case $\mu(\overline{f}) = (x \pm 1)$, the period domain is of dimension $\sigma - 1$ while in the second case the period domain has dimension $(\sigma/n - 1)/2$. If the dimension is zero, then the K3 surface is unique up to isomorphism. Further $F^*\omega = \alpha\omega$, where $0 \neq \omega \in H^0(X, \Omega_X^2)$ and $\mu(\alpha) = 0$.

Remark 5.12. This is in perfect analogy with the case of a complex K3 surface X. There the minimal polynomial of $F^*|T(X)$ is irreducible over $\mathbb{Q}$.

Before we prove the theorem, we note a curious

Corollary 5.13. *There are at most finitely many isomorphism classes of supersingular K3 surfaces X/κ with $\sigma(X) = 2, 4, 8$ admitting an automorphism $F \in Aut(X)$ such that $F^*|H^0(X, \Omega_X^2) \neq \pm\,\mathrm{id}$.*

For the proof, we need the following

Theorem 5.14. **[100]** *Let K be a finite or local field of characteristic $\neq 2$ and D a K-vector space equipped with a non-degenerate inner product. If two unitary operators of D have the same irreducible minimal polynomial, then they are conjugate by a unitary operator.*

Proof of Corollary 5.13. Set $f = F^*$. There are only finitely many possible minimal polynomials of $\overline{f}$. Then, by the previous theorem, there are only finitely many possible actions of $\overline{f}|D_{\mathrm{NS}}$ up to conjugation. Now, the fact that σ/n is odd and divides a power of 2 implies that $n = \sigma$, i.e., the characteristic polynomial of $\overline{f}$ is irreducible. We are in the case of zero dimensional moduli. Then, by our version of the crystalline Torelli theorem, each characteristic polynomial corresponds to a single K3 surface up to isomorphism. □

In order to prove Theorem 5.11, we give the following two lemmas.

Lemma 5.15. *Let X/κ be a supersingular K3 surface of Artin invariant σ, $F \in \mathrm{Aut}(X)$ and set $f = F^*|\,\mathrm{NS}(X)$. Then the minimal polynomial $\mu(\overline{f}|D_{\mathrm{NS}(X)}) \in \mathbb{F}_p[x]$ is irreducible. Either it is $x \pm 1$ or reciprocal of degree $2d$ and σ/d is odd. In the second case*

$$a_i(P_X) = 0 \text{ for } i \not\equiv 0 \mod 2d$$

where $a_i(P_X)$ is defined as in (6).

Proof. Note that $\overline{f}$ preserves the period $P_X \subset D_{\mathrm{NS}} \otimes \kappa$ of X. Since $\overline{f}$ and the semilinear automorphism $\psi = \mathrm{id} \otimes \mathrm{Fr}_\kappa : D_{\mathrm{NS}} \otimes \kappa \to D_{\mathrm{NS}} \otimes \kappa$ commute, the line (cf. Lemma 5.9)

$$l_X = P_X \cap \psi(P_X) \cap \cdots \cap \psi^{\sigma-1}(P_X)$$

is preserved by $\overline{f}$ as well - it is an eigenspace. Let $e_0 \in l_X$ be a basis vector with $\overline{f}e_0 = \alpha_0 e_0$. Since $\sum_i \psi^i(l_X) = D_{\mathrm{NS}} \otimes \kappa$, the vectors $e_i = \psi^i(e_0)$, $i = 0, ..., 2\sigma - 1$ are an eigenbasis of $D \otimes \kappa$. Set $\alpha_i = \mathrm{Fr}^i(\alpha_0)$, $i = 0, \dots, 2d-1 \in \mathbb{Z}/2d\mathbb{Z}$. Then

$$\overline{f}e_i = \alpha_{\bar{i}} e_i$$

where $\bar{i} \in \mathbb{Z}/2d\mathbb{Z}$ is the image of i. This shows that the eigenvalues α_i are roots of a single irreducible polynomial in $\mathbb{F}_p[x]$, the minimal polynomial of $\overline{f}$. In particular $\overline{f}$ is diagonalizable.

$$\langle e_i, e_j \rangle = \langle \overline{f}(e_i), \overline{f}(e_j) \rangle = \alpha_i \alpha_j \langle e_i, e_j \rangle.$$

Hence e_i and e_j are orthogonal unless $\alpha_j = 1/\alpha_i$. By non degeneracy of the bilinear form, the set of roots $\{\alpha_0, ...\alpha_{2d-1}\}$ is invariant under inversion. So unless $\mu = (x \pm 1)$, it is reciprocal and of even degree $2d$. Suppose we are in the second case. Then $\mathrm{Fr}^d(\alpha_0) = \alpha_0^{-1}$ and $a_i = \langle e_0, e_{\sigma+i} \rangle \equiv 0$, unless $e_{\sigma+i}$ has eigenvalue α_d, i.e., $\sigma + i \equiv d \mod 2d$. We know that $a_0 \neq 0$. Hence, $\sigma \equiv d \mod 2d$ and $\sigma/d \equiv 1 \mod 2$. □

REMARK 5.16. It follows from [**82**, (3.20.1)] and the Hodge filtration that

$$\psi^{1-\sigma}(l_X) = F^2_{Hodge}(H^2_{dR}(X,\kappa)) \cong H^0(X,\Omega^2_X).$$

LEMMA 5.17. *Let $\mu \in \mathbb{F}_p[x]$ be irreducible of degree $2d$ and $\sigma \in \{1,\dots 10\}$ such that σ/d is odd. Let $a_1,\dots a_{\sigma-1} \in \overline{\mathbb{F}}_p$ such that $a_i = 0$ for $i \not\equiv 0 \mod 2d$. Then there is an $\mathbb{F}_p$ vector space D of dimension 2σ, an inner product on D, $g \in O(D)$ with minimal polynomial $\mu(g) = \mu$ and a characteristic subspace $P \subseteq D \otimes \overline{\mathbb{F}}_p$. Such that*

$$a_i = a_i\left(D, \langle\cdot,\cdot\rangle, P\right).$$

PROOF. One can use the formulas given in Lemma 5.10 to define $(D,\langle\cdot,\cdot\rangle,P)$ with

$$a_i = a_i\left(D, \langle\cdot,\cdot\rangle, P\right).$$

Choose a root $\alpha \in \overline{\mathbb{F}}_p$ of μ, set $\alpha_i = \mathrm{Fr}^i(\alpha)$ and define g by $g(e_i) = \alpha_i e_i$. By the vanishing of the a_i for $i \not\equiv 0 \mod 2d$ this defines an isometry of $D \otimes \overline{\mathbb{F}}_p$. It remains to check that $g \in O(D)$, i.e., $\psi \circ g = g \circ \psi$. We check this on the basis e_i:

$$g \circ \psi(e_i) = g(e_{i+1}) = \alpha_{i+1} e_{i+1} = \mathrm{Fr}(\alpha_i)\psi(e_i) = \psi(\alpha_i e_i) = \psi \circ g(e_i)$$

for $i < 2\sigma - 1$. With the formulas of Lemma 5.10 we get

$$\psi(e_{2\sigma-1}) = \lambda_0 e_0 + \dots \lambda_{\sigma-1} e_{\sigma-1} + \mu_0 e_\sigma + \dots + \mu_{\sigma-1} e_{2\sigma-1}$$

We have to show that $\psi(e_{2\sigma-1})$ is contained in the eigenspace of $\alpha = \alpha_0$. It is spanned by the e_i where $i \equiv 0 \mod 2d$ and $i \leq 2\sigma - 1$. Hence, we have to show that $\lambda_i = 0$ for $i \not\equiv 0 \mod 2d$ and that $\mu_i = 0$ for $i \not\equiv d \mod 2d$.

- **Claim:** $\lambda_j = 0$ for $j \not\equiv 0 \mod 2d$
  For $j = 0$, there is nothing to show. Suppose the claim holds for $j-1$, if $j \equiv 0 \mod 2d$ the statement holds trivially for j. So let $j \not\equiv 0 \mod 2d$. Then
  $$0 = (\lambda A)_j = \sum_{i=0}^{j} \lambda_i A_{ij} = \lambda_j + \sum_{i=0}^{j-1} \lambda_i A_{ij} = \lambda_j.$$
  Only the last equality needs an explanation: Recall that $A_{ij} = \mathrm{Fr}^i(a_{j-i}) = 0$, if $i \not\equiv j \mod 2d$. By induction hypothesis $\lambda_i = 0$ for $i \not\equiv 0 \mod 2d$. Hence, $\lambda_i A_{ij} = 0$ unless $0 \equiv i \equiv j \mod 2d$, but $j \not\equiv 0 \mod 2d$ by assumption.
- **Claim**: $\mu_j = 0$ for $j \not\equiv d \mod 2d$.
  Since $1 \not\equiv 0 \mod 2d$, $a_1 = 0$ and $\mu_{\sigma-1} = \mathrm{Fr}^{\sigma-1}(a_1) = 0$. Hence the claim holds for $j = \sigma - 1$. Fix some j and suppose the claim is true for $j' > j$. We want to prove that the claim is true for j as well. We may assume $j \not\equiv d \mod 2d$, as else there is nothing to show. The assumption that σ/d is odd is equivalent to $\sigma \equiv d \mod 2d$. Then $\sigma - j \equiv d - j \not\equiv 0 \mod 2d$. Hence $a_{\sigma-j} = 0$, and
  $$0 = \mathrm{Fr}^i(a_{\sigma-j}) = (\mu A^t)_j = \sum_{i=0}^{\sigma-1} \mu_i A_{ji} = \mu_j + \sum_{i>j}^{\sigma-1} \mu_i A_{ji} = \mu_j.$$
  For the last equation, recall $A_{ji} = Fr^j(a_i) = 0$ unless $i \equiv 0 \mod 2d$, but then, by induction, $\mu_i = 0$, for $i > j$, as $0 \equiv i \not\equiv d \mod 2d$.

□

PROOF OF THEOREM 5.11. The if direction is Lemma 5.15. For the other direction, set $N = N_{p,\sigma}$, and suppose that we have $f \in O(N)$ such that $\mu = \mu(\overline{f}|D_N) = 0$. Then we can use Lemma 5.17 to get an inner product space D, $g \in O(D)$ with $\mu(g) = \mu$ and have $(\sigma/d - 1)/2$ degrees of freedom for a strictly characteristic subspace $P \subseteq D \otimes \overline{\mathbb{F}}_p$ preserved by g. To conclude, use Theorem 5.14 to see that $(D,\langle\cdot,\cdot\rangle,g) \cong (D_N, b_N, \overline{f})$ and pull back P. Then the crystalline

Torelli Theorem, in form of Corollary 5.8, does the rest. To get uniqueness in case of $\deg \mu = 2\sigma$ note that all $a_i(P) = 0$. Hence, there is only a single choice for P up to isomorphism. $\square$

CHAPTER 6

Automorphisms and Salem numbers

To a continuous, surjective self-map F of a compact metric space one can associate its topological entropy $h(F)$. Roughly speaking, this number measures how fast general points spread out under the iterations of the automorphism. By work of Gromov [**39**] and Yomdin [**105**], on a compact Kähler manifold X, the topological entropy can be calculated in terms of the action of F on the cohomology groups $H^*(X, \mathbb{Z})$. In this case, the topological entropy is either zero or the logarithm of an algebraic number. If X is moreover a surface and $F : X \to X$ a biholomorphic map, then $h(F)$ is either zero or the logarithm of a Salem number λ, which is precisely the spectral radius of the linear action F^* in $H^2(X, \mathbb{Z})$. For more on the dynamics of surfaces consider the survey [**24**].

A *Salem number* is a real algebraic integer $\lambda > 1$ which is conjugate to $1/\lambda$ and whose other conjugates lie on the unit circle. Its minimal polynomial is called a Salem polynomial and its degree is the degree the Salem number. In each even degree d there is a unique smallest Salem number λ_d. Conjecturally the smallest Salem number is λ_{10} found by Lehmer in 1933 [**53**].

In [**33**] Esnault and Srinivas show that if $F : X \to X$ is an automorphism of a *projective* surface X over a field κ, then the order of F^* on $\mathrm{NS}(X)^\perp \subseteq H^2_{et}(X, \mathbb{Q}_l)$, $l \neq \operatorname{char} \kappa$, is finite. Hence, the spectral radius of F^* is realized already in the Néron-Severi group $\mathrm{NS}(X)$, and by standard arguments for isometries of hyperbolic lattices it is then a Salem number. This leads to the following

THEOREM 6.1. [**62, 33**] *Let X be a smooth projective surface over an algebraically closed field $k = \overline{k}$ and $F \in Aut(X)$ an automorphism. Then the characteristic polynomial of $F^*|H^2_{ét}(X, \mathbb{Q}_l(1))$, (char $k \neq l$) factors into cyclotomic polynomials and at most one Salem polynomial. If a Salem polynomial $s(x)$ occurs, then* $\ker s(F^*) \subseteq \mathrm{NS}(X)$.

We can define the (algebraic) *entropy* $h(F)$ as the logarithm of the spectral radius of $F^*|\,\mathrm{NS}(X)$ and the Salem degree of f as the degree of this Salem number. For complex projective surfaces, the standard comparison results between singular and étale cohomology imply that the algebraic entropy coincides with the topological one.

In fact we know more than just the characteristic polynomial.

THEOREM 6.2. [**37, 23**] *Let $X/\overline{k}$ be an algebraic surface and $F \in Aut(X)$ an automorphism. Then, up to replacing F by F^n for some $n \in \mathbb{N}$, there are 3 cases for the Jordan decomposition of $F^*|H^2_{\acute{e}t}(X, \mathbb{Q}_l(1)) \otimes \overline{\mathbb{Q}_l}$:*

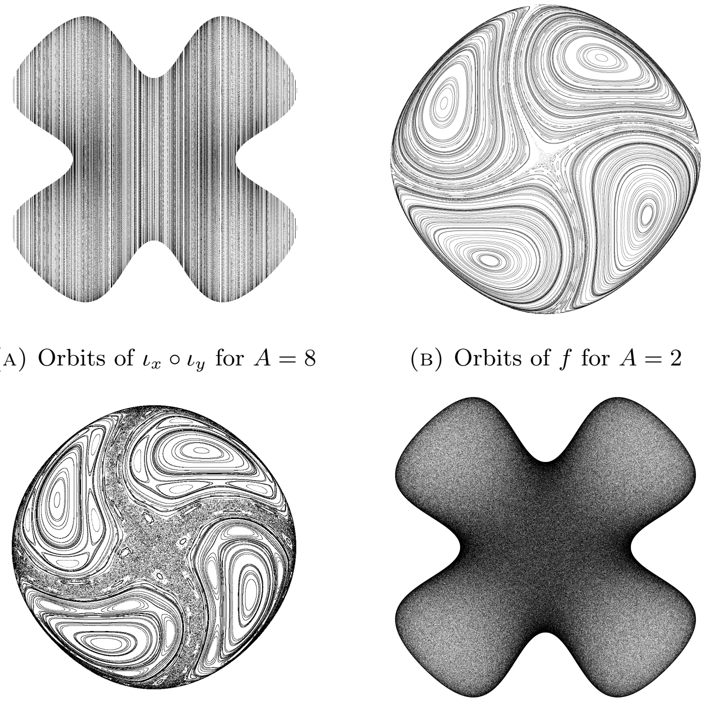
(1) id

(2) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \oplus id$

(3) $diag(\alpha_1, \ldots, \alpha_n, 1, \ldots, 1)$*, where* $s(x) = \prod_{i=1}^n (x - \alpha_i)$ *is a Salem polynomial.*

In the first case if $k = \mathbb{C}$ then F^n is isotopic to the identity. In the second case there is a genus one fibration $\pi : X \to C$ to some curve C and $g \in Aut(C)$ such that $\pi \circ f = g \circ \pi$. This cannot happen in case (3).

FIGURE 1. Dynamics on $X(\mathbb{R})$ seen from above [**61**].



(A) Orbits of $\iota_x \circ \iota_y$ for $A = 8$ (B) Orbits of f for $A = 2$

(C) Orbits of f for $A = 2.5$ (D) Orbits of f for $A = 8$

REMARK 6.3. In the cases (1), (2) the dynamics of F is fairly simple and well understood. For K3 surfaces, we can easily say more. (1) F is finite, (2) $C = \mathbb{P}^1$, g is of finite order and $\pi : X \to \mathbb{P}^1$ is a genus one fibration (cf. § 2). The theorem holds as well for compact Kähler surfaces with singular instead of étale cohomology.

EXAMPLE 6.4. We illustrate the cases (2)(3) by an example of real dynamics on a family of K3 surfaces found in [**62**]. Let

$$X \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

be a smooth hypersurface of degree $(2,2,2)$ defined by

$$(1+x^2)(1+y^2)(1+z^2)+Axyz=2 \quad A\in\mathbb{R}.$$

Since by adjunction $K_X=0$ and by the Lefschetz hyperplane theorem $b_1(X)=0$, X is a K3 surface. Every line through $p\in X$ parallel to the x-axis meets X in a second point $\iota_x(p)$ and similarly ι_y,ι_z for y,z give involutions of X. We get examples, depicted in Figure 1, for the 3 different types:

(1) ι_x is of order 2;
(2) $\iota_x\circ\iota_y$ is of infinite order. We can see the orbits moving along a fibration;
(3) $f=\iota_x\circ\iota_y\circ\iota_z$ is of positive entropy. For $A=2$ the dynamics is dominated by elliptic islands, for $A=8$ the dynamic seems ergodic, while for $A=2.5$ we see a mixture of both behaviors.

The study of the entropy of $F: X\to X$ becomes trivial if X has positive Kodaira dimension (e.g., if X is of general type a power of F is the identity, hence $h(F)=0$). Indeed, if F has positive entropy, then X is either a blow up of $\mathbb{P}^2$ in at least 10 points, or birational to a torus, a K3-surface or an Enriques surface [**22, 71**].

1. Automorphisms of Salem degree 22

In this section we exhibit explicit automorphisms of maximal Salem degree 22 on the supersingular K3 surface of Artin invariant one for all primes $p\equiv 3 \mod 4$ in a systematic way.

Salem numbers of degree 22 were used by McMullen [**62**] to construct Kähler K3 surfaces admitting an automorphism with Siegel disks. These are domains on which f acts by an irrational rotation. Since the Salem degree of a projective surface is bounded by its Picard number $\rho\leq 20$, McMullen's examples cannot be projective. They remain abstract objects.

However, in positive characteristic, there exist projective K3 surfaces with Picard number 22 so there may exist automorphisms of Salem degree 22. As pointed out by Esnault and Oguiso [**32**], a specific feature of such automorphisms is that they do not lift geometrically to any characteristic zero model. One such surface is the supersingular K3 surface $X(p)$ of Artin invariant one defined over $\overline{\mathbb{F}}_p$. Abstractly, Blanc, Cantat [**14**], Esnault, Oguiso and Yu [**34**] proved the existence of automorphisms of Salem degree 22 on $X(p)$ for $p\neq 5,7,13$, while Shimada [**91**] exhibited such automorphisms on every supersingular K3 surface in all odd characteristics $p\leq 7919$ using double plane involutions. Meanwhile, Schütt [**89**] exhibited explicit automorphisms of Salem degree 22 on $X(p)$ for all $p\equiv 2 \mod 3$ using elliptic fibrations.

Building on his methods we obtain the main result of this subsection.

Theorem 6.5. *The supersingular K3 surface $X(p)$ of Artin invariant one admits explicit automorphisms of Salem degree 22 for all primes $p\equiv 3 \mod 4$. Such automorphisms do not lift to any characteristic zero model of $X(p)$.*

Let X be the K3 surface over $\mathbb{Q}$ defined by $y^2=x^3+t^3(t-1)^2x$.
For $p\equiv 3 \mod 4$, its specialization mod p is the supersingular K3 surface $X(p)$ of Artin invariant one. The automorphisms are constructed in the following steps:

(1) Find generators of $\mathrm{NS}(X)$ using the elliptic fibration.
(2) Complement the generators of $\mathrm{NS}(X)$ by two sections P,R to generators of $\mathrm{NS}(X(p))$ using a purely inseparable base change.

(3) Compute the intersection matrix of $\mathrm{NS}(X(p))$.
(4) Search for extended ADE-configurations of (-2)-curves in $\mathrm{NS}(X)$. These induce elliptic fibrations on X.
(5) P and R induce sections of the new elliptic fibration of $X(p)$.
(6) The sections induce automorphisms of $X(p)$. Compute their action on $\mathrm{NS}(X(p))$.
(7) Compose automorphisms obtained from different fibrations to obtain one of the desired Salem degree.

2. Elliptic Fibrations on K3 Surfaces

A *genus one fibration* on a surface X is a surjective map

$$\pi : X \to C$$

to a smooth curve C such that the generic fiber is a smooth curve of genus one. We will call it an *elliptic* fibration, if the additional data of a section O of π is given. Indeed, all genus one fibrations occurring here admit a section. This turns the generic fiber of an elliptic fibration into an elliptic curve E over $K = k(C)$, the function field of C. For a K3 surface, $C = \mathbb{P}^1$ is the only possibility. There is a one to one correspondence between K-rational points of E and sections of π. Both these sets are abelian groups, which we will call the *Mordell-Weil group* of the fibration. This is denoted by $\mathrm{MW}(X, \pi, O)$, where π and O are suppressed from the notation if confusion is unlikely. The addition on MW is denoted by $\oplus$.

A good part of NS is readily available: the *trivial lattice*

$$\mathrm{Triv}(X, \pi, O) := \langle O, \text{fiber components}\rangle_{\mathbb{Z}}.$$

If O and π are understood, we will suppress them from the notation. By results of Kodaira [**49**] and Tate [**99**], the trivial lattice decomposes as an orthogonal direct sum of a hyperbolic plane spanned by O together with the fiber F and negative definite root lattices of type ADE consisting of fiber components disjoint from O. Note that the singular fibers (except in some cases in characteristics 2 and 3) are determined by the j-invariant and discriminant of the elliptic curve E.

An advantage of elliptic fibrations is that they structure the Néron-Severi group into sections and fibers as given by the following theorem.

THEOREM 6.6. [**94**] *There is a group isomorphism*

$$\mathrm{MW}(X) \cong \mathrm{NS}(X)/\,\mathrm{Triv}(X).$$

The Mordell-Weil group can be equipped with a positive definite symmetric $\mathbb{Q}$-valued bilinear form via the orthogonal projection with respect to $\mathrm{Triv}(X)$ in $\mathrm{NS}(X) \otimes \mathbb{Q}$ and switching sign. Explicitly, it is given as follows. Let $P, Q \in \mathrm{MW}(X, O)$, and denote by R the set of singular fibers of the fibration. Then,

$$\langle P, Q\rangle := \chi(\mathcal{O}_X) + P.O + Q.O - P.Q - \sum_{\nu \in R} c_\nu(P, Q)$$

$$\langle P, P\rangle := 2\chi(\mathcal{O}_X) + 2P.O - \sum_{\nu \in R} c_\nu(P, P),$$

where the dot denotes the intersection product on the smooth surface X. The term $c_\nu(P, Q)$ is the local contribution at a singular fiber, given as follows. If one of the two sections involved meets the same component of the singular fiber ν as the zero section, then the contribution at ν is zero. If this is not the case, the contribution is non zero and depends on the fiber type. We only need the types III, III^* and I_0^*; for the others consult [**94**, 8.16]. If ν is of type III^*

(resp. III), the contribution is equal to 3/2 (resp. 1/2) if P and Q meet the same component of ν and zero otherwise. For ν of type I_0^*, we have $c_\nu(P,Q) = 1$ if they meet in the same component and 1/2 otherwise. Equipped with this pairing, $\mathrm{MWL}(X) := \mathrm{MW}(X)/\mathrm{torsion}$ is a positive definite lattice. In general, it is not integral.

3. Isotrivial Fibration

In this section, we compute generators of the Néron-Severi group as well as their intersection matrix.

Proposition 6.7. *Let X be the K3 surface over $\mathbb{C}$ defined by the Weierstrass equation*

$$X: \quad y^2 = x^3 + t^3(t-1)^2 x.$$

Then, its Néron-Severi group is generated by fiber components, the zero section and the 2-torsion section $Q = (0,0)$. It is of rank 20 and determinant -4.

Proof. By the theory of Mordell-Weil lattices, $\mathrm{NS}(X)$ is spanned by fiber components and sections. The elliptic fibration has j-invariant equal to 1728 and discriminant $\Delta = -2^6 t^9 (t-1)^6$. Using the classification of singular fibers by Kodaira and Tate, we get that the fibration has two fibers of type III^* over $t = 0, \infty$ and one fiber of type I_0^* over $t = 1$. Hence, the trivial lattice L is of the form $L \cong U \oplus 2E_7 \oplus D_4$. Since it is of the maximum possible rank 20, the fibration admits no section of infinite order. The determinant -16 of the trivial lattice implies that only 2- or 4-torsion may appear. Obviously, $Q = \{x = y = 0\}$ is the only 2-torsion section and 4-torsion may not occur due to the singular fibers. Alternatively, the reader may note that additional torsion turns $\mathrm{NS}(X)$ into a unimodular lattice of signature $(1, 19)$. Such a lattice does not exist. $\square$

Note that the j-invariant $j = 1728$ is constant. Hence, all smooth fibers are isomorphic - such a fibration is called isotrivial.

Proposition 6.8. *For $p \equiv 3 \mod 4$, the surface $X(p) := X \otimes \overline{\mathbb{F}}_p$ is the supersingular K3 surface of Artin invariant one.*

Proof. The singular K3 surface X has good reduction at $p \neq 2$ and the determinant $\det \mathrm{NS}(X) = -4$. A prime p is inert in $\mathbb{Q}(\sqrt{-1})$ iff $p \equiv 3 \mod 4$. Thus, by Proposition 5.2, for all $p \equiv 3 \mod 4$, the K3 surface $X(p) := X \otimes \overline{\mathbb{F}}_p$ is supersingular; that is, $\mathrm{rk}\,\mathrm{NS}(X(p)) = 22$. It is known that $\mathrm{NS}(X(p))$ is a p-elementary lattice of determinant $-p^{2\sigma}$ where $\sigma \in \{1, ..., 10\}$ is called the Artin invariant of $X(p)$. Following an argument by Shimada [**90**], we show that $\sigma = 1$: Via specialization, we get an embedding $i : \mathrm{NS}(X) \hookrightarrow \mathrm{NS}(X(p))$. Therefore, $\mathrm{NS}(X) \oplus i(\mathrm{NS}(X))^\perp \hookrightarrow \mathrm{NS}(X(p))$. Since $p \neq 2$, the p-part of $D_{\mathrm{NS}(X) \oplus i(\mathrm{NS}(X))^\perp} = D_{\mathrm{NS}(X)} \oplus D_{i(\mathrm{NS}(X))^\perp}$ equals that of $D_{i(\mathrm{NS}(X))^\perp}$. Hence, it is of length at most two. However, the p-part of $D_{\mathrm{NS}(X(p))}$ is a subquotient of this. We conclude that its p-part has length at most two. On the other hand, $\sigma \in \{1, ..., 10\}$, which implies that $2 \leq 2\sigma = l(D_{\mathrm{NS}(X(p))}) \leq 2$. $\square$

Our next task is to work out generators of $\mathrm{NS}(X(p))$ for $p \equiv 3 \mod 4$. By Theorem 6.6, it is generated by sections and fiber components. Reducing j and Δ mod p, we see that the fiber types are preserved mod p (even in case $p = 3$; cf. [**95**, p.365]). Hence, sections of infinite order must appear. Generally, it is hard to compute sections of an elliptic fibration. For this special fibration, there is a trick involving a purely inseparable base change of degree p turning $X(p)$ into a Zariski surface.

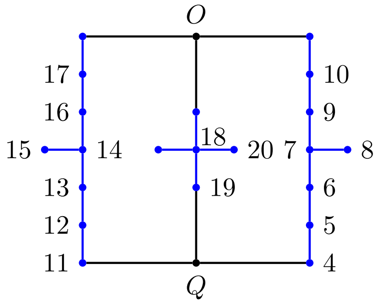


FIGURE 3. 24 (-2)-curves of X supporting singular fibers of type $I_0^*, 2 \times III^*$ and torsion sections O, Q of π.

PROPOSITION 6.9. *Let $p = 4n+3$ be a prime number. Then, the Néron-Severi group of $X(p)$ defined by $y^2 = x^3 + t^3(t-1)^2x$ over $\overline{\mathbb{F}}_p$ is generated by the sections O, Q, P, R and fiber components, where O denotes the section at infinity, $Q = (0,0)$ is the 2-torsion section, $\zeta^4 = -1$ and*

$$P:\ x = \zeta^2 t^{2n+3}(t-1), \qquad y = \zeta^3 t^{n+3}(t-1)^{2n+3},$$
$$R:\ x = -\zeta^2 t^{2n+3}(t-1), \qquad y = -\zeta t^{n+3}(t-1)^{2n+3}.$$

We have the following intersection numbers, symmetric in P and R:

$$O.P = Q.P = O.R = Q.R = n, \quad P.R = 2n.$$

PROOF. One can check directly that P and R are sections of the elliptic fibration, and the patient reader may calculate their intersection numbers by hand. Since $X(p)$ has Artin invariant $\sigma = 1$, $\det \mathrm{NS}(X) = -p^{2\sigma} = -p^2$. All that remains is to compute the intersection matrix of these four sections and the fiber components. One can check that it has a 22×22 minor of determinant $-p^2$. This minor corresponds to a basis of $\mathrm{NS}(X(p))$. □

For later reference, we fix the the following $\mathbb{Z}$-basis of the Néron-Severi group, where the fiber components are sorted as indicated in Figure 3, and e_{20} is distinguished by $e_{20}.P = 1$.

$$(O, F, Q, E_7(t=\infty), E_7(t=0), A_3(\subseteq D_4, t=1), e_{21} = P, e_{22} = R)$$

Replacing Q by one of the missing components of the I_0^* fiber results in a $\mathbb{Q}$ instead of a $\mathbb{Z}$-basis. This is predicted by Theorem 6.6.

For the intersection matrix in this basis, one obtains

$$\begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n & n \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & n & n \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & 0 \\
n & 1 & n & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 2n \\
n & 1 & n & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2n & -2
\end{pmatrix}.$$

In the remaining part of this section, we will explain how the sections P, R are found and give an alternative way of computing their intersection numbers.

Recall that we assume that $p \equiv 3 \mod 4$ and write $p = 4n+3$. Consider the purely inseparable base change $t \mapsto t^p$. This changes the equation as follows:

$$y^2 = x^3 + t^{12n+9}(t-1)^{8n+6}x.$$

We minimize this equation using the birational map

$$(x, y, t) \mapsto \left(\frac{x}{t^{6n+4}(t-1)^{4n+2}}, \frac{y}{t^{9n+6}(t-1)^{6n+3}}, t \right).$$

This leads to the surface Y given by

$$Y: \quad y^2 = x^3 + t(t-1)^2 x.$$

After another base change $t \mapsto t^p$, we get

$$y^2 = x^3 + t^{4n+3}(t-1)^{8n+6}x,$$

and minimizing the fibration

$$\left(\frac{y}{t^{3n}(t-1)^{6n+3}} \right)^2 = \left(\frac{x}{t^{2n}(t-1)^{4n+2}} \right)^3 + t^3(t-1)^2 \frac{x}{t^{2n}(t-1)^{4n+2}}$$

we recover X.

Instead of directly searching for sections on X, we exhibit two sections on Y and pull them back to X. The j-invariant of Y is still equal to 1728, but $\Delta = -2^6 t^3 (t-1)^6$. For $p \neq 3$, this leads to two singular fibers of type III over $t = 0, \infty$ and a singular fiber of type I_0^* over $t = 1$. The Euler number of this surface is $2e(III) + e(I_0^*) = 2 \cdot 3 + 6 = 12$. By general theory, it is rational. The trivial lattice is isomorphic to

$$\operatorname{Triv}(Y) \cong U \oplus 2A_1 \oplus D_4.$$

This lattice is of determinant -16 and rank 8. From the theory of elliptically fibered rational surfaces, we know that $\operatorname{rk} \operatorname{NS}(Y) = 10$, $\det \operatorname{NS}(Y) = -1$, which implies that Y has Mordell-Weil rank 2. Define

$$\operatorname{Triv}'(Y) := \operatorname{Triv}(Y) \otimes_{\mathbb{Z}} \mathbb{Q} \cap \operatorname{NS}(Y).$$

Then, by Theorem 6.6, $\operatorname{Triv}'(Y)/\operatorname{Triv}(Y) \cong \mathrm{MW}_{tors}$. Since $\{x = y = 0\}$ is a 2-torsion section, we know that

$$\det \operatorname{Triv}'(Y) = \det \operatorname{Triv}(Y)/[\operatorname{Triv}'(Y) : \operatorname{Triv}(Y)]^2 \in \{-1, -4\}.$$

As even unimodular lattices of signature $(1, 7)$ do not exist, -1 is impossible. We conclude that $x = y = 0$ is the only torsion section.

To find a section of infinite order, we first determine the Mordell-Weil lattice and then translate this information to equations of the section. Since $j = 1728$, the elliptic curve admits complex multiplication given by $(x, y) \mapsto (-x, iy)$. Obviously, Q and O are the unique sections fixed by this action. Hence, the Mordell-Weil lattice admits an isometry of order four, which, viewed as an element of $O(2)$, may only be a rotation by $\pm\frac{\pi}{2}$. Up to isometry, all positive definite lattices of rank 2 admitting an isometry of order four have a Gram matrix of the form

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

for some $a > 0$. The formula

$$\det \operatorname{NS}(Y) = (-1)^r \det \operatorname{MWL}(Y) \det \operatorname{Triv}'(Y),$$

where $r = rk\,\mathrm{MWL}(Y) = 2$, $\det \mathrm{NS}(Y) = -1$, $\det \mathrm{Triv}'(Y) = 4$, resulting from the definition of $\mathrm{MWL}(Y)$ via the orthogonal projection with respect to $\mathrm{Triv}'(Y)$, yields $a = 1/2$.

$$\mathrm{MWL}(Y) \cong \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

We search for a section P with

$$1/2 = \langle P, P\rangle := 2\chi + 2P.O - c_0(P,P) - c_\infty(P,P) - c_1(P,P),$$

where $\chi = \chi(\mathcal{O}_Y) = 1$ is the Euler characteristic of Y and $c_0, c_\infty \in \{0, 1/2\}$, $c_1 \in \{0,1\}$ are the contributions at the singular fibers; $P.O \in \mathbb{N}$ is the intersection number. This immediately implies $P.O = 0$ and $c_1 = 1$. Let us assume $c_0 = 1/2, c_\infty = 0$. The section P can be given by $(x(t), y(t))$, where x, y are rational functions. Over the chart containing ∞, it is given by $(\hat{x}(s), \hat{y}(s)) = (s^2 x(1/s), s^3 y(1/s))$, where $s = 1/t$. As poles of these functions correspond to intersections with the zero section, we know that $x, y, \hat{x}, \hat{y}$ are actually polynomials. Therefore, $\deg x \leq 2$ and $\deg y \leq 3$. Furthermore, if P intersects the same fiber component as the zero section, then the contribution c_ν of that fiber is zero. The other components arise by blowing up the singularities of the Weierstraß model in $\{x = y = t(t-1) = 0\}$. Hence, $x(0) = y(0) = x(1) = y(1) = 0$. Putting this information together, we obtain $x = at(t-1)$ and $y = t(t-1)b$ for some constant a and a polynomial $b(x)$ of degree one. A quick calculation yields the sections

$$P' : \quad x(t) = \zeta^2 t(t-1), \qquad y(t) = \zeta^3 t(t-1)^2, \tag{7}$$

$$R' : \quad x(t) = -\zeta^2 t(t-1), \qquad y(t) = -\zeta t(t-1)^2, \tag{8}$$

for $\zeta^4 + 1 = 0$. Note that $\mathrm{MWL}(Y)$ contains exactly four sections of height $1/2$. Furthermore, $\langle P', R'\rangle = 0$. (Otherwise, $P = \ominus R$, which is clearly false). The two further Galois conjugates of ζ correspond to the missing two sections $\ominus P$ and $\ominus R$.

Now we pull back P' and R' to X via the map

$$\phi : \; X \to Y, \quad (x, y, t) \mapsto (xt^{2n}(t-1)^{4n+2}, yt^{3n}(t-1)^{6n+3}, t^p),$$

and get the sections

$$P : \quad x = \zeta^2 t^{2n+3}(t-1) \qquad y = \zeta^3 t^{n+3}(t-1)^{2n+3}, \tag{9}$$

$$R : \quad x = -\zeta^2 t^{2n+3}(t-1) \qquad y = -\zeta t^{n+3}(t-1)^{2n+3}. \tag{10}$$

It remains to compute the intersection numbers involving P and R. This can be done by blowing up the singularities and then computing the intersections. However, by applying some more of Shioda's theory, we can avoid the blowing ups. From the behavior of the height pairing under base extension (cf. [**94**, Prop. 8.12]), we know that

$$\langle P, P\rangle = \deg \phi \langle P', P'\rangle = p\langle P', P'\rangle = 2n + 3/2,$$

and get the equation

$$2n + 3/2 = 4 + 2P.O - c_0(P,P) - c_\infty(P,P) - c_1(P,P).$$

Note that a fiber of type III^* has only two simple components. Since P meets the same component over infinity as the zero section and passes through the singularities at $t = 0, 1$, we know that $c_\infty = 0$, $c_0 = 3/2$ and $c_1 = 1$. We conclude that $P.O = n$. The other intersection numbers can be calculated accordingly. In this way, we could calculate the intersection matrix of $\mathrm{NS}(X(p))$ without knowing equations for the extra sections.

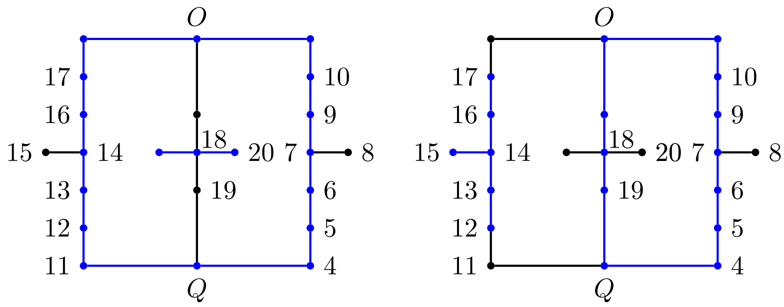

FIGURE 4. π' with I_{16} and I_4 fibers and π'' with I_{12} and IV^* fibers.

The section P induces an automorphism of the surface by fiberwise addition. We shall denote it by $(\oplus P)$. The matrix of its representation on NS is obtained as follows.

- Compute the basis representation of the sections $Q \oplus P, 2P$ and $R \oplus P$.
- Any section S is mapped to $P \oplus S$ under $(\oplus P)_*$, and the fiber is fixed.
- The action of $(\oplus P)$ on the Néron-Severi group preserves each singular fiber. Since it is an isometry, it can be determined by its action on sections.
- Invert the resulting matrix to get $(\oplus P)^* = (\oplus P)_*^{-1}$.

A basis representation of $P \oplus R$ is obtained as follows. Start with $P + R \in \mathrm{NS}$ and subtract nO, such that the resulting divisor D has $D.F = 1$. Add or subtract fiber components until D meets each singular fiber in exactly one simple component. Finally, add multiples of F such that $D^2 = -2$. For example, the basis representation of the section $P \oplus R$ is given by

$$(1, 2, -2, 0, 0, 0, 0, 0, 0, 0, -3, -4, -5, -6, -3, -4, -2, -1, -2, 0, 1, 1).$$

4. Alternative Elliptic Fibrations

The automorphism constructed in the last section has zero entropy. The reason for this is that it fixes the fibers. We overcome this obstruction by combining different fibrations and their sections.

Reducible singular fibers of elliptic fibrations are extended ADE-configurations of (-2)-curves. Conversely, let X be a K3 surface and F an extended ADE configuration of (-2)-curves. Then, the linear system $|F|$ is an elliptic pencil with the extended ADE configuration as singular fiber. See **[50]** and **[84]** for details. We use this fact to detect additional elliptic fibrations in the graph of (-2)-curves. Irreducible curves C are either fiber components, sections or multisections, depending on whether $C.F = 0, 1$ or > 1. Be aware that the (-2) curves occur with multiplicities in F. Hence, it is possible that $C.F > 1$ even if C meets only a single (-2)-curve of the configuration. In general, an elliptic pencil does not necessarily admit a section. However, once its existence is guaranteed, we may choose it as zero section. Then, by Theorem 6.6, multisections still induce sections once modified by fiber components and the zero section, as sketched above.

The first fibration π' is induced by the outer circle of (-2)-curves, which is a singular fiber of type I_{16}. There is a second singular fiber of type I_4. Three of its components are visible in Figure 4. The curve e_8 (= vertex labeled by '8') is a section since it meets I_{16} exactly in a simple component. We take it as zero section. Then, the torsion sections are e_{15}, e_{18} and e_{19}. The second fibration π'' is induced by the right inner circle of (-2)-curves. It has singular fibers of type I_{12} and IV^*, and again we take e_8 as zero section. A simple component of the IV^* fiber is not visible in Figure 4. In both cases P is a multisection and induces a section of each fibration denoted by P' and P''. For example, the class of $P' \in \mathrm{NS}(X(p))$ is given by

$$\begin{aligned} P' = (&n, n, n+1, 2, 2-n, -2n+2, 2-3n, 1-2n, 1-2n, -n, 0, \\ &-2n-2, -3-3n, -2n-2, -2n-2, -1-n, 0, 0, 0, 1, 0). \end{aligned}$$

5. Salem degree 22 automorphism

We consider the automorphism

$$f := (\oplus R) \circ (\oplus P) \circ (\oplus P') \circ (\oplus P'').$$

on $X(p)$ for $p \equiv 3 \mod 4$. Using a computer algebra system, one computes the characteristic polynomial of $f^*|\,\mathrm{NS}(X(p))$:

$$\mu(f^*) = x^{11} g(x + 1/x),$$

where

$$\begin{aligned} g(x) :=& 8n^2 + 88n + 67 \\ &+(-88n^3 - 392n^2 - 976n - 574)x \\ &+(-232n^3 - 1474n^2 - 2854n - 1464)x^2 \\ &+(534n^3 + 2526n^2 + 4605n + 2359)x^3 \\ &+(578n^3 + 3415n^2 + 6196n + 3062)x^4 \\ &+(-568n^3 - 2749n^2 - 4587n - 2245)x^5 \\ &+(-466n^3 - 2689n^2 - 4681n - 2253)x^6 \\ &+(206n^3 + 1014n^2 + 1600n + 770)x^7 \\ &+(148n^3 + 849n^2 + 1426n + 670)x^8 \\ &+(-24n^3 - 120n^2 - 182n - 91)x^9 \\ &+(-16n^3 - 92n^2 - 150n - 69)x^{10} \\ &+x^{11}. \end{aligned}$$

By Theorem 6.1, $\mu(f^*)$ is either a degree 22 Salem polynomial or is divisible by a cyclotomic polynomial of degree at most 22. There are only finitely many cyclotomic polynomials of a given degree. We can exclude the second case directly by computing the remainder after division for each such polynomial. This proves Theorem 6.5.

Remark 6.10. Further compositions of the four automorphisms realize Salem numbers of any even degree between 2 and 22. The full matrix representations of the automorphisms $(\oplus R), (\oplus P), (\oplus P'), (\oplus P'')$ involved are available upon request from the author.

6. Extension to higher Artin invariants

In the last section we proved a part of the following theorem:

Theorem 6.11. [**14, 32, 34, 91, 89, 19**] *The supersingular K3 surface X/k, $k = \overline{k}$, $\operatorname{char} k > 0$, of Artin invariant one has an automorphism of Salem degree 22.*

Recall that supersingular K3 surfaces are classified by their Artin invariant $1 \leq \sigma \leq 10$. For fixed Artin invariant σ, they form a family of dimension $\sigma - 1$, while the supersingular K3 surface of Artin invariant $\sigma = 1$ is unique (cf. [**85, 83**]).

The main purpose of this subsection is to extend Theorem 6.11 to *all* supersingular K3 surfaces.

THEOREM 6.12. *Let Y/k be a supersingular K3 surface over an algebraically closed field such that the crystalline Torelli theorem holds for Y. Then Y has an automorphism of Salem degree 22.*

REMARK 6.13. Set $p = \operatorname{char} k$ and $\sigma = \sigma(Y)$. The crystalline Torelli is proven for $p > 3$ in [**83**, Thm. I] and for $p = 2$ and $\sigma < 10$ and for $p = 3$ and $\sigma < 6$ (at the end of [**85**]). For $p = 3$ the main theorem is proved in [**91**]. Hence the only open case left is $p = 2$ and $\sigma = 10$. The main step in the proof is a reduction to Theorem 6.11.

In a recent preprint [**106**] Yu gives an independent proof of Theorem 6.12 for $p > 3$ using genus one fibrations. However, I believe the new proof to be of independent interest, as it is shorter and characteristic free. In particular the result for $p = 2, \sigma > 1$ is new.

For the readers' convenience, we give a proof of the following well known

LEMMA 6.14. *There is an embedding $N_{p,\sigma} \hookrightarrow N_{p',\sigma'}$ of supersingular K3 lattices if and only if $p = p'$ and $\sigma' \leq \sigma$.*

PROOF. The only if part follows from the fact that if $A \subset B$ are two lattices of the same rank, then
$$\det A = [B : A]^2 \det B.$$
In this situation
$$A \hookrightarrow B \hookrightarrow B^\vee \hookrightarrow A^\vee$$
and B/A is a totally isotropic subspace of $A^\vee/A$. Now, if A is 2-elementary and q_A integral, then, since $B^\vee \subseteq A^\vee$, so are B and q_B. Let $p \neq 2$. Then the quadratic space $N_{p,10}^\vee/N_{p,10} \cong \mathbb{F}_p^{20}$ contains an isotropic line since it is of dimension greater two. As above this line corresponds to an overlattice N of $N_{p,10}$ which is hyperbolic and $|N^\vee/N| = p^{18}$. Since subquotients of vector spaces are vector spaces, we see that $N^\vee/N \cong \mathbb{F}_p^{18}$. Then $N \cong N_{p,9}$ is in fact a supersingular K3 lattice. Continuing in the same way, we get a chain of overlattices
$$N_{p,10} \subseteq N_{p,9} \subseteq \ldots \subseteq N_{p,1}.$$
Note that the process stops at $\sigma = 1$ since there is no isomorphic line in the discriminant group. This is in accordance with the fact that there is no even unimodular lattice of signature $(1, 21)$. For $p = 2$, the discriminant form is isomorphic to a direct sum of forms of type $q(x, y) = x^2 + xy + y^2 \mod 2\mathbb{Z}$ and the existence of an isotropic vector follows as long as there are at least two summands, i.e., $\sigma > 1$. Since everything is contained in $N_{2,10}^\vee$, the discriminant form stays integral. □

Let L be an even lattice of signature $(1, n)$ and denote by $O^+(L)$ the subgroup of isometries preserving the two connected components of the positive cone.

The nef cone of a surface X is denoted by $\operatorname{Nef}(X)$. Classes of smooth rational curves are called *nodal*. By adjunction these are exactly the effective classes of square (-2). Note that if $r^2 = -2$, then by Riemann-Roch either r or $-r$ is effective.

THEOREM 6.15 (Cone conjecture). [**54**, Thm. 6.1] *Let X be a K3 surface over an algebraically closed field k. If X is supersingular, suppose that crystalline Torelli holds for X. Let $\Gamma(X) \subseteq O^+(\mathrm{NS}(X))$ be the subgroup preserving the nef cone. Then $\Gamma(X) \cong O^+(\mathrm{NS}(X))/W(\mathrm{NS}(X))$ and*

(1) The natural map $\operatorname{Aut}(X) \to \Gamma(X)$ has finite kernel and cokernel.
(2) The group $\operatorname{Aut}(X)$ is finitely generated.

(3) *The action of* $\mathrm{Aut}(X)$ *on* $\mathrm{Nef}(X)$ *has a rational polyhedral fundamental domain.*
(4) *The set of orbits of* $\mathrm{Aut}(X)$ *in the nodal classes of* X *is finite.*

Over $\mathbb{C}$ the theorem follows from the strong Torelli theorem by work of Sterk [**96**, Thm. 1]. Then, for K3 surfaces of finite height in arbitrary characteristic one can lift $X, \mathrm{NS}(X)$ and a finite index subgroup of $\mathrm{Aut}(X)$ to characteristic zero and apply the cone theorem there. For supersingular K3 surfaces one has to use the crystalline Torelli Theorem. In this case $\mathrm{Aut}(X) \to \Gamma$ is injective and its image contains the finite index subgroup $\ker(\Gamma \to O(\mathrm{NS}^\vee/\mathrm{NS}))$.

LEMMA 6.16. [**86**, p. 169] *If* λ *is a Salem number of degree* d *then* λ^n, $n \in \mathbb{N}$ *is a Salem number of the same degree.*

PROOF. Denote the Galois conjugates of $\lambda = \lambda_1$ by λ_i $i = 1, ..., n$. Then the Galois conjugates of λ_1^n are the λ_i^n. In particular λ_1^n is a Salem number. It remains to check that its conjugates are all distinct. Suppose that $\lambda_i^n = \lambda_k^n$. After applying a Galois conjugation we may assume that $i = 1$. In particular, $1 < \lambda_1^n = \lambda_k^n$. Now, $|\lambda_k| > 1$ is the unique conjugate of absolute value greater one, i.e. $k = 1$. □

COROLLARY 6.17. *The maximum occurring Salem degree of an automorphism of a K3 surface* X *over an algebraically closed field depends only on the isometry class of* $\mathrm{NS}(X)$*, given that the cone conjecture holds for* X*.*

PROOF. Since any power of a Salem number of degree d remains a Salem number of this degree, we may pass to a finite index subgroup. Combining this with part (1) of the cone conjecture, we get that the maximum occurring Salem degree of an automorphism of X depends only on $\Gamma(X)$. Now, $\Gamma(X)$ depends up to conjugation by an element of the Weyl group only on the isometry class of $\mathrm{NS}(X)$. In particular, the maximal Salem degree of an automorphism of X depends only on $\mathrm{NS}(X)$. □

LEMMA 6.18. *Let* $N \subseteq L$ *be two lattices of the same rank and* $G \subseteq O(L)$ *a subgroup. Then*

$$[G : O(N) \cap G] < \infty$$

where we view $O(N)$ *and* $O(L)$ *as subgroups of* $O(N \otimes \mathbb{R})$.

PROOF. Since the ranks coincide, the index $n = [L : N]$ is finite and

$$nL \subseteq N \subseteq L.$$

Any isometry of L preserves nL. Hence we get a map

$$\varphi \colon G \to \mathrm{Aut}(L/nL).$$

Set $K = \ker \varphi$, which is a finite index subgroup of G. To see that $K \subseteq O(N)$ as well, recall that an isometry f of $O(nL)$ extends to $O(N)$ iff $f(N/nL) = N/nL$. Indeed, $f|L/nL = id|L/nL$ for $f \in K$ by definition. □

The following is an improvement of [**106**, Thm. 1.2]. There the existence of at least one elliptic fibration on X with infinite automorphism group is assumed. We can drop this condition.

THEOREM 6.19. *Let* $X/k, Y/k'$ *be two K3 surfaces over algebraically closed fields* k, k' *satisfying the cone conjecture. Suppose that* $\rho(X) = \rho(Y)$ *and that there is an isometric embedding*

$$\iota : \mathrm{NS}(Y) \hookrightarrow \mathrm{NS}(X).$$

Then $\mathrm{sdeg}(X) \leq \mathrm{sdeg}(Y)$ *where* $\mathrm{sdeg}(X) = \max\{$*Salem degree of* $f | f \in \mathrm{Aut}(X)\}$.

PROOF. Any chamber of the positive cone of $\mathrm{NS}(X)$ is contained in the image of a unique chamber of the positive cone of $\mathrm{NS}(Y)$. Since the Weyl group acts transitively on the chambers in one connected component of the positive cone, we can find an element $\delta \in W(\mathrm{NS}(X))$ of the Weyl group such that $\mathrm{Nef}(X) \subset \iota'_{\mathbb{R}}(\mathrm{Nef}(Y))$ where $\iota' = \pm\delta \circ \iota$. To ease notation we identify $\mathrm{NS}(Y)$ with its image under ι'. By the preceding Lemma $[\Gamma(X) : \Gamma(X) \cap O(\mathrm{NS}(Y))]$ is finite, and since $\mathrm{Nef}(X) \subseteq \mathrm{Nef}(Y)$, we get that $\Gamma(X) \cap O(\mathrm{NS}(Y)) \subseteq \Gamma(Y)$. Now, by the cone Theorem 6.15 and the proof of Corollary 6.17

$$\mathrm{sdeg}(X) = \mathrm{sdeg}(\Gamma(X)) = \mathrm{sdeg}(\Gamma(X) \cap O(\mathrm{NS}(Y))) \leq \mathrm{sdeg}(\Gamma(Y)) = \mathrm{sdeg}(Y).$$

□

PROOF OF THEOREM 6.12. If X/k and Y/k are supersingular K3 surfaces with $\sigma(X) \leq \sigma(Y)$, then $\mathrm{NS}(Y) \hookrightarrow \mathrm{NS}(X)$ by Lemma 6.18. Combining the $\sigma = 1$ case (Thm. 6.11) and the previous theorem we get that

$$22 = \mathrm{sdeg}(X) \leq \mathrm{sdeg}(Y) \leq 22.$$

□

The converse inequality in Theorem 6.19 is false in general. See [**106**, rmk. 7.3] for examples.

CHAPTER 7

Minimal Salem numbers on supersingular K3 surfaces

Instead of considering only the Salem degree of an automorphism, in this chapter we focus on the existence of automorphisms of (supersingular) K3 surfaces with a given entropy. Here we give a strategy to decide whether the logarithm of a given Salem number is realized as entropy of an automorphism of a supersingular K3 surface in positive characteristic. As test case it is proved that $\log \lambda_d$, where λ_d is the minimal Salem number of degree d, is realized in characteristic 5 if and only if $d \leq 22$ is even and $d \neq 18$. In the complex projective setting we settle the case of entropy $\log \lambda_{12}$ left open by McMullen, by giving the construction. A necessary and sufficient test is developed to decide whether a given isometry of a hyperbolic lattice, with spectral radius bigger than one, is positive, i.e. preserves a chamber of the positive cone. This chapter is joint work with Víctor Gonzàlez-Alonso.

For complex projective K3 surfaces, it is proved in [**64**] that λ_d is the spectral radius of an automorphism exactly for $d = 2, 4, 6, 8, 10$ or 18, but not if $d = 14, 16$ or $d \geq 20$, while the case $d = 12$ is left open. As a byproduct of our work, we are able to realize also λ_{12} in the complex case (see § 4), hence proving the following

Theorem 7.1 (Improvement of Theorem 1.2 in [**64**]). *The value* $\log \lambda_d$ *arises as the entropy of an automorphism of a complex projective K3 surface if and only if* $d = 2, 4, 6, 8, 10, 12$ *or* 18.

The proof involves methods from integer linear programming, lattice theory, number fields, reflection groups and the Torelli theorem for complex K3 surfaces.

The main purpose of this work is to extend the tools developed for the proof of this theorem in [**62, 63, 64**] to supersingular K3 surfaces in positive characteristic. The reason to consider the supersingular case is that there is a Torelli theorem readily available while in the non-supersingular case most (all for $p \geq 23$) automorphisms lift to characteristic zero (cf. [**46**]) and can be treated there. In order to illustrate the concepts, we prove the following

Theorem 7.2. *The value* $\log \lambda_d$ *arises as the entropy of an automorphism of a supersingular K3 surface over a field of characteristic* $p = 5$ *if and only if* $d \leq 22$ *is even and* $d \neq 18$.

Here $p = 5$ is chosen because it is the smallest prime for which the crystalline Torelli theorem is fully proven. The same methods apply for any other $p \geq 5$. They handle a single Salem number and one characteristic at a time (sometimes we can deal with p ranging in an arithmetic progression in the spirit of the previous chapter 6 and [**89**]).

In what follows we highlight some of the differences and challenges between the complex and supersingular case. Let λ be a Salem number, $s(x)$ its minimal polynomial. In the complex case let $F : X \to X$ be an automorphism of a projective K3 surface over $\mathbb{C}$ with $h(F) = \log \lambda$. The singular cohomology $H^2(X, \mathbb{Z})$ carries an integral bilinear form turning it into an even *unimodular* lattice of signature $(3, 19)$, on which $f = F^*$ acts as an isometry. It respects further structure such as the Hodge decomposition and the ample cone in $\mathrm{NS}(X) \otimes \mathbb{R} \subseteq H^2(X, \mathbb{R})$. The Torelli Theorem (1.3) states that this datum determines the pair (X, f) up to isomorphism and conversely, that each (good) datum is coming from such a pair. So, in order to construct examples one has to produce a certain lattice together with a (suitable) isometry on it.

The characteristic polynomial of f factors as

$$\chi(f|H^2(X, \mathbb{Z})) = s(x)c(x)$$

where $c(x)$ is a product of cyclotomic polynomials. The *Salem* and *cyclotomic factors* are defined then as

$$S := \ker s(f|H^2(X, \mathbb{Z})) \quad \text{and} \quad C := \ker c(f|H^2(X, \mathbb{Z})).$$

They are lattices of signatures $(1, d-1)$ and $(2, 20-d)$, $C = S^\perp$ and $S \oplus C$ is of finite index in $H^2(X, \mathbb{Z})$. From the unimodularity of the latter we get an isomorphism (called glue map) of discriminant groups

$$D_S \cong D_C$$

compatible with the action of f. It follows that the polynomials $s(x)$ and $c(x)$ have a common factor modulo any prime q dividing $\det S$. Indeed, the minimal polynomials of $f|D_S/qD_S$ and $f|D_C/qD_C$ agree and divide $s(x)$ and $c(x)$ modulo q. The possible values of these *feasible* primes are readily computed from $s(x)$ alone, thus limiting possibilities for S (and C). To reverse the process one first constructs models for S and C by number and lattice theory (Chap. 2 §5) and then *glues* (Chap. 2 §2) them together via the isomorphism $D_S \cong D_C$ to obtain a model for $H^2(X, \mathbb{Z})$ together with an isometry f. It is then checked that f preserves a Hodge structure, represented by a suitable eigenvector of $f|H^2(X, \mathbb{Z}) \otimes \mathbb{C}$. The crucial step is to check whether $f|NS(X) \otimes \mathbb{R}$ preserves a chamber representing the ample cone cut out by the nodal roots. In general it is hard to compute the (infinitely many) nodal roots, hence in **[64]** an integer linear programming test is developed, which gives a sufficient but not necessary condition. To resolve this uncertainty we develop a (convex) quadratic integer program refining the linear one which always gives an answer and yet is fast to compute (see §1).

Let us now consider an algebraically closed field κ of positive characteristic $p = \operatorname{char} \kappa > 0$, and let X/κ be a supersingular K3 surface. Then $\mathrm{NS}(X)$ is an even lattice of signature $(1, 21)$ and determinant $-p^{2\sigma}$ for some $1 \leq \sigma \leq 10$ (the so-called *Artin invariant*). As before, f preserves the ample cone of $\mathrm{NS}(X) \otimes \mathbb{R}$ cut out by the nodal roots, as well as some extra structure (a *crystal*) represented by an eigenvector of $\overline{f}|D_{\mathrm{NS}} \otimes \kappa$. It is proved for $p > 3$ that this datum determines (X, F) and any (good) datum is realized (this is more or less the content of Ogus' *Crystalline Torelli theorem*, see §1).

Thus, in our construction we have to replace $H^2(X, \mathbb{Z})$ by $\mathrm{NS}(X)$ and the Torelli theorem gets a new flavor. The characteristic polynomial of $f|\,\mathrm{NS}(X)$ still factors as

$$\chi(f|\,\mathrm{NS}) = s(x)c(x)$$

where $c(x)$ is a product of cyclotomic polynomials, and the Salem and cyclotomic factors can be analogously defined as

$$S := \ker s(f|\,\mathrm{NS}) \quad \text{and} \quad C := \ker c(f|\,\mathrm{NS}).$$

Notice that the signature of S is still $(1, d-1)$ but now that of C is $(0, 22-d)$. Again $S \oplus C$ is of finite index in $\mathrm{NS}(X)$, but since the latter is not unimodular, there is only a partial gluing between certain subgroups (see § 2)

$$D_S \supseteq G_S \xrightarrow{\phi} G_C \subseteq D_C.$$

One can show that $pD_S \subseteq G_S$, so in this case $s(x)$ and $c(x)$ have a common factor modulo any prime dividing $|pD_S|$. In particular, we take a look again at the feasible primes in §2.

Checking whether f preserves the ample cone of $\mathrm{NS}(X)$ is done exactly as in the complex case. The only difference is that there the failure of necessity of the linear positivity test is less severe, since often one can try a construction with a different $\mathrm{NS}(X)$ and hope for a positive result there. However, in the supersingular case we have less freedom on $\mathrm{NS}(X)$ once deciding for a fixed characteristic p. It was for this reason that we developed the quadratic positivity test described in Theorem 7.9.

Notation. For an even $d > 0$, λ_d denotes the minimal Salem number of degree d, and $s_d(x)$ the corresponding minimal polynomial, which are explicitly included in Appendix 5. Recall that for any integral $k > 0$, $c_k(x)$ denotes the k-th cyclotomic polynomial.

1. Positivity

Recall from the previous section that, given an isometry f of a supersingular K3 lattice N, we are interested in knowing whether f preserves some connected component of V_N. To this end it is useful to consider more general lattices than only supersingular K3 lattices.

In what follows, let L denote an even lattice which is either hyperbolic (has signature $(1, n_-)$) or negative definite (has signature $(0, n_-)$). Recall that

$$\Delta_L = \{\delta \in L \mid \delta^2 = (\delta, \delta) = -2\}$$

is the set of *roots* of L, and that the connected components of

$$V_L = \{x \in L \otimes \mathbb{R} \mid x^2 > 0 \text{ and } (\delta, x) \neq 0 \quad \forall \delta \in \Delta_L\}$$

are the *chambers* of L.

DEFINITION 7.3 (Positive automorphism). We say that an isometry $f \in O(L)$ is *positive* if it preserves some connected component of V_L.

If L is hyperbolic, the *light cone* $\{x \in L \otimes \mathbb{R} \mid x^2 > 0\}$ has two connected components, and any positive isometry $f \in O(L)$ does not interchange them. We denote by $O^+(L)$ the subgroup of isometries with this property.

Here we will summarize some criteria to test the positivity of a given $f \in O^+(L)$. Most of these definitions and results are due to McMullen [**64**]. Note that there signs follow a different convention because the lattices are considered to have signatures $(n_+, 0)$ and $(n_+, 1)$. Since the chamber structure of V_L is given by the roots of L, the positivity of f is naturally related to its action on the set of roots Δ_L, and indeed there are two special kinds of roots.

DEFINITION 7.4. [**64**, Obstructing and cyclic roots] Let $\delta \in \Delta_L$ be a root of a negative definite or hyperbolic lattice L, and $f \in O^+(L)$ an isometry.

- We say that δ is *obstructing* for f if there is no linear form $\phi \in \mathrm{Hom}(L, \mathbb{R})$ such that the bilinear form on $\ker \phi \subset L \otimes \mathbb{R}$ is negative definite and $\phi(f^i(\delta)) > 0$ for all $i \in \mathbb{Z}$.
- We say that δ is *cyclic* for f if $\delta + f(\delta) + f^2(\delta) + \cdots + f^i(\delta) = 0$ for some $i > 0$.

Obviously, any cyclic root is also obstructing. Conversely if L is negative definite, all obstructing roots are cyclic.

REMARK 7.5. To motivate the definition of obstructing roots, suppose that L is the Néron-severi lattice of some projective K3 surface X, f is induced by some automorphism $F : X \to X$, and let $h \in L$ be the class of an ample line bundle. If $\delta \in \Delta_L$ is a root, a standard computation using Riemann-Roch shows that either δ or $-\delta$ is effective. In the first case, also $f^i(\delta)$ is effective for every $i > 0$, and hence $\langle h, f^i(\delta)\rangle > 0$ for every $i > 0$. Thus, the linear form $\phi = \langle h, -\rangle$ shows that δ cannot be obstructing (the negative-definiteness on $\ker\phi$ follows from the Hodge-index theorem). In case $-\delta$ is effective, then $\phi = -\langle h, -\rangle$ leads to the same conclusion. Therefore, an obstructing root is indeed an obstruction to the existence of an ample line bundle on X.

Obstructing roots characterize the positivity of f in the following way.

THEOREM 7.6. [**64**, Theorem 2.2] *An isometry $f \in O^+(L)$ is positive if and only if it has no obstructing roots.*

In order to check whether f has obstucting roots, McMullen [**64**, Section 3] developed the following method. Assume that $f \in O^+(L)$ has positive spectral radius. It can be shown that its characteristic polynomial factors as a product

$$\chi_f = s(x) \cdot c(x)$$

of a Salem polynomial $s(x)$ and a product of cyclotomic polynomials $c(x)$ (cf. [**62**]). As usual, set

$$S = \ker s(f) \qquad \text{and} \qquad C = \ker c(f).$$

Then S is hyperbolic and C negative definite. Note furthermore that f diagonalizes on $L\otimes\mathbb{C}$. Indeed, $f|S$ is semisimple because $s(x)$ is separable, while $f|C$ is of finite order.

First one looks for cyclic roots, which are precisely the roots of the sublattice C. Since C is negative-definite, the set of roots in C is easily computed. We can therefore assume that f has no cyclic root, for otherwise it is not positive. Let $a = f + f^{-1}$ and $A = \mathbb{R}[a] \subset \mathrm{End}_{\mathbb{R}}(L \otimes \mathbb{R})$. Given any $x \in L$, let $\psi_x : A \to \mathbb{R}$ be the *pure state* defined by $\psi_x(a) = \langle a(x), x\rangle$, and consider the lattice of *mixed states* $M \subset \mathrm{Hom}_{\mathbb{R}}(A, \mathbb{R})$ spanned by $\{\psi_x \,|\, x \in L\}$. If $e_1, \ldots, e_n$ is a $\mathbb{Z}$-basis of L, M turns out to be generated by the ψ_{e_i} and $\psi_{e_i+e_j}$ [**64**, Proposition 3.2]. By construction, a diagonalizes with real eigenvalues, which we denote by $\tau_1 > \tau_2 > \ldots > \tau_r$. Then we define $V_i = \ker(a - \tau_i \,\mathrm{Id}) \subset L \otimes \mathbb{R}$, obtaining an f-invariant orthogonal decomposition $L \otimes \mathbb{R} = \bigoplus_{i=1}^r V_i$. Let $p_1, \ldots, p_r$ be the corresponding projections, so that $p_i^2 = p_i$, $a \circ p_i = \tau_i p_i$ and $\sum_{i=1}^r p_i = 1$.

With all these ingredients we can define the following integer linear programming problem: let

$$\mu(f) = \max\{\psi(1) \,|\, \psi \in M, \psi(p_1) < 0 \text{ and } \psi(p_i) \leq 0 \quad \forall\, i > 1\}. \tag{11}$$

Note that by construction $\psi(1) \in 2\mathbb{Z}$ for any $\psi \in M$. Hence in any case $\mu(f) \leq -2$.

THEOREM 7.7. [**64**, Theorem 3.3, Linear positivity test] *Let $f \in O^+(L)$ be an isometry with positive spectral radius on a hyperbolic lattice L. Then it is positive if it has no cyclic roots and $\mu(f) < -2$.*

REMARK 7.8. Note that the statement is not an equivalence. Indeed, there are examples of positive automorphisms with $\mu(f) = -2$ (see [**64**]). The reason for this failure is that the maximizing ψ is not necessarily a pure state ψ_δ for some $\delta \in L$.

Instead, it might be a linear combination of pure states. In order to resolve this matter, we use the following result.

THEOREM 7.9 (Quadratic positivity test). *Fix $y \in V_1$ with $y^2 > 0$ and for $\psi \in M$ set*

$$B_\psi = \{x \in L \otimes \mathbb{R} \mid \langle x, y\rangle \leq 0,\ \langle x, f(y)\rangle \geq 0,\ \text{and } p_i(x)^2 = \psi(p_i),\ \text{for all } i\}.$$

Then f is positive if and only if it has no cyclic roots and for every optimal solution ψ of the linear positivity test (11) with $\psi(1) = -2$, the compact set B_ψ has no integral points, i.e. $B_\psi \cap L = \emptyset$.

PROOF. By the previous discussion it is enough to show that an optimal solution ψ with $\psi(1) = -2$ is pure if and only if $B_\psi \cap L \neq \emptyset$.

Suppose that ψ is pure, i.e., $\psi = \psi_\delta$ for some $\delta \in L$. Observe that

$$\psi_\delta(p_i) = \langle p_i(\delta), \delta\rangle = p_i(\delta)^2.$$

It remains to check the two inequalities in the definition of B_ψ. Since $\langle x, y\rangle = \langle p_1(x), y\rangle$, we only need to consider the situation in V_1. It is an indefinite plane, and there the points of length $\psi(p_1)$ form a (non-compact) hyperbola whose asymptotes $\{x^2 = 0\}$ are the eigenspaces of $f|V_1$. Furthermore f acts by translation along this hyperbola (Figure 1). Since f is an isometry and commutes with $\mathbb{R}[a]$, we get that $\psi_\delta = \psi_{\pm f^n(\delta)}$. Hence, after replacing δ by a suitable $\pm f^n(\delta)$, we can assume that $\delta \in B_\psi$.

We now turn to the compactness of B_ψ. Obviously $B_\psi = \bigoplus_i (V_i \cap B_\psi)$. Recall that V_i is negative definite for $i \geq 2$, hence $(V_i \cap B_\psi)$ is a (compact) sphere of radius $\sqrt{-\psi(p_i)}$. Since $y^2 > 0$, the lines $y^\perp \cap V_1$ and $f(y)^\perp$ intersect each connected component of the hyperbola in a single point. Then $V_1 \cap B_\psi$ is simply the path along one connected component of the hyperbola between these two points, which is thus compact. $\square$

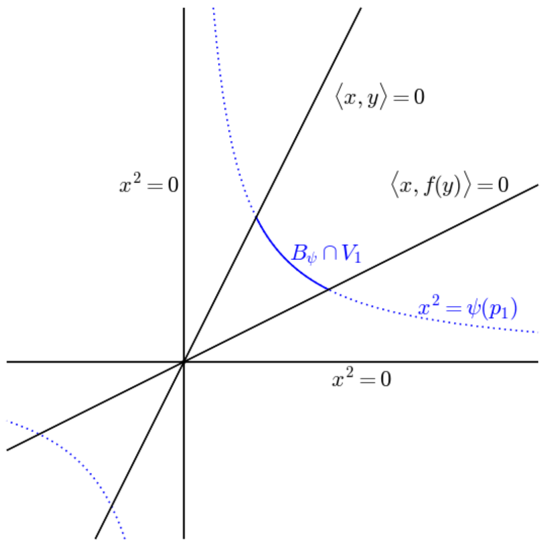

FIGURE 1. Fundamental domain in V_1.

REMARK 7.10. For practical applications, we compute the integral points of the convex hull $\mathrm{Conv}(B_\psi)$ with SCIP [**1**] and CPLEX [**44**] and check which of them are roots. Depending on the rank, computation times vary between seconds and a few minutes.

The following easy Corollary shows that we do not need to care too much about the isometry on the cyclotomic factor as long as it is positive.

COROLLARY 7.11. *Let $S \oplus C \hookrightarrow L$ be a primitive extension of a hyperbolic lattice S and a negative definite lattice C. Let $f_S \in O^+(S), f_1, f_2 \in O(C)$ be positive automorphisms such that $f_S \oplus f_i, i = 1, 2$ extends to L. Then $(L, f_S \oplus f_1)$ is positive if and only if $(L, f_S \oplus f_2)$ is.*

PROOF. Since the f_i are of finite order, we can find $n \in \mathbb{N}$ such that $(f_S \oplus f_1)^n = (f_S \oplus f_2)^n$. To finish the proof note that if an obstructing root is not cyclic it will stay obstructing for all powers of an isometry. □

2. Realizability Conditions

We will now connect our knowledge of the crystalline Torelli theorem and gluing to study automorphisms of positive algebraic entropy on supersingular K3 surfaces defined over an algebraically closed field κ of characteristic $p \geq 5$.

From now on, let X be such a supersingular K3 surface with Néron-Severi lattice NS. Let $F \in \mathrm{Aut}(X)$ be an automorphism, $f = F^* : \mathrm{NS} \to \mathrm{NS}$ the corresponding isometry of NS, and $\overline{f} : D_{\mathrm{NS}} \to D_{\mathrm{NS}}$ the induced isometry of the discriminant group (or its κ-linear extension).

Recall that the characteristic polynomial of f factors as

$$\chi_f = s(x)c(x)$$

where $s(x)$ is a Salem polynomial and and $c(x)$ is a product of cyclotomic polynomials. Moreover, the sublattices

$$S = \ker s(f) \qquad \text{and} \qquad C = \ker c(f).$$

are respectively hyperbolic and negative definite. In particular, the action of f on $\mathrm{NS} \otimes \mathbb{C}$ is semisimple, i.e., the minimal polynomial is separable. The inclusion

$$S \oplus C \hookrightarrow \mathrm{NS}$$

is a primitive extension of S and C. By Theorem 2.13, gluing can occur only over the primes $q \mid \mathrm{res}(s, c)$. We call such primes *feasible for c and s*.

COROLLARY 7.12. *If $\mathrm{char}\,\kappa = p$ is not feasible for c and s, then either $D_{S,p} = 0$ or $D_{C,p} = 0$ (where $D_{S,p}$ resp. $D_{C,p}$ denotes the p-primary part of D_S resp. D_C).*

PROOF. If p is not feasible for c and s, then $p \nmid res(s, c)$ and hence we cannot glue over p, i.e. $D_{\mathrm{NS}} = D_{\mathrm{NS},p} = D_{S,p} \oplus D_{C,p}$. In particular

$$\chi_{\overline{f}|D_{S,p}} \cdot \chi_{\overline{f}|D_{C,p}} = \chi_{\overline{f}|D_{\mathrm{NS}}},$$

which is a perfect power by Theorem 5.11. However, $\chi_{\overline{f}|D_{S,p}} \mid \overline{s(x)}$ and $\chi_{\overline{f}|D_{C,p}} \mid \overline{c(x)}$ are coprime. This is only possible if $D_{S,p} = 0$ or $D_{C,p} = 0$. □

Note that a priori we only know $s(x)$, the minimal polynomial of the Salem number we want to realize as the entropy of f, but there are many possibilities for $c(x)$. As a first constraint, we know that $c(x)$ is a product of cyclotomic polynomials $c_k(x)$ of total degree $22 - \deg s(x)$. Thus we say that a prime number $q \in \mathbb{Z}$ is *feasible* (for $s(x)$) if

$$q \mid \prod_{\varphi(k) \leq 22-d} \mathrm{res}(s, c_k),$$

or equivalently, if the reduction $\overline{s}(x) \in \mathbb{F}_p[x]$ has a cyclotomic factor of degree at most $22 - \deg s$. In particular, we can glue at most over the feasible primes.

The following Theorem gives a list of necessary conditions for an isometry on S to admit an extension to NS and further restrictions on the cyclotomic factor. We denote by $D(n)$ the minimum $D \geq 0$ such that $\mathbb{Z}^D$ has an automorphism of order n. Note that $D(1) = 0$, $D(2) = 1$ and $D(n) = D(n/2)$ if $n \equiv 2 \mod 4$. In any other case we have

$$D(p_1^{e_1} \cdot \ldots \cdot p_s^{e_s}) = \sum \varphi(p_i^{e_i})$$

for the prime decomposition of n.

THEOREM 7.13. *Let f, $s(x)$ and S be defined as above. Then:*

1. *The determinant of S is divisible only by the feasible primes (for s) and the characteristic p.*
2. *The order n of $\overline{f}$ on the subgroup $pD_S \subseteq D_S$ satisfies $D(n) \leq 22 - \deg(s)$.*
3. *There is a product of distinct cyclotomic polynomials $\mu(x)$ with $\deg \mu(x) \leq 22 - \deg s(x)$ and $\mu(\overline{f}|pD_S) = 0$.*
4. *$f|S$ is positive.*

PROOF.

1. By Lemma 2.9 the index $[D_S : G_S]$ is a power of p, while Theorem 2.13 implies that only feasible primes divide $|G_S|$.
2. The order n of $f|C$ satisfies $D(n) \leq 22 - \deg(s)$ and it is a multiple of the order of $\overline{f}|G_C$, which in turn is a multiple of the order on $pD_S \subset G_S \cong G_C$.
3. The isomorphism $G_S \cong G_C$ is compatible with f. Let μ be the minimal polynomial of $f|C$, which is a product of distinct cyclotomic polynomials because f acts semisimply on NS. Then $\mu(f)$ vanishes on C and consequently on D_C. By compatibility of the action it vanishes on $G_C \cong G_S \supseteq pD_S$ as well.
4. f is itself positive (on NS), and therefore so is any restriction.

□

We close this section with a finiteness result on realizable twists of a given lattice.

PROPOSITION 7.14. *Let $s(x)$ be a simple Salem polynomial and L_0 the principal $s(x)$-lattice. Then only a finite number of twists $L_0(a)$, $a \in \mathcal{O}_k/N(\mathcal{O}_K^\times)$ is realizable as Salem factor of an automorphism of a supersingular K3 surface in fixed characteristic p.*

We give two proofs of this fact.

FIRST PROOF. Since the associates of $a \in \mathcal{O}_k$ define only finitely many non-isomorphic $s(x)$-lattices, it suffices to bound the possible prime factorizations of a in $\mathcal{O}_K$ such that

$$S := L_0(a) \cong \ker s(f|\,\mathrm{NS})$$

where $f \in \mathrm{Aut}(X)$ of a supersingular K3 surface in characteristic p. Note that the norm $N(a) \mid \det S$. Hence, it suffices to bound $\det S$. Since NS is p-elementary, we can apply Corollary 2.12 which says that S is $p \cdot \mathrm{res}(s(x), c(x))$-elementary, i.e., $p \cdot \mathrm{res}(s(x), c(x))D_S = 0$. Clearly, this bounds $\det S$. □

SECOND PROOF. According to Lemma 2.20, we can find an ideal $I < \mathcal{O}_K$ such that

$$D_{L_0(a)} \cong \mathcal{O}_K/I$$

as $\mathbb{Z}[f] = \mathcal{O}_K$-modules. By Theorem 7.13 $|D_{L_0(a)}|$ is divisible at most by the feasible primes and p. Thus only finitely many prime ideals $\mathfrak{p}$ are possible divisors of I and hence of $a\mathcal{O}_K$.

By Theorem 7.13 the order of $f|pD_{L_0(a)}$ is bounded. We view $pD_{L_0(a)}$ as an ideal of $\mathcal{O}_K/I$. Using the Chinese remainder theorem we can reduce to the case that $I = \mathfrak{p}^l$ has a single prime divisor and

$$pD_{L_0(a)} \cong p\left(\mathcal{O}_K/\mathfrak{p}^l\right) = \mathfrak{p}^e/\mathfrak{p}^l \cong \mathcal{O}_K/\mathfrak{p}^{l-e}$$

for $e \in \mathbb{N}$ the ramification index of $\mathfrak{p}|p$ which is independent of l. Looking at Lemma 7.17 below we see that the order of f on $\mathcal{O}/\mathfrak{p}^{l-e}$ grows exponentially in l, proving that l is bounded above as wanted. □

In the above proof we needed to control the order of an automorphism of $\mathcal{O}_K/\mathfrak{p}^n$. We may replace $\mathcal{O} = \mathcal{O}_K$ by its completion $\hat{\mathcal{O}}$ at $\mathfrak{p}$ since $\mathcal{O}/\mathfrak{p}^l\mathcal{O} \cong \hat{\mathcal{O}}/\hat{\mathfrak{p}}^l\hat{\mathcal{O}}$. We can thus use the following elementary results from the theory of p-adic numbers. Let K be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}$ its ring of integers with maximal ideal $\mathfrak{p}$, and $\nu_\mathfrak{p}$ the corresponding normalized valuation. Let e be the ramification index of p, that is, $p\mathcal{O} = \mathfrak{p}^e$.

Proposition 7.15. [**73**, II Prop. 3.10 and 5.5] *Let $U^{(n)} = 1 + \mathfrak{p}^n \subset \mathcal{O}^\times$. Then*

$$\mathcal{O}^\times/U^{(n)} \cong (\mathcal{O}/\mathfrak{p}^n)^\times$$

for $n \geq 1$. Furthermore, the power series

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \textit{and} \quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots,$$

yield, for $n > \frac{e}{p-1}$, two mutually inverse isomorphisms

$$\mathfrak{p}^n \rightleftarrows U^{(n)}.$$

Lemma 7.16. *In the setting of the preceding proposition let $f \in U^{(n)} \setminus U^{(n+1)}$. For $l \geq n > \frac{e}{p-1}$, the order of f in $U^{(n)}/U^{(l)}$ is $p^{\lceil \frac{l-n}{e} \rceil}$.*

Proof. By assumption $l \geq n > \frac{e}{p-1}$ so the order of f in $U^{(n)}/U^{(l)}$ is that of $\log(f)$ in $\mathfrak{p}^n/\mathfrak{p}^l$. Write $f = 1 + z$ for $z \in \mathfrak{p}^n \setminus \mathfrak{p}^{n+1}$. It follows from the proof of Proposition 7.15 that

$$\nu_\mathfrak{p}(\log(1+z)) = \nu_\mathfrak{p}(z) = n.$$

Note that $kz \equiv 0 \mod \mathfrak{p}^l$ iff $l \leq \nu_\mathfrak{p}(kz) = \nu_\mathfrak{p}(k) + n$ iff $l - n \leq \nu_\mathfrak{p}(k)$. The smallest such $k \in \mathbb{N}$ is the order of f in $\left(\mathcal{O}/\mathfrak{p}^l\right)^\times$. It equals $p^{\lceil \frac{l-n}{e} \rceil}$. □

Lemma 7.17. *For $f \in \mathcal{O}^\times$, denote by $o(f,l)$ the order of $\overline{f} \in (\mathcal{O}/\mathfrak{p}^l)^\times$. If $l \geq n > \frac{e}{p-1}$, where $f^{o(f,1)} \in U^{(n)} \setminus U^{(n+1)}$, then*

$$o(f,l) = o(f,1)p^{\lceil \frac{l-n}{e} \rceil}.$$

Proof. Let $\alpha = o(f,1)$. With $\alpha \mid o(f,l)$, we get that

$$o(f^\alpha, l) = \mathrm{lcm}(o(f,l),\alpha)/\alpha = o(f,l)/\alpha.$$

Thus, after replacing f by f^α, the conditions of Lemma 7.16 are fulfilled and the order of f is $\alpha p^{\lceil \frac{l-n}{e} \rceil}$ as claimed. □

3. Realized Salem numbers

We summarize now the strategy that can be followed to realize a (simple) Salem number λ as the exponential of the entropy of an automorphism of a supersingular K3 surface. Often the analysis covers all cases, and we can obtain the non-existence as well. Let $s(x)$ be the minimal polynomial of λ, of degree d, and $r(y)$ be the corresponding trace polynomial, then the field $K = \mathbb{Q}[x]/s(x)$ is a quadratic extension of $k = \mathbb{Q}[y]/r(y)$. If s is simple, then $\mathcal{O}_K = \mathbb{Z}[x]/s(x)$ has class number one, and furthermore $\mathcal{O}_k = \mathbb{Z}[y]/r(y)$. For simplicity, we assume also that $h(k) = 1$, since this is the case for every Salem number we are interested in. For $h(k) > 1$, the arguments can be adapted.

(1) Construct the principal isometry $f_0 : L_0 \to L_0$ with characteristic polynomial $s(x)$.
(2) Compute the set P consisting of the primes in $\mathcal{O}_k$ lying over the feasible primes for $s(x)$ and add the primes in $\mathcal{O}_k$ above p.
(3) Let A be the set consisting of those $a \in \mathcal{O}_k$ which are a product of the primes in P and satisfy $D(n) \leq 22-d$, where n is the order of $\overline{f_0}|pD_{L_0(a)}$. This set is finite in virtue of Proposition 7.14.
(4) Replace A with the subset of those $a \in A$ which satisfy $\mu(\overline{f_0}|pD_{L_0(a)}) = 0$ for some product μ of distinct cyclotomic polynomials of degree at most $22-d$.
(5) If p is not feasible, keep only those $a \in A$ such that the minimal polynomial of $\overline{f_0}|(D_{L_0(a)})_p$ is irreducible in $\mathbb{F}_p[x]$.
(6) Denote by $U \subseteq \mathcal{O}_k^\times$ a set of representatives of $\mathcal{O}_k^\times/\mathcal{O}_k^{\times 2}$ and replace A with the set of those $au \in AU$ such that the signature of $L_0(au)$ is $(1, d-1)$.
(7) Replace A with the subset of those $a \in A$ such that $f_0|L_0(a)$ is positive by the quadratic positivity test.
(8) Find an $a \in A$, a negative definite lattice C of rank $22-d$ and a positive $f_C \in O(C)$ that can be glued to $(L_0(a), f_0)$ to obtain a positive isometry of a supersingular K3 lattice.

Steps (1)-(7) are easily implemented on a computer algebra system. While step (8) is finite in principle, computations are only feasible for small ranks of C. Indeed, at this point we have only a finite number of possibilities for the genus of C, each genus contains only a finite number of classes and each class has only a finite number of isometries. Each of these enumerations can be obtained explicitly (there are implementations for example in Magma), but computation times grow rapidly with the rank of C.

To illustrate our results we apply the strategy above to determine which minimal Salem numbers λ_d are realized in characteristic 5. The reason to chose 5 is that it is the smallest for which the Torelli theorems are fully available. In principle any other $p > 3$ is possible. The constructions are mostly carried out with a package developed by the author for the computer algebra system SageMath **[27]**, while computations for positivity are done with SCIP **[1]** and CPLEX **[44]**.

THEOREM 7.18. *The value* $\log \lambda_d$ *arises as the entropy of an automorphism of a supersingular K3 surface over a field of characteristic* $p = 5$ *if and only if* $d \leq 22$ *is even and* $d \neq 18$.

To prove the theorem we consider each minimal Salem number λ_d separately.

PROPOSITION 7.19. *The minimal Salem number λ_{22} in degree 22 is realized on a supersingular K3 surface with Artin invariant $\sigma = 4$ and $\sigma = 7$ in characteristic 5.*

PROOF. Since the Salem factor is of degree 22, no gluing is required. The principal s_{22}-lattice is unimodular and 5 factors in $\mathcal{O}_k$ as a product of two primes p_1p_2 of norms 5^4 and 5^7. Both p_1 and p_2 stay prime in $\mathcal{O}_K$. Indeed, $\overline{s}_{22}$ factors modulo 5 as a product $\overline{g}_1(x)\overline{g}_2(x)$ of irreducible polynomials $g_i(x) \in \mathbb{F}_5[x]$ of degree 8 and 14. Therefore, $p_i\mathcal{O}_K = (5, g_i(x))$ and the characteristic polynomial of $f_0|L_0(p_i)$ is $\overline{g}_i(x)$. In particular it is irreducible. To conclude, one computes units $u_1, u_2 \in \mathcal{O}_k^\times$ such that $(L_0(u_ip_i), f_0)$, $i = 1, 2$ are hyperbolic, and the linear positivity test confirms the positivity of both constructions. To see that the construction yields supersingular K3 lattices, we use Lemma 2.20 and see that the discriminant group $D_{L_0(u_ip_i)}$ is isomorphic to $\mathcal{O}_K/p_i\mathcal{O}_K$. This is indeed a vector space with $|\det L_0 \cdot N^K_{\mathbb{Q}}(p_i)| = 1 \cdot |N^k_{\mathbb{Q}}(p_i)|^2 = 5^8$ or 5^{14} elements. In both cases, Theorem 5.11 provides a supersingular K3 surface over $\overline{\mathbb{F}}_5$ (of Artin invariant 4, respectively 7) together with an automorphism of entropy $\log \lambda_{22}$. □

PROPOSITION 7.20. *The minimal Salem number λ_{20} of degree $d = 20$ is realized in characteristic 5 with Artin invariant $\sigma = 3$ or $\sigma = 7$.*

$$\begin{tikzcd} \boxed{\begin{array}{c} s_{20}(x) \\ \mathbb{F}_{11}\oplus\mathbb{F}_5^6 \end{array}} \arrow[rrr, dash, "\mathbb{F}_{11}"] & & & \boxed{(x^2-1)\ (C, f_C)} \end{tikzcd}$$

FIGURE 2. Gluing for λ_{20}

PROOF. We construct the isometry $f|\,\mathrm{NS}$ following the steps in the general strategy above and gluing together two lattices: the Salem factor S and the cyclotomic factor C.

The Salem factor. Note that s_{20} is simple, hence $S = L_0(a)$ must be a twist of the principal s_{20} lattice L_0, which has determinant $|\det L_0| = |s_{20}(1)s_{20}(-1)| = |-1 \cdot 11|$. In particular, since modulo 11

$$\overline{(x+1)} \mid \overline{s_{20}},$$

we see that 11 is feasible, and in fact it is the only feasible prime. Therefore, the possible twists $a \in A$ must be a product of factors of 11 and $p = 5$ in $\mathcal{O}_k$. In $\mathcal{O}_k$ we have the factorizations $11 = a_1a_2$ into two primes of norm 11 and 11^9, as well as $5 = p_1p_2$ with norms 5^3 and 5^7. On the one hand, $|D_{L_0(a_1)}| = 11^3$ and by Lemma 7.17 (or a direct computation) $f_0|5D_{L_0(a_1)}$ is of order 22. Since $D(22) = 10 > 2 = 22 - \deg(s_{20})$, a cannot be a multiple of a_1, and neither of a_2 by the same reasoning. On the other hand, for any $u \in \mathcal{O}_k^\times/(\mathcal{O}_k^\times)^2$ with $L_0(u)$ hyperbolic the quadratic positivity test shows that $f_0|L_0(u)$ is not positive, hence a must be divisible by either p_1 or p_2. Indeed, for both p_i it is possible to find a unit u_i such that $S = L_0(u_ip_i)$ is hyperbolic and $f_0|L_0(u_ip_i)$ is positive (the linear programming test gives $\mu(f_0|L_0(u_ip_i)) = -4$). Furthermore, the 11-primary part of the discriminant group is $(D_S)_{11} \cong \mathbb{F}_{11}$, the quadratic form is given by $(q_S)_{11}(\overline{x}) = \frac{2}{11}\overline{x}^2$ for a suitable generator $\overline{x}$ and $\left(\overline{f}\right)_{11}$ acts as $-id$.

The cyclotomic factor. Since 5 is not feasible and $\det S$ is divisible by 5, Corollary 7.12 implies that $\det C$ is not divisible by 5. This determines the cyclotomic factor C to be the (unique) negative definite lattice of rank 2 and determinant

11. Its Gram matrix and a positive isometry acting as $-id$ on the discriminant group $D_C \cong \mathbb{F}_{11}$ are given by

$$(C, f_C) = \left[-\begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \right].$$

For the discriminant form, we have $q_C \cong (-2/11x^2)$. Hence there is an isomorphism $\phi_{11} : (D_S)_{11} \to (D_C)_{11}$ such that $q_C \circ \phi_{11} = -(q_S)_{11}$. The gluing of (S, f_0) and (C, f_C) along ϕ_{11} results in a lattice (N, f) of signature $(1, 21)$ and discriminant 5^6 (respectively 5^{14}). This is sketched in Figure 2, where each box represents a sublattice together with its discriminant group and the $\mathbb{F}_{11}$ over the edge represents the glue subgroup. The characteristic polynomial of $\overline{f}$ on D_N is the prime factor of $\overline{s_{20}} \in \mathbb{F}_5[x]$ corresponding to the prime p_1 (resp. p_2), and in particular it is irreducible. Positivity is then verified by the linear programming test.

In both cases Theorem 5.11 provides a supersingular K3 surface over $\overline{\mathbb{F}}_5$ and the automorphism on it.

□

Proposition 7.21. *The minimal Salem number λ_{16} of degree $d = 16$ is realized in characteristic 5 with Artin invariant $\sigma = 5$.*

$$\boxed{\begin{array}{c} s_{16}(x) \\ \mathbb{F}_3 \oplus \mathbb{F}_5^{10} \end{array}} \overset{\mathbb{F}_3}{\rule[0.5ex]{8em}{0.4pt}} \boxed{\begin{array}{c} (x-1)^4(x+1)^2 \\ \mathbb{F}_3,\ (E_6, g) \end{array}}$$

Figure 3. Gluing for λ_{16}

Proof. The feasible primes for s_{16} are 3 and 29. At step (7) of the general strategy we are left with twists of norms $3 \cdot 5^5, 5^5, 29$. We choose the twist a of norm 5^5, so that $S = L_0(a)$ has determinant $-3 \cdot 5^{10}$. In order to remove the 3-primary part of D_S by gluing, C must have determinant 3 and signature $(0, 6)$, which determines it uniquely as E_6 (cf. **[26]**). A direct computation shows that the forms $(q_S)_3 \cong -q_{E_6}$ are opposite and thus a gluing of lattices $N = S \oplus_{\phi_3} C$ exists. Since the action of f_0 on $(D_S)_3$ is given by $-id$, we need a positive isometry of E_6 acting as $-id$ on the discriminant. Looking at the Dynkin diagram of E_6, we consider the symmetry $h \in O(E_6)$ along the center vertex. A computation shows that h has the desired properties, hence $f_0 \oplus h$ extends to an isometry of N whose positivity is verified by the linear test (with $\psi = -6$). The irreducibility of the minimal polynomial on D_N is assured by step (5) of the general strategy and we can apply Theorem 5.11 to conclude the proof. □

Here is why we chose the twist of norm 5^5 for the Salem factor: If instead we twist the Salem factor above 29, the only possibility for the cyclotomic part is $c(x) = c_7(x)$. It is a simple reciprocal polynomial. Hence, C is a twist of the principal c_7-lattice. However, $c_7(1) = 7$, so it is ramified over 7 and $7 \mid \det C$. This leads to a contradiction since 7 is not feasible. Since the principal s_{16}-lattice L_0 is ramified over 3 (has determinant ± 3), it is simpler to twist just above 5^5 than $3 \cdot 5^5$.

Proposition 7.22. *The Salem number λ_{14} is realized on a supersingular K3 surface in characteristic 5 with Artin invariant $\sigma = 6$.*

Proof. The principal $s_{14}(x)$-lattice is unimodular. Now we can twist it by a prime $b \in \mathcal{O}_k$ of norm 5^6 inert in $\mathcal{O}_K$ to get a positive isometry on a $5-$elementary hyperbolic lattice of rank 14. Since the prime is inert, the characteristic polynomial on the discriminant is irreducible. To obtain a hyperbolic lattice of rank 22, we take the direct sum with (E_8, id), obtaining also a positive isometry. As usual Theorem 5.11 provides the supersingular K3 surface and the automorphism. □

PROPOSITION 7.23. *The Salem number λ_{12} is realized on a supersingular K3 surface with Artin invariant $\sigma = 2$ in characteristic* 5.

$c_{30}(x)$ $\mathbb{F}_{31}^2$	$\mathbb{F}_{31}^2$	$s_{12}(x)$ $\mathbb{F}_5^4 \oplus \mathbb{F}_7 \oplus \mathbb{F}_{31}^2$	$\mathbb{F}_7$	(x^2-1) $(C2, g)$, $\mathbb{F}_7$

FIGURE 4. Gluing for λ_{12}

PROOF. The principal $s_{12}(x)$-lattice L_0 has determinant $|s_{12}(1)s_{12}(-1)| = 7$, hence discriminant group $\mathbb{F}_7$, where the isometry acts as $-id$. The feasible primes are $7, 13, 31$. Note that s_{12} and c_{30} have the common factor $(x+7)(x+9)$ modulo 31. Hence, we choose to twist the principal $s_{12}(x)$-lattice with a prime $q_1 \in \mathcal{O}_k$ of norm 5^2 inert in $\mathcal{O}_K$ and a prime q_2 of norm 31 such that $S = L_0(uq_1q_2)$ has signature $(1, 11)$ for a suitable unit u. Then the discriminant group is $D_S = \mathbb{F}_5^4 \oplus \mathbb{F}_7 \oplus \mathbb{F}_{31}^2$. In order to glue over the 7-primary summand, note that the determinant of the form on $(D_S)_7$ is a square. Hence, S can be glued with the negative definite lattice

$$(M, f_M) = \left[-\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\right]$$

because $D_M \cong \mathbb{F}_7$ and $\det(-q_M)$ is a square. Call the resulting lattice (L_1, f_1). For the glue above 31, we take a twist of the principal c_{30}-lattice $(C30, f_{C30})$. Theorem 2.21 guarantees the existence of a twist by a divisor a of 31 such that the characteristic polynomial of $\overline{f_{C30}}$ on $D_{C30(a)}$ is $(x+7)(x+9)$. We can find a unit u such that $C30(ua)$ has signature $(0, 8)$. By construction the characteristic polynomials on the 31-primary parts match, and Theorem 2.14 provides the existence of a glue map $\phi : (D_{L_1})_{31} \to (D_{C30(ua)})_{31}$. Set $(N, f) = (L_1 \oplus_\phi C30(ua), f_1 \oplus f_{C30})$, which is a hyperbolic 5-elementary lattice of determinant -5^4. The linear positivity test of (N, f) fails, since there is an optimal state with objective -2, but the quadratic test does confirm the positivity of (N, f). To apply the crystalline Torelli theorem it suffices to check that the characteristic polynomial on $N^\vee/N \cong \mathbb{F}_5^4$ is irreducible. This is indeed the case, since the twist q_1 remains inert in $\mathcal{O}_K$. □

PROPOSITION 7.24. *Lehmer's number λ_{10} is realized by an automorphism of a supersingular K3 surface in characteristic* 5 *with Artin invariant $\sigma = 2$.*

$s_{10}(x)$ $\mathbb{F}_{13}^2 \oplus \mathbb{F}_5^4$	$\mathbb{F}_{13}^2$	$c_{14}(x)$ $\mathbb{F}_{13}^2 \oplus \mathbb{F}_7$	$\mathbb{F}_7$	$(x^2-1)^3$ (A_6, g), $\mathbb{F}_7$

FIGURE 5. Gluings for λ_{10}.

PROOF. The principal s_{10}-lattice is unimodular and the feasible primes for s_{10} are $3, 5, 13, 23, 29$. There is an element $a \in \mathcal{O}_k$ of norm $5^2 \cdot 13$ such that $S = L_0(a)$ is hyperbolic and $D_S \cong \mathbb{F}_5^4 \oplus \mathbb{F}_{13}^2$ We need to glue S with two negative definite lattices of rank 6 to cancel the 13-primary part of the discriminant group, as follows. The only possibility to glue above 13 is to use the principal c_{14}-lattice $C14$, which has discriminant $\mathbb{F}_7$. Since c_{14} is also simple (with the analogous definition for cyclotomic polynomials), we can find a negative definite twist $C14(b)$ with determinant $7 \cdot 13^2$, and such that the characteristic polynomial of $\overline{f_{14}}$ on the 13-primary part matches with that of $\overline{f_0}$. Call (L_1, f_1) the resulting glue $S \oplus_{\phi_{13}} C12(b)$ over 13. It has rank 16 and determinant $5^4 7$, hence it remains to glue it with a negative definite lattice of rank 6 and determinant 7, i.e. A_6. It also remains to find a good isometry g of A_6. Since $(D_S)_7 \cong \mathbb{F}_7$ and $\overline{f_1}$ acts as $-id$, so should do $\overline{g}$. The obvious choice $g = -id_{A_6}$ glues just fine, however, it is not positive, as any root of A_6 is cyclic, hence we need to look for another one. Let $r_1, \ldots, r_6$ be a be a fundamental root system (corresponding to the nodes of the Dynkin diagram of A_6), then g is given by the central reflection composed with $-id$

$$g : (r_1, \ldots, r_6) \mapsto (-r_6, \ldots, -r_1).$$

A direct computation shows that g has the right properties. Since a is inert in $\mathcal{O}_K$, the resulting isometry has irreducible characteristic polynomial and the proof concludes as the preceding ones. $\square$

PROPOSITION 7.25. *There is a supersingular K3 surface over $\overline{\mathbb{F}}_5$ with Artin invariant $\sigma = 4$ and an automorphism on it realizing λ_8. The characteristic polynomial of the action on* NS *is given by $s_8 c_1^{12} c_2^2$.*

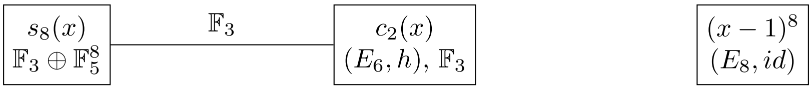

FIGURE 6. Gluings for λ_8.

PROOF. The principal $s_8(x)$-lattice L_0 has discriminant group $\mathbb{F}_3$ and $p = 5$ stays prime in $\mathcal{O}_K$. One can find a unit $u \in \mathcal{O}_K$ such that $S = L_0(u5)$ is hyperbolic. Then $\det S = -3 \cdot 5^8$ and f_0 acts as $-id$ on $(D_S)_3 \cong \mathbb{F}_3$. It turns out we can glue this to (E_6, h) where $h \in O(E_6)$ is given by the central symmetry of the Dynkin diagram of E_6 like for λ_{16}. Now $(S \oplus_\phi E_6, f_0 \oplus h)$ is a lattice of signature $(1, 13)$ and discriminant group $\mathbb{F}_5^8$. Since 5 is prime in $\mathcal{O}_K$, $\overline{s}_8$ is the irreducible characteristic polynomial of the action on the discriminant group. Positivity is confirmed by the linear test, and we conclude by taking the direct sum with (E_8, id) to obtain an hyperbolic 5-elementary lattice of rank 22. $\square$

PROPOSITION 7.26. *There is a supersingular K3 surface over $\mathbb{F}_5$ with Artin invariant $\sigma = 4$ and an automorphism on it realizing λ_6. Its characteristic polynomial on* NS *is given by $s_6(x) c_1^9(x) c_2(x) c_{14}(x)$.*

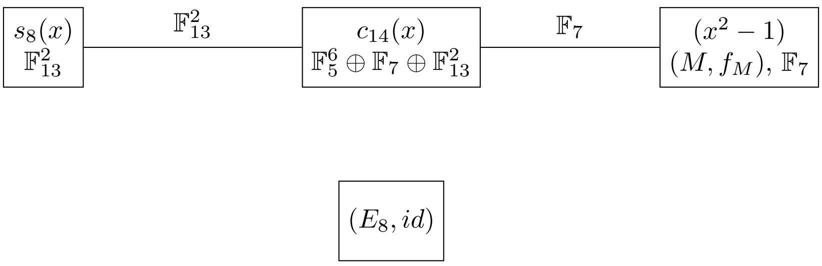

FIGURE 7. Gluings for λ_6.

PROOF. The principal $s_6(x)$-lattice L_0 is unimodular, and the feasible primes are $2, 3, 7, 13, 23, 29, 31, 37, 41, 59, 67$. We choose to twist L_0 by a prime $q \in \mathcal{O}_K$ of norm 13 such that $S = L_0(q)$ is hyperbolic and $\overline{f}|D_S$ has characteristic polynomial

$$x^2 + 8x + 1 = \gcd(\overline{s}_6, \overline{c}_{14}) \mod 13.$$

This suggests to glue S with a twist of the principal c_{14}-lattice $(C14, f_{14})$. By Theorem 2.21, we can find a twist $b \in \mathbb{Z}[\zeta_{14}]$ dividing 13 with the right characteristic polynomial on the 13-primary part of the discriminant. We can even arrange for $C14(b)$ to be negative definite. Since 5 is prime in $\mathbb{Z}[\zeta_{14}]$, we can take the further twist $C14(5b)$ to get $\left(D_{C14(5b)}\right)_5 \cong \mathbb{F}_5^6$ and $\left(\overline{f_{14}}\right)_5$ with irreducible characteristic polynomial. Now Theorem 2.14 provides the existence of a glue map $\phi : D_S \to (D_{C14(5b)})_{13}$ compatible with the actions. Set $N = S \oplus_\phi C14$. It is a hyperbolic lattice of rank 12 and determinant $-5^6 7$ with order 2 action on $(D_N)_7$. Then (N, f_N) turn out to glue to

$$(M, f_M) = \left[-\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \right].$$

We conclude by confirming positivity and filling up the 8 remaining ranks with (E_8, id). □

PROPOSITION 7.27. *The Salem numbers λ_2 and λ_4 are realized on the supersingular K3 surface of Artin invariant $\sigma = 1$ in characteristic 5.*

PROOF. For these Salem numbers of small degree, we will follow a different strategy, along the lines of the proof of [**63**, Theorem 1.3], which gives a more explicit construction of the automorphisms not relying on the Torelli theorem.

First of all, note that the supersingular K3 surface with Artin invariant $\sigma = 1$ over $\overline{\mathbb{F}_5}$ is the Kummer surface associated to the product of any two supersingular elliptic curves. For example, we can consider the reduction modulo 5 of $E = E_{\zeta_3}$, the complex elliptic curve of j-invariant 0. By general theory, if X is any smooth projective variety with an automorphism F defined over $\mathbb{Q}$, the entropy of $F|X(\mathbb{C})$ coincides with the entropy of $F|X_p$ for any prime p of good reduction (this follows from the standard comparison theorems between singular and étale cohomologies and the properties of good reduction).

Therefore, it is enough to construct automorphisms of $\mathrm{Km}(E \times E)$ with entropies λ_2 and λ_4. Moreover, according to the discussion in [**62**, Section 4], it is enough to construct linear maps $F_2, F_4 : \mathbb{C}^2 \to \mathbb{C}^2$ preserving the lattices $\mathbb{Z}[\zeta_3]^2$ whose spectral radii ρ_2, ρ_4 satisfy $|\rho_i|^2 = \lambda_i$. This is achieved, for example, by the matrices

$$F_4 = \begin{pmatrix} 1 & \zeta_3 - 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad F_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

□

PROPOSITION 7.28. *The supersingular K3 surface X with Artin invariant $\sigma = 1$ over $\overline{\mathbb{F}}_{11}$ admits an automorphism $F : X \to X$ such that the characteristic polynomial of $F^*|\,\mathrm{NS}(X)$ is given by $s_{18}(x)c_{12}(x)$. It is not realized on a supersingular K3 surface in characteristic* 5.

PROOF. We begin by proving that λ_{18} is not realized in characteristic $p = 5$. The principal s_{18}-lattice is unimodular, and the feasible primes are 7 and 13. By the time we reach step (7) of the general strategy, we are left with a single twist a (up to units) of norm 13. Then the only possibility for the cyclotomic factor $c(x)$ is $c_{12} = x^4 - x^2 + 1$, which is a simple reciprocal polynomial. Hence, C must be a twist of the principal c_{12}-lattice by factors of 5 and 13. However, 5 is prime in the trace field $k = \mathbb{Q}[y]/r_{18}(y)$, but splits in the Salem field $K = \mathbb{Q}[x]/s_{18}(x)$. This results in the minimal polynomial on the 5-discriminant group being reducible. In consequence λ_{18} is not realizable in characteristic 5.

However in characteristic 11 this is possible. We can find a twist b of the principal c_{12}-lattice $C12$ such that $C12(b)$ is negative definite, $D_{C12(b)} = \mathbb{F}_{11}^2 \oplus \mathbb{F}_{13}^2$, the characteristic polynomials on $\mathbb{F}_{13}^2$ match and the characteristic polynomial of

$\overline{f_1}$ on $\mathbb{F}_{11}^2$ is irreducible. We get the existence of a gluing $N = S \oplus_\psi C12(b)$ along 13^2 such that $D_N = (D_{C12(b)})_{11}$. Positivity of the resulting isometry is confirmed by the (quadratic) test. $\square$

$$\boxed{\begin{matrix} s_{18}(x) \\ \mathbb{F}_{13}^2 \end{matrix}} \overset{\mathbb{F}_{13}^2}{\rule{8em}{0.4pt}} \boxed{\begin{matrix} c_{12}(x) \\ \mathbb{F}_{13}^2 \oplus \mathbb{F}_{11}^2 \end{matrix}}$$

FIGURE 8. Gluing for λ_{18}.

4. Realizing λ_{12} over $\mathbb{C}$

THEOREM 7.29. *There is a complex projective K3-surface X and $F \in Aut(X)$ such that $h(F) = \log \lambda_{12}$, $\mathrm{NS}(X) \cong U(13) \oplus 2E_8$ and the action on the holomorphic 2-form is of order 12.*

PROOF. For s_{12}, we get the 3 feasible primes $7, 13, 31$. Following the general strategy in the complex case, we end up with three twists (up to units) one above each feasible prime. We continue with the twist above 13, as 7 and 31 lead to many dead ends. Modulo 13 we find the commmon factor $\overline{(x+2)(x+7)}$ of s_{12} and c_{12}. By Theorem 2.21, we can find twists a, b above 13 of the principal s_{12}-lattice L_0, and the principal c_{12}-lattice $C12$ such that they have characteristic polynomial $\overline{(x+2)(x+7)}$ on the 13-glue. Then Theorem 2.14 provides the existence of a glue map. It remains to modify a and b by a unit to obtain the right signatures. Indeed for a one can find a unit $u \in \mathcal{O}_k^\times$ such that $S = L_0(ua)$ is of signature $(1, 11)$. For c_{12}, it is not possible to realize glue group $\mathbb{F}_{13}^2$ and signature $(0, 4)$ but it is possible to achieve signature $(2, 2)$. This indicates that we should take $C12(b)$ as transcendental lattice. Since $|\det S| = |\det L_0 \cdot N(a)| = 7 \cdot 13^2$, the only possibility for the remaining part is a negative definite rank 6 lattice of determinant 7, i.e. the A_6 root lattice. And indeed the quadratic forms $(q_S)_7 \cong -(q_{A_6})$ glue. Since the characteristic polynomial of $f|T(X)$ is a perfect power, it must be a part of NS. What remains is to find a good positive isometry g of A_6. Since $\overline{f}|(D_S)_7 = -id$, so is $\overline{g}$ and we can take the pair (A_6, g) from the construction of Lehmer's number.

$$\boxed{\begin{matrix} c_{12}(x) \\ (2,2),\ \mathbb{F}_{13}^2 \end{matrix}} \overset{\mathbb{F}_{13}^2}{\rule{4em}{0.4pt}} \boxed{\begin{matrix} s_{12}(x) \\ (1,11),\ \mathbb{F}_{13}^2 \oplus \mathbb{F}_7 \end{matrix}} \overset{\mathbb{F}_7}{\rule{3em}{0.4pt}} \boxed{\begin{matrix} (x^2-1)^3 \\ (A_6, g),\ (0,6),\ \mathbb{F}_7 \end{matrix}}$$

FIGURE 9. Gluing for λ_{12} in the complex case.

The positvity of the isometry on NS is confirmed by the positivity test. Note that by Corollary 7.11 we could take any other positive $g \in O(A_6)$, acting as $-id$ on the discriminant. The lattice A_6 has only 10080 isometries so a brute-force search is feasible and returns about a hundred suitable isometries. $\square$

5. Minimal Salem polynomials

We include here the minimal polynomials $s_d(x)$ of the minimal Salem numbers λ_d and the determinant $\Delta = |s_d(1)s_d(-1)|$ of their principal lattice for each even degree $d \leq 22$ (cf. [**16**, p. 326], [**64**]).

TABLE 1. Minimal Salem Polynomials

d	λ_d	$s_d(x)$	Δ
2	2.618033988	$x^2 - 3x + 1$	5
4	1.722083805	$x^4 - x^3 - x^2 - x + 1$	3
6	1.401268367	$x^6 - x^4 - x^3 - x^2 + 1$	1
8	1.280638156	$x^8 - x^5 - x^4 - x^3 + 1$	3
10	1.176280818	$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$	1
12	1.240726423	$x^{12} - x^{11} + x^{10} - x^9 - x^6 - x^3 + x^2 - x + 1$	7
14	1.200026523	$x^{14} - x^{11} - x^{10} + x^7 - x^4 - x^3 + 1$	1
16	1.236317931	$x^{16} - x^{15} - x^8 - x + 1$	3
18	1.188368147	$x^{18} - x^{17} + x^{16} - x^{15} - x^{12} + x^{11} - x^{10} + x^9 - x^8 + x^7 - x^6 - x^3 + x^2 - x + 1$	1
20	1.232613548	$x^{20} - x^{19} - x^{15} + x^{14} - x^{11} + x^{10} - x^9 + x^6 - x^5 - x + 1$	11
22	1.235664580	$x^{22} - x^{20} - x^{19} + x^{15} + x^{14} - x^{12} - x^{11} - x^{10} + x^8 + x^7 - x^3 - x^2 + 1$	1

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Curriculum vitae

Simon Brandhorst was born on May 11th 1989 in Bielefeld, Germany. He earned his University Entrance Diploma (*allgemeine Hochschulreife*) in 2008. After which he was conscripted and did nine month of civil service in the Rolf Wagner Haus for youth welfare in Bielefeld.

Then Simon Brandhorst stayed a semester at the Université Paul-Valéry Montpellier III learning French.

In 2009 he started studying mathematics at Leibniz Universität Hannover. Aside his studies he worked as a teaching assistant in calculus, linear algebra and mathematics for engineers. He obtained the scholarship "Niedersachenstipendium" of the German state Lower Saxony.

In 2011 he participated in the Erasmus program at Université Lille I for a semester. After his return he obtained his *Bachelor of Science* under the supervision of Marcel Erné and was awarded a scholarship of the "Studienstiftung des deutschen Volkes".

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