Classification results on skew Schur Q-functions

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Kurzzusammenfassung

In dieser Arbeit werden Eigenschaften von schiefen Schur Q-Funktionen untersucht. Schiefe Schur Q-Funktionen können als erzeugende Funktionen von schiefen verschobenen Tableaux definiert werden. Betrachtet man deren Zerlegung in nichtschiefe Schur Q-Funktionen, so tauchen als Koeffizienten der Konstituenten die verschobenen Littlewood-Richardson-Koeffizienten $f^{\lambda}_{\mu\nu}$ auf. Wir werden in der Arbeit Bedingungen an diese Koeffizienten stellen und untersuchen, welche Klassen von schiefen Schur Q-Funktionen diese Bedingungen erfüllen.

In Kapitel 1 werden die Grundlagen für schiefe Schur Q-Funktionen und verschobene Tableaux bereit gestellt. Auch gibt es einen Abschnitt über die Zerlegung von $Q_{\lambda/\mu}$ für den Fall, dass μ die Länge 1 hat, und es gibt einen Abschnitt über Gleichheit von schiefen Schur Q-Funktionen. Die Eigenschaften vereinfachen die Beweise in späteren Kapiteln.

In Kapitel 2 zeigen wir ein paar Ungleichungen für die Koeffizienten $f^{\lambda}_{\mu\nu}$, die die Beweise in den nachfolgenden Kapitel vereinfachen.

In Kapitel 3 betrachten wir die Q-multiplizitätenfreien schiefen Schur Q-Funktionen. Das sind schiefe Schur Q-Funktionen $Q_{\lambda/\mu}$, bei denen die Koeffizienten $f^{\lambda}_{\mu\nu}$ nur 0 oder 1 sind. Wir werden eine Klassifikation dieser Schur Q-Funktionen angeben.

In Kapitel 4 betrachten wir Q-homogene schiefe Schur Q-Funktionen, bei denen nur ein Koeffizient $f^{\lambda}_{\mu\nu}$ ungleich 0 ist. Auch diese schiefen Schur Q-Funktionen werden wir klassifizieren.

In Kapitel 5 beschäftigen wir uns damit, zwei Konstituenten in der Zerlegung von nicht-Q-homogenen schiefen Schur Q-Funktionen zu finden, welche eine starke Ähnlichkeit miteinander haben.

In Kapitel 6 betrachten wir schiefe Schur Q-Funktionen mit genau zwei homogenen Komponenten und werden auch diese komplett klassifizieren.

In Kapitel 7 werfen wir einen Blick auf offene Fragen und geben Vermutungen zu diesen Fragen ab.

- Schlagwörter: Q-multiplizitätenfrei, Q-homogen, schiefe Schur Q-Funktion

Abstract

In this work properties of skew Schur Q-functions are analysed. Skew Schur Q-functions can be defined as generating functions of skew shifted tableaux. If their decomposition into non-skew Schur Q-functions is considered then the coefficients of the constituents are the shifted Littlewood-Richardson-coefficients $f^{\lambda}_{\mu\nu}$. We will consider special conditions on these coefficients and analyse which classes of skew Schur Q-functions satisfy these conditions.

In Chapter 1, background and some fundamental properties of skew Schur Q-functions and shifted tableaux are given. Additionally there is a section on the decomposition of $Q_{\lambda/\mu}$ for the case that μ has length 1 and there is a section about equality of skew Schur Q-functions. The properties that are shown simplify the proofs in later chapters.

In Chapter 2 we will show some inequalities for the coefficients $f^{\lambda}_{\mu\nu}$ that simplify the proofs in the subsequent chapters.

In Chapter 3 we consider Q-multiplicity-free skew Schur Q-functions. These are skew Schur Q-functions $Q_{\lambda/\mu}$ where the coefficients $f^{\lambda}_{\mu\nu}$ are either equal to 0 or to 1. We will provide a classification of these Schur Q-functions.

In Chapter 4 we consider Q-homogeneous skew Schur Q-functions where only one coefficient $f^{\lambda}_{\mu\nu}$ is non-zero. Again, we will classify these skew Schur Q-functions.

In Chapter 5 we deal with the problem of finding two constituents in the decomposition of a non-Q-homogeneous skew Schur Q-function which are strongly related.

In Chapter 6 we consider skew Schur Q-functions with precisely two homogeneous components and will classify them as well.

In Chapter 7 we take a look at open problems and formulate some conjectures.

- Keywords: Q-multiplicity-free, Q-homogeneous, skew Schur Q-function

Contents

Introduction

The Schur function s_λ on countably many indeterminates can be defined as generating function for the content of semistandard Young tableaux of shape λ, where λ is a partition. The set of all Schur functions is an important basis of the algebra Λ of symmetric functions over $\mathbb{C}$. The decomposition of Schur functions into power sum functions gives information about the character tables of the symmetric groups. The coefficient of the power sum p_μ in the decomposition of s_λ is the value of the character indexed by λ at an element of cycle type μ, divided by the size of the centralizer of a permutation of cycle type μ (see Stanley's book [19] for background). In the decomposition of the induced tensor product of the irreducible characters χ^μ and χ^ν into irreducible characters, the Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$ appear as coefficients of the characters χ^λ (see [19, Appendix A1.3] or the book by James and Kerber [10]). In the decomposition of the product of Schur functions s_μ and s_ν into Schur functions, the very same coefficient $c^\lambda_{\mu\nu}$ appear as coefficient of s_λ. Hence there is a strong connection between irreducible characters of the symmetric groups and Schur functions. The skew Schur function $s_{\lambda/\mu}$ on countably many indeterminates can be defined as generating function for the content of semistandard Young tableaux of skew shape λ/μ. The Littlewood-Richardson coefficient $c^\lambda_{\mu\nu}$ occurs also in the Schur expansion of the skew Schur function $s_{\lambda/\mu}$ as the coefficient of the Schur function s_ν.

The Littlewood-Richardson rule shows that one can obtain the Littlewood-Richardson coefficients by counting semistandard Young tableaux whose reading word is a ballot sequence. Using this, many results concerning (skew) Schur functions have been found, for example, which skew Schur functions are homogeneous (that is, some multiple of a Schur function) or even Schur functions by Bessenrodt and Kleshchev [4], which products of Schur functions are multiplicity-free (that is, the coefficient of each constituent in the decomposition is equal to 1) by Stembridge [21], which skew Schur functions are multiplicity-free by Gutschwager [7] and independently by Thomas and Yong [23] in the context of Schubert calculus.

The Schur Q-function Q_λ on countably many indeterminates can be defined as generating function for the content of shifted tableaux of shifted shape λ, where λ is a partition into distinct parts. For this property and further background and results on Schur Q-functions we refer to the important paper by Stembridge [22]. The $\mathbb{C}$-algebra Ω spanned by all power sum symmetric functions $p_{(i)}$ for odd i is clearly a subalgebra of Λ. The set of all Schur Q-functions is a basis of Ω (see [22, Section 6]). The spin representations of the symmetric groups are the faithful representations of the double cover groups of the symmetric groups; their study is in some sense equivalent to studying the projective representations of the symmetric groups. The Schur Q-functions play an analogous role for the irreducible spin characters of the symmetric groups as the Schur functions do for the ordinary irreducible characters of the symmetric groups. The coefficients of the constituents in the decomposition of a Schur Q-function Q_λ into power sum symmetric functions also give some information about the character values of the irreducible spin character φ^λ or $\varphi^\lambda_\pm$. But this time it is necessary to distinguish the cases where $|\lambda| - \ell(\lambda)$ is even or odd (see [22, Section 7]) and different formulas have to be used to obtain entries in the character table. In the decomposition of reduced Clifford products of spin characters into spin characters, besides the shifted Littlewood-Richardson coefficients $f^\lambda_{\mu\nu}$ also powers of 2 appear (see [22, Theorem 8.1]). Up to powers of 2, these coefficients $f^\lambda_{\mu\nu}$ also appear in the decomposition of products of Schur Q-functions into Schur Q-functions. Hence, there is a connection similar to the one between irreducible characters and Schur functions. The skew Schur Q-function $Q_{\lambda/\mu}$ on countably many indeterminates can be defined as generating function for the content of shifted tableaux of shifted skew shape λ/μ. Analogously, the shifted Littlewood-Richardson coefficients $f^\lambda_{\mu\nu}$ appear in the decomposition of skew Schur Q-functions into Schur Q-functions.

The shifted Littlewood-Richardson coefficients can be obtained by a shifted variant of the Littlewood-Richardson rule. The shifted Littlewood-Richardson rule due to Stembridge [22, Theorem 8.3] uses a lattice property similar to the one occurring in the classical Littlewood-Richardson rule. The shifted Littlewood-Richardson rule as given by Cho [5] uses semistandard decomposition tableaux introduced by Serrano [17]. Seeing so many similarities between Schur functions and Schur Q-functions, it is natural to try to find analogous results for (skew) Schur Q-functions. In [15], Salmasian showed which skew Schur Q-functions are equal to Schur Q-functions; we will expand this result to a classification of Q-homogeneous skew Schur Q-functions. Bessenrodt showed which products of Schur P-functions (where $P_\lambda = 2^{-\ell(\lambda)}Q_\lambda$) are P-multiplicity-free in [2]. This means that a classification of multiplicity-free products of Schur functions, a classification of multiplicity-free skew Schur functions and a classification of P-multiplicity-free products of Schur P-functions were known. What was missing in this context was a shifted analogue of the classification of multiplicity-free skew Schur functions or some skew analogue of the classification of P-multiplicity-free products of Schur P-functions. A main part of this thesis will deal with this problem and will provide the classification of Q-multiplicity-free Schur Q-functions.

Further results concerning (skew) Schur Q-functions will be described now. Barekat and van Willigenburg found relations for equality of skew Schur Q-functions, and they conjectured necessary and sufficient conditions for the equality of ribbon Schur Q-functions in [1]. DeWitt showed which Schur functions are equal to Schur Q-functions, and she also characterized Q-homogeneous skew Schur Q-functions indexed by unshifted diagrams in [6]. Hamel and King proved some bijections concerning certain shifted tableaux and some generalisations of skew Schur Q-functions in [8]. A shifted version of the Robinson-Schensted algorithm was given by Sagan in [13]. Shaw and van Willigenburg classified s-multiplicity-free Schur P-functions in [18]. Stembridge considers enriched P-partitions which are related to shifted tableaux in [20]. Also, the books [9] by Hoffman and Humphreys and [12] by Macdonald provide an introduction to (skew) Schur Q-functions and shifted tableaux.

In this thesis we obtain results on the Q-decomposition of skew Schur Q-functions which are mainly classification results. In Chapter 1 we define skew Schur Q-functions and show properties of skew Schur Q-functions that simplify proofs in the following chapters. In Chapter 2 we prove inequalities for the shifted Littlewood-Richardson coefficients that will also simplify proofs in the following chapters. In Chapter 3 we give a classification of the Q-multiplicity-free skew Schur Q-functions (Theorem 3.58) which is the shifted analogue of Gutschwager's result. In Chapter 4 we give a classification of the Q-homogeneous skew Schur Q-functions (Theorem 4.17); in contrast to the corresponding result on skew Schur functions it turns out that there are Q-homogeneous skew Schur Q-functions that are not equal to some Schur Q-function. In Chapter 5 we find two related non-zero homogeneous components in skew Schur Q-functions that are not Q-homogeneous (Theorem 5.8). In Chapter 6 we give a classification of skew Schur Q-functions with precisely two homogeneous components (Theorem 6.69). In Chapter 7 we give a conjecture concerning certain inequalities of the shifted Littlewood-Richardson coefficients (Conjecture 7.1). Also, we give a conjecture for the number of different reading words of the tableaux that are counted for the shifted Littlewood-Richardson rule using combinatorial arguments that can be proved using algebraical arguments (Proposition 7.9).

1 Preliminaries

In this chapter we will define our object of interest, the skew Schur Q-function, as well as fix notation and state general results that we will use in later chapters.

In Section 1.1 we give the basic definitions needed to define the skew Schur Q-function.

In Section 1.2 we define the skew Schur Q-function and show the shifted Littlewood-Richardson rule that enables us to decompose skew Schur Q-functions into non-skew Schur Q-functions. To classify the skew Schur Q-functions in which this decomposition satisfy some given condition is our main goal in most of the subsequent chapters.

In Section 1.3 we prove some general statements for tableaux, notably Lemma 1.42 which is used in a large number of proofs in later chapters.

In Section 1.4 we prove a formula of the decomposition for some specific family of skew Schur Q-functions.

And in Section 1.5 we prove some statements that show that two skew Schur Q-functions are equal if their respective associated diagrams satisfy some properties.

1.1 Partitions, diagrams and tableaux

The following definitions are based on the papers of Salmasian [15] and Stembridge [22] and the notation will be compatible with both papers except for the fact that a shifted diagram or shifted tableau is called diagram or tableau, respectively, and a classical Young diagram or Young tableau is called unshifted diagram or unshifted tableau, respectively. Also an arbitrary unshifted diagram can be skew or non-skew (see remark and notation after Example 1.6).

A composition is a tuple $\alpha = (\alpha_1, \alpha_2, \ldots)$ of non-negative integers such that $\alpha_i = 0$ for all $i > n$ for some given n. The length of α is $\ell(\alpha) := \min\{n \mid \alpha_i = 0 \text{ for all } i > n\}$.

A **partition** is a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ where $\lambda_i \geq \lambda_{i+1} > 0$ for all $1 \leq i \leq \ell(\lambda) - 1$. A partition λ is called a partition of k if $|\lambda| := \lambda_1 + \lambda_2 + \ldots + \lambda_{\ell(\lambda)} = k$ where $|\lambda|$ is called the **size** of λ. A **partition with distinct parts** is a partition

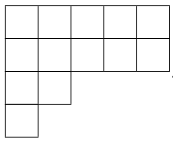
$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ where $\lambda_i > \lambda_{i+1} > 0$ for all $1 \leq i \leq \ell(\lambda) - 1$. The set of partitions of k with distinct parts is denoted by DP_k. By definition the empty partition $\emptyset$ is the only element in DP_0 and it has length 0. The set of all partitions with distinct parts is denoted by $DP := \bigcup_k DP_k$.

Definition 1.1. Let λ be a partition. An **unshifted diagram** $\tilde{D}_\lambda$ is defined by

$$\tilde{D}_\lambda := \{(i,j) \mid 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}$$

and can be depicted as a left-justified arrangement of boxes (i,j) with λ_1 boxes in the uppermost row, λ_2 boxes in the row below etc. The **size** $|\tilde{D}_\lambda|$ is the number of boxes in $\tilde{D}_\lambda$.

Example 1.2. *Let* $\lambda = (5,5,2,1)$. *Then*

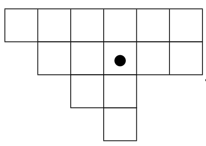
$\tilde{D}_\lambda =$  .

Definition 1.3. Let $\lambda \in DP$. A **(shifted) diagram** D_λ is defined by

$$D_\lambda := \{(i,j) \mid 1 \leq i \leq \ell(\lambda), i \leq j \leq i + \lambda_i - 1\}$$

and can be depicted as the arrangement of boxes we get after shifting the i^{th} row in the unshifted diagram $\tilde{D}_\lambda$ $i-1$ boxes to the right for all i. The **size** $|D_\lambda|$ is the number of boxes in D_λ. The boxes are denoted by (i,j) where i is the row and j is the column of the box. The uppermost leftmost box is denoted by $(1,1)$.

Example 1.4. *Let* $\lambda = (6,5,2,1)$. *Then*

$D_\lambda =$  .

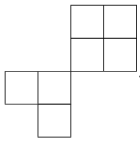
The box marked • is $(2,4)$.

Definition 1.5. Let $\lambda, \mu \in DP$. If $\ell(\mu) \leq \ell(\lambda)$ and $\mu_i \leq \lambda_i$ for all $1 \leq i \leq \ell(\mu)$ then the **skew (shifted) diagram** $D_{\lambda/\mu}$ is defined as the arrangement of boxes obtained by removing the boxes of D_μ from D_λ. The **size** $|D_{\lambda/\mu}| = |D_\lambda| - |D_\mu|$ is the number of boxes remaining. Each edgewise connected part of the diagram is called a **component**.

Analogously define a **skew unshifted diagram** $\tilde{D}_{\alpha/\beta}$ for partitions α and β as arrangement of boxes we get if we take the unshifted diagram $\tilde{D}_\alpha$ and remove all boxes that are also in the unshifted diagram $\tilde{D}_\beta$. The **size** $|\tilde{D}_{\alpha/\beta}| = |\tilde{D}_\alpha| - |\tilde{D}_\beta|$ is again the number of boxes. And also each edgewise connected part of the diagram is called a **component**.

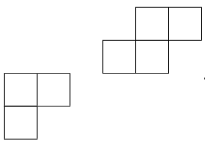
For a given diagram D the number of components of D is denoted by $comp(D)$. If $comp(D) = 1$ the diagram D is called connected, otherwise it is called disconnected.

Example 1.6. *Let* $\lambda = (6,5,2,1)$ *and* $\mu = (4,3)$ *then the diagram is*

$D_{\lambda/\mu} =$  .

We have $|D_{\lambda/\mu}| = 7$ *and the diagram has two components.*

The unshifted diagram is

$\tilde{D}_{\lambda/\mu} =$  .

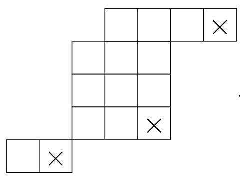
Note that we have $D_{\lambda/\emptyset} = D_\lambda$ and $\tilde{D}_{\lambda/\emptyset} = \tilde{D}_\lambda$.

Remark and **notation**. Every (skew) unshifted diagram $\tilde{D}_{\alpha/\beta}$ can be regarded as a skew shifted diagram $D_{\lambda/\mu}$ where $\ell(\lambda) = \ell(\mu) + 1$ by setting $\lambda = (\alpha_1 + \ell(\alpha) - 1, \alpha_2 + \ell(\alpha) - 2, \ldots, \alpha_{\ell(\alpha)-1} + 1, \alpha_{\ell(\alpha)})$ and $\mu = (\beta_1 + \ell(\alpha) - 1, \beta_2 + \ell(\alpha) - 2, \ldots, \beta_{\ell(\alpha)-1} + 1, \beta_{\ell(\alpha)})$ where $\beta_i = 0$ if $i > \ell(\beta)$ and $\beta_{\ell(\alpha)}$ is omitted if $\beta_{\ell(\alpha)} = 0$. Thus the following Definitions are also satisfied for unshifted diagrams. The difference between (skew or non-skew) shifted and unshifted diagrams is that for an unshifted diagram there are no x, y such that $(x-1,y),(x,y+1) \in \tilde{D}_\lambda$ and $(x,y) \notin \tilde{D}_\lambda$. In the following it only matters if there are such x,y or not; therefore, it does not matter if an unshifted diagram is skew or not. Hence, from now on, if an unshifted diagram is mentioned it can be skew or non-skew unless it is specified whether it is skew or non-skew.

In the following, if components are numbered, the numbering is as follows: the first component is the leftmost component, the second component is the next component to the right of the first component etc.

Definition 1.7. Let D be a diagram. A **corner** of D is a box $(x,y) \in D$ such that $(x+1,y),(x,y+1) \notin D$.

Example 1.8. *Let*

$D =$  .

The corners of D are the boxes marked $\times$.

Definition 1.9. Let $\lambda, \mu \in DP$. A **tableau** T of shape $D_{\lambda/\mu}$ is a map $T : D_{\lambda/\mu} \to \mathcal{A}$ from boxes of $D_{\lambda/\mu}$ to letters from the alphabet $\mathcal{A} = \{1' < 1 < 2' < 2 < \ldots\}$ such that

a) $T(i,j) \leq T(i+1,j)$, $T(i,j) \leq T(i,j+1)$ for all i,j,

b) each column has at most one k $(k = 1,2,3,\ldots)$,

c) each row has at most one k' $(k' = 1', 2', 3', \ldots)$.

Let $c^{(u)}(T) = (c_1^{(u)}, c_2^{(u)}, \ldots)$ where $c_i^{(u)}$ denotes the number of is in the tableau T for each i. Analogously, let $c^{(m)}(T) = (c_1^{(m)}, c_2^{(m)}, \ldots)$ where $c_i^{(m)}$ denotes the number of i's in the tableau T for each i. Then the **content** is defined by $c(T) = (c_1, c_2, \ldots) := c^{(u)}(T) + c^{(m)}(T)$. If there is some k such that $c_k > 0$ but $c_j = 0$ for all $j > k$ then we omit all these c_j from $c(T)$.

Remark. We depict a tableau T of shape $D_{\lambda/\mu}$ by filling the box (x, y) with the letter $T(x, y)$ for all x, y.

Example 1.10. *Let* $\lambda = (8, 6, 5, 3, 2)$ *and* $\mu = (5, 2, 1)$*. Then a tableau of shape* $D_{\lambda/\mu}$ *is*

$$T = \begin{array}{ccccc} & & 1' & 1 & 2 \\ 2' & 2 & 2 & 4 & \\ 2 & 4 & 5 & 5 & \\ 4 & 6' & 6 & & \\ & 6 & 7 & & \end{array} .$$

We have $c(T) = (2, 5, 0, 3, 2, 3, 1)$.

Remark. The letters $1, 2, 3, \dots$ are called unmarked letters and the letters $1', 2', 3', \dots$ are called marked letters. For a letter x of the alphabet $|x|$ denotes the unmarked version of this letter.

1.2 Schur Q-functions

In this section we want to give the definition of (skew) Schur Q-functions as well as show some important properties that will be used in the following chapters. The most important statement is the shifted Littlewood-Richardson rule in Proposition 1.23 due to Stembridge [22] that shows that skew Schur Q-functions can be decomposed into non-skew Schur Q-functions and how the coefficients in this decomposition are related to specific tableaux.

Definition 1.11. Let $\lambda, \mu \in DP$ and $x_1, x_2, \dots$ be a countable set of independent variables. Then the **Schur Q-function** is defined by

$$Q_{\lambda/\mu} := \sum_{T \in T(\lambda/\mu)} x^{c(T)}$$

where $T(\lambda/\mu)$ denotes the set of all tableaux of shape $D_{\lambda/\mu}$ and $x^{(c_1, c_2, \dots, c_{\ell(c)})} := x_1^{c_1} x_2^{c_2} \cdots$ where $c_k := 0$ for $k > \ell(c)$. If $D_\mu \nsubseteq D_\lambda$ then $Q_{\lambda/\mu} = 0$.

Remark. Since $D_{\lambda/\emptyset} = D_\lambda$, we denote $Q_{\lambda/\emptyset}$ by Q_λ. For a given diagram $D = D_{\lambda/\mu}$ for some $\lambda, \mu \in DP$ we denote by Q_D the Schur Q-function $Q_{\lambda/\mu}$.

Definition 1.12. Let a diagram D be such that the y^{th} column has no box but there are boxes to the right of the y^{th} column and after shifting all boxes that are to the right of the y^{th} column one box to the left we obtain a diagram $D_{\alpha/\beta}$ for some $\alpha, \beta \in DP$. Then we call the y^{th} column **empty** and the diagram $D_{\alpha/\beta}$ is obtained by removing the y^{th} column. Similarly, let a diagram D be such that the x^{th} row has no box but there are boxes below the x^{th} row and after shifting all boxes that are below the x^{th} row one box up and then all boxes of the diagram one box to the left we obtain a diagram $D_{\alpha/\beta}$ for some $\alpha, \beta \in DP$. Then we call the x^{th} row **empty** and the diagram $D_{\alpha/\beta}$ is obtained by removing the x^{th} row.

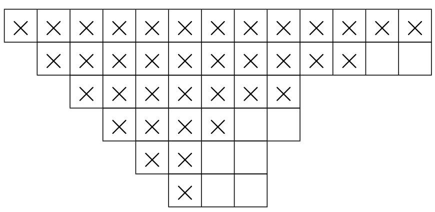
Definition 1.13. For $\lambda, \mu \in DP$ we call the diagram $D_{\lambda/\mu}$ **basic** if it satisfies the following properties for all $1 \leq i \leq \ell(\mu)$:

- $D_\mu \subseteq D_\lambda$,
- $\ell(\lambda) > \ell(\mu)$,
- $\lambda_i > \mu_i$,
- $\lambda_{i+1} \geq \mu_i - 1$.

This means that $D_{\lambda/\mu}$ has no empty rows or columns.

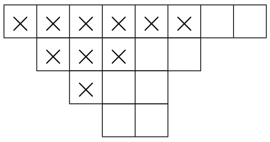
Example 1.14. *Let* $\lambda = (13, 12, 7, 6, 4, 3)$ *and* $\mu = (13, 10, 7, 4, 2, 1)$ *then the diagram*

$D_{\lambda/\mu} =$



is not basic since $\ell(\lambda) = 6 = \ell(\mu)$, $\lambda_1 = 13 = \mu_1$ *and* $\lambda_3 = 7 < 10 = \mu_2$. *In fact, the* 6^{th}, 10^{th} *and* 11^{th} *column and the* 1^{st} *and* 3^{rd} *row are empty.*

Let $\lambda = (8, 5, 3, 2)$ *and* $\mu = (6, 3, 1)$ *then the diagram*

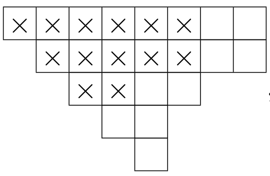
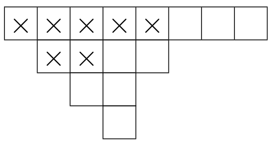
$D_{\lambda/\mu} =$ 

is basic.

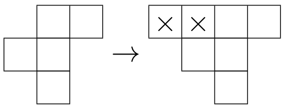
For some given diagram D, let $\bar{D}$ be the diagram obtained by removing all empty rows and columns of the diagram D. Since the restrictions of each entry of the boxes in a diagram are unaffected by removing empty rows and columns, there is a content-preserving bijection between tableaux of a given shape and tableaux of the shape obtained by removing empty rows and columns; thus we have $Q_D = Q_{\bar{D}}$. Hence in considering skew Schur Q-functions $Q_{\lambda/\mu}$ it is enough to consider partitions λ and μ such that $D_{\lambda/\mu}$ is basic.

Notation. In later chapters, we are interested in a subset of boxes U of a given diagram D that also forms a diagram. An example of such a subset is a component. Sometimes, we want to give $\lambda, \mu \in DP$ such that $U = D_{\lambda/\mu}$. Usually, U has empty rows and/or columns. Since these empty rows and columns do not matter for the following problems, we will consider the diagram $\bar{U}$ obtained by removing all empty rows and columns of U. In the following, if we say U has shape $D_{\lambda/\mu}$ for some subset U of D then we mean that $\bar{U} = D_{\lambda/\mu}$ where $D_{\lambda/\mu}$ is a basic diagram. See the following example for a depiction of this notation.

Example 1.15. *For the two diagrams*

$D_{(8,7,4,2,1)/(6,5,2)} =$  , $D_{(8,4,2,1)/(5,2)} =$ 

after removing empty rows and columns first the component C_1 is



$$= D_{(4,2,1)/(2)}.$$

Hence, C_1 has shape $D_{(4,2,1)/(2)}$.

Lemma 1.16. *Let $\lambda, \mu \in DP$ and let $C_1, \dots, C_{comp(D_{\lambda/\mu})}$ denote the components of $D_{\lambda/\mu}$. Then*

$$Q_{\lambda/\mu} = \prod_{i=1}^{comp(D_{\lambda/\mu})} Q_{C_i}.$$

Proof. Let C_1 have shape $D_{\alpha/\beta}$. Let D be the diagram we get after removing the first component C_1 of $D_{\lambda/\mu}$ and let D have shape $D_{\gamma/\delta}$. Since the boxes of C_1 are independent of the boxes of the other components, each tableau of $D_{\lambda/\mu}$ can be constructed by joining a tableau of D on to a tableau of C_1. For each tableau T_1 of C_1 and tableau T_2 of D we obtain a tableau of $D_{\lambda/\mu}$ by filling the boxes of C_1 as in T_1 and the other boxes as in T_2. Two tableaux of $D_{\lambda/\mu}$ are equal if and only if the filling of C_1 and the filling of the remaining boxes are equal. Therefore we obtain

$$\begin{aligned} Q_{\lambda/\mu} = \sum_{T \in T(\lambda/\mu)} x^{c(T)} &= \sum_{T_1 \in T(\alpha/\beta), T_2 \in T(\gamma/\delta)} x^{c(T_1)} \cdot x^{c(T_2)} \\ &= \sum_{T_1 \in T(\alpha/\beta)} x^{c(T_1)} \cdot \sum_{T_2 \in T(\gamma/\delta)} x^{c(T_2)} = Q_{C_1} \cdot Q_D. \end{aligned}$$

Inductively, we obtain $Q_{\lambda/\mu} = \prod_{i=1}^{comp(D_{\lambda/\mu})} Q_{C_i}$. □

Definition 1.17. Let T be a tableau of some diagram D. The **reading word** $w := w(T)$ is the word obtained by reading the rows from left to right beginning with the bottom row and ending with the top row. The **length** $\ell(w)$ is the number of letters and, thus, the number of boxes of D. Let $(x(i), y(i))$ denote the box of the i^{th} letter of the reading word $w(T)$.

Remark. The box $(x(i), y(i))$ is the box that satisfies the property $|\{(u,v) \in D_{\lambda/\mu} \mid$ either we have $u > x(i)$ or we have $u = x(i)$ and $v \leq y(i)\}| = i$.

Example 1.18. *Let*

$$T = \begin{array}{cccccccc} \times & \times & \times & \times & \times & 1' & 1 & 2 \\ & \times & \times & 2' & 2 & 2 & 4 & \\ & & \times & 2 & 4 & 5 & 5 & \\ & & & 4 & 6' & 6 & & \\ & & & & 6 & 7 & & \end{array} .$$

Then $w(T) = 6746'624552'2241'12$ *and* $(x(5), y(5)) = (4, 6)$.

Definition 1.19. Let w be a word of length n consisting of letters from the alphabet $\mathcal{A}$. The **statistics** $m_i(j)$ for $0 \leq j \leq 2n$ are defined as follows:

- $m_i(0) = 0$ for all i.
- For $1 \leq j \leq n$ the statistic $m_i(j)$ is equal to the number of times i occurs in the word $w_{n-j+1} \cdots w_n$.
- For $n+1 \leq j \leq 2n$ we set $m_i(j) := m_i(n) + k(i)$ where $k(i)$ is the number of times i' occurs in the word $w_1 \cdots w_{j-n}$.

Example 1.20. *Let* $w = 322'24'2'1'12$*. Then* $m_2(9) = 3$ *and* $m_2(12) = 4$.

Remark. As Stembridge remarked in [22, before Theorem 8.3], the statistics $m_i(j)$ for some given i can be calculated simultaneously by taking the word $w(T)$ and scan it first from right to left while counting the letters i and afterwards scan it from left to right and adding the number of letters i'. After the j^{th} step of scanning and counting the statistic $m_i(j)$ is calculated.

Note that $c_i^{(u)} = m_i(n)$ and $c_i^{(m)} = m_i(2n) - m_i(n)$.

Definition 1.21. Let $k \in \mathbb{N}$ and w be a word of length n consisting of letters from the alphabet $\mathcal{A}$. The word w is called k**-amenable** if it satisfies the following conditions:

(a) if $m_k(j) = m_{k-1}(j)$ then $w_{n-j} \notin \{k, k'\}$ for all $0 \leq j \leq n-1$,

(b) if $m_k(j) = m_{k-1}(j)$ then $w_{j-n+1} \notin \{k-1, k'\}$ for all $n \leq j \leq 2n-1$,

(c) if j is the smallest number such that $w_j \in \{k', k\}$ then $w_j = k$,

(d) if j is the smallest number such that $w_j \in \{(k-1)', k-1\}$ then $w_j = k-1$.

The word w is called amenable if it is k-amenable for all $k > 1$. A tableau T is called k-amenable if $w(T)$ is k-amenable. A tableau T is called amenable if $w(T)$ is amenable.

Remark. Definition 1.21 a) can be regarded as follows: Suppose that while scanning a word from right to left we have $m_k(j) = m_{k-1}(j)$ for some $j < n$. Then the next letter we scan cannot be a k' or k.

Similarly, Definition 1.21 b) can be regarded as follows: Suppose that while scanning a word from left to right we have $m_k(j) = m_{k-1}(j)$ for some $n \leq j < 2n$. Then the next letter we scan cannot be a $k-1$ or k'.

Example 1.22. *The word* $w = 322'24'2'1'12$ *is not* 2*-amenable since* $m_1(0) = m_2(0) = 0$ *and* $w_9 = 2$*. But* w *is* 3*-amenable.*

The aforementioned shifted analogue of the Littlewood-Richardson rule was proved by Stembridge and will be our next proposition. In the next chapters, whenever we tackle problems concerning the decomposition of skew Schur Q-functions into Schur Q-functions we implicitly use this statement.

Proposition 1.23. *[22, before Proposition 8.2] Let* $\lambda, \mu \in DP$*. Then we have*

$$Q_{\lambda/\mu} = \sum_{\nu \in DP} f^{\lambda}_{\mu\nu} Q_{\nu},$$

where $f^{\lambda}_{\mu\nu}$ *is the number of amenable tableaux* T *of shape* $D_{\lambda/\mu}$ *and content* ν*.*

Remark. If $f^{\lambda}_{\mu\nu} > 0$ then $|D_{\lambda/\mu}| = |D_{\nu}|$.

Definition 1.24. Let $x_1, x_2, \ldots$ be a countable set of independent variables. A **symmetric function** is a formal power series with variables $x_1, x_2, \ldots$ such that for all $i, j \in \mathbb{N}$

such that $i \neq j$ the interchanging of x_i and x_j does not change the formal power series. By iteration of this that means that permuting the variables does not change the formal power series.

Example 1.25. *The formal power series $\sum_{i\in\mathbb{N}} x_i = x_1+x_2+\ldots$ is a symmetric function since interchanging two variables does not change this formal power series.*

Stembridge showed in [22, Corollary 6.2] that the Schur Q-functions Q_λ are symmetric functions using a shifted analogue of Knuth's correspondence due to Sagan [13] and Worley [24]. This is far from obvious by the combinatorial definition used in Definition 1.11. In Proposition 1.23 we see that skew Schur Q-functions $Q_{\lambda/\mu}$ can be written as a linear combination of Schur Q-functions. Hence, we obtain the following proposition.

Proposition 1.26. *For all $\lambda, \mu \in DP$ the skew Schur Q-function $Q_{\lambda/\mu}$ is a symmetric function.*

Remark. This statement implies that for every $Q_{\lambda/\mu}$ the coefficient of a monomial

$$x_1^{c_1} x_2^{c_2} \cdots x_i^{c_i} \cdots x_j^{c_j} \cdots$$

is equal to the coefficient of a monomial

$$x_1^{c_1} x_2^{c_2} \cdots x_j^{c_i} \cdots x_i^{c_j} \cdots = x_1^{c_1} x_2^{c_2} \cdots x_i^{c_j} \cdots x_j^{c_i} \cdots .$$

The first coefficient equals the number of tableaux of shape $D_{\lambda/\mu}$ and content $c = (c_1, c_2, \ldots, c_i, \ldots, c_j, \ldots)$ and the second coefficient equals the number of tableaux of shape $D_{\lambda/\mu}$ and content $\hat{c} = (c_1, c_2, \ldots, c_j, \ldots, c_i, \ldots)$, that is the composition obtained by interchanging the i^{th} and the j^{th} entry of c. It follows that there are as many tableaux of shape $D_{\lambda/\mu}$ and content c as tableaux of shape $D_{\lambda/\mu}$ and content $\hat{c}$.

Iterating this interchanging process implies that there are as many tableaux of shape $D_{\lambda/\mu}$ and content c as tableaux of shape $D_{\lambda/\mu}$ and content $\bar{c}$, where $\bar{c}$ is a composition we get after permuting parts (including the infinity number of parts that are zero) of c.

Since there is only a finite number of tableaux of a given shape and a given content, there is a bijection between the tableaux of shape $D_{\lambda/\mu}$ and content c and the tableaux of shape $D_{\lambda/\mu}$ and content $\bar{c}$.

Proposition 1.27. *[22, before Theorem 8.1] Let $\lambda, \mu, \nu \in DP$. Then*

$$f^{\lambda}_{\mu\nu} = f^{\lambda}_{\nu\mu}.$$

Proposition 1.27 allows us to calculate the numbers $f^{\lambda}_{\mu\nu}$ for given $\lambda, \mu \in DP$ either by finding the possible contents ν of amenable tableaux of shape $D_{\lambda/\mu}$ or by finding the possible shapes $D_{\lambda/\nu}$ of amenable tableaux for the content μ. This yields two approaches to calculate these numbers which are used in the following chapters.

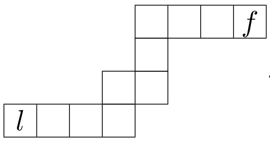
1.3 Properties of tableaux

In this section we show properties of tableaux in general and then take a closer look at amenable tableaux. In particular, we will prove an alternative definition of k-amenability of a tableau in Lemma 1.42 that does not make use of the reading word and which we will use as a checklist for the proof of amenability in later chapters. Also, in this section we will give an algorithm that produces an amenable tableau for all diagrams $D_{\lambda/\mu}$ due to Salmasian [15].

Definition 1.28. A **border strip** is a connected (skew) diagram B such that for each $(x, y) \in B$ we have $(x-1, y-1) \notin B$. The box $(x, y) \in B$ such that $(x-1, y) \notin B$ and $(x, y+1) \notin B$ is called the **first box** of B. The box $(u, v) \in B$ such that $(u+1, v) \notin B$ and $(u, v-1) \notin B$ is called the **last box** of B.

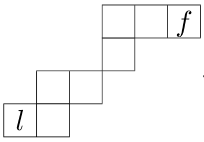
A (possibly disconnected) diagram D where all components are border strips is called a **broken border strip**. Then the first box of the rightmost component is called the first box of D, and the last box of the leftmost component is called the last box of D.

Example 1.29. *For* $\lambda = (11, 7, 6, 4)$ *and* $\mu = (7, 6, 4)$ *the diagram* $D_{\lambda/\mu}$ *is a border strip:*

$D_{\lambda/\mu} =$  .

The box labeled f *is the first box and the box labeled* l *is the last box.*

For $\lambda = (9, 6, 4, 2)$ *and* $\mu = (6, 5, 2)$ *the diagram* $D_{\lambda/\mu}$ *is a broken border strip:*

$D_{\lambda/\mu} =$  .

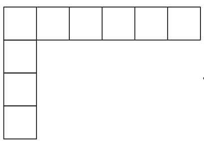
The box labeled f *is the first box of* $D_{\lambda/\mu}$ *and the box labeled* l *is the last box of* $D_{\lambda/\mu}$.

Definition 1.30. A (p, q)**-hook** is a set of boxes

$$\{(u, v + q - 1), \ldots, (u, v + 1), (u, v), (u + 1, v), \ldots, (u + p - 1, v)\}$$

for some $u, v \in \mathbb{N}$. To clarify that we have specific u and v we say that the previous set of boxes is a (p, q)-hook at (u, v).

Example 1.31. *For* $\lambda = (9, 3, 2, 1)$ *and* $\mu = (3, 2, 1)$ *the diagram* $D_{\lambda/\mu}$ *is a* $(4, 6)$*-hook:*

$D_{\lambda/\mu} =$  .

Remark. A (p, q)-hook is a border strip.

Definition 1.32. Let T be a skew shifted tableau of shape $D_{\lambda/\mu}$. Define $T^{(i)}$ by

$$T^{(i)} := \{(x,y) \in D_{\lambda/\mu} \mid |T(x,y)| = i\}.$$

Example 1.33. *Let*

$$T = \begin{array}{cccccccc} \times & \times & \times & \times & \times & 1' & 1 & 2 \\ & \times & \times & 2' & 2 & 2 & 4 & \\ & & \times & 2 & 4 & 5 & 5 & \\ & & & 4 & 6' & 6 & & \\ & & & & 6 & 7 & & \end{array} .$$

Then

$$T^{(2)} = \begin{array}{cccccccc} \times & \times & \times & \times & \times & \times & \times & \square \\ & \times & \times & \square & \square & \square & & \\ & & \times & \square & & & & \end{array} .$$

Lemma 1.34. *[9, before Theorem 13.1] Let T be a tableau of shape $D_{\lambda/\mu}$. Then*

$$|T(x,y)| < |T(x+1,y+1)|$$

for all x, y such that $(x,y), (x+1,y+1) \in D_{\lambda/\mu}$.

Proof. If $(x,y), (x+1,y+1) \in D_{\lambda/\mu}$ then we have $(x,y+1) \in D_{\lambda/\mu}$. If $|T(x,y)| = i$ then $T(x,y+1) \geq i$. For $T(x,y+1) = i$ we have $T(x+1,y+1) > i$ and, therefore, $|T(x,y)| = i < |T(x+1,y+1)|$. For $T(x,y+1) \geq (i+1)'$ we have $T(x+1,y+1) \geq (i+1)'$ and, therefore, $|T(x,y)| = i < |T(x+1,y+1)|$. $\square$

Corollary 1.35. *Let T be a tableau of shape $D_{\lambda/\mu}$. The diagram $T^{(i)}$ is a broken border strip.*

Definition 1.36. Let T be a tableau. If the last box of $T^{(i)}$ is filled with i we call $T^{(i)}$ **fitting**.

Remark. A restatement of 1.21 (c) (respectively, 1.21 (d)) is that $T^{(k)}$ (respectively, $T^{(k-1)}$) is fitting.

Corollary 1.35 as well as the following lemma collect facts that were mentioned by Sagan and Stanley [14, after Corollary 8.6].

Lemma 1.37. *Each component of* $T^{(i)}$ *has two possible fillings with entries from* $\{i', i\}$ *which differ only in the last box of this component.*

Proof. Let $(x, y) \in T^{(i)}$. If $(x+1, y) \in T^{(i)}$ then we have $T(x, y) = i'$, otherwise the y^{th} column is not weakly increasing or contains at least two is. If $(x, y-1) \in T^{(i)}$ then we have $T(x, y) = i$, otherwise the x^{th} row is not weakly increasing or contains at least two i's. If $(x+1, y), (x, y-1) \notin T^{(i)}$ then we have no restrictions and the box (x, y) can be filled with i or i'. Clearly, we have $(x+1, y), (x, y-1) \notin T^{(i)}$ for a given box (x, y) if and only if (x, y) is the last box of a component of $T^{(i)}$. □

The previous lemmas gave statements for tableaux in general. Now we want to see what additional properties arise if the tableau is (k-)amenable.

Lemma 1.38. *Let* T *be an amenable tableau. Then there are no entries greater than* k *in the first* k *rows.*

Proof. Assume the opposite. Let i be the uppermost row with an entry greater than i. Let this entry be x. Then x will be scanned before any $|x| - 1$, contradicting to the amenability of T. □

Lemma 1.39. *[15, Lemma 3.28] Let* w *be a* k*-amenable word for some* $k > 1$. *Let* $n := \ell(w)$. *If* $m_{k-1}(n) > 0$ *then* $m_{k-1}(n) > m_k(n)$.

Proof. If $m_k(n) > m_{k-1}(n)$ then there is some $0 \leq j \leq n-1$ such that $m_k(j) = m_{k-1}(j)$ and $w_{n-j} = k$; a contradiction to the amenability of w. Thus, we have $m_k(n) \leq m_{k-1}(n)$. It suffices to consider $\hat{w} = w|_{\{(k-1)', k-1, k', k\}}$ for k-amenability. Let $\hat{n} = \ell(\hat{w})$. Assume $m_k(\hat{n}) = m_{k-1}(\hat{n}) > 0$. Let $\hat{w}_i$ be the leftmost letter that is not k. This letter is either k' or $k-1$, otherwise the leftmost entry from $\{(k-1)', k-1\}$ in $\hat{w}$ is marked; a contradiction of the k-amenability of $\hat{w}$. Then $m_k(\hat{n}+i-1) = m_k(\hat{n}) = m_{k-1}(\hat{n}) = m_{k-1}(\hat{n}+i-1)$ and $\hat{w}_i \in \{k', k-1\}$; again a contradiction of the k-amenability of $\hat{w}$. $\square$

In the next chapters we will study specific skew Schur Q-functions that have restrictions on the numbers $f^{\lambda}_{\mu\nu}$. Thus, we are interested in the set of amenable tableaux of shape $D_{\lambda/\mu}$ and content ν. Often, we will modify a given amenable tableau by changing some entries. How these changes affect the reading word is not easy to see and, hence, it is hard to analyse the amenability of the modified tableau by using the reading word. Lemma 1.42 gives an equivalent definition for k-amenability that does not resort to the reading word. It may look complicated but in the following chapters usually we will take Corollary 1.44, which has properties that are much easier to check, to show k-amenability for most k and will use Lemma 1.42 only for some k where some entries do not satisfy the properties of Corollary 1.44. We need the following definition to be able to state Lemma 1.42.

Definition 1.40. Let $\lambda, \mu \in DP$ and let T be a tableau of $D_{\lambda/\mu}$. Then

$$\mathcal{S}^{\boxtimes}_{\lambda/\mu}(x,y) := \{(u,v) \in D_{\lambda/\mu} \mid u \leq x, v \geq y\},$$

$$\mathcal{S}^{\boxtimes}_{T}(x,y)^{(i)} := S^{\boxtimes}_{\lambda/\mu}(x,y) \cap T^{-1}(i) \text{ where } T^{-1}(i) \text{ denotes the preimage of } i,$$

$$\mathcal{B}^{(i)}_T := \{(x,y) \in D_{\lambda/\mu} \mid T(x,y) = i' \text{ and } T(x-1,y-1) \neq (i-1)'\},$$

$$\widehat{\mathcal{B}^{(i)}_T} := \{(x,y) \in D_{\lambda/\mu} \mid T(x,y) = i' \text{ and } T(x+1,y+1) \neq (i+1)'\}$$

and $b^{(i)}_T = |\mathcal{B}^{(i)}_T|$ for all i. Then let $\mathcal{B}^{(i)}_T(d)$ denote the set of the first d boxes of $\mathcal{B}^{(i)}_T$.

Remark. Note that, by Lemma 1.34, the diagram $\mathcal{B}^{(i)}_T$ is a broken border strip which is necessary for the definition of $\mathcal{B}^{(i)}_T(d)$.

Example 1.41. *Let* $\lambda = (11, 9, 6, 5, 4, 2, 1)$ *and* $\mu = (8, 6, 5, 4, 1)$ *and let*

$$T = \begin{array}{ccccccccccc} \times & \times & \times & \times & \times & \times & \times & \times & \mathbf{1'} & \mathbf{1} & \mathbf{1} \\ & \times & \times & \times & \times & \times & \times & \mathbf{1'} & \mathbf{2'} & \mathbf{2} & \\ & & \times & \times & \times & \times & \times & \mathbf{1} & & & \\ & & & \times & \times & \times & \times & 2' & & & \\ & & & & \times & 1' & 1 & 2 & & & \\ & & & & & 1 & 2' & & & & \\ & & & & & & 2 & & & & \end{array} .$$

Then $\mathcal{S}^{\boxtimes}_{\lambda/\mu}(3,8)$ *is the set of boxes with boldfaced entries. Also, we have* $\mathcal{S}^{\boxtimes}_{T}(3,8)^{(1)} = \{(1,10),(1,11),(3,8)\}$*,* $\mathcal{B}^{(2)}_T = \{(2,9),(4,8)\}$ *and* $\widehat{\mathcal{B}^{(1)}_T} = \{(1,9),(2,8)\}$*.*

Lemma 1.42. *Let* $\lambda, \mu \in DP$ *and* $n := |D_{\lambda/\mu}|$*. Let* T *be a tableau of* $D_{\lambda/\mu}$*. Then the tableau* T *is* k*-amenable if and only if either* $c(T)_{k-1} = c(T)_k = 0$ *or else it satisfies the following conditions:*

(1) $c(T)^{(u)}_{k-1} > c(T)^{(u)}_k$*;*

(2) when $T(x,y) = k$ *then* $|\mathcal{S}^{\boxtimes}_T(x,y)^{(k-1)}| \geq |\mathcal{S}^{\boxtimes}_T(x,y)^{(k)}|$*;*

(3) for each $(x,y) \in \mathcal{B}^{(k)}_T$ *we have* $|\mathcal{S}^{\boxtimes}_T(x,y)^{(k-1)}| > |\mathcal{S}^{\boxtimes}_T(x,y)^{(k)}|$*;*

(4) if $d := b^{(k)}_T + c^{(u)}_k - c^{(u)}_{k-1} + 1 > 0$ *then there is an injective map* $\phi : \mathcal{B}^{(k)}_T(d) \to \widehat{\mathcal{B}^{(k-1)}_T}$ *such that if* $(x,y) \in \mathcal{B}^{(k)}_T(d)$ *and* $(u,v) = \phi(x,y)$ *then for all* $u < r < x$ *we have* $T(r,s) \notin \{k-1, k'\}$ *for all* s *such that* $(r,s) \in D_{\lambda/\mu}$*;*

(5) $T^{(k-1)}$ *is fitting;*

(6) if $c(T)_k > 0$ *then* $T^{(k)}$ *is fitting.*

Proof. First we want to show that tableaux that satisfy these conditions are indeed k-amenable. Clearly, such a tableau is k-amenable if $c(T)_k = c(T)_{k-1} = 0$. Hence, we assume that $c(T)_k + c(T)_{k-1} \geq 1$.

Lemma 1.42 (2) ensures that if we have $w_i = k$ then $m_{k-1}(n-i) \geq |\mathcal{S}^{\boxtimes}_T(x,y)^{(k-1)}| > |\mathcal{S}^{\boxtimes}_T(x,y)^{(k)}| - 1 = m_k(n-i)$ since, by Lemma 1.34, $T(x-1,y-1) \neq k$ if $(x-1,y-1) \in$

$D_{\lambda/\mu}$. Lemma 1.42 (3) ensures that if $w_i = k'$ and $(x(i), y(i)) \in \mathcal{B}_T^{(k)}$ then $m_{k-1}(n-i) > m_k(n-i)$. If $w_i = k'$ and $(x, y) := (x(i), y(i)) \notin \mathcal{B}_T^{(k)}$ then $T(x-1, y-1) = (k-1)'$. But then $T(x-1, y) \in \{k', k-1\}$. If $T(x-1, y) = k-1$ then we have $m_{k-1}(n-j+1) > m_k(n-j+1)$ if $(x-1, y) = (x(j), y(j))$. But then, by Lemma 1.34, we have $m_{k-1}(n-i) > m_k(n-i)$. If $T(x-1, y) = k'$ then either $(x-1, y) \in \mathcal{B}_T^{(k)}$ or $T(x-2, y-1) = (k-1)'$. Then we can repeat this argument until we find a box (z, y) where $z < x$ such that either $T(z, y) = k-1$ or $(z, y) \in \mathcal{B}_T^{(k)}$. Thus, it is impossible to have $m_{k-1}(i) = m_k(i)$ and $w_{n-i} = k'$ for some i. Hence, we showed that Definition 1.21 (a) is satisfied.

Lemma 1.42 (1) ensures that we always have $m_{k-1}(n) > m_k(n)$. Let i be such that $w_i = k'$, $T(x(i)-1, y(i)-1) = (k-1)'$ and $m_{k-1}(n+i-1) > m_k(n+i-1)$. Then let j be such that $(x(j), y(j)) = (x(i)-1, y(i)-1)$. We have $m_{k-1}(n+i) \geq m_k(n+i)$ and $T(x, z) > k'$ for all $y < z \leq \lambda_x + x - 1$ (the rightmost box of this row is $(x, \lambda_x + x - 1)$). Also, we have $T(x-1, w) < (k-1)'$ for all $\mu_{x-1} + x - 1 \leq w < y$ (the leftmost box of this row is $(x-1, \mu_{x-1} + x - 1)$). Thus, we have $m_{k-1}(n+l) \geq m_k(n+l)$ for all $i \leq l \leq j-1$. Then $m_{k-1}(n+j) \geq m_k(n+j) + 1 > m_k(n+j)$. Hence, Definition 1.21 (b) has not been violated between w_i and w_j. By this argument, k-amenability of T depends on the boxes $(x, y) \in \mathcal{B}_T^{(k)}$. If $w_i = k'$ and $(x(i), y(i))$ is one of the last $c_{k-1}^{(u)} - c_k^{(u)} - 1$ boxes of $\mathcal{B}_T^{(k)}$ then $m_{k-1}(n+i) > m_k(n+i)$ since $m_{k-1}(n) = m_k(n) + c_{k-1}^{(u)} - c_k^{(u)}$. Let $w_i = k'$ and $(x(i), y(i)) \in \mathcal{B}_T^{(k)}(b_T^{(k)} + c_k^{(u)} - c_{k-1}^{(u)} + 1)$. By Lemma 1.42 (4), there is some j such that $w_j = (k-1)'$ and $\phi(x(i), y(i)) = (x(j), y(j))$. We have $m_{k-1}(n+i) - m_k(n+i) \geq c_{k-1}^{(u)} - c_k^{(u)} - (c_{k-1}^{(u)} - c_k^{(u)} - 1) - 1 = 0$ where the last -1 comes from the scanned entry k' in the box $(x(i), y(i))$. Note that pairs of boxes (s, t) and $(s+1, t+1)$ such that $T(s, t) = (k-1)'$ and $T(s+1, t+1) = k'$ do not change the difference $m_{k-1}(n+i) - m_k(n+i)$ because the letter $w_i = k'$ cannot be between these entries in the reading word and, hence, both letters of such pairs are scanned before we scan $w_i = k'$. Also for every box $(v, w) \in \mathcal{B}_T^{(k)}(b_T^{(k)} + c_k^{(u)} - c_{k-1}^{(u)} + 1)$ such that $v > x(i)$

Lemma 1.42 (4) ensures that $\phi(v, w)$ is not in a row above the $x(i)^{\text{th}}$ row or in the $x(i)^{\text{th}}$ row to the right of $(x(i), y(i))$. Hence, $T(v, w) = k'$ and $T(\phi(v, w)) = (k-1)'$ are scanned before $w_i = k'$ and these entries do not change the difference $m_{k-1}(n+i) - m_k(n+i)$. If $x(j) \geq x(i)$ then $m_{k-1}(n+i) - m_k(n+i) > 0$ because $w_j = (k-1)'$ is scanned before $w_i = k'$. If $x(j) < x(i)$ and $m_{k-1}(n+i) - m_k(n+i) = 0$ then $w_l \notin \{k-1, k'\}$ for all $i < l < j$. Thus, there is no i such that $m_{k-1}(n+i-1) = m_k(n+i-1)$ and $w_i \in \{k-1, k'\}$. Hence, we showed that Definition 1.21 (b) is satisfied.

Lemma 1.42 (5) and Lemma 1.42 (6) are restatements of Definition 1.21 (c) and Definition 1.21 (d), respectively (as mentioned in the remark after Definition 1.36). In total these conditions ensure k-amenability.

Now we want to show that if one of these conditions is not satisfied then T is not k-amenable. We may assume that $a + b > 0$.

Suppose Lemma 1.42 (1) is not satisfied. Then we have $m_{k-1}(n) \leq m_k(n)$ which contradicts Lemma 1.39.

Suppose Lemma 1.42 (2) is not satisfied. Let i be such that $w_i = k$ is the first scanned entry k such that $(x, y) := (x(i), y(i))$ violates Lemma 1.42 (2). Then $T(x-1, y) \neq k-1$ and $|\mathcal{S}_T^{\boxtimes}(x, y)^{(k-1)}| = |\mathcal{S}_T^{\boxtimes}(x, y)^{(k)}| - 1$. We have to distinguish the cases $T(x-1, y-1) \neq k-1$ and $T(x-1, y-1) = k-1$. If $T(x-1, y-1) \neq k-1$ then $m_{k-1}(n-i) = |\mathcal{S}_T^{\boxtimes}(x, y)^{(k-1)}| = |\mathcal{S}_T^{\boxtimes}(x, y)^{(k)}| - 1 = m_k(n-i)$ and $w_i = k$ which violates Definition 1.21 (a). If $T(x-1, y-1) = k-1$ then $T(x-1, y) = k'$ and, therefore, $T(x, y+1) \neq k$. Then for j such that $(x(j), y(j)) = (x-1, y)$ we must have $m_{k-1}(n-j) = m_k(n-j)$. But then we have $m_{k-1}(n-j) = m_k(n-j)$ and $w_j = k'$ which also violates Definition 1.21 (a).

Suppose Lemma 1.42 (3) is not satisfied. Let $(x, y) \in \mathcal{B}_T^{(k)}$ be such that $|\mathcal{S}_T^{\boxtimes}(x, y)^{(k-1)}| \leq |\mathcal{S}_T^{\boxtimes}(x, y)^{(k)}|$. If $T(x-1, y-1) = k-1$ then if $(x, y-1) \in D_{\lambda/\mu}$ we have $k-1 = T(x-1, y-1) < T(x, y-1) < T(x, y) = k'$ which is impossible. Hence $(x, y-1) \notin D_{\lambda/\mu}$ and $x = y$. But then $(x, y) = (x, x)$ is the lowermost leftmost box of $T^{(k)}$ and, since

$T(x,x) = k'$, this means that $T^{(k)}$ is not fitting which violates Definition 1.21 (d). Thus, there is no box $(x,y) \in \mathcal{B}_T^{(k)}$ such that $T(x-1,y-1) = k-1$. Hence, if i is such that $(x,y) = (x(i),y(i))$ then $m_{k-1}(n-i) \leq m_k(n-i)$. If $m_{k-1}(n-i) < m_k(n-i)$ then T is not k-amenable. If $m_{k-1}(n-i) = m_k(n-i)$ then $w_i = k'$ which also violates Definition 1.21 (a).

Suppose Lemma 1.42 (4) is not satisfied. Thus, $b_T^{(i)} + c_k^{(u)} - c_{k-1}^{(u)} + 1 > 1$ and there is a box $(x,y) \in \mathcal{B}_T^{(k)}(b_T^{(i)} + c_k^{(u)} - c_{k-1}^{(u)} + 1)$ such that each box of $\mathcal{B}_T^{(k)}(b_T^{(i)} + c_k^{(u)} - c_{k-1}^{(u)} + 1)$ that is below the x^{th} row can be mapped to a different box with the given property of Lemma 1.42 (4) but (x,y) cannot be mapped in this way. If i is such that $(x,y) = (x(i),y(i))$ then $m_{k-1}(n+i) = m_k(n+i)$ since

$$m_{k-1}(n+i) - m_k(n+i) = c_{k-1}^{(u)} - c_k^{(u)} - (b_T^{(i)} - (b_T^{(i)} + c_k^{(u)} - c_{k-1}^{(u)} + 1)) - 1 = 0$$

and, again, pairs of boxes (s,t) and $(s+1,t+1)$ such that $T(s,t) = (k-1)'$ and $T(s+1,t+1) = k'$ do not change the difference $m_{k-1}(i) - m_k(i)$ as well as as each box $(v,w) \in \mathcal{B}_T^{(k)}(b_T^{(k)} + c_k^{(u)} - c_{k-1}^{(u)} + 1)$ such that $v > x$ that can be mapped to a different box with the given property of Lemma 1.42 (4) since $T(u,v) = k'$ and $T(\phi(u,v)) = (k-1)'$ are both scanned before the letter $w_i = k'$. Since the box (x,y) cannot be mapped to a box with the given property of Lemma 1.42 (4), this means that either there is some $l > i$ such that $m_{k-1}(n+l-1) = m_k(n+l-1)$ and $w_l \in \{k-1,k'\}$, which violates Definition 1.21 (b), or we have $m_{k-1}(n-i) = 0$ and $w_i = T(x(i),y(i)) = T(x,y) = k'$ which violates Definition 1.21 (a).

It is clear by definition that a tableau is not k-amenable if Lemma 1.42 (5) and Lemma 1.42 (6) are not satisfied.

Thus, we showed that the k-amenable tableaux are precisely the ones that satisfy the conditions in Lemma 1.42. □

Example 1.43. *Let*

$$T = \begin{array}{ccccccccccc}
\times & \times & \times & \times & \times & \times & \times & \times & \mathbf{1'} & \mathbf{1} & \mathbf{1} \\
 & \times & \times & \times & \times & \times & \times & \mathbf{1'} & \mathbf{2'} & \mathbf{2} & \\
 & & \times & \times & \times & \times & \times & \mathbf{1} & & & \\
 & & & \times & \times & \times & \times & 2' & & & \\
 & & & & \times & 1' & 1 & 2 & & & \\
 & & & & & 1 & 2' & & & & \\
 & & & & & & 2 & & & &
\end{array}$$

be a tableau of shape $D_{(11,9,6,5,4,2,1)/(8,6,5,4,1)}$. *We will check the conditions of Lemma 1.42 for* $k = 2$ *in the following. We have* $c(T)_1^{(u)} = 5 > 3 = c(T)_2^{(u)}$. *Since* $T^{-1}(2) = \{(2,10),(5,8),(7,7)\}$ *we need to check condition (2) of Lemma 1.42 for these boxes. We have* $|\mathcal{S}_T^{\boxtimes}(2,10)^{(1)}| = 2 \geq 1 = |\mathcal{S}_T^{\boxtimes}(2,10)^{(2)}|$, $|\mathcal{S}_T^{\boxtimes}(5,8)^{(1)}| = 3 \geq 2 = |\mathcal{S}_T^{\boxtimes}(5,8)^{(2)}|$ *and* $|\mathcal{S}_T^{\boxtimes}(7,7)^{(1)}| = 4 \geq 3 = |\mathcal{S}_T^{\boxtimes}(7,7)^{(2)}|$. *Since* $\mathcal{B}_T^{(2)} = \{(2,9),(4,8)\}$ *we need to check condition (3) of Lemma 1.42 for these boxes. We have* $|\mathcal{S}_T^{\boxtimes}(2,9)^{(1)}| = 2 > 1 = |\mathcal{S}_T^{\boxtimes}(2,9)^{(2)}|$ *and* $|\mathcal{S}_T^{\boxtimes}(4,8)^{(1)}| = 3 > 1 = |\mathcal{S}_T^{\boxtimes}(4,8)^{(2)}|$. *Since* $d := 2+3-5+1 = 1$ *we have to find a map as in condition (4) of Lemma 1.42 for the box* $(2,9)$. *Such a map is* $\phi((2,9)) = (2,8)$. *Another one is* $\phi((2,9)) = (1,9)$. *Clearly,* $T^{(1)}$ *and* $T^{(2)}$ *are fitting. Hence, the tableau* T *is* 2*-amenable.*

It is easy to check that the conditions in the following corollary are included in the conditions of Lemma 1.42. In particular, it is much easier to check the conditions of this corollary than to check the conditions of Lemma 1.42. Often, it will be enough to use this corollary to show k-amenability for most ks and we have to go back to Lemma 1.42 just for some special cases of k.

Corollary 1.44. *Let* $\lambda, \mu \in DP$. *Let* T *be a tableau of shape* $D_{\lambda/\mu}$ *such that either* $c(T)_k = c(T)_{k-1} = 0$ *or else it satisfies the following conditions:*

(1) there is some box (x,y) *such that* $T(x,y) = k-1$ *and* $T(z,y) \neq k$ *for all* $z > x$*;*

(2) if $T(x,y) = k$ *then there is some* $z < x$ *such that* $T(z,y) = k-1$*;*

(3) if $T(x,y) = k'$ *then* $T(x-1,y-1) = (k-1)'$*;*

(4) $T^{(k-1)}$ *is fitting;*

(5) if $c_k^{(u)} > 0$ *then* $T^{(k)}$ *is fitting.*

Then the tableau is k*-amenable.*

Proof. We may assume that $c(T)_k^{(u)} + c(T)_{k-1}^{(u)} > 0$. Corollary 1.44 (2) states that for every $T(x,y) = k$ we have $|\mathcal{S}_T^{\boxtimes}(x,y)^{(k-1)}| \geq |\mathcal{S}_T^{\boxtimes}(x,y)^{(k)}|$. Thus, Lemma 1.42 (2) is satisfied. Corollary 1.44 (2) and Corollary 1.44 (1) together state that $c(T)_{k-1}^{(u)} > c(T)_k^{(u)}$. Hence, Lemma 1.42 (1) is satisfied. Corollary 1.44 (3) states that the set $\mathcal{B}_T^{(k)}$ is empty, hence, Lemma 1.42 (3) and Lemma 1.42 (4) are trivially satisfied. Corollary 1.44 (4) and Corollary 1.44 (5) are Lemma 1.42 (5) and Lemma 1.42 (6), respectively. □

In many proofs in the subsequent chapters we start with a given amenable tableau and change some entries in such a way that new amenable tableaux are obtained. Using this, we can obtain lower bounds for some $f_{\mu\nu}^{\lambda}$. Thus, it is essential to have a method to gain such amenable tableaux for each diagram. Salmasian found an algorithm that gives an amenable tableau for each skew diagram.

Definition 1.45. [15, before Lemma 3.5] Let $D_{\lambda/\mu}$ be a skew diagram. The tableau $T_{\lambda/\mu}$ is determined by the following algorithm:

(1) Set $k = 1$ and $U_1(\lambda/\mu) = D_{\lambda/\mu}$.

(2) Set $P_k = \{(x,y) \in U_k(\lambda/\mu) \mid (x-1, y-1) \notin U_k(\lambda/\mu)\}$.

(3) For each $(x,y) \in P_k$ set $T_{\lambda/\mu}(x,y) = k'$ if $(x+1,y) \in P_k$, otherwise set $T_{\lambda/\mu}(x,y) = k$.

(4) Let $U_{k+1}(\lambda/\mu) = U_k(\lambda/\mu) \setminus P_k$.

(5) Increase k by one, and go to (2).

Remark. The diagram P_k is a broken border strip.

We have $U_k(\lambda/\mu) = P_k \cup P_{k+1} \cup \ldots \cup P_n$ and will use this notation in the following chapters.

Example 1.46. *For $\lambda = (6, 5, 3, 2)$ and $\mu = (4, 1)$ we have*

$$T_{\lambda/\mu} = \begin{array}{|c|c|c|c|} \hline & & 1' & 1 \\ \hline 1' & 1 & 1 & 2 \\ \hline 1 & 2' & 2 & \\ \hline & 2 & 3 & \\ \hline \end{array}.$$

Salmasian showed the amenability of $T_{\lambda/\mu}$ in [15, Lemma 3.9]. Here, we use Corollary 1.44 to prove amenability.

Lemma 1.47. *For $\lambda, \mu \in DP$ such that $D_{\lambda/\mu}$ is a skew shifted diagram, the tableau $T_{\lambda/\mu}$ is amenable.*

Proof. If $\ell(c(T_{\lambda/\mu})) = 1$ then $T_{\lambda/\mu}$ is amenable since P_1 is fitting. Let $k > 1$ and assume $|P_k| \geq 1$. If (a, b) is the last box of P_k then there are boxes of P_{k-1} in the $(b-1)^{\text{th}}$ column and, hence, there is a box with entry $k-1$ but there is no box with entry k in the $(b-1)^{\text{th}}$ column. Thus, 1.44 (1) is satisfied.

For any $(u, v) \in P_k$ if $w = \max\{u \mid (u, v) \in P_k\}$ then $T_{\lambda/\mu}(w, v) = k$. If $z = \min\{u \mid (u, v) \in P_k\}$ then we have $(z-1, v) \in P_{k-1}$ since $(z-1, v-1) \in P_{k-1}$, $(z-1, v) \in D_{\lambda/\mu}$ and $(z-1, v) \notin P_k$. Thus, for (w, v) such that $T_{\lambda/\mu}(w, v) = k$ there is some $z < w$ such that $T_{\lambda/\mu}(z, v) = k-1$. Thus, 1.44 (2) is satisfied.

If $T_{\lambda/\mu}(x, y) = k'$ then $(x+1, y) \in P_k$ and, therefore, $(x, y-1) \in P_{k-1}$ so that $T_{\lambda/\mu}(x-1, y-1) = (k-1)'$. Thus, 1.44 (3) is satisfied.

The last box of $T^{(i)}_{\lambda/\mu}$ is the last box of P_i and P_i is fitting for each i, in particular, for $i \in \{k-1, k\}$. Thus, 1.44 (4) and 1.44 (5) are satisfied.

In total, Corollary 1.44 states that this tableau is k-amenable for each $k > 1$ and, therefore, amenable. $\square$

The tableau $T_{\lambda/\mu}$ has some special properties. It is always one of the amenable tableaux with the lexicographically largest content which means that every other homogeneous component in the decomposition of $Q_{\lambda/\mu}$ is indexed by some partition lexicographically smaller than $c(T_{\lambda/\mu})$. Also the coefficient of $Q_{c(T_{\lambda/\mu})}$ in the decomposition of $Q_{\lambda/\mu}$ into

Schur Q-functions only depends on the number of components of the P_is. Both statements will be proved in the following.

Definition 1.48. Let $\lambda, \mu \in DP$. The **lexicographical order** $\leq$ in DP is defined as follows: if $\lambda \leq \mu$ then either $\lambda = \mu$ or there is some k such that $\lambda_i = \mu_i$ for $1 \leq i \leq k-1$ and $\lambda_k < \mu_k$ where $\lambda_k := 0$ if $k > \ell(\lambda)$.

Lemma 1.49. *We have $c(T) \leq c(T_{\lambda/\mu})$ for all amenable tableaux T of shape $D_{\lambda/\mu}$. However, if $c(T) = c(T_{\lambda/\mu})$ then $T^{(i)} = P_i$.*

Proof. In order to obtain the lexicographically largest content of an amenable tableau of shape $D_{\lambda/\mu}$, we have to insert the maximal number of $1'$s and 1s in $D_{\lambda/\mu}$, then the maximal number of $2'$s and 2s etc.

By Lemma 1.34, $|T(x,y)| = 1$ implies $(x-1, y-1) \notin D_{\lambda/\mu}$. The set of such boxes is P_1. The algorithm of Definition 1.45 fills these boxes only with $1'$s and 1s. Then the entries $2'$ and 2 must be filled in boxes (x,y) such that $(x-1, y-1) \notin D_{\lambda/\mu} \setminus P_1$. The set of such boxes is P_2 and the algorithm of Definition 1.45 fills these boxes only with $2'$s and 2s. Repeating this argument for all entries greater than 2 gives the statement. □

Proposition 1.50. *Let $D_{\lambda/\mu}$ be a diagram. Let $\nu = c(T_{\lambda/\mu})$. Then we have*

$$f^{\lambda}_{\mu\nu} = \prod_{i=1}^{\ell(\nu)} 2^{comp(P_i)-1}.$$

Proof. Let T be an amenable tableau of $D_{\lambda/\mu}$ with content ν. By Lemma 1.49, we have $T^{(i)} = P_i$. Thus, a tableau T can differ from $T_{\lambda/\mu}$ only by markings of some entries. By Lemma 1.37, for each i each component $C_2, \ldots, C_{comp(P_i)}$ of P_i can be filled in two different ways that differ by the marking of the last box. By Definition 1.21 (c) and (d), the component C_1 must be fitting.

By Corollary 1.44, if (x,y) is the last box of one of the components $C_2, \ldots, C_{comp(P_i)}$ and if $T(x,y) = i'$ then T is amenable because in this case $(x-1, y-1), (x, y-1) \in P_{i-1}$

and, hence, then $T(x-1, y-1) = (i-1)'$. Thus, for each component of P_i except for the first one, there are two possibilities on how to fill the last box and the statement follows. $\square$

1.4 Decomposition of $Q_{\lambda/\mu}$ for partitions μ of length 1

If $\ell(\mu) = 1$ then the decomposition of $Q_{\lambda/\mu}$ can be easily described using Stembridges shifted Littlewood-Richardson rule [22].

Definition 1.51. Let $\lambda \in DP$. Then the **border** is defined by

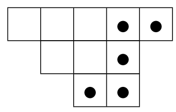
$$B_\lambda := \{(x, y) \in D_\lambda \mid (x+1, y+1) \notin D_\lambda\}.$$

Note that B_λ is a border strip.

Define $B_\lambda^{(n)} := \{D_{\lambda/\mu} \mid D_{\lambda/\mu} \subseteq B_\lambda \text{ and } |D_{\lambda/\mu}| = n\}$.

Remark. The cardinality of the border is given by the first part of λ, that is $|B_\lambda| = \lambda_1$.

Example 1.52. *Let* $\lambda = (5, 3, 2)$. *Then*

$D_{(5,3,2)} =$ 

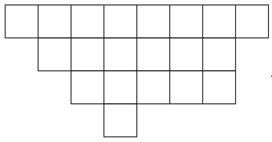
where the boxes denoted with $\boxed{\bullet}$ *are the boxes in* B_λ, *that is*

$$B_{(5,3,2)} = \{(1,5), (1,4), (2,4), (3,4), (3,3)\} = D_{(5,3,2)/(3,2)}.$$

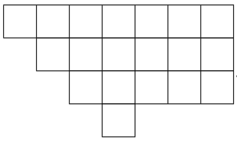
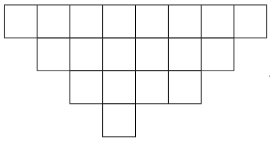
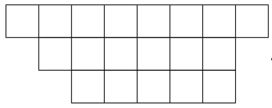
Then we have $B_{(5,3,2)}^{(3)} = \{D_{(5,3,2)/(5,2)}, D_{(5,3,2)/(4,3)}, D_{(5,3,2)/(4,2,1)}\}$.

Definition 1.53. Let $\lambda \in DP$. Define E_λ to be the set of all partitions whose diagram we obtain after removing a corner in D_λ.

Example 1.54. *For $\lambda = (8,6,5,1)$ we have*

$D_{(8,6,5,1)} =$  .

There are three corners in the diagram. We obtain the following three diagrams after removing a corner:

 ,  ,  .

Then we have

$$E_{(8,6,5,1)} := \{(7,6,5,1),(8,6,4,1),(8,6,5)\}.$$

Proposition 1.55. *Let $\lambda \in DP$ and $1 \leq n \leq \lambda_1$ be an integer. Then*

$$Q_{\lambda/(n)} = \sum_{D_{\lambda/\nu} \in B_\lambda^{(n)} \ (D_\nu \subseteq D_\lambda)} 2^{comp(D_{\lambda/\nu})-1} Q_\nu.$$

In particular,

$$Q_{\lambda/(\lambda_1 - 1)} = \sum_{(x,y) \in B_\lambda^\times} c_{B_\lambda}^{(x,y)} Q_{D_\mu \cup \{(x,y)\}}$$

where $D_\mu = D_\lambda \setminus B_\lambda$, $B_\lambda^\times := \{(x,y) \in B_\lambda \mid (x-1,y) \notin B_\lambda$ and $(x,y-1) \notin B_\lambda\}$ and

$$c_{B_\lambda}^{(x,y)} = \begin{cases} 1 & \text{if } (x,y) \text{ is the first or last box of } B_\lambda \\ 2 & \text{otherwise.} \end{cases}$$

and

$$Q_{\lambda/(1)} = \sum_{\nu \in E_\lambda} Q_\nu.$$

Proof. By Proposition 1.27, we have $f^{\lambda}_{(n)\nu} = f^{\lambda}_{\nu(n)}$. Thus, we need to look at tableaux of shape $D_{\lambda/\nu}$ and content (n). These n entries from $\{1', 1\}$ must be in the boxes of B_λ. Hence, $D_{\lambda/\nu} \in B^{(n)}_{\lambda}$. Thus, the constituents of $Q_{\lambda/(n)}$ with a non-zero coefficient are Q_ν such that $D_{\lambda/\nu} \in B^{(n)}_{\lambda}$.

By Lemma 1.37, each component of $D_{\lambda/\nu}$ can be filled in two ways that differ by the marking of the entry of the last box. By definition of amenability, the last box of $D_{\lambda/\nu}$ must contain a 1. Thus, for each component of $D_{\lambda/\nu}$ except for the first one there are two possibilities on how to fill the last box and the coefficient follows. □

Remark. Note that if $D_\mu = D_\lambda \setminus B_\lambda$ for some $\lambda \in DP$ then $\mu = (\lambda_2, \lambda_3, \dots, \lambda_{\ell(\lambda)})$.

1.5 Some conditions for equality of skew Schur Q-functions

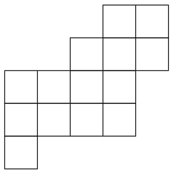
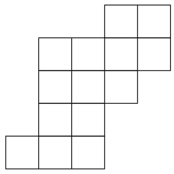
In later chapters we want to classify skew Schur Q-functions with certain properties. Before we start doing this, we want to analyse in what way two diagrams D, D' are related if $Q_D = Q_{D'}$. This will reduce the effort in proving these classifications.

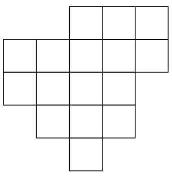
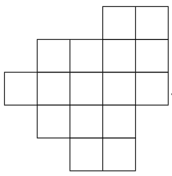
Salmasian proved when a skew Schur Q-function is equal to a non-skew Schur Q-function in [15]. We will see this again in Chapter 3. But this equality relation does not simplify proofs of the subsequent chapters. Barekat and van Willigenburg proved some conditions of equality for skew Schur Q-functions indexed by border strips in [1]. In the same paper one can find some conditions for equality of skew Schur Q-functions indexed by unshifted diagrams. And DeWitt proved the equality condition of Lemma 1.60, which is widely used in this work, in [6].

Lemma 1.56. *Let $D = D_{\lambda/\mu}$ be a diagram and $C_1, \dots, C_k$ be the components of this diagram numbered from left to right. Let D' be a diagram obtained from D by interchanging components of D with the constraint that if C_1 is not an unshifted diagram then C_1 is also the first component of D'. Then we have $Q_D = Q_{D'}$.*

Proof. By Lemma 1.16, $Q_{D'} = \prod_{i=1}^{k} Q_{C_i} = Q_D$. For the case that C_1 is not an unshifted diagram C_1 then has boxes $(x,y),(x+1,y),(x+1,y+1)$ such that $(x,y),(x+1,y+1) \in D$ and $(x+1,y) \notin D$ that can only be in the first component of any shifted diagram. Hence, it is necessary that after interchanging components of D the component C_1 is still the first component of the obtained diagram. □

Definition 1.57. Let D be a diagram. The **orthogonal transpose** of a diagram is obtained as follows: Reflect the boxes of D along the diagonal $\{(z,-z) \mid z \in \mathbb{N}\}$. Move this arrangement of boxes such that the top row with boxes is in the first row and the lowermost box of the leftmost column with boxes is part of the diagonal $\{(z,z) \mid z \in \mathbb{N}\}$. We denote the orthogonal transpose of a diagram by D^{ot}.

Example 1.58. *For* $D_1 =$  *we obtain* $D_1^{ot} =$ .

For $D_2 =$  *we obtain* $D_2^{ot} =$ .

Remark. DeWitt [6] called the diagram D^{ot} the flip of D denoted by D'. We use the notation D^{ot} since *ot* is the abbreviation of orthogonal transpose but in addition for an unshifted diagram D we have $D^{ot} = (D^o)^t$ where D^o is the rotation of Definition 1.66 and D^t is the transpose of Lemma 1.62.

Lemma 1.59. *Let* $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ *and* $n := \ell(\nu)$. *Let* $D^{ot}_{\lambda/\mu}$ *have shape* $D_{\gamma/\delta}$. *Let* $T' = T_{\gamma/\delta}$. *If* $U_i(\lambda/\mu)$ *has shape* $D_{\alpha/\beta}$ *then* $U_i(\gamma/\delta)$ *has shape* $D^{ot}_{\alpha/\beta}$.

Proof. The diagram $U_i(\gamma/\delta)$ is also defined by $\{(x,y) \in D_{\gamma/\delta} \mid (x-i+1, y-i+1) \in D_{\gamma/\delta}\}$ and the image of this set of boxes after orthogonally transposing is given by the set of boxes $\{(u,v) \in D_{\lambda/\mu} \mid (u+i-1, v+i-1) \in D_{\lambda/\mu}\}$ which has the same shape as the set of boxes $\{(u,v) \in D_{\lambda/\mu} \mid (u-i+1, v-i+1) \in D_{\lambda/\mu}\} = U_i(\lambda/\mu)$. □

Remark. For $i = n$ this means that $T'^{(n)}$ has the same shape as P_n^{ot}.

Lemma 1.60. *[6, Proposition IV.13] Let $D = D_{\lambda/\mu}$ be a diagram. There is a content-preserving bijection between the tableaux of shape D and the tableaux of shape D^{ot}. In particular, $Q_D = Q_{D^{ot}}$.*

Proof. Let T be a tableau of shape $D_{\lambda/\mu}$. Let $\nu := c(T)$ and let $n := \ell(\nu)$. Let Λ be the map that maps T to $\Lambda(T)$ where $\Lambda(T)$ is obtained as follows:

- Reflect and move the boxes of T together with their entries along the diagonal $\{(z, -z) \mid z \in \mathbb{N}\}$. Denote the resulting filling of $D_{\lambda/\mu}^{ot}$ by $\bar{T}$.
- For all i do the following:
  - If $\bar{T}(x, y) \in \{i', i\}$ and $\bar{T}(x+1, y) \in \{i', i\}$ then set $\Lambda(T)(x, y) = (n - i + 1)'$.
  - If $\bar{T}(x, y) \in \{i', i\}$ and $\bar{T}(x, y-1) \in \{i', i\}$ then set $\Lambda(T)(x, y) = n - i + 1$.
  - If $\bar{T}(x, y) \in \{i', i\}$ and neither $\bar{T}(x+1, y) \in \{i', i\}$ nor $\bar{T}(x, y-1) \in \{i', i\}$ then if (x, y) is the k^{th} such box counted from the left let (u, v) be the last box of the k^{th} component of $T^{(i)}$. If $T(u, v) = i'$ set $\Lambda(T)(x, y) = (n - i + 1)'$ and if $T(u, v) = i$ set $\Lambda(T)(x, y) = n - i + 1$.

One can see that Λ maps tableaux of D to tableaux of D^{ot}.

After orthogonal transposition, the rows and columns are weakly increasing since we orthogonally transpose the rows and columns and change the entries in reverse order. Clearly, in $\Lambda(T)$ there is at most one i in each column and at most one i' in each row. Hence, the properties of Definition 1.9 are satisfied.

Let a be the unmarked version of the least entry from T and b be the unmarked version of the greatest entry from T. Then

$$c(\Lambda(T)) = \bar{\nu} = (\nu_1, \nu_2 \ldots, \nu_{a-1}, \nu_b, \nu_{b-1}, \nu_{b-2}, \ldots, \nu_{a+1}, \nu_a)$$

where $\nu_1 = \nu_2 = \ldots = \nu_{a-1} = 0$.

Applying Λ to $\Lambda(T)$ gives a tableau of the same shape as T. By Lemma 1.59, we have $\Lambda(\Lambda(T))^{(i)} = T^{(i)}$. The last box of the k^{th} component of $\Lambda(\Lambda(T))^{(i)}$ is marked (respectively, unmarked) if and only if the last box of the k^{th} component of $T^{(i)}$ is marked (respectively, unmarked). Thus, Λ is an involution and hence a bijection.

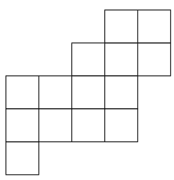
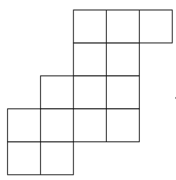
Since $Q_{\lambda/\mu}$ is a symmetric function, there are as many tableaux with content ν as there are with content $\bar{\nu}$. Thus, there is a bijection that maps tableaux of $D_{\lambda/\mu}$ with content ν to tableaux of $D_{\lambda/\mu}$ with content $\bar{\nu}$. Let Θ be such a bijection. Then $\Omega := \Theta \circ \Lambda$ is a content-preserving bijection since Ω is a composition of bijections and each of these two bijections flips the content. $\square$

Remark. The proof of Lemma 1.60 is slightly different from the proof of DeWitt [6] where she showed that the image of free entries (which are the entries of the last boxes of the components of $T^{(i)}$) are also free. Note that $c(\Lambda(T))$ is not the reverse of $c(T)$ if $c(T)_1 = 0$.

Example 1.61. *Let* $T = \begin{array}{ccccc} & & 1' & 1 & 1 \\ 1' & 1 & 3' & 3 & 4 \\ 1 & 3' & 3 & 4' & \\ & 4' & 5' & 5 & \\ & & 5 & & \end{array}$.

Then we have $\bar{T} = \begin{array}{ccccc} & & & 4 & 1 \\ & 5 & 4' & 3 & 1 \\ 5 & 5' & 3 & 3' & 1' \\ & 4' & 3' & 1 & \\ & & 1 & 1' & \end{array}$ *and* $\Lambda(T) = \begin{array}{ccccc} & & & 2 & 5' \\ & 1' & 2' & 3' & 5' \\ 1 & 1 & 3' & 3 & 5' \\ & 2' & 3' & 5' & \\ & & 5 & 5 & \end{array}$.

Definition 1.62. Let D be an unshifted diagram. The **transpose** of a diagram is the unshifted diagram obtained after first reflecting the boxes of D along the diagonal $\{(z, z) \mid z \in \mathbb{N}\}$ and then moving this arrangement of boxes such that the top row with boxes is in the first row and the lowermost box of the leftmost column with boxes is part of the diagonal $\{(z, z) \mid z \in \mathbb{N}\}$. We denote the transpose of a diagram by D^t.

Example 1.63. *For* $D =$  *we obtain* $D^t =$  .

Algebraic proofs of the Lemmas 1.64 and 1.68 were given by Barekat and van Willigenburg in [1, Proposition 3.3]. These proofs use the ring homomorphism θ due to Stembridge [20, Remark 3.2].

Lemma 1.64. *Let $D = D_{\lambda/\mu}$ be an unshifted diagram. Then there is a content-preserving bijection between tableaux of shape D and tableaux of shape D^t. In particular, $Q_D = Q_{D^t}$.*

Proof. Let T be a tableau of shape $D_{\lambda/\mu}$. Let Φ be the map that maps T to $\Phi(T)$ where $\Phi(T)$ is obtained as follows:

- Reflect and move the boxes of T together with their entries along the diagonal $\{(z,z) \mid z \in \mathbb{N}\}$. Denote the resulting filling of $D^t_{\lambda/\mu}$ by $\bar{T}$.
- For all i do the following:
  - If $\bar{T}(x,y) \in \{i', i\}$ and $\bar{T}(x+1,y) \in \{i', i\}$ then set $\Phi(T)(x,y) = i'$.
  - If $\bar{T}(x,y) \in \{i', i\}$ and $\bar{T}(x,y-1) \in \{i', i\}$ then set $\Phi(T)(x,y) = i$.
  - If $\bar{T}(x,y) \in \{i', i\}$ and neither $\bar{T}(x+1,y) \in \{i', i\}$ nor $\bar{T}(x,y-1) \in \{i', i\}$ then if (x,y) is the k^{th} such box counted from the left let (u,v) be the last box of the k^{th} component of $T^{(i)}$. If $T(u,v) = i'$ set $\Phi(T)(x,y) = i'$ and if $T(u,v) = i$ set $\Phi(T)(x,y) = i$.

One can see that Φ maps tableaux of D to tableaux of D^t.

After transposing, the rows and columns are weakly increasing since rows and columns interchange, and rows and columns are weakly increasing in T. Clearly, in $\Phi(T)$ there is at most one i in each column and at most one i' in each row. Hence, the properties of Definition 1.9 are satisfied.

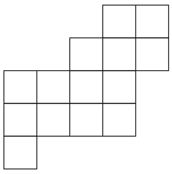
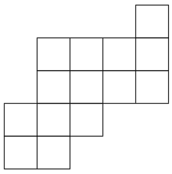
We have $c(\Phi(T)) = c(T)$.

Applying Φ to $\Phi(T)$ gives a tableau of the same shape as T. We have $\Phi(\Phi(T))^{(i)} = T^{(i)}$, and the last box of the k^{th} component of $\Phi(\Phi(T))^{(i)}$ is marked (respectively, unmarked) if and only if the last box of the k^{th} component of $T^{(i)}$ is marked (respectively, unmarked). Thus, Φ is an involution and hence a bijection. $\square$

Example 1.65. *Let* $T = \begin{array}{cccccccc} & & & & 1 & 1 & 4' & 4 \\ & & 1' & 1 & 2' & 2 & 4' & \\ 1' & 1 & 1 & 2 & 2 & 4 & 4 & \\ 1' & 4' & 4 & 4 & & & & \end{array}$.

Then we obtain $\bar{T} = \begin{array}{cccc} & & 1' & 1' \\ & & 1 & 4' \\ & 1' & 1 & 4 \\ & 1 & 2 & 4 \\ 1 & 2' & 2 & \\ 1 & 2 & 4 & \\ 4' & 4' & 4 & \\ 4 & & & \end{array}$ *and* $\Phi(T) = \begin{array}{cccc} & & 1' & 1 \\ & & 1' & 4' \\ & 1' & 1 & 4' \\ & 1 & 2' & 4 \\ 1' & 2' & 2 & \\ 1' & 2 & 4' & \\ 4' & 4 & 4 & \\ 4' & & & \end{array}$.

Definition 1.66. Let D be an unshifted diagram. The **rotation** of a diagram is the unshifted diagram obtained after rotating the boxes of D through $180°$ and moving this arrangement of boxes such that the uppermost row with boxes is in the first row and the lowermost box of the leftmost column with boxes is part of the diagonal $\{(z, z) \mid z \in \mathbb{N}\}$. We denote the rotation of a diagram by D^o.

Example 1.67. *For* $D =$  *we obtain* $D^o =$ .

Lemma 1.68. *Let $D = D_{\lambda/\mu}$ be an unshifted diagram. Then there is a content-preserving bijection between the tableaux of shape D and the tableaux of shape D^o. In particular, $Q_D = Q_{D^o}$.*

Proof. Let T be a tableau of shape $D_{\lambda/\mu}$. Let Φ be as in the proof of Lemma 1.64 and let Ω be as in the proof of Lemma 1.60. Then $\Phi \circ \Omega$ is a content-preserving bijection and the shape of the resulting tableau is D^o since the bijection first reflects along the diagonal $\{(z, -z) \mid z \in \mathbb{N}\}$ and then along the diagonal $\{(z, z) \mid z \in \mathbb{N}\}$, which is the same as a rotation through $180°$. $\square$

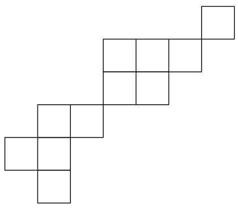
By Lemma 1.16 and Lemma 1.60, if D is a diagram obtained from $D_{\lambda/\mu}$ by orthogonally transposing some components of $D_{\lambda/\mu}$ then we have $Q_D = Q_{D_{\lambda/\mu}}$. Also, by Lemma 1.16

and Lemmas 1.64 and 1.68, if D is a diagram obtained from $D_{\lambda/\mu}$ by transposing and/or rotating some components of $D_{\lambda/\mu}$ except for the first one then we have $Q_D = Q_{D_{\lambda/\mu}}$. If the diagram $D_{\lambda/\mu}$ is unshifted then also the first component can be transposed or rotated.

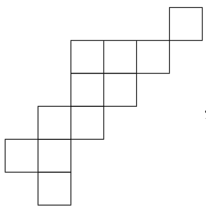
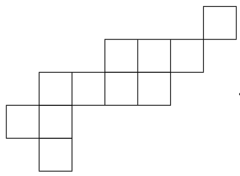
The following definition and lemma is inspired by [1, Section 2.1] where Barekat and van Willigenburg gave some operations on diagrams. In that paper the diagrams $\Delta_i^{\leftarrow}(D_{\lambda/\mu})$ and $\Delta_i^{\downarrow}(D_{\lambda/\mu})$ are defined only for unshifted diagrams and are used to describe border strips.

Definition 1.69. Let $\lambda, \mu \in DP$ be such that $D_{\lambda/\mu}$ is basic and $d = comp(D_{\lambda/\mu}) \geq 2$. Let $1 \leq i \leq d-1$. Then the diagram $\Delta_i^{\leftarrow}(D_{\lambda/\mu})$ is defined by shifting all boxes of the components $C_{i+1}, C_{i+2}, \ldots, C_d$ one box to the left. The diagram $\Delta_i^{\downarrow}(D_{\lambda/\mu})$ is defined by shifting all boxes of the components $C_{i+1}, C_{i+2}, \ldots, C_d$ one box down and removing the first row which is empty.

Example 1.70. *Let* $\lambda = (11, 9, 7, 4, 2, 1)$ *and* $\mu = (10, 6, 5, 2)$. *Then we have*

$D_{\lambda/\mu} =$ 

and obtain

$\Delta_1^{\leftarrow}(D_{\lambda/\mu}) =$  , $\Delta_1^{\downarrow}(D_{\lambda/\mu}) =$  .

Remark. Clearly, the diagrams $\Delta_i^{\leftarrow}(D_{\lambda/\mu})$ and $\Delta_i^{\downarrow}(D_{\lambda/\mu})$ are different.

Lemma 1.71. *Let $\lambda, \mu \in DP$ be such that $D_{\lambda/\mu}$ is basic and $d = comp(D_{\lambda/\mu}) \geq 2$. For some $1 \leq i \leq d-1$ let $\Delta_i^{\leftarrow}(D_{\lambda/\mu}) = D_{\alpha^{(i)}/\beta^{(i)}}$ and $\Delta_i^{\downarrow}(D_{\lambda/\mu}) = D_{\gamma^{(i)}/\delta^{(i)}}$. Then there is a content-preserving bijection between the set $T(\lambda/\mu)$ and the set $T(\alpha^{(i)}/\beta^{(i)}) \dot\cup T(\gamma^{(i)}/\delta^{(i)})$. In particular, $Q_{\lambda/\mu} = Q_{\alpha^{(i)}/\beta^{(i)}} + Q_{\gamma^{(i)}/\delta^{(i)}}$.*

Proof. Let T be a tableau of shape $D_{\lambda/\mu}$. Let (x, y) be the uppermost rightmost box of the component C_i. Let Ξ be the following map:

- If $T(x-1, y+1) < |T(x, y)|$ then shift all boxes above the x^{th} row together with their entries one box to the left.
- If $T(x-1, y+1) \geq |T(x, y)|$ then shift all boxes to the right of the y^{th} column together with their entries one box down.

It is clear that the map Ξ maps each tableau from $T(\lambda/\mu)$ to some tableau from $T(\alpha^{(i)}/\beta^{(i)}) \dot\cup T(\gamma^{(i)}/\delta^{(i)})$. Also, Ξ is a content-preserving map.

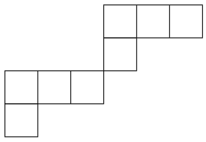
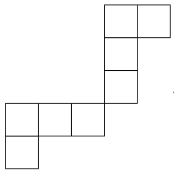
Let $U \in T(\alpha^{(i)}/\beta^{(i)}) \dot\cup T(\gamma^{(i)}/\delta^{(i)})$. The inverse of Ξ is given by the following map:

- If $U \in T(\alpha^{(i)}/\beta^{(i)})$ then shift all boxes above the x^{th} row together with their entries one box to the right.
- If $U \in T(\gamma^{(i)}/\delta^{(i)})$ then shift all boxes to the right of the y^{th} column together with their entries one box up.

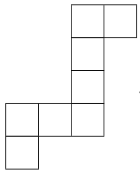
Hence, Ξ is a content-preserving bijection between the two sets of tableaux. □

Lemma 1.72. *Let D be a basic diagram that has two components where both components are the same border strip. Then $Q_D = 2Q_{\Delta_1^{\leftarrow}(\tilde{D})}$ where $\tilde{D}$ is the diagram obtained from D by transposing the second component.*

Proof. By Lemma 1.64, $Q_D = Q_{\tilde{D}}$. Then $\Delta_1^{\leftarrow}(\tilde{D}) = \Delta_1^{\downarrow}(\tilde{D})^t$. Hence, by Lemma 1.71, we have $Q_D = 2Q_{\Delta_1^{\leftarrow}(\tilde{D})}$. □

Example 1.73. *For* $D =$  *we have* $\tilde{D} =$  .

By Lemma 1.72 we have $Q_D = 2Q_{\Delta_1^{\leftarrow}(\tilde{D})}$ *where*

$$\Delta_1^{\leftarrow}(\tilde{D}) =$$  .

2 Inequalities of the coefficients $f^{\lambda}_{\mu\nu}$

Inequalities of the classical Littlewood-Richardson coefficients $c^{\lambda}_{\mu\nu}$ were given by Stembridge [21] and have been generalised by Gutschwager [7]. They help simplify proofs since they give lower or upper bounds for any given $c^{\lambda}_{\mu\nu}$ and allow to restrict problems to smaller cases (and diagrams). Inequalities of the shifted Littlewood-Richardson coefficients were given by Bessenrodt [2]. These inequalities are the shifted analogues of the inequalities appearing in Stembridge's paper. Although the problem of finding shifted analogues of the inequalities of Gutschwager's paper is not solved yet (see Section 7.1 for some work concerning this), we find other inequalities that still allows us to restrict the diagrams that we have to consider.

Lemma 2.1 makes use of the diagrams $U_k(\lambda/\mu)$ of Definition 1.45 and allows sometimes to reduce problems to smaller diagrams in the subsequent chapters. The remaining lemmas of this chapter will also be used to reduce problems to smaller diagrams, mainly in Chapter 6.

Lemma 2.1. *Let $\lambda, \mu \in DP$. Let $\nu = c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let k be such that $U_k(\lambda/\mu)$ has shape $D_{\alpha/\beta}$ for some $\alpha, \beta \in DP$. Then*

$$f^{\alpha}_{\beta\gamma} \leq f^{\lambda}_{\mu(\nu_1,\dots\nu_{k-1},\gamma_1,\dots,\gamma_{\ell(\gamma)})}.$$

Proof. Given m different amenable tableaux of $D_{\alpha/\beta}$ with content $(\gamma_1, \dots, \gamma_{\ell(\gamma)})$, we can obtain m different amenable tableaux of $D_{\lambda/\mu}$ with content $(\nu_1, \dots \nu_{k-1}, \gamma_1, \dots, \gamma_{\ell(\gamma)})$ as follows: For each box of $D_{\alpha/\beta}$ replace its entry i (respectively, i') by $i+k-1$ (respectively, $(i+k-1)'$). Use these as the filling of the boxes of $U_k(\lambda/\mu)$. Fill the other boxes of the diagram $D_{\lambda/\mu}$ as in $T_{\lambda/\mu}$. We only need to show k-amenability, which follows straightforwardly by Corollary 1.44. □

Example 2.2. *Let* $\lambda = (10, 8, 7, 6, 4, 1)$ *and* $\mu = (5, 3, 2, 1)$ *and consider*

$$T_{\lambda/\mu} = \begin{array}{ccccccc} & 1' & 1 & 1 & 1 & 1 & 1 \\ 1' & 1 & 2' & 2 & 2 & 2 & \\ 1' & 2' & 2 & 3' & 3 & 3 & \\ 1' & 2' & 3' & 3 & 4' & 4 & \\ 1 & 2' & 3 & 4 & 4 & & \\ & 2 & & & & & \end{array} .$$

Let $k = 3$. *Then*

$$U_3(\lambda/\mu) = \begin{array}{cccc} & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \end{array} .$$

Two amenable tableaux with the same content are

$$\begin{array}{cccc} & 1' & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & \end{array}, \quad \begin{array}{cccc} & 1 & 1 & 1 \\ 1' & 2 & 2 & 2 \\ 1 & 3 & 3 & \end{array}.$$

We obtain two amenable tableaux of $D_{\lambda/\mu}$ *with the same content:*

$$\begin{array}{ccccccc} & 1' & 1 & 1 & 1 & 1 & 1 \\ 1' & 1 & 2' & 2 & 2 & 2 & \\ 1' & 2' & 2 & 3' & 3 & 3 & \\ 1' & 2' & 3 & 3 & 4 & 4 & \\ 1 & 2' & 4 & 5 & 5 & & \\ & 2 & & & & & \end{array}, \quad \begin{array}{ccccccc} & 1' & 1 & 1 & 1 & 1 & 1 \\ 1' & 1 & 2' & 2 & 2 & 2 & \\ 1' & 2' & 2 & 3 & 3 & 3 & \\ 1' & 2' & 3' & 4 & 4 & 4 & \\ 1 & 2' & 3 & 5 & 5 & & \\ & 2 & & & & & \end{array} .$$

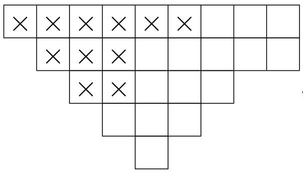
Definition 2.3. Let $\lambda, \mu \in DP$, let $2 \leq a \leq \ell(\mu) + 1$ and let $b \geq \ell(\lambda)$. Let $\Gamma_a^{\rightarrow}(D_{\lambda/\mu})$ be the diagram obtained from $D_{\lambda/\mu}$ by shifting all boxes above the a^{th} row one box to the right. Let $\Gamma_b^{\downarrow}(D_{\lambda/\mu})$ be the diagram obtained from $D_{\lambda/\mu}$ by shifting all boxes (x, y) such that $y < b$ one box down.

Example 2.4. *For* $\lambda = (8, 7, 4, 3, 1)$ *and* $\mu = (5, 2, 1)$ *we have*

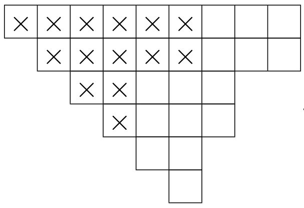
$$D_{\lambda/\mu} = \begin{array}{cccccccc} \times & \times & \times & \times & \times & \square & \square & \square \\ & \times & \times & \square & \square & \square & \square & \square \\ & & \times & \square & \square & \square & & \\ & & & \square & \square & \square & & \\ & & & & \square & & & \end{array}$$

and

$$\Gamma_4^{\rightarrow}(D_{\lambda/\mu}) =$$



$$,\ \Gamma_6^{\downarrow}(D_{\lambda/\mu}) =$$



.

Lemma 2.5. *Let $\lambda, \mu \in DP$ and let $2 \leq a \leq \ell(\mu) + 2$ and $b \geq \ell(\lambda)$. Let $\Gamma_a^{\rightarrow}(D_{\lambda/\mu})$ have shape $D_{\alpha/\beta}$ and let $\Gamma_b^{\downarrow}(D_{\lambda/\mu})$ have shape $D_{\tilde{\alpha}/\tilde{\beta}}$.*

Then $f_{\mu\nu}^{\lambda} \leq f_{\beta\nu}^{\alpha}$ and $f_{\mu\nu}^{\lambda} \leq f_{\tilde{\beta}\nu}^{\tilde{\alpha}}$.

Proof. For every given amenable tableau T of shape λ/μ one can obtain an amenable tableau $\hat{T}$ of shape $D_{\alpha/\beta}$ by setting $\hat{T}(x,y) = T(x, y-1)$ for all $1 \leq x \leq a-1$ and $\hat{T}(x,y) = T(x,y)$ for all $x \geq a$ such that $(x,y) \in D_{\alpha/\beta}$. Since $w(\hat{T}) = w(T)$, the tableau $\hat{T}$ is amenable. If $\hat{T} = \hat{T}'$ for two amenable tableaux T, T' of shape $D_{\lambda/\mu}$ then $T(x,y) = \hat{T}(x, y+1) = \hat{T}'(x, y+1) = T'(x,y)$ for all $1 \leq x \leq a-1$ and $T(x,y) = \hat{T}(x,y) = \hat{T}'(x,y) = T'(x,y)$ for all $x \geq a$ and, hence, $T = T'$. Thus, the statement $f_{\mu\nu}^{\lambda} \leq f_{\beta\nu}^{\alpha}$ follows.

For every given amenable tableau T of shape λ/μ one can obtain an amenable tableau $\tilde{T}$ of shape $D_{\tilde{\alpha}/\tilde{\beta}}$ by setting $\tilde{T}(x,y) = T(x-1, y)$ for all $1 \leq y \leq b-1$ and $\tilde{T}(x,y) = T(x,y)$ for all $y \geq b$ such that $(x,y) \in D_{\tilde{\alpha}/\tilde{\beta}}$. By Lemma 1.42, the tableau $\tilde{T}$ is amenable. If $\tilde{T} = \tilde{T}'$ for two amenable tableaux T, T' of shape $D_{\lambda/\mu}$ then $T(x,y) = \tilde{T}(x+1, y) = \tilde{T}'(x+1, y) = T'(x,y)$ for all $1 \leq y \leq b-1$ and $T(x,y) = \tilde{T}(x,y) = \tilde{T}'(x,y) = T'(x,y)$ for all $y \geq b$ and, hence, $T = T'$. Thus, the statement $f_{\mu\nu}^{\lambda} \leq f_{\tilde{\beta}\nu}^{\tilde{\alpha}}$ follows. □

Remark. The statement $f_{\mu\nu}^{\lambda} \leq f_{\beta\nu}^{\alpha}$ from Lemma 2.5 appeared in the proof of [2, Theorem 2.2] and is, hence, due to Bessenrodt. In the same proof the statement $f_{\mu\nu}^{\lambda} \leq f_{\tilde{\beta}\nu}^{\tilde{\alpha}}$ for $b = \mu_1 + 2$ can be found (without explicitly stating that $\mu_1 + 2 \geq \ell(\lambda)$ is required).

Lemma 2.6. *Let w be an amenable word. Let $\tilde{w}$ be a word such that after removing one letter 1 the word obtained is w (this means that $\tilde{w}$ can be obtained from w by adding a letter 1). Then $\tilde{w}$ is amenable.*

Proof. The number of letters equal to 1 in $\tilde{w}$ is greater than the number of letters equal to 1 in w. Then the word $\tilde{w}$ is not amenable only if there is some $j \geq n := \ell(\tilde{w})$ such that $m_1(j) = m_2(j)$ and w_{j-n+1} is this added 1. But then for the word w we have $m_1(j-2) < m_2(j-2)$; a contradiction to the amenability of T. □

Definition 2.7. Let $\alpha \in DP$ and $a \in \mathbb{N}$. Then

$$\alpha + (1^a) := (\alpha_1 + 1, \alpha_2 + 1, \ldots, \alpha_a + 1, \alpha_{a+1}, \alpha_{a+2}, \ldots, \alpha_\ell(\alpha)).$$

Lemma 2.8. *Let $\lambda, \mu \in DP$ and let $1 \leq a \leq \ell(\mu)$. Then $f^\lambda_{\mu\nu} \leq f^{\lambda+(1^a)}_{\mu+(1^{a-1}),\nu+(1)}$.*

Proof. For this proof we will assume that for a tableau of shape $D_{\lambda/\mu}$ the boxes of D_μ are not removed but instead are filled with 0. Given an amenable tableau T of shape $D_{\lambda/\mu}$ we obtain an amenable tableau $\bar{T}$ of shape $D_{(\lambda+(1^a))/(\mu+(1^{a-1}))}$ as follows. Insert a box with entry zero into each of the first $a-1$ rows such that the rows are weakly increasing from left to right and insert a box with entry 1 into the a^{th} row such that this row is weakly increasing from left to right.

The word $w(\bar{T})$ differs from $w(T)$ only by one added 1. By Lemma 2.6, the word $w(\bar{T})$ is amenable. Clearly, if $T \neq T'$ for some tableaux $T, T' \in T(\lambda/\mu)$ then $\bar{T} \neq \bar{T}'$. □

Remark. Note that $\Gamma^{\rightarrow}_a(D_{\lambda/\mu}) \cup \{(a, a+\mu_a)\}$ has shape $D_{(\lambda+(1^a))/(\mu+(1^{a-1}))}$.

The proof of Lemma 2.8 is inspired by the proof of Theorem 3.1 in [7] where Gutschwager gives a similar statement for Schur functions.

Lemma 2.9. *Let $\lambda, \mu \in DP$ and let $b \geq \ell(\lambda)$. Let $(a, b-1)$ be the uppermost box of $D_{\lambda/\mu}$ in the $(b-1)^{th}$ column. Let $\Gamma^{\downarrow}_b(D_{\lambda/\mu}) \cup \{(a, b-1)\}$ have shape $D_{\alpha/\beta}$.*

Then $f^\lambda_{\mu\nu} \leq f^\alpha_{\beta,\nu+(1)}$.

Proof. For this proof we will assume that for a tableau of shape $D_{\lambda/\mu}$ the boxes of D_μ are not removed but instead are filled with 0. Given an amenable tableau T of shape $D_{\lambda/\mu}$ we obtain an amenable tableau $\bar{T}$ of shape $D_{\alpha/\beta}$ as follows. Insert a box with entry zero into each of the first $b-2$ columns such that the columns are weakly increasing from top to bottom and insert a box with entry 1 into the $(b-1)^{\text{th}}$ column such that this column is weakly increasing from top to bottom if there is no $1'$ or 1 in this column or else insert a box with entry $1'$ into the $(b-1)^{\text{th}}$ column such that this column is weakly increasing from top to bottom.

Let $\tilde{T}$ be the tableau defined by $\tilde{T}(x,y) := T(x-1,y)$ for all $1 \leq y \leq b-1$ and $\tilde{T}(x,y) = T(x,y)$ for all $y \geq b$ such that $(x,y) \in \Gamma_b^{\downarrow}(D_{\lambda/\mu})$. By Lemma 2.5, the tableau $\tilde{T}$ is amenable. The word $w := w(\bar{T})$ differs from $w(\tilde{T})$ only by an added $1'$ or an added 1. If a $1'$ is added then clearly, the tableau $\bar{T}$ is amenable. If a 1 is added then, by Lemma 2.6, the word $w(\bar{T})$ is amenable. Clearly, if $T \neq T'$ for some tableaux $T, T' \in T(\lambda/\mu)$ then $\bar{T} \neq \bar{T}'$. □

3 Classification of Q-multiplicity-free skew Schur Q-functions

The (s-)multiplicity-free products of Schur functions are classified by Stembridge in [21]. Then the (P-)multiplicity-free products of Schur P-function (some multiple of Schur Q-functions) are classified by Bessenrodt in [2]. The (s-)multiplicity skew Schur functions are classified by Gutschwager in [7]. Bessenrodt considered the problem of multiplicity-freeness for the shifted analogue of Schur functions (namely P-functions) while Gutschwager considered the problem of multiplicity-freeness for skew Schur functions. Still open was the problem for the shifted analogue of skew Schur functions, namely the skew Schur Q-functions.

In this chapter we will classify Q-multiplicity-free Schur Q-functions. We will vastly use Lemmas 1.42 and 1.60. The first lemma allows us to easily prove that the tableaux appearing in this chapter are amenable and the latter lemma enables us to always prove a statement for some given diagram and its orthogonal transposition and, hence, cut the work in half.

Note that if a proof of the subsequent lemmas explicitly states how to obtain a tableau then usually it is followed by an example depicting the tableaux obtained in these proofs.

Definition 3.1. A symmetric function $f \in span(Q_\lambda \mid \lambda \in DP)$ is called **Q-multiplicity-free** if the coefficients of the constituents in the decomposition of f into Schur Q-functions are from $\{0, 1\}$. In particular, a skew Schur Q-function $Q_{\lambda/\mu}$ is called Q-multiplicity-free if $f^\lambda_{\mu\nu} \leq 1$ for all $\nu \in DP$.

Our goal is to classify Q-multiplicity-free skew Schur Q-functions given in Theorem 3.58. First we will prove a number of lemmas that exclude all non-Q-multiplicity-free skew Schur Q-functions which results in Proposition 3.33 that is a list of the remaining skew Schur Q-functions. Then we will show that these remaining skew Schur Q-functions are Q-multiplicity-free.

Hypothesis. *We will always assume that* λ *and* μ *are such that* $D_{\lambda/\mu}$ *is basic (see Definition 1.13).*

3.1 Excluding non-Q-multiplicity-free skew Schur Q-functions

Remark. >From now on we will use Corollary 1.44 to prove amenability of a tableau. If some entries of a tableau do not satisfy the properties of Corollary 1.44 then we will show that for these entries the properties of Lemma 1.42 are satisfied and use this lemma to prove amenability.

We will analyse diagrams and show that they are not Q-multiplicity-free by finding two different amenable tableaux with the same content derived by changing some entries in the tableau $T_{\lambda/\mu}$. We are able to find all diagrams that are not Q-multiplicity-free by this way and, hence, the remaining diagrams must be Q-multiplicity-free.

Remark. Let $\lambda, \mu \in DP$ and $\nu = c(T_{\lambda/\mu})$. Proposition 1.50 states that $f^{\lambda}_{\mu\nu} = 1$ is only possible if all the P_is (from Definition 1.45) are connected.

Hypothesis. *>From now on we will consider only diagrams such that each* P_i *is connected.*

Lemma 3.2. *Let* $\lambda, \mu \in DP$*. Let* $\nu = c(T_{\lambda/\mu})$ *and* $n := \ell(\nu)$*. If* P_n *is neither a hook nor a rotated hook then* $Q_{\lambda/\mu}$ *is not* Q*-multiplicity-free.*

Proof. By Lemma 2.1, it is enough to find two amenable tableaux of P_n with the same content. Hence, consider the diagram P_n and let P_n be neither a hook nor a rotated hook. Then we can find a subset of boxes of P_n, U say, such that all but one boxes form a (p, q)-hook where $p, q \geq 2$ and there is either a single box above the rightmost box of the hook, or a single box to the left of the lowermost box of the hook. By Lemmas 1.64, 2.5, 2.8 and 2.9, it is enough to assume that P_n has shape $D_{(4,2)/(2)}$. Since $Q_{(4,2)/(2)} = Q_{(4)} + 2Q_{(3,1)}$, the statement follows. □

Lemma 3.3. *Let $\lambda, \mu \in DP$. Let $\nu = c(T_{\lambda/\mu})$ and $n := \ell(\nu) > 1$. Let P_n be a (p,q)-hook or a rotated (p,q)-hook where $p, q \geq 3$. Suppose the last box of P_{n-1} is not in the row directly above the row of the last box of P_n. Then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. We may assume that P_n is a (p,q)-hook where $p, q \geq 3$. Otherwise, P_n is a rotated (p,q)-hook where $p, q \geq 3$ and we may consider $D^{ot}_{\lambda/\mu}$ since if $D^{ot}_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ then, by Lemma 1.59, the set of boxes $T^{(n)}_{\alpha/\beta}$ is a (q,p)-hook where $p, q \geq 3$.

By Lemma 2.1, we may assume that $n = 2$. Let (x, y) be the last box of P_2. By Lemmas 2.5, 2.8 and 2.9, we may assume that $(x, y-1)$ is the last box of P_1. We get a new tableau T if we set $T(x, y-1) = 3$, $T(x-1, y-1) = 1$, $T(x, y) = 3$, $T(x-1, y) = 2$ and $T(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$.

By Corollary 1.44, this tableau is m-amenable for $m \neq 3$. We have $T(x, y-1) = 3$ but there is no 2 in the $(y-1)^{\text{th}}$ column. However, there are at least two 2s with no 3 below them in the first two boxes of P_2. Hence, by Lemma 1.42, this tableau is amenable.

We get another tableau T' if we set $T'(x, y) = 3$, $T'(x-1, y) = 3'$, $T'(x, y-1) = 2$, $T'(x-1, y-1) = 1$ and $T'(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$.

By Corollary 1.44, this tableau is m-amenable for $m \neq 2, 3$. Since there is a 1 but no 2 in the y^{th} column, 2-amenability follows. We have $T'(x, y) = 3$ but there is no 2 in the y^{th} column. Also, we have $T'(x-1, y) = 3'$ and $T'(x-2, y-1) \neq 3'$. However, in the first two boxes of P_n are 2s with no 3 below. Additionally, there is another 2 with no 3 below in the $(y-1)^{\text{th}}$ column. Thus, by Lemma 1.42, 3-amenability follows. ☐

Example 3.4. *For* $T_{\lambda/\mu} = \begin{array}{|c|c|c|c|}\hline 1' & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & 2 \\ \hline 1' & 2' & \multicolumn{2}{c}{} \\ \cline{1-2} 1 & 2 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$ *we obtain* $T = \begin{array}{|c|c|c|c|}\hline 1' & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & 2 \\ \hline 1 & 2 & \multicolumn{2}{c}{} \\ \cline{1-2} 3 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$, $T' = \begin{array}{|c|c|c|c|}\hline 1' & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & 2 \\ \hline 1 & 3' & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$.

We have $Q_{(7,6,3,2)/(3,2,1)} = Q_{(7,5)} + Q_{(7,4,1)} + Q_{(7,3,2)} + Q_{(6,5,1)} + 2Q_{(6,4,2)} + Q_{(6,3,2,1)} + Q_{(5,4,3)} + Q_{(5,4,2,1)}$.

Lemma 3.5. *Let $\lambda, \mu \in DP$. Let $\nu = c(T_{\lambda/\mu})$ and $n := \ell(\nu) \geq 2$. Let there be some $k < n$ such that the last box of P_k is in a row strictly lower than the last box of P_n and* *some $i < n$ such that the first box of P_i is in a column strictly to the right of the first box of P_n. Then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. Let k, i be maximal with respect to these conditions and let $j := \min\{k, i\}$. By Lemma 2.1, we may assume that $j = 1$. First, we assume that $i \leq k$. Then let $\bar{k}$ be minimal such that the last box of $P_{\bar{k}}$ is in a row strictly lower than the last box of P_n. Let (u, v) be the lowermost box in the rightmost column with a box of $P_{\bar{k}}$ in a row strictly lower than the last box of $P_{\bar{k}+1}$. Let $x := u - \bar{k} + i$ and $y := v - \bar{k} + i$. Then (x, y) is the lowermost box of P_i in the y^{th} column. We get a new tableau T if after the $(i-1)^{\text{th}}$ step of the algorithm of Definition 1.45 we use $P'_i := P_i \setminus \{(x, y)\}$ instead of P_i.

Let $P'_z = T^{(z)}$. Then for $i + 1 \leq r \leq \bar{k}$ if $(x + r - i, y + r - i) \in P_r$ then we have $(x + r - i - 1, y + r - i - 1) \in P'_r$. Hence, $(x, y) \in P'_{i+1}$. Clearly, by Corollary 1.44, this tableau is m-amenable for $m \neq i + 1$. We possibly have $T(x, y) = (i + 1)'$ and $T(x - 1, y - 1) \neq i'$. But there is an i with no $i + 1$ below in the column of the first box of P_i. Thus, by Lemma 1.42, $(i + 1)$-amenability follows.

Let (c, d) be the last box of $P_{\bar{k}+1}$. We get another tableau T' with the same content if we set $T'(c, d) = (\bar{k} + 1)'$ and $T'(e, f) = T_{\lambda/\mu}(e, f)$ for every other box $(e, f) \in D_{\lambda/\mu}$

By Corollary 1.44, it is clear that T' is amenable if T is and we have $c(T') = c(T) = (\nu_1, \dots, \nu_{i-1}, \nu_i - 1, \nu_{i+1}, \dots, \nu_{\bar{k}}, \nu_{\bar{k}+1} + 1, \nu_{\bar{k}+2}, \dots \nu_n)$.

If $k \leq i$ then $U_k(\lambda/\mu)$ is unshifted and we showed that two amenable tableaux of $U_k(\lambda/\mu)^t$ with the same content exist. By Lemma 1.64, the statement follows. ◻

Example 3.6. *For* $T_{\lambda/\mu} = \begin{array}{|c|c|c|c|c|}\hline 1' & 1 & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & 2 & \\ \cline{1-4} 1 & 2' & 3 & 3 & \\ \cline{1-4} & 2 & & & \\ \cline{2-2} \end{array}$ *we get* $T = \begin{array}{|c|c|c|c|c|}\hline 1' & 1 & 1 & 1 & 1 \\ \hline 1 & 2' & 2 & 2 & \\ \cline{1-4} 2 & 2 & 3 & 3 & \\ \cline{1-4} & 3 & & & \\ \cline{2-2} \end{array}$, $T' = \begin{array}{|c|c|c|c|c|}\hline 1' & 1 & 1 & 1 & 1 \\ \hline 1 & 2' & 2 & 2 & \\ \cline{1-4} 2 & 2 & 3' & 3 & \\ \cline{1-4} & 3 & & & \\ \cline{2-2} \end{array}$.

We have $Q_{(7,5,4,1)/(2,1)} = Q_{(7,5,2)} + Q_{(7,4,3)} + Q_{(7,4,2,1)} + 2Q_{(6,5,3)} + Q_{(6,5,2,1)} + Q_{(6,4,3,1)}$.

Lemma 3.7. *Let $\lambda, \mu \in DP$. Let $\nu = c(T_{\lambda/\mu})$ and $n := \ell(\nu) > 1$. Let there be some $k < n$ such that there is a corner, (x, y) say, in P_k above the boxes of P_n and let there be* *some $i \le k$ such that the first box of P_i is above the $(x-k+i)^{th}$ row. Then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. Let k be minimal and i be maximal with respect to these conditions. Then for all $i+1 \le a \le k$ the first box of P_a has no box of P_a below. Let $(x-k+a, y)$ be the first box of P_a for $i+1 \le a \le k-1$ and let $(x-k+i, y)$ be the rightmost box of P_i in the $(x-k+i)^{\text{th}}$ row. We get a new tableau T if we set $T(x-k+i, y) = i+1$, $T(x-k+i-1, y) = i$, for all $i+1 \le a \le k$ set $T(x-k+a, y) = a+1$, $T(x, y) = k+1$ and $T(u, v) = T_{\lambda/\mu}(u, v)$ for every other box $(u, v) \in D_{\lambda/\mu}$. By Corollary 1.44, this tableau is amenable.

We get a new tableau T' if we set $T'(x, y) = (k+1)'$ and $T'(u, v) = T(u, v)$ for every other box $(u, v) \in D_{\lambda/\mu}$. We have $T'(x, y) = (k+1)'$ and $T'(x-1, y-1) \ne k'$. However, we have $T'(x-1, y) = k$ and there is no $k+1$ in the y^{th} column. Hence, by Lemma 1.42, T' is m-amenable for all m.

Clearly, we have $c(T) = c(T') = (\nu_1, \ldots, \nu_{i-1}, \nu_i - 1, \nu_{i+1} \ldots, \nu_k, \nu_{k+1}+1, \nu_{k+2}, \ldots, \nu_n)$.

□

Example 3.8. *For* $T_{\lambda/\mu} = \begin{array}{cccc} & & & 1' \\ 1 & 1 & 1 & 1 \\ & 2 & 2 & \end{array}$ *we get* $T = \begin{array}{cccc} & & & 1 \\ 1 & 1 & 1 & 2 \\ & 2 & 2 & \end{array}$, $T' = \begin{array}{cccc} & & & 1 \\ 1 & 1 & 1 & 2' \\ & 2 & 2 & \end{array}$.

We have $Q_{(5,4,2)/(4)} = Q_{(5,2)} + 2Q_{(4,3)} + Q_{(4,2,1)}$.

For $T_{\lambda/\mu} = \begin{array}{cccccc} & & & & & 1' \\ 1 & 1 & 1 & 1 & 1 & 1 \\ & 2 & 2 & 2 & 2 & 2 \\ & & 3 & 3 & 3 & 3 \\ & & & 4 & 4 & \\ & & & & 5 & \end{array}$ *we get* $T = \begin{array}{cccccc} & & & & & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 \\ & 2 & 2 & 2 & 2 & 3 \\ & & 3 & 3 & 3 & 4 \\ & & & 4 & 4 & \\ & & & & 5 & \end{array}$, $T' = \begin{array}{cccccc} & & & & & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 \\ & 2 & 2 & 2 & 2 & 3 \\ & & 3 & 3 & 3 & 4' \\ & & & 4 & 4 & \\ & & & & 5 & \end{array}$.

We have $Q_{(7,6,5,4,2,1/(6)} = Q_{(7,5,4,2,1)} + 2Q_{(6,5,4,3,1)}$.

Lemma 3.9. *Let $\lambda, \mu \in DP$. Let $\nu = c(T_{\lambda/\mu})$ and $n := \ell(\nu) > 1$. Let there be some $k > 1$ such that the first box of P_{k-1} is to the right of the column of first box of P_k, and P_{k-1} is not a hook. Then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. Let k be maximal with respect to this property. By Lemma 2.1, we may assume that $k = 2$. If the first box of P_1 is not a corner then Lemma 3.7 states that $Q_{\lambda/\mu}$ is not

Q-multiplicity-free. Thus, consider that the first box of P_1 is a corner. If the first box of P_1 is not in the row above the first box of P_2 then an orthogonally transposed version of Lemma 3.7 states that $Q_{\lambda/\mu}$ is not Q-multiplicity-free. Since P_1 is not a hook, there are v, w such that the boxes $(v-1, w), (v, w), (v, w-1) \in P_1$ and the first box of P_1 is not in the w^{th} column. Let v be maximal with respect to this property.

We get a new tableau T if we use $P_1' := P_1 \setminus \{(v, w)\}$ instead of P_1 in the algorithm of Definition 1.45. By Corollary 1.44, it is clear that T is i-amenable for $i \neq 2$. We possibly have $T(v, w) = 2'$ and $T(v-1, w-1) \neq 1'$. However, in the column containing the first box of P_1 there is a 1 and no 2. Thus, by Lemma 1.42, this tableau is amenable.

We get another tableau T' if we set $T'(v-1, w) = 1'$ and $T'(r, s) = T(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. By Corollary 1.44, T' is i-amenable for $i \neq 2$. There is a 2 but no 1 in the w^{th} column. However, in the column containing the first box of P_1 there is a 1 and no 2. We possibly have $T(v, w) = 2'$ and $T(v-1, w-1) \neq 1'$. However, we have $T(v-1, w) = 1'$. Thus, by Lemma 1.42, T' is amenable.

It is easy to see that $c(T) = c(T')$. □

Example 3.10. *For*

$$T_{\lambda/\mu} = \begin{array}{cccccc} & 1' & 1 & 1 & 1 & 1 \\ & 1' & 2' & 2 & 2 & \\ 1 & 1 & 2 & 3 & 3 & \end{array}$$

and $k = 2$ we obtain

$$T = \begin{array}{cccccc} & 1' & 1 & 1 & 1 & 1 \\ & 1 & 2' & 2 & 2 & \\ 1 & 2 & 2 & 3 & 3 & \end{array}, \quad T' = \begin{array}{cccccc} & 1' & 1 & 1 & 1 & 1 \\ & 1' & 2' & 2 & 2 & \\ 1 & 2 & 2 & 3 & 3 & \end{array}.$$

We have $Q_{(8,6,5)/(3,2)} = Q_{(8,4,2)} + 2Q_{(7,5,2)} + 2Q_{(7,4,3)} + 2Q_{(6,5,3)}$.

For

$$T_{\lambda/\mu} = \begin{array}{cccccc} & 1' & 1 & 1 & 1 & 1 \\ & 1' & 2' & 2 & 2 & \\ 1 & 1 & 2' & 3' & 3 & \\ & 2 & 2 & 3 & 4 & \end{array}$$

and $k = 2$ we obtain

$$T = \begin{array}{cccccc} & 1' & 1 & 1 & 1 & 1 \\ & 1 & 2' & 2 & 2 & \\ 1 & 2' & 2 & 3' & 3 & \\ & 2 & 3 & 3 & 4 & \end{array}, \quad T' = \begin{array}{cccccc} & 1' & 1 & 1 & 1 & 1 \\ & 1' & 2' & 2 & 2 & \\ 1 & 2' & 2 & 3' & 3 & \\ & 2 & 3 & 3 & 4 & \end{array}.$$

We have $Q_{(8,6,5,4)/(3,2)} = Q_{(8,6,3,1)} + Q_{(8,5,4,1)} + Q_{(8,5,3,2)} + 2Q_{(7,6,4,1)} + 2Q_{(7,6,3,2)} + 2Q_{(7,5,4,2)}$.

Lemma 3.11. *Let $\lambda, \mu \in DP$. Let $\nu = c(T_{\lambda/\mu})$ and $n := \ell(\nu) > 1$. Let P_n be a (p, q)-hook where $p, q \geq 2$ and let (x, y) be the first box of P_n. Let there be some $k < n$ and some $i \geq y$ such that there are at least two boxes of P_k in the i^{th} column. Then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. Let k be maximal with respect to this property. Let (u, v) be the lowermost box of P_k in the i^{th} column and let (a_r, b_r) be the first box of P_r for all r. We get a new tableau T if we set $T(u, v) = k+1$, $T(u-1, v) = k$, for all $k+1 \leq r \leq n$ set $T(a_r, b_r) = r+1$ and $T(c, d) = T_{\lambda/\mu}(c, d)$ for every other box $(c, d) \in D_{\lambda/\mu}$. By Corollary 1.44, T is amenable.

Let (e, f) be the last box of P_n and let $(x-1, z)$ be the rightmost box of P_{n-1} in the $(x-1)^{\text{th}}$ row. We get another tableau T' if we set $T'(e, f) = n+1$, $T'(e-1, f) = n$, $T'(a_n, b_n) = n$, $T'(x-1, z) = n'$ and $T'(c, d) = T(c, d)$ for every other box $(c, d) \in D_{\lambda/\mu}$. By Corollary 1.44, T' is m-amenable for $m \neq n$. We have $T'(x-1, z) = n'$ and $T'(x-2, z-1) \neq (n-1)'$. However, if (g, h) is the last box of P_n then we have $T'(g-2, h-1) = (n-1)'$ and $T'(g-1, h) \neq n'$. Thus, by Lemma 1.42, amenability follows. ☐

Example 3.12. *For* $T_{\lambda/\mu} = \begin{array}{cccc} & & & 1' \\ 1' & 1 & 1 & 1 \\ 1 & 2' & 2 & 2 \\ & 2 & 3' & 3 \\ & & 3 & \end{array}$ *we obtain* $T = \begin{array}{cccc} & & & 1 \\ 1' & 1 & 1 & 2 \\ 1 & 2' & 2 & 3 \\ & 2 & 3' & 4 \\ & & 3 & \end{array}$, $T' = \begin{array}{cccc} & & & 1 \\ 1' & 1 & 1 & 2 \\ 1 & 2' & 2 & 3' \\ & 2 & 3 & 3 \\ & & 4 & \end{array}$.

We have $Q_{(6,5,4,3,1)/(5,1)} = Q_{(6,4,3)} + Q_{(6,4,2,1)} + 2Q_{(5,4,3,1)}$.

Corollary 3.13. *Let $\lambda, \mu \in DP$. Let $\nu = c(T_{\lambda/\mu})$ and $n := \ell(\nu) > 1$. Let P_n be a (p,q)-hook where $p, q \geq 2$ and let (x, y) be the first box of P_n. Let there be some $k < n$ and some $i \geq x$ such that there are at least two boxes of P_k in the i^{th} row. Then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. The diagram $U_k(\lambda/\mu)$ is unshifted. Then we may transpose $U_k(\lambda/\mu)$ and use Lemma 3.11. □

Now we are able to show an intermediate result that limits the number of corners of $D_{\lambda/\mu}$ and, hence, of D_λ if $\mu \neq \emptyset, (1)$. The number of corners of D_μ is also limited for most $D_{\lambda/\mu}$ because of orthogonal transposition. This restricts the number of cases we have to analyse.

Lemma 3.14. *Let $\lambda, \mu \in DP$ where $\mu \neq \emptyset, (1)$. If λ has more than two corners then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. Assume $Q_{\lambda/\mu}$ is Q-multiplicity-free where D_λ has more than two corners and $\mu \neq \emptyset, (1)$. We will give two amenable tableaux with the same content to show that the assumption of Q-multiplicity-freeness leads to a contradiction. Let $\nu = c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let k be maximal such that $U_k(\lambda/\mu)$ has at least three corners. Thus, at least one corner is in P_k. By Lemma 3.2, which states that P_n must be a hook or a rotated hook, P_n can have at most two corners and, hence, $k < n$. By Lemma 3.5, which states that either the uppermost or the lowermost corner must be in P_n, we only consider diagrams such that the uppermost or the lowermost corner is in P_n. Without loss of generality we may assume that the lowermost corner of $U_k(\lambda/\mu)$ is in P_n, otherwise $U_k(\lambda/\mu)$ is an unshifted diagram and we may transpose $U_k(\lambda/\mu)$. Thus, the uppermost corner is in P_k. By Lemma 3.7, which forbids to have boxes of P_k to the left and above a corner in P_k at once, the uppermost corner is the first box of P_k and it is the only corner of the diagram $U_k(\lambda/\mu)$ that is in P_k.

Case 1: two corners are in P_n.

Then P_n is a (p, q)-hook where $p \geq 2$ and $q \geq 2$. By Lemma 3.11 and Corollary 3.13, which in this case for all $k \leq i \leq n-1$ forbid to have more than one box of P_i in the column of the first box of P_n and in the row of the last box of P_n, all P_i are hooks.

Case 1.1: the last box of P_{n-1} is in the same row as the last box of P_n.

Let (u_a, v_a) be the last box of P_a for all a. We get a new tableau T_1 if for all $k \leq a \leq n$ we set $T_1(u_a, v_a) = a+1$, $T_1(u_a - 1, v_a) = a$ and $T_1(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. By Corollary 1.44, T_1 is m-amenable for $m \neq k+1$. Also by Corollary 1.44, the tableau T_1 is also $(k+1)$-amenable because in the column of the first box of P_k is a k and no $k+1$.

We get another tableau T_1' if we set $T_1'(u_n - 1, v_n) = n'$ and $T_1'(r, s) = T_1(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. By Corollary 1.44, T_1' is m-amenable for $m \neq n+1$. We have $T_1'(u_n, v_n) = n+1$ and $T_1'(u_n - 1, v_n) < n$, however, there is an n with no $n+1$ below in the first box of P_n, and we have $T_1'(u_{n-1}, v_{n-1}) = n$. Thus, by Lemma 1.42, $(n+1)$-amenability follows. We have $c(T_1) = c(T_1')$.

Case 1.2: the last box of P_{n-1} is in the row above the row of the last box of P_n.

For $p = 2$ we get $\mu = (1)$, which is a contradiction. Thus, we have $p > 2$. Let (u_a, v_a) be the last box of P_a for all a. We get a new tableau T_2 if we set $T_2(u_n, v_n) = n+1$, $T_2(u_n - 1, v_n) = (n+1)'$, for all $k \leq a \leq n-1$ set $T_2(u_a, v_a) = a+1$, $T_1(u_a - 1, v_a) = a$ and $T_2(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. By Corollary 1.44, T_2 is m-amenable for $m \neq n+1$. We have $T_2(u_n - 1, v_n) = (n+1)'$ and $T_2(u_n - 2, v_n - 1) \neq n'$. However, we have $T_2(u_n - 2, v_n) = n'$. Thus, by Lemma 1.42, $(n+1)$-amenability follows.

We get another tableau T_2' if we set $T_2'(u_n - 2, v_n) = n$ and $T_2'(r, s) = T_2(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. By Lemma 1.42, it is clear that T_2' is amenable if T_2 is amenable. We have $c(T_2) = c(T_2')$.

Case 2: only one corner is in P_n.

Let the second uppermost corner be in P_i. Then by Lemma 3.7, the second uppermost corner is the first box of P_i and the uppermost corner is the first box of P_k. If P_i has all boxes in a row then $\mu = \emptyset$; a contradiction. Thus, P_i has at least two corners. By Lemma 3.9, P_i is a hook. Then for all $i \leq j < n$ each P_j is a (p,q)-hook for some $p, q \geq 2$.

Case 2.1: The last box of P_{i-1} is in the same row as the last box of P_i.

Let (g,h) be the last box of P_i and (c_a, d_a) be the rightmost box of P_a in the lowermost row with boxes from P_a for all $k \leq a \leq i-1$. We get a new tableau T_3 if for all $k \leq a \leq i-1$ we set $T_3(c_a, d_a) = a+1$ if $(c_a+1, d_a) \notin D_{\lambda/\mu}$ or else set $T_3(c_a, d_a) = (a+1)'$ if $(c_a+1, d_a) \in D_{\lambda/\mu}$, set $T_3(c_a-1, d_a) = a$, $T_3(g,h) = i+1$ and $T_3(r,s) = T_{\lambda/\mu}(r,s)$ for every other box $(r,s) \in D_{\lambda/\mu}$. By Corollary 1.44, the tableau T_3 is m-amenable for $m \neq k+1, i+1$. We possibly have $T_3(c_k, d_k) = (k+1)'$ and $T_3(c_k - 1, d_k - 1) \neq k'$. If not, then there is possibly a $k+1$ in the ${d_k}^{\text{th}}$ column. Anyway, there is a k with no $k+1$ below in the first box of P_k. Thus, by Lemma 1.42, $(k+1)$-amenability follows. We have $T_3(g,h) = i+1$ and $T_3(g-1,h) < i$. However, there is an i with no $i+1$ below in the first box of P_i. Thus, by Lemma 1.42, $(i+1)$-amenability follows.

We get another tableau T_3' if we set $T_3'(g-1,h) = i$ and $T_3'(r,s) = T_3(r,s)$ for every other box $(r,s) \in D_{\lambda/\mu}$. Clearly, T_3' is amenable if T_3 is and we have $c(T_3) = c(T_3')$.

Case 2.2: The last box of P_{i-1} is in the row above the row of the last box of P_i.

If in the column of the last box of P_i there are only two boxes of P_i then we have $\mu = (1)$, which is a contradiction. Thus, there are at least three boxes of P_i in the column of the last box of P_i. Let (c_a, d_a) be the last box of P_a for all $k \leq a \leq i+1$. We get a new tableau T_4 if for all $k \leq a \leq i-1$ we set $T_4(c_a, d_a) = a+1$, $T_4(c_a - 1, d_a) = a$, $T_4(c_i, d_i) = i+1$, $T_4(c_i - 1, d_i) = (i+1)'$, $T_4(c_{i+1}, d_{i+1}) = i+2$, $T_4(c_{i+1} - 1, d_{i+1}) = i+1$ and $T_4(r,s) = T_{\lambda/\mu}(r,s)$ for every other box $(r,s) \in D_{\lambda/\mu}$.

By Corollary 1.44, the tableau T_4 is m-amenable for $m \neq k+1, i+1$. There is a k with no $k+1$ below in the first box of P_k. Thus, by Corollary 1.44, $(k+1)$-amenability follows. We have $T_4(c_i, d_i) = i+1$ and there is no i in the ${d_i}^{\text{th}}$ column. However, there is an i with no $i+1$ below in the first box of P_i. We have $T_4(c_i - 1, d_i) = (i+1)'$ and

$T_4(c_i - 2, d_i - 1) \neq i'$. However, we have $T_4(c_i - 2, d_i) = i'$. Thus, by Lemma 1.42, $(i+1)$-amenability follows.

We get another tableau T_4' if we set $T_4'(c_i - 2, d_i) = i$ and $T_4'(r,s) = T_4(r,s)$ for every other box $(r,s) \in D_{\lambda/\mu}$.

The tableau T_4' is m-amenable for $m \neq i+1$. We have $T_4'(c_i - 1, d_i) = (i+1)'$ and $T_4'(c_i - 2, d_i - 1) \neq i'$. However, there is an i with no $i+1$ below in the first box of P_i. Thus, by Lemma 1.42, $(i+1)$-amenability follows. We have $c(T_4) = c(T_4')$. □

Example 3.15. *For*

$$T_{\lambda/\mu} = \begin{array}{ccccccc} 1' & 1 & 1 & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & 2 & 2 & \\ 1' & 2' & 3' & 3 & 3 & 3 & \\ 1 & 2' & 3' & 4' & 4 & 4 & \\ & 2 & 3' & 4' & 5' & 5 & \\ & & 3 & 4 & 5 & & \end{array}$$

we obtain

$$T_1 = \begin{array}{ccccccc} 1' & 1 & 1 & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & 2 & 2 & \\ 1 & 2' & 3' & 3 & 3 & 3 & \\ 2 & 2 & 3' & 4' & 4 & 4 & \\ & 3 & 3 & 4 & 5 & 5 & \\ & & 4 & 5 & 6 & & \end{array}, \quad T_1' = \begin{array}{ccccccc} 1' & 1 & 1 & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & 2 & 2 & \\ 1 & 2' & 3' & 3 & 3 & 3 & \\ 2 & 2 & 3' & 4' & 4 & 4 & \\ & 3 & 3 & 4 & 5' & 5 & \\ & & 4 & 5 & 6 & & \end{array}.$$

We have $Q_{(10,8,7,6,5,3)/(3,2,1)} = Q_{(10,8,7,5,3)} + Q_{(10,8,7,5,2,1)} + Q_{(10,8,7,4,3,1)} + Q_{(10,8,6,5,3,1)} + Q_{(9,8,7,6,3)} + Q_{(9,8,7,6,2,1)} + Q_{(9,8,7,5,4)} + 3Q_{(9,8,7,5,3,1)} + Q_{(9,8,7,4,3,2)} + Q_{(9,8,6,5,4,1)} + Q_{(9,8,6,5,3,2)}$.

For

$$T_{\lambda/\mu} = \begin{array}{ccccc} 1' & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & \\ 1 & 2' & 3' & 3 & \\ & 2 & 3' & & \\ & & 3 & & \end{array}$$

we obtain

$$T_2 = \begin{array}{ccccc} 1' & 1 & 1 & 1 & 1 \\ 1 & 2' & 2 & 2 & \\ 2 & 2 & 3' & 3 & \\ & 3 & 4' & & \\ & & 4 & & \end{array}, \quad T_2' = \begin{array}{ccccc} 1' & 1 & 1 & 1 & 1 \\ 1 & 2' & 2 & 2 & \\ 2 & 2 & 3 & 3 & \\ & 3 & 4' & & \\ & & 4 & & \end{array}.$$

We have $Q_{(7,5,4,2,1)/(2,1)} = Q_{(7,5,4)} + Q_{(7,5,3,1)} + Q_{(6,4,3,2,1)} + 2Q_{(6,5,3,2)} + Q_{(7,4,3,2)} + Q_{(6,5,4,1)}$.

For

$$T_{\lambda/\mu} = \begin{array}{ccccc} 1' & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & \\ 1' & 2' & 3' & & \\ 1 & 2 & 3 & & \end{array}$$

we obtain

$$T_3 = \begin{array}{ccccc} 1' & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & \\ 1 & 2' & 3' & & \\ 2 & 3 & 3 & & \end{array}, \quad T_3' = \begin{array}{ccccc} 1' & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & \\ 1 & 2 & 3' & & \\ 2 & 3 & 3 & & \end{array}.$$

We have $Q_{(8,6,4,3)/(3,2,1)} = Q_{(8,5,2)} + Q_{(8,4,3)} + Q_{(7,6,2)} + Q_{(8,4,2,1)} + 2Q_{(7,5,3)} + Q_{(6,4,3,2)} + 2Q_{(6,5,3,1)} + Q_{(6,5,4)} + 2Q_{(7,4,3,1)} + 2Q_{(7,5,2,1)}$. *For*

$$T_{\lambda/\mu} = \begin{array}{ccccc} 1' & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & \\ 1 & 2' & 3' & & \\ & 2 & 3 & & \end{array}$$

we obtain

$$T_4 = \begin{array}{ccccc} 1' & 1 & 1 & 1 & 1 \\ 1 & 2' & 2 & 2 & \\ 2 & 3' & 3 & & \\ & 3 & 4 & & \end{array}, \quad T_4' = \begin{array}{ccccc} 1' & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & \\ 2 & 3' & 3 & & \\ & 3 & 4 & & \end{array}.$$

We have $Q_{(7,5,3,2)/(2,1)} = Q_{(7,5,2)} + Q_{(7,4,3)} + Q_{(7,4,2,1)} + Q_{(6,5,3)} + Q_{(6,5,2,1)} + 2Q_{(6,4,3,1)} + Q_{(5,4,3,2)}$.

Corollary 3.16. *Let* $\lambda, \mu \in DP$. *Let* $\nu = c(T_{\lambda/\mu})$ *and* $n := \ell(\nu) > 1$. *If* $D^{ot}_{\lambda/\mu}$ *has shape* $D_{\alpha/\beta}$ *where* $\beta \neq \emptyset, (1)$ *and* α *has more than two corners then* $Q_{\lambda/\mu}$ *is not* Q*-multiplicity-free. If* $D_{\lambda/\mu}$ *is an unshifted diagram and* $D^{o}_{\lambda/\mu}$ *has more than two corners then* $Q_{\lambda/\mu}$ *is not* Q*-multiplicity-free.*

Remark. As it will turn out (and will be proved in Lemma 3.34), for $\mu = \emptyset$ or $\mu = (1)$ the Schur Q-function $Q_{\lambda/\mu}$ is Q-multiplicity-free. Thus, we will only consider the case $\mu \neq \emptyset, (1)$. Since we want to find all λ, μ such that $Q_{\lambda/\mu}$ is Q-multiplicity-free, by Lemma 3.14 from now on we will assume that λ has at most two corners.

The case that the diagrams λ or μ has at most two corners also occurs in the classical setting of Schur functions $s_{\lambda/\mu}$. For instance, Gutschwager proved [7, Theorem 3.5] where the cases in condition (2) have this property. However, this property is not enough in the classical case, where further restrictions need to be imposed for the classification of (s-)multiplicity-free Schur functions.

For the classification of Q-multiplicity-free Schur Q-functions we also need to find further restrictions since the properties from Lemma 3.14 and Corollary 3.16 are not sufficient to guarantee the Q-multiplicity-freeness of a given skew Schur Q-function. We will introduce some new notation for partitions with at most two corners and then obtain restrictions until we can exclude all non-Q-multiplicity-free Schur Q-functions in Proposition 3.33.

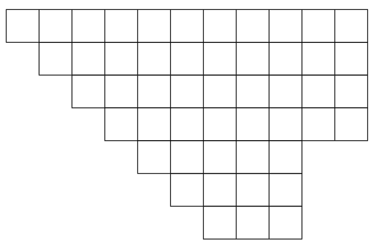
Definition 3.17. Let $DP^{\leq 2} \subseteq DP$ be the set of partitions λ with distinct parts such that D_λ has at most two corners. For a diagram D_λ where $\lambda \in DP^{\leq 2}$ the **shape path** is a 4-tuple defined as follows: Let a be the row of the upper corner. Let

$$b := \begin{cases} \lambda_a \text{ if } a = \ell(\lambda); \\ \lambda_a - \lambda_{a+1} - 1 \text{ otherwise.} \end{cases}$$

If there is a lower corner let $c := \ell(\lambda) - a$ and $d := \lambda_{\ell(\lambda)}$. If there is no lower corner set $c = d := 0$.

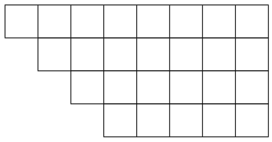
To distinguish it from a partition with four parts, we denote the shape path defined above by $[a, b, c, d]$ for some given $\lambda \in DP^{\leq 2}$.

Example 3.18. *For* $\lambda = (11, 10, 9, 8, 5, 4, 3)$ *we have*

$D_\lambda =$ 

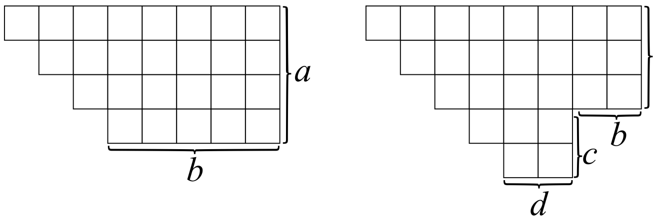
and $[a, b, c, d] = [4, 2, 3, 3]$.

For $\lambda = (8,7,6,5)$ *we have*

$D_\lambda =$ 

and $[a,b,c,d] = [4,5,0,0]$.

Remark. The numbers a,b,c,d of the shape path can be obtained as number of boxes given as follows.



For some $\lambda \in DP^{\leq 2}$ such that D_λ has two corners one can imagine to stand to the right of the first box of B_λ and walk to the first corner and count the boxes that pass. Then one has to turn right to walk until a box is blocking the path and count the boxes that pass on the side. After that, one has to turn left to walk to the second corner and count the boxes that pass on the side. And finally, one has to turn right to walk until the last box of B_λ is arrived and again count the boxes that pass. The four numbers obtained by counting the passing boxes are the numbers of the shape path in the same order. If D_λ has one corner then after turning right to walk after arriving at the corner one counts the boxes that pass until the last box of B_λ is arrived. The walked path is determined by these four numbers and these numbers depend only on the shape of the diagram, hence the name shape path.

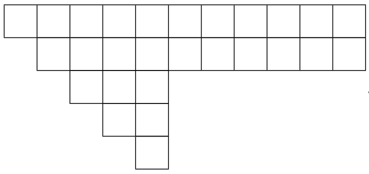
Remark. For a given $\lambda \in DP^{\leq 2}$ the cardinality of the border B_λ can be derived by the shape path. If $\lambda = [a,b,0,0]$ then $|B_\lambda| = a+b-1$. If $\lambda = [a,b,c,d]$ then $|B_\lambda| = a+b+c+d-1$.

Lemma 3.19. *The map* $DP^{\leq 2} \to \mathbb{N}^2 \times (\mathbb{N}^2 \cup \{(0,0)\}) : \lambda \mapsto [a,b,c,d]$ *is a bijection.*

Proof. For some given $[a,b,c,d]$ we get $\lambda = (a+b+c+d-1, a+b+c+d-2, \ldots, b+c+d+1, b+c+d, c+d-1, c+d-2, \ldots, d)$ if $c, d \neq 0$.

If $c = d = 0$ then $\lambda = (a+b-1, a+b-2, \ldots, b)$. Hence, there is an inverse map of the map in Lemma 3.19. □

Example 3.20. *For* $[a,b,c,d] = [2,6,3,1]$ *we get* $\lambda = (11,10,3,2,1)$ *and*

$D_\lambda =$  .

<u>Notation</u>. >From now on we will identify a partition with at most two corners with its shape path. Each letter occurring in a shape path will be considered as a positive integer. This means that in $[a,b,c,d]$ the numbers c and d are positive and we have a partition with two corners while $[a,b,0,0]$ is a partition with one corner.

Lemma 3.21. *Let* $\mu \in DP$ *and suppose* λ *is not equal to* $[a,b,0,0]$ *where* $b \leq 2$*. If* μ *has more than two corners then* $Q_{\lambda/\mu}$ *is not Q-multiplicity-free.*

Proof. For each corner (x,y) of μ except for the lowermost, there is a box $(x+1,z) \in D_{\lambda/\mu}$ such that $(x,z), (x+1,z-1) \notin D_{\lambda/\mu}$. Also there is a box $(1,w) \in D_{\lambda/\mu}$ such that $(1,w-1) \notin D_{\lambda/\mu}$ and there is no box above because $(1,w)$ is in the first row. After transposing this diagram orthogonally, the image of these boxes are corners of $D^{ot}_{\lambda/\mu}$. The diagram $D^{ot}_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ where $\beta \neq \emptyset, (1)$ and α has more than two corners. By Corollary 3.16, $Q_{\lambda/\mu}$ is not Q-multiplicity-free. □

Lemma 3.22. *Let* $\mu \in DP$ *and suppose* λ *is not equal to* $[a,b,0,0]$ *where* $b \leq 2$*. If* $\mu = [w,x,y,z]$ *where* $z > 1$ *then* $Q_{\lambda/\mu}$ *is not Q-multiplicity-free.*

Proof. The leftmost box of the first row of $D_{\lambda/\mu}$, which is $(1, w+x+y+z)$, has no box to the left or above. Also, the leftmost box of the $(w+1)^{\text{th}}$ row of $D_{\lambda/\mu}$, which is

$(w+1, w+y+z)$, has no box to the left or above. Additionally, the leftmost box of the $(w+y+1)^{\text{th}}$ row of $D_{\lambda/\mu}$, which is $(w+y+1, w+y+1)$, has no box to the left or above. After transposing this diagram orthogonally, the image of these boxes are corners of $D^{ot}_{\lambda/\mu}$. Then the diagram $D^{ot}_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ where $\beta \neq \emptyset, (1)$ and α has more than two corners. By Corollary 3.16, $Q_{\lambda/\mu}$ is not Q-multiplicity-free. □

Lemma 3.23. *Suppose* $\lambda = [a, b, c, d]$ *and* $\mu = [w, x, 0, 0]$ *where* $x > 1$ *or* $\mu = [w, x, y, 1]$. *Then* $Q_{\lambda/\mu}$ *is not* Q*-multiplicity-free.*

Proof. Let k be such that $U_k(\lambda/\mu)$ has only one box in the diagonal $\{(s,t) \mid t-s = x-1\}$ for the case $\mu = [w, x, 0, 0]$ or in the diagonal $\{(s,t) \mid t-s = x+y\}$ for the case $\mu = [w, x, y, 1]$. Let this single remaining box be (p, q). Then $(p, q) \in P_k$ and also $(p-1, q), (p, q-1) \in P_k$. Let $n = \ell(c(T_{\lambda/\mu}))$.

Case 1: $k = n$.

If P_n is not a rotated hook, then by Lemma 3.2, $Q_{\lambda/\mu}$ is not Q-multiplicity-free. If P_n is a rotated (l, m)-hook where $l, m \geq 2$ then, since $\lambda = [a, b, c, d]$, there is some $j < n$ such that either the first box of P_j is in a column to the right of the boxes of P_n or the last box of P_j is in a row below the boxes of P_n. Let j be maximal with respect to this condition.

We may assume that the first box of P_j is in a column to the right of the boxes of P_n, otherwise $U_j(\lambda/\mu)$ is unshifted and we may consider $U_j(\lambda/\mu)^t$. By Lemma 1.59, if $D^{ot}_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ then $T^{(n)}_{\alpha/\beta}$ is a (m, l)-hook where $l, m \geq 2$ and the diagram $U_j(\alpha/\beta)$ satisfies the conditions of Lemma 3.11. By Lemma 2.1, it follows that $Q_{\lambda/\mu}$ is not Q-multiplicity-free.

Case 2: $k \neq n$.

If $U_{k+1}(\lambda/\mu)$ has at least two components then, by Lemma 1.37, $Q_{\lambda/\mu}$ is not Q-multiplicity-free. Thus, we may consider that all boxes of $U_{k+1}(\lambda/\mu)$ are either above or below the diagonal $\{(s,t) \mid t-s = x-1\}$ for the case $\mu = [w, x, 0, 0]$ or the diagonal $\{(s,t) \mid t-s = x+y\}$ for the case $\mu = [w, x, y, 1]$.

Case 2.1: P_n is an (l, m)-hook where $l, m \geq 2$.

Then either $U_k(\lambda/\mu)$ or $U_k(\lambda/\mu)^t$ satisfies the conditions of Lemma 3.11 and $Q_{\lambda/\mu}$ is not Q-multiplicity-free.

Case 2.2: only one corner is in P_n.

Let (f, g) be this corner. Then there is some e such that there are two boxes of P_e either in a row weakly below the f^{th} row or in a column weakly to the right of g^{th} column. There is also some h such that either the first box of P_h is to the right of the g^{th} column or the last box of P_h is below the f^{th} row. Let e, h be maximal with respect to these conditions.

By orthogonally transposition, transposition or rotation of $U_{\min\{e,h\}}(\lambda/\mu)$, we may assume that $h \leq e$ and that the first box of P_h is to the right of the g^{th} column. By Lemma 3.7, if $h = e$ then $Q_{\lambda/\mu}$ is not Q-multiplicity-free. Hence, we assume $h < e$.

There is a box $(r, u) \in P_h$ in the diagonal $\{(s, t) \mid t - s = x - 1\}$ for the case $\mu = [w, x, 0, 0]$ or in the diagonal $\{(t, s) \mid t - s = x + y\}$ for the case $\mu = [w, x, y, 1]$.

We get a tableau T if after the $(h-1)^{\text{th}}$ step of the algorithm of Definition 1.45 we use $P'_h := P_h \setminus \{(r, u)\}$ instead of P_h. By Corollary 1.44, this tableau is m-amenable for $m \neq h + 1$. We have $T(r, u) = (h + 1)'$ and $T(r - 1, u - 1) \neq h'$. However, there is a h with no $(h + 1)$ below in the first box of P_h. Thus, by Lemma 1.42, this tableau is amenable.

We get another tableau T' with the same content if we set $T'(r - 1, u) = h'$ and $T'(f, g) = T(f, g)$ for every other box $(f, g) \in D_{\lambda/\mu}$. By Corollary 1.44, this tableau is m-amenable for $m \neq h + 1$.

We have $T'(r, u) = (h+1)'$ and $T'(r-1, u-1) \neq h'$. However, we have $T'(r-1, u) = h'$. In the u^{th} column is a $h + 1$ but no h. However, there are hs with no $(h + 1)$s below in the first box and in the last box of P_1. Thus, by Lemma 1.42, this tableau is amenable.

By Lemma 2.1, $Q_{\lambda/\mu}$ is not Q-multiplicity-free. $\square$

Example 3.24. *For* $\lambda = [1,1,4,1]$ *and* $\mu = [1,1,1,1]$ *we have* $T_{\lambda/\mu} = \begin{array}{cccc} & 1' & 1 & 1 \\ 1' & 1 & 2' & \\ 1 & 2' & 2 & \\ & 2 & 3' & \\ & & 3 & \end{array}$.

Then we obtain $T = \begin{array}{cccc} & 1 & 1 & 1 \\ 1' & 2' & 2 & \\ 1 & 2' & 3' & \\ & 2 & 3' & \\ & & 3 & \end{array}$, $T' = \begin{array}{cccc} & 1' & 1 & 1 \\ 1' & 2' & 2 & \\ 1 & 2' & 3' & \\ & 2 & 3' & \\ & & 3 & \end{array}$.

We have $Q_{(6,4,3,2,1)/(3,1)} = Q_{(6,4,2)} + 2Q_{(5,4,3)} + Q_{(5,4,2,1)}$.

Now for Q-multiplicity-free skew Schur Q-functions the partition μ is restricted to certain families of partitions for some given λ. The following two lemmas and their corollaries restrict λ and μ further until Proposition 3.33 can be proved.

Lemma 3.25. *Let* $\lambda = [a,b,c,1]$ *and* $\mu = [w,1,0,0]$. *If* $a \geq 3$, $b \geq 3$, $c \geq 3$ *and* $4 \leq w \leq a+c-2$ *then* $Q_{\lambda/\mu}$ *is not* Q*-multiplicity-free.*

Proof. We will show that for case $a = 3$ and for case $w = a+c-2$ the statement holds. Afterwards we will explain case $a > 3$ and $w < a+c-2$ by these two cases.

Case 1: $a = 3$.

Let $b \geq 3$, $c \geq 3$ and $4 \leq w \leq a+c-2$. The lowermost box in the leftmost column of the diagram is $(w+1, w+1)$. Since $w < a+c-1$, we have $(w, w+2) \in D_{\lambda/\mu}$.

We get a new tableau T_1 as follows: In the algorithm of Definition 1.45 use $P'_1 := P_1 \setminus \{(w+1,w+1)\}$, $P'_2 := P_2 \setminus \{(w+1,w+2),(w+2,w+2)\}$ and $P'_3 := P_3 \setminus \{(w,w+3), (w+1,w+3),(w+2,w+3),(w+3,w+3)\}$ (for $w = a+c-2$ this means $P'_3 = P_3$) instead of P_1, P_2 and P_3, respectively, and stop after the third step in the algorithm. Then replace the entry 3 in the last box of P'_3 with $3'$ and set $T_1(w+1,w+1) = 3$. Afterwards fill the remaining boxes using the algorithm of Definition 1.45 starting with $k = 4$. By Corollary 1.44, it is clear that T_1 is m-amenable for $m \neq 3,4$. There is a 3 but no 2 in the $(w+1)^{\text{th}}$ column. However, there is a 2 and no 3 in the column of the last box of P'_3 and there is a 2 and no 3 in the column to the left of it. Thus, by Lemma

1.42, this tableau is 3-amenable. In the $(w+2)^{\text{th}}$ column and possibly in the $(w+3)^{\text{th}}$ column, there are 4s and no 3s. However, there are 3s and no 4s in the columns of the first two boxes of P_3'. We have $T_1(w+1, w+2) = 4'$ and $T_1(w, w+1) \neq 3'$. However, if (y, z) is the third box of P_3' then we either have $T_1(y, z) = 3$ and there is no 4 in the z^{th} column or if $w = a + c - 2$ we have $T_1(y, z) = 3'$ and $T_1(y+1, z+1) \neq 4'$. If $w < a + c - 2$ then we have $T_1(w, w+3) = 4'$ and $T_1(w-1, w+2) \neq 3'$. However, we have $T_1(w-1, w+3) = 3'$. Thus, by Lemma 1.42, this tableau is 4-amenable.

We get another tableau T_1' with the same content if we set $T_1'(w+1, w+1) = 3$, $T_1'(w, w+2) = 2$ and $T_1'(r, s) = T_1(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. It is easy to see that, by Corollary 1.44, T_1' is m-amenable for $m \neq 2, 3, 4$. There is a 1 with no 2 below in the $(w+2)^{\text{th}}$ column. Thus, by Lemma 1.42, 2-amenability follows. There is a 3 with no 2 above in the $(w+2)^{\text{th}}$ column. However, there is a 2 with no 3 below in the column of the last box of P_3. Thus, by Lemma 1.42, this tableau is 3-amenable. By Lemma 1.42, it is clear that T_1' is 4-amenable if T_1 is.

Case 2: $w = a + c - 2$.

By Case 1, we may assume $a > 3$. The lowermost box in the leftmost column of the diagram is $(w+1, w+1)$. Since $w < a + c - 1$, we have $(w, w+2) \in D_{\lambda/\mu}$.

Let (y, z) be the last box of P_3. We get a new tableau T_2 if we set $T_2(w+1, w+1) = 3$, $T_2(w, w+1) = 1$, $T_2(w, w+2) = 2$, $T_2(w+1, w+2) = 4$, $T_2(w+2, w+2) = 5$, $T_2(y, z) = 3'$, $T_2(y, z+1) = 4'$, for the case $P_5 \neq \emptyset$ set $T_2(y, z+2) = 5'$ (in this case $(y, z+2)$ is the last box of P_5), and set $T_2(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$.

By Corollary 1.44, T_2 is m-amenable for $m \neq 3, 4, 5$. There is a 3 and no 2 in the $(w+1)^{\text{th}}$ column. However, there is are 2s and no 3s in the z^{th} and in the $(w+2)^{\text{th}}$ column. Thus, by Lemma 1.42, this tableau is 3-amenable. There is a 4 with no 3 above in the $(w+2)^{\text{th}}$ column. However, there are 3s and no 4s in the $(w+1)^{\text{th}}$ column and in the $(z+1)^{\text{th}}$ column. Thus, by Lemma 1.42, 4-amenability follows. The 5-amenability is clear for $P_5 = \emptyset$. If $P_5 \neq \emptyset$ then there is a 4 and no 5 in the $(z+2)^{\text{th}}$ column. Thus, by Lemma 1.42, this tableau is 5-amenable.

We get another tableau T_2' with the same content if we set $T_2'(w+1, w+1) = 2$, $T_2'(w, w+2) = 3$ and $T_2'(r,s) = T_2(r,s)$ for every other box $(r,s) \in D_{\lambda/\mu}$. By Corollary 1.44, T_2' is m-amenable for $m \neq 2,3,4$. There is a 1 and no 2 in the $(w+2)^{\text{th}}$ column. Thus, by Lemma 1.42, 2-amenability follows. There is a 3 and no 2 in the $(w+2)^{\text{th}}$ column. However, there is a 2 with no 3 below in the z^{th} column. Thus, by Lemma 1.42, this tableau is 3-amenable. By Lemma 1.42, it is clear that T_2' is 4-amenable if T_2 is.

Case 3: $a > 3$ and $w < a + c - 2$.

The diagram $U_2(\lambda/\mu)$ has shape $D_{\lambda'/\mu}$ where $\lambda' = [a', b, c, 1]$ where $a' = a - 1$. Either we have $a' = a - 1 = 3$ or $w = a' + c - 2$ or else there is some j such that $U_j(\lambda/\mu)$ has shape $D_{\lambda''/\mu}$ where $\lambda'' = [a'', b, c, 1]$ where $a'' = a - j$ such that either $a'' = 3$ or $w = a'' + c - 2$. Then, by Case 1 and Case 2, we find two different amenable tableaux with the same content and, by Lemma 2.1, $Q_{\lambda/\mu}$ is not Q-multiplicity-free. $\square$

Example 3.26. *For $\lambda = [3,3,6,1]$ and $\mu = [5,1,0,0]$ the tableaux are*

$$T_1 = \begin{array}{|c|c|c|c|c|c|c|}
\hline
1' & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
1' & 2' & 2 & 2 & 2 & 2 & 2 \\ \hline
1' & 2' & 3' & 3 & 3 & 3 & 3 \\ \hline
1' & 2' & 3' & 4' \\ \cline{1-4}
1 & 2 & 4' & 4 \\ \cline{1-4}
3 & 4' & 4 & 5' \\ \cline{1-4}
\multicolumn{1}{c|}{} & 4 & 5' & 5 \\ \cline{2-4}
\multicolumn{2}{c|}{} & 5 & 6' \\ \cline{3-4}
\multicolumn{3}{c|}{} & 6 \\ \cline{4-4}
\end{array}\ , \quad T_1' = \begin{array}{|c|c|c|c|c|c|c|}
\hline
1' & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
1' & 2' & 2 & 2 & 2 & 2 & 2 \\ \hline
1' & 2' & 3' & 3 & 3 & 3 & 3 \\ \hline
1' & 2' & 3' & 4' \\ \cline{1-4}
1 & 3 & 4' & 4 \\ \cline{1-4}
2 & 4' & 4 & 5' \\ \cline{1-4}
\multicolumn{1}{c|}{} & 4 & 5' & 5 \\ \cline{2-4}
\multicolumn{2}{c|}{} & 5 & 6' \\ \cline{3-4}
\multicolumn{3}{c|}{} & 6 \\ \cline{4-4}
\end{array}\ .$$

For $\lambda = [4,5,3,1]$ and $\mu = [5,1,0,0]$ the tableaux are

$$T_2 = \begin{array}{|c|c|c|c|c|c|c|}
\hline
1' & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
1' & 2' & 2 & 2 & 2 & 2 & 2 \\ \hline
1' & 2' & 3' & 3 & 3 & 3 & 3 \\ \hline
1' & 2' & 3' & 4' & 4 & 4 & 4 \\ \hline
1 & 2 \\ \cline{1-2}
3 & 4 \\ \cline{1-2}
\multicolumn{1}{c|}{} & 5 \\ \cline{2-2}
\end{array}\ , \quad T_2' = \begin{array}{|c|c|c|c|c|c|c|}
\hline
1' & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
1' & 2' & 2 & 2 & 2 & 2 & 2 \\ \hline
1' & 2' & 3' & 3 & 3 & 3 & 3 \\ \hline
1' & 2' & 3' & 4' & 4 & 4 & 4 \\ \hline
1 & 3 \\ \cline{1-2}
2 & 4 \\ \cline{1-2}
\multicolumn{1}{c|}{} & 5 \\ \cline{2-2}
\end{array}\ .$$

Corollary 3.27. *Let* $\lambda = [a, b, 0, 0]$ *and* $\nu = [w, x, 0, 0]$*. If* $w \geq 3$*,* $x \geq 4$*,* $a \geq w + 2$*,* $b \geq 5$ *and* $a + b - w - x \geq 3$ *then* $Q_{\lambda/\mu}$ *is not multiplicity-free.*

Proof. If λ, μ satisfy these properties then $D^{ot}_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ where $\alpha = [a', b', c', 1]$ and $\beta = [w', 1, 0, 0]$. Then $b' = w \geq 3$ and $c' = x - 1 \geq 3$. The number a' is the number of boxes of the first row of $D_{\lambda/\mu}$ and can be calculated by $a' = \lambda_1 - \mu_1 = |B_\lambda| - |B_\mu| = a + b - w - x \geq 3$. Since $a \geq w + 2$, we have $a - w - 2 \geq 0$ and, hence, $b \leq a + b - w - 2$. Then we get $4 \leq b - 1 = w' = b - 1 \leq a + b - w - 2 - 1 = a + b - w - x + x - 1 - 2 = a' + c' - 2$. By Lemma 3.25, $Q_{D^{ot}_{\lambda/\mu}}$ is not Q-multiplicity-free and, thus, $Q_{\lambda/\mu}$ is not Q-multiplicity-free. □

Example 3.28. *The smallest diagram satisfying the properties of Corollary 3.27 is* $D_{(9,8,7,6,5)/(6,5,4)}$.

We have $Q_{(9,8,7,6,5)/(6,5,4)} = Q_{(9,8,3)} + Q_{(9,7,4)} + Q_{(9,7,3,1)} + Q_{(9,6,4,1)} + Q_{(9,6,3,2)} + Q_{(9,5,4,2)} + Q_{(8,7,5)} + Q_{(8,7,4,1)} + Q_{(8,7,3,2)} + Q_{(8,6,5,1)} + 2Q_{(8,6,4,2)} + Q_{(8,6,3,2,1)} + Q_{(8,5,4,3)} + Q_{(8,5,4,2,1)} + Q_{(7,6,5,2)} + Q_{(7,6,4,3)} + Q_{(7,6,4,2,1)} + Q_{(7,5,4,3,1)}$.

Lemma 3.29. *Let* $\lambda = [a, b, c, d]$ *and* $\mu = [w, 1, 0, 0]$*. If* $a, b, c, d \geq 2$ *and* $3 \leq w \leq a + c - 1$ *then* $Q_{\lambda/\mu}$ *is not* Q*-multiplicity-free.*

Proof. Let $n = \ell(c(T_{\lambda/\mu}))$. First we prove case $a = 2$ and case $d = 2$ such that we have $w = a + c - 1$. Then we show that case $a, d \geq 3$ such that $w = a + c - 1$ can be explained by case $a = 2$ or $d = 2$ such that $w = a + c - 1$. Afterwards we tackle case $w < a + c - 1$ using case $w = a + c - 1$ while we first prove subcase $d = 2$ and then show how to add boxes with entries to obtain diagrams such that $d > 2$.

Case 1: $w = a + c - 1$ and $2 \in \{a, d\}$.

We may assume $a = 2$, otherwise we can transpose the diagram. If $d = 2$ then P_n is a $(b + 1, c + 1)$-hook where and, by Lemma 3.3, which in this case forbids to have a box directly to the left of the last box of P_n, $Q_{\lambda/\mu}$ is not Q-multiplicity-free. Thus, assume $d \geq 3$.

The box $(w+1, w+1)$ is the last box of P_1. We get a new tableau T_1 if we set $T_1(w, w+1) = 1$, $T_1(w+1, w+1) = 3$, $T_1(w, w+2) = 2$, $T_1(w+1, w+2) = 3$, $T_1(w, w+3) = 3$, $T_1(w+1, w+3) = 4$ and $T_1(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$.

By Corollary 1.44, T_1 is m-amenable for $m \neq 3$. There is a 3 and no 2 in the $(w+1)^{\text{th}}$ column. However, there are 2s and no 3s in the columns of the first two boxes of P_2. Thus, by Lemma 1.42, T_1 is amenable.

We get another tableau T_1' if we set $T_1'(w+1, w+1) = 2$, $T_1'(w, w+2) = 3'$ and $T_1'(r, s) = T_1(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$.

By Corollary 1.44, T_1' is m-amenable for $m \neq 2, 3$. In the $(w+2)^{\text{th}}$ column is a 1 with no 2 below. Thus, by Corollary 1.44, 2-amenability follows. We have $T_1'(w, w+2) = 3'$ and $T_1'(w-1, w+1) \neq 2'$ and there is a 3 and no 2 in the $(w+2)^{\text{th}}$ column. However, there are two 2s and no 3s in the columns of the first two boxes of P_2. Thus, by Lemma 1.42, 3-amenability follows. It is clear that T_1' has the same content as T_1. Hence, $Q_{\lambda/\mu}$ is not Q-multiplicity-free.

Case 2: $w = a + c - 1$ and $a, d \geq 3$.

The diagram $U_2(\lambda/\mu)$ has shape $D_{\alpha/\beta}$ where $\alpha = [a', b, c, d']$ and $\beta = [a'+c-1, 1, 0, 0]$, and $a' = a - 1$ and $d' = d - 1$. If $a' = 2$ or $d' = 2$ then Case 1 proves the statement. Otherwise, there is some j such that $U_j(\lambda/\mu)$ has shape $D_{\alpha'/\beta'}$ where $\alpha = [a'', b, c, d'']$ and $\beta = [a'' + c - 1, 1, 0, 0]$, and $a'' = 2$ or $d'' = 2$. By Lemma 2.1 and Case 1, $Q_{\lambda/\mu}$ is not Q-multiplicity-free.

Case 3: $3 \leq w < a + c - 1$.

Assume $a > 2$. Let (x, y) be the lower corner. Since $w < a+c-1$, the last box of P_1 is not in the x^{th} row. Then the diagram $U_2(\lambda/\mu)$ has shape $D_{\lambda/\mu}$ where $\lambda' = [a-1, b, c, d]$ and $\mu' = [w, 1, 0, 0]$. Then there is some j such that $U_j(\lambda/\mu)$ has shape $D_{\alpha'/\beta'}$ where either $\alpha' = [2, b, c, d]$ and $\beta' = [w, 1, 0, 0]$ or where $\alpha' = [e, b, c, 2]$ and $\beta' = [w', 1, 0, 0]$ where $a > e \geq 3$ and $w' = e + c - 1$. In the latter case the transpose of the diagram is covered in Case 2. Thus, it suffices to consider the case $D_{\alpha/\beta}$ where $\alpha = [2, b, c, d]$ and $\beta = [w, 1, 0, 0]$ and $3 \leq w < 2 + c - 1 = c + 1$.

Case 3.1: $d = 2$.

The box $(w+1, w+1)$ is the last box of P_1. We get a new tableau T_2 as follows: In the algorithm of Definition 1.45 use $P_1' := P_1 \setminus \{(w+1, w+1)\}$ and $P_2' := P_2 \setminus \{(w+1, w+2), (w+2, w+2)\}$ instead of P_1 and P_2, respectively. By Corollary 1.44, T_2 is m-amenable for $m \neq 3$. There is a 3 and no 2 in the $(w+1)^{\text{th}}$ column. However, there are 2s and no 3s in the columns of the first two boxes of P_2. Thus, by Lemma 1.42, 3-amenability follows.

We get another tableau T_2' as follows:

- Set $T_2'(r, s) = T_2(r, s)$ for every $(r, s) \in P_1' \cup (P_2' \setminus \{(w, w+2)\})$ where P_1' and P_2' as above.
- Set $T_2'(w+1, w+1) = 2$.
- Fill the remaining boxes using the algorithm of Definition 1.45 starting with $k = 3$.

By Corollary 1.44, T_2' is m-amenable for $m \neq 2, 3$. There is a 1 and no 2 in the $(w+2)^{\text{th}}$ column. Thus, by Corollary 1.44, 2-amenability follows. There is a 3 and no 2 in the $(w+2)^{\text{th}}$ column. However, there is a 2 and no 3 in the column of the first box of P_2. We have $T_2'(w+1, w+2) = 3'$ and $T_2'(w, w+1) \neq 2'$. However, there is a 2 and no 3 in the column of the second box of P_2. We have $T_2'(w, w+2) = 3'$ and $T_2'(w-1, w+1) \neq 2'$. However, we have $T_2'(w-1, w+2) = 2'$. Thus, by Lemma 1.42, 3-amenability follows.

We have $|T_2(w+1+j, w+1+j)| = j+3$ and $|T_2(w+j, w+2+j)| = j+2$ for $0 \leq j \leq n-2$ and we have $|T_2'(w+1+j, w+1+j)| = j+2$ and $|T_2'(w+j, w+2+j)| = j+3$ for $0 \leq j \leq n-2$. The entries of the other boxes in T_2 and T_2' can only differ by markings. Thus, T_2' has the same content as T_2.

Case 3.2: $d > 2$.

Let (x, y) be the lower corner. We get two tableaux $\tilde{T}_2$ and $\tilde{T}'_2$ of shape $D_{\alpha/\beta}$ where $\alpha = [2, b, c, d]$ and $\beta = [w, 1, 0, 0]$ if we take the two tableaux of Case 3.1 of shape $D_{\alpha'/\beta'}$ where $\alpha' = [2, b, c, 2]$ and $\beta' = [w, 1, 0, 0]$ and add $d - 2$ columns using the following algorithm:

1. Set $\tilde{T}_2(e, f) = T_2(e, f)$ and $\tilde{T}'_2(e, f) = T'_2(e, f)$ for all $f \leq y$ and for all e such that $(e, f) \in D_{\lambda/\mu}$.

2. Set $\tilde{T}_2(p, q) = T_2(p, q - d + 2)$ and $\tilde{T}'_2(p, q) = T'_2(p, q - d + 2)$ for all $q > y$ and for all p such that $(p, q) \in D_{\lambda/\mu}$.

3. For $1 \leq j \leq n$ set $\tilde{T}_2(j, y + 1) = \tilde{T}'_2(j, y + 1) = j$.

4. For $n + 1 \leq r \leq x - 2$ set $\tilde{T}_2(r, y + 1) = \tilde{T}'_2(r, y + 1) = (n + 1)'$.

5. Set $\tilde{T}_2(x-1, y+1) = \tilde{T}'_2(x-1, y+1) = n+1$ and set $\tilde{T}_2(x, y+1) = \tilde{T}'_2(x, y+1) = n+2$.

6. Do the following algorithm:

   (i) Set $i = y + 2$:

   (ii) Scan the $(i - 1)^{\text{th}}$ column of $\tilde{T}_2$ from top to bottom and find the uppermost marked letter, z say. If there is no marked letter in the $(i - 1)^{\text{th}}$ column then set $z := 2 + c$.

   (iii) For $1 \leq r \leq |z|$ set $\tilde{T}_2(r, i) = \tilde{T}'_2(r, i) = r$.

   (iv) For $|z|+1 \leq s \leq 2+c$ set $\tilde{T}_2(s, i) = \tilde{T}'_2(s, i) = t+1$ if $\tilde{T}_2(s-1, i) = \tilde{T}_2(s-1, i) = t$ or else set $\tilde{T}_2(s, i) = \tilde{T}'_2(s, i) = (t + 1)'$ if $\tilde{T}_2(s - 1, i) = \tilde{T}_2(s - 1, i) = t'$.

   (v) Increment i.

   (vi) If $i \leq d - 2$ go to (ii) or else stop.

It is easy to see that these tableaux are amenable if the tableaux for $d = 3$ are amenable. By definition of the algorithm, if we have $T_2(u, y + 1) = T'_2(u, y + 1) = (n + 1)'$ then $T_2(u - 1, y) = T'_2(u - 1, y) = n'$. Hence, by Lemma 1.42, these tableaux are amenable.

For $d > 3$, since the $(y+1)^{\text{th}}$ column has the same entries in both tableaux, the algorithm fills the other $d-3$ columns in the same amenable way. Clearly, the contents of $\tilde{T}_2$ and $\tilde{T}_2'$ are equal. $\square$

Example 3.30. *For $\lambda = [2,2,3,5]$ and $\mu = [4,1,0,0]$ we have*

$$T_1 = \begin{array}{ccccccc} 1' & 1 & 1 & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & 2 & 2 & 2 \\ 1' & 2' & 3' & 3 & 3 & & \\ 1 & 2 & 3 & 4' & 4 & & \\ 3 & 3 & 4 & 4 & 5 & & \end{array}, \quad T_1' = \begin{array}{ccccccc} 1' & 1 & 1 & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & 2 & 2 & 2 \\ 1' & 2' & 3' & 3 & 3 & & \\ 1 & 3' & 3 & 4' & 4 & & \\ 2 & 3 & 4 & 4 & 5 & & \end{array}.$$

For $\lambda = [2,2,6,4]$ and $\mu = [5,1,0,0]$ we first take the tableaux for $\lambda' = [2,2,6,2]$ and $\mu' = [5,1,0,0]$:

$$T_2 = \begin{array}{cccccc} 1' & 1 & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & 2 & 2 \\ 1' & 2' & 3' & 3 & & \\ 1' & 2' & 3' & 4' & & \\ 1 & 2 & 3' & 4' & & \\ 3 & 3 & 3 & 4' & & \\ & 4 & 4 & 4 & & \\ & & 5 & 5 & & \end{array}, \quad T_2' = \begin{array}{cccccc} 1' & 1 & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & 2 & 2 \\ 1' & 2' & 3' & 3 & & \\ 1' & 2' & 3' & 4' & & \\ 1 & 3' & 3 & 4' & & \\ 2 & 3' & 4' & 4 & & \\ & 3 & 4' & 5' & & \\ & & 4 & 5 & & \end{array}.$$

Then we add two columns using the algorithm of Lemma 3.29:

$$\tilde{T}_2 = \begin{array}{cccccccc} 1' & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & 2 & 2 & 2 & 2 \\ 1' & 2' & 3' & 3 & 3 & 3 & & \\ 1' & 2' & 3' & 4' & 4 & 4 & & \\ 1 & 2 & 3' & 4' & 5' & 5 & & \\ 3 & 3 & 3 & 4' & 5' & 6' & & \\ & 4 & 4 & 4 & 5 & 6 & & \\ & & 5 & 5 & 6 & 7 & & \end{array}, \quad \tilde{T}_2' = \begin{array}{cccccccc} 1' & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & 2 & 2 & 2 & 2 \\ 1' & 2' & 3' & 3 & 3 & 3 & & \\ 1' & 2' & 3' & 4' & 4 & 4 & & \\ 1 & 3' & 3 & 4' & 5' & 5 & & \\ 2 & 3' & 4' & 4 & 5' & 6' & & \\ & 3 & 4' & 5' & 5 & 6 & & \\ & & 4 & 5 & 6 & 7 & & \end{array}.$$

Corollary 3.31. *Let $\lambda = [a,b,0,0]$ and $\mu = [w,x,y,1]$. If $w \geq 2$, $x \geq 2$, $b \geq 4$ and $a+b-1-w-x-y \geq 2$ then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. If λ, μ satisfy these properties then $D^{ot}_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ where $\alpha = [a', b', c', d']$ and $\beta = [w', 1, 0, 0]$ where $b' = w \geq 2$, $c' = x \geq 2$, $d' = y+1 \geq 2$ and additionally $a' + c' - 1 \geq w' = b - 1 \geq 3$. The number a' is the number of boxes of the first row of

$D_{\lambda/\mu}$ and can be calculated by $a' = \lambda_1 - \mu_1 = |B_\lambda| - |B_\mu| = a + b - 1 - w - x - y \geq 2$. By Lemma 3.29, $Q_{D^{ot}_{\lambda/\mu}}$ is not Q-multiplicity-free and, thus, $Q_{\lambda/\mu}$ is not Q-multiplicity-free. $\square$

Example 3.32. *The smallest diagram that satisfies the properties of Corollary 3.31 is* $D_{(8,7,6,5,4)/(6,5,1)}$.

We have $Q_{(8,7,6,5,4)/(6,5,1)} = Q_{(8,7,3)} + Q_{(8,6,4)} + Q_{(8,6,3,1)} + Q_{(8,5,4,1)} + Q_{(8,5,3,2)} + Q_{(7,6,4,1)} + Q_{(7,6,3,2)} + 2Q_{(7,5,4,2)} + Q_{(6,5,4,3)} + Q_{(6,5,4,2,1)} + Q_{(7,5,3,2,1)}$.

Now we are able to exclude all non-Q-multiplicity-free Schur Q-functions. The following proposition gives a list of all Schur Q-functions that are possibly Q-multiplicity-free. This is half of the proof of the classification of Q-multiplicity-free Schur Q-functions.

Proposition 3.33. *Let* $\lambda, \mu \in DP$ *such that* $D_{\lambda/\mu}$ *is basic. Let* $a, b, c, d, w, x, y \in \mathbb{N}$. *If* $Q_{\lambda/\mu}$ *is* Q*-multiplicity-free then* λ *and* μ *satisfy one of the following conditions:*

(i) λ *is arbitrary and* $\mu \in \{\emptyset, (1)\}$,

(ii) $\lambda = [a, b, 0, 0]$ *where* $b \in \{1, 2\}$ *and* μ *is arbitrary,*

(iii) $\lambda = [a, b, 0, 0]$ *and* $\mu = [w, x, y, 1]$ *where* $a + b - w - x - y - 1 = 1$ *or* $w = 1$ *or* $x = 1$ *or* $b \leq 3$,

(iv) $\lambda = [a, b, c, d]$ *where* $d \neq 1$ *and* $\mu = [w, 1, 0, 0]$ *where* $1 \in \{a, b, c\}$ *or* $w \leq 2$,

(v) $\lambda = [a, b, c, 1]$ *and* $\mu = [w, 1, 0, 0]$ *where* $a \leq 2$ *or* $b \leq 2$ *or* $c \leq 2$ *or* $w \leq 3$ *or* $w = a + c - 1$.

(vi) $\lambda = [a, b, 0, 0]$ *and* $\mu = [w, x, 0, 0]$ *where* $2 \leq b \leq 4$ *or* $w \leq 2$ *or* $x \leq 3$ *or* $a = w + 1$ *or* $a + b - w - x \leq 2$.

Some of these cases overlap.

The cases (iii) - (vi) are depicted as diagrams in the remark after the proof of this proposition.

We want to note that Case (i) is the orthogonal transposition of Case (ii). Also, Case (iii) is the orthogonal transposition of Case (iv). Case (v) is the orthogonal transposition of Case (vi) for $x > 1$. The orthogonal transposition of Case (vi) for $x = 1$ is also covered in Case (vi).

Proof. If $\mu = \emptyset, (1)$ we have no restrictions for λ. We also have no restrictions for μ if $\lambda = [a, b, 0, 0]$ where $b \in \{1, 2\}$.

Now consider $\mu \notin \{\emptyset, (1)\}$ and if $\lambda = [a, b, 0, 0]$ then $b \geq 3$. Then by Lemma 3.21, Lemma 3.22 and Lemma 3.23, if $Q_{\lambda/\mu}$ is Q-multiplicity-free then λ and μ satisfy one of the following cases:

- $\lambda = [a, b, 0, 0]$ and $\mu = [w, x, 0, 0]$
- $\lambda = [a, b, 0, 0]$ and $\mu = [w, x, y, 1]$
- $\lambda = [a, b, c, d]$ and $\mu = [w, 1, 0, 0]$

for some $a, b, c, d, w, x, y \in \mathbb{N}$. Note that in the last case if $w \geq a + c$ then $\ell(\mu) \geq \ell(\lambda)$ and the diagram $D_{\lambda/\mu}$ is either not defined or is not basic since it has an empty column. Hence, we will only consider $w \leq a + c - 1$.

By Corollary 3.27, for the case $\lambda = [a, b, 0, 0]$ and $\mu = [w, x, 0, 0]$, we have the restriction $b \leq 4$ or $w \leq 2$ or $x \leq 3$ or $a = w + 1$ or $a + b - w - x \leq 2$.

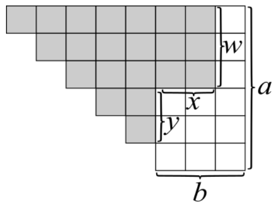
By Corollary 3.31, for the case $\lambda = [a, b, 0, 0]$ and $\mu = [w, x, y, 1]$, we have the restriction $w = 1$ or $x = 1$ or $b \leq 3$ or $a + b - w - x - y - 1 = 1$.

By Lemma 3.29, for the case $\lambda = [a, b, c, d]$ where $d \neq 1$ and $\mu = [w, 1, 0, 0]$, we have the restriction $1 \in \{a, b, c\}$ or $w \leq 2$.

By Lemma 3.25, for the case $\lambda = [a, b, c, 1]$ and $\mu = [w, 1, 0, 0]$, we have the restriction $a \leq 2$ or $b \leq 2$ or $c \leq 2$ or $w \leq 3$ or $w = a + c - 1$. □

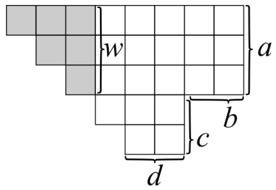
Remark. The following is a depiction of the diagrams of the cases (iii) - (vi) of Proposition 3.33 where all boxes illustrate the diagram of λ and the grey boxes illustrate the diagram of μ:

Case (iii):



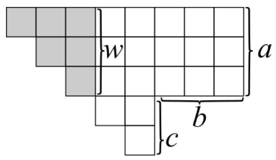
$$a + b - w - x - y - 1 = 1 \text{ or } w = 1 \text{ or } x = 1 \text{ or } b \leq 3.$$

Case (iv):



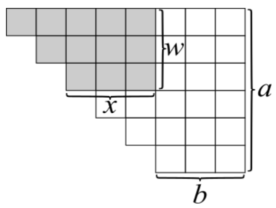
$$\text{If } d \geq 2 \text{ then } 1 \in \{a, b, c\} \text{ or } w \leq 2.$$

Case (v):



$$a \leq 2 \text{ or } b \leq 2 \text{ or } c \leq 2 \text{ or } w \leq 3 \text{ or } w = a + c - 1.$$

Case (vi):



$$2 \leq b \leq 4 \text{ or } w \leq 2 \text{ or } x \leq 3 \text{ or } a = w + 1 \text{ or } a + b - w - x \leq 2.$$

3.2 Proof of Q-multiplicity-freeness

To show that the list in Proposition 3.33 is the classification of Q-multiplicity-free Schur Q-functions we have to prove the Q-multiplicity-freeness of each of these cases. We will do this in the following until stating the classification as Theorem 3.58.

The next lemma shows the Q-multiplicity-freeness of 3.33 (i).

Lemma 3.34. *If λ is arbitrary and $\mu = \emptyset$ then $Q_{\lambda/\mu} = Q_\lambda$ and, thus, $Q_{\lambda/\mu}$ is Q-multiplicity-free.*

If λ is arbitrary and $\mu = (1)$ then

$$Q_{\lambda/\mu} = \sum_{\nu \in E_\lambda} Q_\nu,$$

where E_λ is the set from Definition 1.53. In particular, $Q_{\lambda/\mu}$ is Q-multiplicity-free.

Proof. For $\mu = \emptyset$ we have $Q_{\lambda/\emptyset} = Q_\lambda$. Thus, $f^\lambda_{\emptyset\lambda} = 1$ and $f^\lambda_{\emptyset\nu} = 0$ for $\nu \neq \lambda$. Hence, $Q_{\lambda/\emptyset}$ is Q-multiplicity-free.

The case $\mu = (1)$ is Proposition 1.55. □

Example 3.35. *Since $E_{(8,6,5,1)} := \{(7,6,5,1),(8,6,4,1),(8,6,5)\}$ we have*

$$Q_{(8,6,5,1)/(1)} = Q_{(7,6,5,1)} + Q_{(8,6,4,1)} + Q_{(8,6,5)}.$$

Before showing the Q-multiplicity-freeness of 3.33 (ii) we need to give a definition that allows us to describe the decomposition for a subcase of 3.33 (ii).

Definition 3.36. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}) \in DP$. Let $\mu = (\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_{\ell(\mu)}})$ such that $\{i_1, i_2, \dots, i_{\ell(\mu)}\} \subseteq \{1, 2, \dots, \ell(\lambda)\}$. Then $\lambda \setminus \mu$ is defined as the partition obtained by removing the parts of μ from λ.

Example 3.37. *For $\lambda = (9,7,5,4,3,1)$ and $\mu = (5,3,1)$ we obtain $\lambda \setminus \mu = (9,7,4)$.*

Lemma 3.38. *If $\lambda = [a,b,0,0]$ where $b \in \{1,2\}$ and μ is arbitrary then $Q_{\lambda/\mu}$ is Q-multiplicity-free. In particular, if $\lambda = [a,1,0,0]$ then $Q_{\lambda/\mu} = Q_{\lambda\setminus\mu}$.*

Proof. Case 1: $b = 2$.

Then $D^{ot}_{\lambda/\mu} = D_{\alpha/(1)}$ for some $\alpha \in DP$. By Lemma 1.60, $Q_{\lambda/\mu} = Q_{D^{ot}_{\lambda/\mu}} = Q_{\alpha/(1)}$ which is Q-multiplicity-free by Lemma 3.34.

Case 2: $b = 1$.

Then $D^{ot}_{\lambda/\mu} = D_\alpha$ for some $\alpha \in DP$. By Lemma 1.60, $Q_{\lambda/\mu} = Q_{D^{ot}_{\lambda/\mu}} = Q_\alpha$ which is Q-multiplicity-free. We will show that for all $1 \leq k \leq a$ the number k is either a part of α or a part of μ but it is never a part of both partitions. For this proof only, the diagram $D_{\lambda/\mu}$ is not necessarily basic. This means that in this proof it is possible to have $\lambda_1 = \mu_1$. See Example 3.39 for a depiction of this proof.

The statement clearly holds for $\lambda = [1, 1, 0, 0]$. Let $\lambda = [a, 1, 0, 0]$ where $a > 1$ and consider $D_{\lambda/\mu}$.

Case 2.1: $(1, a) \in \mu$.

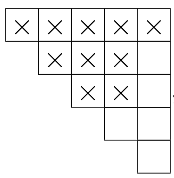
Then $\mu_1 = a$ and the a^{th} column of $D_{\lambda/\mu}$ has at most $a - 1$ boxes. Thus, $\alpha_1 < a$. Let U be the diagram obtained by removing the boxes of the first row.

Case 2.2: $(1, a) \notin \mu$.

Then $\mu_1 < a$ and the a^{th} column of $D_{\lambda/\mu}$ has precisely a boxes. Thus, $\alpha_1 = a$. Let U be the diagram obtained by removing the boxes of the a^{th} column.

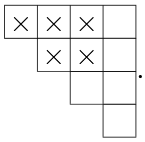
In both cases we have $U = D_{\gamma/\beta}$ for $\gamma = [a - 1, 1, 0, 0]$ and some β. By induction the statement follows. □

Example 3.39. *For $\lambda = [5, 1, 0, 0] = (5, 4, 3, 2, 1)$ and $\mu = (5, 3, 2)$ the diagram is*

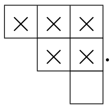
,

where $\boxtimes$ denotes a box from D_μ.

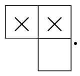
We want to calculate the index α *from* $Q_{\lambda/\mu} = Q_{D^{ot}_{\lambda/\mu}} = Q_\alpha$. *Since* $(1,5) \in D_\mu$, *there cannot be 5 boxes in the first row of* $D^{ot}_{\lambda/\mu} = D_\alpha$. *Thus, there is a part* 5 *in* μ *but not in* α. *After removing the boxes of the first row we obtain*



We have $(1,4) \notin D_\mu$ *and, thus, there is no part* 4 *in* μ *but a part* 4 *in* α. *After removing the fourth column we obtain*



We have $(1,3) \in D_\mu$ *and, thus, there is no part* 3 *in* α *but in* μ. *After removing the boxes of the first row we obtain*



We have $(1,2) \in D_\mu$ *and, thus, there is no part* 2 *in* α *but in* μ. *After removing the boxes of the first row we obtain*

We have $(1,1) \notin D_\mu$ *and, thus, there is no part* 1 *in* μ *but a part* 1 *in* α.

We obtain $\alpha = (4,1) = (5,4,3,2,1) \setminus (5,3,2)$.

We postpone to prove the Q-multiplicity-freeness of 3.33 (iii). We will first show the Q-multiplicity-freeness of 3.33 (iv) and then prove that 3.33 (iii) is just the orthogonally transposed version of 3.33 (iv).

Lemma 3.40. *Let* D *be a basic diagram of shape* $D_{\lambda/[s,1,0,0]}$ *for some* s. *If the first* a *rows of* D *form a diagram* $D_{\alpha/\beta}$ *where* $\alpha = [a,b,0,0]$ *and* $\beta = [w,1,0,0]$ *then the filling of the boxes of the first* a *rows of* D *in any amenable tableau* T *of* D *is the same up to marks.*

Proof. Let the diagram be shifted such that the uppermost leftmost box is $(1,1)$, the uppermost rightmost box is $(1, a+b-w-1)$ and the lowermost rightmost box is the box $(a, a+b-w-1)$. Let T be an amenable tableau of D.

Case 1: $w = a-1$.

Then the uppermost leftmost box is $(1,1)$, the uppermost rightmost box is $(1,b)$, the lowermost leftmost box is $(a,1)$ and the lowermost rightmost box is (a,b). Let $T_{(j)}$ be the subtableau of T consisting of the boxes with their entries of the first j rows. We need to show that $T_{(j)} \cap T^{(i)}$ is a $(j+1-i, b+1-i)$-hook at (i,i) for $1 \leq i \leq \min\{b,j\}$ where $T^{(i)}$ is as in Definition 1.32.

Case 1.1: $T_{(j)} \cap T^{(1)}$ is not a (j,b)-hook at $(1,1)$ but $T_{(j-1)} \cap T^{(i)}$ is a $(j-i, b+1-i)$-hook at (i,i) for all $1 \leq i \leq \min\{b, j-1\}$ for some j.

Then we have $T(j,1) > 1$. Let $t := T(j,1)$. For $t \in \{j', j\}$, by Lemma 1.38, all boxes in the j^{th} row are then filled with entries from $\{j', j\}$. The remark after Definition 1.21 implies that $c^{(u)}(T)_j = b \geq c^{(u)}(T)_{j-1}$; a contradiction to Lemma 1.39. Thus, we have $1 < t < j'$. Then the last box of $T_{(j-1)} \cap T^{(t)}$ contains a $|t|'$, for otherwise, by the remark after Definition 1.21, we have at least as many $|t|$s as $(|t|-1)$s, which contradicts Lemma 1.39. We have $|T(j,2)| > |t|$, or else we would have at least as many $|t|$s as $(|t|-1)$s, which contradicts Lemma 1.39.

Repeating this argument, we get $|T(j,s)| > |T(j,s-1)|$ for $2 \leq s \leq r$ where r is such that $T(j, r+1)$ is the leftmost box with an entry that does not appear in the first $(j-1)^{\text{th}}$ rows.

By Lemma 1.38, $T(j, r+1) \in \{j', j\}$ and, hence $T(j,k) \in \{j', j\}$ for $r+1 \leq k \leq b$. If $T(j-1, r+1) \notin \{(j-1)', j-1\}$ then the remark after Definition 1.21 implies that $c^{(u)}(T)_j > c^{(u)}(T)_{j-1}$; a contradiction to Lemma 1.39. Hence, we have $T(j-1, r+1) \in \{(j-1)', j-1\}$. If $T(j,r) \notin \{(j-1)', j-1\}$ then, again, the remark after Definition 1.21 implies that $c^{(u)}(T)_j \geq c^{(u)}(T)_{j-1}$; a contradiction to Lemma 1.39. If $T(j,r) \in \{(j-1)', j-1\}$ then $T(j-1, r+1) = (j-1)'$. Let $(j, r+1) = (x(l), y(l))$ (from

Definition 1.17). Then $m_{j-1}(n-l) = m_j(n-l)$ and $w(T)_l \in \{j', j\}$, contradicting Definition 1.21 a).

Case 1.2: $T_{(j)} \cap T^{(v)}$ is not a $(j+1-v, b+1-v)$-hook at (v,v) but $T_{(j-1)} \cap T^{(i)}$ is a $(j+1-i, b-i)$-hook at (i,i) for all $1 \leq i \leq \min\{b-1, j\}$ for some j and some minimal $v \leq j-1$.

Let j be minimal with respect to this property. By Case 1.1, we may assume that $v > 1$. Let v be minimal with respect to this property. Then we may take $T_{(j)}$, remove $P_1, P_2, \ldots, P_{v-1}$, and replace each entry x by $x-v+1$ for all $x \geq v$. In this way, we get a tableau U of shape $D_{\alpha'/\beta'}$ where $\alpha' = [a-v+1, b-v+1, 0, 0]$ and $\beta' = [(a-v+1)-1, 1, 0, 0]$ such that $U_{(j-v+1)} \cap U^{(1)}$ is not a $(j-v+1, b-v+1)$-hook at $(1,1)$; a contradiction to the proven fact that $T_{(j)} \cap T^{(1)}$ is a (j,b)-hook at $(1,1)$ for each $1 \leq j \leq \min\{a,b\}$ if T is of shape $D_{[a,b,0,0]/[a-1,1,0,0]}$.

Case 2: $w < a-1$.

The tableau $T_{(w+1)}$ is a tableau of shape $D_{\alpha'/\beta'}$ where $\alpha' = [w+1, a+b-w-1, 0, 0]$ and $\beta' = [w, 1, 0, 0]$. Thus, P_1 is a $(a+b-w-1, b)$-hook at $(1,1)$. After removing P_1 and replacing each entry x by $x-1$ and x' by $(x-1)'$ for all $2 \leq x \leq \ell(c(T))$, we get a tableau of shape $D_{\alpha''/\beta''}$ where $\alpha'' = [a-1, b, 0, 0]$ and $\beta'' = [w, 1, 0, 0]$ where $w \leq a-2$. Using the same argument, P_2 is a $(w+1, a+b-w-2)$-hook at $(2,2)$.

Repeating this argument, we find that all non-empty P_is are hooks at (i,i) and, therefore, the filling of the boxes of the first k rows of D in any amenable tableau T is the same up to marks. □

Remark. Since, by the remark after Definition 1.21, every $T^{(i)}$ must be fitting, this shows that there is only one amenable tableau for diagrams $D_{\lambda/\mu}$ where $\lambda = [a, b, 0, 0]$ and $\mu = [w, 1, 0, 0]$. Different proofs of this fact were given by Salmasian [15, Proposition 3.29] and DeWitt [6, Theorem IV.3].

Lemma 3.41. *Let $\lambda = [a, b, 1, d]$ and $\mu = [w, 1, 0, 0]$. Then $Q_{\lambda/\mu}$ is Q-multiplicity-free.*

Proof. Let the diagram $D = D_{\lambda/\mu}$ be shifted such that the uppermost leftmost box is $(1,1)$. Since case $w = 1$ is shown in Lemma 3.34, we only have to show case $w \geq 2$. The subdiagram consisting of the first a rows is $D_{\alpha/\beta}$ where $\alpha = [a, p, 0, 0]$ and $\beta = [q, 1, 0, 0]$ for some p, q. By Lemma 3.40, it has a unique filling up to marks in the a^{th} row.

Suppose there are two amenable tableaux T_1 and T_2 of D with the same content. Then the difference between these two tableaux are marks since the content of the $(a+1)^{\text{th}}$ row and, therefore, the filling of this row up to marks is determined. Thus, there is a minimal k such that an entry k is in the lowermost row and there is a box (a, k) with entry k' in T_1, say, and with entry k in T_2. Since the k in the $(a+1)^{\text{th}}$ row must be in a column to the left of the k^{th} column, we have $k > 1$. In T_2, if there is no $k-1$ in the $(a+1)^{\text{th}}$ row there are as many unmarked ks as unmarked $(k-1)$s, which is a contradiction to Lemma 1.39. Thus, there is a $k-1$ in the $(a+1)^{\text{th}}$ row in a box to the left of the $(k-1)^{\text{th}}$ column. If there is no $k-2$ in the $(a+1)^{\text{th}}$ row, we have as many unmarked $(k-1)$s as unmarked $(k-2)$s, which is a contradiction to Lemma 1.39. Thus, there is a $k-2$ in the $(a+1)^{\text{th}}$ row in a box to the left of the $(k-2)^{\text{th}}$ column.

Repeating this argument, there must be a 1 in a box to the left of the first column; a contradiction. Thus, there are no two amenable tableaux T_1 and T_2 of D with the same content. □

Example 3.42. *For* $\lambda = [4, 2, 1, 3]$ *and* $\mu = [3, 1, 0, 0]$ *we have*

$$Q_{(9,8,7,6,3)/(3,2,1)} = Q_{(9,8,6,4)} + Q_{(9,8,6,3,1)} + Q_{(9,8,5,4,1)} + Q_{(9,8,5,3,2)} + Q_{(9,7,6,4,1)} + Q_{(9,7,6,3,2)} + Q_{(9,7,5,4,2)}.$$

Corollary 3.43. *Let* $\lambda = [1, b, c, d]$ *and* $\mu = [w, 1, 0, 0]$. *Then* $Q_{\lambda/\mu}$ *is Q-multiplicity-free.*

Proof. For each tableau T of shape $D_{\lambda/\mu}$ let R_T be the diagram of the tableau after removing the boxes of $T^{(1)}$. By Lemma 1.38, the first row has only entries from $\{1', 1\}$. Two amenable tableaux T_1 and T_2 of shape $D_{\lambda/\mu}$ such that $R_{T_1} \neq R_{T_2}$ cannot have the same content because then $c(T_1)_1 \neq c(T_2)_1$. Thus, $R_T = R_{T_1} = R_{T_2}$ has shape $D_{\alpha/\beta}$

where $\alpha = [c, y, 0, 0]$ and $\beta \in \{[v, 1, 0, 0], [v, 2, 0, 0], [z, 1, v, 1]\}$ for some v and z. If for all T the diagram R_T has no two amenable tableaux with the same content then $Q_{\lambda/\mu}$ is Q-multiplicity-free.

We have $R_T^{ot} = D_{\alpha'/\beta'}$ where $\alpha' = [c + y - v - 1, v + 1, 0, 0]$ and $\beta' = [y - 1, 1, 0, 0]$ for $\alpha = [c, y, 0, 0]$ and $\beta = [v, 1, 0, 0]$. We have $R_T^{ot} = D_{\alpha'/\beta'}$ where $\alpha' = [c+y-v-2, v, 1, 1]$ and $\beta' = [y - 1, 1, 0, 0]$ for $\alpha = [c, y, 0, 0]$ and $\beta = [v, 2, 0, 0]$. In addition, we have $R_T^{ot} = D_{\alpha'/\beta'}$ where $\alpha' = [c + y - z - v - 2, z, 1, v + 1]$ and $\beta' = [y - 1, 1, 0, 0]$ for $\alpha = [c, y, 0, 0]$ and $\beta = [z, 1, v, 1]$.

By Lemmas 3.40 and 3.41, in each of these cases R_T^{ot} does not have two amenable tableaux with the same content. Thus, $Q_{\lambda/\mu}$ is Q-multiplicity-free. □

Example 3.44. *For $\lambda = [1, 4, 5, 2]$ and $\mu = [3, 1, 0, 0]$ we have*

$Q_{(11,6,5,4,3,2)/(3,2,1)} = Q_{(11,6,5,3)} + Q_{(10,6,5,4)} + Q_{(10,6,5,3,1)} + Q_{(9,6,5,4,1)} + Q_{(9,6,5,3,2)} + Q_{(8,6,5,4,2)}$.

Lemma 3.45. *Let $\lambda = [a, 1, c, d]$, $d \neq 1$ and $\mu = [w, 1, 0, 0]$. Then $Q_{\lambda/\mu}$ is Q-multiplicity-free.*

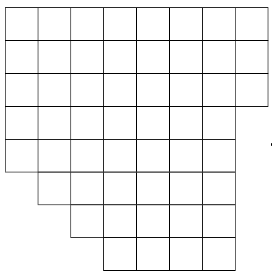
Proof. Consider $D_{\lambda/\mu}^{ot} = D_{\lambda'/\mu'}$ where $\lambda' = [a+c+d-w, w+1, 0, 0]$ and $\mu' = [1, c, d-1, 1]$. Thus, we have $\lambda' = (a+c+d, a+c+d-1, \ldots, w+1)$ and $\mu' = (c+d, d-1, d-2, \ldots, 1)$.

By Proposition 1.27, $f_{\mu'\nu}^{\lambda'} = f_{\nu\mu'}^{\lambda'}$. Thus, we need to look at tableaux of shape $D_{\lambda'/\nu}$ and content μ'. See Example 3.46 for a depiction of the proof.

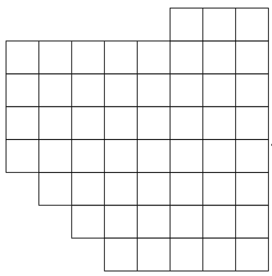
Let T and T' be two different amenable tableaux of shape $D_{\lambda'/\nu}$ and content μ'. By Lemma 1.39, all $2, 3, \ldots, d = \ell(\mu')$ are unmarked. Since d is the largest entry, it must be in a corner. Since there is only one corner, say (x, y), we have $T(x, y) = T'(x, y) = d$. Next insert the $(d - 1)$s. Both $(d - 1)$s must be unmarked and at least one $d - 1$ must be in the y^{th} column, otherwise the tableau is not amenable. Thus, we have $T(x - 1, y) = T'(x - 1, y) = d - 1$ and the other $d - 1$ is in the lowermost box in the $(y - 1)^{\text{th}}$ column. Repeating this argument, we see that the numbers $2, 3, \ldots, d$ are distributed as follows: For $0 \leq i \leq d-2$ in the $(y-i)^{\text{th}}$ column the lowermost boxes are filled from bottom to top with $d-i, d-1-i, \ldots, 2$. This is fixed for all amenable tableaux of the given shape. To get an amenable tableau there must be an unmarked 1 in each column with a 2 and in at least one column with no 2.

If there are two amenable tableaux of the same shape then they differ only by markings on some 1s. Let (u,v) be such that $T(u,v) = 1'$ and $T'(u,v) = 1$ or vice versa. Then $T(u+1,v), T'(u+1,v), T(u,v-1), T'(u,v-1) \notin \{1', 1\}$. Thus, (u,v) is in the lowermost row of the v^{th} column or $T(u+1,v) = T'(u+1,v) = 2$. If $T(u+1,v) = T'(u+1,v) = 2$ then $T(u,v) = T'(u,v) = 1$ as mentioned above. By the remark after Definition 1.36, the leftmost box of the lowermost row with boxes that are filled with entry from $\{1', 1\}$ must contain a 1. Thus, there is no such box (u,v) and, therefore, there are no two amenable tableaux of the same shape. $\square$

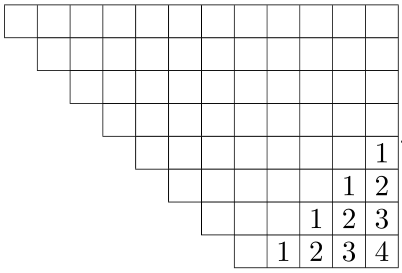
Example 3.46. *Let* $\lambda = (12, 11, 10, 8, 7, 6, 5, 4)$ *and* $\mu = (4, 3, 2, 1)$. *Then we have*

$D_{\lambda/\mu} =$  .

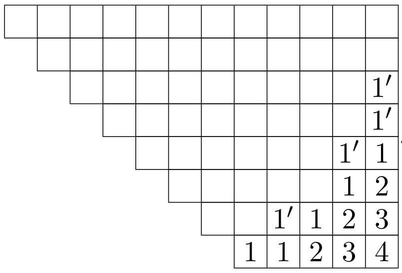
Since $f^{\lambda}_{\mu\nu} = f^{\lambda'}_{\mu'\nu}$ *and* $D^{ot}_{\lambda/\mu} = D_{\lambda'/\mu'}$ *where* $\lambda'/\mu' = (12, 11, 10, 9, 8, 7, 6, 5)/(9, 3, 2, 1)$ *we can consider* $D_{\lambda'/\mu'}$:

$D_{\lambda'/\mu'} =$  .

Since $f^{\lambda'}_{\mu'\nu} = f^{\lambda'}_{\nu\mu'}$ *we can consider amenable tableaux of shape* $D_{(12,11,10,9,8,7,6,5)/\nu}$ *and content* $(9,3,2,1)$*. We know fixed entries:*

$\tilde{T} =$  .

Now we have five entries from $\{1', 1\}$ *left to put into boxes such that we get an amenable tableau. For example we obtain*

$T =$  .

which is the only tableau of shape $D_{(12,11,10,9,8,7,6,5)/(12,11,9,8,6,5,2)}$ *and content* $(9,3,2,1)$*. Thus, we have*

$$f^{(12,11,10,8,7,6,5,4)}_{(4,3,2,1)(12,11,9,8,6,5,2)} = f^{(12,11,10,9,8,7,6,5)}_{(9,3,2,1)(12,11,9,8,6,5,2)} = f^{(12,11,10,9,8,7,6,5)}_{(12,11,9,8,6,5,2)(9,3,2,1)} = 1.$$

Lemma 3.47. *Let* $\lambda = [a,b,c,d]$ *and* $\mu = [w,1,0,0]$ *where* $w \leq 2$*. Then* $Q_{\lambda/\mu}$ *is* Q*-multiplicity-free.*

Proof. Case $w = 1$ follows from Lemma 3.34. Thus, consider case $w = 2$. Since $f^{\lambda}_{\mu\nu} = f^{\lambda}_{\nu\mu}$, we may consider tableaux of shape $D_{\lambda/\nu}$ and content $(2,1)$. There are two words with content $(2,1)$, namely $w^{(1)} = 121$ and $w^{(2)} = 211$. If $Q_{\lambda/\mu}$ is not Q-multiplicity-free then there must be some ν such that $D_{\lambda/\nu}$ is a diagram with two tableaux T_1 and T_2 where $c(T_1) = w^{(1)}$ and $c(T_2) = w^{(2)}$. If $(x(2), y(2)) = (x(3), y(3) - 1)$ then

$T_1(x(2), y(2)) = 2$ and $T_1(x(3), y(3)) = 1$ and T_1 is not a tableau; a contradiction. If $(x(2), y(2)) = (x(3)+1, y(3))$ then $T_2(x(2), y(2)) = 1$ and $T_2(x(3), y(3)) = 1$ and T_2 is not a tableau; a contradiction. Similarly, we have $(x(1), y(1)) \neq (x(2), y(2)-1)$ and $(x(1), y(1)) \neq (x(2)+1, y(2))$. Thus, these three boxes are all in different components consisting of one box. Each component of a diagram has a corner, hence, λ has at least three corners; a contradiction to $\lambda = [a, b, c, d]$. □

Lemma 3.41, Corollary 3.43, Lemma 3.45 and Lemma 3.47 together prove that 3.33 (iv) is Q-multiplicity-free.

Lemma 3.48. *Let* $\lambda = [a, b, 0, 0]$ *and* $\mu = [w, x, y, 1]$ *where* $w = 1$ *or* $x = 1$ *or* $2 \leq b \leq 3$ *or* $a + b - w - x - y - 1 = 1$. *Then* $Q_{\lambda/\mu}$ *is* Q*-multiplicity-free.*

Proof. Let $D := D_{\lambda/\mu}$, where $\lambda = [a, b, 0, 0]$ and $\mu = [w, x, y, 1]$. Then D^{ot} has shape $D_{\alpha/\beta}$ where $\alpha = [a + b - w - x - y - 1, w, x, y + 1]$ and $\beta = [b - 1, 1, 0, 0]$. For each of the given restrictions we have one of the following cases.

Case $w = 1$: Then we have $\alpha = [a + b - x - y - 2, 1, x, y + 1]$ and Lemma 3.45 proves Q-multiplicity-freeness.

Case $x = 1$: Then we have $\alpha = [a + b - w - y - 2, w, 1, y + 1]$ and Lemma 3.41 proves Q-multiplicity-freeness.

Case $2 \leq b \leq 3$: Then we have $\beta = [z, 1, 0, 0]$ where $1 \leq z \leq 2$ and Lemma 3.47 proves Q-multiplicity-freeness.

Case $a + b - w - x - y - 1 = 1$: Then we have $\alpha = [1, w, x, y + 1]$ and Corollary 3.43 proves Q-multiplicity-freeness. □

Thus, we have shown that 3.33 (iii) is Q-multiplicity-free by showing that 3.33 (iii) is the orthogonal transpose of 3.33 (iv). Now we will prove the Q-multiplicity-freeness of 3.33 (vi) and afterwards we will show that the orthogonal transpose of 3.33 (v) is included in 3.33 (vi) which means that the last remaining case of Proposition 3.33 is proved to be Q-multiplicity-free.

Lemma 3.49. *Let $\lambda = [a, b, c, 1]$ and $\mu = [w, 1, 0, 0]$ where $a \leq 2$. Then $Q_{\lambda/\mu}$ is Q-multiplicity-free.*

Proof. Since case $a = 1$ is shown in Corollary 3.43, we only have to show case $a = 2$. For each tableau T of shape $D_{\lambda/\mu}$ let R_T be the diagram of the remaining tableau after removing the boxes with entry from $\{1', 1, 2', 2\}$. By Lemma 1.38, the first two rows only have entries from $\{1', 1, 2', 2\}$. The boxes with entry from $\{1', 1\}$ form a hook. If the boxes with entry from $\{2', 2\}$ form a border strip all the marks of the entries are determined. If the boxes with entry from $\{2', 2\}$ form a diagram with more than one component then it must have precisely two components. The first component has boxes only in the $(w + 1)^{\text{th}}$ column and the second component has boxes in all other columns. In this case the last box of the second component must contain a $2'$ by the remark after Definition 1.36 and by Lemma 1.39. Thus, there are no two tableaux differing just by marks on the entries from $\{1', 1, 2', 2\}$.

If no R_T for any T has two amenable tableaux with the same content then $Q_{\lambda/\mu}$ is Q-multiplicity-free. R_T^{ot} is a diagram of shape $D_{\alpha'}$ for some $\alpha' \in DP$. Such a diagram has only one amenable tableau, namely the one that has just is in the i^{th} row for $1 \leq i \leq \ell(\alpha')$. Thus, $Q_{\lambda/\mu}$ is Q-multiplicity-free. □

Example 3.50. *For $\lambda = [1, 5, 6, 1]$ and $\mu = [4, 1, 0, 0]$ we get*

$Q_{(12,6,5,4,3,2,1)/(4,3,2,1)} = Q_{(12,6,5)} + Q_{(11,6,5,1)} + Q_{(10,6,5,2)} + Q_{(9,6,5,3)} + Q_{(8,6,5,4)}$.

For $\lambda = [2, 5, 5, 1]$ and $\mu = [4, 1, 0, 0]$ we get

$Q_{(12,11,5,4,3,2,1)/(4,3,2,1)} = Q_{(12,11,5)} + Q_{(11,10,5,2)} + Q_{(11,9,5,3)} + Q_{(11,9,5,2,1)} + Q_{(11,8,5,4)} + Q_{(11,8,5,3,1)} + Q_{(11,7,5,4,1)} + Q_{(10,9,5,3,1)} + Q_{(10,8,5,4,1)} + Q_{(10,8,5,3,2)} + Q_{(10,7,5,4,2)} + Q_{(9,8,5,4,2)} + Q_{(9,7,5,4,3)} + Q_{(12,10,5,1)} + Q_{(12,9,5,2)} + Q_{(12,8,5,3)} + Q_{(12,7,5,4)}$.

Lemma 3.51. *Let $\lambda = [a, b, c, 1]$ and $\mu = [w, 1, 0, 0]$ where $b \leq 2$. Then $Q_{\lambda/\mu}$ is Q-multiplicity-free.*

Proof. Case 1: $b = 1$.

The diagram $D^{ot}_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ where $\alpha = [a + c + 1 - w, w + 1, 0, 0]$ and $\beta = [1, c+1, 0, 0]$. Thus, $\alpha = (a+c+1, a+c, \dots, w+1)$ and $\beta = (c+1)$. Then B_α is a rotated hook and every diagram from $B^{(n)}_\alpha$ is connected. By Proposition 1.55, $Q_{\alpha/\beta} = Q_{\lambda/\mu}$ is Q-multiplicity-free.

Case 2: $b = 2$.

The diagram $D^{ot}_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ where $\alpha = [a + c - w + 2, w + 1, 0, 0]$ and $\beta = [2, c + 1, 0, 0]$. Thus, $\alpha = (a + c + 2, a + c + 1, \dots, w + 1)$ and $\beta = (c + 2, c + 1)$. By Proposition 1.27, $f^\alpha_{\beta\nu} = f^\alpha_{\nu\beta}$. Hence, we need to look at amenable tableaux of shape $D_{\alpha/\nu}$ and content $(c + 2, c + 1)$. The boxes with an entry from $\{2', 2\}$ form a border strip (in fact a rotated hook) where marks are determined. In every column with a box of this border strip there is a box filled with 2 and then there must be a box filled with a 1. Above the uppermost box filled with a 1 there cannot be a box filled with a $1'$. Otherwise, if w is the reading word of this tableau and the uppermost box filled with 1 is $(x(j), y(j))$ then $c+1 = m_2(\ell(w)+j-1) \geq m_1(\ell(w)+j-1)$ and $w_j = 1$; a contradiction to the amenability of the tableau.

Suppose we have two amenable tableaux T and T' with the same ν. If there are boxes (x, y) such that $T(x, y) \in \{2', 2\}$ and $T'(x, y) \in \{1', 1\}$ then one of these boxes is either the first or the last box of $T^{(2)}$. But then there is a box (r, s) such that $T(r, s) \in \{1', 1\}$ and $T'(r, s) \in \{2', 2\}$ is the last box or the first box of $T'^{(2)}$, respectively. Without loss of generality we may assume that (x, y) is the first box of $T^{(2)}$. Then $T(x - 1, y) = 1$ and $(x - 2, y)$ is not part of the diagram. Since $T'(x, y) \in \{1', 1\}$, we have $T'(x - 1, y) = 1'$; a contradiction to the fact that there cannot be a box filled with a $1'$ above the uppermost box filled with a 1.

Hence, T and T' differ only by markings on 1s. Let (u, v) be the uppermost rightmost box such that $T'(u, v) = 1'$, say, and $T(u, v) = 1$. Then $(u+1, v), (u, v-1) \notin T^{(1)} = T'^{(1)}$. Thus, either $(u + 1, v) \notin D_{\lambda/\nu}$ or $T(u + 1, v) = T'(u + 1, v) \in T^{(2)} = T'^{(2)}$. Suppose $T(u + 1, v) = T'(u + 1, v) \in T^{(2)} = T'^{(2)}$. If we have $(u + 1, v) = (x(k), y(k))$ then for

$w(T')$ we have $m_2(\ell(w(T')) - k) = m_1(\ell(w(T')) - k)$ and $w_k \in \{2', 2\}$; a contradiction to the amenability of T'. Hence, $(u+1, v) \notin D_{\lambda/\nu}$. By the remark after Definition 1.36, $T^{(1)} = T'^{(1)}$ must be fitting. It follows that there is no box (u, v) and, therefore, there are no two amenable tableaux of $D_{\lambda/\nu}$. □

Example 3.52. *For* $\lambda = [3, 1, 6, 1]$ *and* $\mu = [6, 1, 0, 0]$ *we have*

$Q_{(10,9,8,6,5,4,3,2,1)/(6,5,4,3,2,1)} = Q_{(10,9,8)} + Q_{(10,9,7,1)} + Q_{(10,8,7,2)} + Q_{(9,8,7,3)}$.

For $\lambda = [3, 2, 6, 1]$ *and* $\mu = [6, 1, 0, 0]$ *we have*

$Q_{(11,10,9,6,5,4,3,2,1)/(6,5,4,3,2,1)} = Q_{(11,10,9)} + Q_{(11,10,8,1)} + Q_{(11,10,7,2)} + Q_{(11,9,8,2)} + Q_{(11,9,7,3)} + Q_{(11,9,7,2,1)} + Q_{(11,8,7,3,1)} + Q_{(10,9,8,3)} + Q_{(10,9,7,3,1)} + Q_{(10,9,7,4)} + Q_{(10,8,7,4,1)} + Q_{(10,8,7,3,2)} + Q_{(9,8,7,4,2)}$.

Lemma 3.53. *Let* $\lambda = [a, b, c, 1]$ *and* $\mu = [w, 1, 0, 0]$ *where* $c \leq 2$. *Then* $Q_{\lambda/\mu}$ *is Q-multiplicity-free.*

Proof. Let $n := |D_{\lambda/\mu}|$.

Case 1: $c = 1$.

The only box in the $(a+1)^{\text{th}}$ row is $(a+1, a+1)$. By Lemma 3.40, the filling of the first a rows is unique up to markings. In fact, the filling consists entirely of hooks at the diagonal $\{(s, t) \mid t - s = w\}$. Thus, two different amenable tableaux with the same content differ only by markings. Suppose we have two such tableaux T and T'. Let (y, z) be a box such that $T'(y, z) = k'$, say, and $T(y, z) = k$. Then there must be a box below and to the left of this box with a k. This box is $(a+1, a+1)$ and $y = a$. However, since $T(a, z) = k$, we have $m_{k-1}(n) = m_k(n)$; a contradiction to Lemma 1.39. Thus, there are no two different amenable tableaux with the same content.

Case 2: $c = 2$.

Let T be an amenable tableau of shape $D_{\lambda/\mu}$. By Lemma 3.40, the filling of the first a rows is unique up to markings. In fact, the filling consists entirely of hooks at the diagonal $\{(s,t) \mid t-s=w\}$. The three boxes below the a^{th} row are $(a+1,a+1)$, $(a+1,a+2)$ and $(a+2,a+2)$.

Case 2.1: $|T(a+1,a+1)| = |T(a+1,a+2)| = k$ for some k.

Then, by Lemma 1.34, we have $|T(a+2,a+2)| > k$. Since $(a,a+1) \in D_{\lambda/\mu}$ we have $k > 1$. If k' or k occur in the first a rows, it follows that $m_k(2n) \geq m_{k-1}(2n)$; a contradiction to the amenability of T. Thus, $k = j+1$, where $j = \min\{a, b+3\}$. This is only possible if there are at least three unmarked js, otherwise there is no amenable tableau with these properties. Then $T(a+2,a+2) = k+1 = j+2$ follows and $T(a+1,a+1)$, $T(a+1,a+2)$ and $T(a+2,a+2)$ are unmarked. Additionally, each of the entries in the a^{th} row is unmarked and, therefore, there is no other amenable tableau with the same content.

Case 2.2: $|T(a+1,a+2)| = |T(a+2,a+2)| = k$ for some k.

Since $(a,a+1) \in D_{\lambda/\mu}$ we have $k > 1$. If k' or k occur in the first a rows it follows that $T(a+1,a+1) = k-1$, otherwise $m_k(2n) \geq m_{k-1}(2n)$; a contradiction to the amenability of T. Assume there are two different amenable tableaux T and T' of $D_{\lambda/\mu}$ with the same content such that $|T(a+1,a+1)| = |T'(a+1,a+1)| = k-1$, $|T(a+1,a+2)| = |T'(a+1,a+2)| = k$ and $|T(a+2,a+2)| = |T'(a+2,a+2)| = k$. It follows that these tableaux differ only by markings. Then there is some i such that $T'(y,z) = i'$, say, and $T(y,z) = i$. It follows that $y = a$ since the entries in the other rows are determined. It also follows that there is an i in a box which is lower and to the left of (a,z). Thus, we have $i \in \{k-1,k\}$ and, therefore, $k > 2$. If $i = k-1$, then, since $T(a,z) = k-1$, for $w(T)$ we have $m_{k-2}(n) = m_{k-1}(n)$; a contradiction to Lemma 1.39. Hence, we have $i = k$. If $T(a,z-1) = (k-1)'$, then, since $T(a,z) = k$, for $w(T')$ we have $m_{k-1}(n) = m_k(n)$; again a contradiction to Lemma 1.39. If $T(a,z-1) = k-1$, then we have $m_{k-2}(n) = m_{k-1}(n)$; a contradiction to Lemma 1.39 as well. Thus, there are no such two different amenable tableaux of $D_{\lambda/\mu}$.

Case 2.3: $|T(a+1,a+1)| = u$, $|T(a+1,a+2)| = v$ and $|T(a+2,a+2)| = t$ where $u \neq v$, $u \neq t$ and $v \neq t$.

Then we have $u < v < t$. Assume there are two different amenable tableaux T and T' of $D_{\lambda/\mu}$ with the same content in which the boxes $(a+1,a+1)$, $(a+1,a+2)$ and $(a+2,a+2)$ are filled as above. It follows that these tableaux differ only by markings. Then there is some i such that $T'(y,z) = i'$, say, and $T(y,z) = i$. It follows that $y = a$ since the entries in the other rows are determined. It also follows that there is an i in a box which is lower and to the left of the box (a,z). The only possible case is that $i \in \{u,v,t\}$. Arguing as in the cases above, we see that for T we either have $m_{t-1}(n) = m_t(n)$ or $m_{v-1}(n) = m_v(n)$ or $m_{u-1}(n) = m_u(n)$. This contradicts Lemma 1.39.

Hence, there are no such two different amenable tableaux of $D_{\lambda/\mu}$. □

Example 3.54. *For* $\lambda = [5,3,1,1]$ *and* $\mu = [4,1,0,0]$ *we get*

$Q_{(9,8,7,6,5,1)/(4,3,2,1)} = Q_{(9,8,5,3,1)} + Q_{(9,7,6,3,1)} + Q_{(9,7,5,4,1)} + Q_{(9,7,5,3,2)}$.

For $\lambda = [4,3,2,1]$ *and* $\mu = [4,1,0,0]$ *we get*

$Q_{(9,8,7,6,2,1)/(4,3,2,1)} = Q_{(9,7,5,2)} + Q_{(9,8,4,2)} + Q_{(8,6,5,4)} + Q_{(8,6,5,3,1)} + Q_{(8,6,4,3,2)} + Q_{(8,7,4,3,1)} + Q_{(8,7,5,2,1)} + Q_{(8,7,6,2)} + Q_{(8,7,5,3)} + Q_{(9,6,4,3,1)} + Q_{(9,6,5,2,1)} + Q_{(9,7,4,3)} + Q_{(9,7,4,2,1)} + Q_{(9,6,5,3)}$.

Lemma 3.55. *Let* $\lambda = [a,b,c,1]$ *and* $\mu = [w,1,0,0]$ *where* $w \leq 3$ *or* $w = a+c-1$. *Then* $Q_{\lambda/\mu}$ *is Q-multiplicity-free.*

Proof. Case $w = 1$ follows from Lemma 3.34 and case $w = 2$ follows from Lemma 3.47. For case $w = a+c-1$ the diagram $D^t_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ where $\alpha = [1,c,b,a]$ and $\beta = [b,1,0,0]$ and follows from Corollary 3.43. Thus, we only have to prove case $w = 3$.

By Proposition 1.27, $f^{\lambda}_{\mu\nu} = f^{\lambda}_{\nu\mu}$ and we just need to look at tableaux of shape $D_{\lambda/\nu}$ and content $\mu = (3,2,1)$. By Lemma 1.39, all entries must be unmarked. Assume there are two different amenable tableaux T_1, T_2 of $D_{\lambda/\nu}$ with content μ for some $\nu \in DP$. Thus, we get one tableau from the other by interchanging some entries in certain boxes.

Suppose the 3 is in one of these boxes. Let (a, x) be the upper corner (where $x = a + b + c$) and let (e, e) be the lower corner (where $e = a + c$). Since the 3 is the greatest entry it must be either in (a, x) or in (e, e). Thus, we have $T_1(a, x) = 3$, say, and $T_2(e, e) = 3$. Then, by Lemma 1.38 and since T_1 is amenable, we have $a \geq 3$, $T_1(a - 1, x) = 2$ and $T_2(a - 2, x) = 1$. We have $T_2(a, x) \in \{1, 2\}$. Either way, since all entries are unmarked, we have $T_2(a - 2, x) \leq T_2(a - 1, x) - 1 \leq T_2(a, x) - 2$ and, hence, $T_2(a - 2, x) \notin \{1, 2, 3\}$. Thus, either $T_1(a, x) = T_2(a, x) = 3$ or $T_1(e, e) = T_2(e, e) = 3$.

Suppose $T_1(a, x) = T_2(a, x) = 3$. Then $T_1(a-1, x) = T_2(a-1, x) = 2$ and $T_1(a-2, x) = T_2(a - 1, x) = 1$. Thus, T_1 and T_2 differ only by interchanging one 1 and one 2. Let the boxes containing these entries be (f, t) and (v, g), where $g > t$ and $v < f$. The remaining 1 must be in a box to the right and above (v, g). If $T_1(a-1, x-1) = T_2(a-1, x-1) = 1$ then $T_1(a, x - 1) = T_2(a, x - 1) = 2$ and both tableaux are the same; a contradiction. Thus, we have $T_1(a, x - 1) = T_2(a, x - 1) = 1$. The remaining entries must be in two corners below $(a, x - 1)$. However, there is only one corner (namely (e, e)), thus, there are no two different amenable tableaux such that $T_1(a, x) = T_2(a, x) = 3$. Therefore, we have $T_1(e, e) = T_2(e, e) = 3$.

Suppose $T_1(a, x) = 1$. Then $T_1(e - 1, e) = T_1(e - 1, e - 1) = 2$ and after inserting the 1s the tableau is determined. Thus, if $T_1(a, x) = 1$, there are no two different amenable tableaux.

Therefore, $T_1(a, x) = T_2(a, x) = 2$. By amenability, $T_1(a - 1, x) = T_2(a - 1, x) = 1$. Thus, T_1 and T_2 differ only by interchanging one 1 and one 2. With the same argument as above we see that $T_1(a, x - 1) = T_2(a, x - 1) = 1$. Then we have $T_1(e - 1, e) = T_2(e-1, e) = 2$ and both tableaux are the same; a contradiction. Thus, there are no two different amenable tableaux of shape $D_{\lambda/\nu}$ and content $\mu = (3, 2, 1)$. □

Example 3.56. *For* $\lambda = [3, 3, 3, 1]$ *and* $\mu = [3, 1, 0, 0]$ *we get*

$Q_{(9,8,7,3,2,1)/(3,2,1)} = Q_{(9,8,7)} + Q_{(9,8,6,1)} + Q_{(9,8,5,2)} + Q_{(9,8,4,3)} + Q_{(9,7,6,2)} + Q_{(9,7,5,3)} + Q_{(9,7,5,2,1)} + Q_{(9,7,4,3,1)} + Q_{(9,6,5,3,1)} + Q_{(9,6,4,3,2)} + Q_{(8,7,6,3)} + Q_{(8,7,4,3,2)} + Q_{(8,6,5,3,2)} + Q_{(8,6,4,3,2,1)} + Q_{(8,7,5,3,1)}$.

The Lemmas 3.49, 3.51, 3.53 and 3.55 all together show that 3.33 (v) is Q-multiplicity-free.

Lemma 3.57. *Let $\lambda = [a, b, 0, 0]$ and $\mu = [w, x, 0, 0]$ where $2 \le b \le 4$ or $w \le 2$ or $2 \le x \le 3$ or $a = w + 1$ or $a + b - w - x \le 2$. Then $Q_{\lambda/\mu}$ is Q-multiplicity-free.*

Proof. The diagram $D^{ot}_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ where $\alpha = [a + b - w - x, w, x - 1, 1]$ and $\beta = [b - 1, 1, 0, 0]$. For each of the given restrictions we have one of the following cases.

Case $2 \le b \le 4$: Then we have $\beta = [w', 1, 0, 0]$ where $w' \le 3$ and Lemma 3.55 proves Q-multiplicity-freeness.

Case $w \le 2$: Then we have $\alpha = [a', b', c', 1]$ where $b' \le 2$ and Lemma 3.51 proves Q-multiplicity-freeness.

Case $2 \le x \le 3$: Then we have $\alpha = [a', b', c', 1]$ where $c' \le 2$ and Lemma 3.53 proves Q-multiplicity-freeness.

Case $a = w + 1$: Then we have $\alpha = [a', b', c', 1]$ and $\beta = [w', 1, 0, 0]$ where we have $a' = a+b-w-x = b-x+1$ and, hence, $w' = b-1 = (b-x+1)+(x-1)-1 = a'+c'-1$ and Lemma 3.55 proves Q-multiplicity-freeness.

Case $a + b - w - x \le 2$: Then we have $\alpha = [a', b', c', 1]$ where $a' \le 2$ and Lemma 3.49 proves Q-multiplicity-freeness. □

We have now proven that the cases occurring in Proposition 3.33 are indeed Q-multiplicity-free and are now able to state this result as the following theorem.

Theorem 3.58. *Let $\lambda, \mu \in DP$ and $a, b, c, d, w, x, y \in \mathbb{N}$ such that $D_{\lambda/\mu}$ is basic. $Q_{\lambda/\mu}$ is Q-multiplicity-free if and only if λ and μ satisfy one of the following conditions:*

(i) λ is arbitrary and $\mu \in \{\emptyset, (1)\}$,

(ii) $\lambda = (a+b-1, a+b-2, \dots, b)$, *where* $b \in \{1,2\}$ *and* μ *is arbitrary,*

(iii) $\lambda = (a+b-1, a+b-2, \dots, b)$ *and* $\mu = (w+x+y, w+x+y-1, \dots, x+y+2, x+y+1, y, y-1, \dots, 1)$, *where* $w = 1$ *or* $x = 1$ *or* $b \leq 3$ *or* $a+b-w-x-y-1 = 1$,

(iv) $\lambda = (a+b+c+d-1, a+b+c+d-2, \dots, b+c+d+1, b+c+d, c+d-1, c+d-2, \dots, d)$, *where* $d \neq 1$ *and* $\mu = (w, w-1, \dots, 1)$ *where* $1 \in \{a, b, c\}$ *or* $w \leq 2$,

(v) $\lambda = (a+b+c, a+b+c-1, \dots, b+c+2, b+c+1, c, c-1, \dots, 1)$ *and* $\mu = (w, w-1, \dots, 1)$, *where* $a \leq 2$ *or* $b \leq 2$ *or* $c \leq 2$ *or* $w \leq 3$ *or* $w = a+c-1$,

(vi) $\lambda = (a+b-1, a+b-2, \dots, b)$ *and* $\mu = (w+x-1, w+x-2, \dots, x)$, *where* $2 \leq b \leq 4$ *or* $w \leq 2$ *or* $x \leq 3$ *or* $a = w+1$ *or* $a+b-w-x \leq 2$.

Some of these cases overlap.

Proof. Using the shape path notation of Definition 3.17 we have:

- 3.58 (ii) is the case $\lambda = [a, b, 0, 0]$ where $b \in \{1, 2\}$ and μ is arbitrary.
- 3.58 (iii) is the case $\lambda = [a, b, 0, 0]$ and $\mu = [w, x, y, 1]$ where $w = 1$ or $x = 1$ or $b \leq 3$ or $a+b-w-x-y-1 = 1$.
- 3.58 (iv) is the case $\lambda = [a, b, c, d]$ such that $d \neq 1$ and $\mu = [w, 1, 0, 0]$ where $1 \in \{a, b, c\}$ or $w \leq 2$.
- 3.58 (v) is the case $\lambda = [a, b, c, 1]$ and $\mu = [w, 1, 0, 0]$ where $a \leq 2$ or $b \leq 2$ or $c \leq 2$ or $w \leq 3$ or $w = a+c-1$.
- 3.58 (vi) is the case $\lambda = [a, b, 0, 0]$ and $\mu = [w, x, 0, 0]$ where $2 \leq b \leq 4$ or $w \leq 2$ or $x \leq 3$ or $a = w+1$ or $a+b-w-x \leq 2$.

By Proposition 3.33, only these cases can be Q-multiplicity-free. Lemma 3.34 states that 3.58 (i) is Q-multiplicity-free. Lemma 3.38 states that 3.58 (ii) is Q-multiplicity-free.

Lemmas 3.41, 3.45 and 3.47 and Corollary 3.43 state that 3.58 (iv) is Q-multiplicity-free. Lemma 3.48 states that 3.58 (iii) is Q-multiplicity-free. Lemmas 3.49, 3.51, 3.53 and 3.55 state that 3.58 (v) is Q-multiplicity-free. Lemma 3.57 states that 3.58 (vi) for $x \neq 1$ is Q-multiplicity-free. Lemma 3.40 states that for 3.58 (vi) for $x = 1$ we have $Q_{\lambda/\mu} = Q_\alpha$ for some α (see the remark after Lemma 3.40). Hence, 3.58 (vi) for $x = 1$ is Q-multiplicity-free. Thus, all cases in Theorem 3.58 are Q-multiplicity-free. □

4 Classification of Q-homogeneous skew Schur Q-functions

The classification of (s-)homogeneous skew Schur functions are given by Bessenrodt and Kleshchev [4, Lemma 4.4]. In the classical case the (s-)homogeneous skew Schur functions are equal to some non-skew Schur function. The problem which skew Schur Q-functions are equal to some non-skew Schur Q-function is answered by Salmasian [15]. Clearly, these skew Schur Q-functions are Q-homogeneous. As it turns out these are not the only ones that are Q-homogeneous.

In this chapter we find the Q-homogeneous skew Schur Q-functions that are not equal to some non-skew Schur Q-function to complete the classification of Q-homogeneous skew Schur Q-functions. The statements of this chapter are part of my master's thesis. Using helpful tools of Chapter 1, the proofs of this chapter are shortened compared to the ones in my master's thesis.

Definition 4.1. A symmetric function f is called **Q-homogeneous** if it is a multiple of a single Schur Q-function, that is if $f = k \cdot Q_\nu$ for some $\nu \in DP$ and $k \in \mathbb{N}$.

In the following we will classify the Q-homogeneous skew Schur Q-functions indexed by a disconnected diagram as given in Theorem 4.17, the main theorem. We will exclude non-Q-homogeneous skew Schur Q-functions by finding an amenable tableau with content different from $c(T_{\lambda/\mu})$. Then the decomposition of this skew Schur Q-function has at least two homogeneous components and, hence, is not Q-homogeneous.

If in the following some P_i is mentioned then it is the P_i from Definition 1.45, hence $P_i = T^{(i)}_{\lambda/\mu}$.

Hypothesis. *We will always assume that λ and μ are such that $D_{\lambda/\mu}$ is basic (see Definition 1.13).*

Remark. As in the previous chapter, we use Corollary 1.44 to prove amenability of a tableau. If entries of a tableau do not satisfy the properties of Corollary 1.44 then we will show that for these entries the properties of Lemma 1.42 are satisfied and use this lemma to prove amenability.

4.1 The disconnected case

We will first exclude all non-Q-homogeneous Schur Q-function indexed by a disconnected diagram, and then in Proposition 4.10 we will prove the homogeneity of the remaining skew Schur Q-functions indexed by a disconnected diagram.

Lemma 4.2. *Let comp*$(D_{\lambda/\mu}) > 1$ *and* $\nu = c(T_{\lambda/\mu})$*. If there is a component* C_i *such that* $i > 1$ *and* C_i *has at least two boxes then* $f^{\lambda}_{\mu\bar{\nu}} > 0$ *where* $\bar{\nu} = (\nu_1 - 1, \nu_2 + 1, \nu_3, \nu_4, \ldots)$*. In particular,* $Q_{\lambda/\mu}$ *is not* Q*-homogeneous.*

Proof. We may consider the case that a component which is not the first component has boxes in two rows. Otherwise we may consider the orthogonal transpose of the diagram.

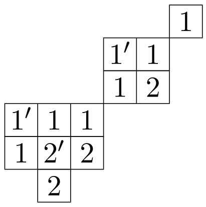
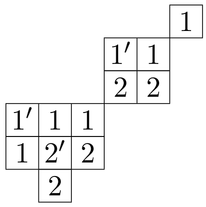
Let C_i where $i > 1$ be a component that has boxes in at least two rows. If (x, y) is the rightmost box of the lowermost row of $C_i \cap P_1$ then $(x-1, y) \in P_1$ and $(x+1, y+1) \notin D_{\lambda/\mu}$. We get a new tableau T if we set $T(x, y) = 2$, $T(x-1, y) = 1$ and $T(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. By Corollary 1.44, T is amenable and has content $c(T) = (\nu_1 - 1, \nu_2 + 1, \nu_3, \nu_4, \ldots)$. ☐

Example 4.3. *For* $T_{\lambda/\mu} = \begin{array}{cccccc} & & & & 1' & 1 \\ & & & 1 & 1 & 2 \\ 1' & 1 & 1 & & & \\ 1 & 2' & 2 & & & \\ & 2 & & & & \end{array}$ *we obtain* $T = \begin{array}{cccccc} & & & & 1 & 1 \\ & & & 1 & 2 & 2 \\ 1' & 1 & 1 & & & \\ 1 & 2' & 2 & & & \\ & 2 & & & & \end{array}$.

Lemma 4.4. *Let comp*$(D_{\lambda/\mu}) > 2$ *and* $\nu = c(T_{\lambda/\mu})$*. Then we have* $f^{\lambda}_{\mu\bar{\nu}} > 0$ *where* $\bar{\nu} = (\nu_1 - 1, \nu_2 + 1, \nu_3, \nu_4, \ldots)$*. In particular,* $Q_{\lambda/\mu}$ *is not* Q*-homogeneous.*

Proof. Let (x, y) be the rightmost box of the lowermost row of $C_2 \cap P_1$. We get a new tableau T if we set $T(x, y) = 2$ and $T(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. By Corollary 1.44, T is m-amenable for $m > 2$. There is a 2 but no 1 in the y^{th} column.

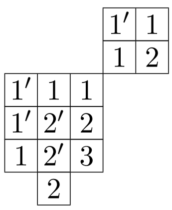
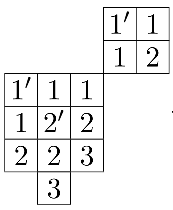
However, there is a 1 in the last box of $C_3 \cap P_1$. Hence, by Lemma 1.42, amenability follows. It is clear that $c(T) = (\nu_1 - 1, \nu_2 + 1, \nu_3, \nu_4, \ldots)$. □

Example 4.5. *For* $T_{\lambda/\mu} =$  *we obtain* $T =$  .

Lemma 4.6. *Let* $comp(D_{\lambda/\mu}) > 1$ *and* $\nu = c(T_{\lambda/\mu})$. *Suppose the leftmost column of* C_1 *(which is the leftmost column of* $D_{\lambda/\mu}$*) contains at least two boxes. Then* $f^{\lambda}_{\mu\bar{\nu}} > 0$ *where* $\bar{\nu} = (\nu_1 - 1, \nu_2, \nu_3, \ldots, \nu_z, \nu_{z+1} + 1, \nu_{z+2}, \ldots)$ *where* $z := \ell(\lambda) - \ell(\mu)$. *In particular,* $Q_{\lambda/\mu}$ *is not Q-homogeneous.*

Proof. Let (x, x) be the last box of P_1. We get a new tableau T if we set $P'_1 := P_1 \backslash \{(x, x)\}$ and use this instead of P_1 in the algorithm of Definition 1.45. Let $P'_i := T^{(i)}$. It is clear that (x, x) is the last box of P'_2. If $(x + 1, x + 1)$ is the last box of P_2 then $(x + 1, x + 1)$ is the last box of P'_3, etc. Thus, the P'_is are distinguished from the P_is by at most one moved or added box. By Corollary 1.44, T is m-amenable for $m > 2$. There is a 1 with no 2 below in the last box of $C_2 \cap P_1$. Thus, by Corollary 1.44, T is 2-amenable and, hence, amenable.

It is clear that $c(T)_1 = \nu_1 - 1$ since $|P'_1| = |P_1| - 1$. The P_is for all $2 \leq i \leq z$ satisfy the property that the last box is part of the main diagonal $\{(a, a) \mid a \in \mathbb{N}\}$. As mentioned above, they differ from P'_is by the fact that the last box is not $(x + i - 1, x + i - 1)$ but instead $(x + i - 2, x + i - 2)$. Thus, $|P'_i| = \nu_i$. Then $(x + z - 1, x + z - 1)$ is the last box of P'_{z+1} but since $(x + z, x + z) \notin D_{\lambda/\mu}$, it follows $|P'_{z+1}| = \nu_{z+1} + 1$. Hence, $c(T) = (\nu_1 - 1, \nu_2, \nu_3, \ldots, \nu_z, \nu_{z+1} + 1, \nu_{z+2}, \ldots)$. □

Example 4.7. *For* $T_{\lambda/\mu} =$  *we obtain* $T =$  .

Lemma 4.8. *Let $comp(D_{\lambda/\mu}) > 1$ and $\nu = c(T_{\lambda/\mu})$. If C_1 has boxes above the row of the uppermost box of the leftmost column then $Q_{\lambda/\mu}$ is not Q-homogeneous.*

Proof. Since Lemma 4.6 states that diagrams which have more than one box in the leftmost column are not Q-homogeneous, it suffices to consider diagrams such that the leftmost column of C_1 has only one box. Let (t, r) be the rightmost box of P_1 in the lowermost row of P_1. Note that the last box of P_1 is to the left of the r^{th} column. We get a new tableau T if we modify the algorithm of Definition 1.45 so that $P_1' := P_1 \setminus \{(t, r)\}$ is used instead of P_1 in the algorithm.

By Corollary 1.44, T is m-amenable for $m > 2$. If $T(t, r) = 2$ then, by Corollary 1.44, this tableau is 2-amenable since $T(t-1, r) = 1$. If $T(t, r) = 2'$ then $T(t-1, r-1) \neq 1'$ since $(t-1, r-1) \notin D_{\lambda/\mu}$. However, there is a 1 with no 2 below it in the last box of $C_2 \cap P_1$. Thus, by Lemma 1.42, this tableau is 2-amenable and, hence, amenable. Since $|P_1'| = |P_1| - 1$, we have $c(T) \neq \nu$. □

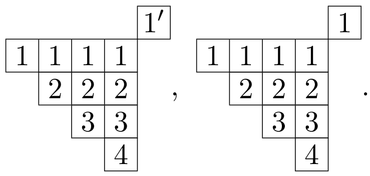
Example 4.9. *For* $T_{\lambda/\mu} = \begin{array}{cccccc} & & & & 1' & 1 \\ & & & & 1 & 2 \\ & & 1' & 1 & & \\ 1 & 1 & 1 & 2' & & \\ & 2 & 2 & 2 & & \end{array}$ *we obtain* $T = \begin{array}{cccccc} & & & & 1' & 1 \\ & & & & 1 & 2 \\ & & 1 & 1 & & \\ 1 & 1 & 2' & 2 & & \\ & 2 & 2 & 3 & & \end{array}$.

Proposition 4.10. *Let $\lambda, \mu \in DP$ be such that $comp(D_{\lambda/\mu}) > 1$ and such that $D_{\lambda/\mu}$ is basic. Then $Q_{\lambda/\mu} = k \cdot Q_\nu$ if and only if $k = 2$, $\lambda = (r+2, r, r-1, \ldots, 1)$, $\mu = (r+1)$ and $\nu = (r+1, r-1, r-2, \ldots, 1)$ for some $r \geq 1$.*

Proof. Let $Q_{\lambda/\mu}$ be Q-homogeneous and $D_{\lambda/\mu}$ be a disconnected diagram. Lemma 4.2 states that for $1 < i \leq comp(D_{\lambda/\mu})$ every component C_i can consist of only one box and Lemma 4.4 states that the diagram must consist of precisely two components. Thus, $D_{\lambda/\mu}$ must have only two components C_1, C_2 where C_2 consists of a single box. Lemma 4.6 states that the leftmost column of C_1 must have only one box and Lemma 4.8 states that this box is in the uppermost row of C_1. This implies that C_1 has shape D_α for some $\alpha \in DP$. The same must be true for the orthogonal transpose of the diagram.

Thus, $\alpha = (r, r-1, \dots, 1)$ for some $r \geq 1$. Therefore, we have $\lambda = (r+2, r, r-1, \dots, 1)$ and $\mu = (r+1) = (\lambda_1 - 1)$. By Proposition 1.55, $B_\lambda^\times = \{(1, r+1)\}$ and we obtain $\nu = (r+1, r-1, r-2, \dots, 1)$ and $k = f_{\mu\nu}^\lambda = 2$. □

Example 4.11. *For $\lambda = (6,4,3,2,1)$ and $\mu = (5)$ the diagram $D_{\lambda/\mu}$ has the following two tableaux:*



Remark. An alternate proof of the Q-multiplicity-freeness of the skew Schur Q-functions appearing in Lemma 4.10 can be obtained by using Lemma 1.71. For the partitions $\lambda = (r+2, r, r-1, \dots, 1)$ and $\mu = (r+1)$ we obtain

$$\begin{aligned} Q_{\lambda/\mu} &= Q_{\Delta_1^{\leftarrow}(D_{\lambda/\mu})} + Q_{\Delta_1^{\downarrow}(D_{\lambda/\mu})} = Q_{(r+1,r,r-1,\dots,1)/(r)} + Q_{(r+1,r-1,r-2,\dots,1)} \\ &= 2 \cdot Q_{(r+1,r-1,r-2,\dots,1)} \end{aligned}$$

by Lemma 1.60.

4.2 The connected case

We have finished the disconnected case and we now consider Q-homogeneous Schur Q-functions indexed by a connected diagram. The following lemmas show the non-Q-homogeneity of $Q_{\lambda/\mu}$ if some P_i in $T_{\lambda/\mu}$ has at least two components. This leads to Lemma 4.16 that shows that in this case for $Q_{\lambda/\mu} = k \cdot Q_\nu$ we obtain $k = 1$ and it gives the conclusion that Salmasian already classified the Q-homogeneous Schur Q-functions indexed by a connected diagram in [15].

Lemma 4.12. *Let $D_{\lambda/\mu}$ be a diagram. Let $\nu := c(T_{\lambda/\mu})$. Let there be some $i > 1$ such that $comp(P_i) \geq 2$ and let $C_1, \dots, C_{comp(P_i)}$ be the components of P_i. Let (x_l, y_l) and (u_l, v_l) be* *the first box and the last box of C_l, respectively. If for some $j \in \{1, 2, \ldots, comp(P_i)-1\}$ we have $v_{j+1} \geq y_j+2$ then $f^{\lambda}_{\mu\tilde{\nu}} > 0$ where $\tilde{\nu} = (\nu_1, \nu_2, \ldots, \nu_{i-2}, \nu_{i-1}-1, \nu_i+1, \nu_{i+1}, \nu_{i+2}, \ldots)$.*

Proof. Let $(u, v) = (u_{j+1}, v_{j+1})$. Then $(u-1, v-1), (u, v-1) \in P_{i-1}$. Let $(s, v-1)$ be the lowermost box of P_{i-1} in the $(v-1)^{\text{th}}$ column. We get a new tableau T if we set $T(s, v-1) = i$, $T(s-1, v-1) = i-1$ and $T(r, t) = T_{\lambda/\mu}(r, t)$ for every other box $(r, t) \in D_{\lambda/\mu}$. If $(s, v) \in D_{\lambda/\mu}$ then $T(s, v) = T_{\lambda/\mu}(s, v) \neq i'$ and this filling is a tableau. By Corollary 1.44, the tableau T is amenable. It is clear that $c(T)_{i-1} = \nu_{i-1} - 1$ and $c(T)_i = \nu_i + 1$ and $c(T)_k = \nu_k$ for $k \neq i-1, i$. □

Example 4.13. *For $\lambda = (9, 8, 5, 3, 2)$ and $\mu = (6, 5, 2, 1)$ the changes are written in boldface:*

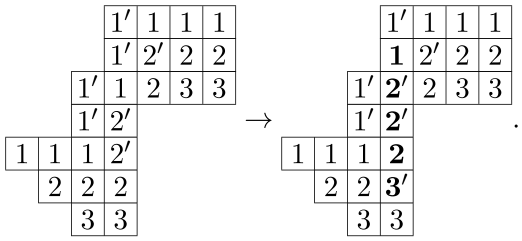
$$\begin{array}{ccccc} & & 1' & 1 & 1 \\ & & 1' & 2 & 2 \\ 1' & 1 & 1 & & \\ 1' & 2' & & & \\ 1 & 2 & & & \end{array} \quad \rightarrow \quad \begin{array}{ccccc} & & 1' & 1 & 1 \\ & & \mathbf{1} & 2 & 2 \\ 1' & 1 & \mathbf{2} & & \\ 1' & 2' & & & \\ 1 & 2 & & & \end{array} .$$

Lemma 4.14. *Let $D_{\lambda/\mu}$ be a diagram. Let $\nu := c(T_{\lambda/\mu})$ where $\nu_j := 0$ for $j > \ell(\nu)$. Let there be some $i > 1$ such that $comp(P_i) \geq 2$ and let $C_1, \ldots, C_{comp(P_i)}$ be the components of P_i. Let (x_l, y_l) and (u_l, v_l) be the first box and the last box of C_l, respectively. If for some $j \in \{1, 2, \ldots, comp(P_i) - 1\}$ we have $v_{j+1} = y_j + 1$ then $f^{\lambda}_{\mu\bar{\nu}} > 0$ where $\bar{\nu} = (\nu_1, \nu_2, \ldots, \nu_{i-2}, \nu_{i-1} - 1, \nu_i, \nu_{i+1} + 1, \nu_{i+2}, \nu_{i+2}, \ldots)$.*

Proof. Let $(x, y) = (x_j, y_j)$ and $(u, y+1) = (u_{j+1}, v_{j+1})$. Then $x > u$ and we have $(x-1, y), (x-2, y) \in P_{i-1}$. Let (s, y) be the lowermost box of P_i in the y^{th} column and let t be such that $T_{\lambda/\mu}(t, y) = i-1$. We get a new tableau T if we set $T(a, y) = T_{\lambda/\mu}(a+1, y)$ for $t-1 \leq a \leq s-1$, $T(s, y) = (i+1)'$ if $(s+1, y) \in P_{i+1}$ or $T(s, y) = i+1$ if $(s+1, y) \notin P_{i+1}$, and $T(e, f) = T_{\lambda/\mu}(e, f)$ for every other box $(e, f) \in D_{\lambda/\mu}$. If $(x-1, y+1) \in D_{\lambda/\mu}$ then $T_{\lambda/\mu}(x-1, y+1) \neq i'$, otherwise $T_{\lambda/\mu}(x, y+1) = i$ and the boxes of C_k and C_{k+1} are in the same component.

By Corollary 1.44, T is m-amenable for $m \neq i, i+1$. There is possibly some b such that $T(b, y) = i'$ and $T(b-1, y-1) \neq (i-1)'$. However, there is some $c \geq b$ such that $T(c, y-1) = (i-1)'$ and $T(c+1, y) \neq i'$. Thus, by Lemma 1.42, i-amenability follows. We possibly have $T(s, y) = (i+1)'$ and $T(s-1, y-1) \neq i'$. However, we have $T(u, y+1) = i$ and there is no $i+1$ in the $(y+1)^{\text{th}}$ column. Hence, by Lemma 1.42, $(i+1)$-amenability follows. It is clear that $c(T)_{i-1} = \nu_{i-1} - 1$ and $c(T)_{i+1} = \nu_{i+1} + 1$ and $c(T)_j = \nu_j$ for $j \neq i-1, i+1$. □

Example 4.15. *For $\lambda = (11, 10, 9, 5, 4, 3, 2)$ and $\mu = (7, 6, 4, 3)$ the changes are written in boldface:*



Lemma 4.16. *Let $Q_{\lambda/\mu} = k \cdot Q_\nu$ for some k. If $comp(D_{\lambda/\mu}) = 1$ then $k = 1$.*

Proof. Clearly, $\nu = c(T_{\lambda/\mu})$. Assume $Q_{\lambda/\mu}$ is Q-homogeneous and there is tableau T of $D_{\lambda/\mu}$ with content ν different from $T_{\lambda/\mu}$. By Lemma 1.49, $T^{(j)} = P_j$ for every j. Then T differs from $T_{\lambda/\mu}$ by markings, say $T(x, y) = j'$ and $T_{\lambda/\mu}(x, y) = j$. Then $(x+1, y), (x, y-1) \notin P_j$ and (x, y) is not the last box of P_j. Then (x, y) is the last box of one of the components $C_2, \ldots, C_{comp(P_j)}$ of P_j. Since $comp(D_{\lambda/\mu}) = 1$, which means $D_{\lambda/\mu}$ is connected, $comp(P_1) = 1$ and $j > 1$. Then by Lemmas 4.12 and 4.14, there is some tableau T' of shape $D_{\lambda/\mu}$ such that $c(T') \neq \nu$. Thus, $Q_{\lambda/\mu}$ is not Q-homogeneous; a contradiction. □

As we mentioned before, Salmasian classified the skew Schur Q-functions $Q_{\lambda/\mu}$ that satisfy $Q_{\lambda/\mu} = Q_\nu$ in [15, Theorem 3.2] and, thus, we get the following theorem.

Theorem 4.17. *Let $\lambda, \mu \in DP$ such that $D_{\lambda/\mu}$ is basic. We have $Q_{\lambda/\mu} = k \cdot Q_\nu$ if and only if one of the following properties is satisfied:*

(i) λ arbitrary, $\mu = \emptyset$ and $\nu = \lambda$ and $k = 1$,

(ii) $\lambda = (r, r-1, \ldots, 1)$ *and* $0 < \ell(\mu) < r-1$ *for some* r *and* $\nu = \lambda \setminus \mu$ *and* $k = 1$*,*

(iii) $\lambda = (p+q+r, p+q+r-1, p+q+r-2, \ldots, p)$*,* $\mu = (q, q-1, \ldots, 1)$*, where* $p, q, r \geq 1$ *and* $\nu = (p+r+q, p+r+q-1, p+r+q-2, \ldots, p+q+1, p+q, p+q-2, p+q-4, \ldots, \max\{p-q, q+2-p\})$ *and* $k = 1$*,*

(iv) $\lambda = (p+q, p+q-1, p+q-2, \ldots, p+1, p)$*,* $\mu = (q, q-1, \ldots, 1)$*, where* $p, q \geq 1$ *and* $\nu = (p+q, p+q-2, p+q-4, \ldots, \max\{p-q, q-p+2\})$ *and* $k = 1$*,*

(v) $\lambda = (r+2, r, r-1, \ldots, 1)$*,* $\mu = (r+1)$ *and* $\nu = (r+1, r-1, r-2, \ldots, 1)$ *for a* $r \geq 1$ *and* $k = 2$*.*

Proof. 4.17 (i) is the trivial case and 4.17 (v) was shown in Proposition 4.10. For 4.17 (ii), 4.17 (iii) and 4.17 (iv) the proof of homogeneity is the main work of Salmasian's paper [15]. We will give the proof of the corresponding ν.

In 4.17 (ii), by Lemma 3.38, $Q_{\lambda/\mu} = Q_{\lambda/\mu^{ot}} = Q_\alpha$ for $\alpha = \lambda \setminus \mu$.

The diagrams of 4.17 (iv) are rectangles with p columns and $q+1$ rows and, hence, the P_is are hooks. Clearly, $|P_1| = p + (q+1) - 1 = p + q$. For each hook P_i let (a_i, b_i) be the first box and let (c_i, d_i) be the last box. Then we have the property that $(a_i+1, b_i+1), (c_i+1, d_i+1) \notin D_{\lambda/\mu}$ for all i such that $P_i \neq \emptyset$. Hence, if $P_i \neq \emptyset$ and $i > 1$ then $|P_i| = |P_{i-1}| - 2$. It is clear that the number of hooks is given by $\min\{p, q+1\}$.

If $p \leq q+1$ then $|P_p| = |P_1| - 2(p-1) = p + q - 2p + 2 = q - p + 2$. Then $q - p + 2 \geq 1 \geq p - q$ and $\max\{p-q, q-p+2\} = q - p + 2$.

If $p \geq q+1$ then $|P_{q+1}| = |P_1| - 2((q+1) - 1) = p + q - 2q = p - q$. Then $p - q \geq 1 \geq (q+1) - p + 1 = q - p + 2$ and $\max\{p-q, q-p+2\} = p - q$.

The diagrams of 4.17 (iii) are rectangles with $p+r$ columns and $q+r+1$ rows where in the j^{th} column the lowermost $r-j+1$ boxes are removed from the diagram for $1 \leq j \leq r$. By the proof of Lemma 3.40, the P_is are hooks. We get ν by taking the ν obtained in case (iii) for a rectangle with $p+r$ columns and $q+r+1$ rows and then lowering ν_j by

$r - j + 1$ for $1 \leq j \leq r$. Note that after removing $P_1, P_2, \ldots, P_r$ the remaining diagram is a rectangle with p columns and $q + 1$ rows. □

Remark. The corresponding ν for 4.17 (iv) is also stated and proved by DeWitt [6, Theorem IV.3].

5 Non-zero coefficients in the decomposition of non-Q-homogeneous skew Schur Q-functions

An algorithm that always gives a non-zero constituent Q_ν in the decomposition of $Q_{\lambda/\mu}$ into Schur Q-functions is Definition 1.45 that is due to Salmasian [15]. As seen in Lemma 1.49 the obtained ν is the lexicographically largest possible partition indexing a non-zero homogeneous component. It is an open problem to find the lexicographically smallest partition indexing a non-zero homogeneous component.

In Chapter 4 we classified the Q-homogeneous skew Schur Q-functions. This means that we are also able to describe all skew Schur Q-functions whose decomposition into Schur Q-functions has least two homogeneous components. In this chapter we will find a second non-zero homogeneous component for these skew Schur Q-functions. For skew Schur Q-functions that decompose into precisely two homogeneous components in this way we obtain the lexicographically smallest possible partition indexing a homogeneous component. Theorem 5.8 is the main theorem of this chapter that lists the second homogeneous component for each non-Q-homogeneous skew Schur Q-function and also shows that the partition that indexes this second homogeneous component is strongly related to ν.

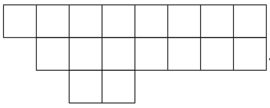
Definition 5.1. The **set of slide down partitions** of λ is defined by

$$SD(\lambda) := \{\mu \in DP \mid |\mu| = |\lambda|, \mu < \lambda \text{ and } |D_\mu \setminus D_\lambda| = 1\}$$

where $<$ means lexicographically lesser than (see Definition 1.48).

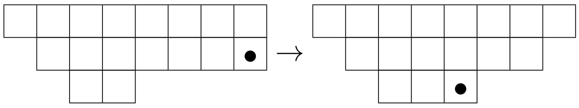
Remark. The set $SD(\lambda)$ is the set of diagrams we obtain by removing a single box and adding a single box in a row below such that the new set of boxes is a valid diagram. If $\mu \in SD(\lambda)$ then the removed box must be a corner of D_λ and the added box must be a corner of D_μ. However, carrying out this procedure at any corner does not necessarily give rise to a valid diagram.

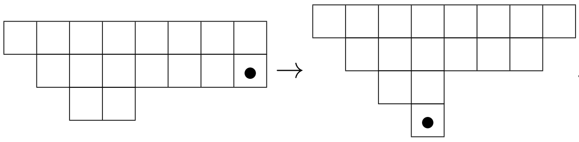
Example 5.2. *Let* $\lambda = (8, 7, 2)$. *Then*

$D_\lambda =$  .

Then $SD(\lambda) = \{(8, 6, 3), (8, 6, 2, 1)\}$.

We get these partitions by sliding down the upper corner:



 .

The lower corner cannot slide down in a valid way.

Lemma 5.3. *Let* $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ *and* $n := \ell(\nu)$. *If* $|P_n| \geq 3$ *and* P_n *has boxes in at least two columns and rows then* $f^{\lambda}_{\mu\bar{\nu}} > 0$ *for* $\bar{\nu} = (\nu_1, \ldots, \nu_{n-1}, \nu_n - 1, 1) \in SD(\nu)$.

Proof. We distinguish the cases whether P_n is connected or not.

Case 1: P_n is connected.

Let (x, y) be the rightmost box of the lowermost row. We get a new tableau T if we set $T(x, y) = n+1$, $T(x-1, y) = n$ and $T(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. By Corollary 1.44, this filling is amenable and has content $\bar{\nu} = (\nu_1, \ldots, \nu_{n-1}, \nu_n - 1, 1)$.

Case 2: P_n is not connected.

By Lemma 2.1, if we find a tableau of P_n with content $(\nu_n - 1, 1)$ then the statement holds. By Lemmas 1.56, 1.64 and 1.68, we can assume that the first component of P_n has at least two boxes or that P_n has at least three components which all consist of one single box. We deal with these subcases in turn. Let P_n have shape $D_{\alpha/\beta}$ for some $\alpha, \beta \in DP$.

Case 2.1: the first component of P_n has at least two boxes.

Let (x, y) be the rightmost box of the lowermost row of the first component. We get a new tableau T if we set $T(x, y) = n + 1$, $T(x - 1, y) = n$ if $(x - 1, y) \in P_n$, and $T(r, s) = T_{\alpha/\beta}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. Since P_n has boxes in at least two rows, there is at least one box in a row above the x^{th} row containing a n and, since P_n has boxes in at least two columns, there is another box containing a n. Hence, by Lemma 1.42, T is amenable and has content $\bar{\nu} = (\nu_1, \ldots, \nu_{n-1}, \nu_n - 1, 1)$.

Case 2.2: P_n has at least three components which all consist of a single box.

Let (x_i, y_i) be the box of the i^{th} component of P_n. We get a new tableau T if we set $T(x_1, y_1) = n + 1$ and $T(r, s) = T_{\alpha/\beta}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. Since $T(x_2, y_2) = T(x_3, y_3) = n$ and there is no $n + 1$ in the ${y_2}^{\text{th}}$ and in the ${y_3}^{\text{th}}$ column, by Lemma 1.42, T is amenable and has content $\bar{\nu} = (\nu_1, \ldots, \nu_{n-1}, \nu_n - 1, 1)$. □

Remark. >From now on we assume that P_n either has boxes only in a single row or a single column or has two components that each consists of a single box. Thus, Q_{P_n} is Q-homogeneous.

Lemma 5.4. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Assume there is some $k > 1$ such that $U_k(\lambda/\mu) = D_{(r+2,r,r-1,\ldots,1)/(r+1)}$ for some r. Then $f^{\lambda}_{\mu\alpha} > 0$ for some $\alpha \in SD(\nu)$.*

Proof. By Lemma 4.12 or Lemma 4.14, we have either $f^{\lambda}_{\mu\bar{\nu}} > 0$ where $\bar{\nu} = (\nu_1, \nu_2, \ldots, \nu_{k-2}, \nu_{k-1} - 1, \nu_k + 1, \nu_{k+1}, \nu_{k+2}, \ldots) \in SD(\nu)$ or $f^{\lambda}_{\mu\hat{\nu}} > 0$ where $\hat{\nu} = (\nu_1, \nu_2, \ldots, \nu_{k-2}, \nu_{k-1} - 1, \nu_k, \nu_{k+1} + 1, \nu_{k+2}, \nu_{k+3}, \ldots) \in SD(\nu)$. □

Lemma 5.5. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Assume there is some $k > 1$ such that the following properties are satisfied:*

- *$U_k(\lambda/\mu)$ has shape $D_{(r+s,r+s-1,\ldots,r)/(t,t-1,\ldots,1)}$ for some $r \geq 2$, $s \geq 0$, $t \geq 1$.*

- *$U_{k-1}(\lambda/\mu)$ has not shape $D_{(r'+s',r'+s'-1,\dots,r')/(t',t'-1,\dots,1)}$ for any $r' \geq 2$, $s' \geq 0$, $t' \geq 1$.*

Then $f^{\lambda}_{\mu\alpha} > 0$ for some $\alpha \in SD(\nu)$.

Proof. Let (x, y) be the first box of P_n. We may assume that P_{k-1} has either at least one box to the right of the y^{th} column or it has at least two boxes in the y^{th} column. Otherwise, the diagram $U_{k-1}(\lambda/\mu)$ is unshifted and $U_{k-1}(\lambda/\mu)^t$ satisfies one of these two properties. We may assume P_{k-1} has at least one box to the right of the y^{th} column. Otherwise, this property is satisfied by $U_{k-1}(\lambda/\mu)^{ot}$.

Let (u, v) be the lowermost box of P_{k-1} in the column to the left of the last box of P_k. Then we get a new tableau T if after the $(k-2)^{\text{th}}$ step of the algorithm of Definition 1.45 we use $P'_{k-1} := P_{k-1} \setminus \{(u, v)\}$ instead of P_{k-1}. Let $P'_i := T^{(i)}$. Since there is a k but no $k+1$ in a column to the right of the y^{th} column, by Lemma 1.42, T is amenable.

If $(u+1, v+1) \in P_k$ then $(u, v) \in P'_k$, if $(u+2, v+2) \in P_{k+1}$ then $(u+1, v+1) \in P'_{k+1}$ and so on. Thus, if $(r, s) \in D_{\lambda/\mu}$ such that $r - s = u - v$ and $r > u$ then if $(r, s) \in P_i$ then $(r-1, s-1) \in P'_i$. Let $j := \max\{i \mid P_i \cap \{(r, s) \mid r - s = u - v\} \neq \emptyset\}$. Then clearly $|P'_{j+1}| = |P_{j+1}| + 1$ and, hence, $c(T) = (\nu_1, \dots, \nu_{k-2}, \nu_{k-1} - 1, \nu_k, \dots, \nu_j, \nu_{j+1} + 1, \nu_{j+2}, \dots, \nu_n) \in SD(\nu)$. □

Lemma 5.6. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Assume there is some $k > 1$ such that the following properties are satisfied:*

- *$U_k(\lambda/\mu)$ has shape D_β for some $\beta \in DP$.*

- *For $D = U_{k-1}(\lambda/\mu)$ the skew Schur Q-function Q_D is not Q-homogeneous.*

Then $f^{\lambda}_{\mu\alpha} > 0$ for some $\alpha \in SD(\nu)$.

Proof. We distinguish the cases whether β is a staircase (which means that we have $\beta = [n-k+1, 1, 0, 0]$ in the shape path notation of Definition 3.17) or not.

Case 1: $\beta = (n-k+1, n-k, \dots, 1)$.

If $U_{k-1}(\lambda/\mu)$ has more than one component then Lemmas 4.2, 4.4, 4.6, and the proof of Lemma 4.8 show that there is some $\alpha \in SD(\nu)$ such that $f^{\lambda}_{\mu\alpha} > 0$. Thus, we only need to consider the case that $U_{k-1}(\lambda/\mu)$ is connected. Let (x, y) be the box of P_n. By Lemma 1.60, we may assume that there are at least two boxes of P_{k-1} in columns to the right of the y^{th} column in at least two rows. Let (u, v) be a box such that $(u+1, v) \notin D_{\lambda/\mu}$, $v > y$ and (u, v) is not the first box of P_{k-1}. Then we get a new tableau T if we set $T(u, v) = k$, $T(u-1, v) = k-1$ if $(u-1, v) \in P_{k-1}$, and $T(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. If $(u-1, v) \notin P_{k-1}$ then there is a k but no $k-1$ in the v^{th} column. However, there is a $k-1$ but no k in the column of the first box of P_{k-1}. Thus, by Lemma 1.42, T is amenable and $c(T) = (\nu_1, \ldots, \nu_{k-2}, \nu_{k-1} - 1, \nu_k + 1, \nu_{k+1}, \ldots, \nu_n) \in SD(\nu)$.

Case 2: $\beta \neq (n-k+1, n-k, \ldots, 1)$.

Let (x, y) be the first box of P_k. If there are at least two boxes of P_{k-1} in columns to the right of the y^{th} column in at least two rows then we can obtain a new tableau the same way as in Case 1. Thus, assume that the rightmost box of P_{k-1} in the $(x-1)^{\text{th}}$ row is $(x-1, y)$ and that $(x-2, y) \in P_{k-1}$. Let (z, y) be the lowermost box of $U_{k-1}(\lambda/\mu)$ in the y^{th} column. We get a new tableau T if we set $T(x-2, y) = k-1$, $T(x-1, y) = k$, $T(x-1+i, y) = k+i$ for all $1 \leq i \leq z-x+1$ and $T(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$.

By Corollary 1.44, T is an amenable tableau since $\beta \neq (n-k+1, n-k \ldots, 1)$ and, hence, $|P_{k+z-x}| \geq |P_{k+z-x+1}| + 2$ where $P_{k+z-x+1} = \emptyset$ if $k+z-x = n$. It is clear that $c(T) = (\nu_1, \ldots, \nu_{k-2}, \nu_{k-1} - 1, \nu_k, \ldots, \nu_{k+z-x}, \nu_{k+z-x+1} + 1, \nu_{k+z-x+2}, \ldots, \nu_n) \in SD(\nu)$. □

Corollary 5.7. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Assume there is some k such that following properties are satisfied:*

- *$U_k(\lambda/\mu)$ has shape $D_{(m,m-1,\ldots,1)/\beta}$ where $0 < \ell(\beta) < m-1$ for some $m > 3$.*
- *For $D = U_{k-1}(\lambda/\mu)$ the skew Schur Q-function Q_D is not Q-homogeneous.*

Then $f^{\lambda}_{\mu\alpha} > 0$ for some $\alpha \in SD(\nu)$.

Proof. Let (x, y) be the lowermost box of $D_{\lambda/\mu}$. Then P_{k-1} has a box to the right of the y^{th} column. Let α, β such that $U_{k-1}(\lambda/\mu)^{ot}$ has shape $D_{\alpha/\beta}$. Then, by Lemma 1.59, $U_2(\alpha/\beta)$ has shape D_γ for some $\gamma \in DP$. Then P_1 of $D_{\alpha/\beta}$ has at least two boxes in the rightmost column with boxes of $D_{\alpha/\beta}$. Thus, this is a diagram satisfying the properties of Lemma 5.6 and the statement follows. □

Theorem 5.8. *Let λ, μ be such that $Q_{\lambda/\mu}$ is not Q-homogeneous, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Then $f^\lambda_{\mu\alpha} > 0$ for some $\alpha \in SD(\nu)$.*

Proof. If $|P_n| \geq 3$ and P_n has boxes in at least two columns and rows then, by Lemma 5.3, $f^\lambda_{\mu\alpha} > 0$ for some $\alpha \in SD(\nu)$. Thus, we may assume that there is some k such that $Q_{U_k(\lambda/\mu)}$ is Q-homogeneous but $Q_{U_{k-1}(\lambda/\mu)}$ is not Q-homogeneous.

If $U_k(\lambda/\mu) = D_{(r+2,r,r-1,\ldots,1)/(r+1)}$ for some r then, by Lemma 5.4, $f^\lambda_{\mu\alpha} > 0$ for some $\alpha \in SD(\nu)$. If $U_k(\lambda/\mu) = D_{(r+s,r+s-1,\ldots,r)/(t,t-1,\ldots,1)}$ for some $r \geq 2$, $s \geq 0$, $t \geq 1$ then, by Lemma 5.5, $f^\lambda_{\mu\alpha} > 0$ for some $\alpha \in SD(\nu)$. If $U_k(\lambda/\mu) = D_\beta$ for a $\beta \in DP$ then, by Lemma 5.6, $f^\lambda_{\mu\alpha} > 0$ for some $\alpha \in SD(\nu)$. And if $U_k(\lambda/\mu) = D_{(m,m-1,\ldots,1)/\beta}$ where $0 < \ell(\beta) < m - 1$ then by Corollary 5.7, $f^\lambda_{\mu\alpha} > 0$ for some $\alpha \in SD(\nu)$. □

Remark. If $Q_{\lambda/\mu}$ is not Q-homogeneous then one partition $\alpha \in SD(c(T_{\lambda/\mu}))$ such that $f^\lambda_{\mu\alpha} > 0$ can explicitly be obtained by the proof of one of the Lemmas 5.3, 5.4, 5.5, 5.6 or Corollary 5.7.

For Schur Q-functions with exactly two homogeneous components Theorem 5.8 restricts the support of partitions of homogeneous components.

6 Classification of skew Schur Q-functions with two homogeneous components

After classifying the Q-homogeneous skew Schur Q-functions in Chapter 4 and finding a second homogeneous component for the non-Q-homogeneous skew Schur Q-functions in Chapter 5, we are interested in the skew Schur Q-functions that have only two homogeneous components. This gives us a bit of insight how the lexicographically smallest homogeneous component of some skew Schur Q-function looks like. Theorem 6.69 is the main theorem of this chapter and classify such skew Schur Q-functions as well as their decomposition.

We will first show that for diagrams that satisfy some given properties we find at least three amenable tableaux with pairwise different content. We will vastly use Lemmas 2.1, 2.5, 2.8 and 2.9 to consider the diagram with the smallest border strip P_1 that satisfies these properties. This is the "worst case" of a diagram satisfying these given properties as more boxes in P_1 can result in more possible fillings (as the aforementioned lemmas state). As it turns out these "worst case" diagrams are actually "best case" diagrams if Proposition 1.27 is used as they or their orthogonally transposition usually have shape $D_{\lambda/\mu}$ where μ has only one or two parts. If μ has one part then these cases can easily be analysed by using Proposition 1.55 and if μ has two parts it is still not too hard to argue why there are three amenable tableaux with pairwise different content. After giving a possible classification in Proposition 6.53, we will show that the list of skew Schur Q-functions in this proposition consists indeed of skew Schur Q-functions with two homogeneous components by often using Proposition 1.55 again.

<u>**Notation**</u>. We will use the same notation as in the previous chapters. To shorten the proofs, we will not mention the use of Lemmas 1.56, 1.60, 1.64 and 1.68. This means whenever the term "without loss of generality" is used without explicitly arguing why, this statement can be obtained by using these lemmas.

The shape path notation from Definition 3.17 will appear again and we will again use the notation that a letter in a shape path is always some positive integer (see the Notation after Lemma 3.19 for this matter).

6.1 Excluding skew Schur Q-functions where P_n is not Q-homogeneous

Similarly to Chapter 3 we will make use of Lemma 2.1 to exclude the skew Schur Q-functions with at least three homogeneous components. First, we consider the case that P_n is not Q-homogeneous.

Lemma 6.1. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let the decomposition of $Q_{\lambda/\mu}$ consists of precisely two homogeneous components. Then P_n satisfy one of the following properties:*

1. *$|P_n| \leq 4$,*
2. *P_n has all boxes in a single row or a single column,*
3. *P_n is a (p,q)-hook or a rotated (p,q)-hook where $p = 2$ or $q = 2$,*
4. *P_n has two components where one consists of one single box and the other one has all boxes in one row or one column.*

Proof. In this proof we will find three tableaux with pairwise different content for any diagram P_n that not included in the list of Lemma 6.1. Then, by Lemma 2.1, $Q_{\lambda/\mu}$ has more than two homogeneous components. Therefore, consider P_n such that $|P_n| \geq 5$ and it is not one of the diagrams of the list of Lemma 6.1. If P_n has shape $D_{\alpha/\beta}$ let $T_{P_n} := T_{\alpha/\beta}$.

Case 1: $comp(P_n) = 1$.

Without loss of generality there is either a column with at least three boxes or there are at least two columns with precisely two boxes.

Case 1.1: P_n has a column with at least three boxes.

Case 1.1.1: P_n has boxes in at least three columns.

Let (x, y) be the lowermost box of a column with at least three boxes. We get a new tableau T_1 if we set $T_1(x, y) = 2$, $T_1(x-1, y) = 1$ and $T_1(r, s) = T_{P_n}(r, s)$ for every other box $(r, s) \in P_n$. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. Since there is a column with 1 and no 2, by Corollary 1.44, T_1 is 2-amenable and, hence, amenable. We get a new tableau T_2 if we set $T_2(x, y) = 2$, $T_2(x - 1, y) = 2'$, $T_2(x - 2, y) = 1$ and $T_2(r, s) = T_{P_n}(r, s)$ for every other box $(r, s) \in P_n$. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. We have $T_2(x-1, y) = 2'$ and $T_2(x-2, y-1) \neq 1'$. However, there are two columns that have an entry 1 and no 2. Thus, by Lemma 1.42, T_2 is 2-amenable and, hence, amenable. Clearly, the contents $c(T_1), c(T_2)$ and $c(T_{P_n})$ are pairwise different.

Case 1.1.2: P_n has boxes only in two columns.

Without loss of generality we may assume that the y^{th} column has at least three boxes and the $(y - 1)^{\text{th}}$ column has at least two boxes. Let (x, y) be the lowermost box of the y^{th} column. We get a new tableau T_3 if we set $T_3(x, y) = 2$, $T_3(x - 1, y) = 1$ and $T_3(r, s) = T_{P_n}(r, s)$ for every other box $(r, s) \in P_n$. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. Since there is a 1 and no 2 in the $(y-1)^{\text{th}}$ column, by Corollary 1.44, T_3 is 2-amenable and, hence, amenable. We get a new tableau T_4 if we set $T_4(x, y) = 2$, $T_4(x-1, y) = 2'$, $T_4(x-2, y) = 1$ and $T_4(r, s) = T_{P_n}(r, s)$ for every other box $(r, s) \in P_n$. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. There is a 1 and no 2 in the $(y - 1)^{\text{th}}$ column. We have $T_4(x - 1, y) = 2'$ and $T_4(x - 2, y - 1) \neq 1'$. However, we have $T_4(x, y-1) = 1'$ and $(x + 1, y) \notin P_n$. Thus, by Lemma 1.42, T_4 is 2-amenable and, hence, amenable. Clearly, the contents $c(T_3), c(T_4)$ and $c(P_n)$ are pairwise different.

Case 1.2: P_n has at least two columns with precisely two boxes.

Let $(x, y), (u, v)$ be the lowermost boxes of two of these columns such that $y < v$. Then there is some $(q, t) \in P_n$ such that $t \neq y, v$. We get a new tableau T_5 if we set $T_5(x, y) = 2$, $T_5(x - 1, y) = 1$ and $T_5(r, s) = T_{P_n}(r, s)$ for every other box $(r, s) \in P_n$. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. Since $T_5(u, v) = 1$, by Corollary 1.44, T_5

is 2-amenable and, hence, amenable. We get a new tableau T_6 if we set $T_6(x,y) = 2$, $T_6(x-1,y) = 1$, $T_6(u,v) = 2'$, $T_6(u-1,v) = 1$ and $T_6(r,s) = T_{P_n}(r,s)$ for every other box $(r,s) \in P_n$. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. We have $T_6(u,v) = 2'$ and $T_6(u-1,v-1) \neq 1'$. However, there is a 1 and no 2 in each the v^{th} column and the t^{th} column. Hence, by Lemma 1.42, T_5 is 2-amenable and, hence, amenable. Clearly, the contents $c(T_5), c(T_6)$ and $c(T_{P_n})$ are pairwise different.

Case 2: $comp(P_n) = 2$.

Case 2.1: One component consists of one single box.

Without loss of generality this box is the first component. Let (x,y) be this box. Without loss of generality the second component has boxes in at least three columns and there is a column with at least two boxes. Let (u,v) be the lowermost box of such a column. We get a new tableau T_7 if we set $T_7(x,y) = 2$ and $T_7(r,s) = T_{P_n}(r,s)$ for every other box $(r,s) \in P_n$. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. Since $T_7(u,v) = 1$, by Corollary 1.44, T_7 is 2-amenable and, hence, amenable. We get a new tableau T_8 if we set $T_8(x,y) = 2$, $T_8(u,v) = 2$, $T_8(u-1,v) = 1$ and $T_8(r,s) = T_{P_n}(r,s)$ for every other box $(r,s) \in P_n$. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. There are 1s and no 2s in two columns with boxes of the second component of P_n that are not the v^{th} column. Thus, by Lemma 1.42, T_8 is 2-amenable and, hence, amenable. Clearly, the contents $c(T_7), c(T_8)$ and $c(T_{P_n})$ are pairwise different.

Case 2.2: Both components consists of at least two boxes.

Without loss of generality we have $|C_1| \leq |C_2|$. Then without loss of generality there is a box $(x,y) \in C_1$ such that $(x+1,y) \notin C_1$, $(x-1,y) \in C_1$ and $(x-1,y+1) \notin C_1$. Also, without loss of generality the second component has either boxes in at least three columns or is equal to $D_{(3,2)/(2)}$.

Case 2.2.1: The second component has boxes in at least three columns.

We get a new tableau T_9 if we set $T_9(x,y) = 2$, $T_9(x-1,y) = 1$ and $T_9(r,s) = T_{P_n}(r,s)$ for every other box $(r,s) \in P_n$. Clearly, by Corollary 1.44, T_9 is amenable. We get a new tableau T_{10} if we set $T_{10}(x,y)=2$, $T_{10}(x-1,y)=2'$, $T_{10}(x-2,y)=1$ if $(x-2,y)\in C_1$, and $T_{10}(r,s)=T_{P_n}(r,s)$ for every other box $(r,s)\in P_n$. By Corollary 1.44, this tableau is m-amenable for $m\neq 2$. We have $T_{10}(x-1,y)=2'$ and $T_{10}(x-2,y-1)\neq 1'$ and possibly a 2 and no 1 in the y^{th} column. However, there are at least three columns with a 1 and no 2 in the second component. Thus, by Lemma 1.42, T_{10} is 2-amenable and, hence, amenable. Clearly, the contents $c(T_9), c(T_{10})$ and $c(T_{P_n})$ are pairwise different.

Case 2.2.2: The second component is equal to $D_{(3,2)/(2)}$.

Let (u,v) be the corner of the second component. We get a new tableau T_{11} if we set $T_{11}(x,y)=2$, $T_{11}(x-1,y)=1$ and $T_{11}(r,s)=T_{P_n}(r,s)$ for every other box $(r,s)\in P_n$. Clearly, by Corollary 1.44, T_{11} is amenable. We get a new tableau T_{12} if we set $T_{12}(x,y)=2$, $T_{12}(x-1,y)=1$, $T_{12}(u,v)=2$, $T_{12}(u-1,v)=1$ and $T_{12}(r,s)=T_{P_n}(r,s)$ for every other box $(r,s)\in P_n$. By Corollary 1.44, this tableau is m-amenable for $m\neq 2$. Since we have $T_{12}(u,v-1)=1$ and $(u+1,v-1)\notin P_n$, by Corollary 1.44, T_{12} is 2-amenable and, hence, amenable. Clearly, the contents $c(T_{11}), c(T_{12})$ and $c(T_{P_n})$ are pairwise different.

Case 3: $comp(P_n)=3$.

Then without loss of generality we may assume $|C_1|\leq|C_2|\leq|C_3|$.

Case 3.1: $|C_2|\geq 2$.

Then without loss of generality C_2 and C_3 each has boxes in at least two columns. Let (x,y) be the rightmost box of the lowermost row of the first component and let (u,v) be the rightmost box of the lowermost row of the second component. We get a new tableau T_{13} if we set $T_{13}(x,y)=2$, $T_{13}(x-1,y)=1$ if $(x-1,y)\in C_1$, and $T_{13}(r,s)=T_{P_n}(r,s)$ for every other box $(r,s)\in P_n$. By Corollary 1.44, this tableau is m-amenable for $m\neq 2$. If $(x-1,y)\notin C_1$ then there is a 2 and no 1 in the y^{th} column. However, there is a 1 and no 2 in each the column of the last box of C_2 and the column of the last box of C_3. Thus, by Lemma 1.42, T_{13} is 2-amenable and, hence, amenable. We get a new tableau T_{14} if we set $T_{14}(x,y)=2$, $T_{14}(x-1,y)=1$ if $(x-1,y)\in C_1$, $T_{14}(u,v)=2$, $T_{14}(u-1,v)=1$ if $(u-1,v)\in C_2$, and $T_{14}(r,s)=T_{P_n}(r,s)$ for every other box $(r,s)\in P_n$. By Corollary

1.44, this tableau is m-amenable for $m \neq 2$. There are at least two columns with entry 1 and no entry 2 in the third component and there is another column with entry 1 and no entry 2 in the second component. Thus, by Lemma 1.42, T_{14} is 2-amenable and, hence, amenable. Clearly, the contents $c(T_{13}), c(T_{14})$ and $c(T_{P_n})$ are pairwise different.

Case 3.2: $|C_2| = 1$.

Then $|C_3| \geq 3$ and without loss of generality the third component has either boxes in at least three columns or is equal to $D_{(3,2)/(2)}$.

Case 3.2.1: the third component has boxes in at least three columns.

Let (x, y) be the box of the first component and let (u, v) be the box of the second component. We get a new tableau T_{15} if we set $T_{15}(x, y) = 2$ and $T_{15}(r, s) = T_{P_n}(r, s)$ for every other box $(r, s) \in P_n$. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. Since there is a 1 and no 2 in each the column of the last box of C_2 and the column of the last box of C_3, by Lemma 1.42, T_{15} is 2-amenable and, hence, amenable. We get a new tableau T_{16} if we set $T_{16}(x, y) = 2$, $T_{16}(u, v) = 2$ and $T_{16}(r, s) = T_{P_n}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. Since there are at least three columns with 1 and no 2 in the third component, by Lemma 1.42, T_{16} is 2-amenable and, hence, amenable. Clearly, the contents $c(T_{15}), c(T_{16})$ and $c(T_{P_n})$ are pairwise different.

Case 3.2.2: The third component is equal to $D_{(3,2)/(2)}$.

Let (x, y) be the box of the first component and let (u, v) be the corner of the third component. We get a new tableau T_{17} if we set $T_{17}(x, y) = 2$ and $T_{17}(r, s) = T_{P_n}(r, s)$ for every other box $(r, s) \in P_n$. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. Since $T_{17}(u, v-1) = T_{17}(u, v) = 1$, by Lemma 1.42, T_{17} is 2-amenable and, hence, amenable. We get a new tableau T_{18} if we set $T_{18}(x, y) = 2$, $T_{18}(u, v) = 2$, $T_{18}(u-1, v) = 1$ and $T_{18}(r, s) = T_{P_n}(r, s)$ for every other box $(r, s) \in P_n$. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. There is a column with a 1 and no 2 in the third component and there is another column with 1 and no 2 in the second component.

Thus, by Lemma 1.42, T_{18} is 2-amenable and, hence, amenable. Clearly, the contents $c(T_{17}), c(T_{18})$ and $c(T_{P_n})$ are pairwise different.

Case 4: $comp(P_n) \geq 4$.

Then without loss of generality we may assume $|C_i| \leq |C_{i+1}|$ for all i. Then without loss of generality the set of boxes $P_n \setminus (C_1 \cup C_2)$ has boxes in at least three columns. Let (x, y) be a corner of the first component and let (u, v) be a corner of the second component. We get a new tableau T_{19} if we set $T_{19}(x, y) = 2$, $T_{19}(x-1, y) = 1$ if $(x-1, y) \in C_1$, and $T_{19}(r, s) = T_{P_n}(r, s)$ for every other box $(r, s) \in P_n$. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. Since there are at least three columns with 1 and no 2 in the remaining components, by Lemma 1.42, T_{19} is 2-amenable and, hence, amenable. We get a new tableau T_{20} if we set $T_{20}(x, y) = 2$, $T_{20}(x-1, y) = 1$ if $(x-1, y) \in C_1$, $T_{20}(u, v) = 2$, $T_{20}(u-1, v) = 1$ if $(u-1, v) \in C_2$, and $T_{20}(r, s) = T_{P_n}(r, s)$ for every other box $(r, s) \in D_{P_n}$. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. Since there are at least three columns with 1 and no 2 in the remaining components, by Lemma 1.42, T_{20} is 2-amenable and, hence, amenable. Clearly, the contents $c(T_{19}), c(T_{20})$ and $c(T_{P_n})$ are pairwise different. □

Remark. The contents of the tableaux in the proof of Lemma 6.1 are as follows: if $|P_n| = k$ then $c(T_{P_n}) = (k)$, the tableaux with an odd index have content $(k-1, 1)$ and the tableaux with an even index have content $(k-2, 2)$. By Lemma 2.1 for diagrams $D_{\lambda/\mu}$ satisfying the conditions of Lemma 6.1 then there are amenable tableaux with content $(\nu_1, \nu_2, \ldots, \nu_n)$, $(\nu_1, \nu_2, \ldots, \nu_{n-1}, \nu_n - 1, 1)$ and $(\nu_1, \nu_2, \ldots, \nu_{n-1}, \nu_n - 2, 2)$.

Lemma 6.2. *Let $\lambda, \mu \in DP$ such that $\ell(\nu) = 1$ where $\nu := c(T_{\lambda/\mu})$. Let $D_{\lambda/\mu}$ satisfy one of the following properties:*

(a) $|D_{\lambda/\mu}| \in \{3, 4\}$ and $comp(D_{\lambda/\mu}) \geq 2$.

(b) $|D_{\lambda/\mu}| \geq 5$ and $D_{\lambda/\mu}$ has two components where one consists of one single box and the other one has all boxes in one row or one column.

Then the decomposition of $Q_{\lambda/\mu}$ consists of precisely two homogeneous components.

Proof. Note that a depiction of the following diagrams is given in Example 6.3.

Case (a):

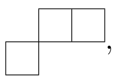
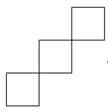
For $|D_{\lambda/\mu}| = 3$ without loss of generality we may consider only $D_{\lambda/\mu} = D_{(4,1)/(2)}$ and $D_{\lambda/\mu} = D_{(5,3,1)/(4,2)}$. Since $Q_{(4,1)/(2)} = 2Q_{(3)} + Q_{(2,1)}$ and $Q_{(5,3,1)/(4,2)} = 4Q_{(3)} + 2Q_{(2,1)}$, the statement holds.

For $|D_{\lambda/\mu}| = 4$ without loss of generality we may consider only $D_{\lambda/\mu} \in \{D_{(5,1)/(2)}, D_{(5,3,1)/(3,2)}, D_{(6,3,1)/(4,2)}, D_{(7,5,3,1)/(6,4,2)}, D_{(5,2)/(3)}\}$. Since $Q_{(5,1)/(2)} = 2Q_{(4)} + Q_{(3,1)}$, $Q_{(5,3,1)/(3,2)} = 2Q_{(4)} + 3Q_{(3,1)}$, $Q_{(6,3,1)/(4,2)} = 4Q_{(4)} + 4Q_{(3,1)}$, $Q_{(7,5,3,1)/(6,4,2)} = 8Q_{(4)} + 8Q_{(3,1)}$ and $Q_{(5,2)/(3)} = 2Q_{(4)} + 2Q_{(3,1)}$, the statement holds.

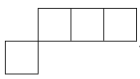
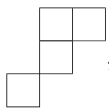
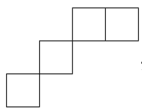
Case (b):

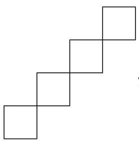
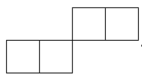
Without loss of generality for $|D_{\lambda/\mu}| = n$ we may consider $D_{\lambda/\mu} = D_{(n+1,n-1)/(n)}$. Using the notation of Proposition 1.55, we have $B_\lambda^\times = \{(1,n),(2,2)\}$. By Proposition 1.55, we obtain $Q_{(n+1,n-1)/(n)} = 2Q_{(n)} + Q_{(n-1,1)}$. □

Example 6.3. *The diagrams for the case $|D_{\lambda/\mu}| = 3$ of Lemma 6.2 are*

$D_{(4,1)/(2)} =$ , $D_{(5,3,1)/(4,2)} =$ .

The diagrams for the case $|D_{\lambda/\mu}| = 4$ of Lemma 6.2 are

$D_{(5,1)/(2)} =$ , $D_{(5,3,1)/(3,2)} =$ , $D_{(6,3,1)/(4,2)} =$ ,

$D_{(7,5,3,1)/(6,4,2)} =$ , $D_{(5,2)/(3)} =$ .

We want to show that for some of the cases of P_n in the list of Lemma 6.1 if $n \geq 2$ then the decomposition of the skew Schur Q-function consists of more than two homogeneous components. Similar to Chapter 3, we can shorten proofs by orthogonally transposing diagrams.

Lemma 6.4. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some k such that $U_k(\lambda/\mu)$ is not connected and $Q_{U_k(\lambda/\mu)}$ is not Q-homogeneous. If $k \geq 2$ then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $k = 2$. Since the skew Schur Q-function $Q_{U_2(\lambda/\mu)}$ is not Q-homogeneous and by Lemma 2.1, there are two tableaux T and T' of $D_{\lambda/\mu}$ such that $c(T) \neq c(T')$ and $c(T)_1 = c(T')_1 = \nu_1$. By Lemma 4.14, there is an amenable tableau $\tilde{T}$ such that $c(\tilde{T})_1 = \nu_1 - 1$. Thus, the decomposition of $Q_{\lambda/\mu}$ has at least three homogeneous components. □

Lemma 6.5. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let (x, y) be the last box of P_n. If there is some $k < n$ such that there are at least two boxes of P_k below the x^{th} row in different columns then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. Let k be maximal with this properties. By Lemma 2.1, it is enough to consider the case $k = 1$. By Lemma 6.4, we may assume that $U_2(\lambda/\mu)$ is connected. Let (e, f) be the last box of P_2. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider that $(e, f-1) \in P_1$ and that $(e, f-2)$ is the last box of P_1 if $e > x$ or else $(e+1, f-1) \in P_1$ and that $(e+1, f-2)$ is the last box of P_1. We denote the last box of P_1 by (u, v) to treat both cases at once. Then $(u, v+1), (u-1, v+1), (u-2, v+1) \in P_1$. We need to find two tableaux different from $T_{\lambda/\mu}$ that have pairwise different content and have content different from ν.

We get a new tableau T_1 if we set $T_1(u, v+1) = 2$, $T_1(u-1, v+1) = 1$ and $T_1(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. Clearly, by Corollary 1.44, this tableau is amenable and we have $c(T_1) = (\nu_1 - 1, \nu_2 + 1, \nu_3, \nu_4 \ldots, \nu_n)$.

Let $(u, v+2) \notin D_{\lambda/\mu}$. We get another amenable tableau T_2 if we set $T_2(u, v+1) = 3$, $T_2(u-1, v+1) = 2$, $T_2(u-2, v+1) = 1$ and $T_2(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. By Corollary 1.44, this tableau is amenable and has content $(\nu_1 - 2, \nu_2 + 1, \nu_3 + 1, \nu_4, \nu_5, \ldots, \nu_n)$.

Let $(u, v+2) \in D_{\lambda/\mu}$. We get another amenable tableau T_3 if we set $T_3(u, v+1) = 2$, $T_3(u-1, v+1) = 1$, $T_3(u, v+2) = 3$, $T_3(u-1, v+2) = 2$ and $T_3(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. By Corollary 1.44, this tableau is amenable and has content $(\nu_1 - 1, \nu_2, \nu_3 + 1, \nu_4, \nu_5, \ldots, \nu_n)$.

□

Example 6.6. *For $\lambda = (5, 4, 2)$ and $\mu = (3, 2)$ we obtain*

$$T_{\lambda/\mu} = \begin{array}{ccc} & 1' & 1 \\ & 1' & 2 \\ 1 & 1 & \end{array}, \quad T_1 = \begin{array}{ccc} & 1' & 1 \\ & 1 & 2 \\ 1 & 2 & \end{array}, \quad T_2 = \begin{array}{ccc} & 1 & 1 \\ & 2 & 2 \\ 1 & 3 & \end{array}.$$

For $\lambda = (7, 6, 5, 3)$ and $\mu = (4, 3, 2)$ we obtain

$$T_{\lambda/\mu} = \begin{array}{cccc} & 1' & 1 & 1 \\ & 1' & 2' & 2 \\ & 1' & 2' & 3 \\ 1 & 1 & 2 & \end{array}, \quad T_1 = \begin{array}{cccc} & 1' & 1 & 1 \\ & 1' & 2' & 2 \\ & 1 & 2' & 3 \\ 1 & 2 & 2 & \end{array}, \quad T_3 = \begin{array}{cccc} & 1' & 1 & 1 \\ & 1' & 2' & 2 \\ & 1 & 2 & 3 \\ 1 & 2 & 3 & \end{array}.$$

Lemma 6.7. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let P_n have shape $D_{(4,2)/(2)}$ or $D_{(4,3,1)/(3,1)}$. If $n \geq 2$ then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $n = 2$. By Lemma 1.59, we may assume that $P_n = D_{(4,2)/(2)}$. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider $D_{\alpha/\beta} = D_{(5,4,2)/(2)}$. Since $Q_{(5,4,2)/(2)} = Q_{(5,4)} + 2Q_{(5,3,1)} + Q_{(4,3,2)}$, the statement holds. □

Lemma 6.8. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let P_n be a $(p, 2)$-hook or an orthogonally transposed $(p, 2)$-hook where $p \geq 3$. If $n \geq 2$ then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $n = 2$. By Lemma 1.59, we may consider that P_2 is an orthogonally transposed $(p, 2)$-hook. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider $D_{\alpha/\beta} = D_{(p+2,p+1,p)/(p)}$. Using the notation of Proposition 1.55, the following diagrams are in $B_\alpha^{(p)}$:

- $B_\alpha \setminus \{(1, p + 2), (2, p + 2)\}$,
- $B_\alpha \setminus \{(1, p + 2), (3, 3)\}$,
- $B_\alpha \setminus \{(3, 3), (3, 4)\}$.

Then, by Proposition 1.55, the decomposition of $Q_{(p+2,p+1,p)/(p)}$ has at least three homogeneous components and, hence, so does the decomposition of $Q_{\lambda/\mu}$. □

Remark. Lemmas 6.1, 6.4, 6.7 and 6.8 show that for a skew Schur Q-function $Q_{\lambda/\mu}$ with precisely two components $n = \ell(c(T_{\lambda/\mu})) > 1$ is only possible if P_n satisfy one of the following properties:

- $|P_n| \leq 2$,
- P_n has all boxes in a single row or a single column,
- P_n is a $(2, q)$-hook or an orthogonally transposed $(2, q)$-hook.

Now we will consider the case that P_n is a $(2, q)$-hook or an orthogonally transposed $(2, q)$-hook and will find further restrictions.

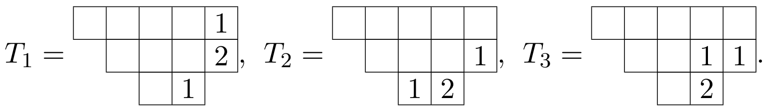
Lemma 6.9. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let $n \geq 2$, $q \geq 2$ and P_n be a $(2, q)$-hook or an orthogonally transposed $(2, q)$-hook. Let (x, y) be the last box of P_n. If $(x, y - 1) \in D_{\lambda/\mu}$ then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $n = 2$. Let $|P_1| = k$. By Lemma 1.59, we may consider that P_2 is a $(2, q)$-hook. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider $D_{\alpha/\beta} = D_{(q+3,q+2,2)/(2,1)}$. By Proposition 1.27, $f^{\alpha}_{\beta\nu} = f^{\alpha}_{\nu\beta}$ and we just need to look at tableaux of shape $D_{\alpha/\nu}$ and content $\beta = (2,1)$. Then we obtain three tableaux as follows:

- $T_1(1, q+3) = 1$, $T_1(2, q+3) = 2$, $T_1(3,4) = 1$;
- $T_2(2, q+3) = 1$, $T_2(3,3) = 1$, $T_2(3,4) = 2$;
- $T_3(2, q+3) = 1$, $T_3(2, q+2) = 1$, $T_3(3,4) = 2$.

Since $w(T_1) = w(T_2) = 121$ and $w(T_3) = 211$, these tableaux are amenable. Then the decomposition of $Q_{(q+3,q+2,2)/(2,1)}$ has at least three homogeneous components and, hence, so does the decomposition of $Q_{\lambda/\mu}$. $\square$

Example 6.10. *For $\lambda = (5,4,2)$ and $\mu = (2,1)$ we obtain*



The following three lemmas are more general statments that also restrict the case that P_n is a $(2,q)$-hook or an orthogonally transposed $(2,q)$-hook.

Lemma 6.11. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k > 1$ such that $U_k(\lambda/\mu)$ has shape $D_{[a,1,c,1]/[w,1,0,0]}$. Let (x,y) be the first box of P_k. If there are boxes of P_{k-1} to the right of the y^{th} column then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $k = 2$. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider that $(x-1, y+1)$ is the first box of P_1.

By Theorem 4.17, $Q_{U_2(\lambda/\mu)}$ is not Q-homogeneous and, thus, there are at least two amenable tableaux of $U_2(\lambda/\mu)$ with different content. By Lemma 2.1, there are at least two amenable tableaux T_1, T_2 such that $c(T_1) \neq c(T_2)$ and $c(T_1)_1 = c(T_2)_1 = \nu_1$.

Let (u,v) be rightmost box in the row of the last box of P_1. We get a new tableau T if we set $P_1' := P_1 \setminus \{(u,v)\}$ and use this instead of P_1 in the algorithm of Definition 1.45. Clearly, by Corollary 1.44, this tableau is m-amenable for $m \neq 2$. Since there is a 1 with no 2 below in the $(y+1)^{\text{th}}$ column, by Corollary 1.44, this tableau is 2-amenable and, hence, amenable. Since $|P_1'| = \nu_1 - 1$, we have $c(T) \notin \{c(T_1), c(T_2)\}$. $\square$

Example 6.12. *For $\lambda = (8,6,5,4,3,1)$ and $\mu = (3,2,1)$ we obtain*

$$T_1 = \begin{array}{|c|c|c|c|c|} \hline 1' & 1 & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & 2 & \\ \hline 1' & 2' & 3' & 3 & \\ \hline 1 & 2' & 3' & 4' & \\ \hline & 2 & 3' & 4 & \\ \hline & & 3 & & \\ \hline \end{array}\ , \quad T_2 = \begin{array}{|c|c|c|c|c|} \hline 1' & 1 & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & 2 & \\ \hline 1' & 2' & 3' & 3 & \\ \hline 1 & 2' & 3' & 4' & \\ \hline & 2 & 3 & 4' & \\ \hline & & 4 & & \\ \hline \end{array}\ , \quad T = \begin{array}{|c|c|c|c|c|} \hline 1' & 1 & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & 2 & \\ \hline 1 & 2' & 3' & 3 & \\ \hline 2 & 2 & 3' & 4' & \\ \hline & 3 & 3 & 4 & \\ \hline & & 4 & & \\ \hline \end{array}\ .$$

Lemma 6.13. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k > 1$ such that $U_k(\lambda/\mu)$ has shape $D_{[a,1,c,1]/[w,1,0,0]}$. Let (x,y) be the first box of P_k. If there are boxes of P_{k-1} above the $(x-1)^{th}$ row then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $k = 2$. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider that $(x-2, y)$ is the first box of P_1. By Theorem 4.17, $Q_{U_2(\lambda/\mu)}$ is not Q-homogeneous and, by Lemma 2.1, there are at least two amenable tableaux T_1, T_2 of $U_2(\lambda/\mu)$ such that $c(T_1) \neq c(T_2)$ and $c(T_1)_1 = c(T_2)_1 = \nu_1$. We get a new tableau T if we set $T(x+i-2, y) = i$ for all $1 \leq i \leq n-1$, $T(x+n-2, y) = n'$ if $(x+n, y) \in D_{\lambda/\mu}$ or else we set $T(x+n-2, y) = n$ as well as $T(x+n-1, y) = n+1$ if $(x+n, y) \notin D_{\lambda/\mu}$, and set $T(r,s) = T_{\lambda/\mu}(r,s)$ for every other box $(r,s) \in D_{\lambda/\mu}$. By Corollary 1.44, this tableau is m-amenable for $m \neq n$. We possibly have $T(x+n-2, y) = n'$ and $T(x+n-3, y-1) \neq (n-1)'$. If $(u, y-1)$ is the second to last box of P_{n-1} then we have $T(u, y-1) = (n-1)'$ and $(u+1, y) \notin D_{\lambda/\mu}$. Thus, by Lemma 1.42, this tableau is 2-amenable and, hence, amenable. Since $|P_1'| = \nu_1 - 1$, we have $c(T) \notin \{c(T_1), c(T_2)\}$. $\square$

Example 6.14. *For $\lambda = (5,4,3,1)$ and $\mu = (4,1)$ we obtain*

$$T_1 = \begin{array}{ccc} & & 1' \\ 1' & 1 & 1 \\ 1 & 2' & 2 \\ & 2 & \end{array}, \quad T_2 = \begin{array}{ccc} & & 1' \\ 1' & 1 & 1 \\ 1 & 2 & 2 \\ & 3 & \end{array}, \quad T = \begin{array}{ccc} & & 1 \\ 1' & 1 & 2 \\ 1 & 2' & 3 \\ & 2 & \end{array}.$$

For $\lambda = (8,7,6,5,4,3,1)$ and $\mu = (7,3,2,1)$ we obtain

$$T_1 = \begin{array}{cccc} & & & 1' \\ 1' & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 \\ 1' & 2' & 3' & 3 \\ 1 & 2' & 3' & 4' \\ & 2 & 3' & 4 \\ & & 3 & \end{array}, \quad T_2 = \begin{array}{cccc} & & & 1' \\ 1' & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 \\ 1' & 2' & 3' & 3 \\ 1 & 2' & 3' & 4' \\ & 2 & 3 & 4' \\ & & 4 & \end{array}, \quad T = \begin{array}{cccc} & & & 1 \\ 1' & 1 & 1 & 2 \\ 1' & 2' & 2 & 3 \\ 1' & 2' & 3' & 4' \\ 1 & 2' & 3' & 4' \\ & 2 & 3' & 4 \\ & & 3 & \end{array}.$$

Lemma 6.15. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k > 1$ such that $U_k(\lambda/\mu)$ has shape $D_{[a,b,c,d]/[1,1,0,0]}$. Let (x, y) be the first box of P_k. If P_{k-1} has boxes above the $(x-1)^{th}$ row then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $k = 2$. By Proposition 1.55, there are two tableaux with different content of the diagram $D_{[a,b,c,d]/[1,1,0,0]}$ and we obtain two tableaux of $U_2(\lambda/\mu)$ with different content. By Lemma 2.1, there are two tableaux T_1, T_2 such that $c(T_1) \neq c(T_2)$ and $c(T_1)_1 = c(T_2)_1 = \nu_1$. Let (u, v) be the lowest box in the column of the first box of P_1. We get a new tableau T if we set $T(u-i, v) = a+1-i$ for $0 \leq i \leq a$ and $T(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. By Corollary 1.44, this tableau is amenable and, since $c(T)_1 = \nu_1 - 1$, we have $c(T) \notin \{c(T_1), c(T_2)\}$. $\square$

Example 6.16. *For $\lambda = (7,6,5,4,2)$ and $\mu = (6,1)$ we obtain*

$$T_1 = \begin{array}{ccccc} & & & & 1' \\ 1' & 1 & 1 & 1 & 1 \\ 1 & 2' & 2 & 2 & 2 \\ & 2 & 3' & 3 & 3 \\ & & 3 & 4 & \end{array}, \quad T_2 = \begin{array}{ccccc} & & & & 1' \\ 1' & 1 & 1 & 1 & 1 \\ 1 & 2' & 2 & 2 & 2 \\ & 2 & 3 & 3 & 3 \\ & & 4 & 4 & \end{array}, \quad T = \begin{array}{ccccc} & & & & 1 \\ 1' & 1 & 1 & 1 & 2 \\ 1 & 2' & 2 & 2 & 3 \\ & 2 & 3' & 3 & 4 \\ & & 3 & 4 & \end{array}.$$

Lemma 6.17. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k > 1$ such that $U_k(\lambda/\mu)$ has shape $D_{[a,b,c,d]/[1,1,0,0]}$. Let (x, y) be the first box of P_k. If P_{k-1} has boxes to the right of the y^{th} column then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $k = 2$. By Lemmas 2.5, 2.8 and 2.9, we may assume that the first box of P_1 is $(x-1, y+1)$. Then, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ consists of three homogeneous components. □

6.2 Excluding skew Schur Q-functions where P_n is Q-homogeneous

We now want to consider the case $|P_n| \leq 2$ and the case that P_n has all boxes in a single row or in a single column. This means that Q_{P_n} is Q-homogeneous. Hence, we will always find some minimal k such that $Q_{U_k(\lambda/\mu)}$ is Q-homogeneous. Since we want to exclude all skew Schur Q-functions with more or less than two homogeneous components in the decomposition into Schur Q-functions and Q-homogeneous skew Schur Q-functions have only one homogeneous component, we may assume that $k > 1$. We will find restrictions for these cases. We start with the case that $U_k(\lambda/\mu)$ is disconnected and that $Q_{U_k(\lambda/\mu)}$ is Q-homogeneous.

Lemma 6.18. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k > 1$ such that $U_k(\lambda/\mu)$ has shape $D_{(m+2,m,m-1,\ldots,1)/(m+1)}$ for some m. If there is an empty column or row between the components of $U_k(\lambda/\mu)$ then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $k = 2$. We may assume that there is an empty column between the components of $U_k(\lambda/\mu)$. Otherwise, by Lemma 1.59, we may consider $D^{ot}_{\lambda/\mu}$.

Let (x, y) be the box of the second component of $U_2(\lambda/\mu)$, $(z, y-1)$ be the lowermost box of P_1 in the $(y-1)^{\text{th}}$ column and let (u, v) be the rightmost box of the uppermost row of the first component of $U_2(\lambda/\mu)$. We get a tableau T if we set $T(z, y-1) = 2$, $T(z-1, y-1) = 1$ and $T(r,s) = T_{\lambda/\mu}(r,s)$ for every other box $(r,s) \in D_{\lambda/\mu}$. Clearly, by Corollary 1.44, this tableau is amenable.

We get another tableau T' if we set $T'(u,v) = 3'$ if $2 < n$ or else $T'(u,v) = 3$, and $T'(r,s) = T(r,s)$ for every other box $(r,s) \in D_{\lambda/\mu}$. By Corollary 1.44, this tableau is m-amenable for $m \neq 3$. There is a 3 but no 2 in the v^{th} column. However, we have $T'(x,y) = 2$ and $(x+1, y) \notin D_{\lambda/\mu}$. If $T'(u,v) = 3'$ then $T'(u-1, v-1) \neq 2'$. However, we have $T'(z, y-1) = 2$ and $(z+1, y-1) \notin D_{\lambda/\mu}$ and if (a,b) is the last box of P_2 then $T'(a,b) = 2$ and $(a+1, b) \notin D_{\lambda/\mu}$. If $T'(u,v) = 3$ then $T'(u-1, v) < 2$ but $T'(z, y-1) = 2$ and $(z+1, y-1) \notin D_{\lambda/\mu}$. Either way, by Lemma 1.42, 3-amenability and, hence, amenability follows. Clearly, $c(T) \neq c(T')$ and $c(T_{\lambda/\mu}) \notin \{c(T), c(T')\}$. □

Example 6.19. *For $\lambda = (5,4,1)$ and $\mu = (3)$ we obtain*

$$T_{\lambda/\mu} = \begin{array}{cccc} & & 1' & 1 \\ 1 & 1 & 1 & 2 \\ & 2 & & \end{array}, \quad T = \begin{array}{cccc} & & 1 & 1 \\ 1 & 1 & 2 & 2 \\ & 2 & & \end{array}, \quad T' = \begin{array}{cccc} & & 1 & 1 \\ 1 & 1 & 2 & 2 \\ & 3 & & \end{array}.$$

For $\lambda = (6,5,2,1)$ and $\mu = (4)$ we obtain

$$T_{\lambda/\mu} = \begin{array}{ccccc} & & & 1' & 1 \\ 1 & 1 & 1 & 1 & 2 \\ & 2 & 2 & & \\ & & 3 & & \end{array}, \quad T = \begin{array}{ccccc} & & & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 \\ & 2 & 2 & & \\ & & 3 & & \end{array}, \quad T' = \begin{array}{ccccc} & & & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 \\ & 2 & 3' & & \\ & & 3 & & \end{array}.$$

Lemma 6.20. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k > 1$ such that $U_k(\lambda/\mu)$ has shape $D_{(m+2,m,m-1,\ldots,1)/(m+1)}$ for some m and let (x,y) be the box of the second component of $U_k(\lambda/\mu)$. If there is some $i < k$ such that $(x-k+i, y)$ is not the first box of P_i then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

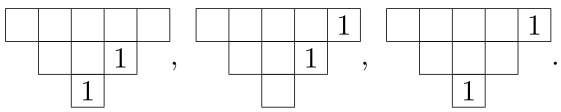
Proof. Let i be maximal with respect to these properties. By Lemma 2.1, we may assume that $i = 1$. By Lemma 6.18, we may assume that there are no empty rows or columns between the components of $U_k(\lambda/\mu)$. Without loss of generality we may assume that $(x-k+1, y+1) \in P_1$. Otherwise, by Lemma 1.59, we may consider $D^{ot}_{\lambda/\mu}$. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider that $(x-k+1, y+1)$ is the first box of P_1.

Since $U_k(\lambda/\mu)$ has shape $D_{(m+2,m,m-1,\ldots,1)/(m+1)}$, we have $\lambda = (m+k+2, m+k, m+k-1, \ldots, m+2, m, m-1, \ldots, 1)$ and $\mu = (m+1)$ where $r \geq 0$. Then $(x+m, y-1)$ is the lowermost box of $D_{\lambda/\mu}$. Using the notation of Proposition 1.55, the following diagrams are in $B^{(m+1)}_{\lambda}$:

- $\{(x,y), (x+1, y-1)\} \cup \{(t, y-1) \mid x+2 \leq t \leq x+m\}$,
- $\{(x-k+1, y+1+r), (x,y)\} \cup \{(t, y-1) \mid x+2 \leq t \leq x+m\}$,
- $\{(x-k+1, y+1+r), (x+1, y-1)\} \cup \{(t, y-1) \mid x+2 \leq t \leq x+m\}$.

Then, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ has at least three homogeneous components. □

Example 6.21. *For $\lambda = (5,3,1)$ and $\mu = (2)$ we need to find tableaux of shape $D_{(5,3,1)/\nu}$ with content (2). The three tableaux in the proof of Lemma 6.20 are*



Lemma 6.22. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k \geq 3$ such that $U_k(\lambda/\mu)$ has shape $D_{(m+2,m,m-1,\ldots,1)/(m+1)}$ for some $m > 1$. Then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

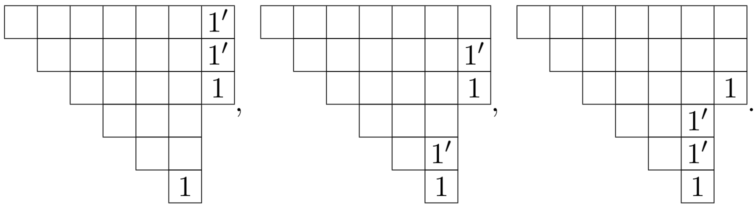
Proof. By Lemma 2.1, we may assume that $k = 3$. By Lemma 6.18, we may assume that there are no empty rows or columns between the components of $U_3(\lambda/\mu)$. Let (x,y) be the box of the second component of $U_3(\lambda/\mu)$. By Lemma 6.20 or an orthogonally transposed version of Lemma 6.20, we just need to consider diagrams such that the box $(x-2, y)$ is the first box of P_1. Since $U_3(\lambda/\mu)$ has shape $D_{(m+2,m,m-1,\ldots,1)/(m+1)}$, we have

$\lambda = (m+4, m+3, m+2, m, m-1, \ldots, 1)$ and $\mu = (m+1)$. Then $(x+m, y-1)$ is the lowermost box of $D_{\lambda/\mu}$. Using the notation of Proposition 1.55, the following diagrams are in $B_\lambda^{(m+1)}$:

- $\{(x-2, y), (x-1, y), (x, y)\} \cup \{(t, y-1) \mid x+3 \leq t \leq x+m\}$,
- $\{(x-1, y), (x, y)\} \cup \{(t, y-1) \mid x+2 \leq t \leq x+m\}$,
- $\{(x, y)\} \cup \{(t, y-1) \mid x+1 \leq t \leq x+m\}$.

Then, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ has at least three homogeneous components. ◻

Example 6.23. *For $\lambda = (7, 6, 5, 3, 2, 1)$ and $\mu = (4)$ we need to find tableaux of shape $D_{(7,6,5,3,2,1)/\nu}$ with content (4). The three tableaux in the proof of Lemma 6.22 are*



Lemma 6.24. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k > 1$ such that $U_k(\lambda/\mu)$ has shape $D_{[a,1,1,1]/[1,2,0,0]}$ for some a. Let $U_{k-1}(\lambda/\mu)$ not have shape $D_{[a+1,1,1,1]/[1,2,0,0]}$. Then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. First consider case $a > 1$:

By Lemma 2.1, we may assume that $k = 2$. Let (x, y) be the first box of P_2. We may assume that $(x-1, y+1) \in P_1$, otherwise, by Lemma 1.59, we may consider $D^{ot}_{\lambda/\mu}$. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider that $(x-1, y+1)$ is the first box of P_1.

Then $D_{\lambda/\mu}$ has three corners, $(x-1, y+1)$, $(x+a-1, y)$ and $(x+a, y-1)$. Using the notation of Proposition 1.55, the following diagrams are in $B_\lambda^{(2)}$:

- $\{(x-1,y+1),(x+a-1,y)\}$,
- $\{(x-1,y+1),(x+a,y-1)\}$,
- $\{(x+a-1,y),(x+a,y-1)\}$.

Then, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ has at least three homogeneous components.

Now consider case $a=1$:

Let (x,y) be the first box of P_2. If $(x-1,y+1)\in P_1$ or $(x-2,y)\in P_1$ then, by Lemmas 2.5, 2.8 and 2.9 and by the same argument as in case $a>1$, the statement holds. Thus, consider $(x-1,y)$ is the first box of P_1. If $(x,y-2)$ is the last box of P_1 the diagram $D_{\lambda/\mu}$ has shape $D_{[2,1,1,1]/[1,2,0,0]}$; a contradiction. Hence, $(x+1,y-2)\in P_1$. By transposition and the argument of case $a>1$, the box $(x+1,y-2)$ is the last box of P_1. Then $D_{\lambda/\mu}$ has shape $D_{(5,4,2)/(3,1)}$ and, since $Q_{(5,4,2)/(3,1)} = 2Q_{(5,2)} + 2Q_{(4,3)} + 2Q_{(4,2,1)}$, the statement holds. $\square$

Now we will tackle the case that $U_k(\lambda/\mu)$ is connected and $Q_{U_k(\lambda/\mu)}$ Q-homogeneous and will find further restrictions. We first start with the case that $U_k(\lambda/\mu)$ or $U_k(\lambda/\mu)^{ot}$ has shape D_α for some $\alpha \in DP$.

Lemma 6.25. *Let $\lambda,\mu\in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k>1$ such that $U_k(\lambda/\mu)$ has shape D_α where $\alpha \neq [a,b,0,0],[a,b,c,1]$. If P_{k-1} has boxes in at least two rows then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $k=2$. Let (x,y) be the first box of P_2. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider that $(x-2,y)$ is the first box of P_1. Then $\mu=(\lambda_1-1)$. By Proposition 1.55 and since $|B_\lambda|=\lambda_1$, we need to remove one box from B_λ such that the remaining set of boxes is still a valid diagram to obtain diagrams of $B_\lambda^\times$. Since the uppermost box of B_λ in a column of a corner of B_λ can be removed, if λ has at least three corners then the statement holds. If $\lambda = [a, b, c, d]$ such that $d \geq 2$ then the uppermost boxes in the columns of the corners can be removed and also the last box of B_λ (which is not a corner) can be removed. Thus, the statement holds. $\square$

The previous lemma states that if there is some $k > 1$ such that $U_k(\lambda/\mu)$ has shape D_α for some $\alpha \neq [a, b, 0, 0], [a, b, c, 1]$ then the boxes of P_{k-1} must be in a row. But then $U_{k-1}(\lambda/\mu)$ has shape D_β for some $\beta \neq [a, b, 0, 0], [a, b, c, 1]$. Hence, if there is some $k > 1$ such that $U_k(\lambda/\mu)$ has shape D_α for some $\alpha \neq [a, b, 0, 0], [a, b, c, 1]$ then either $Q_{\lambda/\mu}$ is Q-homogeneous or the decomposition of $Q_{\lambda/\mu}$ into Schur Q-functions consists of at least three homogeneous components.

Lemma 6.26. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k > 1$ such that $U_k(\lambda/\mu)$ has shape $D_{\alpha/\beta}$ where $\alpha = [m, 1, 0, 0]$ for some $m > 1$ and $\beta = [a, b, 0, 0]$ for some a, b. Let (x, y) be the first box of P_k. If there are boxes of P_{k-1} in rows above the $(x-1)^{th}$ row in at least two columns then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $k = 2$. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider that $(x-2, y+1)$ is the first box of P_1 and that $(x-2, y) \in P_1$. Then we consider $D_{\lambda/\mu}$ where $\lambda = [1, 1, m+1, 1]$ and $\mu = [1, m-a-b+1, a, b]$.

Then $D^{ot}_{\lambda/\mu} = D_{\lambda'/\mu'}$ where $\mu' = (\lambda'_1 - 1)$. By Proposition 1.55, to obtain diagrams of $B^{\times}_{\lambda'}$ we need to remove one box from $B_{\lambda'}$ such that the remaining set of boxes is still a valid diagram. Let (s, t) be the first box of $B_{\lambda'}$ and let (u, v) the uppermost box of $B_{\lambda'}$ in the column of the last box of P_1. Using the notation of Proposition 1.55, we have $(s, t), (s+1, t-1), (u, v) \in B^{\times}_{\lambda'}$. Then, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ has at least three homogeneous components. $\square$

Lemma 6.27. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k > 1$ such that $U_k(\lambda/\mu)$ has shape $D_{\alpha/\beta}$ where $\alpha = [m, 1, 0, 0]$ for some $m > 1$ and $\beta = [a, b, 0, 0]$*

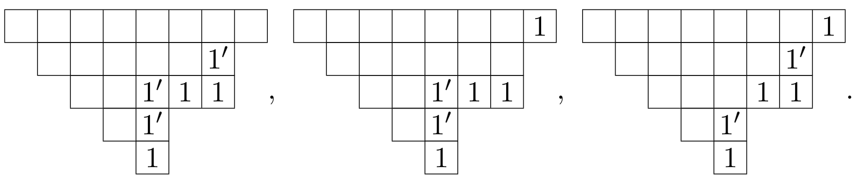
for some a, b. Let (x, y) be the first box of P_k. If there are boxes of P_{k-1} in columns to the right of the y^{th} column in at least two rows then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.

Proof. By Lemma 2.1, it is enough to consider the case $k = 2$. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider $(x-2, y+1)$ is the first box of P_1 and that $(x-1, y+1) \in P_1$. Then $D^{ot}_{\lambda/\mu}$ has shape $D_{\lambda'/\mu'}$ where $\mu' = (\lambda' - 2)$. Using the notation of Proposition 1.55, let (u, v) be the first box of $B_{\lambda'}$ and let (s, t) be the uppermost box in the column of the last box of $B_{\lambda'}$. Note that $(u+2, v-1) \in B_{\lambda'}$, for otherwise, $(x-1, y+1) \notin D_{\lambda/\mu}$. Then the following diagrams are in $B^{(\lambda'-2)}_{\lambda'}$:

- $B_{\lambda'} \setminus \{(u, v-1), (u, v)\}$,
- $B_{\lambda'} \setminus \{(u, v-1), (u+1, v-1)\}$,
- $B_{\lambda'} \setminus \{(u, v-1), (s, t)\}$.

Thus, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ has at least three homogeneous components. □

Example 6.28. *For $\lambda = (8, 7, 5, 4, 3, 2, 1)$ and $\mu = (7, 4, 3)$ the diagram $D^{ot}_{\lambda/\mu}$ has shape $D_{(8,6,5,2,1)/(6)}$ and we need to find tableaux of $D_{(8,6,5,2,1)/\nu}$ with content (6). The three tableaux in the proof of Lemma 6.27 are*



Lemma 6.29. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k > 1$ such that $U_k(\lambda/\mu)$ has shape $D_{\alpha/\beta}$ where $\alpha = [m, 1, 0, 0]$ for some $m > 2$ and $\beta = [a, b, 0, 0]$ for some a, b such that $(a, b) \neq (1, 1)$. Let (x, y) be the first box of P_k. If there are boxes of P_{k-1} to the right of the y^{th} column in at least two columns then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $k = 2$. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider that $(x-1, y+2)$ is the first box of P_1. Then $\lambda = [1, 2, m, 1]$ and $\mu = [a, b, 0, 0]$.

We need to find two amenable tableaux of $D_{\lambda/\mu}$ with pairwise different content and content different from ν.

Let (s, t) be the lowermost corner of P_1. We get a new tableau T_1 if we set $P_1' := P_1 \setminus \{(s, t)\}$ and use this instead of P_1 in the algorithm of Definition 1.45. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. Possibly we have $T_1(s, t) = 2'$ and $T_1(s-1, t-1) \neq 1'$. However, we have $T_1(x-1, y+2) = 1$ and $(x, y+2) \notin D_{\lambda/\mu}$. Thus, by Lemma 1.42, this filling is 2-amenable and, hence, amenable. Since $c(T_1)_1 = \nu_1 - 1$, we have $c(T_1) \neq \nu$.

Now we have to distinguish the cases $b > 1$ and $b = 1$.

Case 1: $b > 1$.

We get another tableau T_2 if we set $P_1' := P_1 \setminus \{(s, t-1), (s, t)\}$ and use this instead of P_1 in the algorithm of Definition 1.45. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. If $T_2(s, t-1) = 2'$ then $T_2(s-1, t-2) \neq 1'$. If $T_2(s, t-1) = 2$ then $(s, t-1)$ is the last box of P_1. Either way, we have $T_2(x-1, y+2) = T_2(x-1, y+1) = 1$ and $(x, y+2), (x, y+1) \notin D_{\lambda/\mu}$. Thus, by Lemma 1.42, this filling is 2-amenable and, hence, amenable. Since $c(T_1)_1 = \nu_1 - 2$, we have $c(T_2) \notin \{c(T_1), \nu\}$.

Case 2: $b = 1$.

We get another tableau T_3 if we set $P_1' := P_1 \setminus \{(s-1, t), (s, t)\}$ and use this instead of P_1 in the algorithm of Definition 1.45. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. We have $T_3(s-1, t) = 2'$ and $T_3(s-2, t-1) \neq 1'$. However, we have $T_3(x-1, y+2) = T_3(x-1, y+1) = 1$ and $(x, y+2), (x, y+1) \notin D_{\lambda/\mu}$. Thus, by Lemma 1.42, this filling is 2-amenable and, hence, amenable. Since $c(T_1)_1 = \nu_1 - 2$, we have $c(T_3) \notin \{c(T_1), \nu\}$. □

Example 6.30. *For $\lambda = (7,4,3,2,1)$ and $\mu = (3)$ we have $U_2(\lambda/\mu) = D_{(4,3,2,1)/(3)}$ and*

$$T_{\lambda/\mu} = \begin{array}{|c|c|c|c|c|c|} & & 1' & 1 & 1 & 1 \\ 1 & 1 & 1 & 2' & & \\ & 2 & 2 & 2 & & \\ & & 3 & 3 & & \\ & & & 4 & & \end{array}, \quad T_1 = \begin{array}{|c|c|c|c|c|c|} & & 1 & 1 & 1 & 1 \\ 1 & 1 & 2' & 2 & & \\ & 2 & 2 & 3' & & \\ & & 3 & 3 & & \\ & & & 4 & & \end{array}, \quad T_2 = \begin{array}{|c|c|c|c|c|c|} & & 1 & 1 & 1 & 1 \\ 1 & 2' & 2 & 2 & & \\ & 2 & 3' & 3 & & \\ & & 3 & 4' & & \\ & & & 4 & & \end{array}.$$

For $\lambda = (7,4,3,2,1)$ and $\mu = (2)$ we have $U_2(\lambda/\mu) = D_{(4,3,2,1)/(2)}$ and

$$T_{\lambda/\mu} = \begin{array}{|c|c|c|c|c|c|} & 1' & 1 & 1 & 1 & 1 \\ 1 & 1 & 2' & 2 & & \\ & 2 & 2 & 3' & & \\ & & 3 & 3 & & \\ & & & 4 & & \end{array}, \quad T_1 = \begin{array}{|c|c|c|c|c|c|} & 1 & 1 & 1 & 1 & 1 \\ 1 & 2' & 2 & 2 & & \\ & 2 & 3' & 3 & & \\ & & 3 & 4' & & \\ & & & 4 & & \end{array}, \quad T_2 = \begin{array}{|c|c|c|c|c|c|} & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & & \\ & 3 & 3 & 3 & & \\ & & 4 & 4 & & \\ & & & 5 & & \end{array}.$$

For $\lambda = (6,3,2,1)$ and $\mu = (2,1)$ we have $U_2(\lambda/\mu) = D_{(3,2,1)/(2,1)}$ and

$$T_{\lambda/\mu} = \begin{array}{|c|c|c|c|} 1' & 1 & 1 & 1 \\ 1' & 2' & & \\ 1 & 2' & & \\ & 2 & & \end{array}, \quad T_1 = \begin{array}{|c|c|c|c|} 1' & 1 & 1 & 1 \\ 1 & 2' & & \\ 2 & 2 & & \\ & 3 & & \end{array}, \quad T_3 = \begin{array}{|c|c|c|c|} 1 & 1 & 1 & 1 \\ 2' & 2 & & \\ 2 & 3' & & \\ & 3 & & \end{array}.$$

Lemma 6.31. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k > 1$ such that $U_k(\lambda/\mu)$ has shape $D_{\alpha/\beta}$ where $\alpha = [m,1,0,0]$ for some $m > 1$ and $\beta = [a,b,0,0]$ for some a, b such that $(a,b) \neq (1,1)$. Let (x,y) be the first box of P_k and let $(x-1,y+1) \in P_{k-1}$. If $k \geq 3$ then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $k = 3$. By Lemmas 6.26 and 6.29, we may assume that $(x-1,y+1)$ is the first box of P_2. We need to find two amenable tableaux of $D_{\lambda/\mu}$ with pairwise different content and content different from ν.

Let (s,t) be the lowermost corner of P_2. We get a new tableau T_1 if after the first step of the algorithm of Definition 1.45 we use $P_2' := P_2 \setminus \{(s,t)\}$ instead of P_2. By Corollary 1.44, this tableau is m-amenable for $m \neq 3$. If $T_1(s,t) = 3'$ then $T_1(s-1,t-1) \neq 2'$. If $T_1(s,t) = 3$ then (s,t) is the last box of P_2. Either way, we have $T_1(x-1,y+1) = 2$

and $(x, y+1) \notin D_{\lambda/\mu}$. Hence, by Lemma 1.42, this tableau is 3-amenable and, hence, amenable. Since $c(T_1)_2 = \nu_2 - 1$, we have $c(T_1) \neq \nu$.

We get another tableau T_2 if we set $P_1' := P_1 \setminus \{(s-1, t-1)\}$ and use this instead of P_1 in the algorithm of Definition 1.45. Stop after the second step of the algorithm and let P_2' be the set of boxes filled with entries from $\{2', 2\}$. Let (u, v) be the lowermost corner of P_2'. Remove the entry of (u, v) and if this box is the last box of P_2' then fill $(u-1, v)$ with 2. Then add entries to the remaining empty boxes as the algorithm of Definition 1.45 does for entries greater than 2. By Corollary 1.44, this tableau is m-amenable for $m \neq 3$. If $T_2(u, v) = 3'$ then $T_2(u-1, v-1) \neq 2'$. In this case we have $T_2(u-1, v) = 2'$ since (u, v) cannot be the last box of P_2'. If $T_2(u, v) = 3$ then $T_2(x-1, y+1) = 2$ and $(x, y+1) \notin D_{\lambda/\mu}$. Hence, by Lemma 1.42, this tableau is 3-amenable and, hence, amenable. Since $c(T_2)_1 = \nu_1 - 1$, we have $c(T_2) \notin \{c(T_1), \nu\}$. $\square$

Example 6.32. *For $\lambda = (7, 6, 4, 3, 2, 1)$ and $\mu = (3)$ we have $U_3(\lambda/\mu) = D_{(4,3,2,1)/(3)}$ and*

$$T_{\lambda/\mu} = \begin{array}{cccccc} & & 1' & 1 & 1 & 1 \\ 1 & 1 & 1 & 2' & 2 & 2 \\ & 2 & 2 & 2 & 3' & \\ & & 3 & 3 & 3 & \\ & & & 4 & 4 & \\ & & & & 5 & \end{array}, \quad T_1 = \begin{array}{cccccc} & & 1' & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ & 2 & 2 & 3' & 3 & \\ & & 3 & 3 & 4' & \\ & & & 4 & 4 & \\ & & & & 5 & \end{array}, \quad T_2 = \begin{array}{cccccc} & & 1 & 1 & 1 & 1 \\ 1 & 1 & 2' & 2 & 2 & 2 \\ & 2 & 3' & 3 & 3 & \\ & & 3 & 4' & 4 & \\ & & & 4 & 5' & \\ & & & & 5 & \end{array}.$$

For $\lambda = (6, 5, 3, 2, 1)$ and $\mu = (2)$ we have $U_3(\lambda/\mu) = D_{(3,2,1)/(2)}$ and

$$T_{\lambda/\mu} = \begin{array}{ccccc} & 1' & 1 & 1 & 1 \\ 1 & 1 & 2' & 2 & 2 \\ & 2 & 2 & 3' & \\ & & 3 & 3 & \\ & & & 4 & \end{array}, \quad T_1 = \begin{array}{ccccc} & 1' & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 2 \\ & 2 & 3' & 3 & \\ & & 3 & 4' & \\ & & & 4 & \end{array}, \quad T_2 = \begin{array}{ccccc} & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ & 3 & 3 & 3 & \\ & & 4 & 4 & \\ & & & 5 & \end{array}.$$

Lemma 6.33. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k > 1$ such that $U_k(\lambda/\mu)$ has two components where the first component is $D_{\alpha/\beta}$ where $\alpha = [m, 1, 0, 0]$ for some $m > 1$ and $\beta = [a, b, 0, 0]$ for some a, b and the second component consists of* *a single box. Then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $k = 2$. Let $D = U_2(\lambda/\mu)$. By Lemma 2.1 and Theorem 4.17, the skew Schur Q-function Q_D is not Q-homogeneous and there are two amenable tableaux T and T' such that $c(T) \neq c(T')$ and $c(T)_1 = c(T')_2 = \nu_1$. Either by Lemma 4.12 or by Lemma 4.14, there is an amenable tableau T'' such that $c(T'')_1 = \nu_1 - 1$. Thus, the decomposition of $Q_{\lambda/\mu}$ has at least three homogeneous components. □

Lemma 6.34. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k > 1$ such that $U_k(\lambda/\mu)$ has shape D_α where $\alpha = [m, 1, 0, 0]$ for some m. Let (x, y) be the first box of P_k. If there are at least three boxes of P_{k-1} to the right of the y^{th} column in at least two rows and at least two columns and at least two boxes are in a row above the x^{th} row then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $k = 2$. By orthogonal transposition of $D_{\lambda/\mu}$ as well as Lemmas 2.5, 2.8 and 2.9, it is enough to consider that $(x-2, y+2), (x-2, y+1), (x-1, y+1) \in P_1$ and $(x-2, y+2)$ is the first box of P_1.

Using the notation of Definition 1.51, the following diagrams are in $B_\lambda^{(\lambda-2)}$:

- $B_\lambda \setminus \{(x-2, y+1), (x-1, y)\}$,
- $B_\lambda \setminus \{(x-2, y+1), (x-2, y+2)\}$,
- $B_\lambda \setminus \{(x-1, y), (x, y)\}$.

Then, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ has at least three non-zero homogeneous components. □

Lemma 6.35. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k > 1$ such that $U_k(\lambda/\mu)$ has shape D_α where $\alpha = [m, 1, 0, 0]$ for some m. Let (x, y) be the first box* *of P_k. If there are boxes of P_{k-1} to the right of the y^{th} column in at least three rows then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $k = 2$. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider that $(x-3, y+1), (x-2, y+1), (x-1, y+1) \in P_1$ and $(x-3, y+1)$ is the first box of P_1.

Then $D^{ot}_{\lambda/\mu}$ has shape $D_{\gamma/\delta}$ where $\gamma = (n+3, n, n-1, n-2, \ldots, 1)$ and $\delta = (n)$. Using the notation of Definition 1.51, let (u, v) be the first box of B_γ. Using the notation of Proposition 1.55, the following diagrams are in $B_\gamma^{(\gamma-3)}$:

- $B_\gamma \setminus \{(u, v), (u, v-1), (u, v-2)\}$,
- $B_\gamma \setminus \{(u, v-1), (u, v-2), (u+1, v-2)\}$,
- $B_\gamma \setminus \{(u, v-2), (u+1, v-2), (u+2, v-2)\}$.

Then, by Proposition 1.55, the decomposition of $Q_{\gamma/\delta}$ has at least three homogeneous components and, hence, so does the decomposition of $Q_{\lambda/\mu}$. □

Lemma 6.36. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k > 1$ such that $U_k(\lambda/\mu)$ has shape D_α where $\alpha = [m, 1, 0, 0]$ for some m. Let (x, y) be the first box of P_k. Let there be at least two boxes of P_{k-1} to the right of the y^{th} column and at least one box above the $(x-1)^{th}$ row. If $k \geq 3$ then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $k = 3$. By orthogonal transposition of $D_{\lambda/\mu}$ as well as Lemmas 1.59, 6.34 and 6.35, we may assume that there are precisely two boxes, (r_1, s_1) and (r_2, s_2) say, to the right of the y^{th} column, such that $r_1 < r_2$. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider that $(x-1, y+1) \in P_2$, that $(x-2, y+1)$ is the first box of P_2 and that $(x-3, y+1)$ is the first box of P_1. Then $\lambda = [3, 1, n-2, 1]$ and $\mu = (n) = (\lambda_1 - 2)$. Using the notation of Definition 1.51, let (u, v) be the first box of B_λ. Then the following diagrams are in $B_\lambda^{(\lambda-2)}$:

- $B_\lambda \setminus \{(u,v),(u+1,v)\}$,
- $B_\lambda \setminus \{(u,v),(u+2,v-1)\}$,
- $B_\lambda \setminus \{(u+2,v-1),(u+3,v-1)\}$.

Then, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ has at least three homogeneous components. □

Next, we consider the case that $U_k(\lambda/\mu)$ has shape $D_{[a,b,0,0]/[w,1,0,0]}$ for some $a,b,w \in \mathbb{N}$.

Lemma 6.37. *Let $\lambda,\mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k > 1$ such that $U_k(\lambda/\mu)$ has shape $D_{\alpha/\beta}$ where $\alpha = [a,b,0,0]$ such that $a \geq 3$, $b \geq 2$ and $\beta = [w,1,0,0]$ such that $a-1 \geq w \geq 2$. Let (x,y) be the first box of P_k. If P_{k-1} has a box to the right of the y^{th} column then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $k=2$. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider that $(x-1,y+1)$ is the first box of P_1. Let (u,v) be the last box of P_2 and let $(e,v-1)$ be the lowermost box of P_1 in the $(v-1)^{\text{th}}$ column. We need to find two tableaux with content different from ν.

We get a new tableau T_1 if we set $P_1' := P_1 \setminus \{(e,v-1)\}$ and use this instead of P_1 in the algorithm of Definition 1.45. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. Let $P_i' := T_1^{(i)}$ for all i. We have $T_1(x-1,y+1) = 1$ and $(x,y+1) \notin D_{\lambda/\mu}$. Thus, by Corollary 1.44, this tableau is 2-amenable and, hence, amenable. Since $c(T_1)_1 = \nu_1 - 1$, we have $c(T_1) \neq \nu$. Now we have to distinguish two cases for the third tableau.

Case 1: $e \geq u$.

Then we get another tableau T_2 if we set $T_2(u,v) = 3$, $T_2(u-1,v) = 2$ and $T_2(r,s) = T_1(r,s)$ for every other box $(r,s) \in D_{\lambda/\mu}$. By Corollary 1.44, this tableau is amenable. Since $c(T_2)_1 = \nu_1 - 1$ and $c(T_2)_2 = c(T_1)_2 - 1$, we have $c(T_2) \notin \{c(T_1),\nu\}$.

Case 2: $e = u-1$.

The last box of P'_i is the last box of P_{i-1} for $2 \leq i \leq a-w+1$ and the last box of P_{a-w} is the leftmost box in the lowermost row with boxes. Let (f,t) be the last box of P'_{a-w+1}. Then $(f-1,t+1) \in P'_{a-w+1}$. Otherwise, $P_{a-w+1} = P_n$ has boxes only in one row and the last box of P_{n-2} is in the row above the last box of P_n. Then $U_k(\lambda/\mu)$ has shape $D_{\alpha/\beta}$ where $\beta = [1,1,0,0]$; a contradiction. We get another tableau T_3 if we set $T_3(f,t+1) = a-w+2$, $T_3(f-1,t+1) = a-w+1$ and $T_3(r,s) = T_1(r,s)$ for every other box $(r,s) \in D_{\lambda/\mu}$. By Corollary 1.44, this tableau is amenable. Since $c(T_3)_1 = \nu_1 - 1$ and $c(T_3)_{a-w+1} = c(T_1)_{a-w+1} - 1$, we have $c(T_3) \notin \{c(T_1), \nu\}$. ☐

Example 6.38. *For $\lambda = (9,7,6,5,4)$ and $\mu = (4,3,2,1)$ we obtain*

$$T_{\lambda/\mu} = \begin{array}{ccccc} 1' & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & \\ 1' & 2' & 3' & 3 & \\ 1' & 2' & 3' & 4' & \\ 1 & 2 & 3 & 4 & \end{array}, \quad T_1 = \begin{array}{ccccc} 1' & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & \\ 1' & 2' & 3' & 3 & \\ 1 & 2' & 3' & 4' & \\ 2 & 2 & 3 & 4 & \end{array}, \quad T_2 = \begin{array}{ccccc} 1' & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & \\ 1' & 2' & 3' & 3 & \\ 1 & 2 & 3' & 4' & \\ 2 & 3 & 3 & 4 & \end{array}.$$

For $\lambda = (7,5,4,3)$ and $\mu = (2,1)$ we obtain

$$T_{\lambda/\mu} = \begin{array}{ccccc} 1' & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & \\ 1 & 2' & 3' & 3 & \\ & 2 & 3 & 4 & \end{array}, \quad T_1 = \begin{array}{ccccc} 1' & 1 & 1 & 1 & 1 \\ 1 & 2' & 2 & 2 & \\ 2 & 2 & 3' & 3 & \\ & 3 & 3 & 4 & \end{array}, \quad T_3 = \begin{array}{ccccc} 1' & 1 & 1 & 1 & 1 \\ 1 & 2' & 2 & 2 & \\ 2 & 2 & 3 & 3 & \\ & 3 & 4 & 4 & \end{array}.$$

Now we consider the case that $U_k(\lambda/\mu)$ has two components where the first component has shape $D_{[a,b,0,0]/[1,1,0,0]}$ and the second component consists of a single box.

Lemma 6.39. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some k such that $U_k(\lambda/\mu)$ consists of two components where the first component is $D_{[a,b,0,0]/[1,1,0,0]}$ where $a \geq 2$, $b \geq 3$ and the second component consists of a single box. Then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $k = 1$. By Lemma 1.71, we have $Q_{\lambda/\mu} = Q_{([1,1,a-1,b]/[1,1,0,0])} + Q_D$ for some diagram D and, by Proposition 1.55, there are two tableaux T and T' of $D_{\lambda/\mu}$ such that $c(T) \neq c(T')$ and $\ell(c(T)) = \ell(c(T')) = n$.

Let (x, y) be the corner of the first component. We get a new tableau $\tilde{T}$ if we set $\tilde{T}(x-i, y) = n - i + 1$ for $0 \leq i \leq n-1$ and $T(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. There is a 2 but no 1 in the y^{th} column. However, there is a 1 and no 2 in the box of the second component. Hence, by Lemma 1.42, this filling is 2-amenable and, hence, amenable. Since $\ell(c(\tilde{T})) = n + 1$, we have $c(\tilde{T}) \notin \{c(T), c(T')\}$. $\square$

Example 6.40. *For $\lambda = (7, 5, 4, 3)$ and $\mu = (6, 1)$ we obtain*

$$T = \begin{array}{ccccc} & & & & 1 \\ 1' & 1 & 1 & 1 & \\ 1 & 2' & 2 & 2 & \\ & 2 & 3 & 3 & \end{array}, \quad T' = \begin{array}{ccccc} & & & & 1 \\ 1 & 1 & 1 & 1 & \\ 2 & 2 & 2 & 2 & \\ & 3 & 3 & 3 & \end{array}, \quad \tilde{T} = \begin{array}{ccccc} & & & & 1 \\ 1' & 1 & 1 & 2 & \\ 1 & 2' & 2 & 3 & \\ & 2 & 3 & 4 & \end{array}.$$

Lemma 6.41. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k > 1$ such that $U_k(\lambda/\mu)$ has shape $D_{[a,b,0,0]/[1,1,0,0]}$ where $a, b \geq 2$ and let (x, y) be the first box of P_k. If there are boxes of P_{k-1} above the $(x-1)^{th}$ row in at least two columns then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $k = 2$. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider that $(x-2, y+1), (x-2, y) \in P_1$ and that $(x-2, y+1)$ is the first box of P_1. By the same lemmas, it is enough to consider that if (e, f) is the last box of P_2 then $(e-1, f-1)$ is the last box of P_1. We need to find two tableaux with content different from ν.

We get a new tableau T_1 if we set $T_1(x-1, y) = 2'$, $T_1(x-2, y) = 1$ and $T_1(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. We have $T_1(x-1, y) = 2'$ and $T_1(x-2, y-1) \neq 1'$. However, we have $T_1(x-2, y+1) = 1$ and $(x-1, y+1) \notin D_{\lambda/\mu}$. Thus, by Lemma 1.42, this tableau is 2-amenable and, hence, amenable. Its content is given by $(\nu_1 - 1, \nu_2 + 1, \nu_3, \nu_4, \ldots, \nu_n)$.

We get a new tableau T_2 if for all $1 \leq i \leq n$ we set $T_2(x+i-1, u) = i$ for all u such that $(x+i-1, u) \in D_{\lambda/\mu}$ and $T_2(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. By

Corollary 1.44, this tableau is m-amenable for $m \neq 2$. We have $T_2(x-2, y+1) = 1$ and $(x-1, y+1) \notin D_{\lambda/\mu}$. Thus, by Corollary 1.44, this tableau is 2-amenable and, hence, amenable. It has content $(\nu_1 - 1, \nu_2, \nu_3, \nu_4, \ldots, \nu_n + 1)$. $\square$

Example 6.42. *For $\lambda = (6, 4, 3, 2)$ and $\mu = (4, 1)$ we get*

$$T_{\lambda/\mu} = \begin{array}{cccc} & & 1' & 1 \\ 1' & 1 & 1 & \\ 1 & 2' & 2 & \\ & 2 & 3 & \end{array} \quad T_1 = \begin{array}{cccc} & & 1 & 1 \\ 1' & 1 & 2' & \\ 1 & 2' & 2 & \\ & 2 & 3 & \end{array} \quad T_2 = \begin{array}{cccc} & & 1' & 1 \\ 1 & 1 & 1 & \\ 2 & 2 & 2 & \\ & 3 & 3 & \end{array}.$$

The following lemmas will be needed for the case that P_n has all boxes in a row or column and $Q_{U_{n-1}(\lambda/\mu)}$ is not Q-homogeneous. After that, we are able to prove Proposition 6.53 that gives a list of all skew Schur Q-functions that possibly decompose into precisely two homogeneous components.

Lemma 6.43. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu) > 1$. Let P_n have shape $D_{(c)}$ for some $c > 1$. Let (x, y) be the last box of P_n. If the last box of P_{n-1} is below the x^{th} row then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $n = 2$. By Lemma 1.64, we may consider $D^t_{\lambda/\mu}$. Let (s, t) be the last box of P_2 of $D^t_{\lambda/\mu}$. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider that $(s, t-1)$ is the last box of P_1 of $D^t_{\lambda/\mu}$. We need to find two amenable tableaux different from $T_{\lambda/\mu}$ that have pairwise different content and have content different from ν.

We get a new tableau T_1 if we set $T_1(s, t-1) = 2$, $T_1(s-1, t-1) = 1$ and $T_1(r, v) = T(r, v)$ for every other box $(r, v) \in D^t_{\lambda/\mu}$. By Corollary 1.44, this tableau is m-amenable for $m \neq 2$. There is a 1 and no 2 in the column of the first box of P_1 (which is to the right of the t^{th} column). Thus, by Corollary 1.44, this tableau is 2-amenable and, hence, amenable. It has content $c(T_1) = (\nu_1 - 1, \nu_2 + 1)$.

We get another tableau T_2 if we set $T_2(s, t) = 3$, $T_2(s-1, t) = 2$ and $T_2(r, v) = T_1(r, v)$ for every other box $(r, v) \in D^t_{\lambda/\mu}$. By Corollary 1.44, this tableau is m-amenable for

$m \neq 2$. There is a 1 and no 2 in the column of the first box of P_1 (which is to the right of the t^{th} column). Thus, by Corollary 1.44, this tableau is 2-amenable and, hence, amenable. It has content $c(T_2) = (\nu_1 - 1, \nu_2, 1)$. $\square$

Example 6.44. *For $\lambda = (5,3,2)$ and $\mu = (2,1)$ we obtain*

$$T_{\lambda/\mu} = \begin{array}{|c|c|c|}\hline 1' & 1 & 1 \\ \hline 1' & 2' \\ \cline{1-2} 1 & 2 \\ \cline{1-2}\end{array}\ , \quad T_1 = \begin{array}{|c|c|c|}\hline 1' & 1 & 1 \\ \hline 1 & 2' \\ \cline{1-2} 2 & 2 \\ \cline{1-2}\end{array}\ , \quad T_2 = \begin{array}{|c|c|c|}\hline 1' & 1 & 1 \\ \hline 1 & 2 \\ \cline{1-2} 2 & 3 \\ \cline{1-2}\end{array}\ .$$

Lemma 6.45. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let P_n have all boxes in one row. Let (x, y) be the first box of P_n. If $n > 1$ and P_{n-1} has a box in a row below the x^{th} row and a box in a column to the right of the y^{th} column then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, we may assume that $n = 2$. Let $|P_n| = k$. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider $\lambda = (k+4, k+2, 1)$ and $\mu = (2,1)$. By Proposition 1.27, $f^{\lambda}_{\mu\nu} = f^{\lambda}_{\nu\mu}$ and we just need to look at tableaux of shape $D_{\lambda/\nu}$ and content $\mu = (2,1)$. Then we obtain three tableaux as follows:

- $T_1(1, k+4) = 1$, $T_1(2, k+3) = 1$, $T_1(3,3) = 2$;
- $T_2(2, k+3) = 1$, $T_2(2, k+2) = 1$, $T_2(3,3) = 2$;
- $T_3(1, k+4) = 1$, $T_3(1, k+3) = 1$, $T_3(2, k+3) = 2$.

Since $w(T_1) = w(T_2) = w(T_3) = 211$, these tableaux are amenable. Then the decomposition of $Q_{(k+4,k+2,1)/(2,1)}$ has at least three homogeneous components and, hence, so does the decomposition of $Q_{\lambda/\mu}$. $\square$

Example 6.46. *For $\lambda = (5,3,1)$ and $\mu = (2,1)$ we obtain*

$$T_1 = \begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & 1 \\ \hline \multicolumn{1}{c|}{} & \ & \ & 1 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 \\ \cline{3-3}\end{array}\ , \quad T_2 = \begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & 1 & 1 \\ \hline \multicolumn{1}{c|}{} & \ & \ & 2 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \ \\ \cline{3-3}\end{array}\ , \quad T_3 = \begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & \ \\ \hline \multicolumn{1}{c|}{} & \ & 1 & 1 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 \\ \cline{3-3}\end{array}\ .$$

Lemma 6.47. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu) > 1$. Let P_n have all boxes in one row. Let (x, y) be the first box of P_n and let (x, z) be the last box of P_n. Let the last box of P_{n-1} be to the left of the $(z-1)^{th}$ column and the first box of P_{n-1} is to the right of the y^{th} column. Let one of the following properties be satisfied:*

(a) $n = 2$ and $(x, z-2)$ is not the last box of P_1,

(b) $n \geq 3$.

Then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.

Proof. By Lemmas 6.5 and 6.43, we may assume that the last box of P_{n-1} is in the x^{th} row.

Case (a):

Let $|P_2| = k$. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider $\lambda = (k+5, k+3)$ and $\mu = (3)$. By Proposition 1.27, $f^{\lambda}_{\mu\nu} = f^{\lambda}_{\nu\mu}$ and we just need to look at tableaux of shape $D_{\lambda/\nu}$ and content $\mu = (3)$. Using the notation of Proposition 1.55, the following diagrams are in $B^{(3)}_{\lambda}$:

- $\{(1, k+4), (1, k+5), (2, k+4)\}$,
- $\{(1, k+5), (2, k+3), (2, k+4)\}$,
- $\{(2, k+2), (2, k+3), (2, k+4)\}$.

Then, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ has at least three homogeneous components.

Case (b):

By Lemma 2.1, we may assume $n = 3$. Let $|P_2| = k$. By case (a) and and a rotated version of case (a), we may assume that $U_2(\lambda/\mu)$ has shape $D_{(k+4,k+2)/(2)}$. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider $\lambda = (k+5, k+4, k+2)$ and $\mu = (2)$. Using the notation of Proposition 1.55, the following diagrams are in $B^{(2)}_{\lambda}$:

- $\{(1, k+5), (2, k+5)\}$,
- $\{(3, k+4), (2, k+5)\}$,
- $\{(3, k+3), (3, k+4)\}$.

Then, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ has at least three homogeneous components. $\square$

Lemma 6.48. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let $n \geq 3$ and let $U = U_{n-1}(\lambda/\mu)$. Let U, U^t, U^{ot} or U^o have shape $D_{(a,b)/(1)}$ where $a \geq b+2$ and let (x, y) be the last box of P_{n-1}. If $(x, y-1) \in P_{n-2}$ then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, we may assume that $n = 3$. Let $|P_n| = k$. Without loss of generality and by Lemmas 2.5, 2.8 and 2.9, it is enough to consider $\lambda = (a+2, a+1, b+1)$ and $\mu = (2, 1)$. By Proposition 1.27, $f^{\lambda}_{\mu\nu} = f^{\lambda}_{\nu\mu}$ and we just need to look at tableaux of shape $D_{\lambda/\nu}$ and content $\mu = (2, 1)$. Then we obtain three tableaux as follows:

- $T_1(1, a+2) = 1$, $T_1(2, a+2) = 2$, $T_1(3, b+3) = 1$;
- $T_2(2, a+2) = 1$, $T_2(2, a+1) = 1$, $T_2(3, b+3) = 2$;
- $T_3(2, a+2) = 1$, $T_3(3, b+3) = 2$, $T_3(3, b+2) = 1$.

Since $w(T_1) = w(T_3) = 121$ and $w(T_2) = 211$, these tableaux are amenable. Then the decomposition of $Q_{(a+2,a+1,b+1)/(2,1)}$ has at least three homogeneous components and, hence, so does the decomposition of $Q_{\lambda/\mu}$. $\square$

Example 6.49. *For $\lambda = (6, 5, 3)$ and $\mu = (2, 1)$ we obtain*

$$T_1 = \begin{array}{|c|c|c|c|c|c|} \hline \; & \; & \; & \; & \; & 1 \\ \hline & \; & \; & \; & \; & 2 \\ \hline & & \; & \; & 1 & \\ \hline \end{array}, \quad T_2 = \begin{array}{|c|c|c|c|c|c|} \hline \; & \; & \; & \; & \; & \; \\ \hline & \; & \; & \; & 1 & 1 \\ \hline & & \; & \; & 2 & \\ \hline \end{array}, \quad T_3 = \begin{array}{|c|c|c|c|c|c|} \hline \; & \; & \; & \; & \; & \; \\ \hline & \; & \; & \; & \; & 1 \\ \hline & & \; & 1 & 2 & \\ \hline \end{array}.$$

Lemma 6.50. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu) > 1$. Let $U_{n-1}(\lambda/\mu)$ have shape $D_{[a,1,c,1]/[a+c-1,1,0,0]}$ for some $a, c \geq 2$. If $n \geq 3$ then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $n = 3$. By Lemmas 2.5, 2.8 and 2.9, it is enough to consider that $D_{\lambda/\mu}$ has shape $D_{[a+1,1,c,1]/[a+c-1,1,0,0]}$. Then $D^{ot}_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ where $\alpha = [3, a+c-1, 0, 0]$ and $\beta = [1, c+1, 0, 0]$. Using the notation of Proposition 1.55, the following diagrams are in $B^{(c+1)}_{\alpha}$:

- $\{(3, a+c+1), (3, a+c), \ldots, (3, a+1)\}$,
- $\{(2, a+c+1), (3, a+c+1), (3, a+c), \ldots, (3, a+2)\}$,
- $\{(1, a+c+1), (2, a+c+1), (3, a+c+1), (3, a+c), \ldots, (3, a+3)\}$.

Then, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ has at least three homogeneous components. □

Lemma 6.51. *Let $\lambda, \mu \in DP$, $\nu := c(T_{\lambda/\mu})$ and $n := \ell(\nu)$. Let there be some $k > 1$ such that $U_k(\lambda/\mu)$ has shape $D_{\alpha/\beta}$ where $\alpha = [a, b, 0, 0]$ such that $a, b \geq 2$ and $\beta = [1, 2, 0, 0]$. If $D(\lambda/\mu)$ is not equal to $D_{\alpha'/\beta'}$ where $\alpha' = [c, d, 0, 0]$ such that $c, d \geq 2$ and $\beta' = [1, 2, 0, 0]$ then the decomposition of $Q_{\lambda/\mu}$ consists of more than two homogeneous components.*

Proof. By Lemma 2.1, it is enough to consider the case $k = 2$. Let (x, y) be the first box of P_2.

By Lemmas 2.5, 2.8 and 2.9, it is enough to consider that the last box of P_1 is in the row above the last box of P_2.

If there are boxes of P_1 to the right of the y^{th} column then, by Lemmas 2.5, 2.8 and 2.9, it is enough to consider that $(x-1, y+1)$ is the first box of P_1. Using the notation of Definition 1.51, then the following diagrams are in $B^{(2)}_{\lambda}$:

- $\{(a+1, y), (a, y)\}$,

- $\{(a+1,y),(a+1,y-1)\}$,

- $\{(a+1,y),(x-1,y+1)\}$.

Then, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ has at least three homogeneous components.

If there are no boxes of P_1 to the right of the y^{th} column then, by Lemmas 2.5, 2.8 and 2.9, it is enough to consider that $(x-2,y)$ is the first box of P_1. By Theorem 4.17, the skew Schur Q-function $Q_{U_2(\lambda/\mu)}$ is not Q-homogeneous and, by Lemma 2.1, there are two tableaux T and T' of $D_{\lambda/\mu}$ with different content such that $c(T)_1 = c(T')_1 = \nu_1$. We get another tableau, if we set $\bar{T}(u,y) = T_{\lambda/\mu}(u+1,y)$ for $1 \le u \le x+a-1$ and either set $\bar{T}(x+a-2,y) = T_{\lambda/\mu}(x+a-1,y)$ and $\bar{T}(x+a-1,y) = n+1$ if $|P_n| > 1$ or else set $\bar{T}(x+a-2,y) = n'$ and $\bar{T}(x+a-1,y) = n$ if $|P_n| = 1$ and $\bar{T}(r,s) = T_{\lambda/\mu}(r,s)$ for every other box $(r,s) \in D_{\lambda/\mu}$. By Lemma 1.42, this filling is amenable for the case $|P_n| > 1$ since if $\bar{T}(u,e) = i'$ and $\bar{T}(u-1,e-1) \neq (i-1)'$ then there is some $f > e$ such that $\bar{T}(u-1,f) = (i-1)'$ and $\bar{T}(u,f+1) \neq i'$. For the case $|P_n| = 1$ we have $\bar{T}(x+a-2,y) = n'$ and $\bar{T}(x+a-3,y-1) \neq (n-1)'$. However, we have $\bar{T}(x+a-2,y-1) = (n-1)'$ and $\bar{T}(x+a-1,y) \neq n'$. Thus, by Lemma 1.42, this filling is amenable. Clearly, $c(\bar{T}) = \nu_1 - 1$. □

Example 6.52. *For $\lambda/\mu = (6,5,4,3,2)/(5,2)$ we have*

$$T = \begin{array}{cccc} & & & 1' \\ & 1' & 1 & 1 \\ 1 & 1 & 2' & 2 \\ & 2 & 2 & 3' \\ & & 3 & 3 \end{array}, \quad T' = \begin{array}{cccc} & & & 1' \\ & 1' & 1 & 1 \\ 1 & 1 & 2' & 2 \\ & 2 & 2 & 3 \\ & & 3 & 4 \end{array}, \quad \bar{T} = \begin{array}{cccc} & & & 1 \\ & 1' & 1 & 2 \\ 1 & 1 & 2' & 3' \\ & 2 & 2 & 3 \\ & & 3 & 4 \end{array}.$$

For $\lambda/\mu = (7,6,5,4,3)/(6,2)$ we have

$$T = \begin{array}{ccccc} & & & & 1' \\ & 1' & 1 & 1 & 1 \\ 1 & 1 & 2' & 2 & 2 \\ & 2 & 2 & 3' & 3 \\ & & 3 & 3 & 4 \end{array}, \quad T' = \begin{array}{ccccc} & & & & 1' \\ & 1' & 1 & 1 & 1 \\ 1 & 1 & 2' & 2 & 2 \\ & 2 & 2 & 3 & 3 \\ & & 3 & 4 & 4 \end{array}, \quad \bar{T} = \begin{array}{ccccc} & & & & 1 \\ & 1' & 1 & 1 & 2 \\ 1 & 1 & 2' & 2 & 3 \\ & 2 & 2 & 3' & 4' \\ & & 3 & 3 & 4 \end{array}.$$

Proposition 6.53. *Let $\lambda, \mu \in DP$ be such that $D_{\lambda/\mu}$ is basic, let $\nu := c(T_{\lambda/\mu})$ and let $n := \ell(\nu)$. If the decomposition of $Q_{\lambda/\mu}$ consists of precisely two homogeneous components then the diagram $D_{\lambda/\mu}$ satisfies one of the following conditions up to transposing and orthogonally transposing of the diagram:*

(i) $\lambda = [a, b, c, d]$ *where* $a, b, c, d > 0$ *and* $\mu = (1)$,

(ii) $\lambda = [a, b, 0, 0]$ *where* $a \geq 2$, $b \geq 2$ *or* $\lambda = [e, 1, 1, 1]$ *where* $e \geq 2$ *or* $\lambda = [1, k, 1, l]$ *where* $1 \in \{k, l\}$ *but* $(k, l) \neq (1, 1)$ *and* $\mu = (2)$,

(iii) $|D_{\lambda/\mu}| \in \{3, 4\}$ *and* $D_{\lambda/\mu}$ *is a union of at least two border strips,*

(iv) $\lambda = [2, 1, c, 1]$ *and* $\mu = [1, c + 1, 0, 0]$

(v) $\lambda = [1, 1, c, d]$ *where* $d \geq 2$ *and* $\mu = [1, c + d, 0, 0]$,

(vi) $\lambda = [1, 1, c, 2]$ *and* $\mu = [1, c, 1, 1]$ *for some* $c \geq 2$,

(vii) $\lambda = [1, 1, c, 1]$ *where* $c \geq 2$ *and* $\mu = [s, t, 0, 0]$ *where* $t \leq c$,

(viii) $D_{\lambda/\mu}$ *has two components where the first component is* $D_{[a,b,c,1]}$ *and the second component consists of a single box.*

(ix) $D_{\lambda/\mu}$ *has two components where the first component is* $D_{[a,1,0,0]}$ *and the second component consists of two boxes in a row.*

(x) $D_{\lambda/\mu}$ *has three components where the first component is* $D_{[a,1,0,0]}$ *and the other components each consists of a single box.*

Some of these cases overlap.

Proof. We suppose that the decomposition of $Q_{\lambda/\mu}$ consists of precisely two homogeneous components and consider the possible diagrams $D_{\lambda/\mu}$.

We first consider the case $|D_{\lambda/\mu}| \leq 4$. Clearly skew Schur Q-functions $Q_{\lambda/\mu}$ with $|D_{\lambda/\mu}| \in \{1, 2\}$ have only one homogeneous component, namely $Q_{(1)}$ or $Q_{(2)}$. For the case $|D_{\lambda/\mu}| \in \{3,4\}$, by Theorem 4.17, $Q_{\lambda/\mu}$ has only one homogeneous component if after removing empty rows and columns the diagram $D_{\lambda/\mu}$ is contained in the set $\{D_{(3)}, D_{(4)}, D_{(4,2,1)/(3)}, D_{(3,2)/(1)}\}$. The remaining shapes are covered by the 6.53 (i), 6.53 (ii) and 6.53 (iii).

$>$From now on we consider $|D_{\lambda/\mu}| \geq 5$. By Lemma 6.1, we only need to consider the cases

- $|P_n| \leq 4$,
- P_n has all boxes in one row or one column,
- P_n is a (p,q)-hook or an orthogonally transposed (p,q)-hook where $p = 2$ or $q = 2$,
- P_n has two components where one consists of one single box and the other one has all boxes in one row or one column.

Case 1: $|P_n| \in \{3,4\}$ and P_n consists of at least two border strips. By Lemma 6.4, $Q_{\lambda/\mu}$ has more than two homogeneous components if $n \geq 2$. The case $n = 1$ is covered by 6.53 (iii).

Case 2: P_n has two components where one consists of one single box and the other one has all boxes in one row or one column.

By Lemma 6.4, $Q_{\lambda/\mu}$ has more than two homogeneous components if $n \geq 2$. The case $n = 1$ is covered by 6.53 (ii).

Case 3: Up to transposing and orthogonally transposing and after removing empty rows or columns $P_n = D_{(4,2)/(2)}$.

By Lemma 6.7, $Q_{\lambda/\mu}$ has more than two homogeneous components if $n \geq 2$. The case $n = 1$ is covered by 6.53 (ii).

Case 4: P_n is a $(p,2)$-hook or an orthogonally transposed $(p,2)$-hook where $p \geq 3$.

By Lemma 6.8, $Q_{\lambda/\mu}$ has more than two homogeneous components if $n \geq 2$. The case $n = 1$ is covered by 6.53 (i).

Case 5: P_n is a $(2,q)$-hook or an orthogonally transposed $(2,q)$-hook where $q \geq 2$.

By Lemmas 6.9, 6.13, 6.15 and 6.17, the diagram P_{n-1} can only be a border strip where its first box is the box above the first box of P_n and its last box is in the row above the row of the last box of P_n. Repeating this argument for $P_{n-1}, \ldots, P_1$ we obtain diagrams covered by 6.53 (i).

The last remaining possibility for P_n is that it has all boxes in one row or one column. This means that there is some $k \geq 2$ such that $Q_{U_k(\lambda/\mu)}$ is Q-homogeneous and $Q_{U_{k-1}(\lambda/\mu)}$ is not Q-homogeneous.

Case 6: $U_k(\lambda/\mu)$ has shape $D_{[1,1,c,1]/[1,c+1,0,0]}$ for some $c > 0$.

Let the box of the second component of $U_k(\lambda/\mu)$ be (x,y). By Lemma 6.18, the uppermost rightmost box of the first component of $U_k(\lambda/\mu)$ is $(x+1, y-1)$. By Lemmas 6.20, 6.22 and 6.24, if $P_n \neq D_{(3,1)/(2)}$ then we have $k = 2$ and the first box of P_1 is $(x-1, y)$. This case is covered by 6.53 (iv). If P_n has shape $D_{(3,1)/(2)}$ then, by the same lemmas, the first box of P_i must be $(x-n+i, y)$ for $1 \leq i \leq n-1$ and the last box of P_{n-1} is in the x^{th} row. This case is covered by 6.53 (ii).

Case 7: $U_k(\lambda/\mu)$ has shape $D_{[a,b,0,0]/[c,1,0,0]}$ where $a, b \geq 2$.

Note that $a \geq 2$ is mandatory for $c \geq 1$ and case $b = 1$ is covered by Case 8 of this proof. Let (x,y) be the first box of P_k.

Case 7.1: $c \geq 2$.

Then $a \geq 3$. By Lemma 6.37 and its orthogonally transposed version, we have $b = 2$ and the first box of P_{k-1} must be in the y^{th} column. Then $U_{k-1}(\lambda/\mu)^{ot} = D_{[a',b',c',d']/(1)}$ for some a', b', c', d'. By orthogonally transposed versions of Lemmas 6.13, 6.15 and 6.17, the diagram $D^{ot}_{\lambda/\mu}$ must have shape $D_{[a'',b'',c'',d'']/(1)}$ for some a'', b'', c'', d'' and is covered by 6.53 (i).

Case 7.2: $c = 1$.

By Lemma 6.41 and its orthogonally transposed version, the diagram $U_{k-1}(\lambda/\mu)$ or $U_{k-1}(\lambda/\mu)^{ot}$ either has shape $D_{[a',b',c',d']/(1)}$ and, by the same argument as in Case 7.1, is covered by 6.53 (i) or has shape $D_{[1,1,c',d']/[1,c'+d'-2,1,1]}$ where $d' \geq 2$. In the later case, by Lemmas 6.4 and 6.39, we have $d' = 2$ and $k = 2$ and this case is covered by 6.53 (vi).

Case 8: Up to transposing and orthogonally transposing $U_k(\lambda/\mu)$ has shape D_α for some partition α.

By Lemma 6.25, we have $\alpha = [a, b, 0, 0]$ or $\alpha = [a, b, c, 1]$.

Case 8.1: The diagram $U_k(\lambda/\mu)$ or $U_k(\lambda/\mu)^{ot}$ has shape $D_{[a,b,c,1]}$.

Without loss of generality we assume that $U_k(\lambda/\mu) = D_{[a,b,c,1]}$. Let (x, y) be the first box of P_k. By orthogonally transposed versions of Lemmas 6.26, 6.27, 6.29, 6.31 and 6.33, we have $k = 2$ and either $(x-2, y)$ is the first box of P_1 or the diagram $D_{\lambda/\mu}$ has two components where the first component is $D_{[a+1,b,c,1]}$ and the second component consists of a single box. The first case is covered by 6.53 (vii) and the second case is covered by 6.53 (viii).

Case 8.2: The diagram $U_k(\lambda/\mu)$ is equal to $D_{[a,b,0,0]}$.

Case 8.2.1: $a > 1$.

Case 8.2.1.1: $b = 1$.

Let (x, y) be the first box of P_k. By Lemmas 6.34, 6.35 and 6.36 and their orthogonally transposed versions, we have $k = 2$ and one of the following cases:

(a) there is only one box in the $(x-2)^{\text{th}}$ row which is the only box above the $(x-1)^{\text{th}}$ row and the rightmost box of the $(x-1)^{\text{th}}$ row is to the right of the y^{th} column,

(b) $D_{\lambda/\mu}$ has two components where the first component is $D_{[a+1,1,0,0]}$ and the other components consists of two boxes in a row,

(c) $D_{\lambda/\mu}$ has three components where the first component is $D_{[a+1,1,0,0]}$ and the other components each consists of a single box.

Case (a) is covered by 6.53 (vii) if the diagram is connected and it is covered by 6.53 (viii) if the diagram is disconnected. Case (b) is covered by 6.53 (ix). And Case (c) is covered by 6.53 (x).

Case 8.2.1.2: $b > 1$.

Let (x, y) be the first box of P_k. By orthogonally transposed versions of Lemmas 6.26, 6.27, 6.29, 6.31 and 6.33, we have $k = 2$ and either $(x - 2, y)$ is the first box of P_1 or the diagram $D_{\lambda/\mu}$ has two components where the first component is $D_{[a+1,b,0,0]}$ and the second component consists of a single box. The first case is covered by 6.53 (vii) and the second case is covered by 6.53 (v).

Case 8.2.2: $a = 1$.

Case 8.2.2.1: $b > 1$.

Let (x, y) be the first box of P_n and let (x, z) be the last box of P_n. By Lemma 6.43, the last box of P_{n-1} is in the $(x - 1)^{\text{th}}$ row or in the x^{th} row.

Case 8.2.2.1.1: The last box of P_{n-1} is in the $(x - 1)^{\text{th}}$ row.

By Lemma 6.4 and orthogonally transposed versions of Lemmas 6.26, 6.27, 6.29, 6.31 and 6.33, we have $k = 2$ and either $(x - 2, y)$ is the first box of P_1 or the diagram $D_{\lambda/\mu}$ has two components where the first component is $D_{[2,b,0,0]}$ and the second component consists of a single box. The first case is covered by 6.53 (vii) and the second case is covered by 6.53 (v).

Case 8.2.2.1.2: The last box of P_{n-1} is in the x^{th} row.

By a rotated version of Lemma 6.43, the first box of P_{n-1} must be in the $(x - 1)^{\text{th}}$ row.

Case 8.2.2.1.2.1: The last box of P_{n-1} is $(x, z - 1)$.

Since $Q_{U_{n-1}(\lambda/\mu)}$ is not Q-homogeneous, the first box of P_{n-1} is to the right of the y^{th} column. If $n = 2$ then this case is covered by 6.53 (i). If $n \geq 3$ then, by Lemma 6.48, the last box of P_{n-2} must be $(x - 1, z - 2)$. By Lemmas 6.15 and 6.17, the first box of P_{n-2} must be the box above the first box of P_{n-1}. Repeating this argument, we obtain diagrams covered by 6.53 (i).

Case 8.2.2.1.2.2: The last box of P_{n-1} is to the left of $(z - 2)^{\text{th}}$ column.

By an orthogonally transposed version of Lemma 6.45, and Lemma 6.47, the first box of P_{n-1} is $(x-1, y)$. By an orthogonally transposed version of Lemma 6.50, we have $n = 2$ which is covered by 6.53 (i).

Case 8.2.2.1.2.3: The last box of P_{n-1} is $(x, z-2)$.

By an orthogonally transposed version of Lemma 6.45, and an rotated version of Lemma 6.47, the first box of P_{n-1} is either $(x-1, y)$ or $(x-1, y+1)$. If the first box of P_{n-1} is $(x-1, y+1)$ then, by Lemma 6.47, we have $n = 2$ which is covered by 6.53 (ii). If the first box of P_{n-1} is $(x-1, y)$ then either $n = 2$ which is covered by 6.53 (ii) or if $n \geq 3$, by an orthogonally transposed version of Lemma 6.48, the last box of P_{n-2} is $(x-1, z-3)$. Then, by Lemma 6.51, the first box of P_{n-2} is the box above the first box of P_{n-1}. Repeating this argument, we obtain diagrams covered by 6.53 (ii).

Case 8.2.2.2: $b = 1$.

This means that $|P_n| = 1$. Let (x, y) be the box of P_n.

Case 8.2.2.2.1: The last box of P_{n-1} is in the $(x-1)^{\text{th}}$ row.

By Lemmas 6.34, 6.35 and 6.36 and their orthogonally transposed versions, we have $k = 2$ and one of the following cases:

(a) there is only one box in the $(x-2)^{\text{th}}$ row which is the only box above the $(x-1)^{\text{th}}$ row and the rightmost box of the $(x-1)^{\text{th}}$ row is to the right of the y^{th} column,

(b) $D_{\lambda/\mu}$ has two components where the first component is $D_{[2,1,0,0]}$ and the other components consists of two boxes in a row,

(c) $D_{\lambda/\mu}$ has three components where the first component is $D_{[2,1,0,0]}$ and the other components each consists of a single box.

Case (a) is covered by 6.53 (vii) if the diagram is connected and it is covered by 6.53 (viii) if the diagram is disconnected. Case (b) is covered by 6.53 (ix). And Case (c) is covered by 6.53 (x).

Case 8.2.2.2.2: The last box of P_{n-1} is in a row below the x^{th} row.

By Lemma 6.45, the first box of P_{n-1} is in the y^{th} column. By Lemma 6.5 and its orthogonal transposed version, the last box of P_{n-1} is in the $(y-1)^{\text{th}}$ column or $U_{n-1}(\lambda/\mu)$ has two components where one component is $D_{(3,2)/(1)}$ and the other component consists of a single box. The orthogonal transposition of the first case is considered in Case 8.2.2.2.3 of this proof. For the latter case, by Lemma 6.4, we have $n = 2$ and this case is covered by 6.53 (v).

Case 8.2.2.2.3: The last box of P_{n-1} is in the x^{th} row.

Case 8.2.2.2.3.1: The last box of P_{n-1} is $(x, y-1)$.

Since $Q_{U_{n-1}(\lambda/\mu)}$ is not Q-homogeneous, $(x-1, y)$ is not the first box of P_{n-1}.

Case 8.2.2.2.3.1.1: The first box of P_{n-1} is in the $(x-1)^{\text{th}}$ row.

Then the first box of P_{n-1} is in a column to the right of the y^{th} column. If $n = 2$ then this case is covered by 6.53 (i). If $n \geq 3$ then, by Lemma 6.48, the last box of P_{n-2} must be $(x-1, y-2)$. By Lemmas 6.15 and 6.17, the first box of P_{n-2} must be the box above the first box of P_{n-1}. Repeating this argument, we obtain diagrams covered by 6.53 (i).

Case 8.2.2.2.3.1.2: The first box of P_{n-1} is above the $(x-1)^{\text{th}}$ row.

By a transposed version of Lemma 6.5, either the first box of P_{n-1} is in the y^{th} column above the $(x-1)^{\text{th}}$ row or the diagram $U_{n-1}(\lambda/\mu)$ has two components where the first component is $D_{(3,2)/(1)}$ and the second component consists of a single box. In the first case if $n = 2$ this case is covered by 6.53 (i). If $n \geq 3$ then, by an orthogonally transposed version of Lemma 6.48, the last box of P_{n-2} must be $(x-1, y-2)$. By orthogonally transposed versions of Lemmas 6.15 and 6.17, the first box of P_{n-2} must be the box above the first box of P_{n-1}. Repeating this argument, we obtain diagrams covered by 6.53 (i). In the latter case, by Lemma 6.4, we have $n = 2$ which is covered by 6.53 (vi).

Case 8.2.2.2.3.2: The last box of P_{n-1} is to the left of $(y-2)^{\text{th}}$ column.

By an orthogonally transposed version of Lemma 6.45, and Lemma 6.47, the first box of P_{n-1} is $(x-1, y)$. By Lemma 6.50, we have $n = 2$ which is covered by 6.53 (i).

Case 8.2.2.2.3.3: The last box of P_{n-1} is $(x, y-2)$.

By an orthogonally transposed version of Lemma 6.45, and an rotated version of Lemma 6.47, the first box of P_{n-1} is either $(x-1, y)$ or $(x-1, y+1)$. If the first box of P_{n-1} is $(x-1, y+1)$ then, by Lemma 6.47, we have $n = 2$ which is covered by 6.53 (ii). If the first box of P_{n-1} is $(x-1, y)$ then either $n = 2$ which is covered by 6.53 (ii) or if $n \geq 3$, by an orthogonally transposed version of Lemma 6.48, the last box of P_{n-2} is $(x-1, y-3)$. Then, by Lemma 6.51, the first box of P_{n-2} is the box above the first box of P_{n-1}. Repeating this argument, we obtain diagrams covered by 6.53 (ii). □

6.3 Proof that the decomposition of the remaining skew Schur Q-functions consists of precisely two homogeneous components

Now we will show case by case that the decomposition of the skew Schur Q-functions appearing in Proposition 6.53 consists of precisely two homogeneous components. We will also give the constituents and their coefficients.

Hypothesis. *We will always assume that λ and μ are such that $D_{\lambda/\mu}$ is basic (see Definition 1.13).*

Lemma 6.54. *Let $\lambda = [a, b, c, d]$ where $a, b, c, d > 0$ and $\mu = (1)$. Let $\alpha = (a+b+c+d-1, a+b+c+d-2, \ldots, b+c+d+2, b+c+d+1, b+c+d-1, c+d-1, c+d-2, \ldots, d+1, d)$ and $\beta = (a+b+c+d-1, a+b+c+d-2, \ldots, b+c+d+1, b+c+d, c+d-1, c+d-2, \ldots, d+2, d+1, d-1)$. Then $Q_{\lambda/\mu} = Q_\alpha + Q_\beta$.*

Proof. By Proposition 1.55, the partitions occurring in the decomposition are partitions obtained by the diagrams we obtain by removing a corner of D_λ. The partitions obtained by this way are α and β. Also by Proposition 1.55, the coefficients are one for both constituents. □

Lemma 6.55. *Let $\lambda = [a, b, 0, 0]$ where $a \geq 2$, $b \geq 2$ and $\mu = (2)$. Let $\alpha = (a+b-1, a+b-2, \ldots, b+2, b+1, b-2)$ and $\beta = (a+b-1, a+b-2, \ldots, b+3, b+2, b, b-1)$. Then $Q_{\lambda/\mu} = Q_\alpha + Q_\beta$.*

Proof. Using the notation of Definition 1.51, let (x, y) be the corner of B_λ. Then $B_\alpha^{(2)} = \{\{(x, y), (x, y-1)\}, \{(x, y), (x-1, y)\}\}$. Since $D_\lambda \setminus \{(x, y), (x, y-1)\} = D_\alpha$ and $D_\lambda \setminus \{(x, y), (x-1, y)\} = D_\beta$, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ consists only of the constituents Q_α and Q_β. Since both diagrams of $B_\alpha^{(2)}$ have only one component, the coefficients are one for both constituents. □

Lemma 6.56. *Let* $\lambda = [e, 1, 1, 1]$ *where* $e \geq 2$ *and* $\mu = (2)$. *Let* $\alpha = (4, 2)$ *and* $\beta = (3, 2, 1)$ *if* $e = 2$ *or let* $\alpha = (e+2, e+1, \ldots, 5, 4, 2)$ *and* $\beta = (e+2, e+1, \ldots, 5, 3, 2, 1)$ *if* $e \geq 3$. *Then* $Q_{\lambda/\mu} = 2Q_\alpha + Q_\beta$.

Proof. Using the notation of Definition 1.51, let (x, y) be the lowermost box of B_λ. Then we have $B_\alpha^{(2)} = \{\{(x, y), (x-1, y+1)\}, \{(x-1, y+1), (x-2, y+1)\}\}$. Since $D_\lambda \setminus \{(x, y), (x-1, y+1)\} = D_\alpha$ and $D_\lambda \setminus \{(x-1, y+1), (x-2, y+1)\} = D_\beta$, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ consists only of the constituents Q_α and Q_β. Since $D_{\lambda/\alpha} = \{(x, y), (x-1, y+1)\}$ has two components, the coefficient is two for the constituent Q_α. Since $D_{\lambda/\beta} = \{(x-1, y+1), (x-2, y+1)\}$ has only one component, the coefficient is one for the constituent Q_β. □

Lemma 6.57. *Let* $\lambda = [1, 1, 1, l]$ *where* $l \geq 2$ *and* $\mu = (2)$. *Let* $\alpha = (l+2, l-2)$ *and* $\beta = (l+1, l-1)$. *Then* $Q_{\lambda/\mu} = Q_\alpha + 2Q_\beta$.

Proof. Using the notation of Definition 1.51, we have $B_\lambda^{(2)} = \{\{(2, l+1), (2, l)\}, \{(2, l+1), (1, l+2)\}\}$. Since $D_\lambda \setminus \{(2, l+1), (2, l)\} = D_\alpha$ and $D_\lambda \setminus \{(2, l+1), (1, l+2)\} = D_\beta$, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ consists only of the constituents Q_α and Q_β. Since $D_{\lambda/\alpha} = \{(2, l+1), (2, l)\}$ has only one component, the coefficient is one for the constituent Q_α. Since $D_{\lambda/\beta} = \{(2, l+1), (1, l+2)\}$ has two components, the coefficient is two for the constituent Q_β. □

Lemma 6.58. *Let* $\lambda = [1, k, 1, 1]$ *where* $k \geq 2$ *and* $\mu = (2)$. *Let* $\alpha = (k, 1)$ *and* $\beta = (k+1)$. *Then* $Q_{\lambda/\mu} = Q_\alpha + 2Q_\beta$.

Proof. Using the notation of Definition 1.51, we have $B_\lambda^{(2)} = \{\{(1, k+2), (1, k+1)\}, \{(1, k+2), (2, 2)\}\}$. Since $D_\lambda \backslash \{(1, k+2), (1, k+1)\} = D_\alpha$ and $D_\lambda \backslash \{(1, k+2), (2, 2)\} = D_\beta$, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ consists only of the constituents Q_α and Q_β. Since $D_{\lambda/\alpha} = \{(1, k+2), (1, k+1)\}$ has only one component, the coefficient is one for the constituent Q_α. Since $D_{\lambda/\beta} = \{(1, k+2), (2, 2)\}$ has two components, the coefficient is two for the constituent Q_β. □

Lemma 6.59. *Let $|\lambda/\mu| \in \{3, 4\}$ and $D_{\lambda/\mu}$ is a union of at least two border strips. If $|\lambda/\mu| = 3$ then $Q_{\lambda/\mu}$ is equal to one of the following Q-functions:*

(a) $Q_{(4,1)/(2)} = Q_{(2,1)} + 2Q_{(3)}$,

(b) $Q_{(5,3,1)/(4,2)} = 2Q_{(2,1)} + 4Q_{(3)}$.

If $|\lambda/\mu| = 4$ then $Q_{\lambda/\mu}$ is equal to one of the following Q-functions:

(i) $Q_{(5,1)/(2)} = Q_{(3,1)} + 2Q_{(4)}$,

(ii) $Q_{(5,3,1)/(3,2)} = 3Q_{(3,1)} + 2Q_{(4)}$,

(iii) $Q_{(5,2)/(3)} = 2Q_{(3,1)} + 2Q_{(4)}$,

(iv) $Q_{(6,3,1)/(4,2)} = 4Q_{(3,1)} + 4Q_{(4)}$,

(v) $Q_{(7,5,3,1)/(6,4,2)} = 8Q_{(3,1)} + 8Q_{(4)}$.

Proof. These decompositions can easily be verified.

For $|\lambda/\mu| = 3$ either $D_{\lambda/\mu}$ has two components where one component has two boxes and the other consists of one single box or $D_{\lambda/\mu}$ has three components that consist of single boxes. These Q-functions are covered by case (a) or (b), respectively.

Now consider the case $|\lambda/\mu| = 4$. If $D_{\lambda/\mu}$ has four components then these components consist of single boxes. These Q-functions are covered by case (v). If $D_{\lambda/\mu}$ has three components then one component has two boxes and the other components consist of single boxes. These Q-functions are covered by case (iv). If $D_{\lambda/\mu}$ has two components and there is no component that consists of a single box then both components must have two boxes. These Q-functions are covered by case (iii). If $D_{\lambda/\mu}$ has two components and there is a component that consists of a single box then the other component consists of three boxes. If these boxes form a $(2,2)$-hook or an orthogonally transposed $(2,2)$-hook then these Q-functions are covered by case (ii). If these boxes are in a row or column then these Q-functions are covered by case (i). □

Lemma 6.60. *Let* $\lambda = (k, k-1, k-3, k-4, \ldots, 1)$ *and* $\mu = (k-2)$ *for some* $k \geq 5$. *Let* $\alpha = (k, k-2, k-4, k-5, \ldots, 1)$ *and* $\beta = (k-1, k-2, k-3, k-5, k-6, \ldots, 1)$. *Then* $Q_{\lambda/\mu} = 2Q_\alpha + 2Q_\beta$.

Proof. Using the notation of Definition 1.51, we have $B_\lambda^{(\lambda_1-2)} = \{B_\lambda \setminus \{(1,k),(2,k-1)\}, B_\lambda \setminus \{(2,k-1),(3,k-1)\}\}$. Since $D_\lambda \setminus (B_\lambda \setminus \{(1,k),(2,k-1)\}) = D_\alpha$ and $D_\lambda \setminus (B_\lambda \setminus \{(2,k-1),(3,k-1)\}) = D_\beta$, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ consists only of the constituents Q_α and Q_β. Since $D_{\lambda/\alpha}$ has two components, the coefficient is two for the constituent Q_α. Since $D_{\lambda/\beta}$ has two components, the coefficient is two for the constituent Q_β. □

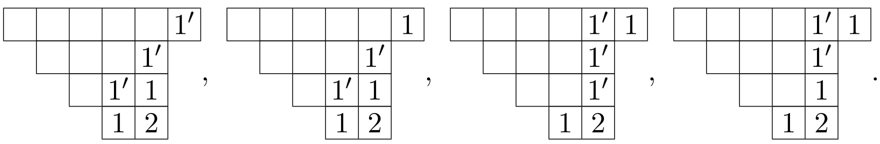
Lemma 6.61. *Let* $\lambda = [1,1,c,d]$ *and* $\mu = [1, c+d, 0, 0]$ *where* $d \geq 2$. *Let* $\alpha = (c+d-1, c+d-2, \ldots, d+1, d, 1)$ *and* $\beta = (c+d, c+d-2, c+d-3, \ldots, d+1, d)$. *Then* $Q_{\lambda/\mu} = Q_\alpha + 2Q_\beta$.

Proof. Using the notation of Proposition 1.55, we have $B_\lambda^\times = \{(d+1,d+1),(1,c+d)\}$. Since $D_\lambda \setminus (B_\lambda \setminus \{(d+1,d+1)\}) = D_\alpha$ and $D_\lambda \setminus (B_\lambda \setminus \{(1,c+d)\}) = D_\beta$, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ consists only of the constituents Q_α and Q_β. Since $D_{\lambda/\alpha}$ has one component, the coefficient is one for the constituent Q_α. Since $D_{\lambda/\beta}$ has two components, the coefficient is two for the constituent Q_β. □

Lemma 6.62. *Let* $\lambda = [1,1,c,2]$ *and* $\mu = [1,c,1,1]$ *for some* $c \geq 2$. *Let* $\alpha = (c+2, c, c-1, \ldots, 3, 1)$ *and* $\beta = (c+1, c, \ldots, 3, 2)$. *Then* $Q_{\lambda/\mu} = 2Q_\alpha + 2Q_\beta$.

Proof. We want to find the coefficients $f_{\mu\nu}^{\lambda}$ that are non-zero. By Proposition 1.27, we may consider tableaux of shape $D_{\lambda/\nu}$ and content $(c+2,1)$ for some $\gamma \in DP$. Clearly, for every tableau T of shape $D_{\lambda/\nu}$ for some ν we have $T(c+1,c+2) = 2$ (which is the lower corner of D_λ). Let $\hat{\lambda} = (c+3, c+1, c, \ldots, 3, 1)$. Using the notation of Lemma 1.55, for every T of shape $D_{\lambda/\nu}$ the set of boxes $T^{(1)}$ must be a subset of $B_{\hat{\lambda}}$. If $T^{(1)} = B_{\hat{\lambda}} \setminus \{(1, c+2)\}$ then the filling of $T^{(1)}$ is uniquely determined except for the box $(1, c+3)$. Since $T(c, c+2) = 1$ and $T(c+1, c+1) = 1$, we have $T(1, c+3) \in \{1', 1\}$ and both choices give an amenable tableau. Since $D_\lambda \setminus ((B_{\hat{\lambda}} \setminus \{(1, c+2)\}) \cup \{(c+1, c+2)\}) = D_\alpha$, we have precisely two tableaux with content $\nu = \alpha$. If $T^{(1)} = B_{\hat{\lambda}} \setminus \{(c, c+1)\}$ then the filling of $T^{(1)}$ is uniquely determined except for the box $(c, c+2)$. Since $T(1, c+3) = 1$ and $T(c+1, c+1) = 1$, we have $T(c, c+2) \in \{1', 1\}$ and both choices give an amenable tableau. Since $D_\lambda \setminus ((B_{\hat{\lambda}} \setminus \{(c, c+1)\}) \cup \{(c+1, c+2)\}) = D_\beta$, we have precisely two tableaux with $\nu = \beta$. □

Example 6.63. *For* $\lambda/\mu = (6,4,3,2)/(5,1)$ *the tableaux appearing in the proof of Lemma 6.62 are*



Remark. An alternate proof of Lemma 6.62 can be obtained by using Lemma 1.71. The diagram $\Delta_1^{\leftarrow}(D_{\lambda/\mu})$ has shape $D_{[1,1,c-1,2]/[1,1,0,0]}$ and the diagram $\Delta_1^{\downarrow}(D_{\lambda/\mu})$ has shape $D_{[c+1,2,0,0]/[1,c-1,1,1]}$. We obtain

$$Q_{\lambda/\mu} = Q_{\Delta_1^{\leftarrow}(D_{\lambda/\mu})} + Q_{\Delta_1^{\downarrow}(D_{\lambda/\mu})} = 2Q_{\Delta_1^{\leftarrow}(D_{\lambda/\mu})} = 2 \cdot (Q_\alpha + Q_\beta) = 2Q_\alpha + 2Q_\beta$$

by Lemma 1.60 and Lemma 3.34.

Lemma 6.64. *Let* $\lambda = [1, 1, c, 1]$ *where* $c \geq 2$ *and* $\mu = [s, t, 0, 0]$ *where* $1 < t \leq c$. *Let* $\alpha = (c+2, c, c-1, \ldots, s+t, t-1, t-2, \ldots, 1)$ *and* $\beta = (c+1, c, \ldots, s+t, t, t-2, t-3 \ldots, 1)$. *Then* $Q_{\lambda/\mu} = Q_\alpha + 2Q_\beta$.

Proof. The diagram $D^{ot}_{\lambda/\mu}$ has shape $D_{\hat{\lambda}/\hat{\mu}}$ where $\hat{\lambda} = (c+2, c+1, \ldots, s+t, t-1, t-2, \ldots, 1)$ and $\hat{\mu} = (c + 1)$. Using the notation of Proposition 1.55, we have $B^\times_{\hat{\lambda}} = \{(1, c + 2), (c - s - t + 3, c - s + 2)\}$. Since $(D_\lambda \setminus B_\lambda) \cup \{(c - s - t + 3, c - s + 2)\} = D_\beta$ and $(D_\lambda \setminus B_\lambda) \cup \{(1, c + 2)\} = D_\alpha$, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ consists only of the constituents Q_α and Q_β. Since $D_{\lambda/\alpha}$ has one component, the coefficient is one for the constituent Q_α. Since $D_{\lambda/\beta}$ has two components, the coefficient is two for the constituent Q_β. □

Lemma 6.65. *Let* $\lambda = [1, 1, c, 1]$ *where* $c \geq 2$ *and* $\mu = [s, 1, 0, 0]$. *Let* $\alpha = (c + 2, c, c - 1, \ldots, s + 1)$ *and* $\beta = (c + 1, c, \ldots, s + 1, 1)$. *Then* $Q_{\lambda/\mu} = Q_\alpha + Q_\beta$.

Proof. The diagram $D^{ot}_{\lambda/\mu}$ has shape $D_{\hat{\lambda}/\hat{\mu}}$ where $\hat{\lambda} = (c+2, c+1, \ldots, s+1)$ and $\hat{\mu} = (c+1)$. Using the notation of Proposition 1.55, we have $B^\times_{\hat{\lambda}} = \{(1, c + 2), (c - s + 2, c - s + 2)\}$. Since $(D_\lambda \setminus B_\lambda) \cup \{(1, c + 2)\} = D_\alpha$ and $(D_\lambda \setminus B_\lambda) \cup \{(c - s + 2, c - s + 2)\} = D_\beta$, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ consists only of the constituents Q_α and Q_β. Since $D_{\lambda/\alpha}$ has one component, the coefficient is one for the constituent Q_α. Since $D_{\lambda/\beta}$ has one component, the coefficient is one for the constituent Q_β. □

Lemma 6.66. *Let* $D_{\lambda/\mu}$ *have two components where the first component is* $D_{[a,b,c,1]}$ *and the second component consists of a single box. Let* $\alpha = (a + b + c + 1, a + b + c - 1, a + b + c - 2, \ldots, b + c + 1, c, c - 1, \ldots, 2, 1)$ *and* $\beta = (a + b + c, a + b + c - 1, \ldots, b + c + 1, c + 1, c - 1, c - 2, \ldots, 2, 1)$. *Then* $Q_{\lambda/\mu} = 2Q_\alpha + 2Q_\beta$.

Proof. Using the notation of Proposition 1.55, we have $B^\times_{\hat{\lambda}} = \{(1, a + b + c + 1), (a + 1, a + c + 1)\}$. Since $(D_\lambda \setminus (B_\lambda \setminus \{(1, a + b + c + 1)\})) = D_\alpha$ and $(D_\lambda \setminus (B_\lambda \setminus \{(a+1, a+c+1)\})) = D_\beta$, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ consists only of the constituents Q_α and Q_β. Since $D_{\lambda/\alpha}$ has two components, the coefficient is two for the constituent Q_α. Since $D_{\lambda/\beta}$ has two components, the coefficient is two for the constituent Q_β. □

Lemma 6.67. *Let $D_{\lambda/\mu}$ have two components where the first component is $D_{[a,1,0,0]}$ where $a \geq 2$ and the other component consists of two boxes in a row. Let $\alpha = (a+2, a-1, a-2, \ldots, 1)$ and $\beta = (a+1, a, a-2, a-3, \ldots, 1)$. Then $Q_{\lambda/\mu} = 2Q_\alpha + 2Q_\beta$.*

Proof. Using the notation of Definition 1.51, we have $B_\lambda^{(\lambda_1-2)} = \{B_\lambda \setminus \{(1, a+1), (1, a+2)\}, B_\lambda \setminus \{(1, a+1), (2, a+1)\}\}$. Since $D_\lambda \setminus (B_\lambda \setminus \{(1, a+1), (1, a+2)\}) = D_\alpha$ and $D_\lambda \setminus (B_\lambda \setminus \{(1, a+1), (2, a+1)\}) = D_\beta$, by Proposition 1.55, the decomposition of $Q_{\lambda/\mu}$ consists only of the constituents Q_α and Q_β. Since both diagrams of $B_\lambda^{(\lambda_1-2)}$ have two components, the coefficients are two for both constituents. □

Lemma 6.68. *Let $D_{\lambda/\mu}$ have three components where the first component is $D_{[a,1,0,0]}$ where $a \geq 2$ and the other components each consists of a single box. Let $\alpha = (a+2, a-1, a-2, \ldots, 1)$ and $\beta = (a+1, a, a-2, a-3, \ldots, 1)$. Then $Q_{\lambda/\mu} = 4Q_\alpha + 4Q_\beta$.*

We will give a proof in style of the previous proofs that make use of Proposition 1.27. We do this because it shows that this lemma can also be useful if μ is not a partition of length 1 (as in the previous proofs). In Lemma 6.62 we already saw that Proposition 1.27 is helpful if μ has two parts and the second part is 1. Like in Lemma 6.62, a much shorter proof that uses Lemma 1.71 will be added as a remark.

Proof of Lemma 6.68. We have $\lambda = (a+4, a+2, a, a-1, \ldots, 1)$ and $\mu = (a+3, a+1)$. By Proposition 1.27, we may consider tableaux of shape $D_{\lambda/\gamma}$ and content $(a+3, a+1)$ for some $\gamma \in DP$. Let $S_2 := \{(x, y) \in D_\lambda \mid x \geq 2 \text{ and } (x+1, y+1) \notin D_\lambda\}$. The $a+1$ entries from $\{2', 2\}$ must be in the boxes of P_2. Since $|S_2| = a+2$ we must remove a box from S_2 such that the remaining set of boxes is a valid diagram. The box $(2, a+2)$ is the only box of S_2 that can be removed. Set $S_2' := S_2 \setminus \{(2, a+2)\}$. By Lemma 1.37 and since the last box of S_2' must be unmarked, all entries in S_2' are fixed except for the box of the second component which is $(2, a+3)$. Let $S_1 := \{(x, y) \in D_\lambda \setminus S_2' \mid (x+1, y+1) \notin D_\lambda \setminus S_2'\}$. The $a+3$ entries from $\{1', 1\}$ must be in the boxes of S_1. Since $|S_1| = a+4$ we must remove a box from S_1 such that the remaining set of boxes is a valid diagram. The only possibilities to remove one box from S_1 such that the remaining boxes form a valid diagram is either to remove $(1, a+2)$ or to remove $(2, a+1)$. If we remove $(1, a+2)$ we have $D_{\lambda/\mu} \setminus (S_1 \cup S_2') = D_\alpha$. If we remove $(2, a+1)$ we have $D_{\lambda/\mu} \setminus (S_1 \cup S_2') = D_\beta$. For all tableaux T obtained as above we have $T(1, a+4) = 1$ and $(2, a+4) \notin D_{\lambda/\mu}$. If $(1, a+2) \in S_1$ then $T(1, a+3) = 1$ and if $(2, a+1) \in S_1$ then $T(2, a+2) = 1$. Either way, the tableaux are amenable regardless of the markings of the last boxes of the second components of S_1 and S_2'. There are two possible markings for the last box of the second component of S_1 and there are two possible markings for the last box of the second component of S_2'. Thus, the coefficient for each Q_α and Q_β is $2 \cdot 2 = 4$. $\square$

Remark. An alternative proof of Lemma 6.68 can be obtained by using Lemma 1.71. The diagram $\Delta_2^{\leftarrow}(D_{\lambda/\mu})$ has two components where the first component is $D_{[a,1,0,0]}$ and the other component consists of two boxes in a row and the diagram $\Delta_2^{\downarrow}(D_{\lambda/\mu})$ has two components where the first component is $D_{[a,1,0,0]}$ and the other component consists of two boxes in a column. We obtain

$$Q_{\lambda/\mu} = Q_{\Delta_2^{\leftarrow}(D_{\lambda/\mu})} + Q_{\Delta_2^{\downarrow}(D_{\lambda/\mu})} = 2Q_{\Delta_2^{\leftarrow}(D_{\lambda/\mu})} = 2 \cdot (2Q_\alpha + 2Q_\beta) = 4Q_\alpha + 4Q_\beta$$

by Lemma 1.60 and Lemma 6.67.

Theorem 6.69. *Let $\lambda, \mu \in DP$ such that $D_{\lambda/\mu}$ is basic. The decomposition of $Q_{\lambda/\mu}$ consists of precisely two homogeneous components if and only if $D_{\lambda/\mu}$ satisfies one of the following conditions up to transposing and orthogonally transposing:*

(i) $\lambda = (a+b+c+d-1, a+b+c+d-2, \ldots, b+c+d+1, b+c+d, c+d-1, c+d-2, \ldots, d)$ *where $a, b, c, d > 0$ and $\mu = (1)$. Let* $\alpha = (a+b+c+d-1, a+b+c+d-2, \ldots,$

$b+c+d+2, b+c+d+1, b+c+d-1, c+d-1, c+d-2, \ldots, d+1, d)$ *and* $\beta = (a+b+c+d-1, a+b+c+d-2, \ldots, b+c+d+1, b+c+d, c+d-1, c+d-2, \ldots, d+2, d+1, d-1)$.

Then $Q_{\lambda/\mu} = Q_\alpha + Q_\beta$.

(ii) $\lambda = (a+b-1, a+b-2, \ldots, b)$ *where* $a \geq 2$, $b \geq 2$ *and* $\mu = (2)$. *Let* $\alpha = (a+b-1, a+b-2, \ldots, b+2, b+1, b-2)$ *and* $\beta = (a+b-1, a+b-2, \ldots, b+3, b+2, b, b-1)$.

Then $Q_{\lambda/\mu} = Q_\alpha + Q_\beta$.

(iii) $\lambda = (e+2, e+1, \ldots, 4, 3, 1)$ *where* $e \geq 2$ *and* $\mu = (2)$ *Let* $\alpha = (4,2)$ *and* $\beta = (3,2,1)$ *if* $e = 2$ *or let* $\alpha = (e+2, e+1, \ldots, 5, 4, 2)$ *and* $\beta = (e+2, e+1, \ldots, 5, 3, 2, 1)$ *if* $e \geq 3$.

Then $Q_{\lambda/\mu} = 2Q_\alpha + Q_\beta$.

(iv) $\lambda = (l+2, l)$ *where* $l \geq 2$ *and* $\mu = (2)$. *Let* $\alpha = (l+2, l-2)$ *and* $\beta = (l+1, l-1)$.

Then $Q_{\lambda/\mu} = Q_\alpha + 2Q_\beta$.

(v) $\lambda = (k+2, 1)$ *where* $k \geq 2$ *and* $\mu = (2)$. *Let* $\alpha = (k,1)$ *and* $\beta = (k+1)$.

Then $Q_{\lambda/\mu} = Q_\alpha + 2Q_\beta$.

(vi) $Q_{(4,1)/(2)} = Q_{(2,1)} + 2Q_{(3)}$,

$Q_{(5,3,1)/(4,2)} = 2Q_{(2,1)} + 4Q_{(3)}$,

$Q_{(5,1)/(2)} = Q_{(3,1)} + 2Q_{(4)}$,

$Q_{(5,3,1)/(3,2)} = 3Q_{(3,1)} + 2Q_{(4)}$,

$Q_{(5,2)/(3)} = 2Q_{(3,1)} + 2Q_{(4)}$,

$Q_{(6,3,1)/(4,2)} = 4Q_{(3,1)} + 4Q_{(4)}$,

$Q_{(7,5,3,1)/(6,4,2)} = 8Q_{(3,1)} + 8Q_{(4)}$.

(vii) $\lambda = (k, k-1, k-3, k-4, \ldots, 1)$ *and* $\mu = (k-2)$ *for some* $k \geq 3$. *Let* $\alpha = (k, k-2, k-4, k-5, \ldots, 1)$ *and* $\beta = (k-1, k-2, k-3, k-5, k-6, \ldots, 1)$ *for* $k \geq 4$

and $\alpha = (5,3,1)$ *and* $\beta = (4,3,2)$ *for* $k = 3$.

Then $Q_{\lambda/\mu} = 2Q_\alpha + 2Q_\beta$.

(viii) $\lambda = (c+d+1, c+d-1, c+d-2, \ldots, d)$ *and* $\mu = (c+d)$ *where* $d \geq 2$. *Let* $\alpha = (c+d-1, c+d-2, \ldots, d+1, d, 1)$ *and* $\beta = (c+d, c+d-2, c+d-3, \ldots, d+1, d)$.

Then $Q_{\lambda/\mu} = Q_\alpha + 2Q_\beta$.

(ix) $\lambda = (c+3, c+1, c, \ldots, 2)$ *and* $\mu = (c+2, 1)$. *Let* $\alpha = (c+2, c, c-1, \ldots, 3, 1)$ *and* $\beta = (c+1, c, \ldots, 3, 2)$.

Then $Q_{\lambda/\mu} = 2Q_\alpha + 2Q_\beta$.

(x) $\lambda = (c+2, c, c-1, \ldots, 1)$ *where* $c \geq 2$ *and* $\mu = (s+t-1, s+t-2, \ldots, t)$ *where* $1 < t \leq c$. *Let* $\alpha = (c+2, c, c-1, \ldots, s+t, t-1, t-2, \ldots, 1)$ *and* $\beta = (c+1, c, \ldots, s+t, t, t-2, t-3 \ldots, 1)$.

Then $Q_{\lambda/\mu} = Q_\alpha + 2Q_\beta$.

(xi) $\lambda = (c+2, c, c-1, \ldots, 1)$ *where* $c \geq 2$ *and* $\mu = (s, s-1, \ldots, 1)$. *Let* $\alpha = (c+2, c, c-1, \ldots, s+1)$ *and* $\beta = (c+1, c, \ldots, s+1, 1)$.

Then $Q_{\lambda/\mu} = Q_\alpha + Q_\beta$.

(xii) $\lambda = (a+b+c+2, a+b+c, a+b+c-1, \ldots, b+c+2, b+c+1, c, c-1, \ldots, 1)$ *and* $\mu = (a+b+c+1)$ *where* $a, b, c > 0$. *Let* $\alpha = (a+b+c+1, a+b+c-1, a+b+c-2, \ldots, b+c+1, c, c-1, \ldots, 2, 1)$ *and* $\beta = (a+b+c, a+b+c-1, \ldots, b+c+1, c+1, c-1, c-2, \ldots, 2, 1)$.

Then $Q_{\lambda/\mu} = 2Q_\alpha + 2Q_\beta$.

(xiii) $\lambda = (a+3, a, a-1, \ldots, 1)$ *and* $\mu = (a+1)$ *where* $a \geq 2$. *Let* $\alpha = (a+2, a-1, a-2, \ldots, 1)$ *and* $\beta = (a+1, a, a-2, a-3, \ldots, 1)$.

Then $Q_{\lambda/\mu} = 2Q_\alpha + 2Q_\beta$.

(xiv) $\lambda = (a+4, a+2, a, a-1, \ldots, 1)$ *and* $\mu = (a+3, a+1)$ *where* $a \geq 2$. *Let* $\alpha = (a+2, a-1, a-2, \ldots, 1)$ *and* $\beta = (a+1, a, a-2, a-3, \ldots, 1)$.

Then $Q_{\lambda/\mu} = 4Q_\alpha + 4Q_\beta$.

Some of these cases overlap.

Proof. Proposition 6.53 states that the skew Schur Q-functions that decomposes into precisely two homogeneous components are included in this list. Lemma 6.54 states that the decomposition of case (i) is true. Lemma 6.55 states that the decomposition of case (ii) is true. Lemma 6.56 states that the decomposition of case (iii) is true. Lemma 6.57 states that the decomposition of case (iv) is true. Lemma 6.58 states that the decomposition of case (v) is true. Lemma 6.59 states that the decomposition of case (vi) is true. Lemma 6.60 states that the decomposition of case (vii) is true. Lemma 6.61 states that the decomposition of case (viii) is true. Lemma 6.62 states that the decomposition of case (ix) is true. Lemma 6.64 states that the decomposition of case (x) is true. Lemma 6.65 states that the decomposition of case (xi) is true. Lemma 6.66 states that the decomposition of case (xii) is true. Lemma 6.67 states that the decomposition of case (xiii) is true. Lemma 6.68 states that the decomposition of case (xiv) is true. $\square$

7 Open problems and conjectures

As mentioned in Chapter 2 there are open problems but there are some conjectures concerning these problems. This chapter is about stating these conjectures and arguing why these conjectures are reasonable and what are the problems in proving them.

In Section 7.1 we want to find a shifted analogue of the inequalities given by Gutschwager [7, Theorem 3.1]. The desired statement we want to prove is Conjecture 7.1. We show what problems occur if one tries to prove this conjecture in the way Gutschwager did. Then we give some numerical data to support Conjecture 7.1.

Section 7.2 is about the number of amenable words of a given length n. We will use a shifted analogue of the Robinson-Schensted correspondence to mimic the proof for the classical case (see [19, Section 7.13.9]). This led to a conjecture for this number that was then proven algebraically (Proposition 7.9). But first, we describe a bijective approach similar to the classical one and discuss why this approach is not enough to prove the conjecture. Finally, we provide a high power of 2 dividing the number of amenable words of a given length.

7.1 Further inequalities of the coefficients $f^{\lambda}_{\mu\nu}$

As we could see (in particular in Chapter 6) inequalities for shifted Littlewood-Richardson coefficients can shorten and simplify proofs. Chapter 2 gives some inequalities for shifted Littlewood-Richardson coefficients $f^{\lambda}_{\mu\nu}$ that only change the first part of the corresponding partition ν. In this chapter we are interested to find such inequalities where not just the first part of the corresponding partition ν is changed.

Conjecture 7.1. *Let $\lambda, \mu, \nu \in DP$. Let a, b be such that $a \leq \ell(\lambda) + 1$, $b \leq \ell(\mu) + 1$ and $c = a - b \leq \ell(\nu)$. Then*

$$f^{\lambda}_{\mu\nu} \leq f^{\lambda+(1^a)}_{\mu+(1^b)\nu+(1^c)}.$$

Remark. Of course, if $f^{\lambda}_{\mu\nu} \neq 0$ then $c = a - b$, $a \leq \ell(\lambda) + 1$, $b \leq \ell(\mu) + 1$ and $c \leq \ell(\nu) + 1$ is necessary to have $f^{\lambda+(1^a)}_{\mu+(1^b)\nu+(1^c)} \neq 0$.

In [2, proof of Theorem 2.2] Bessenrodt showed the case $a = b \leq \ell(\mu) + 1$ and $c = 0$. Thus, the remaining case that needs to be considered is the case $a > b$. Lemma 2.8 shows that Conjecture 7.1 holds for $a = b + 1$.

The natural approach would be to add entries in the same way as Gutschwager does in the proof of [7, Theorem 3.1] in the classical setting. As usual, the shifted case is more complicated (as can be seen by the fact that we have upper bounds for the letters a, b and c in Conjecture 7.1 while there are no such upper bounds in the classical case) and problems occur that do not occur in the classical setting.

One problem is that added entries can be less than or equal to the entries in the box directly above. This only happens if some added entry ends up in the main diagonal $\{(x, x) \mid x \in \mathbb{N}\}$. This can be corrected to obtain an amenable tableau by replacing such entry with its marked version and sorting the columns (and possibly switching markings if the added entry is the leftmost entry in the reading word of the obtained diagram).

Example 7.2. *For $\lambda = (5, 4, 1)$, $\mu = (3, 1)$ and $\nu = (3, 2, 1)$ the tableau*

$$T = \begin{array}{|c|c|c|c|c|} \hline \times & \times & \times & 1 & 1 \\ \hline & \times & 1 & 2 & 2 \\ \hline & & 3 & & \\ \hline \end{array}$$

has shape $D_{\lambda/\mu}$ and content $c(T) = \nu$. For $a = 4$, $b = 2$ and $c = 2$ we obtain

$$T' = \begin{array}{|c|c|c|c|c|c|} \hline \times & \times & \times & \times & 1 & 1 \\ \hline & \times & \times & 1 & 2 & 2 \\ \hline & & \mathbf{1} & 3 & & \\ \hline & & & \mathbf{2} & & \\ \hline \end{array}$$

where the added entries are in boldface. Then we have $T'(4,4) = 2 < 3 = T'(3,4)$. We can obtain an amenable tableau if we do the following changes (highlighted in boldface):

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
\times & \times & \times & \times & 1 & 1 \\
\hline
 & \times & \times & 1 & 2 & 2 \\
\cline{2-6}
\multicolumn{1}{c}{} & & 1 & 3 & \multicolumn{2}{c}{} \\
\cline{3-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & & \mathbf{2'} & \multicolumn{2}{c}{} \\
\cline{4-4}
\end{array}
\;\rightarrow\;
\begin{array}{|c|c|c|c|c|c|}
\hline
\times & \times & \times & \times & 1 & 1 \\
\hline
 & \times & \times & 1 & 2 & 2 \\
\cline{2-6}
\multicolumn{1}{c}{} & & 1 & \mathbf{2'} & \multicolumn{2}{c}{} \\
\cline{3-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & & \mathbf{3} & \multicolumn{2}{c}{} \\
\cline{4-4}
\end{array}
\;\rightarrow\;
\begin{array}{|c|c|c|c|c|c|}
\hline
\times & \times & \times & \times & 1 & 1 \\
\hline
 & \times & \times & 1 & \mathbf{2'} & 2 \\
\cline{2-6}
\multicolumn{1}{c}{} & & 1 & \mathbf{2} & \multicolumn{2}{c}{} \\
\cline{3-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & & 3 & \multicolumn{2}{c}{} \\
\cline{4-4}
\end{array}.
$$

A much bigger problem is that some added entry k can violate the amenability of the obtained tableau T'. This happens if it is in a row such that there is an entry $(k+1)'$ in a row below and an entry k' in a row above, between these both entries the only entry from $\{k, (k+1)'\}$ is the added entry k and if the entry $(k+1)'$ is in the box $(x(j), y(j))$ then $m_k(n+j) = m_{k+1}(n+j)$ for $n = \ell(w(T'))$. This can only happen if also some $k+1$ has been added to the tableau, for otherwise we have $m_k(n+j) > m_{k+1}(n+j)$ for all $1 \le j \le n$. This is the reason why this problem does not appear in Lemma 2.8.

Example 7.3. *For $\lambda = (11, 10, 6, 4, 2)$, $\mu = (8, 5, 4, 2)$ and $\nu = (6, 5, 3)$ the tableau*

$$
T = \begin{array}{ccccccccccc}
\times & \times & \times & \times & \times & \times & \times & \times & 1 & 1 & 1 \\
 & \times & \times & \times & \times & \times & 1' & 1 & 2 & 2 & 2 \\
 & & \times & \times & \times & \times & 3' & 3 & & & \\
 & & & \times & \times & 2' & 3 & & & & \\
 & & & & 1 & 2 & & & & &
\end{array}
$$

has shape $D_{\lambda/\mu}$ and content $c(T) = \nu$. For $a = 4$, $b = 2$ and $c = 2$ we obtain

$$
T' = \begin{array}{cccccccccccc}
\times & \times & \times & \times & \times & \times & \times & \times & \times & 1 & 1 & 1 \\
 & \times & \times & \times & \times & \times & \times & 1' & 1 & 2 & 2 & 2 \\
 & & \times & \times & \times & \times & \mathbf{1} & 3' & 3 & & & \\
 & & & \times & \times & 2' & \mathbf{2} & 3 & & & & \\
 & & & & 1 & 2 & & & & & &
\end{array}
$$

where the added entries are in boldface. Then the reading word is given by $w = w(T') = 122'2313'31'1222111$ and $\ell(w) = 16$. The tableau T' is not amenable because $m_2(21) = 6 = m_1(21)$ and $w_6 = 1$.

Despite having problems mapping the amenable tableau T from Example 7.3 to an amenable tableau, Conjecture 7.1 still holds for these values of λ, μ, ν, a, b and c:

$$f^{(11,10,6,4,2)}_{(8,5,4,2)(6,5,3)} = 107 \leq 448 = f^{(12,11,7,5,2)}_{(9,6,4,2)(7,6,3)}.$$

Hence, Conjecture 7.1 seems to hold not only for the cases $a = b$ and $a = b + 1$. As an example we calculate the corresponding shifted Littlewood-Richardson coefficients for all possible values of a, b and c for $\lambda = (6, 4, 3)$, $\mu = (3, 1)$ and $\nu = (5, 3, 1)$.

Example 7.4. *For* $\lambda = (6, 4, 3)$, $\mu = (3, 1)$ *and* $\nu = (5, 3, 1)$ *we have* $1 \leq a \leq 4$, $1 \leq b \leq 3$ *and* $1 \leq c \leq 4$. *We have* $f^{(6,4,3)}_{(3,1)(5,3,1)} = 3$.

(a, b, c)	$\lambda + (1^a)$	$\mu + (1^b)$	$\nu + (1^c)$	value of $f^{\lambda+(1^a)}_{\mu+(1^b)\nu+(1^c)}$
$(1, 0, 1)$	$(7, 4, 3)$	$(3, 1)$	$(6, 3, 1)$	3
$(2, 1, 1)$	$(7, 5, 3)$	$(4, 1)$	$(6, 3, 1)$	5
$(3, 2, 1)$	$(7, 5, 4)$	$(4, 2)$	$(6, 3, 1)$	4
$(4, 3, 1)$	$(7, 5, 4, 1)$	$(4, 2, 1)$	$(6, 3, 1)$	5
$(2, 0, 2)$	$(7, 5, 3)$	$(3, 1)$	$(6, 4, 1)$	4
$(3, 1, 2)$	$(7, 5, 4)$	$(4, 1)$	$(6, 4, 1)$	3
$(4, 2, 2)$	$(7, 5, 4, 1)$	$(4, 2)$	$(6, 4, 1)$	8
$(3, 0, 3)$	$(7, 5, 4)$	$(3, 1)$	$(6, 4, 2)$	3
$(4, 1, 3)$	$(7, 5, 4, 1)$	$(4, 1)$	$(6, 4, 2)$	8
$(4, 0, 4)$	$(7, 5, 4, 1)$	$(3, 1)$	$(6, 4, 2, 1)$	3

As we can see, for all $\lambda+(1^a)$, $\mu+(1^b)$ *and* $\nu+(1^c)$ *we have* $f^{(6,4,3)}_{(3,1)(5,3,1)} \leq f^{\lambda+(1^a)}_{\mu+(1^b)\nu+(1^c)}$.

7.2 The number of amenable words of a given length

Another interesting problem is the number of amenable words of length n for some given n. The number of lattice words (or ballot sequences) appearing in the Littlewood-Richardson rule for Schur functions is well known (see `http://oeis.org/A000085`). It is equal to the number of involutions in S_n (see [19, Corollary 7.13.9 and the comment after its proof]) and is given by $\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2k)! \cdot 2^k \cdot k!}$. Note that the k-th summand is the length of the conjugacy class of S_n that has cycle type $(2^k, 1^{n-2k})$. This length can be obtained by dividing the order of the group S_n by the order of the centralizer of an element of the given cycle type (see [11, Chapter 12] by James and Liebeck).

Every amenable word of length n appears as a reading word for a tableau of the diagram with n components which all consist of a single box. By Lemma 1.16, the decomposition of the Q-function indexed by this diagram is equal to the decomposition of $Q_{(1)}^n$.

Definition 7.5. Let $n \in \mathbb{N}$ and $\lambda \in DP_n$. Let $a(n, \lambda)$ be the **number of amenable words of length** n **and content** λ. Let $a(n) := \sum_{\lambda \in DP_n} a(n, \lambda)$ be the **number of amenable words of length** n.

With help of the QF package for Maple made by Stembridge (`http://www.math.lsa.umich.edu/~jrs/maple.html`) the number $a(n)$ can be calculated by calculating the decomposition of $Q_{(1)}^n$ and then by replacing the constituents with 1 such that the sum of coefficients is obtained which is the number of amenable words of length n. Clearly, this method is inefficient and the calculation time increases vastly.

In the classical case the number of lattice words is obtained by giving a bijection between these words w and Standard Young Tableaux (SYT) T via the condition that if $w_i = j$ then there shall be a box filled with i in the j^{th} row. Using these SYT as (unshifted) tableaux P and Q in the Robinson-Schensted correspondence (see [19, Chapter 7.11] by Stanley) this correspondence provides a bijection between SYT with n entries and involutions in S_n (see [19, Corollary 7.13.9] by Stanley).

There exists a shifted analogue of the Robinson-Schensted correspondence due to Sagan [13] and Worley [24]. In this algorithm the tableau P is a shifted Standard Young Tableau (sSYT) and Q is an sSYT where entries that are not in the main diagonal $\{(x, x) \mid x \in \mathbb{N}\}$ can be marked. Let the set of such sSYT with marked entries but unmarked main diagonal of shape D_λ be $sSYT'(\lambda)$ and let $sSYT'(n) = \bigcup_{\lambda \in DP_n} sSYT'(\lambda)$. If some entries in Q are marked then we cannot have $P = Q$ as in the classical Robinson-Schensted correspondence. But if we set P to be the tableau obtained from Q by removing all markings then the possible pairings (P, Q) depend only on Q and, hence, the number of such pairings is the number of tableaux Q. In the classical case the number of pairings (P, Q) also depends only on the number of tableaux Q.

The number of tableaux Q in $sSYT'(n)$ can be obtained as follows. For every sSYT of shape D_λ there are $2^{|\lambda|-\ell(\lambda)}$ ways to mark some of the $|\lambda| - \ell(\lambda)$ letters that are not on the main diagonal. The number of sSYT of a given shape D_λ is denoted by g_λ and can be obtained by using the shifted hook formula (see [9, Proposition 10.6]) or using the formula [9, Proposition 10.4] given by $g_\lambda = \frac{n!}{\lambda!} \prod_{1 \le i < j \le \ell(\lambda)} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}$ where $\lambda! := \prod_{i=1}^{\ell(\lambda)} \lambda_i!$ that is due to Schur ([16, Proposition IX in §41, p. 235]). Then we have

$$|sSYT'(n)| = \sum_{\lambda \in DP_n} 2^{|\lambda|-\ell(\lambda)} g_\lambda.$$

Again using the QF package, these numbers can be calculated. Computations showed that the obtained numbers are equal for $1 \le n \le 29$ (see Figure 1 for the numbers). This led to the conjecture that $a(n) = \sum_{\lambda \in DP_n} 2^{|\lambda|-\ell(\lambda)} g_\lambda$. Calculating $|sSYT'(n)|$ for $1 \le n \le 29$ was a matter of a few minutes while calculating $a(n)$ took more than a day. Hence, it is desirable to prove that our conjecture is true.

In the proof for the classical case a lattice word with content λ can bijectively be mapped to an SYT of shape λ. We want to find an analogous map that maps amenable words w with content λ to $sSYT'(\lambda)$. A correlation between such a word w and tableaux from $sSYT'(\lambda)$ is that the leftmost i that appears in w must be unmarked and the leftmost letter in the i^{th} row must also be unmarked. Using this correlation, a natural map Ψ is to scan w from right to left and add the box with entry i in the j^{th} row if $w_{n-i} \in \{j', j\}$ and then mark the l^{th} entry of the k^{th} row if the l^{th} entry of $w|_{\{k',k\}}$ is marked.

Example 7.6. *Let* $w = 212'1'1$ *which is an amenable word of length* 5. *We obtain the tableau* $\Psi(w) = \begin{array}{|c|c|c|} \hline 1 & 2' & 4 \\ \hline \end{array}$ over $\begin{array}{c|c|c|} & 3 & 5' \\ \hline \end{array}$.

It is easy to see how to obtain w for some given $\Psi(w)$. Clearly, this gives a bijection between amenable words with content (n) and the set $sSYT'((n))$ (the set of such tableaux of the partition (n) that has only one part).

Let $n_i(j)$ be the number of letters from $\{i', i\}$ in $w_{n-j+1}w_{n-j+2}\dots w_n$. The map Ψ does not map to a tableaux of $sSYT'(c(w))$ if for some $j \in \{1, 2, \dots \ell(w)\}$ we have $w_{n-j} \in \{(i+1)', i+1\}$ and $n_i(j) \leq n_{i+1}(j) + 1$. This follows from the fact that if $n_i(j) = n_{i+1}(j) + 1$ and $w_{n-j} \in \{(i+1)', i+1\}$ then in the $(i+1)^{\text{th}}$ row there is a box with entry j' or j and the box directly above will be filled with a greater entry. The following example depicts this fact.

Example 7.7. *The word* $w = 121$ *gives the filling* $\Psi(121) = \begin{array}{cc} \boxed{1} & \boxed{3} \\ & \boxed{2} \end{array}$ *which is not a tableau. The only other amenable word with content* $(2, 1)$ *is* $w' = 211$ *and we have* $\Psi(211) = \begin{array}{cc} \boxed{1} & \boxed{2} \\ & \boxed{3} \end{array}$. *We have* $sSYT'((2, 1)) = \{\begin{array}{cc} \boxed{1} & \boxed{2} \\ & \boxed{3} \end{array}, \begin{array}{cc} \boxed{1} & \boxed{2'} \\ & \boxed{3} \end{array}\}$.

As we see in Example 7.7 the number $a(3, (2, 1))$ is equal to the number of tableaux in $sSYT'((2, 1))$ but the word 121 should be mapped to the tableau $\begin{array}{cc} \boxed{1} & \boxed{2'} \\ & \boxed{3} \end{array}$. Also, if we set $\Psi(w) = \begin{array}{cc} \boxed{1} & \boxed{2'} \\ & \boxed{3} \end{array}$ then we obtain $w = 211'$ which is not amenable by Lemma 1.39. It is an open problem to find a modification of Ψ such that each amenable word of length n and content λ is mapped to a tableau from $sSYT'(\lambda)$. However, if such a modification is found then it provides a combinatorial proof of both statements in Proposition 7.8 below.

It remains open to give a shifted analogue of the bijection between lattice words and standard Young tableaux to mimic the bijective proof of the classical case.

However, a short algebraic proof based on results in Stembridge's paper [22] was found by Bessenrodt [3]. Hence we can state the following result.

Proposition 7.8. *We have*

$$a(n, \lambda) = |sSYT'(\lambda)| = 2^{|\lambda| - \ell(\lambda)} g_\lambda$$

and, therefore,

$$a(n) = \sum_{\lambda \in DP_n} 2^{|\lambda| - \ell(\lambda)} g_\lambda.$$

As closing statement we will give a factor of the numbers $a(n)$ and $a(n, \lambda)$.

Proposition 7.9. *Let $n \in \mathbb{N}$ and $\lambda \in DP_n$. Let $c(n) = \prod_{i=1}^{\infty} 2^{d(i)}$ where $d(i) = \lfloor \frac{n}{2^i} \rfloor$.*

Then $c(n)$ is a factor of $a(n, \lambda)$ and in particular $c(n)$ is a factor of $a(n)$.

Proof. For

$$Q_{(1)}^n = (Q_{(1)}^2)^{d(1)} \cdot Q_{(1)}^{2(\frac{n}{2} - d(1))}$$

we can use Lemma 1.72 for each $Q_{(1)}^2$. This means

$$Q_{(1)}^n = 2^{d(1)} Q_{(2)}^{d(1)} \cdot Q_{(1)}^{2(\frac{n}{2} - d(1))}.$$

But

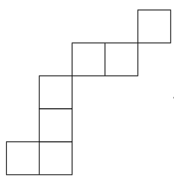
$$Q_{(2)}^{d(1)} = (Q_{(2)}^2)^{d(2)} \cdot Q_{(2)}^{2(\frac{d(1)}{2} - d(2))}.$$

Then we may use Lemma 1.72 again for each $Q_{(2)}^2$ to obtain

$$Q_{(1)}^n = 2^{d(1)} \cdot 2^{d(2)} Q_{(4,3,2)/(3,2)}^{d(2)} \cdot Q_{(2)}^{2(\frac{d(1)}{2} - d(2))} \cdot Q_{(1)}^{2(\frac{n}{2} - d(1))}.$$

Repeating this argument over and over, we obtain $Q_{(1)}^n = c(n) \cdot Q_{\hat{D}}$ for some diagram $\hat{D}$ and the statement follows. □

Example 7.10. *For $n = 7$ we have $(Q_1)^7 = 16 \cdot Q_D = 16 \cdot (4Q_{(7)} + 10Q_{(6,1)} + 18Q_{(5,2)} + 10Q_{(4,3)} + 7Q_{(4,2,1)})$ where*

$D =$  .

Remark. The number $c(n)$ is the largest power of two that is a factor of $n!$ and is equal to $2^{t(n)}$ where $t(n) = n -$ number of non-zero summands in the 2-adic expansion.

See Figure 1 for the numbers $\frac{a(n)}{c(n)}$ for $1 \leq n \leq 29$.

n	$a(n)$	$\frac{a(n)}{c(n)}$
1	1	1
2	2	1
3	6	3
4	16	2
5	56	7
6	192	12
7	784	49
8	3200	25
9	14464	113
10	66560	260
11	326656	1276
12	1656832	1618
13	8776704	8571
14	48304128	23586
15	274083840	133830
16	1613561856	49242
17	9724035072	296754
18	60597796864	924649
19	385368260608	5880253
20	2525806198784	9635186
21	16873294659584	64366511
22	115812134289408	220894116
23	809558929833984	1544111118
24	5797011295043584	1382115196
25	42242383802269696	10071369124
26	314466188543393792	37487290924
27	2380321071178973184	283756383798
28	18364956037989007360	547318340480
29	143971055333544034304	4290671805547

Figure 1: The numbers $a(n)$ and the numbers $\frac{a(n)}{c(n)}$ for $1 \leq n \leq 29$.

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