Affine rank two Nichols algebras of diagonal type

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Kurzfassung. Wurzelsysteme und kristallographische Coxeter-Gruppen sind zentrale Werkzeuge in der Theorie der halbeinfachen Lie-Algebren. Eine ähnliche Rolle sollen Weyl-Gruppoide in der Theorie der Nichols-Algebren spielen: Die endlich-dimensionalen Nichols-Algebren vom diagonalem Typ wurden mit Hilfe von Weyl-Gruppoiden klassifiziert. In dieser Dissertation sollen affine Nichols-Algebren diagonalen Typs vom Rang zwei durch affine Weyl-Gruppoide, welche als eine Symmetriestruktur betrachtet werden können, klassifizert werden.

Wir führen affine Weyl-Gruppoide, affine kristallographische Arrangements und die Korrespondenz zwischen ihnen ein. Wir definieren affine Weyl-Gruppoide vom Rang zwei mit Grenzwert 1 und charakterisieren affine kristallographische Arrangements vom Rang zwei mit Grenzwert 1.

Mithilfe von zwei Sätzen von M. Cuntz erhalten wir alle Teilsequenzen der Länge vier von potenziellen Perioden affiner charakteristischer Sequenzen. Wir bestimmen deren zugehörige Bicharaktere und Cartan-Graphen. Dies schließt die Klassifikation aller affinen Nichols-Algebren vom Rang zwei ab.

Abstract. Root systems and crystallographic Coxeter groups are key tools in the theory of semisimple Lie algebras. Weyl groupoids play a similar role in the theory of Nichols algebras: the finite dimensional Nichols algebras of diagonal type were classified using Weyl groupoids. In this thesis, affine rank two Nichols algebras of diagonal type are classified by rank two affine Weyl groupoids, which can be seen as symmetry structures.

We introduce affine Weyl groupoids, affine crystallographic arrangements, and the correspondence between them. We then proceed to give the definitions of affine rank two Weyl groupoids with limit 1 and the characterization of affine rank two crystallographic arrangements with limit 1.

Using two theorems by M. Cuntz, we obtain all of the length four subsequences of potential periods of affine characteristic sequences. We compute their corresponding bicharacters and Cartan graphs. This completes the classification of all affine rank two Nichols algebras of diagonal type.

Schlagworte: Nichols-Algebra, Weyl-Gruppoid, affines Arrangement
Keywords: Nichols algebra, Weyl groupoid, affine arrangement

Contents

Standard Notation

Introduction

A Hopf algebra H over a field $\Bbbk$ is called *pointed* if all its simple left or right comodules are one-dimensional ([**2**]). In order to make a classification of pointed Hopf algebras, we need to study the structure of Nichols algebras. Theories of Nichols algebras have a closed relation to that of semi-simple Lie algebras. Many examples of Nichols algebras come from quantum groups.

One of the most important branches is the Nichols algebras of diagonal type. Just like root systems and crystallographic Coxeter groups in the theory of semi-simple Lie algebra, Weyl groupoids play an important role in the theory of Nichols algebras. The finite-dimensional Nichols algebras of diagonal type can be classified by using the Weyl groupoids. In this thesis, we classify infinite dimensional Nichols algebras of diagonal type of rank two using the affine Weyl groupoids of rank two.

Weyl groupoids are the symmetry objects of Nichols algebras of finite group type, which were first reported in [**17**] and later in [**1**] in a more general setting. An introduction about Weyl groupoids and their root systems was given in [**19**] and [**10**]. To classify finite dimensional Nichols algebras of diagonal type, it is important to determine whether a given Cartan matrix (a categorical generalization of the notion of a generalized Cartan matrix) has a finite root system. A number of papers ([**9**], [**12**], [**11**], [**13**]) reported the characterization of these finite root systems and a complete classification of finite Weyl groupoids has been made.

The theorems required for the classification share some common features: finite Weyl groupoids correspond to certain simplicial arrangements called *crystallographic* [**6**]: Assume that $\mathcal{A}$ is a simplicial arrangement of finitely many real hyperplanes in a Euclidean space V and R is a set of nonzero covectors such that $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$. For all $\alpha \in R$, assume that $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$. The triple $(\mathcal{A}, T, R)$ is called crystallographic, see [**6**, Def. 2.3], if for any chamber K the elements of R are integer linear combinations of the covectors defining the walls of K. The connection between a set of crystallographic arrangements of rank two with n hyperplanes and the triangulations of a convex n-gon by non-intersecting diagonals can be expressed by one bijection. It produces η-sequences, or on the other hand, produces continued fractions (see Def. 3.10, Ch. 1).

In this thesis, we classify the *affine Weyl groupoids of rank two* who are more general Weyl groupoids. These are meant to be the symmetry structures of Nichols algebras which are not finite dimensional but almost, similarly to affine Kac-Moody algebras. And we believe that our definition (see Def. 1.3, Ch. 2) is appropriate: the axioms are the same as for 'finite' Weyl groupoids except that we replace the finiteness of the root system by assuming that there is exactly one imaginary root and that there does not exist other root system of the same type with more than one imaginary root. These axioms correspond to conditions on arrangements (see Def. 1.1, Ch. 2). The imaginary root corresponds a hyperplane which could be viewed as the limit of the hyperplanes of the arrangement. Further, the notion of Tits cone is introduced for connected Cartan graphs. For a Weyl groupoid, the Tits cone is the whole space if and only if the associated Cartan graph's root system is finite. For an affine Cartan graph, its corresponding Tits cone is a half space minus one hyperplane.

As for finite Weyl groupoids of rank two, a parametrization is achieved by certain characteristic sequences. Under some further natural conditions on the imaginary root, an affine Weyl groupoid of rank two can be determined by a sequence of η-sequences which are concatenated together (see Thm. 2.2, Ch. 3). The sequence is periodic if and only if a covering of the affine Weyl groupoid with multiple finitely objects exists.

In the last part of this thesis, we give the main result and its proof.

THEOREM 0.1. *If χ is a bicharacter (see Def. 1.8, Ch. 1) of an affine rank two Nichols algebra of diagonal type, then χ is one of the following types of bicharacters:*

(1) $\overset{\zeta^2}{\circ}\overset{\zeta}{\rule[2pt]{40pt}{0.4pt}}\overset{\zeta^4}{\circ}$, $\zeta \in \mu_6$, *its corresponding period is* $[2,2,2,2,2,2]$*;*

(2) $\overset{\zeta}{\circ}\overset{\zeta}{\rule[2pt]{40pt}{0.4pt}}\overset{\zeta}{\circ}$, $\zeta \in \mu_3$, *its corresponding period is* $[2,2,2,2,2,2]$*;*

(3) $\overset{\zeta^4}{\circ}\overset{\zeta^{10}}{\rule[2pt]{40pt}{0.4pt}}\overset{\zeta}{\circ}$, $\zeta \in \mu_{12}$, *its corresponding period is* $[2,2]$*;*

(4) $\overset{\zeta^{-2}}{\circ}\overset{\zeta^{-2}}{\rule[2pt]{40pt}{0.4pt}}\overset{\zeta}{\circ}$, $\zeta \in \mu_6$, *its corresponding period is* $[2,2]$*;*

(5) $\overset{\zeta}{\circ}\overset{\zeta^{-2}}{\rule[2pt]{40pt}{0.4pt}}\overset{\zeta^4}{\circ}$, $\zeta \in \mu_{12}$, *its corresponding period is* $[2,2]$*;*

(6) $\overset{\zeta}{\circ}\overset{\zeta^{-2}}{\rule[2pt]{40pt}{0.4pt}}\overset{\zeta^4}{\circ}$, $\zeta \in \mu_6$, *its corresponding period is* $[2,2]$*;*

(7) $\overset{q^{-2}}{\circ}\overset{q}{\rule[2pt]{40pt}{0.4pt}}\overset{q^{-2}}{\circ}$, $q \in \mathbb{C}\backslash\{\pm 1\}$, *it is a one-dimensional type bicharacter and its corresponding period is* $[2,2]$*;*

(8) $\overset{\zeta^{-4}}{\circ}\overset{\zeta}{\rule[2pt]{40pt}{0.4pt}}\overset{\zeta^{-1}}{\circ}$, $\zeta \in \mu_{10}$, *its corresponding period is* $[1,4]$*;*

(9) $\overset{\zeta}{\circ}\overset{\zeta}{\rule[2pt]{40pt}{0.4pt}}\overset{\zeta^{-1}}{\circ}$, $\zeta \in \mu_5$, *its corresponding period is* $[1,4]$*;*

(10) $\overset{\zeta}{\circ}\overset{-1}{\rule[2pt]{40pt}{0.4pt}}\overset{-1}{\circ}$, $\zeta \in \mu_8$, *its corresponding period is* $[1,4]$*;*

(11) $\overset{q}{\circ}\overset{q^{-4}}{\rule[2pt]{40pt}{0.4pt}}\overset{q^{-4}}{\circ}$, $q \in \mathbb{C}\backslash\{1 \cup \mu_2 \cup \mu_3 \cup \mu_4\}$, *it is a one-dimensional type bicharacter and its corresponding period is* $[1,4]$*;*

(12) $\overset{\zeta^2}{\circ}\overset{\zeta^{-6}}{\rule[2pt]{40pt}{0.4pt}}\overset{-1}{\circ}$, $\zeta \in \mu_{18}$, *its corresponding period is* $[1,3,1,6]$*;*

(13) $\overset{\zeta^3}{\circ}\overset{\zeta^{-4}}{\rule[2pt]{40pt}{0.4pt}}\overset{\zeta^4}{\circ}$, $\zeta \in \mu_{12}$, *its corresponding period is* $[1,3,2,3]$*;*

(14) $\overset{\zeta^3}{\circ}\overset{\zeta^4}{\rule[2pt]{40pt}{0.4pt}}\overset{-1}{\circ}$, $\zeta \in \mu_{12}$, *its corresponding period is* $[1,3,3,1,4,1,3,3,1,4]$.

where μ_m denotes the set of primitive mth roots of unity, $\mathbb{C}$ is the set of complex numbers.

REMARK 0.2. With constraints of q_{ii} and $q_{ij}q_{ji}$, we can denote the bicharacters of affine rank two Nichols algebras of diagonal type by generalized Dynkin diagrams (see Def. 1.11, Ch. 1). As for a generalized Dynkin diagram itself, it is only an invariant.

Firstly apply Thm. 1.8 (Ch. 4) four times, it yields a set of η-sequences E_5 and a set of sequences F_5, both sets consist of elements whose length are at least five. Apply Thm. 1.8 (Ch. 4) again, we obtain the η-sequences set E_6 and the set of sequences F_6. For every element from E_6 or F_6, its length is at least six. The η-sequences with lengths less or equal to five also should not be forgotten. By Thm. 2.2 (Ch. 3), all of length four subsequences of the potential affine characteristic sequences are given. For these subsequences of length four, we computed their associated bicharacters and Cartan graphs. Then we obtain the main result.

We organize the thesis in the following way: The first chapter are the preliminaries consist of the basic notions, proposition and theorems of Nichols algebras, Cartan graphs, Weyl groupoids, crystallographic arrangements, and so on. In the second chapter, we recall the definition of affine crystallographic arrangements, affine Cartan graphs and the correspondence between them. In the Chapter 3, the notion of affine crystallographic arrangements with limit 1 and the interpretation of the periodicity of the affine crystallographic arrangements with limit 1 are introduced. After the section of subsequences of η-sequences and the section of bicharacters with entries not roots of unity, we give our main result and its proof in the last chapter.

In this thesis, many associated lemmas, theorems and propositions come from an unpublished manuscript of which I am the co-author [**14**]. And there are also some parts come from my supervisor Prof. Cuntz [**7**]. I got these materials from him through private communications in the past three years.

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CHAPTER 1

Preliminaries

1. Nichols algebras

In this section, we briefly introduce Nichols algebra and its history. Then we recall the notion of bicharacter. More details of Nichols algebras are in paper [**2**]. Before giving the definition of Nichols algebra formally, we need to recall the notion of Hopf algebra.

Definition 1.1. A *Hopf algebra* is a 6-tuple $H = (H, \mu, 1, \mathit{\Delta}, \varepsilon, S)$, where $(H, \mu, 1)$ is a (unital associative) algebra, $(H, \mathit{\Delta}, \varepsilon)$ is a coalgebra, and $S : H \to H$ is a linear map such that $\mathit{\Delta} : H \to H \otimes H$ and $\varepsilon : H \to \Bbbk 1$ are algebra maps and S satisfies

$$\begin{tikzcd} H \otimes H \arrow[d, "(\mathrm{id}\otimes S)"'] & H \arrow[l, "\mathit{\Delta}"'] \arrow[r, "\mathit{\Delta}"] \arrow[d, "1\varepsilon"] & H \otimes H \arrow[d, "(S\otimes \mathrm{id})"] \\ H \otimes H \arrow[r, "\mu"'] & H & H \otimes H \arrow[l, "\mu"] \end{tikzcd}$$

Using Sweedler notation, this means that $S(h_{(1)})h_{(2)} = h_{(1)}S(h_2) = \varepsilon(h)$ for all $h \in H$.

Definition 1.2. [**2**, Section 1.2] Let V be a vector space,

$$c : V \otimes V \to V \otimes V$$

a linear isomorphism with

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

Then c is a *braiding*, and (V, c) is a *braided vector space.*

Then we give the notion of Nichols algebra.

Definition 1.3. [**7**] Define a map $\rho : S_n \to \mathrm{End}(V^{\otimes n})$ by:
For a transposition $(i, i+1) \in S_n$ let

$$\rho((i, i+1)) := \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes c \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id},$$

where c acts in the copies i and $i+1$ of V.
If $\omega = \tau_1 \ldots \tau_\ell$ is a reduced expression of $\omega \in S_n$, then

$$\rho(\omega) := \rho(\tau_1) \ldots \rho(\tau_\ell).$$

Let $\mathfrak{S}_n := \sum_{\omega \in S_n} \rho(\omega)$. The algebra

$$\mathfrak{B}(V) := \bigoplus_{n \geq 0} T^n(V) / \ker(\mathfrak{S}_n)$$

is called the *Nichols algebra* of (V, c).

There are some examples of Nichols algebras which were given in M. Cuntz's talk in Soltau, Germany (Mar. 2014).

EXAMPLE 1.4.
- $c(x \otimes y) = y \otimes x \quad$ for all $x, y \in V$:
  $\mathfrak{B}(V) = S(V)$ symmetric algebra
- $c(x \otimes y) = -y \otimes x \quad$ for all $x, y \in V$:
  $\mathfrak{B}(V) = \Lambda(V)$ exterior algebra

The Nichols algebra was first developed by W. D. Nichols in 1978. From then on, considerable attention has been paid to this area and the subject is well developed. Some important achievements for the research of Nichols algebra was present below:

- In 1978, Nichols constructed examples of Hopf algebras.
- In 1988, Woronowicz built a quantum differential calculus.
- Abstract definition of quantized universal enveloping algebras was developed by Lusztig (1993), Rosso (1994) and Schauenburg (1996).
- In 1998, Andruskiewitsch-Schneider developed essential tool in the classification of pointed Hopf algebras

Actually, Nichols algebra is a special kind of Hopf algebra. There exists a theorem about this fact.

THEOREM 1.5. [**2**, Prop. 2.2] *The Nichols algebra is an* $\mathbb{N}_0$*-graded braided Hopf algebra.*

To date, Nichols algebras have various branches (see [**2**]), such as the Hecke type, the diagonal type, the non-Cartan diagonal type and finite non-abelian group type. In this thesis, our attention focus on the Nichols algebras of diagonal type. It is defined as follows.

DEFINITION 1.6. [**2**, Def. 1.6] Let $\{x_1, \ldots, x_r\}$ be basis of V,

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \quad q_{ij} \in \mathbb{C}.$$

Then c and $\mathfrak{B}(V)$ are called of *diagonal type.*

DEFINITION 1.7. [**2**, Def. 1.2] Let H be a Hopf algebra and V a module and a comodule over H. Then V is called a *Yetter-Drinfeld module* if

$$\delta_V(hv) = h_1 v_{-1} S(h_3) \otimes h_2 v_0 \quad \forall h \in H, v \in V.$$

A Yetter-Drinfeld module V is a braided vector space via

$$c : V \otimes V \to V \otimes V, \quad v \otimes w \mapsto v_{-1} w \otimes v_0.$$

In the following, we introduce the notion of bicharacter who plays an important role in this thesis.

DEFINITION 1.8. [**18**, Section 2] For a Nichols algebra of diagonal type, the numbers q_{ij}, $i, j = 1, \dots, r$ define a *bicharacter*

$$\chi : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{C}, \quad ((a_1, \dots, a_r), (b_1, \dots, b_r)) \mapsto \prod_{i,j=1}^{r} q_{ij}^{a_i b_j}.$$

THEOREM 1.9 (see [**21**]). *Let (V, c) be of diagonal type. There exists a totally ordered index set $(L, \leq)$ and $\mathbb{Z}^r$-homogeneous elements $X_\ell \in \mathfrak{B}(V)$, $\ell \in L$ such that*

$$\begin{aligned}\{X_{\ell_1}^{m_1} \cdots X_{\ell_\nu}^{m_\nu} \quad | \quad & \nu \geq 0, \quad \ell_1, \dots, \ell_\nu \in L, \quad \ell_1 > \dots > \ell_\nu, \\ & 0 \leq m_i < h_{\ell_\nu} \ \forall i = 1, \dots, \nu\}\end{aligned}$$

is a vector space basis of $\mathfrak{B}(V)$, where

$$h_\ell = \min\{m \in \mathbb{N} \mid 1 + q_\ell + \dots + q_\ell^{m-1} = 0\} \cup \{\infty\}$$

and $q_\ell = \chi(\deg X_\ell, \deg X_\ell)$, $\ell \in L$.

And this fact also means that h_l depends only on the $\mathbb{Z}^n$-degree of $\deg X_l$ because $\mathfrak{B}(V)$ is $\mathbb{Z}^n$-graded.

For finite cases, there exists a theorem of PBW generators of Nichols algebras and root system of Weyl groupoids.

THEOREM 1.10 (see [**17**]). *Let $\mathfrak{B}$ be a finite dimensional Nichols algebra of diagonal type. Let R_+ be the set of degrees of the PBW generators of $\mathfrak{B}$. Then $R_+ \cup -R_+$ is a root system of a finite Weyl groupoid.*

As for the 'Weyl groupoids', we will give their definitions and related details in next section.

In 2008, Heckenberger gives the definition of generalized Dynkin diagram of bicharacter. We find that it is convenient to denote a bicharacter by a generalized Dynkin diagram.

DEFINITION 1.11. Let χ be a bicharacter on $\mathbb{Z}^n$. The *generalized Dynkin diagram* of χ is a non-oriented graph with n vertices $v_1, \dots, v_n$, where the vertex v_i is labeled by q_{ii}. Let $i, j \in \{1, \dots, n\}$ with $i \neq j$. If $q_{ij} q_{ji} = 1$, then there is no edge between v_i and v_j. Otherwise there is precisely one edge between them, and it is labeled by $q_{ij} q_{ji}$. Notice that any Dynkin diagram itself is only an invariant.

DEFINITION 1.12 (compare to [**2**, Def. 1.6]). If (V, c) is of diagonal type, then we say that it is *indecomposable* if for all $i \neq j$, there exists a sequence $i = i_1, i_2, \dots, i_t = j$ of elements of $\{1, \dots, \theta\}$ such that $q_{i_s, i_{s+1}} q_{i_{s+1}, i_s} \neq 1, 1 \leq s \leq t-1$. Otherwise, we say that the matrix is *decomposable.* For a Nichols algebra of rank $= 2$, we denote its bicharacter by

$$\chi = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}.$$

In this thesis, the cases of decomposable are not involved. We only consider the cases that (V, c) are indecomposable. Thus our assumption is $q_{12} q_{21} \neq 1$. According to the formula of bicharacter:

$c_{ij}^a = -\min\{m \in \mathbb{N}_0 \mid 1 + q_{ii} + q_{ii}^2 + \cdots + q_{ii}^m = 0 \ \text{ or } \ q_{ii}^m q_{ij} q_{ji} = 1\}$
if $i \neq j$ and $c_{ii}^a = 2$ (we will introduce this formula with more details later), we find when we fix the minus minimum of m, the positions of entires q_{12} and q_{21} have nothing to do with the value of m. It implies that q_{12} and q_{21} can be considered as one part. Thus for convenience, we can write $q_{12}q_{21}$ in the lower left corner of the matrix:

$$\begin{pmatrix} q_{11} & 1 \\ q_{12}q_{21} & q_{22} \end{pmatrix}.$$

2. Cartan graphs and Weyl groupoids

The general notion of a Weyl groupoid was introduced by Heckenberger and Yamane [**19**] and redeveloped by Cuntz and Heckenberger in [**10**].

Definition 2.1. [**20**, Section 1.1] Let $I := \{1, \ldots, r\}$ and $\{\alpha_i \,|\, i \in I\}$ the standard basis of $\mathbb{Z}^I$. A *generalized Cartan matrix* $C = (c_{ij})_{i,j\in I}$ is a matrix in $\mathbb{Z}^{I\times I}$ such that

(M1) $c_{ii} = 2$ and $c_{jk} \leq 0$ for all $i, j, k \in I$ with $j \neq k$,
(M2) if $i, j \in I$ and $c_{ij} = 0$, then $c_{ji} = 0$.

Definition 2.2. [**10**, Def. 2.1] Let A be a non-empty set, $\rho_i : A \to A$ a map for all $i \in I$, and $C^a = (c_{jk}^a)_{j,k\in I}$ a generalized Cartan matrix in $\mathbb{Z}^{I\times I}$ for all $a \in A$. The quadruple

$$\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$$

is called a *Cartan graph* if

(C1) $\rho_i^2 = \mathrm{id}$ for all $i \in I$,
(C2) $c_{ij}^a = c_{ij}^{\rho_i(a)}$ for all $a \in A$ and $i, j \in I$.

In the left of this section, all of the notions and remark without notes come from the paper [**10**].

Definition 2.3. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan graph. For all $i \in I$ and $a \in A$ define $\sigma_i^a \in \mathrm{Aut}(\mathbb{Z}^I)$ by

$$\sigma_i^a(\alpha_j) = \alpha_j - c_{ij}^a \alpha_i \qquad \text{for all } j \in I. \tag{2.1}$$

The *Weyl groupoid of* $\mathcal{C}$ is the category $\mathcal{W}(\mathcal{C})$ such that $\mathrm{Ob}(\mathcal{W}(\mathcal{C})) = A$ and the morphisms are compositions of maps σ_i^a with $i \in I$ and $a \in A$, where σ_i^a is considered as an element in $\mathrm{Hom}(a, \rho_i(a))$. We call the cardinality of I is the *rank of* $\mathcal{W}(\mathcal{C})$.

Definition 2.4. A Cartan graph is called *connected* if its Weyl groupoid is connected, that is, if for all $a, b \in A$ there exists $w \in \mathrm{Hom}(a, b)$. The Cartan graph is called *simply connected*, if $\mathrm{Hom}(a, a) = \{\mathrm{id}^a\}$ for all $a \in A$. And two Cartan graphs $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ and $\mathcal{C}' = \mathcal{C}'(I', A', (\rho_i')_{i\in I'}, (C'^a)_{a\in A'})$ are termed *equivalent* if there are bijections $\varphi_0 : I \to I'$ and $\varphi_1 : A \to A'$ such that

$$\varphi_1(\rho_i(a)) = \rho'_{\varphi_0(i)}(\varphi_1(a)), \quad c_{\varphi_0(i)\varphi_0(j)}^{\varphi_1(a)} = c_{ij}^a,$$

for all $i, j \in I$ and $a \in A$. We then write $\mathcal{C} \cong \mathcal{C}'$.

Let $\mathcal{C}$ be a Cartan graph. For all $a \in A$ let

$$(R^{\mathrm{re}})^a = \{\mathrm{id}^a \sigma_{i_1} \cdots \sigma_{i_k}(\alpha_j) \mid k \in \mathbb{N}_0,\, i_1, \dots, i_k, j \in I\} \subseteq \mathbb{Z}^I.$$

The elements of the set $(R^{\mathrm{re}})^a$ are called *real roots* (at a). The pair $(\mathcal{C}, ((R^{\mathrm{re}})^a)_{a \in A})$ is denoted by $\mathcal{R}^{\mathrm{re}}(\mathcal{C})$. A real root $\alpha \in (R^{\mathrm{re}})^a$, where $a \in A$, is called positive (resp. negative) if $\alpha \in \mathbb{N}_0^I$ (resp. $\alpha \in -\mathbb{N}_0^I$).

Definition 2.5. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A})$ be a Cartan graph. For all $a \in A$ let $R^a \subseteq \mathbb{Z}^I$, and define $m^a_{i,j} = |R^a \cap (\mathbb{N}_0 \alpha_i + \mathbb{N}_0 \alpha_j)|$ for all $i, j \in I$ and $a \in A$. We say that

$$\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a \in A})$$

is a *root system of type* $\mathcal{C}$, if it satisfies the following axioms.

(R1) $R^a = R^a_+ \cup -R^a_+$, where $R^a_+ = R^a \cap \mathbb{N}_0^I$, for all $a \in A$.
(R2) $R^a \cap \mathbb{Z}\alpha_i = \{\alpha_i, -\alpha_i\}$ for all $i \in I$, $a \in A$.
(R3) $\sigma^a_i(R^a) = R^{\rho_i(a)}$ for all $i \in I$, $a \in A$.
(R4) If $i, j \in I$ and $a \in A$ such that $i \neq j$ and $m^a_{i,j}$ is finite, then $(\rho_i \rho_j)^{m^a_{i,j}}(a) = a$.

The root system $\mathcal{R}$ is *finite* if for all $a \in A$ the set R^a is finite. By [**10**, Prop. 2.12], if $\mathcal{R}$ is a finite root system of type $\mathcal{C}$, then $\mathcal{R} = \mathcal{R}^{\mathrm{re}}$, and hence $\mathcal{R}^{\mathrm{re}}$ is a root system of type $\mathcal{C}$ in that case. The roots are called *imaginary roots* provided they are not the real roots.

Remark 2.6. If $\mathcal{C}$ is a Cartan graph and there exists a root system of type $\mathcal{C}$, then $\mathcal{C}$ satisfies

(C3) If $a, b \in A$ and $\mathrm{id} \in \mathrm{Hom}(a, b)$, then $a = b$.

The following definition constructs a connection between Weyl groupoids and braided vector spaces.

Definition 2.7. Let (V^a, c) be a braided vector space of diagonal type with braiding defined by a bicharacter $\chi^a = (q_{ij})_{i,j=1,\dots,r}$ as above (with respect to a basis $x_1, \dots x_n$ of V).

We call (V^a, c) *locally finite* if there is a well defined matrix $(c^a_{ij})_{1 \le i,j \le n} \in \mathbb{Z}^{n \times n}$ such that the following formula of bicharacter holds:

(2.2)
$$c^a_{ij} = -\min\{m \in \mathbb{N}_0 \mid 1 + q_{ii} + q_{ii}^2 + \cdots + q_{ii}^m = 0 \text{ or } q_{ii}^m q_{ij} q_{ji} = 1\} \text{ if } i \neq j \text{ and } c^a_{ii} = 2.$$

Such a matrix (c^a_{ij}) is called a *Cartan matrix*.

Let $\alpha_1, \dots, \alpha_n$ be the standard basis of $\mathbb{Z}^n$. Now we explain the construction of the Weyl groupoid of a diagonal type Nichols algebra.

Definition 2.8. Let $\sigma_i^{\chi_a} \in \mathrm{Aut}(\mathbb{Z}^I)$ acts on a bicharacter χ_a. Then we define a new bicharacter $\chi_b = (\sigma_i^{\chi_a}) * \chi_a$ by

$$\chi_b(\alpha, \beta) = \chi_a((\sigma_i^{\chi_a})^{-1}(\alpha), (\sigma_i^{\chi_a})^{-1}(\beta)).$$

Notice that $\sigma_i^{\chi_a} = \sigma_i^{\chi_b}$.

Assume that the dimension of V is two. The following definition offers a technique to construct Cartan graph and characteristic sequence for a Nichols algebra.

DEFINITION 2.9. Repeatedly applying the above steps, we yield a sequence

$$\cdots \xrightarrow{\sigma_1^{\chi_{a_{-2}}}} \chi_{a_{-1}} \xrightarrow{\sigma_2^{\chi_{a_{-1}}}} \chi_{a_0} \xrightarrow{\sigma_1^{\chi_{a_0}}} \chi_{a_1} \xrightarrow{\sigma_2^{\chi_{a_1}}} \cdots$$

where each bicharacter χ_{a_i} should be locally finite and produces a well defined Cartan matrix $C^{\chi_{a_i}}$. The sequence of Cartan matrices connected by edges labeled by the reflections is called the *Cartan graph* of χ_{a_0}. We call the infinite sequence

$$\mathbf{c} = (\dots, c_{21}^{\chi_{a_{-1}}}, c_{12}^{\chi_{a_0}}, c_{21}^{\chi_{a_1}}, c_{12}^{\chi_{a_2}}, \dots)$$

the characteristic sequence of $\mathcal{C}^{\chi_{a_0}}$ or $\mathfrak{B}(V)$.

DEFINITION 2.10 (compare [**9**, Def. 3.1]). Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ and $\mathcal{C}' = \mathcal{C}'(I, A', (\rho'_i)_{i\in I}, (C'^a)_{a\in A'})$ be connected Cartan graphs. Let $\pi : A \to A'$ be a map such that $C'^{\pi(a)} = C^a$ for all $a \in A$ and the diagrams

$$\begin{array}{ccc} A & \xrightarrow{\rho_i} & A \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ A' & \xrightarrow{\rho'_i} & A' \end{array}$$

commute for all $i \in I$. We say that $\pi : \mathcal{C} \to \mathcal{C}'$ is a *covering*, and that $\mathcal{C}'$ is a *covering* of $\mathcal{C}$.

The composition of two coverings is still a covering. Any covering $\pi : \mathcal{C} \to \mathcal{C}'$ of Cartan graphs $\mathcal{C}$, $\mathcal{C}'$ is surjective by the diagrams above, since A is non-empty and $\mathcal{C}'$ is connected.

3. Simplicial and crystallographic arrangements

The notions of (simplicial, crystallographic) hyperplane arrangements in T will be recalled in this section. They were introduced in [**15**]. In order to apply topological arguments, we fix an inner product $(\, , \,)$ of V and V^*. Further, denote $\alpha^\perp = \ker \alpha$ for $\alpha \in V^*$. In this note, $0 \in \mathbb{N}$.

DEFINITION 3.1 (see [**15**]). A *hyperplane arrangement (of rank r)* is a pair $(\mathcal{A}, T)$, where T is a convex open cone in $V = \mathbb{R}^r$, and $\mathcal{A}$ is a (possibly infinite) set of linear hyperplanes such that $H \cap T \neq \emptyset$ for all $H \in \mathcal{A}$.

The arrangement $(\mathcal{A}, T)$ is called *locally finite*, if for every $x \in T$ there exists a neighbourhood $U_x \subset T$, such that $\sec(U_x)$ is a finite set, where

$$\sec_{\mathcal{A}}(X) := \bigcup_{x\in X} \operatorname{supp}_{\mathcal{A}}(x) = \{H \in \mathcal{A} \mid H \cap X \neq \emptyset\}.$$

If $(\mathcal{A}, T)$ is locally finite, the connected components of $T \setminus \bigcup_{H\in\mathcal{A}} H$ are open sets and will be called *chambers*, and denoted with $\mathcal{K}$=$\mathcal{K}(\mathcal{A})$.

Let $\mathcal{A}$ be a locally finite hyperplane arrangement, we associate to a chamber $K \in \mathcal{K}$ the *walls of K*

$$W^K := \{H \in \mathcal{H} \mid \langle H \cap \overline{K} \rangle = H, H \cap K = \emptyset\}.$$

DEFINITION 3.2 (compare [**15**]). We call a hyperplane arrangement $(\mathcal{A}, T)$ *simplicial*, if

(1) $\mathcal{A}$ is locally finite,
(2) every $K \in \mathcal{K}(\mathcal{A})$ is an open simplicial cone.

A locally finite hyperplane arrangement $(\mathcal{A}, T)$ is called *thin*, if $W^K \subset \mathcal{A}$ holds for all $K \in \mathcal{K}(\mathcal{A})$.

DEFINITION 3.3 (see [**15**]). We call a locally finite hyperplane arrangement $(\mathcal{A}, T)$ *spherical*, if $T = \mathbb{R}^r$. We call it *affine*, if

$$T = \{v \in V \mid \gamma(v) > 0\}$$

For some $0 \neq \gamma \in V^*$. Such a γ is unique up to multiples and called the *imaginary root* of the arrangement.

REMARK 3.4. The cone of a hyperplane arrangement $(\mathcal{A}, T)$ is called T as it resembles the Tits cone for Coxeter groups. If T satisfies the following assumption, the requirement of T to be convex can be omitted:

- T is an open and connected cone,
- T is the interior of its closure,
- $\mathcal{A}$ is thin,
- for $H \in \mathcal{A}$, $T \setminus H$ has exactly two connected components.

DEFINITION 3.5 (compare [**15**]). Let $V = \mathbb{R}^r$ and $T \subset V$ be an open convex cone. A *root system (in T)* is a subset $R \subset V^*$ such that

(1) $(\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}, T)$ is a simplicial arrangement in T,
(2) $\langle \alpha \rangle \cap R = \{\pm\alpha\}$ for all $\alpha \in R$.

We call $(\mathcal{A}, T)$ the simplicial arrangement *associated to* R. Let K be a chamber. The *root basis of* K is the set

$$B^K := \{\alpha \in R \mid \alpha^\perp \in W^K, \alpha(x) \geq 0 \text{ for all } x \in K\}.$$

LEMMA 3.6 (compare [**6**, Lemma 2.2]). *Let $(\mathcal{A}, T, R)$ be a simplicial arrangement, K a chamber. Then $R \subset \pm \sum_{\alpha \in B^K} \mathbb{R}_{\geq 0}\alpha$. That is, every root is a non-negative or non-positive linear combination of B^K.*

DEFINITION 3.7. [**6**, Def. 2.3] Let $(\mathcal{A}, T, R)$ be simplicial as above. Then by Lemma 3.6 (Ch. 1), fixing a chamber K we may define

$$R_+ := \{\beta \in R \mid \beta \in \sum_{\alpha \in B^K} \mathbb{R}_{\geq 0}\alpha\}.$$

Notice that $R = R_+ \mathbin{\dot\cup} -R_+$.

DEFINITION 3.8. [**8**, Def. 3.2] A *crystallographic arrangement* is a simplicial arrangement with root system $(\mathcal{A}, T, R)$ such that:

$$R \subseteq \sum_{\alpha \in B^K} \mathbb{Z}\alpha. \tag{3.1}$$

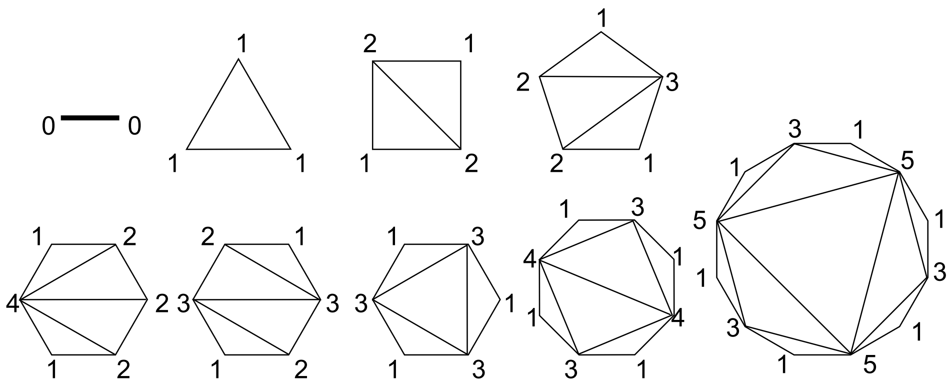

Figure 1. *Triangulations associated to and η-sequences.*

for each $K \in \mathcal{K}(\mathcal{A})$.

REMARK 3.9. By 3.6 (Ch. 1), if $\mathcal{A}$ is crystallographic, then in fact we even have $R \subseteq \pm \sum_{\alpha \in B^K} \mathbb{N}\alpha$ for all $K \in \mathcal{K}(\mathcal{A})$.

By the notions of triangulations of polygons, the spherical crystallographic arrangements of rank two can be parametrized. We briefly recall this correspondence.

DEFINITION 3.10 (compare [**11**, Def. 3.2]). We define the set $\mathcal{E}$ of *η-sequences* recursively by:

(1) $(0,0) \in \mathcal{E}$.
(2) If $(c_1, \dots, c_n) \in \mathcal{E}$, then $(c_2, c_3, \dots, c_{n-1}, c_n, c_1) \in \mathcal{E}$ and $(c_n, c_{n-1}, \dots, c_2, c_1) \in \mathcal{E}$.
(3) If $(c_1, \dots, c_n) \in \mathcal{E}$, then $(c_1+1, 1, c_2+1, c_3, \dots, c_n) \in \mathcal{E}$.

The triangulations of convex polygons by non-intersecting diagonals correspond to η-sequences which are also called *quiddity cycles* in [**5**]: To each vertex i we attach the number of triangles adjacent to this vertex and obtain an η-sequence $(c_1, c_2, \dots, c_n) \in \mathbb{N}^n$ (see Figure 1).

Finite crystallographic arrangements of rank two are in one-to-one correspondence with η-sequences (see for example [**11**]). They could be expressed in the following way. A given η-sequence is constructed recursively by definition. We construct the set R_+ of the arrangements using the same recursion:

- The sequence $(0,0)$ corresponds to $((1,0),(0,1))$.
- If $(c_1, \dots, c_n)$ corresponds to $(\alpha_1, \dots, \alpha_n)$, then $(c_1+1, 1, c_2+1, \dots, c_n)$ corresponds to $(\alpha_1, \alpha_1+\alpha_2, \alpha_2, \dots, \alpha_n)$.

The coordinates of the elements of R_+ may also be computed using the triangulation. To each vertex i, attach numbers $\varphi_i(j), j = 1, 2, \dots, n$, write 0 at vertex i, 1 at each adjacent vertex, and if two vertices of a triangle have labels a and b then write $a+b$ at the third vertex. The number written at vertex j is $\varphi_i(j)$ (see Figure 1 (ii)). The numbers $\varphi_i(j)$ are coordinates of roots of the associated *Weyl groupoid* of rank two. Choose two neighbouring vertices i, $i+1$ and write $(1,0)$ at vertex i and $(0,1)$ at vertex $i+1$. Then if two vertices of a triangle have labels α and β, then write $\alpha+\beta$ at the third vertex (see Figure 1 (ii)). The set R_+ is then

$$R_+ = \{(\varphi_i(j), \varphi_{i+1}(j)) \mid j = 1, \dots, n\}.$$

For some fixed i (see Figure 1 (iii)).

The η-sequences also appeared earlier as the third line of *frieze patterns* (see for example [**5**]). The numbers $\varphi_i(j)$ are all entries of its associated frieze pattern. We will illustrate this by an example in Section 2 of Chapter 3.

4. Weyl groupoids and crystallographic arrangements

Connected simply connected Cartan graphs are in one-to-one correspondence with crystallographic arrangements (see [**15**] for more details).

PROPOSITION 4.1. [**14**] *Let $(\mathcal{A}, T, R)$ be a crystallographic arrangement. Then there exists a connected simply connected Cartan graph $\mathcal{C}$ with root system $\mathcal{R}$ such that $R = R^a$ for some $a \in A$.*

PROOF. A construction of the desired Cartan graph $\mathcal{C}$ for R works in the same way as for finite crystallographic arrangements, see [**6**]. The resulting Cartan graph depends on a chosen starting chamber K, but all other choices for K yield equivalent Cartan graphs. $\square$

The Cartan graph constructed in Prop. 4.1 (Ch. 1) is denoted by $\mathcal{C}(\mathcal{A}, T, R, K)$.

THEOREM 4.2. [**6**, Thm. 5.4] *Let $\mathfrak{A}$ be the set of all crystallographic arrangements and $\mathfrak{C}$ be the set of all connected simply connected Cartan graphs for which the real roots are a root system. Then the map*

$$\Lambda : \mathfrak{A}/_{\cong} \to \mathfrak{C}/_{\cong}, \quad \overline{(\mathcal{A}, T, R)} \mapsto \overline{\mathcal{C}(\mathcal{A}, T, R, K)},$$

where K is any chamber of $\mathcal{A}$, is a bijection.

CHAPTER 2

Affine crystallographic arrangements

In this chapter, we introduce affine crystallographic arrangements and affine Cartan graphs. The notion of characteristic sequences of affine crystallographic arrangements is given. Then we recall the SL_2-patterns of the half-plane. At last, the correspondence between affine crystallographic arrangements and affine Cartan graphs will be introduced.

1. Affine crystallographic arrangements and affine Cartan graphs

Affine crystallographic arrangements were introduced in [**15**] and affine Cartan graphs were defined in [**14**].

Definition 1.1. We call $(\mathcal{A}, T, R, \gamma_1)$ an *affine crystallographic arrangement of rank* r if $(\mathcal{A}, T, R)$ is a crystallographic arrangement and it satisfies one of the following equivalent conditions:

(1) $V = \overline{T} \cup -\overline{T}$, i. e. T is a half-space.
(2) Up to signs, there exists exactly one element $\gamma_1 \in R$ such that the hyperplane $\gamma^\perp$ is not a wall of a chamber of $\mathcal{A}$. We call γ_1 the *imaginary root* of R.

Remark 1.2. Let $(\mathcal{A}, T, R, \gamma_1)$ be an *affine crystallographic arrangement of rank* r.

(1) Since R is infinite, $\dim\langle R\rangle_{\mathbb{R}} \geq 2$.
(2) The hyperplane $\gamma_1^\perp$ is not a wall of a chamber. Every neighborhood $U \subseteq \mathbb{P}(V^*)$ of $p(\gamma_1)$ contains almost all elements of $p(R)$.

Definition 1.3. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a connected Cartan graph of rank r ($|I| = r$) and assume that there exists a root system $\mathcal{R}$ of type $\mathcal{C}$. Let $a \in A$ be an object. Assume that there is exactly one imaginary root in R^a_+, and that there is no root system of type $\mathcal{C}$ with more than one positive imaginary at a. Then we call $\mathcal{C}$ an *affine Cartan graph* and the Weyl groupoid $\mathcal{W}(\mathcal{C})$ an *affine Weyl groupoid.*

Remark 1.4. Since an affine Cartan graph is connected, the assumption on the nummers of imaginary roots at a does not depend on the chosen object a.

Proposition 1.5. *Let* $\mathcal{C}$ *be an affine Cartan graph, then*

(1) *Any root system of type* $\mathcal{C}$ *is infinite.*
(2) *There are infinitely many real roots at each object of* $\mathcal{C}$*.*
(3) *The universal cover of* $\mathcal{C}$ *has infinitely many objects.*

Proof. By [**10**, Prop. 2.12], any root system of type $\mathcal{C}$ is infinite. By [**10**, Prop. 2.9], the real roots of $\mathcal{C}$ also form a root system of type $\mathcal{C}$, thus there are infinitely many real roots at each object of $\mathcal{C}$. In particular, the universal cover of $\mathcal{C}$ has infinitely many objects. □

2. Crystallographic arrangements of rank two

In the following, we will give the notion of characteristic sequences from which we can get the corresponding Weyl groupoids. Except for some noted notions, this section comes from [**14**].

Let $(\mathcal{A}, T, R)$ be a crystallographic arrangements of rank two. Assume that $\mathcal{A}$ is not spherical, i. e. T is an open convex cone of the form

$$T = \{v \in V \mid \gamma_1(v) > 0 \text{ and } \gamma_2(v) < 0\}$$

for suitable $\gamma_1, \gamma_2 \in V^*$ (possibly $\gamma_1 = -\gamma_2$). The linear forms γ_1 and γ_2 are not in R, they are *imaginary roots.*

Now fix an $\alpha \in R$, without loss of generality (choose $-\alpha$ otherwise)

$$\{v \in V \mid \alpha(v) > 0 \text{ and } \gamma_1(v) > 0\} \subseteq T$$

Thus

$$R \subseteq \{k\alpha + m\gamma_1 \mid k, m \in \mathbb{R}\}$$

Definition 2.1. [**7**] We call

$$R_{c\ell} := \{k\alpha + m\gamma_1 \in R \mid k, m \in \mathbb{R}, k > 0\}$$

the set of *clockwise roots* of R (remark that $R = R_{c\ell} \,\dot{\cup}\, -R_{c\ell}$).

Definition 2.2. [**7**] Define a total ordering $\leq$ on $R_{c\ell}$ by

$$k\alpha + m\gamma_1 \leq k'\alpha + m'\gamma_1 \quad :\Longleftrightarrow \quad mk' \leq m'k.$$

Notice that $\leq$ does not depend on the length of γ_1.

Definition 2.3. [**7**] Denote

$$R_{c\ell} = \{\ldots, \beta_{-2}, \beta_{-1}, \alpha = \beta_0, \beta_1, \beta_2, \ldots\},$$

where $-\gamma_1 < \ldots < \beta_{-1} < \beta_0 < \beta_1 < \ldots < \gamma_1$. The set $R_{c\ell}$ defines a sequence of adjacent chambers dual to the sequence of cones

$$\ldots, \langle\beta_0, -\beta_{-1}\rangle_{>0}, \langle\beta_1, -\beta_0\rangle_{>0}, \langle\beta_2, -\beta_1\rangle_{>0}, \langle\beta_3, -\beta_2\rangle_{>0}, \ldots$$

By Axiom of crystallographic arrangements 3.8 (Ch. 1), every base change between two consecutive cones is of the form

$$\begin{pmatrix} a & -1 \\ b & 0 \end{pmatrix}$$

for some $a, b \in \mathbb{N}$. By Axiom of crystallographic arrangements 3.8 (Ch. 1) again, the determinant of such a base change is -1 (because of our choice of ordering of the basis elements), so it is of the form

$$\eta(a) := \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$$

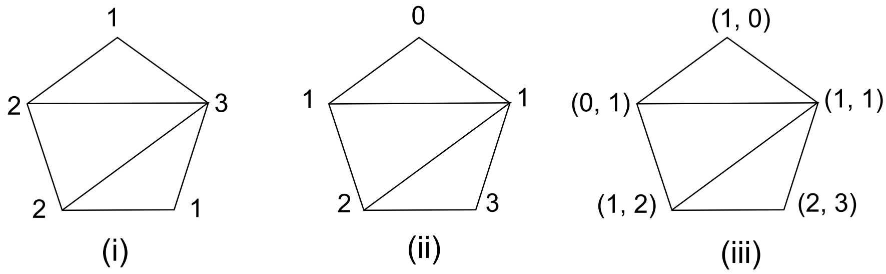

Figure 1. *Triangulation associated to (2, 1, 3, 1, 2).*

for some $a \in \mathbb{N}$. Thus $(\mathcal{A}, T, R, \alpha)$ defines a sequence of natural numbers (*Cartan entries*) $(c_i)_{i\in\mathbb{Z}}$ such that the base change from $\langle \beta_i, -\beta_{i-1}\rangle_{>0}$ to $\langle \beta_{i+1}, -\beta_i\rangle_{>0}$ is $\eta(c_{i-1})$.
We call the sequence $(c_i)_{i\in\mathbb{Z}}$ the *characteristic sequence* of R.

Then we can have the following proposition naturally:

PROPOSITION 2.4. [**7**] *Up to base change, an affine crystallographic arrangement is uniquely determined by its characteristic sequence.*

From the work what we have done, we find that maybe it is very difficult to classify the characteristic sequences $(c_i)_{i\in\mathbb{Z}}$ corresponding to affine crystallographic arrangements completely. However, when we consider this question in the context of continued fractions, we find that Tietze's theorem offers a very useful and hopeful interpretation. Then let us recall the following theorem:

THEOREM 2.5 (Tietze, [**23**], see also [**22**, Satz 2.18]). *Let $a_1, a_2, \ldots \in \mathbb{Z}$ and $b_0, b_1, \ldots \in \mathbb{Z}$ and consider the continued fraction*

$$b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \ldots = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}}.$$

If $|b_i| \geq |a_i| + 1$ for all i, then the continued fraction is convergent with limit ξ, where ξ is irrational except if

$$a_i < 0, \quad b_i = |a_i| + 1$$

for all i greater than some fixed i_0.

COROLLARY 2.6. *Let $(c_i)_{i\in\mathbb{Z}}$ be the characteristic sequence of an affine crystallographic arrangement. Then $c_i > 0$ for all $i \in \mathbb{Z}$, and one of the following holds:*

(1) $\exists j, k \in \mathbb{Z} \,:\, c_i = 2 \;\forall i \in \mathbb{Z}$ *with* $i < j$ *or* $i > k$, *or*
(2) *there exists* $j \in \mathbb{Z}$ *with* $c_j = 1$.

PROOF. If there is an i with $c_i = 0$, then $\mathcal{A}$ has only four chambers; this is excluded. Write

$$\eta(c_\nu)\cdots\eta(c_1) = \begin{pmatrix} B_\nu & A_\nu \\ B_{\nu-1} & A_{\nu-1} \end{pmatrix}$$

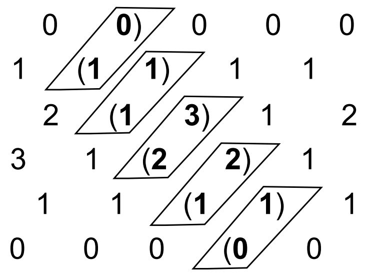

Figure 2. *The frieze pattern to the sequence (2, 1, 3, 1, 2).*

for suitable A_ν, B_ν, $\nu \in \mathbb{N}$. Then

$$b_0 + \frac{-1|}{|b_1} + \frac{-1|}{|b_2} + \ldots + \frac{-1|}{|b_\nu} = \frac{A_\nu}{B_\nu}.$$

Since γ_1 is rational with respect to for instance β_0, β_1, the above continued fraction cannot converge to an irrational number. Thus by Thm. 2.5 (Ch. 2), either $(c_{i_0}, c_{i_0+1}, \ldots) = (2, 2, \ldots)$ for some $i_0 \in \mathbb{Z}$, or there exists a $j \in \mathbb{Z}$ with $c_j < 2$. Similarly, $(c_{i_0}, c_{i_0-1}, \ldots) = (2, 2, \ldots)$ for some $i_0 \in \mathbb{Z}$, or there exists a $j \in \mathbb{Z}$ with $c_j < 2$. $\square$

REMARK 2.7. Since we have the rule (compare [**9**, Lemma 5.2])

$$\eta(a)\eta(b) = \eta(a+1)\eta(1)\eta(b+1)$$

for all a, b, we can "remove" and "insert" finitely many 1's from a characteristic sequence (equivalently remove elements of R) without affecting the fact that we have an affine crystallographic arrangement. After removing finitely many entries in the middle, the only characteristic sequence without a 1 is $(\ldots, 2, 2, 2, \ldots)$; It is the sequence of the affine Weyl group of type A. Certainly there are sequences with infinitely many 1's, for which the continued fraction is not convergent, and which are also characteristic sequences of affine crystallographic arrangements. The situation is much easier provided the characteristic sequence to be periodic.

3. SL_2-Patterns of the half-plane

The correspondence between SL_2-patterns and crystallographic arrangements of rank two will be completed in this section.

DEFINITION 3.1. An SL_2-*pattern* is a map $z : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ such that

$$\det \begin{pmatrix} z(i,j) & z(i,j+1) \\ z(i+1,j) & z(i+1,j+1) \end{pmatrix} = 1$$

for all $(i,j) \in \mathbb{Z} \times \mathbb{Z}$. (This is called the *unimodular rule* in [**5**].)

Notice that for historical reasons (because of the special case of *friezes*), the map z is usually expressed in the form of a lattice rotated as for example in Fig. 3. Friezes were classified in [**5**] by Conway and Coxeter, more general types of SL_2-patterns were classified in [**4**].

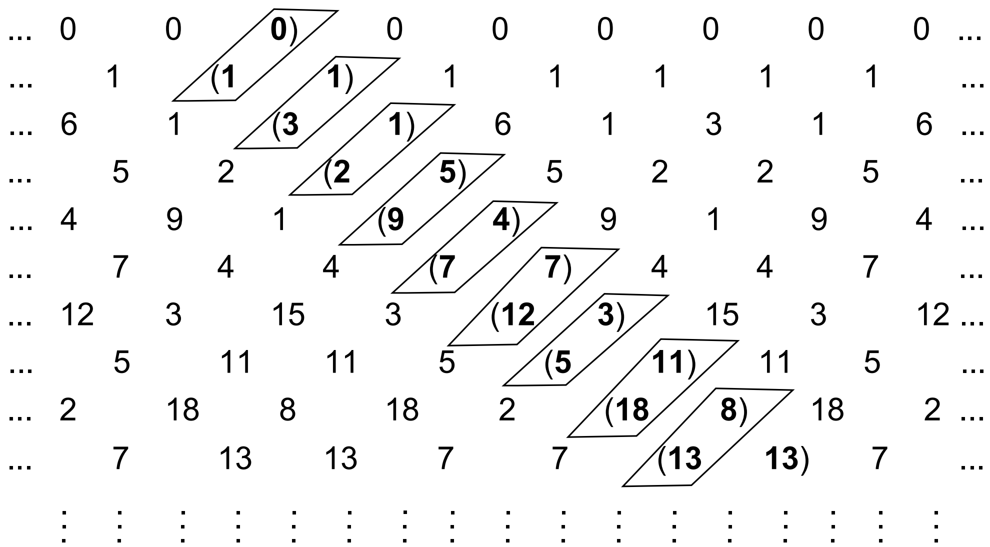

Figure 3. *The frieze pattern to the periodic characteristic sequence ..., 6, 1, 3, 1, 6, 1, 3, 1, 6, 1, 3, 1, ...*

Definition 3.2 (see [**3**]). Let $Z := \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid i \geq j\}$. A *half-plane* SL_2-*pattern* is a map $z : Z \to \mathbb{Z}$ satisfying the unimodular rule where it is defined and such that

$$z(i,i) = 0, z(i+1,i) = 1, z(i,j) \in \mathbb{N}^+$$

for all $i \in \mathbb{Z}$ and $i > j$.

See Fig. 3 for an example of a half-plane SL_2-pattern.

Remark 3.3. If z is a half-plane SL_2-pattern, then it is uniquely determined by the entries

$$c_i := z(i+2,i), \quad i \in \mathbb{Z}.$$

Proposition 3.4. [**7**] *Let $c_i, i \in \mathbb{Z}$ be arbitrary integers. Then there exists a map $z : Z \mapsto \mathbb{Z}$ satisfying the unimodular rule with*

$$z(i,i) = 0, z(i+1,i) = 1, \text{ and } z(i+2,i) = c_i, \quad i \in \mathbb{Z}.$$

Proof. We prove the existence of the entry $z(j,i)$ by induction over $j-i$. We need to go back four steps to perform the induction step, thus we extend our map z to

$$z(i-1,i) := -1$$

for $i \in \mathbb{Z}$. Now consider the following local part of the pattern,

$$\begin{array}{ccccc} \cdots & & a & & \cdots \\ & b & & c & \\ d & & m & & e \\ & r & & s & \\ & & t & & \end{array}$$

where $a, b, c, d, m, e, r, s \in \mathbb{Z}$ and $t = z(j, i)$ is to be shown to be in $\mathbb{Z}$ by the unimodular rule.

Assume first that $b, c, r, s, m \neq 0$. Notice first that b, c, r, s are all coprime to m. Now $t \in \mathbb{Z}$ if and only if $rs - 1$ is divisible by m, or $rs \equiv 1 \pmod{m}$. We get

$$bc \equiv -br \equiv -cs \equiv 1 \pmod{m}$$

But then $c \equiv -r$ and thus $rs \equiv 1$, hence $t \in \mathbb{Z}$.

Now assume that $m = 0$. Then $bc - a \cdot 0 = 1$, thus $b = c = 1$ or $b = c = -1$. In both cases, $r = s \in \{\pm 1\}$ since $dm - br = 1$ and $me - cs = 1$. Thus for $m = 0$ we may choose $t \in \mathbb{Z}$ arbitrary.

The last case is when one of b, c, r, s is 0. But then $m = \pm 1$, thus $t = m(rs - 1) \in \mathbb{Z}$ is valid. ☐

For arbitrary given c_i, usually, the resulting pattern will not be a half-plane SL_2-pattern although the map above always exists. The following example illustrate this:

$$\begin{array}{cccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & 1 & & 2 & & 1 & & 1 & & 2 & & 1 \\
0 & & 1 & & 1 & & 0 & & 1 & & 1 & \\
 & -1 & & 0 & & -1 & & -1 & & 0 & & -1 \\
0 & & -1 & & -1 & & 0 & & -1 & & -1 &
\end{array}$$

However, by using the results from the last section, we can obtain the following correspondence.

THEOREM 3.5. [**7**] *Let $c_i \in \mathbb{Z}$ for $i \in \mathbb{Z}$, $z : Z \to \mathbb{Z}$ satisfy the unimodular rule where it is defined, and*

$$z(i, i) = 0, z(i + 1, i) = 1, z(i + 2, i) = c_i \text{ for all } i \in \mathbb{Z}$$

(1) *The map z defines a frieze pattern if and only if $(c_i)_{i \in \mathbb{Z}}$ is the characteristic sequence of a finite crystallographic arrangement of rank two.*
(2) *The map z is a half-plane SL_2-pattern if and only if $(c_i)_{i \in \mathbb{Z}}$ is the characteristic sequence of an infinite crystallographic arrangement of rank two.*

4. The correspondence

The connection between affine crystallographic arrangement and Cartan graphs will be explained in this section. The content of this section comes from [**14**] except for some noted notions. For any affine crystallographic arrangement, we firstly need to show that it defines a Cartan graph which is unique up to base changes.

PROPOSITION 4.1. *Let $(\mathcal{A}, T, R, \gamma_1)$ be an affine crystallographic arrangement. Then there exists a simply connected affine Cartan graph $\mathcal{C}$ with root system $\mathcal{R}$ such that $R = R^a$ for some $a \in A$.*

PROOF. A construction of the desired Cartan graph $\mathcal{C}$ for R works in the same way as for finite crystallographic arrangements, see [**6**]. It remains to check the additional assumption on imaginary roots. The imaginary root γ_1 of R becomes an imaginary root for $\mathcal{C}$. Now assume that we have a root system $\mathcal{R}$ of type $\mathcal{C}$ such that R^a contains two different imaginary roots γ_1 and γ_1' for some object a. But then either $\gamma_1^\perp$ or $\gamma_1'^\perp$ will intersect a chamber of R non-trivially, and thus axiom (R1) will be violated at the object corresponding to the class of paths to this chamber. $\square$

REMARK 4.2. Notice that (2) of Def. 1.1 (Ch. 2) is really stronger than the fact that every neighborhood $U \subseteq \mathbb{P}(V^*)$ of $p(\gamma_1)$ contains almost all elements of $p(R)$: Without (2) of Def. 1.1 (Ch. 2), $\mathcal{A}$ could have a chamber with wall $\gamma_1^\perp$. Such an arrangement would not allow the construction of a corresponding Cartan graph.

In order to complete the correspondence, a convex cone associated to any connected Cartan graph need to be constructed.

DEFINITION 4.3. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a connected Cartan graph with root system $\mathcal{R}$. Let $a \in A$ be an object and

$$\mathcal{A}^a := \{\alpha^\perp \mid \alpha \in R^a\}$$

Let B denote the standard basis of $\mathbb{R}^I$ and $B^\vee$ its dual basis. Then $B \subseteq R^a$ and thus

$$K_0 := \langle B^\vee \rangle_{>0} \in \mathcal{K}(\mathcal{A}^a)$$

because no hyperplane of $\mathcal{A}$ intersects K_0. Using the definition of σ_i^a at $\rho_i(a)$ we see that

$$\sigma_i^a(K_0) \in \mathcal{K}(\mathcal{A}^a), \quad i \in I,$$

as well. But σ_i^a is a linear automorphism, so the chambers of $\mathcal{A}^a$ and $\mathcal{A}^{\rho_i(a)}$ are isomorphic via σ_i^a. We call the set

$$M_{\mathcal{C}}^a := \bigcup_{m\in\mathbb{N}} \bigcup_{\mu_1,\ldots,\mu_m\in I} \overline{\sigma_{\mu_m} \cdots \sigma_{\mu_1}^a(K_0)}$$

the *real cone* of $\mathcal{C}$. By induction every subset $\sigma_{\mu_m} \cdots \sigma_{\mu_1}^a(K_0)$ is a chamber of $\mathcal{A}^a$, and all of them are open simplicial cones. Notice further that choosing another object b, there exists a morphism $w \in \mathrm{Hom}(a, b)$ since $\mathcal{C}$ is connected, and we have $M_{\mathcal{C}}^b = w(M_{\mathcal{C}}^a)$. Thus the real cone of $\mathcal{C}$ is unique up to base change.

Some associated notions and theorems of convex set are also needed. Let us denote the *boundary* of a set A by $\mathrm{bd}\, A$ and *relative interior* of S by $\mathrm{relint}\, S$ and *interior* of A by $\mathrm{int}\, A$. We call $\mathrm{aff}\, A$ the *affine hull* of a set A. The relative interior may be defined as $\mathrm{relint}\, S := \{x \in \mathbb{R}^r : (\mathrm{aff}\, A) \cap (x + \varepsilon B^r) \subset A$ for some $\varepsilon > 0, B$ is the unit ball of $\mathbb{R}^r\}$ and the *closure* of a set A is defined as $\mathrm{Cl}\, A$.

DEFINITION 4.4 ([**16**], p.11). Two subsets A and A' of $\mathbb{R}^r$ are said to be *separated* by a hyperplane H provided A is contained in one of the closed halfspaces determined by H while A' is contained in the other. The sets A and A' are *strictly separaed* by H if they are separated and $A \cap H = A' \cap H = \phi$. In other words, A

and A' are strictly separated by H provided they are contained in different open halfspaces determined by H.

THEOREM 4.5 ([**16**], p.11, Thm. 2). *If A and A' are convex subsets of $\mathbb{R}^r$ such that $\operatorname{aff}(A \cup A') = \mathbb{R}^r$ then A and A' may be separated by a hyperplane if and only if*

$$\operatorname{relint} A \cap \operatorname{relint} A' = \phi.$$

From the Thm. 4.5 (Ch. 2) above, we have the following proposition which is given in the book ([**16**], p.12, Prop. 4) without proof. Here we give the full proof.

PROPOSITION 4.6. *If K is a convex set in $\mathbb{R}^r$ and if C is a convex subset of $\operatorname{bd} K$ (in particular, if C is a single point of $\operatorname{bd} K$) then there is a hyperplane separating K and C. In other words, there exists a supporting hyperplane of K which contains C.*

PROOF. Assume that $\operatorname{aff} C = \mathbb{R}^r$. In this case C is nonempty and $K \subseteq \operatorname{aff} C$. Since $C \subseteq \operatorname{bd} K$, then the interior of K is empty. If C is nonempty, then $\operatorname{relint} C$ is never empty. By the definition of relative interior, $\phi \neq \operatorname{relint} C = \{x \in \mathbb{R}^r \mid \text{for some } \varepsilon > 0 : x + \varepsilon B^r \subset C\} = \operatorname{int} C$. Since $C \subseteq \operatorname{bd} K = \operatorname{Cl} K \cap \operatorname{Cl}(\sim K)$, the fact $\operatorname{int} C \neq \phi$ leads to a contradiction (here $\sim K$ denotes the *complement* of K). It implies that $\operatorname{aff} C \neq \mathbb{R}^r$.

Then there are three cases:

(1) $\operatorname{aff} C \neq \operatorname{aff} K, \operatorname{aff} K = \mathbb{R}^r$: If $\operatorname{aff} K = \mathbb{R}^r$, then $\operatorname{relint} K$ is the interior of K in $\mathbb{R}^r$. Then $\operatorname{relint} C \subset C \cap \operatorname{relint} K = \phi$. Since C and K are convex subsets of $\mathbb{R}^r$, then Thm. 4.5 (Ch. 2) is satisfied. There exists a hyperplane $H_1 \in \mathbb{R}^r$ separates K and C.
(2) $\operatorname{aff} C \neq \operatorname{aff} K, \operatorname{aff} K \neq \mathbb{R}^r$: For any subset A of $\mathbb{R}^r$, its dimension $\dim A$ is defined by $\dim A = \dim(\operatorname{aff} A)$. If $\operatorname{aff} C \neq \operatorname{aff} K, \operatorname{aff} K \neq \mathbb{R}^r$, i. e., $\dim C = l < r, \dim K = m < r, l \neq m$, then there exists a hyperplane $H_2 \in \mathbb{R}^r$ contains K and C ($\dim H_2 \geq l, m$). Then H_2 separates K and C.
(3) $\operatorname{aff} C = \operatorname{aff} K$: If $\operatorname{aff} C = \operatorname{aff} K$, i. e. $\dim C = \dim K = p < r$, then there is a hyperplane $H_3 \in \mathbb{R}^r$ such that $C \subseteq H_3, K \subseteq H_3(\dim H_3 \geq p)$. Thus H_3 satisfies the definition of separating hyperplane.

□

The following proposition completes the correspondence.

PROPOSITION 4.7. *Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be an affine Cartan graph with root system $\mathcal{R}$, and $a \in A$. Then*

$$(\mathcal{A}^a := \{\alpha^\perp \mid \alpha \in R^a\}, M^a_{\mathcal{C}}, R^a)$$

is an affine crystallographic arrangement.

PROOF. Let γ_1 be the (positive) imaginary root of of $\mathcal{C}$ and $T := M^a_{\mathcal{C}} \neq V$ be the real cone of $\mathcal{C}$. By construction, T is connected and by definition, $\gamma_1(u) > 0$

for all $u \in T$. Further, by Remark 3.4 (Ch. 1), T is convex. Thus $(\mathcal{A}^a, T, R^a)$ is crystallographic. It remains to prove that it is affine,

$$T = \{u \in V \mid \gamma_1 > 0\},$$

i. e., that $u \in T$ if $\gamma_1(u) > 0$. Thus let $u \in V$ be such that $\gamma_1(u) > 0$ and assume $u \notin T$. Then $u \in \overline{T} \backslash T$. By Prop. 4.6 (Ch. 2), there exists a hyperplane $H \in V$ which contains u and $H \cap T = \phi$. But then we find a $\beta \in V^*, \beta \neq \gamma_1$ such that $H = \beta^\perp$ and $\beta(v) > 0$ for all $v \in T$. In the lattice spanned by R^a, we can choose an element

$$\delta = \lambda\gamma_1 + \mu\beta \in V^* \text{ with } \lambda, \mu > 0.$$

Then still, $\delta(u) > 0$ for all $u \in T$, and $\delta \neq \gamma_1$. But then including $\pm\delta$ into the root system $\mathcal{R}$ yields a root system of type $\mathcal{C}$ with two imaginary roots contradicting the assumption. □

REMARK 4.8. The Cartan graphs corresponding to affine crystallographic arrangements do not necessarily have a covering with finitely many objects. See Section 2 of Chapter 3 for some examples of rank two.

CHAPTER 3

Affine Weyl groupoids of rank two with limit 1

In this chapter a large class of affine crystallographic arrangements are discussed. Those affine crystallographic arrangements include the ones with periodic characteristic sequence (as well as all the classical types). This chapter is one part of [**14**].

1. Affine Weyl groupoids of rank two with limit 1

Definition 1.1. Let $(\mathcal{A}, T, R, \gamma_1)$ be an affine crystallographic arrangement of rank $r = 2$ with imaginary root γ_1. We say that $(\mathcal{A}, T, R, \gamma_1)$ has *limit* 1 if there exists an $\alpha \in R$ with

$$\langle R\rangle_{\mathbb{Z}} = \mathbb{Z}\alpha + \mathbb{Z}\gamma_1.$$

Remark 1.2. The classical affine Weyl groups of rank two define affine crystallographic arrangements with limit 1; see 2.8, 2.9, 2.10 of Ch. 3 for further examples.

Throughout this section, assume that $(\mathcal{A}, T, R, \gamma_1)$ has limit 1 with respect to α. Let

$$N := N(R, \alpha) := \{k \in \mathbb{Z} \mid k\alpha + m\gamma_1 \in R \text{ for some } m \in \mathbb{Z}\}.$$

By Axiom of crystallographic arrangements, $N = -N$. Let $N_+ := N \cap \mathbb{R}_{\geq 0}$. Notice that

$$R \subseteq \{k\alpha + m\gamma_1 \mid k \in N,\ m \in \mathbb{Z}\}\backslash\{0\},$$

and that equality is not possible because $\gamma_1 \in R$ implies $0 \in N$ and $m\gamma_1 \in R$ implies $m = \pm 1$ by Axiom of crystallographic arrangement 3.8 (Ch. 1).

Definition 1.3. As in Def. 2.2 (Ch. 2), we have a total ordering $\leq$ on R_+. We write

$$[\beta, \gamma] := \{\delta \in R_+ \mid \beta \leq \delta \leq \gamma\}$$

and $R_0 := [\alpha, \alpha + \gamma_1]$.

Lemma 1.4. *If $k\alpha + m\gamma_1 \in R_+$, then* $\operatorname{Gcd}(k, m) = 1$.

Proof. Let $\beta := k\alpha + m\gamma_1 \in R_+$ and $g := \operatorname{Gcd}(k, m)$. Choose any $\gamma \in R$ such that the dual cone to $\langle \beta, \gamma\rangle_{>0}$ is a chamber of $\mathcal{A}$. Then by Axiom of crystallographic arrangement 3.8 (Ch. 1), the determinant d of a base change from $\{\alpha, \gamma_1\}$ to $\{\beta, \gamma\}$ is an integer divisible by g. But the inverse base change also has an integral determinant, so $d = \pm 1$ and hence $g = 1$. $\square$

LEMMA 1.5. *Let R, α, γ_1 be as above. Then for all $n \in \mathbb{Z}$, $\alpha + n\gamma_1 \in R$ and*

$$R_+ \backslash \{\gamma_1\} = \bigcup_{n \in \mathbb{Z}} [\alpha + n\gamma_1,\ \alpha + (n+1)\gamma_1].$$

PROOF. It suffices to prove that $\alpha + n\gamma_1 \in R$ for all $n \in \mathbb{Z}$. Assume there is an $n \in \mathbb{Z}$ with $\alpha + n\gamma_1 \notin R$. Let $\beta_1, \beta_2 \in R$ be such that

$$\beta_1 < \alpha + n\gamma_1 < \beta_2, \quad |[\beta_1, \alpha + n\gamma_1] \cap R_+| = |[\alpha + n\gamma_1, \beta_2] \cap R_+| = 1.$$

Hence $\beta_1, -\beta_2$ is the dual basis of a chamber and by Axiom of crystallographic arrangement 3.8 (Ch. 1),

$$|n_1 m_2 - m_1 n_2| = 1 \tag{1.1}$$

if $\beta_1 = n_1\alpha + n_2\gamma_1$, $\beta_2 = m_1\alpha + m_2\gamma_1$. But $\beta_1 < \alpha + n\gamma_1 < \beta_2$ implies that $n_2 < nn_1$, $nm_1 < m_2$, so $n_1 m_2 > m_1 n_2 + 1$ contradicting Equation [1.1]. □

2. The periodicity

Now we characterize affine crystallographic arrangements with limit 1 of rank two by their characteristic sequences.

DEFINITION 2.1. Let $(c_i)_{i \in \mathbb{Z}}$ be the characteristic sequence of R with respect to

$$\{\dots, \beta_{-2}, \beta_{-1}, \alpha = \beta_0, \beta_1, \beta_2, \dots\}.$$

By Lemma 1.5 (Ch. 3), for any $k \in \mathbb{Z}$ there exists an index $u_k \in \mathbb{Z}$ such that $\alpha + k\gamma_1 = \beta_{u_k}$. The characteristic sequence decomposes into sequences

$$\mathbf{c}_k := (c_{u_k}, \dots, c_{u_{k+1}})$$

for $k \in \mathbb{Z}$.

THEOREM 2.2. *Let $(\mathcal{A}, T, R, \gamma_1)$ be an affine crystallographic arrangement of rank two with limit 1 with characteristic sequence $(c_i)_{i \in \mathbb{Z}}$ decomposed into sequences $\mathbf{c}_k$, $k \in \mathbb{Z}$ as in Definition 2.1 (Ch. 3). Then for all $k \in \mathbb{Z}$, there exist $\mathbf{c}'_k = (c'_{k,1}, \dots, c'_{k,r_k}) \in \mathcal{E}$, $r_k \in \mathbb{N}^+$ such that*

$$\mathbf{c}_k = (c'_{k-1,r_{k-1}} + c'_{k,1} + 2, c'_{k,2}, \dots, c'_{k,r_k-1}, c'_{k,r_k} + c'_{k+1,1} + 2).$$

PROOF. Assume first that $R_{c\ell} = \{\alpha + k\gamma_1 \mid k \in \mathbb{Z}\}$, i.e. $R_{c\ell}$ is of type A, so $\beta_k = \alpha + k\gamma_1$. The base change from $\{\beta_{k+2}, -\beta_{k+1}\}$ to $\{\beta_{k+1}, -\beta_k\}$ is then $\eta(2)$ and does not depend on k. Thus in this case, the characteristic sequence is $(\dots, 2, 2, 2, \dots)$.

Consider now an arbitrary $R_{c\ell}$ and let $\mathbf{c}_k$, $k \in \mathbb{Z}$ be one of the components of the characteristic sequence. Let $R^+_{fin} \subseteq \mathbb{Z}^2$ be the set of coordinate vectors of $[\beta_{u_k}, \beta_{u_{k+1}}]$ with respect to $\{\beta_{u_k}, \beta_{u_{k+1}}\}$. Then R^+_{fin} is a set of positive roots of a finite Weyl groupoid of rank two because of Axiom of crystallographic arrangement 3.8 (Ch. 1) and because $R \subseteq \mathbb{Z}\beta_{u_k} + \mathbb{Z}\beta_{u_{k+1}}$. Let $\mathbf{c}'_k = (c'_{k,1}, \dots, c'_{k,r_k}) \in \mathcal{E}$, $r_k \in \mathbb{N}_{>1}$ be its corresponding η-sequence. If $|[\beta_{u_k}, \beta_{u_{k+1}}]| > 2$, then by [**9**, Cor. 4.2] the sequence $\mathbf{c}'_k$ contains an entry 1, or in other words there is a label $u_k < j < u_{k+1}$

such that $\beta_j = \beta_{j-1} + \beta_{j+1}$. The set $R_+ \backslash \{\beta_j\}$ is still a set of positive roots of an affine crystallographic arrangement, the new component $\mathbf{c}'_k$ of this arrangement is

$$(c'_{k,1}, \dots, c'_{k,j-1} - 1, c'_{k,j+1} - 1, \dots, c'_{k,r_k}).$$

Thus each component $\mathbf{c}_k$, $k \in \mathbb{Z}$ may be reduced to the sequence $(2,2)$ of length 2 by excluding roots and thereby preserving the axioms of an affine crystallographic arrangement. Performing this for all k yields the above arrangement $R_{c\ell}$ of type A. □

Definition 2.3. Let t_k be the linear map

$$t_k : V^* \to V^*, \quad \alpha \mapsto \alpha + k\gamma_1, \;\; \alpha + \gamma_1 \mapsto \alpha + (k+1)\gamma_1.$$

We call R *periodic* if

$$t_k(R_0) = [\alpha + k\gamma_1, \; \alpha + (k+1)\gamma_1]$$

for all $k \in \mathbb{Z}$. We call R *N-complete* if

$$R_+ = \{k\alpha + m\gamma_1 \mid k \in N_+, \; m \in \mathbb{Z}, \; \mathrm{Gcd}(k,m) = 1\}.$$

Lemma 2.4. *Any N-complete affine crystallographic arrangement of rank two is periodic.*

Proof. Let R be N-complete, and $\beta = n\alpha + m\gamma_1 \in R_0$, i.e. $\mathrm{Gcd}(n,m) = 1$ and $\alpha \le \beta \le \alpha + \gamma_1$ which implies $0 \le m \le n$. For $k \in \mathbb{Z}$,

$$t_k(\beta) = n\alpha + (kn+m)\gamma_1.$$

But $n \in N_+$ and $\mathrm{Gcd}(n, kn+m) = \mathrm{Gcd}(n,m) = 1$, so $t_k(\beta) \in R$. Further, $\alpha + k\gamma_1 \le t_k(\beta) \le \alpha + (k+1)\gamma_1$. □

Lemma 2.5. *Let R be a periodic affine crystallographic arrangement of rank two. Then $N(R)$ is finite.*

Proof. Consider the set $S := \{\frac{m}{k} \mid k\alpha + m\gamma_1 \in R_0\}$. If N_+ is infinite, then S is infinite and has an accumulation point $q = \frac{m}{k}$ with $0 \le q \le 1$, i.e. any neighborhood of $p(k\alpha + m\gamma_1)$ contains infinitely many elements of R. But then $p(k\alpha + m\gamma_1) \ne p(\gamma_1)$ and since $\mathbb{P}(V)$ is Hausdorff, Axiom of affine crystallographic arrangement cannot be fulfilled. □

Finally, we complete the interpretation about periodicity for affine Cartan graphs.

Theorem 2.6. *Let $\mathcal{C}$ be an affine Cartan graph with corresponding affine crystallographic arrangement $(\mathcal{A}, T, R, \gamma_1)$. Then there exists a covering of $\mathcal{C}$ with finitely many objects if and only if the characteristic sequence of R is periodic. In this case, R has limit* 1 *and is periodic (in the sense of Def. 2.3 (Ch. 3)).*

Proof. If the set of objects of $\mathcal{C}$ is finite, then the object change diagram of $\mathcal{C}$ is either a chain or a cycle (see [**9**, Section 6]). Thus the first claim is obvious (see Def. 2.3, Ch. 2).

Now assume that the characteristic sequence of R is periodic, and assume that R is not of type A. Then by Cor. 2.6 (Ch. 2) and Rem. 2.7 (Ch. 2) it is possible to simultaneously remove a one in each period of the characteristic sequence. Since the period has finite length, repeatedly removing ones will eventually yield the sequence of type A. However, removing a one does not change the imaginary root. Thus by Thm. 2.2(Ch. 3), R has limit 1. Again by Thm. 2.2 (Ch. 3), the structure of the positive roots between $\alpha+k\gamma_1$ and $\alpha+(k+1)\gamma_1$ is the same for all k, hence R is periodic. □

As an application, we get the full description of N-complete affine crystallographic arrangements of rank two.

Corollary 2.7. *Let R be an N-complete affine crystallographic arrangement of rank two. Then there exists an $n \in \mathbb{N}$ such that*

$$R_0 = \{k\alpha + m\gamma_1 \mid 1 \le k \le n,\ 0 \le m \le n,\ \mathrm{Gcd}(k,m) = 1,\ m \le k\},$$

i.e. $\{\frac{m}{k} \mid k\alpha + m\gamma_1 \in R_0\}$ is the Farey sequence of order n.

Proof. Let R be N-complete. By Lemma 2.4 (Ch. 3), R is periodic and by Lemma 2.5 (Ch. 3), $N(R)$ is finite. Let $n := \max N(R)$. Then $n\alpha + \gamma_1 \in R_+$ by definition. But by Thm. 2.2 (Ch. 3) and Thm. 2.6 (Ch. 3), $R_0 = [\alpha, \alpha+\gamma_1]$ gives rise to an η-sequence. Thus $n\alpha+\gamma_1$ is the sum of two elements of R_+. But the only decomposition allowed by Axiom (R2) is $((n-1)\alpha+\gamma_1)+\alpha$. Using induction we obtain $N(R) = \{-n,\dots,n\}$. □

Example 2.8. The N-complete affine crystallographic arrangement R with $n = 1$ is the affine Weyl group of type denoted $A_1^{(1)}$ in [**20**, Ch. 4].

Example 2.9. For the N-complete affine crystallographic arrangement R with $n = 2$, we have

$$R_0 = \{\alpha, 2\alpha+\gamma_1, \alpha+\gamma_1\}.$$

The characteristic sequence of R is $(\dots,1,4,1,4,1,\dots)$. The components $\mathbf{c}'_k$ of Thm. 2.2 (Ch. 3) are all equal to $(1,1,1)$ because R is periodic. There exists a connected Cartan graph $\mathcal{C}'$ whose covering is the Cartan graph of R. $\mathcal{C}'$ is *standard*, i.e. it has only one object. Its only Cartan matrix is

$$\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$

This is the affine Weyl group denoted $A_2^{(2)}$ in [**20**, Ch. 4].

Example 2.10. Choose the η-sequence $(2,1,3,1,2)$. Then by Thm. 2.2 (Ch. 3),

$$\dots,6,1,3,1,6,1,3,1,6,1,3,1,6,\dots$$

is the characteristic sequence of an affine crystallographic arrangement with limit 1 (which is N-complete, $n = 3$). A covering of the corresponding simply connected Cartan graph is:

$$^{\sigma_1}\circlearrowright \begin{pmatrix} 2 & -6 \\ -1 & 2 \end{pmatrix} \overset{\sigma_2}{\text{—}} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1}.$$

Let us finally complete the interpretation about the correspondence between frieze patterns and arrangements (finite case and affine case).

We consider again the η-sequence $(2, 1, 3, 1, 2)$ from Example 2.10 (Ch. 3) corresponding to a triangulated pentagon (see Figure 1). The η-sequence is the third line of a frieze pattern (which satisfies the *unimodular rule* (see [**5**]). The ordered pairs in the quadrangles (see Figure 2) are the positive roots of the associated finite crystallographic arrangement.

Theorem 2.2 (Ch. 3) associates a periodic characteristic sequence of an affine crystallographic arrangement to the η-sequence $(2, 1, 3, 1, 2)$:

$$\dots, 6, 1, 3, 1, 6, 1, 3, 1, 6, 1, 3, 1, 6, \dots$$

We get this periodic sequence's associated frieze pattern (see Figure 3) by writing down this periodic sequence below the first two horizontal rows of 0s and 1s and using the unimodular rule.

The affine crystallographic arrangement's associated frieze pattern has infinitely many horizontal lines with infinitely many ordered pairs in the quadrangles on a diagonal. Those pairs are the positive roots of the associated affine crystallographic arrangement.

CHAPTER 4

Classification of affine rank 2 Nichols algebras of diagonal type

At the beginning of this part, the notions, theorems and properties of subsequences of η-sequences are introduced. Then in the second section, we introduce the bicharacters which have at least one entry not root of unity. Apply Thm. 1.8 (Ch. 4) and Thm. 2.2 (Ch. 3), we give the set of all length 4 subsequence of potential affine characteristic sequences. By computing bicharacters and associated Cartan graphs for the subsequences of the set, we obtain the main result of this thesis (Thm. 3.7, Ch. 4). In the last section, the proof of the main result is given. It consist of the computations of bicharacters for periods of affine characteristic sequences.

1. Subsequences of η-sequences

This section comes from M. Cuntz's manuscript. I got these materials from private communications between us in the past three years. Thm. 1.8 (Ch. 4) is a very important theorem by which we can produce affine sequences' subsequence of length more than 4. It plays a key role in the construction of potential periods of affine characteristic sequences.

DEFINITION 1.1. Let $\mathcal{F}_n := \mathbb{N}_0^n$ and

$$\mathcal{D}_n := \mathcal{F}_n / \sim,$$

where $(c_1, \dots, c_n) \sim (d_1, \dots, d_n)$ if and only if there exists a permutation σ in the dihedral group with $(d_1, \dots, d_n) = (c_{\sigma(1)}, \dots, c_{\sigma(n)})$, i. e. $\sim$ is the equivalence relation obtained by setting

$$(c_1, \dots, c_n) \sim (c_2, c_3, \dots, c_{n-1}, c_n, c_1)$$

$$(c_1, \dots, c_n) \sim (c_n, c_{n-1} \dots, c_2, c_1)$$

We write

$$\mathcal{D} := \bigcup_{n\in\mathbb{N}} \mathcal{D}_n, \quad \mathcal{F} := \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$$

We further write $(c_1, \dots, c_n)$ for the equivalence class of $(c_1, \dots, c_n)$.

REMARK 1.2. Whenever we consider elements of $\mathcal{D}$ in this section, all arguments and definitions are using representatives and well-defined nevertheless. For instance, speaking of positions $i, i+1$ can be meant to be the last and first position for a representative.

DEFINITION 1.3. We say that $c = (c_1, \dots, c_n) \in \mathcal{D}$ *contains* $d = (d_1, \dots, d_m) \in \mathcal{F}$ and write $d \subseteq c$ if there exists a $k \in \mathbb{N}_0$ and a representative $(c_1, \dots, c_n)$ of c such that

$$c_{k+i} = d_i \text{ for all } i = 1, \dots, m.$$

DEFINITION 1.4. For a $a \in \mathbb{Z}$, let

$$\eta(a) := \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$$

Then there is the rule (compare [**9**, Lemma 5.2]):

$$\eta(a)\eta(b) = \eta(a+1)\eta(1)\eta(b+1), \quad \text{for all } a, b.$$

Now we define a map on $\mathcal{F}$.

DEFINITION 1.5. Let

$$\psi : \mathcal{F} \to \mathcal{F}, \quad (c_1, c_2, \dots) \mapsto (c_1 + 2, 1, c_2 + 2, 1, \dots)$$

Denote

$$\mathcal{E}' := \{(c_1, \dots, c_n) \in \mathcal{E} \mid n \text{ is even and } |\{i \mid c_i = 1\}| = \frac{n}{2}\}.$$

Since two consecutive 1's in a $c \in \mathcal{E}$ imply $c = (1, 1, 1)$, every $c \in \mathcal{E}'$ may be written as

$$c = (*, 1, *, 1, \dots, *, 1)$$

For $a \in \mathbb{Z}$ let

$$\xi(a) := \eta(a)\eta(1).$$

Then we have the rule

$$\xi(a)\xi(3)\xi(b) = \xi(a-1)\xi(b-1)$$

for all a, b, and thus obtain that ψ induces a map

$$\overline{\psi} : \mathcal{E} \to \mathcal{E}', \quad (c_1, c_2, \dots) \mapsto (c_1 + 2, 1, c_2 + 2, 1, \dots)$$

which is a bijection ($\overline{\psi}^{-1}$ corresponds to removing all ears at once in the triangulation). For $c = (c_1, 1, c_3, 1, \dots) \in \mathcal{F}_n$, let

$$\iota(c) = (1, c_1, 1, c_3, 1, \dots) \in \mathcal{F}_{n+1}.$$

Only in this definition, the $*$ represent c_i in the odd number positions.

The following definition offers two maps of the η-sequences and sequences.

DEFINITION 1.6. For $c = (c_1, \dots, c_n) \in \mathcal{F}$, construct a new sequence c' by repeatedly replacing

(1) $(\dots, c_i, c_{i+1}, \dots)$ by $(\dots, c_i + 1, 1, c_{i+1} + 1, \dots)$ if $c_i, c_{i+1} > 1$,
(2) $(c_1, \dots)$ by $(1, c_1 + 1, \dots)$ if $c_1 > 1$, and
(3) $(\dots, c_n)$ by $(\dots, c_n + 1, 1)$ if $c_n > 1$

until none of these rules apply anymore. We write

$$\rho : \mathcal{F} \to \mathcal{F}, \quad c \mapsto c'.$$

Similarly, define a map $\delta : \mathcal{D} \backslash \{(0,0), (1,1,1)\} \to \mathcal{E}'$, $c \mapsto c'$, where c' is obtained from c repeatedly applying the rules

(1) $(\dots, c_i, c_{i+1}, \dots) \mapsto (\dots, c_i + 1, 1, c_{i+1} + 1, \dots)$ if $c_i, c_{i+1} > 1$,
(2) $(c_1, \dots, c_n) \mapsto (c_1 + 1, \dots, c_n + 1, 1)$ if $c_1, c_n > 1$.

Example 1.7.

$$\rho((3,1,2,2,1)) = (1,4,1,3,1,3,1),$$
$$\rho((3,1,2,3,1,2)) = (1,4,1,3,1,4,1,3,1),$$
$$\delta((3,1,2,3,1,2)) = (4,1,3,1,4,1,3,1),$$
$$\delta((4,1,2,2,2,1)) = (4,1,3,1,4,1,3,1)$$

The following theorem can be used to produce longer subsequences for the corresponding η-sequences and sequences.

Theorem 1.8. *Assume that $E \in \mathcal{E}, F \in \mathcal{F}$ are finite sets such that:*
(1) $\{(0,0), (1,1,1)\} \subseteq E$,
(2) *For all $c \in \mathcal{E} : c \in E$ or there exists $f \in F : f \subsetneq c$,*
(3) *Every $f \in F$ contains a* 1.

Then

$$E' := E \cup \delta^{-1}(\overline{\psi}(E)) \in \mathcal{E},$$
$$F' := \rho^{-1}(\iota(\psi(F))) \in \mathcal{F},$$

are finite sets satisfying the same properties (1), (2), (3) *as E, F, and such that*

$$\min\{\textit{length of } c \mid c \in F\} < \min\{\textit{length of } c \mid c \in F'\}.$$

Proof. The finiteness of E', F' is obvious. Notice first that every $d \in \mathcal{E}' \backslash \overline{\psi}(E)$ contains an element of $\psi(F)$ since $\overline{\psi}$ is a bijection. Now let $c \in \mathcal{E} \backslash E$. Then $\delta(c) \in \mathcal{E}'$, hence either $\delta(c) \in \overline{\psi}(E)$ or $\delta(c)$ contains an element of $\psi(F)$. If $\delta(c) \in \overline{\psi}(E)$, then $c \in E'$. Otherwise, $\delta(c)$ strictly contains an element of $\psi(F)$, say $\psi(f)$; but $\psi(f)$ begins with an entry greater than 1, thus $\delta(c)$ contains $\iota(\psi(f))$. This implies $c \in \rho^{-1}(\iota(\psi(f)))$. Now every element of $\rho^{-1}(\iota(\psi(f)))$ has length greater than the length of f, because f contains a 1. □

Corollary 1.9. *Every η-sequence contains at least one of*

$$(0,0), (1,1), (1,2), \textit{ or } (1,3)$$

Proof. If $(1,1) \subseteq c \in \mathcal{E}$ then $c = (1,1,1)$. Further, any η-sequence which is not $(0,0)$ contains a one. Thus we may apply Thm. 1.8 (Ch. 4) to the set

$$E = \{(0,0), (1,1,1)\}, \quad F = \{(1)\}.$$

And obtain

$$E' = \{(0,0), (1,1,1), (1,2,1,2), (1,2,2,1,3), (1,3,1,3,1,3)\},$$
$$F' = \{(1,2), (2,1), (1,3,1)\},$$

which implies the claim. □

2. Bicharacters with entries not roots of unity

In Chapter 1, the notion of bicharacter has been given. We find that there also exists a special sort of bicharacters which have at least one entry not the root of unity. A theorem of this special type bicharacter is given as follows. In next section, the first case of this theorem can be used in the proof of Prop. 3.5 (Ch. 4) of the one-dimensional type bicharacters.

THEOREM 2.1. *For any bicharacter* $\chi_1 = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} q & 1 \\ r & s \end{pmatrix}$*, where* $q_{11} = q, q_{22} = s, q_{12}q_{21} = r$ *(see Def. 1.12, Ch. 1), if at least one of its entries is not a root of unity, then its associated Cartan graph is one of the following :*

$$^{\sigma_2}\circlearrowright \underset{\chi_1}{\begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix}} \circlearrowleft^{\sigma_1},$$

$$^{\sigma_1}\circlearrowright \begin{pmatrix} 2 & -i \\ -n & 2 \end{pmatrix} \overset{\sigma_2}{\text{—}} \underset{\chi_1}{\begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix}} \circlearrowleft^{\sigma_1},$$

$$^{\sigma_2}\circlearrowright \begin{pmatrix} 2 & -i \\ -j & 2 \end{pmatrix} \overset{\sigma_1}{\text{—}} \begin{pmatrix} 2 & -i \\ -n & 2 \end{pmatrix} \overset{\sigma_2}{\text{—}} \underset{\chi_1}{\begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix}} \circlearrowleft^{\sigma_1}$$

(where $i, j, m, n \in \mathbb{N}^+$*).*

PROOF. For χ_1: $\overset{q}{\circ}\overset{r}{\text{————}}\overset{s}{\circ}$, assume that q and s are roots of unity, r is not a root of unity. Here we are not planning to consider this case further. Because we find it is contained in the following part of the proof with more details (χ_1 of this case corresponds to $\chi_2 = \begin{pmatrix} qr^n s & 1 \\ r^{-1}s^2 & s \end{pmatrix}$ of the case that $qr^n s, s \in \mu_n, r^{-1}s^2 \notin \mu_n$).

Now assume that q is not a root of unity. We have the formula of bicharacter:

(2.2, p. 15) $c_{ij}^{\chi} = -\min\{m \in \mathbb{N}_0 \mid 1 + q_{ii} + q_{ii}^2 + \cdots + q_{ii}^m = 0 \text{ or } q_{ii}^m q_{ij} q_{ji} = 1\}$ if $i \neq j$ and $c_{ii}^{\chi} = 2$.

Let the Cartan matrix associated to χ_1 be $C^{\chi_1} = \begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix}$. Then the two reflections of C^{χ_1} are $\sigma_1^{\chi_1} = \begin{pmatrix} -1 & m \\ 0 & 1 \end{pmatrix}$, $\sigma_2^{\chi_1} = \begin{pmatrix} 1 & 0 \\ n & -1 \end{pmatrix}$. By the notation of generalized Dynkin diagram:

for $\chi_1 : \overset{q}{\circ}\overset{r}{\text{————}}\overset{s}{\circ}$, let the reflection $\sigma_1^{\chi_1}$ acts on χ_1. According to the formula (2.2, p. 15), there are two cases:

$$\begin{cases} i) : \ q^m \cdot r = 1 \Rightarrow \overset{q}{\circ}\overset{r}{\text{————}}\overset{s}{\circ} \\ \text{or } ii) : \ q^{m+1} = 1, q \neq 1 \Rightarrow \overset{q}{\circ}\overset{q^2 r^{-1}}{\text{————}}\overset{qr^m s}{\circ} \end{cases};$$

For $\chi_1 : \overset{q}{\circ}\overset{r}{\text{———}}\overset{s}{\circ}$, let the reflection $\sigma_2^{\chi_1}$ acts on χ_1. Because of the formula of bicharacter as above, there also are two cases:

$$\begin{cases} iii) : \ s^n \cdot r = 1 \Rightarrow \overset{q}{\circ}\overset{r}{\text{———}}\overset{s}{\circ} \\ \text{or } iv) : \ s^{n+1} = 1, s \neq 1 \Rightarrow \overset{qr^ns}{\circ}\overset{r^{-1}s^2}{\text{———}}\overset{s}{\circ} \end{cases}.$$

If q is not a root of unity, then r is not a a root of unity. Otherwise if $r^k = 1$, $q \cdot r$ is also a root of unity, then assume that $(q \cdot r)^l = 1$. From the assumption $(q \cdot r)^{lk} = 1$, we obtain $q^{lk} = 1$. It leads to a contradiction.

Hence we get the following diagrams:

(1) $\overset{q}{\circ}\overset{r}{\text{———}}\overset{s}{\circ}$, q, r and s are not roots of unity;

For this case, it satisfies $i)$ and $iii)$. Then we obtain the following picture:

$$\sigma_2^{\chi_1} \circlearrowright \underset{\chi_1}{\begin{pmatrix} q & 1 \\ r & s \end{pmatrix}} \circlearrowleft \sigma_1^{\chi_1}$$

This diagram's associated Cartan graph is

$$\sigma_2 \circlearrowright \begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix} \circlearrowleft \sigma_1 .$$

(2) $\overset{q}{\circ}\overset{r}{\text{———}}\overset{s}{\circ}$, q and r are not roots of unity, s is the root of unity. For this case, it satisfies $i)$ and $iv)$. Then we can get a part of the bicharacter's diagram:

$$\xrightarrow{?} \underset{\chi_2}{\begin{pmatrix} qr^ns & 1 \\ r^{-1}s^2 & s \end{pmatrix}} \xrightarrow{\sigma_2^{\chi_1}} \underset{\chi_1}{\begin{pmatrix} q & 1 \\ r & s \end{pmatrix}} \circlearrowleft \sigma_1 .$$

For χ_2, if $r^{-1}s^2$ is not a root of unity, we have the following cases:

(1) qr^ns is not a root of unity. Then χ_2 satisfies $i)$ and $iv)$ and the picture of bicharacters is as follows.

$$\sigma_1 \circlearrowright \underset{\chi_2}{\begin{pmatrix} qr^ns & 1 \\ r^{-1}s^2 & s \end{pmatrix}} \xrightarrow{\sigma_2} \underset{\chi_1}{\begin{pmatrix} q & 1 \\ r & s \end{pmatrix}} \circlearrowleft \sigma_1 .$$

The associated Cartan graph of this diagram is

$$\sigma_1 \circlearrowright \underset{\chi_2}{\begin{pmatrix} 2 & -i \\ -n & 2 \end{pmatrix}} \xrightarrow{\sigma_2} \underset{\chi_1}{\begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix}} \circlearrowleft \sigma_1, \text{ for some suitable } i, m, n \in \mathbb{N}^+.$$

(2) qr^ns is a root of unity. Thus χ_2 satisfies $ii)$ and $iv)$,

$$\xrightarrow{?} \underset{\chi_3}{\begin{pmatrix} qr^ns & 1 \\ q^2r^{2n+1} & (qr^ns)(r^{-1}s^2)^is \end{pmatrix}} \xrightarrow{\sigma_1^{\chi_2}} \underset{\chi_2}{\begin{pmatrix} qr^ns & 1 \\ r^{-1}s^2 & s \end{pmatrix}} \xrightarrow{\sigma_2^{\chi_1}} \underset{\chi_1}{\begin{pmatrix} q & 1 \\ r & s \end{pmatrix}} \circlearrowleft \sigma_1,$$

χ_3 satisfies $ii)$ and $iii)$, then we have

$$\sigma_2 \circlearrowright \underset{\chi_3}{\begin{pmatrix} qr^n s & 1 \\ q^2 r^{2n+1} & (qr^n s)(r^{-1}s^2)^i s \end{pmatrix}} \overset{\sigma_1}{\text{———}} \underset{\chi_2}{\begin{pmatrix} qr^n s & 1 \\ r^{-1}s^2 & s \end{pmatrix}} \overset{\sigma_2}{\text{———}} \underset{\chi_1}{\begin{pmatrix} q & 1 \\ r & s \end{pmatrix}} \circlearrowleft^{\sigma_1} .$$

Then the corresponding Cartan graph of this diagram is

$$\sigma_2 \circlearrowright \begin{pmatrix} 2 & -i \\ -j & 2 \end{pmatrix} \overset{\sigma_1}{\text{———}} \begin{pmatrix} 2 & -i \\ -n & 2 \end{pmatrix} \overset{\sigma_2}{\text{———}} \underset{\chi_1}{\begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix}} \circlearrowleft^{\sigma_1},$$

for some suitable $i, j, m, n \in \mathbb{N}^+$.

□

3. Main theorem

Apply the Thm. 1.8 (Ch. 4) given by M. Cuntz finitely many times, we find all η-sequences containing some subsequences with given length. Construct potential periods by η-sequences containing these subsequences, we get some length four subsequences of the potential affine characteristic sequences.

Compute associated bicharacters and Cartan graphs of all these length four subsequences, we can classify the affine rank two Nichols algebras of diagonal type by some suitable bicharacters.

These zero-dimensional type bicharacters (we will give the corresponding definition later) and their corresponding Weyl groupoids can be obtained from the length four subsequences by the program implemented by M. Cuntz. These computations are not included in this thesis. From some examples, we find that one-dimensional type bicharacters always correspond to the Weyl groupoids with characteristic sequences containing $[a, b, a, b]$ type subsequences. Thus we give a proposition of subsequences of $[a, b, a, b]$ type from which the one-dimensional type bicharacters are given. There are some different length four subsequences which can be dealt with by program. Here we compute their corresponding bicharacters and Weyl groupoids by hand.

In order to get all of length four subsequences of potential affine characteristic sequences, we need to apply Thm. 1.8 (Ch. 4) four times. Then it yields a set of η-sequences and a set of sequences, both consist of sequences of length at least five.

In the first three times, replace each sequence in F by its lexicographically smallest subsequence of length four containing a 1. After the third time, remove from E those η-sequences which contain an element of F as a subsequence. After the fourth application of Thm. 1.8 (Ch. 4), replace each sequence in F by its lexicographically smallest subsequence of length five containing a 1. Now we have a set of η-sequences corresponding to E and a set of sequences corresponding to F, both consist of sequences of length at least five. The first set is denoted by E_5 and the second one is denoted by F_5. These have 21 respectively 176 elements (up to symmetries).

Elements of E_5 containing an element of F_5 are removed. And we denote the new set by E_5'. Apply Thm. 1.8 (Ch. 4) again on E_5' and F_5, then we get a set E_6

of η-sequences and a set of sequences denoted by F_6'. Replace every sequence of F_6' by its lexicographically smallest subsequence of length six containing a 1. The new set corresponding to F_6' is defined as F_6. E_6 and F_6 consist of sequences of length at least six.

After applying Thm. 1.8 (Ch. 4) twice, the computation work become fairly difficult by hand. So M. Cuntz has implemented a program in **MAGMA** that computes E_5, F_5, E_6 and F_6 and all of the zero-dimensional type bicharacters with potential affine characteristic sequences.

DEFINITION 3.1. By Thm. 2.2 (Ch. 3), we need η-sequences $\mathbf{c}_k' = (c_{k,1}', \dots, c_{k,r_k}')$, $r_k \in \mathbb{N}^+$ to build potential subsequences of affine characteristic sequences. When we fix the potential period of affine characteristic sequences, subsequences containing at least 4 entries are needed. Choose any η-sequence $\mathbf{c}_i'$, let it be the *first η-sequence*. And let the η-sequence in the neighbor row of $\mathbf{c}_i'$ be the *second η-sequence*.

There are two cases:

(I) For any $i \in \mathbb{Z}$, the length of $\mathbf{c}_i'$ is less or equal to five;

(II) There exists at least one $i \in \mathbb{Z}$, such that the length of $\mathbf{c}_i'$ is greater than five.

For the case (I), lengths of the η-sequences constructing potential periods are less than or equal to 5. Then they are contained in the following set S:
$S = \{[0,0], [1,1,1], [1,2,1,2], [2,1,2,1], [1,2,2,1,3], [2,2,1,3,1], [2,1,3,1,2], [1,3,1,2,2], [3,1,2,2,1]\}$.

Choose any element of S as the first η-sequence and then choose another element from S as the second η-sequence. Apply Thm. 2.2 (Ch. 3), if the length of subsequence is less than 4, then we need to choose an element of S again till the length of subsequence is at least 4. All of the possible cases should be contained. Then a set of 176 subsequences of the potential affine characteristic sequences is given. These subsequences' length are at least 4. For different elements of this set, they may have the same subsequence of length 4. We collect all of such kind subsequences and obtain a set of 74 elements.

$$\{[1,2,2,3], [1,2,2,4], [1,2,2,5], [1,2,3,2], [1,2,3,3], [1,2,3,4], [1,2,4,1], [1,2,4,2], [1,2,5,1], [1,3,1,4], [1,3,1,5], [1,3,1,6], [1,3,2,2], [1,3,2,3], [1,3,2,4], [1,3,3,1], [1,3,3,2], [1,3,4,1], [1,4,1,3], [1,4,1,4], [1,4,1,5], [1,4,2,1], [1,5,1,2], [2,1,3,3], [2,1,3,4], [2,1,3,5], [2,1,4,2], [2,1,4,3], [2,1,4,4], [2,1,5,1], [2,1,5,2], [2,1,6,1], [2,2,1,5], [2,2,1,6], [2,2,1,7], [2,2,1,8], [2,2,2,2], [2,2,2,3], [2,2,2,4], [2,2,3,1], [2,2,3,2], [2,2,4,1], [2,3,1,3], [2,3,1,4], [2,3,1,5], [2,3,2,1], [2,4,1,2], [3,1,2,4], [3,1,2,5], [3,1,2,6], [3,1,3,2], [3,1,3,3], [3,1,3,4], [3,2,1,4], [3,2,1,5], [3,2,1,6], [3,2,2,1], [3,3,1,2], [4,1,2,3], [4,1,2,4], [4,1,2,5], [4,1,3,1], [4,2,1,3], [4,2,2,1], [4,3,1,2], [5,1,2,2], [5,1,3,1], [5,2,1,3], [5,2,2,1], [5,3,1,2], [6,1,2,2], [6,1,3,1], [6,2,1,3], [7,1,2,2]\}.$$

Every element of the above set is a length four subsequence of potential affine characteristic sequences.

For the case (II), without loss of generality, assume that $\mathbf{c}'_k, k \in \mathbb{Z}$ be the first η-sequence and $|\mathbf{c}'_k| > 5$. For any η-sequence except for elements from the set $S = \{[0,0],[1,1,1],[1,2,1,2],[2,1,2,1],[1,2,2,1,3],[2,2,1,3,1],[2,1,3,1,2],[1,3,1,2,2],[3,1,2,2,1]\}$, it contains at least one subsequence of length five from E_5 or F_5. Let such a subsequence be $[a_1,a_2,a_3,a_4,a_5]$. By Def. 1.1 (Ch. 4), the η-sequence $[\dots,a_1,a_2,a_3,a_4,a_5,\dots]$ represents an equivalence class, it implies that the positions of $[a_1,a_2,a_3,a_4,a_5]$ are not fixed in this η-sequence.

For any η-sequence except for the elements of S, it also contains at least one subsequence of length 6 from E_6 or F_6. Let such a subsequences be $[b_1,b_2,b_3,b_4,b_5,b_6]$. By Thm. 2.2 (Ch. 3), we build potential periods.

In the following, when the subsequences of length 5 or 6 are cut into two pieces, we call each of the pieces *part*. If a part in a η-sequence with more (respectively less) entries than the other, then we call it *longer part* (respectively *shorter part*).

The following different cases should be considered.

(1) The first η-sequence contains an element of F_5 and this subsequence of length 5 is not cut into two parts.

$$\begin{array}{lllllllllllll}
,j] & & & & & & & & & & & \\
[k, & \dots, & a_1, & a_2, & a_3, & a_4, & a_5, & \dots, & l] & & & \\
2 & & & & & & & & [g, & \dots, & &], \\
 & & & & & & & & 2 & & &
\end{array}$$

$j,k,l,g \in \mathbb{N}^+$.

In this case, without considering the second η-sequence, we can obtain subsequences of affine characteristic sequences of length 4:
$[a_1,a_2,a_3,a_4]$,$[a_2,a_3,a_4,a_5]$, $a_i \in \{1,2,\dots,m\}, i = 1,2,\dots,5$,
$m = \{\text{maximum entry of } F_5\}$.

(2) The subsequence of length 5 of the first η-sequence is cut into two parts while the subsequence of length 6 of the second η-sequence is not.

$$\begin{array}{llllllllllllll}
,j] & & & & & & & & & & & & & \\
[(& \dots &), & \dots, & (& \dots &)] & & & & & & & \\
2 & & & & & & [k, & \dots, & b_1, & b_2, & b_3, & b_4, & b_5, & b_6, \;\dots, \;], \\
 & & & & & & 2 & & & & & & &
\end{array}$$

$j,k \in \mathbb{N}^+$.

In this case, we obtain subsequences of affine characteristic sequences of length 4:
$[b_1,b_2,b_3,b_4]$,$[b_2,b_3,b_4,b_5]$,$[b_3,b_4,b_5,b_6]$; $b_i \in \{1,2,\dots,m\}, i = 1,2,\dots,6$,
$m = \{\text{maximum entry of } F_6\}$.

(3) Both of the subsequence of length 5 and the subsequence of length 6 are cut into two parts. For the first η-sequence, the longer part of subsequence of length 5 has 4 or 3 entries. Here we only need to consider the cases that longer parts are on the right border because η-sequences can be transformed by reversing (see Def. 1.1, Ch. 4).

(a) For the first η-sequence, the longer part with 4 entries is on the right border. Let the shorter part be (c_5) and the longer part be $c = (c_1,c_2,c_3,c_4)$. Then $c = \{(a_4,a_3,a_2,a_1),(a_2,a_3,a_4,a_5)\}$. For the second η-sequence, the part on the left border has at least one entry.

$, j]$
$[(c_5), \quad \ldots, \quad (c_1, \quad c_2, \quad c_3, \quad c_4)]$
$2 \qquad [(u, \quad \ldots), \quad \ldots, \quad (\ldots, \quad)],$
2

$j \in \mathbb{N}^+$.
In this case, we get subsequences of length 4 of affine characteristic sequences:
$[c_1, c_2, c_3, c_4 + u + 2], u \in \{1, 2, \ldots, m\}, m = \{\text{maximum entry of } F_6\}$;

(b) In the first η-sequence, the longer part with 3 entries is on the right border. Let the shorter part be (c_4, c_5) and the longer part be $c = (c_1, c_2, c_3)$. Then $c = \{(a_3, a_2, a_1), (a_3, a_4, a_5)\}$. For the second η-sequence, the part on the left border has at least two entries.
$, j]$
$[(c_4, \quad c_5) \quad , \ldots, \quad (c_1, \quad c_2, \quad c_3)]$
$2 \qquad [(u, \quad v \quad \ldots), \quad \ldots, \quad (\ldots, \quad)],$
2

$j \in \mathbb{N}^+$.
From this case, we get subsequences of length 4 of affine characteristic sequences:
$[c_1, c_2, c_3 + u + 2, v], u, v \in \{1, 2, \ldots, m\}, m = \{\text{maximum entry of} F_6\}$;

(c) The longer part of the first η-sequence has 3 entries and the part on left border of second η-sequence has only one entry. Let the longer part of second η-sequence be $d = (d_1, d_2, d_3, d_4, d_5)$. Then $d = \{(b_5, b_4, b_3, b_2, b_1), (b_2, b_3, b_4, b_5, b_6)\}$. Thus the subsequence of length 4 contained by affine characteristic sequences can be yielded in another way.
$, j]$
$[(c_4, \quad c_5) \quad , \ldots \quad (c_1, \quad c_2, \quad c_3)]$
$2 \qquad [(u), \quad \ldots, \quad (d_1, \quad d_2, \quad d_3 \quad d_4 \quad d_5)]$
$2 \qquad [k$
2

$j, k \in \mathbb{N}^+, u \in \{1, 2, \ldots, m\}, m = \{\text{maximum entry of } F_6\}$.
From this case, we have subsequences of length 4 of affine characteristic sequences:
$[b_5, b_4, b_3, b_2], [b_2, b_3, b_4, b_5]$; $b_i \in \{1, 2, \ldots, m\}, i = 1, 2, \ldots, 6$,
$m = \{\text{maximum entry of } F_6\}$.

(4) The first η-sequence contains at least one element of F_5 and the second η-sequence $\mathbf{c}'_j, j \in \mathbb{Z}$ comes from E_6. For the second sequence, its length is at least 6, we remove two entries on both borders. The left part of $\mathbf{c}'_j$ is denoted by c'. Choose subsequence of length four on c'. Then we obtain subsequences of affine characteristic sequence:
$[c_1, c_2, c_3, c_4], c_1, c_2, c_3, c_4 \in \{1, 2, \ldots, m\}$, $m = \{\text{maximum entry of } E_6\}$.

(5) Assume that the first η-sequence is $\mathbf{c}'_k, k \in \mathbb{Z}$ and it contains at least one element of F_5. The second η-sequence comes from the set S of η-sequences whose length are less or equal to 5:

$S = \{[0,0],[1,1,1],[1,2,1,2],[2,1,2,1],[1,2,2,1,3],[2,2,1,3,1],[2,1,3,1,2],[1,3,1,2,2],[3,1,2,2,1]\}$.

Because of the fact that we can reverse an η-sequence (see Def. 1.1, Ch. 4), if a subsequence from F_5 is divided into two parts, we only need to discuss the cases that longer parts are on the light border of $\mathbf{c}'_k$. Thus there are the cases as follows.

(a) For $\mathbf{c}'_k$, the subsequences of length 5 are not divided into two parts. In this case, we need not to know what the second η-sequence are.

$$\begin{array}{lllllllllll}
,j] & & & & & & & & & & \\
[k, & \ldots, & a_1, & a_2, & a_3, & a_4, & a_5, & \ldots, & l] & & \\
2 & & & & & & & & [g, & \ldots, &], \\
 & & & & & & & & 2 & &
\end{array}$$

$j,k,l,g \in \mathbb{N}^+$. It yields subsequences of affine characteristic sequences of length 4:
$[a_1,a_2,a_3,a_4]$,$[a_2,a_3,a_4,a_5]$, $a_i \in \{1,2,\ldots,m\}, i = 1,2,\ldots,5$,
$m = \{\text{maximum entry of } F_5\}$.

(b) $\mathbf{c}'_k$ is cut into two parts. The longer part of length 4 is on the right border of C_k.

$$\begin{array}{llllllll}
,j] & & & & & & & \\
[(c_5), & \ldots, & (c_1, & c_2, & c_3, & c_4)] & & \\
2 & & & & & [u, & \ldots, &] \\
 & & & & & 2 & &
\end{array}$$

$j \in \mathbb{N}^+$, $[u,\ldots,] \in S$.
In this case, we obtain subsequences of length 4 of affine characteristic sequences: $[c_1,c_2,c_3,c_4+u+2], u \in \{0,1,2,3\}$;

(c) The longer part with 3 entries is on the right border of the first η-sequence.

$$\begin{array}{lllllllll}
,j] & & & & & & & & \\
[(c_4, & c_5) & \ldots, & (c_1, & c_2, & c_3] & & & \\
2 & & & & & [u, & v, & \ldots, &] \\
 & & & & & 2 & & &
\end{array}$$

$j \in \mathbb{N}^+$, $[u,v,\ldots,] \in S\backslash\{(0,0)\}$.
In this case, it yields subsequences of length 4 of affine characteristic sequences: $[c_1,c_2,c_3+u+2,v], u,v \in \{1,2,3\}$;

(d) In this case, the longer part with 3 entries is on the right border of the first η-sequence and the second η-sequence is $[0,0]$.

$$\begin{array}{lllllllll}
,j] & & & & & & & & \\
[(c_4, & c_5) & \ldots, & (c_1, & c_2, & c_3] & & & \\
2 & & & & & [0, & 0] & & \\
 & & & & & 2 & [u, & \ldots, &] \\
 & & & & & & 2 & &
\end{array}$$

$j \in \mathbb{N}^+$, $[u,\ldots,] \in S$.
In this case, we get subsequences of length 4 of affine characteristic sequences: $[c_1,c_2,c_3+2,u+2], u \in \{0,1,2,3\}$.

(6) For the first η-sequence $\mathbf{c}'_k$ of length greater than 5 from E_5, it still has at least four entries after removing two entries from both borders. We call the corresponding sequence c'. Choose subsequences of length 4 from c', then it produces subsequences of length 4 of the affine characteristic sequences:
$[c_1, c_2, c_3, c_4], c_1, c_2, c_3, c_4 \in \{1, 2, \ldots, m\}$, $m = \{\text{maximum entry of } E_5\}$.

From the steps as above, we obtain all of the length four subsequences of potential affine characteristic sequences. They are all collected in a set L. For every element c in L, compute the constraints for a potential bicharacter which would produce this part of a characteristic sequence: compute an ideal I in $\mathbb{Q}[q_1, q, q_2]$ generated by the constraints. If the sequence has length k, then we obtain 2^k cases and thus 2^k ideals. These ideals have dimension at most 1 because c has length greater than three. We find that one-dimensional ideals all originate from elements of L with the following structure:

$$[a, b, a, b], a, b \in \mathbb{N}^+.$$

Collect the ideals of the primary decompositions of all zero-dimensional ideals in a set Z. For each zero-dimensional ideal in Z, we compute a cyclotomic field large enough to contain all the points on its variety. For each point (up to Galois conjugates) we compute the corresponding Cartan graph.

Definition 3.2. A bicharacter corresponding to zero-dimensional ideal (respectively one-dimensional ideal) is called *zero-dimensional tpye bicharacter* (respectively *one-dimensional tpye bicharacter*).

For one-dimensional type bicharacters, we consider the general case of $[a, b, a, b]$ type subsequences which are contained by the period of potential affine characteristic sequences. There is a proposition as follows.

Proposition 3.3. *If a characteristic sequence contains $[a, b, a, b]$ as a subsequence, then the Cartan graph with this characteristic sequence has only one one-dimensional type bicharacter.*

Before beginning the proof of this proposition, it is necessary to explain the method by which we can obtain bicharacters with affine characteristic sequences. This method will be used many times in this thesis. Firstly, for any subsequence or period of potential affine characteristic sequences, it is expressed as a diagram. From the known entries contained in subsequence or period, the corresponding Cartan graph is constructed partly (because there are only finitely many fixed entries).

We choose one Cartan matrix of this Cartan graph and denote its possible bicharacter by χ_1. Let this chosen generalized Cartan matrix be $\mathcal{C}^{\chi_1}$. $\mathcal{C}^{\chi_1}$ has two reflections $\sigma_1^{\chi_1}$ and $\sigma_2^{\chi_1}$ (by Def. 2.3 of Ch. 1). We also have a formula of bicharacter (in Def. 2.7, p. 15):
$c_{ij}^{\chi} = -\min\{m \in \mathbb{N}_0 \mid 1 + q_{ii} + q_{ii}^2 + \cdots + q_{ii}^m = 0 \text{ or } q_{ii}^m q_{ij} q_{ji} = 1\}$ if $i \neq j$ and $c_{ii}^{\chi} = 2$.

Then we construct χ_1's set of constraints which consists of inequalities and equations. Further, let both reflections of $\mathcal{C}^{\chi_1}$ act on χ_1. Thus we have two new neighbor bicharacters. Apply the formula (2.2, p. 15) again, it yields new corresponding Cartan matrices and reflections. Repeatedly applying the above steps a few times, it produces enough potential bicharacters expressed by the entries of χ_1 and their corresponding sets of constraints. Then we construct a set of equations whose elements come from the sets of constraints.

For the set of equations, we should consider all of the different cases of the potential bicharacters. Those inequalities in sets of constraints also should be considered. In this step, some cases maybe lead to contradictions. The correct bicharacters are given in the form of generalized Dynkin diagram (see Def. 1.11, Ch. 1). And the corresponding Cartan graphs and characteristic sequences of the bicharacters can be given entirely by the same technique as above.

Remark 3.4. The rows of sets of constraints and equations are independent of each other. According to the formula of bicharacter (2.2, p. 15), some rows of the sets contain a word 'or'. For each row of such kind, it has two equations and at least one of the two holds. That is the reason why we need to discuss the different cases. If a set of equations has k rows, then there are 2^k cases should be considered.

Proof.

$$\cdots \overset{}{\bullet}\!\!\!-\!\!\!\overset{e}{—}\!\!\!\bullet\!\!\!\overset{a}{—}\!\!\!\bullet\!\!\!\overset{b}{—}\!\!\!\bullet\!\!\!\overset{a}{—}\!\!\!\bullet\!\!\!\overset{b}{—}\!\!\!\bullet\!\!\!\overset{f}{—}\!\!\!\bullet \cdots, \quad (1 \le e, a, b, f \in \mathbb{N}^+),$$

the associated Cartan graphs are

$$\cdots \xrightarrow{\sigma_1} \begin{pmatrix} 2 & -e \\ -a & 2 \end{pmatrix} \xrightarrow{\sigma_2} \underset{\chi_1}{\begin{pmatrix} 2 & -b \\ -a & 2 \end{pmatrix}} \xrightarrow{\sigma_1} \underset{\chi_2}{\begin{pmatrix} 2 & -b \\ -a & 2 \end{pmatrix}} \xrightarrow{\sigma_2} \underset{\chi_3}{\begin{pmatrix} 2 & -b \\ -a & 2 \end{pmatrix}} \xrightarrow{\sigma_1} \begin{pmatrix} 2 & -b \\ -f & 2 \end{pmatrix} \xrightarrow{\sigma_2} \cdots$$

Assume that: $\chi_1 = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} q_1 & 1 \\ q & q_2 \end{pmatrix}$, where $q_1 = q_{11}$, $q_2 = q_{22}$, $q = q_{12} \cdot q_{21}$.

$C^{\chi_1} = \begin{pmatrix} 2 & -b \\ -a & 2 \end{pmatrix}$, $\sigma_1^{\chi_1} = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}$, $\sigma_2^{\chi_1} = \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix}$.

By formula (2.2, p. 15), we get the set of constraints of χ_1:

$$I): \begin{cases} q \neq 1 \\ q_1^i \neq 1, i = 1, 2, \dots, b \\ q_1^j \cdot q \neq 1, j = 1, 2, \dots, b-1 \\ q_1^{b+1} = 1 \text{ or } q_1^b \cdot q = 1 \\ q_2^k \neq 1, k = 1, 2, \dots, a \\ q_2^l q \neq 1, l = 1, 2, \dots, a-1 \\ q_2^{a+1} = 1 \text{ or } q_2^a \cdot q = 1 \end{cases}$$

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} q_1 & 1 \\ q_1^{-2b} q^{-1} & q_1^{b^2} q^b q_2 \end{pmatrix}$,

$C^{\chi_2} = \begin{pmatrix} 2 & -b \\ -a & 2 \end{pmatrix}$, $\sigma_1^{\chi_2} = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}$, $\sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix}$.

By formula (2.2, p. 15), we get the set of constraints of χ_2:

$$II):\begin{cases} q_1^{-2b}q^{-1}\neq 1\\ q_1^g\neq 1, g=1,2,\dots,b\\ q_1^h\cdot q\neq 1, h=1,2,\dots,b-1\\ q_1^{b+1}=1 \text{ or } q_1^b\cdot q_1^{-2b}q^{-1}=1\\ (q_1^{b^2}q^bq_2)^c\neq 1, c=1,2,\dots,a\\ (q_1^{b^2}q^bq_2)^dq_1^{-2b}q^{-1}\neq 1, d=1,2,\dots,a-1\\ (q_1^{b^2}q^bq_2)^{a+1}=1 \text{ or } (q_1^{b^2}q^bq_2)^aq_1^{-2b}q^{-1}=1 \end{cases}$$

$\chi_3=\sigma_2^{\chi_2}*\chi_2=\begin{pmatrix} q_1^{(ab-1)^2}q^{a(ab-1)}q_2^{a^2} & 1\\ q_1^{-2b(ab-1)}q^{-2ab+1}q_2^{-2a} & q_1^{b^2}q^bq_2\end{pmatrix}$,

$C^{\chi_3}=\begin{pmatrix}2 & -b\\ -a & 2\end{pmatrix}$, $\sigma_1^{\chi_3}=\begin{pmatrix}-1 & b\\ 0 & 1\end{pmatrix}$, $\sigma_2^{\chi_3}=\begin{pmatrix}1 & 0\\ a & -1\end{pmatrix}$.

By using the formula (2.2, p. 15), we will get the set of constraints of χ_3:

$$III):\begin{cases} q_1^{-2b(ab-1)}q^{-2ab+1}q_2^{-2a}\neq 1\\ (q_1^{(ab-1)^2}q^{a(ab-1)}q_2^{a^2})^u\neq 1, u=1,2,\dots,b\\ (q_1^{(ab-1)^2}q^{a(ab-1)}q_2^{a^2})^vq_1^{-2b(ab-1)}q^{-2ab+1}q_2^{-2a}\neq 1, v=1,2,\dots,b-1\\ (q_1^{(ab-1)^2}q^{a(ab-1)}q_2^{a^2})^{b+1}=1 \text{ or } (q_1^{(ab-1)^2}q^{a(ab-1)}q_2^{a^2})^bq_1^{-2b(ab-1)}q^{-2ab+1}q_2^{-2a}=1\\ (q_1^{b^2}q^bq_2)^x\neq 1, x=1,2,\dots,a\\ (q_1^{b^2}q^bq_2)^yq_1^{-2b(ab-1)}q^{-2ab+1}q_2^{-2a}\neq 1, y=1,2,\dots,a-1\\ (q_1^{b^2}q^bq_2)^{a+1}=1 \text{ or } (q_1^{b^2}q^bq_2)^aq_1^{-2b(ab-1)}q^{-2ab+1}q_2^{-2a}=1 \end{cases}$$

From $I)$, $II)$ and $III)$ above, the set of constraints of bicharacters is fixed partly. These equations and inequalities in this set of constraints are enough to decide the possible bicharacters. Here we give only the set of equations. When we need some constraints of inequalities in the following, we will choose them from $I), II)$ and $III)$ directly.

$$\begin{cases} q_1^{b+1}=1 \text{ ① or } q_1^bq=1 \text{ ②}\\ q_2^{a+1}=1 \text{ ③ or } q_2^aq=1 \text{ ④}\\ (q_1^{b^2}q^bq_2)^{a+1}=1 \text{ ⑤ or } q_1^{ab^2-2b}q^{ab-1}q_2^a=1 \text{ ⑥}\\ (q_1^{(ab-1)^2}q^{a(ab-1)}q_2^{a^2})^{b+1}=1 \text{ ⑦ or } q_1^{b(ab-1)^2-2b(ab-1)}q^{ab(ab-1)-2ab+1}q_2^{a^2b-2a}=1 \text{ ⑧}\end{cases}$$

For the set of equations above, there are 16 different cases:
(①,③,⑤,⑦), (①,③,⑤,⑧), (①,③,⑥,⑦), (①,③,⑥,⑧), (①,④,⑤,⑦), (①,④,⑤,⑧), (①,④,⑥,⑦), (①,④,⑥,⑧), (②,③,⑤,⑦), (②,③,⑤,⑧), (②,③,⑥,⑦), (②,③,⑥,⑧), (②,④,⑤,⑦), (②,④,⑤,⑧), (②,④,⑥,⑦), (②,④,⑥,⑧).

(1): $\begin{cases} ①\\ ③\\ ⑤\\ ⑦\end{cases} \Rightarrow \begin{cases} q_1^{b+1}=1\\ q_2^{a+1}=1\\ (q_1^{b^2}q^b)^{a+1}\\ q^{a(ab-1)(b+1)}q_2^{a^2(b+1)}=1\end{cases} \Rightarrow \begin{cases} q_1^{b+1}=1\\ q_2^{a+1}=1\\ q^{b(a+1)(b+1)}=1\\ q^{b(a+1)(b+1)}=1\end{cases}.$

(2): $\begin{cases} ①\\ ③\\ ⑤\\ ⑧\end{cases} \Rightarrow \begin{cases} q_1^{b+1}=1\\ q_2^{a+1}=1\\ q^{b(a+1)(b+1)}=1\\ q^{(a+1)(b+1)}=1\end{cases} \Rightarrow \begin{cases} q_1^{b+1}=1\\ q_2^{a+1}=1\\ q^b=1\\ q^{a+1}=1\end{cases}.$

$$(3)\text{:}\ \left\{\begin{array}{l} \textcircled{1}\\ \textcircled{3}\\ \textcircled{6}\\ \textcircled{7} \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^{b+1}=1\\ q_2^{a+1}=1\\ q^{(ab-1)(a+1)(b+1)}=1\\ q^{a(ab-1)(a+1)(b+1)}=1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^{b+1}=1\\ q_2^{a+1}=1\\ q^{a}=1\\ q^{b+1}=1 \end{array}\right. .$$

$$(4)\text{:}\ \left\{\begin{array}{l} \textcircled{1}\\ \textcircled{3}\\ \textcircled{6}\\ \textcircled{8} \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^{b+1}=1\\ q_2^{a+1}=1\\ q^{(ab-1)(a+1)(b+1)}=1\\ q^{(a+1)(b+1)}=1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^{b+1}=1\\ q_2^{a+1}=1\\ q^{(a+1)(b+1)}=1\\ q^{ab-1}=1 \end{array}\right. .$$

$$(5)\text{:}\ \left\{\begin{array}{l} \textcircled{1}\\ \textcircled{4}\\ \textcircled{5}\\ \textcircled{7} \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^{b+1}=1\\ q_2^{a}q=1\\ q^{(ab-1)(a+1)(b+1)}=1\\ q^{a(a^2-2)(b+1)}=1 \end{array}\right. .$$

$$(6)\text{:}\ \left\{\begin{array}{l} \textcircled{1}\\ \textcircled{4}\\ \textcircled{5}\\ \textcircled{8} \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^{b+1}=1\\ q_2^{a}q=1\\ q^{(ab-1)(a+1)(b+1)}=1\\ q^{[ab(ab-1)-2ab+1](b+1)-(ab-2)(b+1)}=1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^{b+1}=1\\ q_2^{a}q=1\\ q^{(ab-1)(a+1)(b+1)}=1\\ q^{(ab-1)(b+3)(b+1)}=1 \end{array}\right. .$$

$$(7)\text{:}\ \left\{\begin{array}{l} \textcircled{1}\\ \textcircled{4}\\ \textcircled{6}\\ \textcircled{7} \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^{b+1}=1\\ q_2^{a}q=1\\ q^{(ab-2)(b+1)}=1\\ q^{a(a^2-2)(b+1)}=1 \end{array}\right. .$$

$$(8)\text{:}\ \left\{\begin{array}{l} \textcircled{1}\\ \textcircled{4}\\ \textcircled{6}\\ \textcircled{8} \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^{b+1}=1\\ q_2^{a}q=1\\ q^{(ab-2)(b+1)}=1\\ q^{(ab-1)(b+1)(b+3)}=1 \end{array}\right. .$$

$$(9)\text{:}\ \left\{\begin{array}{l} \textcircled{2}\\ \textcircled{3}\\ \textcircled{5}\\ \textcircled{7} \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^{b}q=1\\ q_2^{a+1}=1\\ q_1^{(ab-1)(a+1)(b+1)}=1 \end{array}\right. .$$

$$(10)\text{:}\ \left\{\begin{array}{l} \textcircled{2}\\ \textcircled{3}\\ \textcircled{5}\\ \textcircled{8} \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^{b}q=1\\ q_2^{a+1}=1\\ q_1^{b(ab-1)(a+1)}=1 \end{array}\right. .$$

$$(11)\text{:}\ \left\{\begin{array}{l} \textcircled{2}\\ \textcircled{3}\\ \textcircled{6}\\ \textcircled{7} \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^{b}q=1\\ q_2^{a+1}=1\\ q_1^{b(a+1)}=1\\ q_1^{(ab-1)(a+1)(b+1)}=1 \end{array}\right. .$$

$$(12)\text{:}\ \left\{\begin{array}{l} \textcircled{2}\\ \textcircled{3}\\ \textcircled{6}\\ \textcircled{8} \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^{b}q=1\\ q_2^{a+1}=1\\ q_1^{b(a+1)}=1\\ q_1^{b(ab-1)(a+1)}=1 \end{array}\right. .$$

$$(13): \begin{cases} ② \\ ④ \\ ⑤ \\ ⑦ \end{cases} \Rightarrow \begin{cases} q_1^b q = 1 \\ q_2^a q = 1 \\ q_2^{a+1} = 1 \\ q_2^{b+1} = 1 \end{cases}.$$

$$(14): \begin{cases} ② \\ ④ \\ ⑤ \\ ⑧ \end{cases} \Rightarrow \begin{cases} q_1^b q = 1 \\ q_2^a q = 1 \\ q_2^{a+1} = 1 \end{cases}.$$

$$(15): \begin{cases} ② \\ ④ \\ ⑥ \\ ⑦ \end{cases} \Rightarrow \begin{cases} q_1^b q = 1 \\ q_2^a q = 1 \\ q_2^{b+1} = 1 \end{cases}.$$

$$(16): \begin{cases} ② \\ ④ \\ ⑥ \\ ⑧ \end{cases} \Rightarrow \begin{cases} q_1^b q = 1 \\ q_2^a q = 1 \end{cases} \Rightarrow \overset{q_1}{\circ}\!\overset{q}{\text{————}}\!\overset{q_2}{\circ}\ , q = q_1^{-b}, q_1^b = q_2^a, ieqs$$

(here *ieqs* denotes the inequalities from the sets of constraints).

For each potential bicharacter of the first 15 cases, although we can not obtain the bicharacter itself, we get its more new constraints. According to these new constraints, the potential bicharacters are of zero-dimensional type.

When we consider these new constraints together with the other constraints from set $I), II)$ and $III)$, some of the 15 cases maybe lead to contradictions (for example, see the contradictions in the computation of $[1, 4, 1, 4]$).

The bicharacter of case (16) is the only one-dimensional type bicharacter:
$\overset{q_1}{\circ}\!\overset{q}{\text{————}}\!\overset{q_2}{\circ}\ , q = q_1^{-b}, q_1^b = q_2^a, ieqs$
(here *ieqs* denotes the inequalities from the sets of constraints). □

In the following, we give a proposition of one-dimensional type bicharacters.

PROPOSITION 3.5. *If χ_1 is a one-dimensional type bicharacter with characteristic sequence containing $[a, b, a, b]$, then the Weyl groupoid of χ_1 is standard with Cartan matrix*

$$\begin{pmatrix} 2 & -b \\ -a & 2 \end{pmatrix} \quad or \quad \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}.$$

PROOF. Let χ_1 be

$$\begin{pmatrix} q_1 & 1 \\ q & q_2 \end{pmatrix}.$$

By Prop. 3.3 (Ch. 4), a characteristic sequence containing $[a, b, a, b]$ has only one one-dimensional type bicharacter and we write it in the form of Dykin diagram:
$\overset{q_1}{\circ}\!\overset{q}{\text{————}}\!\overset{q_2}{\circ}\ , q = q_1^{-b}, q_1^b = q_2^a.$

We find that the constraints of this bicharacter satisfy the first one of the three cases of Thm. 2.1 (Ch. 4). According to Thm. 2.1 (Ch. 4) and its proof, we obtain the associated Cartan graph $\mathcal{C}_1$:

$$\begin{array}{c} {}^{\sigma_2}\circlearrowright \begin{pmatrix} 2 & -b \\ -a & 2 \end{pmatrix} \circlearrowleft^{\sigma_1} . \\ \chi_1 \end{array}$$

Then for the one-dimensional type bicharacter χ_1, its Weyl groupoid is standard with Cartan matrix

$$\begin{pmatrix} 2 & -b \\ -a & 2 \end{pmatrix}.$$

Construct the associated Cartan graphs differently,

$$\cdots \overset{\sigma_2}{\text{---}} \begin{pmatrix} 2 & -a \\ -e & 2 \end{pmatrix} \overset{\sigma_1}{\text{---}} \underset{\chi_1'}{\begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}} \overset{\sigma_2}{\text{---}} \underset{\chi_2'}{\begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}} \overset{\sigma_1}{\text{---}} \underset{\chi_3'}{\begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}} \overset{\sigma_2}{\text{---}} \begin{pmatrix} 2 & -f \\ -b & 2 \end{pmatrix} \overset{\sigma_1}{\text{---}} \cdots$$

by similar technique, we can also get a one-dimensional type bicharacter χ_1':
$\overset{q_1}{\circ}\overset{q}{\text{———}}\overset{q_2}{\circ}$, $q = q_1^{-a}, q_1^a = q_2^b$.
And its corresponding Weyl groupoid is standard with one Cartan matrix

$$\begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}.$$

$\square$

REMARK 3.6. From M. Cuntz's computations, the following subsequences of $[a, b, a, b]$ type yield one-dimensional type bicharacters:
$[1, 3, 1, 3], [1, 4, 1, 4], [1, 5, 1, 5], [1, 6, 1, 6], [1, 7, 1, 7], [1, 8, 1, 8], [2, 2, 2, 2], [2, 3, 2, 3],$
$[2, 4, 2, 4], [2, 5, 2, 5], [2, 6, 2, 6], [3, 1, 3, 1], [3, 3, 3, 3], [3, 4, 3, 4], [3, 5, 3, 5], [4, 4, 4, 4],$
where the subsequences $[1, 4, 1, 4]$ and $[2, 2, 2, 2]$ produce affine Weyl groupoids.

For all of the length four subsequences of potential periods, we compute their bicharacters for the potential periods of affine characteristic sequences. And we obtain the associated Cartan graphs by program and by hand. Then we complete the classification of affine rank two Nichols algebras of diagonal type by suitable bicharacters. The main result is given as follows:

THEOREM 3.7. *If χ is a bicharacter of an affine rank two Nichols algebra of diagonal type, then χ is one of the following types of bicharacters (in form of generalized Dynkin diagram, see Def. 1.11, Ch. 1):*

(1) $\overset{\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{\zeta^4}{\circ}$, $\zeta \in \mu_6$, *its corresponding period is* $[2, 2, 2, 2, 2, 2]$;

(2) $\overset{\zeta}{\circ}\overset{\zeta}{\text{———}}\overset{\zeta}{\circ}$, $\zeta \in \mu_3$, *its corresponding period is* $[2, 2, 2, 2, 2, 2]$;

(3) $\overset{\zeta^4}{\circ}\overset{\zeta^{10}}{\text{———}}\overset{\zeta}{\circ}$, $\zeta \in \mu_{12}$, *its corresponding period is* $[2, 2]$;

(4) $\overset{\zeta^{-2}}{\circ}\overset{\zeta^{-2}}{\text{———}}\overset{\zeta}{\circ}$, $\zeta \in \mu_6$, *its corresponding period is* $[2, 2]$;

(5) $\overset{\zeta}{\circ}\overset{\zeta^{-2}}{\text{———}}\overset{\zeta^4}{\circ}$, $\zeta \in \mu_{12}$, *its corresponding period is* $[2, 2]$;

(6) $\overset{\zeta}{\circ}\overset{\zeta^{-2}}{\text{――――}}\overset{\zeta^{4}}{\circ}$, $\zeta \in \mu_6$, *its corresponding period is* $[2,2]$*;*

(7) $\overset{q^{-2}}{\circ}\overset{q}{\text{――――}}\overset{q^{-2}}{\circ}$, $q \in \mathbb{C}\backslash\{\pm 1\}$, *it is a one-dimensional type bicharacter and its corresponding period is* $[2,2]$*;*

(8) $\overset{\zeta^{-4}}{\circ}\overset{\zeta}{\text{――――}}\overset{\zeta^{-1}}{\circ}$, $\zeta \in \mu_{10}$, *its corresponding period is* $[1,4]$*;*

(9) $\overset{\zeta}{\circ}\overset{\zeta}{\text{――――}}\overset{\zeta^{-1}}{\circ}$, $\zeta \in \mu_5$, *its corresponding period is* $[1,4]$*;*

(10) $\overset{\zeta}{\circ}\overset{-1}{\text{――――}}\overset{-1}{\circ}$, $\zeta \in \mu_8$, *its corresponding period is* $[1,4]$*;*

(11) $\overset{q}{\circ}\overset{q^{-4}}{\text{――――}}\overset{q^{-4}}{\circ}$, $q \in \mathbb{C}\backslash\{1 \cup \mu_2 \cup \mu_3 \cup \mu_4\}$, *it is a one-dimensional type bicharacter and its corresponding period is* $[1,4]$*;*

(12) $\overset{\zeta^{2}}{\circ}\overset{\zeta^{-6}}{\text{――――}}\overset{-1}{\circ}$, $\zeta \in \mu_{18}$, *its corresponding period is* $[1,3,1,6]$*;*

(13) $\overset{\zeta^{3}}{\circ}\overset{\zeta^{-4}}{\text{――――}}\overset{\zeta^{4}}{\circ}$, $\zeta \in \mu_{12}$, *its corresponding period is* $[1,3,2,3]$*;*

(14) $\overset{\zeta^{3}}{\circ}\overset{\zeta^{4}}{\text{――――}}\overset{-1}{\circ}$, $\zeta \in \mu_{12}$, *its corresponding period is* $[1,3,3,1,4,1,3,3,1,4]$.

where μ_m *denotes the set of primitive* m*th roots of unity,* $\mathbb{C}$ *is the set of complex numbers.*

Remark 3.8. With constraints of q_{ii} and $q_{ij}q_{ji}$, we can denote the bicharacters of affine rank two Nichols algebras of diagonal type by generalized Dynkin diagrams (see Def. 1.11, Ch. 1). As for a generalized Dynkin diagram itself, it is only an invariant.

4. Proof of the main theorem

This section is the proof of Thm. 3.7 (Ch. 4) in last section. For subsequences of $[a,b,a,b]$ type, we can fix their corresponding one-dimensional type bicharacters and standard Weyl groupoids by Prop. 3.3 (Ch. 4)and Prop. 3.5 (Ch. 4). From the proof of Prop. 3.3 (Ch. 4), we find there exist also some cases which yield zero-dimensional type bicharacters or contradictions (with more constraints of inequalities). Naturally, these zero-dimensional type bicharacters can be dealt with by program. From Rem. 3.6 (Ch. 4), we know there are two subsequences $[2,2,2,2]$ and $[1,4,1,4]$ with bicharacters producing affine Weyl groupoids.

Most of the length four subsequences not of $[a,b,a,b]$ type can be dealt with program. There are finitely many subsequences with bicharacters producing affine Weyl groupoids: $[1,3,1,6]$, $[1,3,2,3]$ and $[1,3,3,1]$. These three subsequences correspond to three periods of affine characteristic sequences: $(1,3,1,6)$, $(1,3,2,3)$, $(1,3,3,1,4)$.

Thus it is sufficient to consider only the above two subsequences of $[a,b,a,b]$ type and three periods of affine characteristic sequences. We compute their corresponding bicharacters and Cartan graphs and collect the bicharacters with affine

Weyl groupoids. Then we complete the classification of affine rank two Nichols algebras of diagonal type by bicharacters.

4.1. Subsequences of $[a, b, a, b]$ type. In this part, computations of $[a, b, a, b]$ type subsequences of affine characteristic sequences are given. We compute the bicharacters and their associated Cartan graphs corresponding to the subsequences: $[2, 2, 2, 2], [1, 4, 1, 4]$.

1. For the affine characteristic sequences contain subsequence $[2, 2, 2, 2]$:

$$\cdots \frac{\quad}{}\bullet \frac{e}{\quad\quad} \bullet \frac{2}{\quad\quad} \bullet \frac{2}{\quad\quad} \bullet \frac{2}{\quad\quad} \bullet \frac{2}{\quad\quad} \bullet \frac{f}{\quad\quad} \bullet \frac{\quad}{} \cdots, \quad (1 \le e, f \in \mathbb{N}^+),$$

the associated Cartan graphs are

$$\cdots \xrightarrow{\sigma_1} \begin{pmatrix} 2 & -e \\ -2 & 2 \end{pmatrix} \xrightarrow{\sigma_2} \underset{\chi_1}{\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}} \xrightarrow{\sigma_1} \underset{\chi_2}{\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}} \xrightarrow{\sigma_2} \underset{\chi_3}{\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}} \xrightarrow{\sigma_1} \begin{pmatrix} 2 & -2 \\ -f & 2 \end{pmatrix} \xrightarrow{\sigma_2} \cdots$$

Assume that: $\chi_1 = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} q_1 & 1 \\ q & q_2 \end{pmatrix}$, where $q_1 = q_{11}$, $q_2 = q_{22}$, $q = q_{12} \cdot q_{21}$.

$C^{\chi_1} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, $\sigma_1^{\chi_1} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$, $\sigma_2^{\chi_1} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$.

By formula (2.2, p. 15), we get the set of constraints of χ_1:

$$I): \begin{cases} q \neq 1 \\ q_1 \neq \pm 1 \\ q_1 q \neq 1 \\ q_1^3 = 1 \text{ or } q_1^2 \cdot q = 1 \\ q_2 \neq \pm 1 \\ q_2 q \neq 1 \\ q_2^3 = 1 \text{ or } q_2^2 \cdot q = 1 \end{cases}$$

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} q_1 & 1 \\ q_1^{-4} q^{-1} & q_1^4 q^2 q_2 \end{pmatrix}$,

$C^{\chi_2} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, $\sigma_1^{\chi_2} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$, $\sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$.

By formula (2.2, p. 15), we get the set of constraints of χ_2:

$$II): \begin{cases} q_1^{-4} q^{-1} \neq 1 \\ q_1 \neq \pm 1 \\ q_1 \cdot (q_1^{-4} q^{-1}) \neq 1 \\ q_1^3 = 1 \text{ or } q_1^2 \cdot (q_1^{-4} q^{-1}) = 1 \\ q_1^4 q^2 q_2 \neq \pm 1 \\ q_1^4 q^2 q_2 \cdot q_1^{-4} q^{-1} \neq 1 \\ (q_1^4 q^2 q_2)^3 = 1 \text{ or } (q_1^4 q^2 q_2)^2 \cdot q_1^{-4} q^{-1} = 1 \end{cases}$$

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} q_1^9 q^6 q_2 & 1 \\ q_1^{-12} q^{-7} q_2^{-4} & q_1^4 q^2 q_2 \end{pmatrix}$,

$C^{\chi_3} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, $\sigma_1^{\chi_3} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$, $\sigma_2^{\chi_3} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$.

By using the formula (2.2, p. 15), we can get the set of constraints of χ_3:

$$III):\begin{cases} q_1^{-12}q^{-7}q_2^{-4}\neq 1\\ q_1^9q^6q_2^4\neq \pm 1\\ (q_1^9q^6q_2^4)\cdot q_1^{-12}q^{-7}q_2^{-4}\neq 1\\ (q_1^9q^6q_2^4)^3=1 \text{ or } (q_1^9q^6q_2^4)^2\cdot q_1^{-12}q^{-7}q_2^{-4}=1\\ q_1^4q^2q_2\neq\pm 1\\ (q_1^4q^2q_2)\cdot q_1^{-12}q^{-7}q_2^{-4}=\neq 1\\ (q_1^4q^2q_2)^3=1 \text{ or } (q_1^4q^2q_2)^2\cdot q_1^{-12}q^{-7}q_2^{-4}=1 \end{cases}$$

From $I)$, $II)$ and $III)$ above, the set of constraints of bicharacters is fixed partly. These equations and inequalities in this set of constraints are enough to decide the possible bicharacters. Here we give only the set of equations. When we need some constraints of inequalities in the following, we will choose them from $I), II)$ and $III)$ directly.

$$\begin{cases} q_1^3=1 \text{ ① or } q_1^2\cdot q=1 \text{ ②}\\ q_2^3=1 \text{ ③ or } q_2^2q=1 \text{ ④}\\ (q_1^4q^2q_2)^3=1 \text{ ⑤ or } q_1^4q^3q_2^2=1 \text{ ⑥}\\ (q_1^9q^6q_2^4)^3=1 \text{ ⑦ or } q_1^6q^5q_2^4=1 \text{ ⑧}\end{cases}$$

For the set of equations above, there are 16 different cases:
(①,③,⑤,⑦), (①,③,⑤,⑧), (①,③,⑥,⑦), (①,③,⑥,⑧), (①,④,⑤,⑦), (①,④,⑤,⑧), (①,④,⑥,⑦), (①,④,⑥,⑧), (②,③,⑤,⑦), (②,③,⑤,⑧), (②,③,⑥,⑦), (②,③,⑥,⑧), (②,④,⑤,⑦), (②,④,⑤,⑧), (②,④,⑥,⑦), (②,④,⑥,⑧).

$$(1): \begin{cases} \text{①}: q_1^3=1\\ \text{③}: q_2^3=1\\ \text{⑤}: (q_1^4q^2q_2)^3=1\\ \text{⑦}: (q_1^9q^6q_2^4)^3=1\end{cases} \Rightarrow \begin{cases} q_1^3=1\\ q_2^3=1\\ q^6=1\\ q^{18}=1\end{cases} \Rightarrow \begin{cases} q_1^3=1\\ q_2^3=1\\ q^6=1\end{cases} \Rightarrow \overset{t}{\circ}\overset{r}{\text{———}}\overset{s}{\circ},$$

$t^3=1, r^6=1, s^3=1, t\neq\pm1, s\neq\pm1, r\neq 1, tr\neq 1, sr\neq 1, tr^2s\neq\pm1.$

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1=\begin{pmatrix} t & 1\\ r & s\end{pmatrix}, \mathcal{C}^{\chi_1}=\begin{pmatrix} 2 & -2\\ -2 & 2\end{pmatrix}, \sigma_1^{\chi_1}=\begin{pmatrix} -1 & 2\\ 0 & 1\end{pmatrix};$

$\chi_2=\sigma_1^{\chi_1}*\chi_1=\begin{pmatrix} t & 1\\ t^{-1}r^{-1} & tr^2s\end{pmatrix},$

$\mathcal{C}^{\chi_2}=\begin{pmatrix} 2 & -2\\ -2 & 2\end{pmatrix}, \sigma_2^{\chi_2}=\begin{pmatrix} 1 & 0\\ 2 & -1\end{pmatrix};$

$\chi_3=\sigma_2^{\chi_2}*\chi_2=\begin{pmatrix} s & 1\\ r^{-1}s^{-1} & tr^2s\end{pmatrix},$

$\mathcal{C}^{\chi_3}=\begin{pmatrix} 2 & -2\\ -2 & 2\end{pmatrix}, \sigma_1^{\chi_3}=\begin{pmatrix} -1 & 2\\ 0 & 1\end{pmatrix};$

$\chi_4=\sigma_1^{\chi_3}*\chi_3=\begin{pmatrix} s & 1\\ r & t\end{pmatrix}, \mathcal{C}^{\chi_4}=\begin{pmatrix} 2 & -2\\ -2 & 2\end{pmatrix}, \sigma_2^{\chi_4}=\begin{pmatrix} 1 & 0\\ 2 & -1\end{pmatrix};$

$\chi_5=\sigma_2^{\chi_4}*\chi_4=\begin{pmatrix} sr^2t & 1\\ r^{-1}t^{-1} & t\end{pmatrix}, \mathcal{C}^{\chi_5}=\begin{pmatrix} 2 & -2\\ -2 & 2\end{pmatrix}, \sigma_1^{\chi_5}=\begin{pmatrix} -1 & 2\\ 0 & 1\end{pmatrix};$

$\chi_6 = \sigma_1^{\chi_5} * \chi_5 = \begin{pmatrix} sr^2t & 1 \\ s^{-1}r^{-1} & s \end{pmatrix}, \mathcal{C}^{\chi_6} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \sigma_2^{\chi_6} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix};$

$\chi_7 = \sigma_2^{\chi_6} * \chi_6 = \begin{pmatrix} t & 1 \\ r & s \end{pmatrix} = \chi_1.$

Then the diagram of bicharacters is given as follows:

$$\begin{tikzcd}
\begin{pmatrix} sr^2t & 1 \\ s^{-1}r^{-1} & s \end{pmatrix} \arrow[r, dash, "\sigma_1"] \arrow[d, dash, "\sigma_2"] & \begin{pmatrix} sr^2t & 1 \\ r^{-1}t^{-1} & t \end{pmatrix} \arrow[r, dash, "\sigma_2"] & \begin{pmatrix} s & 1 \\ r & t \end{pmatrix} \arrow[d, dash, "\sigma_1"] \\
\underset{\chi_1}{\begin{pmatrix} t & 1 \\ r & s \end{pmatrix}} \arrow[r, dash, "\sigma_1"] & \begin{pmatrix} t & 1 \\ t^{-1}r^{-1} & tr^2s \end{pmatrix} \arrow[r, dash, "\sigma_2"] & \begin{pmatrix} s & 1 \\ r^{-1}s^{-1} & tr^2s \end{pmatrix}.
\end{tikzcd}$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$\begin{tikzcd}
\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \arrow[r, dash, "\sigma_1"] \arrow[d, dash, "\sigma_2"] & \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \arrow[r, dash, "\sigma_2"] & \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \arrow[d, dash, "\sigma_1"] \\
\underset{\chi_1}{\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}} \arrow[r, dash, "\sigma_1"] & \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \arrow[r, dash, "\sigma_2"] & \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.
\end{tikzcd}$$

$$(2)\text{:} \begin{cases} ①: \ q_1^3 = 1 \\ ③: \ q_2^3 = 1 \\ ⑤: \ (q_1^4 q^2 q_2)^3 = 1 \\ ⑧: \ q_1^6 q^5 q_2^4 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 = 1 \\ q_2^3 = 1 \\ q^6 = 1 \\ q_1^5 q_2 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 = 1 \\ q_2^3 = 1 \\ q_2 = q \end{cases}$$

$\Rightarrow \overset{t}{\circ}\overset{s}{\rule{4em}{0.4pt}}\overset{s}{\circ}, t^3 = 1, s^3 = 1, t \neq \pm 1, s \neq \pm 1, st \neq 1.$

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1 = \begin{pmatrix} t & 1 \\ s & s \end{pmatrix}, \mathcal{C}^{\chi_1} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \sigma_1^{\chi_1} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix};$

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} t & 1 \\ t^{-1}s^{-1} & t \end{pmatrix},$

$\mathcal{C}^{\chi_2} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix};$

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} s & 1 \\ s & t \end{pmatrix},$

$\mathcal{C}^{\chi_3} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \sigma_1^{\chi_3} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix};$

$\chi_4 = \sigma_1^{\chi_3} * \chi_3 = \begin{pmatrix} s & 1 \\ s & t \end{pmatrix}, \mathcal{C}^{\chi_4} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \sigma_2^{\chi_4} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix};$

$\chi_5 = \sigma_2^{\chi_4} * \chi_4 = \begin{pmatrix} t & 1 \\ s^{-1}t^{-1} & t \end{pmatrix}, \mathcal{C}^{\chi_5} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \sigma_1^{\chi_5} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix};$

$\chi_6 = \sigma_1^{\chi_5} * \chi_5 = \begin{pmatrix} t & 1 \\ s & s \end{pmatrix}, \mathcal{C}^{\chi_6} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \sigma_2^{\chi_6} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix};$

$\chi_7 = \sigma_2^{\chi_6} * \chi_6 = \begin{pmatrix} t & 1 \\ s & s \end{pmatrix} = \chi_1.$

Then the diagram of bicharacters is given as follows:

$$^{\sigma_2}\circlearrowright \underset{\chi_1}{\begin{pmatrix} t & 1 \\ s & s \end{pmatrix}} \overset{\sigma_1}{\text{—}} \begin{pmatrix} t & 1 \\ t^{-1}s^{-1} & t \end{pmatrix} \overset{\sigma_2}{\text{—}} \begin{pmatrix} s & 1 \\ s & t \end{pmatrix} \circlearrowleft^{\sigma_1}.$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$^{\sigma_2}\circlearrowright \underset{\chi_1}{\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}} \overset{\sigma_1}{\text{—}} \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \overset{\sigma_2}{\text{—}} \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1}.$$

$$(3){:}\ \begin{cases} ①: q_1^3 = 1 \\ ③: q_2^3 = 1 \\ ⑥: q_1^4 q^3 q_2^2 = 1 \\ ⑦: (q_1^9 q^6 q_2^4)^3 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 = 1 \\ q_2^3 = 1 \\ q_1 q^3 q_2^2 = 1 \\ q^9 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 = 1 \\ q_2^3 = 1 \\ q_1 = q^{-3} q_2 \end{cases}$$

$\Rightarrow \overset{t^{-3}s}{\circ}\overset{t}{\text{———}}\overset{s}{\circ}, t^9 = 1, s^3 = 1, t \neq 1, s \neq \pm 1, st \neq \pm 1, t^{-2}s \neq 1, t^{-3}s \neq \pm 1.$

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1 = \begin{pmatrix} t^{-3}s & 1 \\ t & s \end{pmatrix}, \mathcal{C}^{\chi_1} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \sigma_1^{\chi_1} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix};$

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} t^{-3}s & 1 \\ t^2 s^{-1} & t^{-1}s^{-1} \end{pmatrix},$

$\mathcal{C}^{\chi_2} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix};$

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} t^{-3}s & 1 \\ t^2 s^{-1} & t^{-1}s^{-1} \end{pmatrix},$

$\mathcal{C}^{\chi_3} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \sigma_1^{\chi_3} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix};$

$\chi_4 = \sigma_1^{\chi_3} * \chi_3 = \begin{pmatrix} t^{-3}s & 1 \\ t & s \end{pmatrix}, \mathcal{C}^{\chi_4} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \sigma_2^{\chi_4} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix};$

$\chi_5 = \sigma_2^{\chi_4} * \chi_4 = \begin{pmatrix} t^{-1}s^{-1} & 1 \\ t^{-1}s^{-1} & s \end{pmatrix}, \mathcal{C}^{\chi_5} = \begin{pmatrix} 2 & -8 \\ -2 & 2 \end{pmatrix}, \sigma_1^{\chi_5} = \begin{pmatrix} -1 & 8 \\ 0 & 1 \end{pmatrix};$

$\chi_6 = \sigma_1^{\chi_5} * \chi_5 = \begin{pmatrix} t^{-1}s^{-1} & 1 \\ t^{-1}s^{-1} & s \end{pmatrix}, \mathcal{C}^{\chi_6} = \begin{pmatrix} 2 & -8 \\ -2 & 2 \end{pmatrix}, \sigma_2^{\chi_6} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix};$

$\chi_7 = \sigma_2^{\chi_6} * \chi_6 = \begin{pmatrix} t^{-3}s & 1 \\ t & s \end{pmatrix} = \chi_1.$

Then the diagram of bicharacters is given as follows:

$$^{\sigma_1}\circlearrowright \begin{pmatrix} t^{-1}s^{-1} & 1 \\ t^{-1}s^{-1} & s \end{pmatrix} \overset{\sigma_2}{\text{—}} \underset{\chi_1}{\begin{pmatrix} t^{-3}s & 1 \\ t & s \end{pmatrix}} \overset{\sigma_1}{\text{—}} \begin{pmatrix} t^{-3}s & 1 \\ t^2 s^{-1} & t^{-1}s^{-1} \end{pmatrix} \circlearrowleft^{\sigma_2}.$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$ {}^{\sigma_1}\circlearrowright \begin{pmatrix} 2 & -8 \\ -2 & 2 \end{pmatrix} \overset{\sigma_2}{\text{———}} \underset{\chi_1}{\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}} \overset{\sigma_1}{\text{———}} \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \circlearrowleft^{\sigma_2} . $$

$$ (4): \begin{cases} ① : q_1^3 = 1 \\ ③ : q_2^3 = 1 \\ ⑥ : q_1^4 q^3 q_2^2 = 1 \\ ⑧ : q_1^6 q^5 q_2^4 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 = 1 \\ q_2^3 = 1 \\ q_1 q^3 q_2^2 = 1 \\ q^5 q_2 = 1 \\ q_1^2 q^2 q_2^2 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 = 1 \\ q_2^3 = 1 \\ q_1 = q \\ q^5 q_2 = 1 \end{cases} $$

$$ \Rightarrow \begin{cases} q_1^3 = 1 \\ q_2^3 = 1 \\ q_1 = q \\ q_1 = q_2 \end{cases} \Rightarrow \overset{t}{\circ}\!\overset{t}{\text{————}}\!\overset{t}{\circ}, t^3 = 1, t \neq \pm 1. $$

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1 = \begin{pmatrix} t & 1 \\ t & t \end{pmatrix}, \mathcal{C}^{\chi_1} = \begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix}, \sigma_1^{\chi_1} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$;

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} t & 1 \\ t & t \end{pmatrix} = \chi_1, \mathcal{C}^{\chi_2} = \begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix}, \sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$;

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} t & 1 \\ t & t \end{pmatrix} = \chi_1$.

Then the diagram of bicharacters is given as follows:

$$ {}^{\sigma_2}\circlearrowright \underset{\chi_1}{\begin{pmatrix} t & 1 \\ t & t \end{pmatrix}} \circlearrowleft^{\sigma_1} . $$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$ {}^{\sigma_2}\circlearrowright \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1} . $$

$$ (5): \begin{cases} ① : q_1^3 = 1 \\ ④ : q_2^2 q = 1 \\ ⑤ : (q_1^4 q^2 q_2)^3 = 1 \\ ⑦ : (q_1^9 q^6 q_2^4)^3 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 = 1 \\ q_2^2 q = 1 \\ q^6 q_2^3 = 1 \\ q^{18} q_2^{12} = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 = 1 \\ q_2^2 q = 1 \\ q^5 q_2 = 1 \\ q^6 = 1 \end{cases} $$

$$ \Rightarrow \begin{cases} q_1^3 = 1 \\ q_2 = q \\ q_2^3 = 1 \end{cases} \Rightarrow \overset{t}{\circ}\!\overset{s}{\text{————}}\!\overset{s}{\circ}, t^3 = 1, s^3 = 1, t \neq \pm 1, s \neq \pm 1, st \neq 1. $$

For this bicharacter, it is as same as (2). The Cartan graph associated to χ_1 is

$$ {}^{\sigma_2}\circlearrowright \underset{\chi_1}{\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}} \overset{\sigma_1}{\text{———}} \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \overset{\sigma_2}{\text{———}} \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1} . $$

$$(6)\colon \begin{cases} ①: q_1^3 = 1 \\ ④: q_2^2 q = 1 \\ ⑤: (q_1^4 q^2 q_2)^3 = 1 \\ ⑧: q_1^6 q^5 q_2^4 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 = 1 \\ q_2^2 q = 1 \\ q^5 q_2 = 1 \\ q^3 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 = 1 \\ q_2^2 q = 1 \\ q^2 q_2 = 1 \\ q^3 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} q_1^3 = 1 \\ q = q_2 \\ q^3 = 1 \end{cases} \Rightarrow \overset{t}{\circ}\overset{s}{\text{———}}\overset{s}{\circ}, t^3 = 1, s^3 = 1, t \neq \pm 1, s \neq \pm 1, st \neq 1.$$

For this bicharacter, it is as same as (2). The Cartan graph associated to χ_1 is

$$^{\sigma_2} \circlearrowright \underset{\chi_1}{\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}} \overset{\sigma_1}{\text{——}} \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \overset{\sigma_2}{\text{——}} \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1}.$$

$$(7)\colon \begin{cases} ①: q_1^3 = 1 \\ ④: q_2^2 q = 1 \\ ⑥: q_1^4 q^3 q_2^2 = 1 \\ ⑦: (q_1^9 q^6 q_2^4)^3 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 = 1 \\ q_2^2 q = 1 \\ q_1 q^2 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 = 1 \\ q_2^2 q = 1 \\ q^6 = 1 \\ q_1 q^2 = 1 \end{cases}$$

$$\Rightarrow \overset{t^4}{\circ}\overset{t^{-2}}{\text{———}}\overset{t}{\circ}, t^{12} = 1, t \neq \pm 1, t^4 \neq \pm 1.$$

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1 = \begin{pmatrix} t^4 & 1 \\ t^{-2} & t \end{pmatrix}, \mathcal{C}^{\chi_1} = \begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix}, \sigma_1^{\chi_1} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix};$

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} t^4 & 1 \\ t^{-2} & t \end{pmatrix} = \chi_1, \mathcal{C}^{\chi_2} = \begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix}, \sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix};$

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} t^4 & 1 \\ t^{-2} & t \end{pmatrix} = \chi_1.$

Then the diagram of bicharacters is given as follows:

$$^{\sigma_2} \circlearrowright \underset{\chi_1}{\begin{pmatrix} t^4 & 1 \\ t^{-2} & t \end{pmatrix}} \circlearrowleft^{\sigma_1}.$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$^{\sigma_2} \circlearrowright \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1}.$$

$$(8)\colon \begin{cases} ①: q_1^3 = 1 \\ ④: q_2^2 q = 1 \\ ⑥: q_1^4 q^3 q_2^2 = 1 \\ ⑧: q_1^6 q^5 q_2^4 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 = 1 \\ q_2^2 q = 1 \\ q_1 q^2 = 1 \\ q^3 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 = 1 \\ q_2^2 q = 1 \\ q_1 = q \end{cases}$$

$$\Rightarrow \overset{t^{-2}}{\circ}\overset{t^{-2}}{\text{———}}\overset{t}{\circ}, t^6 = 1, t^2 \neq \pm 1.$$

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1 = \begin{pmatrix} t^{-2} & 1 \\ t^{-2} & t \end{pmatrix}, \mathcal{C}^{\chi_1} = \begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix}, \sigma_1^{\chi_1} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix};$

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} t^{-2} & 1 \\ t^{-2} & t \end{pmatrix} = \chi_1, \mathcal{C}^{\chi_2} = \begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix}, \sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix};$

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} t^{-2} & 1 \\ t^{-2} & t \end{pmatrix} = \chi_1.$

Then the diagram of bicharacters is given as follows:

$$\begin{array}{c} {}^{\sigma_2} \circlearrowright \begin{pmatrix} t^{-2} & 1 \\ t^{-2} & t \end{pmatrix} \circlearrowleft^{\sigma_1} . \\ \chi_1 \end{array}$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$${}^{\sigma_2} \circlearrowright \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1} .$$

$$(9): \left\{ \begin{array}{l} ②: \ q_1^2 q = 1 \\ ③: \ q_2^3 = 1 \\ ⑤: \ (q_1^4 q^2 q_2)^3 = 1 \\ ⑦: \ (q_1^9 q^6 q_2^4)^3 = 1 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} q_1^2 q = 1 \\ q_2^3 = 1 \\ q_1^{12} q^6 = 1 \\ q_1^{27} q^{18} = 1 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} q_1^2 q = 1 \\ q_2^3 = 1 \\ q_1^9 = 1 \end{array} \right.$$

$\Rightarrow \overset{t}{\circ} \overset{t^{-2}}{\rule[0.5ex]{5em}{0.4pt}} \overset{s}{\circ}, t^9 = 1, s^3 = 1, t^2 \neq 1, t^3 \neq 1, s \neq \pm 1, t^2 s \neq 1, t^2 s^2 \neq 1, t^3 s^2 \neq 1.$

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1 = \begin{pmatrix} t & 1 \\ t^{-2} & s \end{pmatrix}, \mathcal{C}^{\chi_1} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \sigma_1^{\chi_1} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix};$

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} t & 1 \\ t^{-2} & s \end{pmatrix}, \mathcal{C}^{\chi_2} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix};$

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} t^{-3}s & 1 \\ t^2 s^{-1} & s \end{pmatrix}, \mathcal{C}^{\chi_3} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \sigma_1^{\chi_3} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix};$

$\chi_4 = \sigma_1^{\chi_3} * \chi_3 = \begin{pmatrix} t^{-3}s & 1 \\ t & t \end{pmatrix}, \mathcal{C}^{\chi_4} = \begin{pmatrix} 2 & -2 \\ -8 & 2 \end{pmatrix}, \sigma_2^{\chi_4} = \begin{pmatrix} 1 & 0 \\ 8 & -1 \end{pmatrix};$

$\chi_5 = \sigma_2^{\chi_4} * \chi_4 = \begin{pmatrix} t^{-3}s & 1 \\ t & t \end{pmatrix}, \mathcal{C}^{\chi_5} = \begin{pmatrix} 2 & -2 \\ -8 & 2 \end{pmatrix}, \sigma_1^{\chi_5} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix};$

$\chi_6 = \sigma_1^{\chi_5} * \chi_5 = \begin{pmatrix} t^{-3}s & 1 \\ t^2 s^{-1} & s \end{pmatrix}, \mathcal{C}^{\chi_6} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \sigma_2^{\chi_6} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix};$

$\chi_7 = \sigma_2^{\chi_6} * \chi_6 = \begin{pmatrix} t & 1 \\ t^{-2} & s \end{pmatrix} = \chi_1.$

Then the diagram of bicharacters is given as follows:

$$\begin{array}{c} {}^{\sigma_2} \circlearrowright \begin{pmatrix} t^{-3}s & 1 \\ t & t \end{pmatrix} \overset{\sigma_1}{\text{———}} \begin{pmatrix} t^{-3}s & 1 \\ t^2 s^{-1} & s \end{pmatrix} \overset{\sigma_2}{\text{———}} \underset{\chi_1}{\begin{pmatrix} t & 1 \\ t^{-2} & s \end{pmatrix}} \circlearrowleft^{\sigma_1} . \end{array}$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$\begin{array}{c} {}^{\sigma_2} \circlearrowright \begin{pmatrix} 2 & -2 \\ -8 & 2 \end{pmatrix} \overset{\sigma_1}{\text{———}} \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \overset{\sigma_2}{\text{———}} \underset{\chi_1}{\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}} \circlearrowleft^{\sigma_1} . \end{array}$$

$$(10)\text{:}\begin{cases} ②: \ q_1^2 q = 1 \\ ③: \ q_2^3 = 1 \\ ⑤: \ (q_1^4 q^2 q_2)^3 = 1 \\ ⑧: \ q_1^6 q^5 q_2^4 = 1 \end{cases} \Rightarrow \begin{cases} q_1^2 q = 1 \\ q_2^3 = 1 \\ q^2 q_2 = 1 \end{cases} \Rightarrow \overset{t}{\circ}\overset{t^{-2}}{\text{———}}\overset{t^4}{\circ}\,,$$

$t^{12} = 1, t \neq \pm 1, t^4 \neq \pm 1$.

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1 = \begin{pmatrix} t & 1 \\ t^{-2} & t^4 \end{pmatrix}, \mathcal{C}^{\chi_1} = \begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix}, \sigma_1^{\chi_1} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$;

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} t & 1 \\ t^{-2} & t^4 \end{pmatrix} = \chi_1, \mathcal{C}^{\chi_2} = \begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix}, \sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$;

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} t & 1 \\ t^{-2} & t^4 \end{pmatrix} = \chi_1$.

Then the diagram of bicharacters is given as follows:

$$^{\sigma_2}\circlearrowright \underset{\chi_1}{\begin{pmatrix} t & 1 \\ t^{-2} & t^4 \end{pmatrix}} \circlearrowleft^{\sigma_1} .$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$^{\sigma_2}\circlearrowright \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1} .$$

$$(11)\text{:}\begin{cases} ②: \ q_1^2 q = 1 \\ ③: \ q_2^3 = 1 \\ ⑥: \ q_1^4 q^3 q_2^2 = 1 \\ ⑦: \ (q_1^9 q^6 q_2^4)^3 = 1 \end{cases} \Rightarrow \begin{cases} q_1^2 q = 1 \\ q_2^3 = 1 \\ q_2^2 q = 1 \\ q_1^{27} q^{18} = 1 \end{cases} \Rightarrow \begin{cases} q_1^2 q = 1 \\ q_2^3 = 1 \\ q = q_2 \\ q_1 q^5 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} q_1^2 q = 1 \\ q_2^3 = 1 \\ q_2 = q \\ q_1 = q \end{cases} \Rightarrow \overset{t}{\circ}\overset{t}{\text{———}}\overset{t}{\circ}, t^3 = 1, t \neq \pm 1.$$

For this bicharacter, it is as same as (4). The Cartan graph associated to χ_1 is

$$^{\sigma_2}\circlearrowright \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1} .$$

$$(12)\text{:}\begin{cases} ②: \ q_1^2 q = 1 \\ ③: \ q_2^3 = 1 \\ ⑥: \ q_1^4 q^3 q_2^2 = 1 \\ ⑧: \ q_1^6 q^5 q_2^4 = 1 \end{cases} \Rightarrow \begin{cases} q_1^2 q = 1 \\ q_2^3 = 1 \\ q q_2^2 = 1 \\ q^2 q_2 = 1 \end{cases} \Rightarrow \begin{cases} q_1^2 q = 1 \\ q_2^3 = 1 \\ q_2 = q \end{cases}$$

$\Rightarrow \overset{t}{\circ}\overset{t^{-2}}{\text{———}}\overset{t^{-2}}{\circ}\,, t^6 = 1, t^2 \neq \pm 1$.

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1 = \begin{pmatrix} t & 1 \\ t^{-2} & t^{-2} \end{pmatrix}, \mathcal{C}^{\chi_1} = \begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix}, \sigma_1^{\chi_1} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$;

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} t & 1 \\ t^{-2} & t^{-2} \end{pmatrix} = \chi_1, \mathcal{C}^{\chi_2} = \begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix}, \sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix};$
$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} t & 1 \\ t^{-2} & t^{-2} \end{pmatrix} = \chi_1.$

Then the diagram of bicharacters is given as follows:

$$^{\sigma_2}\circlearrowright \underset{\chi_1}{\begin{pmatrix} t & 1 \\ t^{-2} & t^{-2} \end{pmatrix}} \circlearrowleft^{\sigma_1}.$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$^{\sigma_2}\circlearrowright \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1}.$$

$$(13)\text{:} \begin{cases} ②: \ q_1^2 q = 1 \\ ④: \ q_2^2 q = 1 \\ ⑤: \ (q_1^4 q^2 q_2)^3 = 1 \\ ⑦: \ (q_1^9 q^6 q_2^4)^3 = 1 \end{cases} \Rightarrow \begin{cases} q_1^2 q = 1 \\ q_2^2 q = 1 \\ q_2^3 = 1 \\ q_1^3 = 1 \end{cases} \Rightarrow \begin{cases} q_1 = q \\ q_2 = q \\ q_1^3 = 1 \\ q_2^3 = 1 \end{cases}$$

$\Rightarrow \overset{t}{\circ}\overset{t}{\text{———}}\overset{t}{\circ}, t^3 = 1, t \neq \pm 1.$

For this bicharacter, it is as same as (4). The Cartan graph associated to χ_1 is

$$^{\sigma_2}\circlearrowright \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1}.$$

$$(14)\text{:} \begin{cases} ②: \ q_1^2 q = 1 \\ ④: \ q_2^2 q = 1 \\ ⑤: \ (q_1^4 q^2 q_2)^3 = 1 \\ ⑧: \ q_1^6 q^5 q_2^4 = 1 \end{cases} \Rightarrow \begin{cases} q_1^2 q = 1 \\ q_2^2 q = 1 \\ q_2^3 = 1 \end{cases} \Rightarrow \begin{cases} q_1^2 q = 1 \\ q_2 = q \\ q_2^3 = 1 \end{cases}$$

$\Rightarrow \overset{t}{\circ}\overset{t^{-2}}{\text{———}}\overset{t^{-2}}{\circ}, t^6 = 1, t^2 \neq \pm 1.$

The associated Cartan graph is

$$^{\sigma_2}\circlearrowright \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1}$$

$$(15)\text{:} \begin{cases} ②: \ q_1^2 q = 1 \\ ④: \ q_2^2 q = 1 \\ ⑥: \ q_1^4 q^3 q_2^2 = 1 \\ ⑦: \ (q_1^9 q^6 q_2^4)^3 = 1 \end{cases} \Rightarrow \begin{cases} q_1^2 q = 1 \\ q_2^2 q = 1 \\ q_1^3 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 = 1 \\ q_1 = 1 \\ q_2^2 q = 1 \end{cases}$$

$\Rightarrow \overset{t^{-2}}{\circ}\overset{t^{-2}}{\text{———}}\overset{t}{\circ}, t^6 = 1, t^2 \neq \pm 1.$

For this bicharacter, it is as same as (8). The Cartan graph associated to χ_1 is

$$^{\sigma_2}\circlearrowright \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1}.$$

$$(16)\text{:}\ \left\{\begin{array}{l} 2:\ q_1^2q=1 \\ 4:\ q_2^2q=1 \\ 6:\ q_1^4q^3q_2^2=1 \\ 8:\ q_1^6q^5q_2^4=1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^2q=1 \\ q_2^2q=1 \end{array}\right. \Rightarrow \overset{t}{\circ}\overset{r}{\text{——————}}\overset{s}{\circ},$$

$t^2r=1, s^2r=1, r\neq 1, t\neq \pm 1, s\neq \pm 1, tr\neq 1, sr\neq 1, st^{-2}\neq 1, ts^{-2}\neq 1.$

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1=\begin{pmatrix} t & 1\\ r & s\end{pmatrix}, \mathcal{C}^{\chi_1}=\begin{pmatrix} 2 & -2\\ -2 & 1\end{pmatrix}, \sigma_1^{\chi_1}=\begin{pmatrix} -1 & 2\\ 0 & 1\end{pmatrix};$

$\chi_2=\sigma_1^{\chi_1}*\chi_1=\begin{pmatrix} t & 1\\ r & s\end{pmatrix}=\chi_1, \mathcal{C}^{\chi_2}=\begin{pmatrix} 2 & -2\\ -2 & 1\end{pmatrix}, \sigma_2^{\chi_2}=\begin{pmatrix} 1 & 0\\ 2 & -1\end{pmatrix};$

$\chi_3=\sigma_2^{\chi_2}*\chi_2=\begin{pmatrix} t^{-3}s^4 & 1\\ t^2s^{-4} & s\end{pmatrix}=\begin{pmatrix} t & 1\\ r & s\end{pmatrix}=\chi_1.$

Then the diagram of bicharacters is given as follows:

$$^{\sigma_2}\circlearrowright \underset{\chi_1}{\begin{pmatrix} t & 1\\ r & s\end{pmatrix}} \circlearrowleft^{\sigma_1}.$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$^{\sigma_2}\circlearrowright \begin{pmatrix} 2 & -2\\ -2 & 2\end{pmatrix} \circlearrowleft^{\sigma_1}.$$

From the above 16 cases, it yields the following bicharacters whose corresponding characteristic sequences are the periods of affine characteristic sequences.

(1) $\overset{\zeta^2}{\circ}\overset{\zeta}{\text{——————}}\overset{\zeta^4}{\circ}$, $\zeta\in\mu_6$;

(2) $\overset{\zeta}{\circ}\overset{\zeta}{\text{——————}}\overset{\zeta}{\circ}$, $\zeta\in\mu_3$;

(3) $\overset{\zeta^4}{\circ}\overset{\zeta^{10}}{\text{——————}}\overset{\zeta}{\circ}$, $\zeta\in\mu_{12}$;

(4) $\overset{\zeta^{-2}}{\circ}\overset{\zeta^{-2}}{\text{——————}}\overset{\zeta}{\circ}$, $\zeta\in\mu_6$;

(5) $\overset{\zeta}{\circ}\overset{\zeta^{-2}}{\text{——————}}\overset{\zeta^4}{\circ}$, $\zeta\in\mu_{12}$;

(6) $\overset{\zeta}{\circ}\overset{\zeta^{-2}}{\text{——————}}\overset{\zeta^4}{\circ}$, $\zeta\in\mu_6$;

(7) $\overset{q^{-2}}{\circ}\overset{q}{\text{——————}}\overset{q^{-2}}{\circ}$, $q\in\mathbb{C}\backslash\{\pm 1\}$, it is a one-dimensional type bicharacter;

where μ_m denotes the set of primitive mth roots of unity, $\mathbb{C}$ is the set of complex numbers. The first two bicharacters have the same characteristic sequences: $[2,2,2,2,2,2]$; the other five bicharacters have the same characteristic sequences: $[2,2]$.

2. For the affine characteristic sequences containing subsequence $[1,4,1,4]$:

$$\cdots\text{—}\bullet\overset{e}{\text{———}}\bullet\overset{1}{\text{———}}\bullet\overset{4}{\text{———}}\bullet\overset{1}{\text{———}}\bullet\overset{4}{\text{———}}\bullet\overset{f}{\text{———}}\bullet\text{—}\cdots,\quad (1\leq e,f\in\mathbb{N}^+),$$

the associated Cartan graphs are

$$\cdots \xrightarrow{\sigma_1} \begin{pmatrix} 2 & -e \\ -1 & 2 \end{pmatrix} \xrightarrow{\sigma_2} \underset{\chi_1}{\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}} \xrightarrow{\sigma_1} \underset{\chi_2}{\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}} \xrightarrow{\sigma_2} \underset{\chi_3}{\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}} \xrightarrow{\sigma_1} \begin{pmatrix} 2 & -4 \\ -f & 2 \end{pmatrix} \xrightarrow{\sigma_2} \cdots$$

Assume that: $\chi_1 = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} q_1 & 1 \\ q & q_2 \end{pmatrix}$, where $q_1 = q_{11}$, $q_2 = q_{22}$, $q = q_{12} \cdot q_{21}$.

$C^{\chi_1} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$, $\sigma_1^{\chi_1} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}$, $\sigma_2^{\chi_1} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$.

By formula (2.2, p. 15), we get the set of constraints of χ_1:

$$I): \begin{cases} q \neq 1 \\ q_1^i \neq 1, (i = 2, 3, 4) \\ q_1^j \cdot q \neq 1, (j = 1, 2, 3) \\ q_1^5 = 1 \text{ or } q_1^4 \cdot q = 1 \\ q_2 = -1 \text{ or } q_2 \cdot q = 1 \end{cases}$$

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} q_1 & 1 \\ q_1^{-8} q^{-1} & q_1^{16} q^4 q_2 \end{pmatrix}$,

$C^{\chi_2} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$, $\sigma_1^{\chi_2} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}$, $\sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$.

By formula (2.2, p. 15), we get the set of constraints of χ_2:

$$II): \begin{cases} q_1^{-8} q^{-1} \neq 1 \\ q_1^i \neq 1, (i = 2, 3, 4) \\ q_1^j \cdot (q_1^{-8} q^{-1}) \neq 1, (j = 1, 2, 3) \\ q_1^5 = 1 \text{ or } q_1^4 \cdot (q_1^{-8} q^{-1}) = 1 \\ q_1^{16} q^4 q_2 = -1 \text{ or } q_1^{16} q^4 q_2 \cdot q_1^{-8} q^{-1} = 1 \end{cases}$$

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} q_1^9 q^3 q_2 & 1 \\ q_1^{-24} q^{-7} q_2^{-2} & q_1^{16} q^4 q_2 \end{pmatrix}$,

$C^{\chi_3} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$, $\sigma_1^{\chi_3} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}$, $\sigma_2^{\chi_3} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$.

By using the formula (2.2, p. 15), we can get the set of constraints of χ_3:

$$III): \begin{cases} q_1^{-24} q^{-7} q_2^{-2} \neq 1 \\ (q_1^9 q^3 q_2)^i \neq 1, (i = 2, 3, 4) \\ (q_1^9 q^3 q_2)^j \cdot q_1^{-24} q^{-7} q_2^{-2} \neq 1, (j = 1, 2, 3) \\ (q_1^9 q^3 q_2)^5 = 1 \text{ or } (q_1^9 q^3 q_2)^4 \cdot q_1^{-24} q^{-7} q_2^{-2} = 1 \\ q_1^{16} q^4 q_2 = -1 \text{ or } q_1^{16} q^4 q_2 \cdot q_1^{-24} q^{-7} q_2^{-2} = 1 \end{cases}$$

From $I)$, $II)$ and $III)$ above, the set of constraints of bicharacters is fixed partly. These equations and inequalities in this set of constraints are enough to decide the possible bicharacters. Here we give only the set of equations. When we need some constraints of inequalities in the following, we will choose them from $I), II)$ and $III)$ directly.

$$\begin{cases} q_1^5 = 1 \text{ ① or } q_1^4 \cdot q = 1 \text{ ②} \\ q_2 = -1 \text{ ③ or } q_2 \cdot q = 1 \text{ ④} \\ q_1^{16} q^4 q_2 = -1 \text{ ⑤ or } q_1^8 q^3 q_2 = 1 \text{ ⑥} \\ (q_1^9 q^3 q_2)^5 = 1 \text{ ⑦ or } q_1^{12} q^5 q_2^2 = 1 \text{ ⑧} \end{cases}$$

For the set of equations above, there are 16 different cases:
(①,③,⑤,⑦), (①,③,⑤,⑧), (①,③,⑥,⑦), (①,③,⑥,⑧), (①,④,⑤,⑦), (①,④,⑤,⑧), (①,④,⑥,⑦), (①,④,⑥,⑧), (②,③,⑤,⑦), (②,③,⑤,⑧), (②,③,⑥,⑦), (②,③,⑥,⑧), (②,④,⑤,⑦), (②,④,⑤,⑧), (②,④,⑥,⑦), (②,④,⑥,⑧).

$$(1): \begin{cases} ①: \ q_1^5 = 1 \\ ③: \ q_2 = -1 \\ ⑤: \ q_1^{16} q^4 q_2 = -1 \\ ⑦: \ (q_1^9 q^3 q_2)^5 = 1 \end{cases} \Rightarrow \begin{cases} q_1^5 = 1 \\ q_2 = -1 \\ q_1 q^4 = 1 \\ q^{15} = -1 \end{cases} \Rightarrow \begin{cases} q_1^5 = 1 \\ q_2 = -1 \\ q_1 = -q \end{cases}$$

$\Rightarrow \overset{t}{\circ}\!\!\xrightarrow{\quad -t \quad}\!\!\overset{-1}{\circ}$, $t^5 = 1, t^2 \neq \pm 1$.

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1 = \begin{pmatrix} t & 1 \\ -t & -1 \end{pmatrix}$,

$\mathcal{C}^{\chi_1} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_1} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_1} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$;

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} t & 1 \\ -t & -1 \end{pmatrix}$,

$\mathcal{C}^{\chi_2} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_2} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$;

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} t^2 & 1 \\ -t^{-1} & -1 \end{pmatrix}$,

$\mathcal{C}^{\chi_3} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_3} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_3} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$;

$\chi_4 = \sigma_1^{\chi_3} * \chi_3 = \begin{pmatrix} t^2 & 1 \\ -1 & -t^3 \end{pmatrix}$,

$\mathcal{C}^{\chi_4} = \begin{pmatrix} 2 & -4 \\ -5 & 2 \end{pmatrix}, \sigma_1^{\chi_4} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_4} = \begin{pmatrix} 1 & 0 \\ 5 & -1 \end{pmatrix}$;

$\chi_5 = \sigma_2^{\chi_4} * \chi_4 = \begin{pmatrix} t^2 & 1 \\ -1 & -t^3 \end{pmatrix}$,

$\mathcal{C}^{\chi_5} = \begin{pmatrix} 2 & -4 \\ -5 & 2 \end{pmatrix}, \sigma_1^{\chi_5} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_5} = \begin{pmatrix} 1 & 0 \\ 5 & -1 \end{pmatrix}$

$\chi_6 = \sigma_1^{\chi_5} * \chi_5 = \begin{pmatrix} t^2 & 1 \\ -t^{-1} & -1 \end{pmatrix}$,

$\mathcal{C}^{\chi_6} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_6} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_6} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$;

$\chi_7 = \sigma_2^{\chi_6} * \chi_6 = \begin{pmatrix} t & 1 \\ -t & -1 \end{pmatrix} = \chi_1$.

Then the diagram of bicharacters is given as follows:

$$\sigma_2 \circlearrowright \begin{pmatrix} t^2 & 1 \\ -1 & -t^3 \end{pmatrix} \overset{\sigma_1}{\text{——}} \begin{pmatrix} t^2 & 1 \\ -t^{-1} & -1 \end{pmatrix} \overset{\sigma_2}{\text{——}} \underset{\chi_1}{\begin{pmatrix} t & 1 \\ -t & -1 \end{pmatrix}} \circlearrowleft \sigma_1 .$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$^{\sigma_2}\circlearrowright \begin{pmatrix} 2 & -4 \\ -5 & 2 \end{pmatrix} \overset{\sigma_1}{\text{———}} \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \overset{\sigma_2}{\text{———}} \underset{\chi_1}{\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}} \circlearrowleft^{\sigma_1} .$$

$$(2): \begin{cases} ①: \ q_1^5 = 1 \\ ③: \ q_2 = -1 \\ ⑤: \ q_1^{16} q^4 q_2 = -1 \\ ⑧: \ q_1^{12} q^5 q_2^2 = 1 \end{cases} \Rightarrow \begin{cases} q_1^5 = 1 \\ q_2 = -1 \\ q_1 q^4 = 1 \\ q_1^2 q^5 = 1 \end{cases} \Rightarrow \begin{cases} q_1^5 = 1 \\ q_2 = -1 \\ q_1 q = 1 \\ q_1^j \cdot q \neq 1, (j = 1, 2, 3) \end{cases} \Rightarrow \lightning .$$

$$(3): \begin{cases} ①: \ q_1^5 = 1 \\ ③: \ q_2 = -1 \\ ⑥: \ q_1^8 q^3 q_2 = 1 \\ ⑦: \ (q_1^9 q^3 q_2)^5 = 1 \end{cases} \Rightarrow \begin{cases} q_1^5 = 1 \\ q_2 = -1 \\ q_1^3 q^3 = -1 \\ q^{15} = -1 \end{cases} \Rightarrow \begin{cases} q_1^5 = 1 \\ q_2 = -1 \\ q_1 = q^6 \end{cases}$$

$\Rightarrow \overset{t^{-6}}{\circ} \overset{t}{\text{————}} \overset{-1}{\circ}$, $t^{30} = 1, t \neq 1, t^i \neq -1, (i = 2, 3, 4), t^5 \neq 1, t^6 \neq \pm 1.$

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1 = \begin{pmatrix} t^{-6} & 1 \\ t & -1 \end{pmatrix}$,

$\mathcal{C}^{\chi_1} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_1} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_1} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$;

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} t^{-6} & 1 \\ -t^2 & -t^{-2} \end{pmatrix}$,

$\mathcal{C}^{\chi_2} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_2} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$;

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} t^{-6} & 1 \\ -t^2 & -t^{-2} \end{pmatrix}$,

$\mathcal{C}^{\chi_3} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_3} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_3} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$;

$\chi_4 = \sigma_1^{\chi_3} * \chi_3 = \begin{pmatrix} t^{-6} & 1 \\ t & -1 \end{pmatrix}$,

$\mathcal{C}^{\chi_4} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_4} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_4} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$;

$\chi_5 = \sigma_2^{\chi_4} * \chi_4 = \begin{pmatrix} -t^{-5} & 1 \\ t^{-1} & -1 \end{pmatrix}$,

$\mathcal{C}^{\chi_5} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_5} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_5} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$;

$\chi_6 = \sigma_1^{\chi_5} * \chi_5 = \begin{pmatrix} -t^{-5} & 1 \\ -t^6 & t^{-7} \end{pmatrix}$,

$\mathcal{C}^{\chi_6} = \begin{pmatrix} 2 & -2 \\ -3 & 2 \end{pmatrix}, \sigma_1^{\chi_6} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_6} = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}$;

$\chi_7 = \sigma_2^{\chi_6} * \chi_6 = \begin{pmatrix} -t^{-5} & 1 \\ -t^6 & t^{-7} \end{pmatrix}$,

$\mathcal{C}^{\chi_7} = \begin{pmatrix} 2 & -2 \\ -3 & 2 \end{pmatrix}, \sigma_1^{\chi_7} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_7} = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix};$

$\chi_8 = \sigma_1^{\chi_7} * \chi_7 = \begin{pmatrix} -t^{-5} & 1 \\ t^{-1} & -1 \end{pmatrix},$

$\mathcal{C}^{\chi_8} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_8} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_8} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix};$

$\chi_9 = \sigma_2^{\chi_6} * \chi_6 = \begin{pmatrix} t^{-6} & 1 \\ t & -1 \end{pmatrix} = \chi_1.$

Then the diagram of bicharacters is given as follows:

$$\sigma_2 \circlearrowright \begin{pmatrix} -t^{-5} & 1 \\ -t^{6} & t^{-7} \end{pmatrix} \overset{\sigma_1}{\text{——}} \begin{pmatrix} -t^{-5} & 1 \\ t^{-1} & -1 \end{pmatrix} \overset{\sigma_2}{\text{——}} \underset{\chi_1}{\begin{pmatrix} t^{-6} & 1 \\ t & -1 \end{pmatrix}} \overset{\sigma_1}{\text{——}} \begin{pmatrix} t^{-6} & 1 \\ -t^{2} & -t^{-2} \end{pmatrix} \circlearrowleft^{\sigma_2}.$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$\sigma_2 \circlearrowright \begin{pmatrix} 2 & -2 \\ -3 & 2 \end{pmatrix} \overset{\sigma_1}{\text{——}} \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \overset{\sigma_2}{\text{——}} \underset{\chi_1}{\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}} \overset{\sigma_1}{\text{——}} \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \circlearrowleft^{\sigma_2}$$

$$(4): \begin{cases} ①: \ q_1^5 = 1 \\ ③: \ q_2 = -1 \\ ⑥: \ q_1^8 q^3 q_2 = 1 \\ ⑧: \ q_1^{12} q^5 q_2^2 = 1 \end{cases} \Rightarrow \begin{cases} q_1^5 = 1 \\ q_2 = -1 \\ q_1^3 q^3 = -1 \\ q_1^2 q^5 = 1 \end{cases} \Rightarrow \begin{cases} q_1^5 = 1 \\ q_2 = -1 \\ q_1 = -q^2 \end{cases}$$

$\Rightarrow \overset{-t^2}{\circ} \overset{t}{\text{——————}} \overset{-1}{\circ}, t^{10} = -1, t^i \neq \pm 1, (i = 2, 3, 4), t^5 \neq 1.$

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1 = \begin{pmatrix} -t^{2} & 1 \\ t & -1 \end{pmatrix},$

$\mathcal{C}^{\chi_1} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_1} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_1} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix};$

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} -t^{2} & 1 \\ t^3 & t^6 \end{pmatrix},$

$\mathcal{C}^{\chi_2} = \begin{pmatrix} 2 & -4 \\ -9 & 2 \end{pmatrix}, \sigma_1^{\chi_2} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 9 & -1 \end{pmatrix};$

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} t^{5} & 1 \\ -t^{-1} & t^6 \end{pmatrix},$

$\mathcal{C}^{\chi_3} = \begin{pmatrix} 2 & -3 \\ -9 & 2 \end{pmatrix}, \sigma_1^{\chi_3} = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_3} = \begin{pmatrix} 1 & 0 \\ 9 & -1 \end{pmatrix};$

$\chi_4 = \sigma_1^{\chi_3} * \chi_3 = \begin{pmatrix} t^{5} & 1 \\ t & t^{-2} \end{pmatrix},$

$\mathcal{C}^{\chi_4} = \begin{pmatrix} 2 & -3 \\ -9 & 2 \end{pmatrix}, \sigma_1^{\chi_4} = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_4} = \begin{pmatrix} 1 & 0 \\ 9 & -1 \end{pmatrix};$

$\chi_5 = \sigma_2^{\chi_4} * \chi_4 = \begin{pmatrix} -t^{2} & 1 \\ t^{-5} & t^{-2} \end{pmatrix},$

$\mathcal{C}^{\chi_5} = \begin{pmatrix} 2 & -4 \\ -9 & 2 \end{pmatrix}, \sigma_1^{\chi_5} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_5} = \begin{pmatrix} 1 & 0 \\ 9 & -1 \end{pmatrix};$

$\chi_6 = \sigma_1^{\chi_5} * \chi_5 = \begin{pmatrix} -t^2 & 1 \\ -t^{-1} & -1 \end{pmatrix},$

$\mathcal{C}^{\chi_6} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_6} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_6} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix};$

$\chi_7 = \sigma_2^{\chi_6} * \chi_6 = \begin{pmatrix} -t & 1 \\ -t & -1 \end{pmatrix},$

$\mathcal{C}^{\chi_7} = \begin{pmatrix} 2 & -19 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_7} = \begin{pmatrix} -1 & 19 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_7} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix};$

$\chi_8 = \sigma_1^{\chi_7} * \chi_7 = \begin{pmatrix} -t & 1 \\ -t & -1 \end{pmatrix},$

$\mathcal{C}^{\chi_8} = \begin{pmatrix} 2 & -19 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_8} = \begin{pmatrix} -1 & 19 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_8} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix};$

$\chi_9 = \sigma_2^{\chi_8} * \chi_8 = \begin{pmatrix} -t^2 & 1 \\ -t^{-1} & -1 \end{pmatrix},$

$\mathcal{C}^{\chi_9} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_9} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_9} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix};$

$\chi_{10} = \sigma_1^{\chi_9} * \chi_9 = \begin{pmatrix} -t^2 & 1 \\ t^{-5} & t^{-2} \end{pmatrix},$

$\mathcal{C}^{\chi_{10}} = \begin{pmatrix} 2 & -4 \\ -9 & 2 \end{pmatrix}, \sigma_1^{\chi_{10}} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_{10}} = \begin{pmatrix} 1 & 0 \\ 9 & -1 \end{pmatrix};$

$\chi_{11} = \sigma_2^{\chi_{10}} * \chi_{10} = \begin{pmatrix} t^5 & 1 \\ t & t^{-2} \end{pmatrix},$

$\mathcal{C}^{\chi_{11}} = \begin{pmatrix} 2 & -3 \\ -9 & 2 \end{pmatrix}, \sigma_1^{\chi_{11}} = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_{11}} = \begin{pmatrix} 1 & 0 \\ 9 & -1 \end{pmatrix};$

$\chi_{12} = \sigma_1^{\chi_{11}} * \chi_{11} = \begin{pmatrix} t^5 & 1 \\ -t^{-1} & t^6 \end{pmatrix},$

$\mathcal{C}^{\chi_{12}} = \begin{pmatrix} 2 & -3 \\ -9 & 2 \end{pmatrix}, \sigma_1^{\chi_{12}} = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_{12}} = \begin{pmatrix} 1 & 0 \\ 9 & -1 \end{pmatrix};$

$\chi_{13} = \sigma_2^{\chi_{12}} * \chi_{12} = \begin{pmatrix} -t^2 & 1 \\ t^3 & t^6 \end{pmatrix},$

$\mathcal{C}^{\chi_{13}} = \begin{pmatrix} 2 & -4 \\ -9 & 2 \end{pmatrix}, \sigma_1^{\chi_{13}} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_{13}} = \begin{pmatrix} 1 & 0 \\ 9 & -1 \end{pmatrix};$

$\chi_{14} = \sigma_1^{\chi_{13}} * \chi_{13} = \begin{pmatrix} -t^2 & 1 \\ t & -1 \end{pmatrix},$

$\mathcal{C}^{\chi_{14}} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_{14}} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_{14}} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix};$

$\chi_{15} = \sigma_2^{\chi_{14}} * \chi_{14} = \begin{pmatrix} t^3 & 1 \\ t^{-1} & -1 \end{pmatrix},$

$\mathcal{C}^{\chi_{15}} = \begin{pmatrix} 2 & -7 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_{15}} = \begin{pmatrix} -1 & 7 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_{15}} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix};$

$\chi_{16} = \sigma_1^{\chi_{15}} * \chi_{15} = \begin{pmatrix} t^3 & 1 \\ t^{-1} & -1 \end{pmatrix}$,
$\mathcal{C}^{\chi_{16}} = \begin{pmatrix} 2 & -7 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_{16}} = \begin{pmatrix} -1 & 7 \\ 0 & 1 \end{pmatrix}, \sigma_2^{\chi_{16}} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$;
$\chi_{17} = \sigma_2^{\chi_{16}} * \chi_{16} = \begin{pmatrix} -t^2 & 1 \\ t & -1 \end{pmatrix} = \chi_1$.

Then the diagram of bicharacters is given as follows:

$$\begin{array}{l}
\sigma_1 \circlearrowright \begin{pmatrix} t^3 & 1 \\ t^{-1} & -1 \end{pmatrix} \overset{\sigma_2}{\text{——}} \overset{\chi_1}{\begin{pmatrix} -t^2 & 1 \\ t & -1 \end{pmatrix}} \overset{\sigma_1}{\text{——}} \begin{pmatrix} -t^2 & 1 \\ t^3 & t^6 \end{pmatrix} \overset{\sigma_2}{\text{——}} \begin{pmatrix} t^5 & 1 \\ -t^{-1} & t^6 \end{pmatrix} \\
\hfill |\sigma_1 \\
\sigma_1 \circlearrowright \begin{pmatrix} -t & 1 \\ -t & -1 \end{pmatrix} \overset{\sigma_2}{\text{——}} \begin{pmatrix} -t^2 & 1 \\ -t^{-1} & -1 \end{pmatrix} \overset{\sigma_1}{\text{——}} \begin{pmatrix} -t^2 & 1 \\ t^{-5} & t^{-2} \end{pmatrix} \overset{\sigma_2}{\text{——}} \begin{pmatrix} t^5 & 1 \\ t & t^{-2} \end{pmatrix}.
\end{array}$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$\begin{array}{l}
\sigma_1 \circlearrowright \begin{pmatrix} 2 & -7 \\ -1 & 2 \end{pmatrix} \overset{\sigma_2}{\text{——}} \overset{\chi_1}{\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}} \overset{\sigma_1}{\text{——}} \begin{pmatrix} 2 & -4 \\ -9 & 2 \end{pmatrix} \overset{\sigma_2}{\text{——}} \begin{pmatrix} 2 & -3 \\ -9 & 2 \end{pmatrix} \\
\hfill |\sigma_1 \\
\sigma_1 \circlearrowright \begin{pmatrix} 2 & -19 \\ -1 & 2 \end{pmatrix} \overset{\sigma_2}{\text{——}} \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \overset{\sigma_1}{\text{——}} \begin{pmatrix} 2 & -4 \\ -9 & 2 \end{pmatrix} \overset{\sigma_2}{\text{——}} \begin{pmatrix} 2 & -3 \\ -9 & 2 \end{pmatrix}.
\end{array}$$

$$(5): \left\{\begin{array}{l} ①: q_1^5 = 1 \\ ④: q_2 q = 1 \\ ⑤: q_1^{16} q^4 q_2 = -1 \\ ⑦: (q_1^9 q^3 q_2)^5 = 1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^5 = 1 \\ q_2 q = 1 \\ q_1 q^3 = -1 \\ q^{10} = 1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^5 = 1 \\ q_2 q = 1 \\ q_1^3 = -q \\ q_1^9 q^3 q_2 \neq \pm 1 \end{array}\right. \Rightarrow \lightning.$$

$$(6): \left\{\begin{array}{l} ①: q_1^5 = 1 \\ ④: q_2 q = 1 \\ ⑤: q_1^{16} q^4 q_2 = -1 \\ ⑧: q_1^{12} q^5 q_2^2 = 1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^5 = 1 \\ q_2 q = 1 \\ q_1 q^3 = -1 \\ q_1^2 q^3 = 1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1 = -1 \\ q_2 q = 1 \\ q_1^i \neq 1, (i = 2, 3, 4) \end{array}\right. \Rightarrow \lightning.$$

$$(7): \left\{\begin{array}{l} ①: q_1^5 = 1 \\ ④: q_2 q = 1 \\ ⑥: q_1^8 q^3 q_2 = 1 \\ ⑦: (q_1^9 q^3 q_2)^5 = 1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^5 = 1 \\ q_2 q = 1 \\ q_1^3 q^2 = 1 \\ q^{10} = 1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^5 = 1 \\ q_2 q = 1 \\ q_1 = q^{-4} \\ q^{10} = 1 \end{array}\right.$$

$\Rightarrow \overset{t^{-4}}{\circ}\overset{t}{\text{————}}\overset{t^{-1}}{\circ}, t^{10} = 1, t \neq \pm 1, t^3 \neq 1, t^4 \neq \pm 1.$

Now we are trying to construct the entire Cartan graph associated to this bicharacter.
$\chi_1 = \begin{pmatrix} t^{-4} & 1 \\ t & t^{-1} \end{pmatrix}, \mathcal{C}^{\chi_1} = \begin{pmatrix} 2 & -4 \\ -1 & 1 \end{pmatrix}, \sigma_1^{\chi_1} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}$;

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} t^{-4} & 1 \\ t & t^{-1} \end{pmatrix} = \chi_1, \mathcal{C}^{\chi_2} = \begin{pmatrix} 2 & -4 \\ -1 & 1 \end{pmatrix}, \sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix};$

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} t^{-4} & 1 \\ t & t^{-1} \end{pmatrix} = \chi_1.$

Then the diagram of bicharacters is given as follows:

$$\sigma_2 \circlearrowright \underset{\chi_1}{\begin{pmatrix} t^{-4} & 1 \\ t & t^{-1} \end{pmatrix}} \circlearrowleft^{\sigma_1}.$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$\sigma_2 \circlearrowright \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1}.$$

$$(8)\colon \begin{cases} ①: \ q_1^5 = 1 \\ ④: \ q_2 q = 1 \\ ⑥: \ q_1^8 q^3 q_2 = 1 \\ ⑧: \ q_1^{12} q^5 q_2^2 = 1 \end{cases} \Rightarrow \begin{cases} q_1^5 = 1 \\ q_2 q = 1 \\ q_1^4 q^2 q_2 = 1 \end{cases} \Rightarrow \begin{cases} q_1^5 = 1 \\ q_2 q = 1 \\ q_1 = q \end{cases}$$

$\Rightarrow \overset{t}{\circ}\overset{t}{\rule[0.5ex]{5em}{0.4pt}}\overset{t^{-1}}{\circ}, t^5 = 1, t^2 \neq 1.$

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1 = \begin{pmatrix} t & 1 \\ t & t^{-1} \end{pmatrix}, \mathcal{C}^{\chi_1} = \begin{pmatrix} 2 & -4 \\ -1 & 1 \end{pmatrix}, \sigma_1^{\chi_1} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix};$

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} t & 1 \\ t & t^{-1} \end{pmatrix} = \chi_1, \mathcal{C}^{\chi_2} = \begin{pmatrix} 2 & -4 \\ -1 & 1 \end{pmatrix}, \sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix};$

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} t & 1 \\ t & t^{-1} \end{pmatrix} = \chi_1.$

Then the diagram of bicharacters is given as follows:

$$\sigma_2 \circlearrowright \underset{\chi_1}{\begin{pmatrix} t & 1 \\ t & t^{-1} \end{pmatrix}} \circlearrowleft^{\sigma_1}.$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$\sigma_2 \circlearrowright \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1}.$$

$$(9)\colon \begin{cases} ②: \ q_1^4 q = 1 \\ ③: \ q_2 = -1 \\ ⑤: \ q_1^{16} q^4 q_2 = -1 \\ ⑦: \ (q_1^9 q^3 q_2)^5 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 q = 1 \\ q_2 = -1 \\ q_1 q^4 = -1 \end{cases} \Rightarrow \begin{cases} q_1^4 q = 1 \\ q_2 = -1 \\ q_1^{15} = -1 \end{cases}$$

$\Rightarrow \overset{t}{\circ}\overset{t^{-4}}{\rule[0.5ex]{5em}{0.4pt}}\overset{-1}{\circ}, t^{15} = -1, t^2 \neq 1, t^3 \neq \pm 1, t^4 \neq 1, t^5 \neq -1, t^6 \neq 1.$

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1 = \begin{pmatrix} t & 1 \\ t^{-4} & -1 \end{pmatrix},$

$\mathcal{C}^{\chi_1}=\begin{pmatrix}2&-4\\-1&2\end{pmatrix},\sigma_1^{\chi_1}=\begin{pmatrix}-1&4\\0&1\end{pmatrix},\sigma_2^{\chi_1}=\begin{pmatrix}1&0\\1&-1\end{pmatrix};$

$\chi_2=\sigma_1^{\chi_1}*\chi_1=\begin{pmatrix}t&1\\t^{-4}&-1\end{pmatrix},$

$\mathcal{C}^{\chi_2}=\begin{pmatrix}2&-4\\-1&2\end{pmatrix},\sigma_1^{\chi_2}=\begin{pmatrix}-1&4\\0&1\end{pmatrix},\sigma_2^{\chi_2}=\begin{pmatrix}1&0\\1&-1\end{pmatrix};$

$\chi_3=\sigma_2^{\chi_2}*\chi_2=\begin{pmatrix}-t^{-3}&1\\t^4&-1\end{pmatrix},$

$\mathcal{C}^{\chi_3}=\begin{pmatrix}2&-4\\-1&2\end{pmatrix},\sigma_1^{\chi_3}=\begin{pmatrix}-1&4\\0&1\end{pmatrix},\sigma_2^{\chi_3}=\begin{pmatrix}1&0\\1&-1\end{pmatrix};$

$\chi_4=\sigma_1^{\chi_3}*\chi_3=\begin{pmatrix}-t^{-3}&1\\-t^5&-t^{-2}\end{pmatrix},$

$\mathcal{C}^{\chi_4}=\begin{pmatrix}2&-4\\-10&2\end{pmatrix},\sigma_1^{\chi_4}=\begin{pmatrix}-1&4\\0&1\end{pmatrix},\sigma_2^{\chi_4}=\begin{pmatrix}1&0\\10&-1\end{pmatrix};$

$\chi_5=\sigma_2^{\chi_4}*\chi_4=\begin{pmatrix}-t^{-3}&1\\-t^5&-t^{-2}\end{pmatrix},$

$\mathcal{C}^{\chi_5}=\begin{pmatrix}2&-4\\-10&2\end{pmatrix},\sigma_1^{\chi_5}=\begin{pmatrix}-1&4\\0&1\end{pmatrix},\sigma_2^{\chi_5}=\begin{pmatrix}1&0\\10&-1\end{pmatrix};$

$\chi_6=\sigma_1^{\chi_5}*\chi_5=\begin{pmatrix}-t^{-3}&1\\t^4&-1\end{pmatrix},$

$\mathcal{C}^{\chi_6}=\begin{pmatrix}2&-4\\-1&2\end{pmatrix},\sigma_1^{\chi_6}=\begin{pmatrix}-1&4\\0&1\end{pmatrix},\sigma_2^{\chi_6}=\begin{pmatrix}1&0\\1&-1\end{pmatrix};$

$\chi_7=\sigma_2^{\chi_6}*\chi_6=\begin{pmatrix}t&1\\t^{-4}&-1\end{pmatrix}=\chi_1.$

Then the diagram of bicharacters is given as follows:

$$^{\sigma_2}\circlearrowright\begin{pmatrix}-t^{-3}&1\\-t^5&-t^{-2}\end{pmatrix}\overset{\sigma_1}{\text{———}}\begin{pmatrix}-t^{-3}&1\\t^4&-1\end{pmatrix}\overset{\sigma_2}{\text{———}}\underset{\chi_1}{\begin{pmatrix}t&1\\t^{-4}&-1\end{pmatrix}}\circlearrowleft^{\sigma_1}.$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$^{\sigma_2}\circlearrowright\begin{pmatrix}2&-4\\-10&2\end{pmatrix}\overset{\sigma_1}{\text{———}}\begin{pmatrix}2&-4\\-1&2\end{pmatrix}\overset{\sigma_2}{\text{———}}\underset{\chi_1}{\begin{pmatrix}2&-4\\-1&2\end{pmatrix}}\circlearrowleft^{\sigma_1}.$$

$$(10)\colon\begin{cases}②:\ q_1^4q=1\\③:\ q_2=-1\\⑤:\ q_1^{16}q^4q_2=-1\\⑧:\ q_1^{12}q^5q_2^2=1\end{cases}\Rightarrow\begin{cases}q_1^4q=1\\q_2=-1\\q^2=1\\q\neq 1\end{cases}\Rightarrow\begin{cases}q_1^4q=1\\q_2=-1\\q=-1\end{cases}$$

$\Rightarrow \overset{t}{\circ}\overset{-1}{\text{———}}\overset{-1}{\circ}$ $,t^4=-1,t^2\neq\pm1,t^3\neq\pm1.$

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1=\begin{pmatrix}t&1\\-1&-1\end{pmatrix},\mathcal{C}^{\chi_1}=\begin{pmatrix}2&-4\\-1&1\end{pmatrix},\sigma_1^{\chi_1}=\begin{pmatrix}-1&4\\0&1\end{pmatrix};$

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} t & 1 \\ -1 & -1 \end{pmatrix} = \chi_1, \mathcal{C}^{\chi_2} = \begin{pmatrix} 2 & -4 \\ -1 & 1 \end{pmatrix}, \sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix};$

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} t & 1 \\ -1 & -1 \end{pmatrix} = \chi_1.$

Then the diagram of bicharacters is given as follows:

$$^{\sigma_2} \circlearrowright \underset{\chi_1}{\begin{pmatrix} t & 1 \\ -1 & -1 \end{pmatrix}} \circlearrowleft^{\sigma_1}.$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$^{\sigma_2} \circlearrowright \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1}.$$

$$(11)\colon \begin{cases} ②: \ q_1^4 q = 1 \\ ③: \ q_2 = -1 \\ ⑥: \ q_1^8 q^3 q_2 = 1 \\ ⑦: \ (q_1^9 q^3 q_2)^5 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 q = 1 \\ q_2 = -1 \\ q = -1 \\ q_1^5 q^5 = -1 \end{cases} \Rightarrow \begin{cases} q_1^4 q = 1 \\ q_2 = -1 \\ q = -1 \\ q_1^5 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} q_1 = -1 \\ q = -1 \\ q_2 = -1 \\ q_1^i \neq 1, (i = 2, 3, 4) \end{cases} \Rightarrow \lightning.$$

$$(12)\colon \begin{cases} ②: \ q_1^4 q = 1 \\ ③: \ q_2 = -1 \\ ⑥: \ q_1^8 q^3 q_2 = 1 \\ ⑧: \ q_1^{12} q^5 q_2^2 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 q = 1 \\ q_2 = -1 \\ q = -1 \end{cases} \Rightarrow \overset{t}{\circ}\overset{-1}{\text{———}}\overset{-1}{\circ},$$

$t^4 = -1, t^2 \neq \pm 1, t^3 \neq \pm 1.$

For this bicharacter, it is as same as that in (10). The Cartan graph associated to χ_1 is

$$^{\sigma_2} \circlearrowright \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1}.$$

$$(13)\colon \begin{cases} ②: \ q_1^4 q = 1 \\ ④: \ q_2 q = 1 \\ ⑤: \ q_1^{16} q^4 q_2 = -1 \\ ⑦: \ (q_1^9 q^3 q_2)^5 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 q = 1 \\ q_2 q = 1 \\ q_2 = -1 \\ q_1^5 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_1^5 = 1 \\ q = -1 \\ q_2 = -1 \end{cases}$$

$$\Rightarrow \begin{cases} q_1 = 1 \\ q_2 = -1 \\ q = -1 \\ q_1^i \neq 1, (i = 2, 3, 4) \end{cases} \Rightarrow \lightning.$$

$$(14)\colon \begin{cases} ②: \ q_1^4 q = 1 \\ ④: \ q_2 q = 1 \\ ⑤: \ q_1^{16} q^4 q_2 = -1 \\ ⑧: \ q_1^{12} q^5 q_2^2 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 q = 1 \\ q_2 q = 1 \\ q_2 = -1 \end{cases} \Rightarrow \begin{cases} q_1^4 = -1 \\ q_2 = -1 \\ q = -1 \end{cases}$$

$$\Rightarrow \overset{t}{\circ}\overset{-1}{\text{———}}\overset{-1}{\circ}, t^4 = -1, t^2 \neq \pm 1, t^3 \neq \pm 1.$$

For this bicharacter, it is as same as that in (10). The Cartan graph associated to χ_1 is

$$^{\sigma_2}\circlearrowright \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1} .$$

$$(15): \begin{cases} ②: \ q_1^4 q = 1 \\ ④: \ q_2 q = 1 \\ ⑥: \ q_1^8 q^3 q_2 = 1 \\ ⑦: \ (q_1^9 q^3 q_2)^5 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 q = 1 \\ q_2 q = 1 \\ q^5 = 1 \end{cases} \Rightarrow \begin{cases} q_1 = q \\ q_2 q = 1 \\ q^5 = 1 \end{cases}$$

$\Rightarrow \overset{t}{\circ}\!\overset{t}{\text{———}}\!\overset{t^{-1}}{\circ}$, $t^5 = 1, t^2 \neq 1$.

For this bicharacter, it is as same as that in (8). The Cartan graph associated to χ_1 is

$$^{\sigma_2}\circlearrowright \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1} .$$

$$(16): \begin{cases} ②: \ q_1^4 q = 1 \\ ④: \ q_2 q = 1 \\ ⑥: \ q_1^8 q^3 q_2 = 1 \\ ⑧: \ q_1^{12} q^5 q_2^2 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 q = 1 \\ q_2 q = 1 \end{cases} \Rightarrow \overset{t}{\circ}\!\overset{t^{-4}}{\text{———}}\!\overset{t^4}{\circ} , t^i \neq 1, i = 2, 3, 4.$$

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1 = \begin{pmatrix} t & 1 \\ t^{-4} & t^4 \end{pmatrix}, \mathcal{C}^{\chi_1} = \begin{pmatrix} 2 & -4 \\ -1 & 1 \end{pmatrix}, \sigma_1^{\chi_1} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix};$

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} t & 1 \\ t^{-4} & t^4 \end{pmatrix} = \chi_1, \mathcal{C}^{\chi_2} = \begin{pmatrix} 2 & -4 \\ -1 & 1 \end{pmatrix}, \sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix};$

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} t & 1 \\ t^{-4} & t^4 \end{pmatrix} = \chi_1.$

Then the diagram of bicharacters is given as follows:

$$^{\sigma_2}\circlearrowright \underset{\chi_1}{\begin{pmatrix} t & 1 \\ t^{-4} & t^4 \end{pmatrix}} \circlearrowleft^{\sigma_1} .$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$^{\sigma_2}\circlearrowright \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1} .$$

From the above 16 cases, it yields the following bicharacters whose corresponding characteristic sequences are the periods of affine characteristic sequences.

(1) $\overset{\zeta^{-4}}{\circ}\!\overset{\zeta}{\text{———}}\!\overset{\zeta^{-1}}{\circ}$, $\zeta \in \mu_{10}$;

(2) $\overset{\zeta}{\circ}\!\overset{\zeta}{\text{———}}\!\overset{\zeta^{-1}}{\circ}$, $\zeta \in \mu_5$;

(3) $\overset{\zeta}{\circ}\!\overset{-1}{\text{———}}\!\overset{-1}{\circ}$, $\zeta \in \mu_8$;

(4) $\overset{q}{\circ}\!\!\overset{q^{-4}}{\text{———————}}\!\!\overset{q^{-4}}{\circ}$, $q \in \mathbb{C}\backslash\{1 \cup \mu_2 \cup \mu_3 \cup \mu_4\}$, it is a one-dimensional type bicharacter;

where μ_m denotes the set of primitive mth roots of unity, $\mathbb{C}$ is the set of complex numbers. All of the four bicharacters above have the same characteristic sequences: $[1, 4]$.

4.2. Computations of periods of affine characteristic sequences. In this part, we compute the bicharacters and Cartan graphs corresponding to the periods of the affine characteristic sequences:$(1, 3, 1, 6)$, $(1, 3, 2, 3)$ and $(1, 3, 3, 1, 4)$.

(1). For the affine characteristic sequences whose period is $[1, 3, 1, 6]$:

$$\cdots \text{—}\bullet\overset{e}{\text{——}}\bullet\overset{1}{\text{——}}\bullet\overset{3}{\text{——}}\bullet\overset{1}{\text{——}}\bullet\overset{6}{\text{——}}\bullet\overset{f}{\text{——}}\bullet\text{—}\cdots,$$

($1 \leq e, f \in \mathbb{N}^+$), the associated Cartan graphs are

$$\cdots\xrightarrow{\sigma_1}\begin{pmatrix} 2 & -e \\ -1 & 2 \end{pmatrix}\xrightarrow{\sigma_2}\underset{\chi_1}{\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}}\xrightarrow{\sigma_1}\underset{\chi_2}{\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}}\xrightarrow{\sigma_2}\underset{\chi_3}{\begin{pmatrix} 2 & -6 \\ -1 & 2 \end{pmatrix}}\xrightarrow{\sigma_1}\begin{pmatrix} 2 & -6 \\ -f & 2 \end{pmatrix}\xrightarrow{\sigma_2}\cdots$$

Assume that: $\chi_1 = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} q_1 & 1 \\ q & q_2 \end{pmatrix}$, where $q_1 = q_{11}$, $q_2 = q_{22}$, $q = q_{12} \cdot q_{21}$.

$C^{\chi_1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$, $\sigma_1^{\chi_1} = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}$, $\sigma_2^{\chi_1} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$.

By formula (2.2, p. 15), we get the set of constraints of χ_1:

$$I): \begin{cases} q \neq 1 \\ q_1^i \neq 1, (i = 2, 3) \\ q_1^j \cdot q \neq 1, (j = 1, 2) \\ q_1^4 = 1 \text{ or } q_1^3 \cdot q = 1 \\ q_2 = -1 \text{ or } q_2 \cdot q = 1 \end{cases}$$

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} q_1 & 1 \\ q_1^{-6}q^{-1} & q_1^9q^3q_2 \end{pmatrix}$,

$C^{\chi_2} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$, $\sigma_1^{\chi_2} = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}$, $\sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$.

By formula (2.2, p. 15), we get the set of constraints of χ_2:

$$II): \begin{cases} q_1^{-6}q^{-1} \neq 1 \\ q_1^i \neq 1, (i = 2, 3) \\ q_1^j \cdot (q_1^{-6}q^{-1}) \neq 1, (j = 1, 2) \\ q_1^4 = 1 \text{ or } q_1^3 \cdot (q_1^{-6}q^{-1}) = 1 \\ q_1^9q^3q_2 = -1 \text{ or } q_1^9q^3q_2 \cdot q_1^{-6}q^{-1} = 1 \end{cases}$$

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} q_1^4q^2q_2 & 1 \\ q_1^{-12}q^{-5}q_2^{-2} & q_1^9q^3q_2 \end{pmatrix}$,

$C^{\chi_3} = \begin{pmatrix} 2 & -6 \\ -1 & 2 \end{pmatrix}$, $\sigma_1^{\chi_3} = \begin{pmatrix} -1 & 6 \\ 0 & 1 \end{pmatrix}$, $\sigma_2^{\chi_3} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$.

By using the formula (2.2, p. 15), we can get the set of constraints of χ_3:

$$III):\begin{cases} q_1^{-12}q^{-5}q_2^{-2}\neq 1\\ (q_1^4q^2q_2)^i\neq 1,(i=2,3,\dots,6)\\ (q_1^4q^2q_2)^j\cdot q_1^{-12}q^{-5}q_2^{-2}\neq 1,(j=1,2,\dots,5)\\ (q_1^4q^2q_2)^7=1 \text{ or } (q_1^4q^2q_2)^6\cdot q_1^{-12}q^{-5}q_2^{-2}=1\\ q_1^9q^3q_2=-1 \text{ or } q_1^9q^3q_2\cdot q_1^{-12}q^{-5}q_2^{-2}=1\end{cases}$$

From $I)$, $II)$ and $III)$ above, the set of constraints of bicharacters is fixed partly. These equations and inequalities in this set of constraints are enough to decide the possible bicharacters. Here we give only the set of equations. When we need some constraints of inequalities in the following, we will choose them from $I), II)$ and $III)$ directly.

$$\begin{cases} q_2=-1\\ q_1^4=1 \text{ ① or } q_1^3q=1 \text{ ②}\\ q_1^9q^3q_2=-1 \text{ ③ or } q_1^3q^2q_2=1 \text{ ④}\\ (q_1^4q^2q_2)^7=1 \text{ ⑤ or } q_1^{12}q^7q_2^4=1 \text{ ⑥}\end{cases}$$

For the set of equations above, there are 8 different cases:
(①,③,⑤), (①,③,⑥),(①,④,⑤), (①,④,⑥),(②,③,⑤), (②,③,⑥),
(②,④,⑤), (②,④,⑥).

$$(1):\begin{cases} q_2=-1\\ ①: q_1^4=1\\ ③: q_1^9q^3q_2=-1\\ ⑤: (q_1^4q^2q_2)^7=1\end{cases}\Rightarrow\begin{cases} q_1^4=1\\ q_1q^3=1\\ q^{14}=-1\\ q_2=-1\end{cases}\Rightarrow\begin{cases} q_1^4=1\\ q_1=q\\ q^2=-1\\ q_2=-1\\ (q_1^4q^2q_2)^i\neq 1,(i=2,\dots,6)\end{cases}$$

$\Rightarrow \lightning$.

$$(2):\begin{cases} q_2=-1\\ ①: q_1^4=1\\ ③: q_1^9q^3q_2=-1\\ ⑥: q_1^{12}q^7q_2^4=1\end{cases}\Rightarrow\begin{cases} q_1^4=1\\ q_1q^3=1\\ q^7=1\\ q_2=-1\end{cases}\Rightarrow\begin{cases} q_1^4=1\\ q_1^7=1\\ q_2=-1\\ q^7=1\end{cases}$$

$$\Rightarrow\begin{cases} q_1=1\\ q_2=-1\\ q^7=1\\ q_1^i\neq 1,(i=2,3)\end{cases}\Rightarrow \lightning.$$

$$(3):\begin{cases} q_2=-1\\ ①: q_1^4=1\\ ④: q_1^3q^2q_2=1\\ ⑤: (q_1^4q^2q_2)^7=1\end{cases}\Rightarrow\begin{cases} q_1^4=1\\ q_1^3q^2=-1\\ q_1^7=1\\ q_2=-1\end{cases}\Rightarrow\begin{cases} q_1=1\\ q^2=-1\\ q_2=-1\\ q_1^i\neq 1,(i=2,3)\end{cases}\Rightarrow \lightning.$$

$$(4):\begin{cases} q_2=-1\\ ①: q_1^4=1\\ ④: q_1^3q^2q_2=1\\ ⑥: q_1^{12}q^7q_2^4=1\end{cases}\Rightarrow\begin{cases} q_1^4=1\\ q_2=-1\\ q=1\\ q\neq 1\end{cases}\Rightarrow \lightning.$$

$$(5):\ \begin{cases} q_2=-1 \\ ②:\ q_1^3q=1 \\ ③:\ q_1^9q^3q_2=-1 \\ ⑤:\ (q_1^4q^2q_2)^7=1 \end{cases} \Rightarrow \begin{cases} q_1^3q=1 \\ q_2=-1 \\ q_1^7q^7=-1 \end{cases} \Rightarrow \begin{cases} q_1^{14}=-1 \\ q_1^3q=1 \\ q_2=-1 \end{cases}$$

$\Rightarrow \overset{t}{\circ}\overset{t^{-3}}{\rule[2pt]{6em}{0.4pt}}\overset{-1}{\circ}$ $, t^{14}=-1, t^2\neq\pm1, t^3\neq\pm1, t^i\neq 1, (i=4,5,7,8)$.

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1=\begin{pmatrix} t & 1 \\ t^{-3} & -1 \end{pmatrix}, \mathcal{C}^{\chi_1}=\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_1}=\begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix};$

$\chi_2=\sigma_1^{\chi_1}*\chi_1=\begin{pmatrix} t & 1 \\ t^{-3} & -1 \end{pmatrix}, \mathcal{C}^{\chi_2}=\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \sigma_2^{\chi_2}=\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix};$

$\chi_3=\sigma_2^{\chi_2}*\chi_2=\begin{pmatrix} -t^{-2} & 1 \\ t^3 & -1 \end{pmatrix}, \mathcal{C}^{\chi_3}=\begin{pmatrix} 2 & -6 \\ -2 & 2 \end{pmatrix}, \sigma_1^{\chi_3}=\begin{pmatrix} -1 & 6 \\ 0 & 1 \end{pmatrix};$

$\chi_4=\sigma_1^{\chi_3}*\chi_3=\begin{pmatrix} -t^{-2} & 1 \\ -t^7 & -t^2 \end{pmatrix}, \mathcal{C}^{\chi_4}=\begin{pmatrix} 2 & -6 \\ -6 & 2 \end{pmatrix}, \sigma_2^{\chi_4}=\begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix};$

$\chi_5=\sigma_2^{\chi_4}*\chi_4=\begin{pmatrix} -1 & 1 \\ -t^{-3} & -t^2 \end{pmatrix}, \mathcal{C}^{\chi_5}=\begin{pmatrix} 2 & -1 \\ -6 & 2 \end{pmatrix}, \sigma_1^{\chi_5}=\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix};$

$\chi_6=\sigma_1^{\chi_5}*\chi_5=\begin{pmatrix} -1 & 1 \\ -t^3 & -t^{-1} \end{pmatrix}, \mathcal{C}^{\chi_6}=\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \sigma_2^{\chi_6}=\begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix};$

$\chi_7=\sigma_2^{\chi_6}*\chi_6=\begin{pmatrix} -1 & 1 \\ -t^3 & -t^{-1} \end{pmatrix}, \mathcal{C}^{\chi_7}=\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \sigma_1^{\chi_7}=\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix};$

$\chi_8=\sigma_1^{\chi_7}*\chi_7=\begin{pmatrix} -1 & 1 \\ -t^{-3} & -t^2 \end{pmatrix}, \mathcal{C}^{\chi_8}=\begin{pmatrix} 2 & -1 \\ -6 & 2 \end{pmatrix}, \sigma_2^{\chi_8}=\begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix};$

$\chi_9=\sigma_2^{\chi_8}*\chi_8=\begin{pmatrix} -t^{-2} & 1 \\ -t^7 & -t^2 \end{pmatrix}, \mathcal{C}^{\chi_9}=\begin{pmatrix} 2 & -6 \\ -6 & 2 \end{pmatrix}, \sigma_1^{\chi_9}=\begin{pmatrix} -1 & 6 \\ 0 & 1 \end{pmatrix};$

$\chi_{10}=\sigma_1^{\chi_9}*\chi_9=\begin{pmatrix} -t^{-2} & 1 \\ t^3 & -1 \end{pmatrix}, \mathcal{C}^{\chi_{10}}=\begin{pmatrix} 2 & -6 \\ -1 & 2 \end{pmatrix}, \sigma_2^{\chi_{10}}=\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix};$

$\chi_{11}=\sigma_2^{\chi_{10}}*\chi_{10}=\begin{pmatrix} t & 1 \\ t^{-3} & -1 \end{pmatrix}=\chi_1.$

Then the diagram of bicharacters is given as follows:

$$\sigma_1 \circlearrowright \underset{\chi_1}{\begin{pmatrix} t & 1 \\ t^{-3} & -1 \end{pmatrix}} \overset{\sigma_2}{\rule[2pt]{2em}{0.4pt}} \begin{pmatrix} -t^{-2} & 1 \\ t^3 & -1 \end{pmatrix} \overset{\sigma_1}{\rule[2pt]{2em}{0.4pt}} \begin{pmatrix} -t^{-2} & 1 \\ -t^7 & -t^2 \end{pmatrix} \overset{\sigma_2}{\rule[2pt]{2em}{0.4pt}} \begin{pmatrix} -1 & 1 \\ -t^{-3} & -t^2 \end{pmatrix} \overset{\sigma_1}{\rule[2pt]{2em}{0.4pt}} \begin{pmatrix} -1 & 1 \\ -t^3 & -t^{-1} \end{pmatrix} \circlearrowleft^{\sigma_2}.$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$\sigma_1 \circlearrowright \underset{\chi_1}{\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}} \overset{\sigma_2}{\rule[2pt]{2em}{0.4pt}} \begin{pmatrix} 2 & -6 \\ -1 & 2 \end{pmatrix} \overset{\sigma_1}{\rule[2pt]{2em}{0.4pt}} \begin{pmatrix} 2 & -6 \\ -6 & 2 \end{pmatrix} \overset{\sigma_2}{\rule[2pt]{2em}{0.4pt}} \begin{pmatrix} 2 & -1 \\ -6 & 2 \end{pmatrix} \overset{\sigma_1}{\rule[2pt]{2em}{0.4pt}} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \circlearrowleft^{\sigma_2}.$$

$$(6): \begin{cases} q_2 = -1 \\ ②: \ q_1^3 q = 1 \\ ③: \ q_1^9 q^3 q_2 = -1 \\ ⑥: \ q_1^{12} q^7 q_2^4 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q^3 = 1 \end{cases} \Rightarrow \overset{t}{\circ}\!\!\xrightarrow{\quad t^{-3} \quad}\!\!\overset{-1}{\circ},$$

$t^9 = 1, t^2 \neq \pm 1, t^3 \neq \pm 1, t^4 \neq 1$.

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1 = \begin{pmatrix} t & 1 \\ t^{-3} & -1 \end{pmatrix}, \mathcal{C}^{\chi_1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_1} = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix};$

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} t & 1 \\ t^{-3} & -1 \end{pmatrix}, \mathcal{C}^{\chi_2} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix};$

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} -t^{-2} & 1 \\ t^3 & -1 \end{pmatrix}, \mathcal{C}^{\chi_3} = \begin{pmatrix} 2 & -6 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_3} = \begin{pmatrix} -1 & 6 \\ 0 & 1 \end{pmatrix};$

$\chi_4 = \sigma_1^{\chi_3} * \chi_3 = \begin{pmatrix} -t^{-2} & 1 \\ t^3 & -1 \end{pmatrix}, \mathcal{C}^{\chi_4} = \begin{pmatrix} 2 & -6 \\ -1 & 2 \end{pmatrix}, \sigma_2^{\chi_4} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix};$

$\chi_5 = \sigma_2^{\chi_4} * \chi_4 = \begin{pmatrix} t & 1 \\ t^{-3} & -1 \end{pmatrix} = \chi_1.$

Then the diagram of bicharacters is given as follows:

$$^{\sigma_1}\circlearrowright \underset{\chi_1}{\begin{pmatrix} t & 1 \\ t^{-3} & -1 \end{pmatrix}} \overset{\sigma_2}{\text{——}} \begin{pmatrix} -t^{-2} & 1 \\ t^3 & -1 \end{pmatrix} \circlearrowleft^{\sigma_1}.$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1. The associated Cartan graph is

$$^{\sigma_1}\circlearrowright \underset{\chi_1}{\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}} \overset{\sigma_2}{\text{——}} \begin{pmatrix} 2 & -6 \\ -1 & 2 \end{pmatrix} \circlearrowleft^{\sigma_1}$$

$$(7): \begin{cases} q_2 = -1 \\ ②: \ q_1^3 q = 1 \\ ④: \ q_1^3 q^2 q_2 = 1 \\ ⑤: \ (q_1^4 q^2 q_2)^7 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q_2 q = 1 \\ (q_1^4 q^2 q_2)^7 = 1 \\ q_2 q \neq 1 \end{cases} \Rightarrow \lightning.$$

$$(8): \begin{cases} q_2 = -1 \\ ②: \ q_1^3 q = 1 \\ ④: \ q_1^3 q^2 q_2 = 1 \\ ⑥: \ q_1^{12} q^7 q_2^4 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q_2 q = 1 \\ q_1^{12} q^7 q_2^4 = 1 \\ q_2 q \neq 1 \end{cases} \Rightarrow \lightning.$$

From the above 8 cases, it yields one bicharacter whose corresponding characteristic sequence is the period of affine characteristic sequence.

(1) $\overset{\zeta^2}{\circ}\!\!\xrightarrow{\quad \zeta^{-6} \quad}\!\!\overset{-1}{\circ}$, $\zeta \in \mu_{18}$, where μ_m denotes the set of primitive mth roots of unity.

This bicharacter has the characteristic sequence: $[1, 3, 1, 6]$.

2. For the affine characteristic sequence whose period is $[1, 3, 2, 3]$:

$$\cdots \bullet \overset{e}{\text{---}} \bullet \overset{1}{\text{---}} \bullet \overset{3}{\text{---}} \bullet \overset{2}{\text{---}} \bullet \overset{3}{\text{---}} \bullet \overset{f}{\text{---}} \bullet \cdots$$

($1 \leq e, f \in \mathbb{N}^+$), the associated Cartan graphs are

$$\cdots \xrightarrow{\sigma_1} \begin{pmatrix} 2 & -e \\ -1 & 2 \end{pmatrix} \xrightarrow{\sigma_2} \underset{\chi_1}{\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}} \xrightarrow{\sigma_1} \underset{\chi_2}{\begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}} \xrightarrow{\sigma_2} \underset{\chi_3}{\begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}} \xrightarrow{\sigma_1} \begin{pmatrix} 2 & -3 \\ -f & 2 \end{pmatrix} \xrightarrow{\sigma_2} \cdots$$

Assume that: $\chi_1 = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} q_1 & 1 \\ q & q_2 \end{pmatrix}$, where $q_1 = q_{11}$, $q_2 = q_{22}$, $q = q_{12} \cdot q_{21}$.

$C^{\chi_1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$, $\sigma_1^{\chi_1} = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}$, $\sigma_2^{\chi_1} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$.

By formula (2.2, p. 15), we get the set of constraints of χ_1:

$$I): \begin{cases} q \neq 1 \\ q_1^i \neq 1, (i = 2, 3) \\ q_1^j \cdot q \neq 1, (j = 1, 2) \\ q_1^4 = 1 \text{ or } q_1^3 \cdot q = 1 \\ q_2 = -1 \text{ or } q_2 \cdot q = 1 \end{cases}$$

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} q_1 & 1 \\ q_1^{-6} q^{-1} & q_1^9 q^3 q_2 \end{pmatrix}$,

$C^{\chi_2} = \begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}$, $\sigma_1^{\chi_2} = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}$, $\sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$.

By formula (2.2, p. 15), we get the set of constraints of χ_2:

$$II): \begin{cases} q_1^{-6} q^{-1} \neq 1 \\ q_1^i \neq 1, (i = 2, 3) \\ q_1^j \cdot q_1^{-6} q^{-1} \neq 1, (j = 1, 2) \\ q_1^4 = 1 \text{ or } q_1^3 \cdot q_1^{-6} q^{-1} = 1 \\ q_1^9 q^3 q_2 \neq \pm 1 \\ q_1^9 q^3 q_2 \cdot q_1^{-6} q^{-1} \neq 1 \\ (q_1^9 q^3 q_2)^3 = 1 \text{ or } (q_1^9 q^3 q_2)^2 \cdot q_1^{-6} q^{-1} = 1 \end{cases}$$

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} q_1^{25} q^{10} q_2^4 & 1 \\ q_1^{-30} q^{-11} q_2^{-4} & q_1^9 q^3 q_2 \end{pmatrix}$,

$C^{\chi_3} = \begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}$, $\sigma_1^{\chi_3} = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}$, $\sigma_2^{\chi_3} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$.

By using the formula (2.2, p. 15), we can get the set of constraints of χ_3:

$$III): \begin{cases} q_1^{-30} q^{-11} q_2^{-4} \neq 1 \\ (q_1^{25} q^{10} q_2^4)^i \neq 1, (i = 2, 3) \\ (q_1^{25} q^{10} q_2^4)^j \cdot q_1^{-30} q^{-11} q_2^{-4} \neq 1, (j = 1, 2) \\ (q_1^{25} q^{10} q_2^4)^4 = 1 \text{ or } (q_1^{25} q^{10} q_2^4)^3 \cdot q_1^{-30} q^{-11} q_2^{-4} = 1 \\ q_1^9 q^3 q_2 \neq \pm 1 \\ q_1^9 q^3 q_2 \cdot q_1^{-30} q^{-11} q_2^{-4} \neq 1 \\ (q_1^9 q^3 q_2)^3 = 1 \text{ or } (q_1^9 q^3 q_2)^2 \cdot q_1^{-30} q^{-11} q_2^{-4} = 1 \end{cases}$$

From I), II) and III) above, the set of constraints of bicharacters is fixed partly. These equations and inequalities in this set of constraints are enough to decide the possible bicharacters. Here we give only the set of equations. When we need some constraints of inequalities in the following, we will choose them from $I), II)$ and $III)$ directly.

$$\begin{cases} q_1^4=1 \text{ ① or } q_1^3q=1 \text{ ②}\\ q_2=-1 \text{ ③ or } q_2q=1 \text{ ④}\\ (q_1^9q^3q_2)^3=1 \text{ ⑤ or } q_1^{12}q^5q_2^2=1 \text{ ⑥}\\ (q_1^{25}q^{10}q_2^4)^4=1 \text{ ⑦ or } q_1^{45}q^{19}q_2^8=1 \text{ ⑧}\end{cases}$$

For the set of equations above, there are 16 different cases:
(①,③,⑤,⑦), (①,③,⑤,⑧), (①,③,⑥,⑦), (①,③,⑥,⑧), (①,④,⑤,⑦), (①,④,⑤,⑧), (①,④,⑥,⑦), (①,④,⑥,⑧), (②,③,⑤,⑦), (②,③,⑤,⑧), (②,③,⑥,⑦), (②,③,⑥,⑧), (②,④,⑤,⑦), (②,④,⑤,⑧), (②,④,⑥,⑦), (②,④,⑥,⑧).

$$(1): \begin{cases} \text{①}: q_1^4=1\\ \text{③}: q_2=-1\\ \text{⑤}: (q_1^9q^3q_2)^3=1\\ \text{⑦}: (q_1^{25}q^{10}q_2^4)^4=1\end{cases} \Rightarrow \begin{cases} q_1^4=1\\ q_2=-1\\ q_1^{-1}q^9=-1\\ q_1^8q^{-4}=1\end{cases} \Rightarrow \begin{cases} q_1^4=1\\ q_2=-1\\ q_1=-q\\ q_1^9q^3q_2\neq\pm1\end{cases} \Rightarrow \lightning.$$

$$(2): \begin{cases} \text{①}: q_1^4=1\\ \text{③}: q_2=-1\\ \text{⑤}: (q_1^9q^3q_2)^3=1\\ \text{⑧}: q_1^{45}q^{19}q_2^8=1\end{cases} \Rightarrow \begin{cases} q_1^4=1\\ q_2=-1\\ q_1^{-1}q^9=-1\\ q_1q^{19}=1\end{cases} \Rightarrow \begin{cases} q_1^4=1\\ q_2=-1\\ q^{28}=1\\ q_1q^{19}=1\end{cases}$$

$$\Rightarrow \begin{cases} q_1^4=1\\ q_2=-1\\ q_1^3q=1\\ q_1^9q^3q_2\neq\pm1\end{cases} \Rightarrow \lightning.$$

$$(3): \begin{cases} \text{①}: q_1^4=1\\ \text{③}: q_2=-1\\ \text{⑥}: q_1^{12}q^5q_2^2=1\\ \text{⑦}: (q_1^{25}q^{10}q_2^4)^4=1\end{cases} \Rightarrow \begin{cases} q_1^4=1\\ q_2=-1\\ q^5=1\end{cases} \Rightarrow \overset{t}{\circ}\overset{r}{\text{——————}}\overset{-1}{\circ},$$

$t^4=1, r^5=1, t\neq\pm1, r\neq\pm1, tr\neq1, tr^2\neq1, tr^3\neq\pm1.$

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1=\begin{pmatrix} t & 1\\ r & -1\end{pmatrix}, \mathcal{C}^{\chi_1}=\begin{pmatrix} 2 & -3\\ -1 & 2\end{pmatrix}, \sigma_1^{\chi_1}=\begin{pmatrix} -1 & 3\\ 0 & 1\end{pmatrix};$

$\chi_2=\sigma_1^{\chi_1}*\chi_1=\begin{pmatrix} t & 1\\ t^{-2}r^{-1} & -tr^3\end{pmatrix},$

$\mathcal{C}^{\chi_2}=\begin{pmatrix} 2 & -3\\ -2 & 2\end{pmatrix}, \sigma_2^{\chi_2}=\begin{pmatrix} 1 & 0\\ 2 & -1\end{pmatrix};$

$\chi_3=\sigma_2^{\chi_2}*\chi_2=\begin{pmatrix} t & 1\\ t^{-2}r^{-1} & -tr^3\end{pmatrix},$

$\mathcal{C}^{\chi_3}=\begin{pmatrix} 2 & -3\\ -2 & 2\end{pmatrix}, \sigma_1^{\chi_3}=\begin{pmatrix} -1 & 3\\ 0 & 1\end{pmatrix};$

$\chi_4=\sigma_1^{\chi_3}*\chi_3=\begin{pmatrix} t & 1\\ r & -1\end{pmatrix}, \mathcal{C}^{\chi_4}=\begin{pmatrix} 2 & -3\\ -1 & 2\end{pmatrix}, \sigma_2^{\chi_4}=\begin{pmatrix} 1 & 0\\ 1 & -1\end{pmatrix};$

$\chi_5 = \sigma_2^{\chi_4} * \chi_4 = \begin{pmatrix} -tr & 1 \\ r^{-1} & -1 \end{pmatrix}, \mathcal{C}^{\chi_5} = \begin{pmatrix} 2 & -16 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_5} = \begin{pmatrix} -1 & 16 \\ 0 & 1 \end{pmatrix};$

$\chi_6 = \sigma_1^{\chi_5} * \chi_5 = \begin{pmatrix} -tr & 1 \\ r^{-1} & -1 \end{pmatrix}, \mathcal{C}^{\chi_6} = \begin{pmatrix} 2 & -16 \\ -1 & 2 \end{pmatrix}, \sigma_2^{\chi_6} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix};$

$\chi_7 = \sigma_2^{\chi_6} * \chi_6 = \begin{pmatrix} t & 1 \\ r & -1 \end{pmatrix} = \chi_1.$

Then the diagram of bicharacters is given as follows:

$$^{\sigma_1} \circlearrowright \begin{pmatrix} -tr & 1 \\ r^{-1} & -1 \end{pmatrix} \overset{\sigma_2}{\text{———}} \underset{\chi_1}{\begin{pmatrix} t & 1 \\ r & -1 \end{pmatrix}} \overset{\sigma_1}{\text{———}} \begin{pmatrix} t & 1 \\ t^{-2}r^{-1} & -tr^3 \end{pmatrix} \circlearrowleft^{\sigma_2}.$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$^{\sigma_1} \circlearrowright \begin{pmatrix} 2 & -16 \\ -1 & 2 \end{pmatrix} \overset{\sigma_2}{\text{———}} \underset{\chi_1}{\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}} \overset{\sigma_1}{\text{———}} \begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix} \circlearrowleft^{\sigma_2}.$$

$$(4): \begin{cases} ①: q_1^4 = 1 \\ ③: q_2 = -1 \\ ⑥: q_1^{12}q^5q_2^2 = 1 \\ ⑧: q_1^{45}q^{19}q_2^8 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q^5 = 1 \\ q_1q^{19} = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q^5 = 1 \\ q_1 = q \end{cases}$$

$$\Rightarrow \begin{cases} q_1 = 1 \\ q_2 = -1 \\ q = 1 \\ q \neq 1 \end{cases} \Rightarrow \lightning.$$

$$(5): \begin{cases} ①: q_1^4 = 1 \\ ④: q_2q = 1 \\ ⑤: (q_1^9q^3q_2)^3 = 1 \\ ⑦: (q_1^{25}q^{10}q_2^4)^4 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2q = 1 \\ q_1^{-1}q^6 = 1 \\ q^{24} = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2q = 1 \\ q_1 = q^6 \\ (q_1^{25}q^{10}q_2^4)^i \neq 1, (i = 2, 3) \end{cases}$$

$\Rightarrow \lightning.$

$$(6): \begin{cases} ①: q_1^4 = 1 \\ ④: q_2q = 1 \\ ⑤: (q_1^9q^3q_2)^3 = 1 \\ ⑧: q_1^{45}q^{19}q_2^8 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2q = 1 \\ q_1^{-1}q^6 = 1 \\ q_1q^{11} = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2q = 1 \\ q_1^{-1}q^6 = 1 \\ q^{17} = 1 \end{cases}$$

$$\Rightarrow \begin{cases} q_1^4 = 1 \\ q_2q = 1 \\ q_1^{-1}q^6 = 1 \\ q_1q = 1 \\ q_1q \neq 1 \end{cases} \Rightarrow \lightning.$$

$$(7): \begin{cases} ①: q_1^4 = 1 \\ ④: q_2q = 1 \\ ⑥: q_1^{12}q^5q_2^2 = 1 \\ ⑦: (q_1^{25}q^{10}q_2^4)^4 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2q = 1 \\ q^3 = 1 \end{cases} \Rightarrow \overset{t}{\circ} \overset{s^{-1}}{\text{————}} \overset{s}{\circ},$$

$t^4 = 1, s^3 = 1, t \neq \pm 1, s \neq \pm 1, ts \neq \pm 1, ts^{-1} \neq 1.$

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1 = \begin{pmatrix} t & 1 \\ s^{-1} & s \end{pmatrix}, \mathcal{C}^{\chi_1} = \begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}, \sigma_1^{\chi_1} = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix};$

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} t & 1 \\ t^{-2}s & ts \end{pmatrix}, \mathcal{C}^{\chi_2} = \begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}, \sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix};$

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} t & 1 \\ t^{-2}s & ts \end{pmatrix}, \mathcal{C}^{\chi_3} = \begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}, \sigma_1^{\chi_3} = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix};$

$\chi_4 = \sigma_1^{\chi_3} * \chi_3 = \begin{pmatrix} t & 1 \\ s^{-1} & s \end{pmatrix}, \mathcal{C}^{\chi_4} = \begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}, \sigma_2^{\chi_4} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix};$

$\chi_5 = \sigma_2^{\chi_4} * \chi_4 = \begin{pmatrix} t & 1 \\ s^{-1} & s \end{pmatrix} = \chi_1.$

Then the diagram of bicharacters is given as follows:

$$^{\sigma_2} \circlearrowright \underset{\chi_1}{\begin{pmatrix} t & 1 \\ s^{-1} & s \end{pmatrix}} \overset{\sigma_1}{\text{———}} \begin{pmatrix} t & 1 \\ t^{-2}s & ts \end{pmatrix} \circlearrowleft^{\sigma_2} .$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$^{\sigma_2} \circlearrowright \underset{\chi_1}{\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}} \overset{\sigma_1}{\text{———}} \begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix} \circlearrowleft^{\sigma_2} .$$

$$(8)\colon \begin{cases} \textcircled{1}: \ q_1^4 = 1 \\ \textcircled{4}: \ q_2 q = 1 \\ \textcircled{6}: \ q_1^{12} q^5 q_2^2 = 1 \\ \textcircled{8}: \ q_1^{45} q^{19} q_2^8 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 q = 1 \\ q^3 = 1 \\ q_1 q^{11} = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 q = 1 \\ q^3 = 1 \\ q_1 = q \\ q_1^2 \neq 1 \end{cases} \Rightarrow \lightning .$$

$$(9)\colon \begin{cases} \textcircled{2}: \ q_1^3 q = 1 \\ \textcircled{3}: \ q_2 = -1 \\ \textcircled{5}: \ (q_1^9 q^3 q_2)^3 = 1 \\ \textcircled{7}: \ (q_1^{25} q^{10} q_2^4)^4 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q_2^3 = 1 \\ q_1 q^7 = 1 \end{cases} \Rightarrow \lightning .$$

$$(10)\colon \begin{cases} \textcircled{2}: \ q_1^3 q = 1 \\ \textcircled{3}: \ q_2 = -1 \\ \textcircled{5}: \ (q_1^9 q^3 q_2)^3 = 1 \\ \textcircled{8}: \ q_1^{45} q^{19} q_2^8 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q_2^3 = 1 \\ q^4 = 1 \end{cases} \Rightarrow \lightning .$$

$$(11)\colon \begin{cases} \textcircled{2}: \ q_1^3 q = 1 \\ \textcircled{3}: \ q_2 = -1 \\ \textcircled{6}: \ q_1^{12} q^5 q_2^2 = 1 \\ \textcircled{7}: \ (q_1^{25} q^{10} q_2^4)^4 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q = 1 \\ q_1 q^7 = 1 \\ q \neq 1 \end{cases} \Rightarrow \lightning .$$

$$(12)\colon \begin{cases} \textcircled{2}: \ q_1^3 q = 1 \\ \textcircled{3}: \ q_2 = -1 \\ \textcircled{6}: \ q_1^{12} q^5 q_2^2 = 1 \\ \textcircled{8}: \ q_1^{45} q^{19} q_2^8 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q = 1 \\ q \neq 1 \end{cases} \Rightarrow \lightning .$$

$$(13)\colon \begin{cases} \textcircled{2}:\ q_1^3q=1 \\ \textcircled{4}:\ q_2q=1 \\ \textcircled{5}:\ (q_1^9q^3q_2)^3=1 \\ \textcircled{7}:\ (q_1^{25}q^{10}q_2^4)^4=1 \end{cases} \Rightarrow \begin{cases} q_1^3q=1 \\ q_2q=1 \\ (q_1^3)^3=1 \\ (q_1^7)^4=1 \end{cases} \Rightarrow \begin{cases} q_1^3q=1 \\ q_2q=1 \\ q_1=1 \\ q_1^2\neq 1 \end{cases} \Rightarrow \lightning.$$

$$(14)\colon \begin{cases} \textcircled{2}:\ q_1^3q=1 \\ \textcircled{4}:\ q_2q=1 \\ \textcircled{5}:\ (q_1^9q^3q_2)^3=1 \\ \textcircled{8}:\ q_1^{45}q^{19}q_2^8=1 \end{cases} \Rightarrow \begin{cases} q_1^3q=1 \\ q_2q=1 \\ q_1^9=1 \\ q_1^{12}=1 \end{cases} \Rightarrow \begin{cases} q_1^3q=1 \\ q_2q=1 \\ q=1 \\ q\neq 1 \end{cases} \Rightarrow \lightning.$$

$$(15)\colon \begin{cases} \textcircled{2}:\ q_1^3q=1 \\ \textcircled{4}:\ q_2q=1 \\ \textcircled{6}:\ q_1^{12}q^5q_2^2=1 \\ \textcircled{7}:\ (q_1^{25}q^{10}q_2^4)^4=1 \end{cases} \Rightarrow \begin{cases} q_1^3q=1 \\ q_2q=1 \\ q_2=1 \\ q_1q^7q_2^{16}=1 \end{cases} \Rightarrow \begin{cases} q_1=1 \\ q_2=1 \\ q=1 \\ q\neq 1 \end{cases} \Rightarrow \lightning.$$

$$(16)\colon \begin{cases} \textcircled{2}:\ q_1^3q=1 \\ \textcircled{4}:\ q_2q=1 \\ \textcircled{6}:\ q_1^{12}q^5q_2^2=1 \\ \textcircled{8}:\ q_1^{45}q^{19}q_2^8=1 \end{cases} \Rightarrow \begin{cases} q_1^3q=1 \\ q_2q=1 \\ q_2=1 \\ q_2^4=1 \end{cases} \Rightarrow \begin{cases} q_1^3=1 \\ q_2=1 \\ q=1 \\ q\neq 1 \end{cases} \Rightarrow \lightning.$$

From the above 16 cases, it yields one bicharacter whose corresponding characteristic sequence is the period of affine characteristic sequence.

(1) $\overset{\zeta^3}{\circ}\overset{\zeta^{-4}}{\text{———}}\overset{\zeta^4}{\circ}$, $\zeta \in \mu_{12}$, where μ_m denotes the set of primitive mth roots of unity.

This bicharacter has the characteristic sequence: $[1, 3, 2, 3]$.

3. For the affine characteristic sequences whose period is $[1, 3, 3, 1, 4, 1, 3, 3, 1, 4]$:

$$\cdots\text{—}\bullet\overset{e}{\text{——}}\bullet\overset{1}{\text{——}}\bullet\overset{3}{\text{——}}\bullet\overset{3}{\text{——}}\bullet\overset{1}{\text{——}}\bullet\overset{4}{\text{——}}\bullet\overset{f}{\text{——}}\bullet\text{—}\cdots$$

$(1 \le e, f \in \mathbb{N}^+)$,
the associated Cartan graphs are

$$\overset{\sigma_1}{\text{—}}\begin{pmatrix} 2 & -e \\ -1 & 2 \end{pmatrix}\overset{\sigma_2}{\text{—}}\underset{\chi_1}{\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}}\overset{\sigma_1}{\text{—}}\underset{\chi_2}{\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}}\overset{\sigma_2}{\text{—}}\underset{\chi_3}{\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}}\overset{\sigma_1}{\text{—}}\underset{\chi_4}{\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}}\overset{\sigma_2}{\text{—}}\begin{pmatrix} 2 & -f \\ -4 & 2 \end{pmatrix}\overset{\sigma_1}{\text{—}}$$

Assume that: $\chi_1 = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} q_1 & 1 \\ q & q_2 \end{pmatrix}$, where $q_1 = q_{11}$, $q_2 = q_{22}$, $q = q_{12} \cdot q_{21}$.

$C^{\chi_1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$, $\sigma_1^{\chi_1} = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}$, $\sigma_2^{\chi_1} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$.

By the formula of bicharacter (2.2, p. 15), we can get the set of constraints:

$$I):\begin{cases} q \neq 1 \\ q_1^i \neq 1, (i=2,3) \\ q_1^j \cdot q \neq 1, (j=1,2) \\ q_1^4 = 1 \text{ or } q_1^3 \cdot q = 1 \\ q_2 = -1 \text{ or } q_2 \cdot q = 1 \end{cases}$$

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} q_1 & 1 \\ q_1^{-6}q^{-1} & q_1^9q^3q_2 \end{pmatrix}$,

$$C^{\chi_2} = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix},\ \sigma_1^{\chi_2} = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix},\ \sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}.$$

By formula (2.2, p. 15), we get the set of constraints of χ_2:

$$II):\begin{cases} q_1^{-6}q^{-1} \neq 1 \\ q_1^k \neq 1, g = 2,3 \\ q_1^l \cdot q \neq 1, l = 1,2 \\ q_1^4 = 1 \text{ or } q_1^3 \cdot q_1^{-6}q^{-1} = 1 \\ (q_1^9 q^3 q_2)^g \neq 1, g = 2,3 \\ (q_1^9 q^3 q_2)^h q_1^{-6} q^{-1} \neq 1, h = 1,2 \\ (q_1^9 q^3 q_2)^4 = 1 \text{ or } (q_1^9 q^3 q_2)^3 q_1^{-6} q^{-1} = 1 \end{cases}$$

$$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} q_1^{64} q^{24} q_2^9 & 1 \\ q_1^{-48} q^{-17} q_2^{-6} & q_1^9 q^3 q_2 \end{pmatrix},$$

$$C^{\chi_3} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix},\ \sigma_1^{\chi_3} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix},\ \sigma_2^{\chi_3} = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}.$$

By using the formula (2.2, p. 15), we obtain the set of constraints of χ_3:

$$III):\begin{cases} q_1^{-48} q^{-17} q_2^{-6} \neq 1 \\ q_1^{64} q^{24} q_2^9 = -1 \text{ or } q_1^{64} q^{24} q_2^9 \cdot q_1^{-48} q^{-17} q_2^{-6} = 1 \\ (q_1^9 q^3 q_2)^x \neq 1, x = 2,3 \\ (q_1^9 q^3 q_2)^y q_1^{-48} q^{-17} q_2^{-6} \neq 1, y = 1,2 \\ (q_1^9 q^3 q_2)^4 = 1 \text{ or } (q_1^9 q^3 q_2)^3 q_1^{-48} q^{-17} q_2^{-6} = 1 \end{cases}$$

$$\chi_4 = \sigma_1^{\chi_3} * \chi_3 = \begin{pmatrix} q_1^{64} q^{24} q_2^9 & 1 \\ q_1^{-80} q^{-31} q_2^{-12} & q_1^{25} q^{10} q_2^4 \end{pmatrix},$$

By using the formula (2.2, p. 15), we obtain the set of constraints of χ_4:

$$IV):\begin{cases} q_1^{-80} q^{-31} q_2^{-12} \neq 1 \\ q_1^{64} q^{24} q_2^9 = -1 \text{ or } q_1^{64} q^{24} q_2^9 \cdot q_1^{-80} q^{-31} q_2^{-12} = 1 \\ (q_1^{25} q^{10} q_2^4)^u \neq 1, u = 2,3,4 \\ (q_1^{25} q^{10} q_2^4)^v q_1^{-80} q^{-31} q_2^{-12} \neq 1, v = 1,2,3 \\ (q_1^{25} q^{10} q_2^4)^5 = 1 \text{ or } (q_1^{25} q^{10} q_2^4)^4 q_1^{-80} q^{-31} q_2^{-12} = 1 \end{cases}$$

From $I)$, $II)$,$III)$ and $IV)$ above, the set of constraints of bicharacters is fixed partly. These equations and inequalities in this set of constraints are enough to decide the possible bicharacters. Here we give only the set of equations. When we need some constraints of inequalities in the following, we will choose them from $I), II)$ and $III)$ directly.

$$\begin{cases} q_1^4 = 1 \text{ ① or } q_1^3 q = 1 \text{ ②} \\ q_2 = -1 \text{ ③ or } q_2 q = 1 \text{ ④} \\ q_1^{36} q^{12} q_2^4 = 1 \text{ ⑤ or } q_1^{21} q^8 q_2^3 = 1 \text{ ⑥} \\ q_1^{64} q^{24} q_2^9 = -1 \text{ ⑦ or } q_1^{16} q^7 q_2^3 = 1 \text{ ⑧} \\ q_1^{125} q^{50} q_2^{20} = 1 \text{ ⑨ or } q_1^{20} q^9 q_2^4 = 1 \text{ ⑩} \end{cases}$$

For the set of equations above, there are 32 different cases:

(①,③,⑤,⑦,⑨), (①,③,⑤,⑦,⑩), (①,③,⑤,⑧,⑨), (①,③,⑤,⑧,⑩),
(①,③,⑥,⑦,⑨), (①,③,⑥,⑦,⑩), (①,③,⑥,⑧,⑨), (①,③,⑥,⑧,⑩),
(①,④,⑤,⑦,⑨), (①,④,⑤,⑦,⑩), (①,④,⑤,⑧,⑨), (①,④,⑤,⑧,⑩),
(①,④,⑥,⑦,⑨), (①,④,⑥,⑦,⑩), (①,④,⑥,⑧,⑨), (①,④,⑥,⑧,⑩),

(②,③,⑤,⑦,⑨), (②,③,⑤,⑦,⑩), (②,③,⑤,⑧,⑨), (②,③,⑤,⑧,⑩),
(②,③,⑥,⑦,⑨), (②,③,⑥,⑦,⑩), (②,③,⑥,⑧,⑨), (②,③,⑥,⑧,⑩),
(②,④,⑤,⑦,⑨), (②,④,⑤,⑦,⑩), (②,④,⑤,⑧,⑨), (②,④,⑤,⑧,⑩),
(②,④,⑥,⑦,⑨), (②,④,⑥,⑦,⑩), (②,④,⑥,⑧,⑨), (②,④,⑥,⑧,⑩),

$$(1):\ \begin{cases} ① \\ ③ \\ ⑤ \\ ⑦ \\ ⑨ \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q^{12} = 1 \\ q^{24} = 1 \\ q_1 q^{50} = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q_1 q^2 = 1 \\ q^4 = 1 \\ q_1^2 = 1 \\ q_1 \neq \pm 1 \end{cases} \Rightarrow \lightning .$$

$$(2):\ \begin{cases} ① \\ ③ \\ ⑤ \\ ⑦ \\ ⑩ \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q^{12} = 1 \\ q^{24} = 1 \\ q^9 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q^3 = 1 \end{cases} \Rightarrow \overset{t}{\circ}\!\!\overset{s}{\text{———————}}\!\!\overset{-1}{\circ}\ ,$$

$t^4 = 1, s^3 = 1, t \neq \pm 1, s \neq 1, ts \neq \pm 1, t^2 s \neq 1.$

Now we are trying to construct the entire Cartan graph associated to this bicharacter.

$\chi_1 = \begin{pmatrix} t & 1 \\ s & -1 \end{pmatrix}, \mathcal{C}^{\chi_1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_1} = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix};$

$\chi_2 = \sigma_1^{\chi_1} * \chi_1 = \begin{pmatrix} t & 1 \\ -s^{-1} & -t \end{pmatrix}, \mathcal{C}^{\chi_2} = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}, \sigma_2^{\chi_2} = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix};$

$\chi_3 = \sigma_2^{\chi_2} * \chi_2 = \begin{pmatrix} -1 & 1 \\ s & -t \end{pmatrix}, \mathcal{C}^{\chi_3} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \sigma_1^{\chi_3} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix};$

$\chi_4 = \sigma_1^{\chi_3} * \chi_3 = \begin{pmatrix} -1 & 1 \\ s^2 & ts \end{pmatrix}, \mathcal{C}^{\chi_4} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}, \sigma_2^{\chi_4} = \begin{pmatrix} 1 & 0 \\ 4 & -1 \end{pmatrix};$

$\chi_5 = \sigma_2^{\chi_4} * \chi_4 = \begin{pmatrix} -1 & 1 \\ s^{-1} & ts \end{pmatrix}, \mathcal{C}^{\chi_5} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}, \sigma_1^{\chi_5} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix};$

$\chi_6 = \sigma_1^{\chi_5} * \chi_5 = \begin{pmatrix} -1 & 1 \\ s & -t \end{pmatrix}, \mathcal{C}^{\chi_6} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \sigma_2^{\chi_6} = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix};$

$\chi_7 = \sigma_2^{\chi_6} * \chi_6 = \begin{pmatrix} t & 1 \\ -s^{-1} & -ts \end{pmatrix}, \mathcal{C}^{\chi_7} = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}, \sigma_1^{\chi_7} = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix};$

$\chi_8 = \sigma_1^{\chi_7} * \chi_7 = \begin{pmatrix} t & 1 \\ s & -1 \end{pmatrix}, \mathcal{C}^{\chi_8} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \sigma_2^{\chi_8} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix};$

$\chi_9 = \sigma_2^{\chi_8} * \chi_8 = \begin{pmatrix} -ts & 1 \\ s^2 & -1 \end{pmatrix}, \mathcal{C}^{\chi_9} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \sigma_1^{\chi_9} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix};$

$\chi_{10} = \sigma_1^{\chi_9} * \chi_9 = \begin{pmatrix} -ts & 1 \\ s^{-1} & -1 \end{pmatrix}, \mathcal{C}^{\chi_{10}} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \sigma_2^{\chi_{10}} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix};$

$\chi_{11} = \sigma_2^{\chi_{10}} * \chi_{10} = \begin{pmatrix} t & 1 \\ s & -1 \end{pmatrix} = \chi_1, \mathcal{C}^{\chi_{11}} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$

Then the diagram of bicharacters is given as follows:

$$
{}^{\sigma_1}\circlearrowright \begin{pmatrix} -ts & 1 \\ s^2 & -1 \end{pmatrix} \frac{\sigma_2}{\quad} \underset{\chi_1}{\begin{pmatrix} t & 1 \\ s & -1 \end{pmatrix}} \frac{\sigma_1}{\quad} \begin{pmatrix} t & 1 \\ -s^{-1} & -t \end{pmatrix} \frac{\sigma_2}{\quad} \begin{pmatrix} -1 & 1 \\ s & -t \end{pmatrix} \frac{\sigma_1}{\quad} \begin{pmatrix} -1 & 1 \\ s^2 & ts \end{pmatrix} \circlearrowleft^{\sigma_2}.
$$

By using the formula (2.2, p. 15), it is not difficult to get the entire Cartan graph associated to χ_1:

$$
{}^{\sigma_1}\circlearrowright \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \frac{\sigma_2}{\quad} \underset{\chi_1}{\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}} \frac{\sigma_1}{\quad} \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix} \frac{\sigma_2}{\quad} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \frac{\sigma_1}{\quad} \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \circlearrowleft^{\sigma_2}.
$$

$$
(3)\colon \begin{cases} 1 \\ 3 \\ 5 \\ 8 \\ 9 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q^{12} = 1 \\ q^7 = -1 \\ q_1 q^{50} = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q^7 = -1 \\ q^2 = 1 \\ q_1 q^2 = 1 \\ q_1 \neq \pm 1 \end{cases} \Rightarrow \lightning.
$$

$$
(4)\colon \begin{cases} 1 \\ 3 \\ 5 \\ 8 \\ 10 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q^{12} = 1 \\ q^7 = -1 \\ q^9 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q^7 = -1 \\ q = 1 \\ q \neq 1 \end{cases} \Rightarrow \lightning.
$$

$$
(5)\colon \begin{cases} 1 \\ 3 \\ 6 \\ 7 \\ 9 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q_1 q^8 = -1 \\ q^{24} = 1 \\ q_1 q^{50} = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q_1 q^8 = -1 \\ q^{24} = 1 \\ q_1 q^2 = 1 \\ q_1^3 = -1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q_1 q^8 = -1 \\ q^{24} = 1 \\ q_1 q^2 = 1 \\ q_1^3 = -1 \\ q_1 = -1 \\ q_1 \neq \pm 1 \end{cases} \Rightarrow \lightning.
$$

$$
(6)\colon \begin{cases} 1 \\ 3 \\ 6 \\ 7 \\ 10 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q_1 q^8 = -1 \\ q^{24} = 1 \\ q^9 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q_1 q^8 = -1 \\ q^{24} = 1 \\ q^9 = 1 \\ q_1^3 = -1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q_1 q^8 = -1 \\ q^9 = 1 \\ q_1^3 = -1 \\ q_1 = -1 \\ q_1 \neq \pm 1 \end{cases} \Rightarrow \lightning.
$$

$$
(7)\colon \begin{cases} 1 \\ 3 \\ 6 \\ 8 \\ 9 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q_1 q^8 = -1 \\ q^7 = -1 \\ q_1 q^{50} = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q_1 q^8 = -1 \\ q^7 = -1 \\ q_1 q^{50} = 1 \\ q_1 q = 1 \\ q_1 q \neq 1 \end{cases} \Rightarrow \lightning.
$$

$$(8): \begin{cases} ① \\ ③ \\ ⑥ \\ ⑧ \\ ⑩ \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q_1 q^8 = -1 \\ q^7 = -1 \\ q^9 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 = -1 \\ q_1 q^8 = -1 \\ q^7 = -1 \\ q^9 = 1 \\ q_1 q = 1 \\ q_1 q \neq 1 \end{cases} \Rightarrow \lightning.$$

$$(9): \begin{cases} ① \\ ④ \\ ⑤ \\ ⑦ \\ ⑨ \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 q = 1 \\ q^8 = 1 \\ q^{15} = 1 \\ q_1 q^{30} = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 q = 1 \\ q^8 = 1 \\ q = 1 \\ q_1 q^{30} = 1 \\ q \neq 1 \end{cases} \Rightarrow \lightning.$$

$$(10): \begin{cases} ① \\ ④ \\ ⑤ \\ ⑦ \\ ⑩ \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 q = 1 \\ q^8 = 1 \\ q^{15} = 1 \\ q^5 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 q = 1 \\ q = 1 \\ q \neq 1 \end{cases} \Rightarrow \lightning.$$

$$(11): \begin{cases} ① \\ ④ \\ ⑤ \\ ⑧ \\ ⑨ \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 q = 1 \\ q^8 = 1 \\ q^4 = 1 \\ q_1 q^{30} = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 q = 1 \\ q_1 q^2 = 1 \\ q^4 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 q = 1 \\ q_1 q^2 = 1 \\ q^4 = 1 \\ q_1^2 = 1 \\ q_1 \neq \pm 1 \end{cases} \Rightarrow \lightning.$$

$$(12): \begin{cases} ① \\ ④ \\ ⑤ \\ ⑧ \\ ⑩ \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 q = 1 \\ q^8 = 1 \\ q^4 = 1 \\ q^5 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 q = 1 \\ q = 1 \\ q \neq 1 \end{cases} \Rightarrow \lightning.$$

$$(13): \begin{cases} ① \\ ④ \\ ⑥ \\ ⑦ \\ ⑨ \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 q = 1 \\ q_1 q^5 = 1 \\ q^{15} = 1 \\ q_1 q^{30} = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 q = 1 \\ q_1 q^5 = 1 \\ q^{15} = 1 \\ q_1 q^{30} = 1 \\ q_1^3 = 1 \\ q_1^3 \neq 1 \end{cases} \Rightarrow \lightning.$$

$$(14): \begin{cases} ① \\ ④ \\ ⑥ \\ ⑦ \\ ⑩ \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 q = 1 \\ q_1 q^5 = 1 \\ q^{15} = 1 \\ q^5 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 q = 1 \\ q_1 q^5 = 1 \\ q^5 = 1 \\ q_1 = 1 \\ q_1 \neq 1 \end{cases} \Rightarrow \lightning.$$

$$(15)\colon \begin{cases} ① \\ ④ \\ ⑥ \\ ⑧ \\ ⑨ \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 q = 1 \\ q_1 q^5 = 1 \\ q^4 = 1 \\ q_1 q^{30} = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 q = 1 \\ q_1 q = 1 \\ q^4 = 1 \\ q_1 q^{30} = 1 \\ q_1 q \neq 1 \end{cases} \Rightarrow \lightning.$$

$$(16)\colon \begin{cases} ① \\ ④ \\ ⑥ \\ ⑧ \\ ⑩ \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 q = 1 \\ q_1 q^5 = 1 \\ q^4 = 1 \\ q^5 = 1 \end{cases} \Rightarrow \begin{cases} q_1^4 = 1 \\ q_2 q = 1 \\ q_1 q^5 = 1 \\ q = 1 \\ q \neq 1 \end{cases} \Rightarrow \lightning.$$

$$(17)\colon \begin{cases} ② \\ ③ \\ ⑤ \\ ⑦ \\ ⑨ \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q_1 q^3 = 1 \\ q_1^2 q^9 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q_1 q^3 = 1 \\ q_1^2 q^9 = 1 \\ q_1 = 1 \\ q_1 \neq 1 \end{cases} \Rightarrow \lightning.$$

$$(18)\colon \begin{cases} ② \\ ③ \\ ⑤ \\ ⑦ \\ ⑩ \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q_1 q^3 = 1 \\ q_1^2 q^3 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q_1 q^3 = 1 \\ q_1^2 q^3 = 1 \\ q_1 = 1 \\ q_1 \neq 1 \end{cases} \Rightarrow \lightning.$$

$$(19)\colon \begin{cases} ② \\ ③ \\ ⑤ \\ ⑧ \\ ⑨ \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q_1 q^2 = -1 \\ q_1^2 q^9 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q_1 q^2 = -1 \\ q_1^2 q^9 = 1 \\ q_1^5 = -1 \\ q_1^{25} = 1 \end{cases} \Rightarrow \lightning.$$

$$(20)\colon \begin{cases} ② \\ ③ \\ ⑤ \\ ⑧ \\ ⑩ \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q_1 q^2 = -1 \\ q_1^2 q^3 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q_1 q^2 = -1 \\ q_1^2 q^3 = 1 \\ q_1 q = -1 \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q = 1 \\ q_1 = -1 \\ q_1 \neq \pm 1 \end{cases} \Rightarrow \lightning.$$

$$(21)\colon \begin{cases} ② \\ ③ \\ ⑥ \\ ⑦ \\ ⑨ \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q = -1 \\ q_1 q^3 = 1 \\ q_1^2 q^9 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q = -1 \\ q_1 = -1 \\ q_1^2 = -1 \\ q_1 \neq \pm 1 \end{cases} \Rightarrow \lightning.$$

$$(22): \begin{cases} ② \\ ③ \\ ⑥ \\ ⑦ \\ ⑩ \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q = -1 \\ q_1 q^3 = 1 \\ q_1^2 q^3 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q = -1 \\ q_1 = -1 \\ q_1 \neq \pm 1 \end{cases} \Rightarrow \lightning.$$

$$(23): \begin{cases} ② \\ ③ \\ ⑥ \\ ⑧ \\ ⑨ \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q = -1 \\ q_1 q^2 = -1 \\ q_1^2 q^9 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q = -1 \\ q_1 = -1 \\ q_1^2 q^9 = 1 \\ q_1 \neq \pm 1 \end{cases} \Rightarrow \lightning.$$

$$(24): \begin{cases} ② \\ ③ \\ ⑥ \\ ⑧ \\ ⑩ \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q = -1 \\ q_1 q^2 = -1 \\ q_1^2 q^3 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 = -1 \\ q = -1 \\ q_1 = -1 \\ q_1^2 q^3 = 1 \\ q_1 \neq \pm 1 \end{cases} \Rightarrow \lightning.$$

$$(25): \begin{cases} ② \\ ④ \\ ⑤ \\ ⑦ \\ ⑨ \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 q = 1 \\ q_2^4 = 1 \\ q_1^{19} = -1 \\ q_1^{35} = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 q = 1 \\ q_2^4 = 1 \\ q_1^{19} = -1 \\ q_1^{35} = 1 \\ q_1^3 = 1 \\ q_1^3 \neq 1 \end{cases} \Rightarrow \lightning.$$

$$(26): \begin{cases} ② \\ ④ \\ ⑤ \\ ⑦ \\ ⑩ \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 q = 1 \\ q_2^4 = 1 \\ q_1^{19} = -1 \\ q_1^5 = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 q = 1 \\ q_2^4 = 1 \\ q_1^{19} = -1 \\ q_1^5 = 1 \\ q_1 = -1 \\ q_1 \neq \pm 1 \end{cases} \Rightarrow \lightning.$$

$$(27): \begin{cases} ② \\ ④ \\ ⑤ \\ ⑧ \\ ⑨ \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 q = 1 \\ q_2^4 = 1 \\ q_1^4 = 1 \\ q_1^{35} = 1 \end{cases} \Rightarrow \begin{cases} q_1^3 q = 1 \\ q_2 q = 1 \\ q_2^4 = 1 \\ q_1^4 = 1 \\ q_1^{35} = 1 \\ q_1 = 1 \\ q_1 \neq \pm 1 \end{cases} \Rightarrow \lightning.$$

$$(28)\colon \left\{\begin{array}{l} \textcircled{2} \\ \textcircled{4} \\ \textcircled{5} \\ \textcircled{8} \\ \textcircled{10} \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^3 q = 1 \\ q_2 q = 1 \\ q_2^4 = 1 \\ q_1^4 = 1 \\ q_1^5 = 1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^3 q = 1 \\ q_2 q = 1 \\ q_2^4 = 1 \\ q_1^4 = 1 \\ q_1^5 = 1 \\ q_1 = 1 \\ q_1 \neq \pm 1 \end{array}\right. \Rightarrow \lightning.$$

$$(29)\colon \left\{\begin{array}{l} \textcircled{2} \\ \textcircled{4} \\ \textcircled{6} \\ \textcircled{7} \\ \textcircled{9} \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^3 q = 1 \\ q_2 q = 1 \\ q_2^2 = 1 \\ q_1^{19} = -1 \\ q_1^{35} = 1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^3 q = 1 \\ q_2 q = 1 \\ q_2^2 = 1 \\ q_1^{19} = -1 \\ q_1^{35} = 1 \\ q_1^3 = 1 \\ q_1^3 \neq 1 \end{array}\right. \Rightarrow \lightning.$$

$$(30)\colon \left\{\begin{array}{l} \textcircled{2} \\ \textcircled{4} \\ \textcircled{6} \\ \textcircled{7} \\ \textcircled{10} \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^3 q = 1 \\ q_2 q = 1 \\ q_2^2 = 1 \\ q_1^{19} = -1 \\ q_1^5 = 1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^3 q = 1 \\ q_2 q = 1 \\ q_2^2 = 1 \\ q_1^{19} = -1 \\ q_1^5 = 1 \\ q_1 = -1 \\ q_1 \neq \pm 1 \end{array}\right. \Rightarrow \lightning.$$

$$(31)\colon \left\{\begin{array}{l} \textcircled{2} \\ \textcircled{4} \\ \textcircled{6} \\ \textcircled{8} \\ \textcircled{9} \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^3 q = 1 \\ q_2 q = 1 \\ q_2^2 = 1 \\ q_1^4 = 1 \\ q_1^{35} = 1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^3 q = 1 \\ q_2 q = 1 \\ q_2^2 = 1 \\ q_1^4 = 1 \\ q_1^{35} = 1 \\ q_1 = 1 \\ q_1 \neq \pm 1 \end{array}\right. \Rightarrow \lightning.$$

$$(32)\colon \left\{\begin{array}{l} \textcircled{2} \\ \textcircled{4} \\ \textcircled{6} \\ \textcircled{8} \\ \textcircled{10} \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^3 q = 1 \\ q_2 q = 1 \\ q_2^2 = 1 \\ q_1^4 = 1 \\ q_1^5 = 1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} q_1^3 q = 1 \\ q_2 q = 1 \\ q_2^2 = 1 \\ q_1^4 = 1 \\ q_1^5 = 1 \\ q_1 = 1 \\ q_1 \neq \pm 1 \end{array}\right. \Rightarrow \lightning.$$

From the above 32 cases, it yields one bicharacter whose corresponding characteristic sequence is the period of affine characteristic sequence.

(1) $\overset{\zeta^3}{\circ}\!\!\overset{\zeta^4}{\rule[0.5ex]{5em}{0.4pt}}\!\!\overset{-1}{\circ}$, $\zeta \in \mu_{12}$, where μ_m denotes the set of primitive mth roots of unity.

This bicharacter has the characteristic sequence: $[1, 3, 3, 1, 4, 1, 3, 3, 1, 4]$.

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