Flux Backreaction in Supergravity: Heterotic Calabi-Yau Compactifications and New Type II Anti-de-Sitter Vacua

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Fabio Apruzzi
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Korreferent: Prof. Daniel Waldram.

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Kurzzusammenfassung

In der vorliegenden Arbeit werden die Rückkopplungen von Flüssen in bestimmten Hintergrundgeometrien von ausgewählten heterotischen und Typ II Stringkompaktifizierungen systematisch untersucht. In heterotischen Calabi-Yau-Kompaktifizierungen sind *Wilson Lines* ein Grundbaustein um Standard Modell ähnliche effektive Theorien zu konstruieren. Wilson Lines sind dabei nicht-triviale, flache Zusammenhänge auf nicht einfach zusammenhängenden Räumen. Zum Beispiel können 3-Zykeln eines nicht einfach zusammenhängenden Calabi-Yau Raumes Wilson Lines definieren. In diesem Fall können Wilson Lines einen nicht-trivialen H-Fluss mittels des Chern-Simons-Terms induzieren und die Rückkopplung des H-Flusses kann die Widerspruchsfreiheit der Hintergrundgeometrie in führender Ordnung beeinflussen. Außerdem können H-Flüsse in heterotischen Kompaktifizierungen eine entscheidende Rolle bei der Moduli Stabilisierung spielen sowie die Supersymmetriebrechungsskala stark einschränken.

Im ersten Teil dieser Arbeit zeige ich wie man, ausgehend von einer Wahl einer Wilson Line in *Complete Intersection Calabi-Yau* Mannigfaltigkeiten, den H-Fluss sowie das zugehörige Superpotential explizit konstruiert. Dies erfolgt durch die Identifikation einer großen Klasse von sLags (*special Lagrangian submanifolds*) innerhalb der Calabi-Yau Mannigfaltigkeit, dem Verständnis wie sich die flachen Zusammenhänge auf den dreidimensionalen sLags verhalten und der Berechnung der Chern-Simons Invarianten. Um die benutzten Methoden zu verdeutlichen, betrachte ich die *quintic hypersurface* sowie die *split-bicubic*, da diese potentiell realistische Drei-Generationen-Modelle erlauben.

Im darauf folgenden zweiten Teil beschränke ich mich hauptsächlich auf höher-dimensionale ($d > 4$) $AdS_d \times M_{10-d}$ Vakua in Typ II Supergravitation mit allen erlaubten (RR und NSNS) Flüssen. Dabei ermittle ich explizit die Geometrie der internen Mannigfaltigkeit M_{10-d} unter Berücksichtigung der Rückkopplung der Flüsse. Dank der AdS/CFT-Korrespondenz können mittels der Anti-de-Sitter Vakua einige Aspekte von stark gekoppelten und höher-dimensionalen ($d \geq 4$) Systemen untersucht werden, welche durch konforme Feldtheorie beschrieben werden. Diese Feldtheorien sind nur schwer zugänglich mittels der Standard QFT-Methoden.

Für den Hintergrund $AdS_6 \times M_4$ in Typ II Supergravitation sind bis jetzt nur wenige Lösungen bekannt: eine eindeutige in Typ IIA sowie zwei dazu duale Lösungen in Typ IIB. Ich reduziere die Supersymmetriegleichungen auf zwei partielle Differentialgleichungen. Geometrisch ist M_4 eine S^2-Faserung über einem zweidimensionalen Raum Σ.

Für den Fall $AdS_7 \times M_3$ zeige ich zum einen das keine supersymmetrischen Lösungen in Typ IIB existieren und zum anderen klassifiziere ich alle supersymmetrischen Kompaktifizierungen (massiv und masselos) in Typ IIA, wobei unendlich viele neue Lösungen entdeckt werden. Topologisch gleicht M_3 dabei einer 3-Sphäre, aber die Geometrie wird durch Quellen von D6-Branen/O6-Ebenen an den Polen sowie von D8-/O8-Ebenen, welche 2-Sphären innerhalb von M_3 aufwickeln, deformiert. Obwohl die Hintergründe mit Quellen von D6-Branen/O6-Ebenen an den Polen einige Singularitäten aufweisen, habe ich eine Klasse von Lösungen gefunden, welche vollständig regulär ist und nur Stapel von D8-Branen beinhaltet. Die globale Geometrie von M_3 sowie die Gegenwart von regulären supersymmetrischen Vakua mit D8-Branen klärt einige Aspekte zu Stabilität und Auflösungen von Hintergründen mit D6-Branen an den Polen auf. Diese Sachverhalte konnten bis dahin nur lokal studiert werden.

Zum Ende analysiere ich das supersymmetrische System von Gleichungen für $AdS_5 \times M_5$ in Typ IIA Supergravitation. Das Problem kann auf mehrere partielle Differentialgleichungen reduziert werden und ich finde einige bereits bekannte Lösungen für massenlose Typ IIA Supergravitation wieder. Die kompakte interne Mannigfaltigkeit M_5 entspricht einer M_3-Faserung über einer Riemannschen Fläche, wobei M_3 eine dreidimensionale Mannigfaltigkeit ist. Ein Unterklasse dieser Vakua steht in Bijektion mit den $AdS_7 \times S^3$ Vakua, welche ich bereits vorher in massiver/massloser Typ IIA Supergravitation gefunden habe. Nebenbei erhalte ich eine analytische Version dieser Lösungen, welche vorher nur numerische im AdS_7-Rahmen bekannt waren. Diese analytischen Lösungen erlauben die Berechnung der freien Energie in einigen Beispielen für die dualen konformen Feldtheorien mittels der AdS/CFT-Regeln.

Schlüsselwörter: STRINGTHEORIE, FLÜSSEN, KOMPAKTIFIZIERUNGEN

Abstract

In this thesis, we systematically analyze the issue of the backreaction of fluxes on the background geometry of certain compactifications of heterotic and type II string theory. In heterotic Calabi-Yau compactifications, Wilson lines are basic ingredients for the construction of Standard Model-like effective theories. Wilson lines are non-trivial flat gauge connections on a non-simply connected space. Three-cycles of a non-simply connected Calabi-Yau can sometimes support Wilson lines. If this is the case, they can induce a non-trivial H-flux via the Chern-Simons term, and the backreaction of the H-flux may affect the consistency of the leading order background geometry. On the other hand, the H-flux in heterotic compactifications can also play a crucial role for moduli stabilization and could strongly constrain the supersymmetry breaking scale.

In the first part of this thesis, we show how to explicitly derive the H-flux and the corresponding superpotential given a choice of Wilson lines in complete intersection Calabi-Yau manifolds. We do so by identifying a large class of special Lagrangian submanifolds (sLags) in the Calabi-Yau, understanding how the flat gauge connection is supported on these three-dimensional submanifolds and computing the Chern-Simons invariants. We illustrate our methods with the quintic hypersurface as well as the split-bicubic, which provides a potentially realistic three generation model.

In the second part of the thesis, we are mainly focused on higher dimensional ($d > 4$) $\mathrm{AdS}_d \times M_{10-d}$ vacua of type II supergravity with all the fluxes (RR and NSNS) turned on. We explicitly determine the geometry of the internal manifold M_{10-d} taking the backreaction of the fluxes into account. Anti-de-Sitter vacua are very useful for probing, via the AdS/CFT correspondence, some aspects of strongly coupled and higher-dimensional ($d \geq 4$) systems governed by conformal field theories, which are very difficult to study with standard QFT techniques. We start from $\mathrm{AdS}_6 \times M_4$ backgrounds in type II supergravity, where few solutions are already known: a unique one in type IIA, and two type IIB solutions dual to it. We reduce the supersymmetry equations of type IIB supergravity to two PDEs. The geometry of M_4 is given by an S^2-fiberation over a two-dimensional space Σ.

Regarding $\mathrm{AdS}_7 \times M_3$ backgrounds, we show that there are no supersymmetric solutions in type IIB, whereas we classify all the supersymmetric $\mathrm{AdS}_7 \times M_3$ compactifications in type IIA (massive and massless), discovering infinitely many new solutions. The topology of M_3 is that of an S^3, and its geometry gets distorted by D6-brane/O6-planes sources at the poles and D8-branes/O8-planes wrapping S^2's inside M_3. Whereas the backgrounds with D6-branes/O6-planes at the poles manifest singularities, we found a class of completely regular solutions with only stacks of D8-branes. The global geometry of M_3, together with the presence of regular supersymmetric vacua with D8-branes, clarifies some issues on the stability and the resolutions of backgrounds, which were locally studied before near D6 sources.

Finally, we analyze the supersymmetric system of equations for $\mathrm{AdS}_5 \times M_5$ backgrounds in type IIA supergravity. We reduce the problem to several PDEs and recover some already known solutions of type IIA supergravity. The geometry of the compact internal space, M_5, is a three-dimensional manifold, M_3, fibered over a Riemann surface. A subclass of these vacua that is in one-to-one correspondence with the $\mathrm{AdS}_7 \times S^3$ vacua mentioned above. As a by-product, we obtain the analytic version of such solutions, which, in the AdS_7 case, were just known numerically. The analytic expressions allow us to explicitly compute some examples of free energy for the dual conformal field theories by using the AdS/CFT dictionary.

Keywords: STRING THEORY, FLUXES, COMPACTIFICATION

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Contents

Introduction

General relativity and quantum field theory are the cornerstones of modern theoretical physics. In fact, they provide the theoretical frameworks in which all observed fundamental interaction processes can be described with astonishing accuracy. On one hand, general relativity describes the geometry of our space-time at large distances. On the other hand, quantum field theory is an experimentally well-tested framework for the short distance behaviour of the microscopic objects of our universe and it describes the non-gravitational fundamental interactions between elementary particles. Both theories are nowadays largely confirmed by high-precision experiments. However, they are not compatible at energy/distance regimes where both gravity and quantum effects become significant, as is the case, e.g., in the vicinity of black hole singularities or in the early universe shortly after the big bang.

In the last decades, many efforts in theoretical high energy physics have been devoted to the quest for a unified and quantum description of all the known interactions, including gravity. Finding a quantum description of gravity would mean to develop a theory that includes the geometry of the space-time itself as a dynamical quantum variable. Moreover, in a unified theory of interactions the degrees of freedom that encode the geometry can mix with the matter and gauge fields at a quantum level and at high energies.

String theory is a promising candidate for an unified theory of all the forces, including gravity. It describes vibrating one-dimensional objects (strings) moving in a ten-dimensional space-time. The quantization of such string gives a spectrum which contains an infinite tower of fields with higher and higher masses and spins. For this reason, unfortunately, a second quantized (or path-integral) formulation of this theory is technically very difficult. Due to the relevance for realistic experiments, however, it is primarily that the massless particles and their scattering amplitudes are of most direct physical interest. This is in particular true for the Graviton, which arises as a massless spin 2 excitation of the quantized closed string. Moreover, it is crucial that, in order to have fermions in the string spectrum, we need to consider string theory with an additional ingredient: supersymmetry. The effective theory that consistently captures the dynamic of the massless spectrum of the supersymmetric string (or superstring) is ten-dimensional supergravity, which is the low-energy limit of supersymmetric string theory (Actually there are five different perturbatively consistent superstring theories: type I, type IIA/B, SO(32) and $E_8 \times E_8$ heterotic with low-energy limit described by ten-dimensional supergravities). On the other hand, ten-dimensional supergravity does not describe the complete string theory set-up; for instance, one cannot study directly non-perturbative phenomena with this low-energy approximation. However, due to the presence of supersymmetry and to the non-linearity of supergravity, one can still probe non-perturbative degrees of freedom and higher order effects.

So far, we have described string theory as a nice mathematical model, but one of the main questions that physicists should ask is: can it describe the real world? A consistent supersymmetric string quantization requires a ten-dimensional space-time, which is obviously incompatible with the four dimensions we observe. In fact, the standard procedure for resolving this apparent contradiction consists in compactifing supergravity from ten dimensions to four dimensions, by assuming that six of the ten dimensions form a compact and small manifold. Studying the underling compact geometry of string theory backgrounds as well as the small deformations around these classical solutions allow one to develop a toolkit for "string phenomenology", which attempts to bridge the gap between mathematics and measurable physics. For example, one of the feature, one would like to obtain in the effective four-dimensional theory, is the gauge group of the standard model. Another important feature of the compactification procedure is that different geometries could lead to different properties of the physical effective theories in four dimensions. Moreover, from a generic compactification one obtains a lot of unwanted degrees of freedom, such us exotic fields or additional scalar fields, which need to be sufficiently decoupled from the standard model degrees of freedom. A very natural framework for a four-dimensional effective gauge theory is the $\mathrm{E}_8\times\ \mathrm{E}_8$ heterotic string, which contains automatically a gauge group compatible with the ideas of Grand Unification Theories (GUTs). There are also attempts to construct realistic models from type IIA/B with D-branes (which are higher-dimensional extended objects on which the strings can end) carrying a gauge group. More recently, phenomenologically interesting models have also been developed in M/F-theory (eleven/twelve-dimensional theories), which are non-perturbative extensions of superstring theory. To conclude, the past decades have seen mutual interplay between sophisticated techniques of algebraic/differential geometry and string theory, driven by the types of above mentioned geometrical set-ups that string theory requires.

Geometry appears to be a very useful tool to study also strongly coupled systems. Strongly interacting quantum field theories appear many times in physics. For instance, a very relevant example is QCD, the SU(3) gauge theory which describes the strong interactions between quarks. There are strongly coupled systems in condensed matter physics as well. More abstractly, there is a theoretical interest in higher dimensional field theories, $d > 3$, including conformally invariant field theories. In fact, these theories turn out to be good candidates for modelling the world-volume dynamic of the D/M-branes and other extended objects, which frequently appears in String/M-Theory. For instance, M5-branes on top of each other are believed to be governed by a six-dimensional conformal field theory. Moreover, it is sometimes considered that many QFTs in different dimensions descend from supersymmetric conformal field theories (sCFTs), by the process of compactification on certain spaces, by Renormalization Group (RG) flow and by symmetry breaking. Very often these kind of theories are very difficult to be explicitly analyzed, and sometimes they even lack a Lagrangian description. Here is were the AdS/CFT correspondence plays a crucial role; in fact, it is a conjecture that provides a very useful tool to study strongly coupled system. This correspondence is believed to be a subclass of more general set of dualities between quantum gravity and gauge theories. Its original and best understood formulation relates strongly coupled conformal theories with weakly coupled gravitational theories on asymptotically AdS space-time. Every entity in one theory has a counterpart in the other theory. For example, a single field in the gravitational theory corresponds to a certain operator in the conformal theory. Many difficult computations in field theory can now be related to much simpler ones in a weakly coupled gravitational theory, i.e. there is a "dictionary" between various quantities of the two different theories. Moreover, the conformal field theory can be viewed as living on the conformal boundary of the anti-de-Sitter space. This boundary is one dimension less than the anti-de Sitter space itself, which is the reason why AdS/CFT is often described as a "holographic duality". It is generally believed also that the existence of a certain AdS background of a quantum theory of gravity, such as string theory, means that there is a dual conformal field theory. It is then interesting to find and classify AdS solutions of string theory. In order to do so, we will analyze classes of AdS backgrounds of ten-dimensional supergravity using powerful techniques of differential geometry.

In Calabi-Yau compactifications to Minkowski space, as well as compactifications to AdS external space, fluxes and their backreaction on the geometry play a crucial role in what we have discussed so far, and they will be the main common topic throughout this thesis. Fluxes are non-trivial background field strengths of certain massless bosonic field coming from the first quantization of the various string theories. In heterotic supergravity (the low energy effective field theory of the heterotic string), it is important to control the H-flux given by the Chern-Simons term of a gauge connection in $\mathrm{E}_8\times\mathrm{E}_8$, because it can backreact and spoil the Calabi-Yau condition, which comes together with vanishing H-flux. This is different in IIB supergravity, where certain fluxes $(F_3,\ H)$ can co-exist with a conformal Calabi-Yau space. In AdS compactifications, it is important and interesting to introduce all the backreacting fluxes and classify the underling geometry in order to get new types of vacua, believed to be dual to new CFTs. Moreover, fluxes are closely related to branes of various kinds. As a by-product of studying systematically supersymmetric AdS vacua with fluxes and branes, we will give new insights into the issue of backreaction and stability of such solutions, which has recently been a subject of intense debate.

Outline of the thesis: In chapter 1 of this thesis, we will give an overview of ten-dimensional supergravity, in order to introduce some basic concepts and fix the notation. In chapter 2 we will briefly review some basics on the geometrical techniques that we use in this thesis. The rest of the thesis is divided in two parts, where the distinction is made according to the two main physical applications of the geometrical toolkit: phenomenology and AdS/CFT. Although they are quite different topic, they share the issue of the backreaction of fluxes on the geometry of some low-energy solutions. In the first part, we deal with the consistency of having H-flux in heterotic Calabi-Yau compactifications, and in the second part, we look for new higher-dimensional ($d > 4$) AdS backgrounds with all fluxes (NSNS and RR) turned on.

In chapter 3, which is work based on [1], we address one of the problem that occurs in $\mathrm{E}_8\times$ E_8 heterotic model building. Namely, we study to what extent Wilson lines in heterotic Calabi-Yau compactifications lead to non-trivial H-flux via Chern-Simons terms. Wilson lines are basic ingredients for Standard Model constructions, but their induced H-flux may have a backreaction on the Calabi-Yau geometry, affecting the consistency of the leading order background and of the two-dimensional worldsheet theory. Moreover, H-flux in heterotic compactifications also plays an important role for moduli stabilization and could strongly constrain the supersymmetry breaking scale. The main result of chapter 3 is that we show how to compute H-flux and the corresponding superpotential, given an explicit complete intersection Calabi-Yau compactification and choice of Wilson lines. We do so by identifying large classes of special Lagrangian submanifolds in the Calabi-Yau, understanding how the Wilson lines project onto these submanifolds, and computing their Chern-Simons invariants. We illustrate our procedure with the quintic hypersurface as well as the split-bicubic, which can provide a potentially realistic three generation model, and where the backreaction of Wilson lines via the Chern-Simons flux was not know before.

In the rest of the thesis, we present new solutions in type II supergravity with an external AdS space, and where the internal geometry is non-Calabi-Yau consistent with the full backreaction of all the fluxes. One of the main motivations for investigating in these types of vacua is to probe supersymmetric conformal field theories in various dimensions as well as gain new insights into the issue of backreaction and stability solutions with fluxes. In chapter 4, which is based on [2], we will start begin by studying the supersymmetric system of equations for $\mathrm{AdS}_6 \times M_4$ in type IIB supergravity. Very few supersymmetric solutions are known: one in massive IIA supergravity, and two IIB solutions dual to it. The IIA solution is known to be unique. We use the pure spinor approach to give a classification for IIB supergravity, where only some solutions are known, and a complete classification is missing. The main achievement of chapter 4 is the reduction of the pure spinors system to two PDEs on a two-dimensional space Σ. The classification is given by means of the two PDEs; in the sense that each supersymmetric AdS_6 solution of type IIB supergravity is a solution of our system of PDE's. The internal geometry, M_4, is a fiberation of S^2 over Σ; the metric and fluxes are completely determined in terms of the solution to the PDEs. The results seem likely to accommodate near-horizon limits of (p,q)-fivebrane webs studied in the literature to generate five-dimensional CFTs. We also demonstrate that there are no AdS_6 solutions in eleven-dimensional supergravity. We started with the AdS_6 vacua also because they allow us to introduce the pure spinor techniques reminiscent of generalized complex geometry, which will be extensively used in the next chapters, 5 and 6. The new and explicit result of chapter 5, which is based on [3], is the construction and classification of new supersymmetric solutions of the type $\mathrm{AdS}_7 \times M_3$ in type II supergravity. In M-theory, the only supersymmetric AdS_7 backgrounds are $\mathrm{AdS}_7 \times S^4$ and its orbifolds. We show that in IIB no such vacua exist, whereas in IIA with Romans mass (which does not lift to M-theory) we find many new ones. Without the need for any ansatz, the system determines uniquely the form of the metric and fluxes, up to solving a system of ODEs. The metric on M_3 is that of an S^2 fibered over an interval, topologically $M_3 \cong S^3$; this is consistent with the Sp(1) R-symmetry of the holographically dual $(1,0)$ theories. By including D8-brane sources, one can numerically obtain regular solutions. Finally, in chapter 6, which is based on [4, 5], we classify AdS_5 solutions in massive IIA supergravity, finding infinitely many new analytical examples. The first main result is the reduction of the general problem to a set of PDEs, determining the local internal metric, which is a fiberation over a surface. Under a certain simplifying assumption, we are then able to analytically solve the PDEs and give a complete list of all solutions. Among these, one class is new and regular. These spaces can be related to the

AdS_7 solutions mentioned above, via a simple universal map for the metric, the dilaton and the fluxes. The natural interpretation of this map is that the dual CFT_6 and CFT_4 are related by twisted compactification on a Riemann surface Σ_g. The ratio of their free energy coefficients is proportional to the Euler characteristic of Σ_g. The second explicit and new result is a by-product of our analysis, namely an analytic expression for the AdS_7 solutions mentioned above, which were previously known only numerically. We determine the free energy for simple examples: it is a simple cubic function of the flux integers.

Chapter 1

Supergravity: the low-energy theory of superstrings

There are five perturbatively consistent superstring theories with low-energy limits described by ten-dimensional supergravities. In this thesis, we will be mainly focused on three types of low-energy effective theories: heterotic and IIA/B supergravity. For completeness, we now briefly explain what we mean by the low-energy limit of superstring theory and then discuss the field content of these effective theories as well as the corresponding supersymmetry variations.

1.1 The low-energy limit and effective theories

A string is the generalization of a point particles to a one-dimensional object with an assumed length scale, l_s, below the current experimental resolution, perhaps of the same order as the Planck Length, $l_s \sim l_p$. The bosonic string has an action which is given by the area of the world-sheet swept out by the string embedded in an external D-dimensional target space-time. This so-called Nambu-Goto action can equivalently be rewritten in terms of a standard non-linear sigma model action (the Polyakov action). The bosonic string can be closed or open with different boundary conditions (Dirichlet and Neumann boundary condition) at the end points. Moreover, in string theory, the possible boundary loci for the open strings define new extended higher dimensional objects, called D-branes. The excitation modes of the open strings ending on D-branes can be viewed as degrees of freedom living on the D-brane world-volume. The key feature of the non-linear sigma model for the bosonic string is its invariance under conformal transformations, which makes the quantization of the theory consistent. This conformal symmetry has a quantum anomaly, unless, the dimension of the target space, D, is 26, which fixes the number of space-time dimensions for a consistent bosonic string theory, [6–9].

The bosonic string contains only bosons in the spectrum, one of which is a tachyon of negative mass-square. Superstring theory is its extension, and it includes fermions as superpartners of all the bosonic fields. Superstrings are perturbatively described by two-dimensional superconformal theories embedded in a D-dimensional target space-time, which have no tachyon in the spectrum. Here again the vanishing of the superconformal anomaly restricts the dimension, this time to $D = 10$. There are different types of supersymmetric string theories depending on the different type of world-sheet content of fields. The five perturbatively consistent superstring theories are type I, type II A/B, $E_8 \times E_8$ and SO(32) heterotic. All the five superstring theories are related by dualities, for example, T-duality between type IIA and type IIB physically relates the two theories when compactified, respectively, on a circle of radius R and $1/R$. There are other types of dualities among the superstring theories, such as S-duality, however, they are not relevant for this thesis and we will not review them here, (see [6–9]).

In particular, all the superstring theories have a finite massless spectrum, which contain also a spin 2 particle, the graviton, whereas the other infinite excitations have mass which is $M \geq \frac{1}{\sqrt{\alpha'}}$, where $\alpha' = l_s^2$ is the Regge-slope parameter. The low-energy limit of superstring theory is defined as the effective field theory that describes the low-energy interactions of the massless modes in the ten-dimensional space-time. In fact, at low-energies, the massive states are too heavy to be observable, and hence, they are decoupled. The low-energy theories of superstrings are supersymmetric and contain ten-dimensional gravity and for this reason they are called ten-dimensional supergravities. As they contain a finite number of fields they are more easy to handle, although they are just an effective approximation of the entire string framework. Finally, there is also an eleven-dimensional supergravity, which is related with the ten-dimensional ones by dimensional reduction along a circle and by dualities. This theory is believed to be the low-energy limit of M-theory, which should represent the non-perturbative completion of the superstring theories.

In the following subsections, we will describe the content of fields for ten-dimensional $E_8 \times E_8$ heterotic and type IIA/B supergravity together with their effective actions.

1.2 Type II supergravity

The massless spectrum of type II supergravity is given by two fermions fields, the gravitino and the dilatino:

$$\psi^a_M \text{ (gravitino)}, \quad \lambda^a \text{ (dilatino)} \quad \begin{cases} \text{IIA} & a = 1,2 \quad \text{opposite chirality} \\ \text{IIB} & a = 1,2 \quad \text{same chirality} \end{cases} \tag{1.2.1}$$

where $M = 0, ..., 9$, in type IIA: ψ^1_M, λ^1 have chiralities $+$; ψ^2_M, λ^2 have chiralities $-$ and in type IIB all have positive chiralities. Furthermore, they satisfy the Majorana (reality) condition:

$(\psi_M^a)^* = \psi_M^a$, $(\lambda^a)^* = \lambda^a$.

We have some gauge bosons, called NSNS fields: the dilaton, the graviton (metric) and the B-field

$$\phi \text{ (dilaton)}, \quad g_{MN} \text{ (metric)}, \quad B_{MN} \text{ (2-form NSNS field)}. \tag{1.2.2}$$

Other bosons called Ramond-Ramond fields:

$$C_p \text{ (p-form RR field)}. \tag{1.2.3}$$

The field strengths of the C-fields are

$$F_{p+1} = dC_p - H \wedge C_{p-2} \text{ (p-form RR flux)}, \tag{1.2.4}$$

where H is the field strength of the B-field:

$$H = dB \text{ (3-form NSNS flux)}. \tag{1.2.5}$$

These fields and their strengths, as differential form, have hodge dualities, namely

$$F_k = (-1)^{\lfloor \frac{k}{2} \rfloor} *_{10} F_{10-k} \iff \lambda F_k = *_{10} F_{10-k}. \tag{1.2.6}$$

where the λ is an operator which acts on forms as follows $\lambda\omega_k = (-1)^{\lfloor \frac{k}{2} \rfloor}\omega_k$, for each k-form ω_k. The indices p takes odd integer values for type IIA and even for type IIB:

$$p = \begin{cases} 1,3,5,7,9 & \text{IIA} \\ 0,2,4,6,8 & \text{IIB} \end{cases} \tag{1.2.7}$$

Finally, for simplicity of calculation, we can recollect all forms and field strengths in the following fashion, defining the so-called polyforms:

$$C = \sum_p C_p \tag{1.2.8}$$

and poliflux F

$$F = \sum_k F_k \quad \text{is such that} \quad F = \lambda * F. \tag{1.2.9}$$

In type IIA supergravity, we have a pair of 16-components Majorana-Weyl gravitinos of opposite chiralities and a pair of two Majorana-Weyl dilatinos of opposite chiralities. The fermionic part of the action is given by the kinetic terms of these fermions:

$$S_\Psi \sim \int \bar{\Psi}_M \Gamma^{MNP} \partial_N \Psi_P d^{10}x, \qquad S_\lambda \sim \int \bar{\lambda} \Gamma^M \partial_M \lambda d^{10}x. \tag{1.2.10}$$

The bosonic action of D=10 type IIA supergravity theory contains three distinct type of terms:

$$S = S_{NS} + S_R + S_{CS}. \tag{1.2.11}$$

The first term comes from the NS sector fields and it is

$$S_{NS} = \frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{-g}e^{-2\phi}\left(R + 4\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}|H_3|^2\right). \tag{1.2.12}$$

where the coupling constant κ is related to the string length and the string coupling constant g_s:

$$2\kappa^2 = \frac{1}{2\pi}(2\pi l_s)^8, \quad \text{where } \ g_s^{3/2}l_p = g_s l_s. \tag{1.2.13}$$

S_R term of the bosonic action involves the R sector fields:

$$S_R = -\frac{1}{4\kappa^2}\int d^{10}x\,\sqrt{-g}\left(|F_2|^2 + |\tilde{F}_4|^2\right) \tag{1.2.14}$$

where we have redefined the Ramond-Ramond fields: $C = e^{-\phi}\tilde{C}$. The last term is called Chern-Simons term and this term involves the RR fields and the B-field:

$$S_{CS} = -\frac{1}{4\kappa^2}\int d^{10}x\; B_2 \wedge F_4 \wedge F_4. \tag{1.2.15}$$

In type IIB supergravity, we can write in a similar way an action, with the issue of the self-duality of the five-form flux, $F_5 = *_{10}F_5$. This consists in an obstruction to formulating the action in a manifestly covariant form. The strategy is to focus on the field equations instead, since they can be written covariantly. Otherwise, one can write an action which looks similar to (1.2.11). However, it needs to be supplemented by a self-duality constraint and a lagrangian multiplier.

1.2.1 Supersymmetry Variations

The supersymmetric action has to be invariant under the supersymmetric variations of all the fields. Since supersymmetry relates bosonic object with fermions, it is convenient here to introduce a useful map that send differential forms to object with two spinor indices, the so-called "Bispinors". For a "α" poliform, one can associate a poliform to a bispinor in the following way:

$$\alpha \equiv \sum_k \frac{1}{k!}\alpha_{i_1\ldots i_k}dx^{i_1}\wedge \ldots \wedge dx^{i_k} \longleftrightarrow \not{\alpha} \equiv \sum_k \frac{1}{k!}\alpha_{i_1\ldots i_k}\gamma^{i_1\ldots i_k}_{\alpha\beta}.\text{[1]} \tag{1.2.16}$$

[1]In what follows, we will drop the slash on forms whenever it should lead to confusion.

This map is called Clifford map. Finally, after having introduced a little bit of notation, we are able to write down the supersymmetry variations on fermions:

$$\delta\psi^1_M = \left(D_M + \frac{1}{4}H_M\right)\epsilon^1 + \frac{e^\phi}{16}F\Gamma_M\Gamma\epsilon^2, \tag{1.2.17a}$$

$$\delta\psi^2_M = \left(D_M - \frac{1}{4}H_M\right)\epsilon^2 - \frac{e^\phi}{16}\lambda(F)\Gamma_M\Gamma\epsilon^1; \tag{1.2.17b}$$

$$\Gamma^M\delta\psi^1_M - \delta\lambda^1 = \left(D - \partial\phi - \frac{1}{4}H\right)\epsilon^1 \tag{1.2.17c}$$

$$\Gamma^M\delta\psi^2_M - \delta\lambda^2 = \left(D - \partial\phi + \frac{1}{4}H\right)\epsilon^2, \tag{1.2.17d}$$

where ϵ^1, ϵ^2 are the supersymmetry spinor parameters, they are ten-dimensional Majorana-Weyl spinors with opposite chiralities in type IIA, and the same in IIB supergravity. Spinors, which satisfy equations of the type (1.2.17), are called Killing spinors. D_M is the covariant derivative (with respect to the spin-connection of a spin bundle on M_{10}), $H_M = \frac{1}{2}H_{MNP}\Gamma^{NP}$, where Γ_M are elements in the ten dimensional Clifford algebra, with Γ the chiral gamma matrix.

The most important point is that if we are looking for supersymmetric solutions of the equations of motion, we can set to zero the expectation values of the gravitino and the dilatino: Ψ^a_M and λ^a. Invariance under supersymmetry means that all variations (1.2.17) should be set to zero. Similarly to Yang-Mills theories, we have some Bianchi identities for the field strengths,

$$(d - H\wedge)F = 0, \quad dH = 0. \tag{1.2.18}$$

These equations are valid almost everywhere, it means that we could have sources, where these equations are not valid. Finally, one can prove that, imposing supersymmetry equations on Killing spinors, and, separately, the Bianchi identities, we can find supersymmetric solutions of equation of motion. We will discuss later on how to solve these Killing spinor equations.

1.3 Heterotic supergravity

The low energy effective action of the heterotic string written in the 10D string frame takes the form [10] (we use the conventions of [8])

$$S = \int \mathrm{e}^{-2\phi}\left\{R + 4|\mathrm{d}\phi|^2 - \frac{1}{2}|T|^2 - \frac{\alpha'}{4}\mathrm{tr}(|F|^2 + 2\bar{\chi}D\chi)\right\}, \tag{1.3.1}$$

where χ is the gaugino, $D \equiv \Gamma^M\partial_M$. ϕ is the dilaton, R is the Ricci scalar curvature of the 10D metric G, $F = dA - A\wedge A$ is not to be confused with an RR flux, but in heterotic supergravity it is the Yang-Mills field strength, which transforms under the adjoint of SO(32) or $E_8\times E_8$. The trace in (1.3.1) is the trace over the adjoint representation of the gauge group. In this thesis from now on we will consider only the $E_8\times E_8$ heterotic supergravity, where a phenomenological model building is already extensively been developed. In this action, the factor $2\kappa^2$ has been reabsorbed into the definition of the metric. H is the heterotic 3-form field strength. The form T is the corrected NS H-flux,

$$T = H - \Sigma/2 \tag{1.3.2}$$

and the 3-form Σ is the gaugino bilinear

$$\Sigma = \frac{1}{24}\alpha' \mathrm{tr}(\bar{\chi}\Gamma_{MNR}\chi)\mathrm{d}x^{MNR}, \tag{1.3.3}$$

where Γ_{MNR} is the antisymmetrization of three 10D Γ-matrices. For the heterotic string, the 3-form H, i.e. the gauge invariant field strength for the Kalb-Ramond 2-form B, is given not simply by $\mathrm{d}B$, but rather as:

$$H = \mathrm{d}B - \frac{\alpha'}{4}\left(\omega_{3\mathrm{Y}} - \omega_{3\mathrm{L}}\right), \tag{1.3.4}$$

where the 3-form $\omega_{3\mathrm{Y}}$ is the Chern-Simons form

$$\omega_{3\mathrm{Y}} = \mathrm{tr}\left(A \wedge F - \frac{1}{3}A \wedge A \wedge A\right), \tag{1.3.5}$$

which locally satisfies $\mathrm{d}\omega_{3\mathrm{Y}} = \mathrm{tr}F \wedge F$, and similar expressions can be written down for the Lorentz Chern-Simons form $\omega_{3\mathrm{L}}$. The Bianchi identity for H therefore has a non-trivial contribution on the right hand side:

$$\mathrm{d}H = \frac{\alpha'}{4}\left(\mathrm{tr}R \wedge R - \mathrm{tr}F \wedge F\right). \tag{1.3.6}$$

1.3.1 Supersymmetry variations

A supersymmetric solution of the action (1.3.1) requires the vanishing of all supersymmetry variations, which for the dilatino λ, gaugino χ and gravitino ψ_M, are [10, 11]

$$\delta\lambda = -\frac{1}{2}\partial_M\phi\Gamma^M\epsilon + \frac{1}{24}\left(H_{MNR} + \frac{1}{4}\Sigma_{MNR}\right)\Gamma^{MNR}\epsilon, \tag{1.3.7a}$$

$$\delta\chi = -\frac{1}{4}F_{MN}\Gamma^{MN}\epsilon, \tag{1.3.7b}$$

$$\delta\psi_M = D_M\epsilon - \frac{1}{8}H_{MNR}\Gamma^{NR}\epsilon + \frac{1}{96}\Sigma_{NRS}\Gamma^{NRS}\Gamma_M\epsilon. \tag{1.3.7c}$$

Solving the condition $\delta\lambda = 0$, $\delta\chi = 0$ and $\delta\psi_M = 0$ is not enough for having a supersymmetric solution of heterotic supergravity, but we should solve the Bianchi identity (1.3.6) as well.

This system has been studied extensively in the literature (see e.g. [11–16]) for Kähler and non-Kähler internal spaces. In this section, our focus will be on CY internal spaces, which is the most studied case.

Chapter 2

Compactification and geometry

"Compactification" is an extensively used word in the string theory literature. This word has different meanings, which we would like to clarify. First, a compactification is a background solution of ten-dimensional supergravity with a compact factor. As a second meaning, it refers to the process of reducing a ten-dimensional theory over the internal compact space to get, for example, a four-dimensional theory. This is also called Kaluza-Klein reduction. To this end, one starts from a particular ten-dimensional vacuum solution and studies the fluctuations around this background. Usually this computation proceeds in two steps. In the first step, one identifies some differential operators in the internal manifold whose eigenvalues give the effective lower-dimensional masses of the fluctuations. For instance, for scalars (Kaluza-Klein tower of scalars) in the lower-dimensional action, this operator is given by the internal Laplacian, $g^{mn}\nabla_n\nabla_m$. In a second step, one computes the spectrum of this differential operator. Doing this computation explicitly can be very hard for complicated manifolds, such as Calabi-Yau spaces[1]. If one is only interested in a low-energy effective theory, however, one usually only keeps the lightest

[1] For a review on Kaluza-Klein-reduction and compactification see [17, 18].

fluctuation modes and derives an action for them by dimensional reduction of ten-dimensional supergravity on the compact space.

A particularly interesting class of light fields is provided by the so-called moduli fields of a compactification. For instance sometimes one is able to identify a finite-dimensional space of internal metrics with a feature that distinguishes them from all other internal metrics. For example, Ricci-flat metrics usually come in finite-dimensional families. Letting the parameters, which describe the families, depend on the coordinates of the external space, defines a natural set of finitely many moduli fields in the effective lower-dimensional theory. This lower-dimensional supergravity theory has a potential, V, for these scalar fields, and in the Ricci-flat case it vanishes identically. Ideally we would like to fix the masses of the moduli to sufficiently large values in order to decouple them from a realistic effective theory in four dimensions. For instance, there could be some mechanisms, such as non-perturbative corrections, flux contributions and string-loop and α'-corrections, which introduce a moduli dependence. V can then have some critical points, and the masses of the moduli will depend on the value of the second derivative of the scalar potential at those points. It would also be nice if all of these solutions of a 4d effective theory corresponded to genuine vacuum solutions of the ten-dimensional theory. In fact, one would like to be able to "lift" any classical solution of the four-dimensional theory to the full solution of the ten-dimensional supergravity. A theory for which this is always true is called a consistent truncation of the higher-dimensional theory. Unfortunately, most reductions are not consistent truncations, and finding solutions in a lower-dimensional effective theory might not guarantee a self-consistent ten-dimensional solution. This is one of the main motivations for taking a fully ten-dimensional point of view and study the backreaction of fluxes for the different set-ups studied in this thesis.

2.1 Vacua in any dimensions

We will look here at vacuum solutions of both type II/Heterotic supergravity. In supergravity we usually look for 10D vacuum solutions of (1.2.17) or (1.3.7) in a compactified ten-dimensional space-time. A compactified ten-dimensional space means that M_{10} ia a fiberation over a four-dimensional external manifold M_d:

$$M_{10-d} \hookrightarrow M_d \rightarrow M_{10},$$

where M_{10-d} is the internal compact manifold. In general, a vacuum of type II and heterotic supergravity is a solution of its equations of motion and Bianchi identities, such that M_{10-d} is fibered over a space-time M_d and, in addition, the whole solution has maximal symmetry in four dimensions. The groups of transformation, under which the external spaces are invariant, are the Poincaré Groups, SO$(d-1,2)$ and SO$(d-1,1)$. Each of these isometry groups defines a different M_d:

- ISO$(d-1,1)$: the space which admits this symmetry is Mink$_d$;
- SO$(d-1,2)$: with this symmetry group we have M_d = AdS$_d$;

- SO$(d-1,1)$: this symmetry implies M_d = dS$_d$.

We will consider only the first two cases, because the dS$_d$ as external space breaks supersymmetry [2]. In this frame, as a consequence of previous definitions, the metric[3] of M_{10} is

$$ds^2_{10} = e^{2A} ds^2_d + ds^2_{10-d} \tag{2.1.1}$$

where $g_{\mu m} = 0$, with $\mu = 0,..,d-1$ and $m = 1,...10-d$. The external metric ds^2_{10-d} is unconstrained, whereas the internal metric ds^2_d is totally constrained and it is the metric of Mink$_d$ or AdS$_d$. e^{2A} is the warp factor, which depends on internal coordinates: $\{y^m\}$. The compactification and the vacuum ansatz entail some consequences on fluxes. The Dilaton ϕ depends only on internal coordinates. The H-flux has only internal components. In type II we have also RR fluxes, F_p, with $p < d$ they have purely internal legs like H, but if $p \geq d$ we can have external indices. This corresponds to the following ansatz for $F_{p\geq d}$, $F_{0\ldots d m_1 \ldots} \sim \mathrm{vol}_{\mathrm{Mink/AdS}_d} \wedge \ldots$.

2.2 Minkowski$_4$ solutions of ten-dimensional supergravity

Heterotic supergravity comes with a very compelling feature of a non-abelian gauge group. In the first part of this thesis, we will study a simple class of four-dimensional vacua of the heterotic theory. The metric is of the form (2.1.1) with $d = 4$, and, for simplicity, we assume that the warp factor is constant. We can then generally set $A = 0$ or rescale the external metric to absorb a constant warp factor. The compactified ten-dimensional space now is

$$M_{10} = \mathrm{Mink}_4 \times Y, \tag{2.2.1}$$

where Y is the six-dimensional internal compact manifold. The Killing spinor ϵ and the gamma matrices that acts on them are decomposed according to the factorized space (2.2.1). The gamma matrices are represented as tensor product:

$$\Gamma_\mu = \gamma_\mu \otimes 1_6 \qquad \Gamma_m = \gamma_5 \otimes \gamma_m, \tag{2.2.2}$$

where γ_μ, with $(\mu = 1, \ldots, 4)$, are the four-dimensional gamma with $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}_{\mathrm{Mink}_4}$, γ_m, with $(m = 1, \ldots, 6)$, are the six-dimensional gamma matrices with $\{\gamma^m, \gamma^n\} = 2g^{mn}_Y$ (see A for the conventions on the indices), and, γ_5 is the four-dimensional chirality matrix. On the other hand, in the case of 4d, $\mathcal{N} = 1$, supersymmetry, the ten-dimensional Majorana-Weyl spinor, ϵ, is decomposed in the following way

$$\epsilon = \zeta \otimes \eta, \tag{2.2.3}$$

[2] This is justified by, for example, a 4d $\mathcal{N} = 1$ supergravity argument, indeed, we have to set to zero the auxiliary fields, in order to preserve supersmmetry, this implies a negative or zero cosmological constant.

[3] Sometimes instead of the fiberation diagram (2.1.1), we will use the direct product notation, $M_{10} = M_d \times M_{10-d}$, also for warped product. It will be clear from the context when we have trivial or non-trivial warping.

where ζ is a spinor of Minkowski$_4$ and η is a Majorana-Weyl spinor of M_6. Finally, we can substitute this decomposition in equations (1.3.7), where the Killing spinor equation[4] in Mink$_4$ is $D_\mu \zeta = 0$. The results are the supersymmetry variations rewritten only in terms of the internal spinor η,

$$\delta\lambda = -\frac{1}{2}\partial_m \phi \gamma^m \eta + \frac{1}{24}\left(H_{mnr} + \frac{1}{4}\Sigma_{mnr}\right)\gamma^{mnr}\eta, \tag{2.2.4a}$$

$$\delta\chi = -\frac{1}{4}F_{mn}\gamma^{mn}\eta, \tag{2.2.4b}$$

$$\delta\psi_m = D_m\eta - \frac{1}{8}H_{mnr}\Gamma^{nr}\eta + \frac{1}{96}\Sigma_{nrs}\gamma^{nrs}\gamma_m\eta, \tag{2.2.4c}$$

where all the indices are now internal and referred to the manifold Y, and the fields are all defined in 1.3. The best understood solution of (2.2.4) is specified by $D_m\eta = 0$ and $H = \Sigma = 0$ as well as some conditions on the field strength F, which we will discuss later on. Moreover, (2.2.4a) and $H = \Sigma = 0$ implies that the dilaton is constant. In the next subsection, we will describe the geometry of this solution in a more detailed way.

2.2.1 Calabi-Yau threefolds

Y is a compact manifold, and in order to solve (2.2.4), we take $D_m\eta = 0$. In the tangent bundle of the orientable manifold M_6, we have a natural SO(6) frame bundle. In order to define a Majorana-Weyl spinor spinor on M_6, we need to lift the frame bundle to a spin bundle, Spin(6). Because of the isomorphism Spin(6) $\simeq$ SU(4), a Majorana-Weyl spinor transforms under the fundamental representation of SU(4). The stabilizer of one globally nowhere-vanishing spinor is the subgroup SU(3), hence, M_6 has a SU(3)-structure. If, moreover, the spinor is covariantly constant under the spin connection induced by the Levi-Civita connection, M_6 is equipped with an SU(3)-holonomy.

However, before we proceed to study the geometry of the internal manifold, we introduce some important basic concepts. A real $2d$-dimensional manifold can be regarded a d-dimensional almost complex manifold only if it admits an almost complex structure (ACS), namely a globally defined tensor $I^m_{\ n}$ such that $I^2 = -1$ (in indices $I^m_{\ p} I^p_{\ n} = -\delta^m_{\ n}$). If in addition I satisfies

$$N^m_{np} = \partial_{[p} I^m_{i]} - I^i_{[n} I^j_{p]} \partial_i I^k_j = 0, \tag{2.2.5}$$

where N^m_{np} is called the Niejenhuis tensor, the manifold M then is a d-dimensional complex manifold and I is integrable. The complexified tangent space, $T^*M_p \otimes \mathbb{C}$, at each point $p \in M$ decomposes into the (i)-eigenspace and the $(-i)$-eigenspace of the ACS I . Since I is a global object we can extend this concept to the entire complexified tangent bundle $T^*M \otimes \mathbb{C}$ defined as the union $\bigsqcup_p T^*M_p \otimes \mathbb{C}$, $\forall p \in M$. Then the sections of these bundles, locally constructed as (i)-eigenspace and $(-i)$-eigenspace of I, define the $(1,0)$-forms and the $(0,1)$-forms respectively.

[4]As a comment, in general D_μ and D_m always indicate the convariant derivatives (with respect to the spin connection associated to the Levi Civita connection) acting on spinors respectively on the external and internal space for any decomposition of M_{10}.

In a similar way, considering the complexified extended bundle $\bigwedge^k T^*M \otimes \mathbb{C}$ of $k-$forms, we can generalize this definition to any (p, q)-form.

(2.2.5) can be translated into a condition on a symplectic real two-form. In order to see this, since M_d is a complex manifold, we define some local complex holomorphic coordinates $\{z^a\}, \quad a = 1, .., d$. Moreover, we assume the presence of a nowhere degenerate Hermitian metric on M_d, $g = g_{a\bar{b}} dz^a d\bar{z}^{\bar{b}}$, where Hermitian means that $g(v, \bar{w}) = \overline{g(w, \bar{v})}$, $\forall\, v, w \in TM_6$. We can then define a $(1, 1)$-form, J on M_d using the following relation,

$$g_{c\bar{b}} = I^a_c J_{a\bar{b}}, \tag{2.2.6}$$

where

$$J = J_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}. \tag{2.2.7}$$

It can be proven that the integrability condition (2.2.5) on J now reads $dJ = 0$. A manifold with an Hermitian metric and a symplectic real two-form is called Kähler manifold. [5] A Calabi-Yau manifold is a Kähler manifold obeying a constraint on one of the topological invariants of the tangent bundle defined on this manifold, namely the first-chern class vanishes, $c_1(TY) = 0$. For such manifolds, a powerful theorem by Yau [19, 20] (see page 72 [21] for a little review), guarantees the existence of a Ricci-flat metric with SU(d)-holomomy. In this section, we have introduced Calabi-Yau manifold in complex d dimensions, however, from now on in this thesis, we will consider Calabi-Yau 3-folds.

Coming back to our covariantly constant spinor, η, it restricts the holonomy to SU(3), and, hence, the internal manifold Y is a Calabi-Yau 3-fold. In fact, using the Clifford map (1.2.16), J can be written as

$$J \leftrightarrow \eta^\dagger \gamma_{mn} \eta, \tag{2.2.8}$$

and we can define a holomorphic three form:

$$\Omega \leftrightarrow \eta^\dagger \gamma_{mnp} \eta. \tag{2.2.9}$$

The existence of a nowhere-vanishing $(3, 0)$-form, Ω, says that the canonical bundle (the subbundle of $\bigwedge^3 T^*M$ consisting in the $(3, 0)$-forms) is topologically trivial. The condition $D_m\eta = 0$, implies that $dJ = 0$, i.e. we have a Kähler manifold, and, moreover, $d\Omega = 0$. Conversely a pair of integrable (J, Ω) defines an SU(3)-holonomy on the manifold. We then see that the Calabi-Yau condition can be written also as differential equations on differential forms.

2.2.2 Supersymmetric vector bundles

The condition $D_m\eta = 0$, together with $H = \Sigma = 0$ is not sufficient to satisfy (2.2.4). The gaugino variation (2.2.4b) implies, $\gamma^{mn} F^j_{mn} \eta = 0$, where j are indices in the adjoint representation

[5] Note that the exterior differential d in this case has not to be confused with the number of complex dimension above.

of $E_8 \times E_8$. We need to construct a vector bundle with field strength $F = dA + A \wedge A$, where the gauge field[6], A, satisfies (2.2.4b) that can be rewritten in the following way

$$F_{ab} = F_{\bar{a}\bar{b}} = 0 \tag{2.2.10}$$

$$g^{a\bar{b}} F_{\bar{b}a} = 0. \tag{2.2.11}$$

The first condition (2.2.10) (called holomorphicity condition) simply implies that the vector bundle on Y has to be holomorphic. Finding explicit solution for A is a very hard task with the techniques known so far. However, for Calabi-Yau manifolds Y there exists a powerful way of transforming this question into an problem of algebraic geometry, namely the vector bundle has to satisfy a so-called *stability condition*. Reviewing the beautiful result by Donaldson, Uhlenbeck and Yau is not one of our goals here, so we refer to [22, 23] for the theorem and to [21] for a short explanation of the poly-stability condition of vector bundles on Calaby-Yau manifolds.

Standard and non-standard embeddings

In order to have a final and complete solution, we need to satisfy also the Bianchi identity (1.3.6). Having set H to zero it reads

$$\text{tr}\,(R \wedge R) = \text{tr}(F \wedge F)\ . \tag{2.2.12}$$

Let us forget about one of the E_8 factor of the gauge group. The simplest non-trivial solution is to set the vector bundle $\mathcal{V}$ to be the tangent bundle of Y. Moreover, one can prove that the tangent bundle TY of a Ricci-flat and Kähler manifold is automatically stable and, hence, $\mathcal{V}$ satisfies equations (2.2.10-2.2.11). With this choice of $\mathcal{V}$, it is clear that over the internal space the structure group is no longer E_8, but rather the SU(3). Namely, we have broken structure group E_8 to the subgroup $E_8 \to E_6 \times$ SU(3), where E_6 is structure group of the vector bundle on the external space. In general, if we solve (2.2.10-2.2.11) by using only a subset of the E_8 group indices (setting $F = 0$ for the other E_8 factor), then the E_8 bundle breaks into a product bundle with structure group H$\times$ G where[7] G, H $\in E_8$. H is identified with the Yang-Mills gauge group, which connection has components only on the physical space Minkowski$_4$, but it can depend on the coordinates of the internal space, Y, as background function, i.e. the H is identified with an H gauge bundle on Minkowski$_4 \times Y$. G is the structure group of $\mathcal{V}$, the bundle over the internal space. So, for a given choice of bundle and structure group, G, we can determine the maximal H such that G$\times$ H $\in E_8$. H is called the "commutant" of G in E_8. For the above choice of G = SU(3), we have H = E_6 and this is defined as standard embedding. The vacuum would have a E_6 gauge group in the four dimensional Minkowski space, but its connection can still depend on the coordinates of the internal space, Y, feature that is fundamental to break further the gauge group by using Wilson lines, given by non-trivial loops in the internal Calabi-Yau space. All in

[6] A here is the gauge connection and not the warp factor, which we already have gotten rid. It will be clear from the context that A will be the gauge connection when we talk of heterotic supergravity and the warp factor when we talk about more general vacuum solutions in type II.

[7] H in this paragraph has not to be confused with the three form flux.

all, this means that one could, in principle, construct a E_6 GUT theory starting from this vacuum solution.

It is clearly of interest to ask whether there are more general solutions to (2.2.10-2.2.11). It was realized in [24] that such constructions are indeed possible by choosing $\mathcal{V}$ not as the tangent bundle, TX, like in the Standard Embedding above, but $\mathcal{V}$ can be a more general holomorphic vector bundle over Y with structure group G. For instance, if we are interested in obtaining an SO(10) theory in four-dimensions, we must select G to be SU(4). Similarly, a bundle $\mathcal{V}$ with SU(5) symmetry will lead to an SU(5) GUT theory. Such choices of holomorphic vector bundles are known as general embeddings. However, the stability condition of $\mathcal{V}$ coming from (2.2.10-2.2.11) is not automatically satisfied now, and it needs to be verified, giving some constraints on the topological data of the vector bundle, $\mathcal{V}$, of the Calabi-Yau Y.

2.2.3 A playground: quick introduction to CICY

In this work, we will construct explicit subclasses of Calabi-Yau manifolds as submanifold of compact Kähler manifold, with the right topological requirement $c_1(Y) = 0$. More concretely, we will consider complex projective spaces as ambient spaces. An n-dimensional complex projective space $\mathbb{C}P^n$ (or simply $\mathbb{P}^n$) is defined by

$$\mathbb{C}P^n \equiv \frac{\mathbb{C}^{n+1} - 0}{\mathbb{C}^*}, \tag{2.2.13}$$

by which one reads of the space of equivalence classes $[z] = [z_0, z_1, \ldots, z_n]$, where we mod out the multiplicative action of $\lambda \in \mathbb{C}^*$, namely $z = (z_0, z_1, \ldots, z_n) = (\lambda z_0, \lambda z_1, \ldots, \lambda z_n)$ with $(z_0, z_1, \ldots, z_n)$ coordinates of $\mathbb{C}^{n+1}$, which in $\mathbb{C}P^n$ will be called homogeneous coordinates. Basically, one could think of $\mathbb{C}P^n$ as the space of complex lines through the origin of $\mathbb{C}^{n+1}$. The Kähler form locally reads

$$J = i\partial_i \partial_{\bar{j}} K dZ^a \wedge d\bar{Z}^{\bar{b}}, \qquad K = \log(1 + \sum_{i=1}^{n} |Z_a|^2), \tag{2.2.14}$$

where $Z_a = z_a/z_0$ are the affine coordinates, and the first chern-class of $\mathbb{C}P^n$ is

$$c_1(\mathbb{C}P^n) = (n+1)J. \tag{2.2.15}$$

For a p-degree polynomial in a projective space we have

$$c_1(Y) = (n+1-d)J = 0. \tag{2.2.16}$$

One of the most famous examples is the quintic hypersurface in $\mathbb{P}^4$:

$$\left\{ z \in \mathbb{C}P^4 \;\middle|\; p^{a_1 a_2 a_3 a_4 a_5} z_{a_1} z_{a_2} z_{a_3} z_{a_4} z_{a_5} = 0 \right\}. \tag{2.2.17}$$

where $a_{1,2,3,4,5} = 0, \ldots 4$, and $p_{a_1 a_2 a_3 a_4 a_5}$ are the complex parameters that actually parameterize a family of this kind of algebraic variety. The quintic does satisfy (2.2.16), with $d = 5$ and

$n = 4$. We can generalize this manifold to a more general and reach class. This class is called complete intersection Calabi-Yau manifolds (CICY). Here we sketch the relevant information from the vast literature on complete intersection CICYs. Much more detailed discussion can be found in the pioneering papers [25, 26] and in the textbook [27]. A CY manifold may be constructed as the set of homogeneous solutions to a set of polynomials determined by the configuration matrix

$$\left[\begin{array}{c|cccc} \mathbb{C}P^{n_1} & m_{11} & m_{12} & \cdots & m_{1l} \\ \mathbb{C}P^{n_2} & m_{21} & m_{22} & \cdots & m_{2l} \\ \vdots & \vdots & & \ddots & \\ \mathbb{C}P^{n_k} & m_{k1} & m_{k2} & \cdots & m_{kl} \end{array}\right]. \tag{2.2.18}$$

This matrix specifies a class of l polynomials in the ambient space

$$\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \cdots \times \mathbb{C}P^{n_k}. \tag{2.2.19}$$

We call each polynomial P_i, where $i = 1, \ldots, l$ corresponds to the ith column of the configuration matrix, and the entries in the matrix specify that each term in the ith polynomial must contain m_{ji} powers of the coordinates from $\mathbb{C}P^{n_j}$. The set of simultaneous homogeneous solutions to all the polynomials is a compact and smooth Kähler subspace of the ambient space provided that the polynomials are transverse, that is $\mathrm{d}P_1 \wedge \cdots \wedge \mathrm{d}P_l \neq 0$ at all points of intersection, $P_i = 0$. The subspace is a three-fold if

$$\sum_{i=1}^{k} n_i = l + 3, \tag{2.2.20}$$

and furthermore it is Ricci-flat with vanishing first Chern Class and therefore CY if the configuration matrix satisfies

$$\sum_i m_{ji} = n_j + 1, \qquad \forall j = 1, \ldots, k. \tag{2.2.21}$$

Of course for each configuration matrix there are many different choices of polynomials, most of which correspond to smooth CY manifolds. All smooth complete intersections corresponding to the same configuration matrix are diffeomorphic and therefore topologically equivalent as real manifolds. Finally, the coordinate expression of the holomorphic three-form has an implicit definition, given in [26]. It is a contour integral on a contour, γ, in the ambient space around the zero-locus of the polynomials, P_i,

$$\Omega = \lim_{\delta \to 0} \int_\gamma \frac{\varepsilon_{A_1 A_2 \ldots A_{k+4}} z^{A_1} dz^{A_2} \wedge dz^{A_3} \wedge \ldots \wedge dz^{A_{k+4}}}{P^1 P^2 \ldots P^l}, \tag{2.2.22}$$

where δ is the radius of the contour around each zero-locus of the polynomials, and z^{A_1}, z^{A_2}, $\ldots z^{A_{k+4}}$ are the homogeneous coordinates in the ambient space. The Kähler form is instead locally given by the expression (2.2.14) intersected with the polynomials P_i.

All CICYs are simply connected, whereas model building requires multiply connected CYs in order to allow GUT symmetry breaking by Wilson lines. Multiply connected CYs can be obtained by quotienting a CICY by some freely-acting discrete symmetry group Γ. The fundamental group of the quotient CICY is then non-trivial, $\pi_1(Y_3/\Gamma) = \Gamma$. When quotienting a given CICY configuration by Γ, one must of course consider only polynomials that respect this symmetry. This significantly lowers the dimensionality of the moduli space of the CY.

2.2.4 Moduli spaces and Hodge numbers

At the beginning of this chapter we mentioned that some vacuum solutions comes in parameterized continuous families, such as e.g. Calabi-Yau manifolds, and within the same family the manifold are topologically the same. These families are parameterized by the so-called moduli. When we allow these parameters to depend on the external space coordinates, they become fields. For Calabi-Yau spaces the Hodge numbers count the dimensions of the moduli spaces, i.e. the number of independent parameters of a given family, and, more explicitly, there are only two types of moduli spaces associated to deformations of the holomorphic three-form (complex structure moduli) and deformations of the Kähler form (Kähler moduli).

For example, the quintic in (2.2.17) depends on a choice of the coefficients $p^{a_1 \cdots a_5}$ of the degree 5 polynomial. This is a $\binom{5+4}{5} = 126$-dimensional space. However, many of these polynomials are equivalent under linear changes of coordinates, which is the group $Gl(5, \mathbb{C})$, with dimension 25. So we actually have a 101-dimensional space.. From the point of view of the four-dimensional theory, we expect each of these quintics to give rise to a supersymmetric Minkowski vacuum. So the effective potential in four dimensions should have 101 flat directions, which means 101 massless scalars. For any phenomenological application, this is a disaster: these would be mediated by long-range scalar forces, which are not observed. This is a common feature of all Calabi–Yau compactifications. Fortunately, things get better once we give masses to the moduli. Indeed, one of the main issues in string compactification is the moduli stabilization problem and to find the right mechanism to give them a sufficiently large mass by fixing their VEV's (Vacuum Expectation Values) dynamically. In any case, there are many good reasons to keep studying Calabi–Yau manifolds, even without immediately tackling their moduli stabilization. The 101 moduli we have just found for the quintic are deformations of its complex structure: they change the way we define our complex coordinates z_i out of the real coordinates x_m. Variations δz imply deformations of $\delta\Omega$, more precisely these deformations belong to the space,

$$\delta\Omega \in H^{2,1}(Y) \equiv \Lambda^{2,1} \cap H^3(Y). \tag{2.2.23}$$

$H^{p,q}(Y)$ are the Dolbeault cohomology groups of the Calabi-Yau Y, $\Lambda^{p,q} = \{\omega_{a_1,\ldots,a_p\bar{b}_1\ldots\bar{b}_q} dz^{a_1} \wedge \ldots \wedge dz^{a_p} \wedge d\bar{z}^{\bar{b}_1} \wedge \ldots \wedge d\bar{z}^{\bar{b}_q}\}$, and $H^m(Y)$ are the de Rham cohomology groups defined as follows

$$H^m(Y) = \frac{Z^m(Y)}{B^m(Y)}, \tag{2.2.24}$$

with $Z^m = \{\omega_m \,\|\, d\omega_m = 0\}$, the space of closed m-forms and $B^m = \{\omega_m, \,|\, d\omega_{m-1}\}$, the space of exact m-forms ($m = 1, \ldots, 6$ and $p, q = 1, \ldots, 3$). The cohomology groups have always a finite set of independent Harmonic forms (that means they are solution of the Laplace equations, see [18] for a review), which are representatives of these cohomology groups. In general, the complex structure moduli space of a Calabi–Yau is a complicated manifold, and we cannot say much at this stage. There is, however, a subspace of moduli which we should be careful about: the so-called singular locus where Y is singular. Fortunately, algebraic geometry gives us a simple criterion to check whether a manifold defined by l polynomials $P_i(z_0, ..., z_n)$ is singular. The singular locus is defined as the sublocus of the manifold given by the intersecting system of polynomials, P_i, where

$$\mathrm{Rank}\left(\frac{\partial P_i}{\partial z_a}\right) < k \qquad i = 1, \ldots, l, \;\; a = 0, \ldots, n. \tag{2.2.25}$$

with $(\partial_a P_i)$ the $(l \times (n+1))$ Jacobian matrix.

Finally, we can relate these Harmonic forms to 3-cycles. In general m-cycles are particular m-dimensional submanifolds of Y and they generate the homology group $H_m(Y)$. The connection between homology and cohomology is given by Poincaré duality: given any m-cycle, Q, there exists a closed $(6-m)$-form α, called Poincaré dual of Q such that for any closed m-form, ω,

$$\int_Q \omega = \int_Y \alpha \wedge \omega. \tag{2.2.26}$$

Since ω is closed α is only defined up to an exact form. Therefore, we can give to the complex structure moduli a beautiful geometric interpretation, namely these moduli fields are deformation of certain 3-dimensional submanifolds of Y.

Another class of deformations allowed in Calabi-Yau manifolds are Kähler moduli δJ, related to deformations of the Kähler form (2.2.7). These are elements of a cohomology group as well,

$$\delta J \in H^{1,1}(Y) \equiv \Lambda^{1,1} \cap H^2(Y). \tag{2.2.27}$$

As for the complex structure moduli, we have a nice geometric interpretation of the Kähler moduli. They are deformation of two-dimensional holomorphic cycles associated via Poincaré duality to Harmonic representatives of $H^{1,1}(Y)$. For example δJ represents changes in the volume of the cycles c_i of dimension 2, with their volume being given by $v^i \int_{c_i} J$.

All the dimensions of these moduli spaces are counted by Hodge numbers defined as

$$h^{p,q} = \dim_{\mathbb{C}}(H^{p,q}(Y)). \tag{2.2.28}$$

For Calabi-Yau manifolds, we have the following non-trivial Hodge numbers:

$$h^{0,0} = h^{3,3} = 1, \quad h^{3,0} = h^{0,3} = 1, \qquad h^{1,1}, \; h^{2,1} = h^{1,2}, \tag{2.2.29}$$

Some of these topological data are related by complex conjugation

$$h^{p,q} = h^{q,p} \tag{2.2.30}$$

and Hodge duality

$$h^{p,q} = h^{3-p,3-q}. \tag{2.2.31}$$

$h^{3,3} = h^{0,0}$, $h^{3,0} = h^{0,3}$ are the same in every Calabi-Yau, whereas $h^{1,1}$, $h^{2,1} = h^{1,2}$ can be different in different families of Calabi-Yau. Different Hodge numbers describe topologically different Calabi-Yau's, on the other hand, different manifolds in the same moduli space are diffeomorphic to each other.

2.2.5 4d effective theory

In order to construct an effective field theory in four dimensions, we need to have a concrete expression the moduli. For the complex structure moduli we begin by introducing a basis of $H^3(Y)$ with generators α_a, β^a (with $a, b = 0, \dots h^{2,1}$) which are Poincaré dual to a canonical homology basis (Q_a, S^b) of $H_3(Y)$ with intersection numbers defined by

$$S^a \cdot S^b = \int_Y \beta^a \wedge \beta^b = 0, \tag{2.2.32}$$

$$Q_a \cdot Q_b = \int_Y \alpha_a \wedge \alpha_b = 0, \tag{2.2.33}$$

$$Q_a \cdot S^b = \int_Y \alpha_a \wedge \beta^b = \delta_a^b. \tag{2.2.34}$$

This implies

$$\int_{S^b} \alpha_a = \int_Y \alpha_a \wedge \beta^b = - \int_{Q_a} \beta^b = \delta_a^b. \tag{2.2.35}$$

We can expand every closed three-form, ω_2, in terms of the canonical Harmonic basis

$$\omega_3 = \zeta^a \alpha_a + \tilde{\zeta}_a \beta^a \tag{2.2.36}$$

Furthermore, under a change of complex structure the holomorphic 3-form, Ω, which was pure $(3,0)$ to start with, becomesa mixture of $(3,0)$ and $(2,1)$, and can be expanded in the basis of Harmonic forms as (2.2.36). Finally, $\zeta^a, \tilde{\zeta}_a$ are the real $(2h^{2,1}+2)$ moduli, from which we can define $(h^{2,1}+1)$ complex structure moduli as complex combinations, where we allow $\zeta^a, \tilde{\zeta}_a$ to depend on the external coordinates.

Similarly to the complex structure moduli, the Kähler moduli come from expanding the closed two form in a basis of Harmonic forms $\{b_\alpha\}$, which are representatives of the cohomology group $H^{1,1}(X)$,

$$\omega_2 = t^\alpha b_\alpha, \qquad \alpha = 1, \dots, h^{1,1}+1 \tag{2.2.37}$$

We can associate 2-cycles that are elements of the Homology, $H_2(X)$, via Poincaré duality and by computing the triple intersection numbers. t^α are $h^{1,1}$ real moduli, and they can be combined with the moduli coming from the expansion of the NSNS two form field B_2 in terms of Harmonic forms, in order to get complex combinations. These are the complexified Kähler moduli.

Let us briefly describe the main quantities related to the moduli fields of a 4-dimensional, $\mathcal{N}=1$, effective theory coming from a compactified ten-dimensional supergravity. We can start from a Calabi-Yau solution of heterotic supergravity and dimensionally reduce the ten-dimensional theory on the underling compact internal space. Ignoring higher massive KK-modes, we are left with the massless scalars, which are identified with the above introduced moduli fields, ϕ_i, $i = 1, \dots (h^{1,1}+h^{2,1}+2)$. A general scalar potential of a 4d, $\mathcal{N}=1$, effective field theory has the following restricted form:

$$V = e^K (D_i W D^i W - 3|W|^2), \qquad D_i = \frac{\partial}{\partial \phi_i} + \frac{\partial K}{\partial \phi_i}, \tag{2.2.38}$$

where K is the so-called Kähler potentials, a real function of the complex and Kähler moduli moduli fields, which defines the kinetic term in a 4d supergravity theory. The superpotential W is an holomorphic function of the complex moduli. For the unstabilized moduli, W vanishes identically, and hence V does. In [28], a non-trivial H-flux was conjectured to result in a flux superpotential,

$$W = \int_Y H \wedge \Omega. \tag{2.2.39}$$

If we expand Ω and H in the Harmonic basis and we integrate over Y, we get the dependence on the complex structure moduli. However, the consistency of turning on an H-flux on a supersymmetric Calabi-Yau background has to be verified, we will deal with this issue in 3 in the context of heterotic supergravity. There can exist other contributions to K and W, such as non-Kähler deformations, $\alpha^{'}$-corrections and non-perturbative effects. Finally, the standard way to stabilize the moduli is to search for minima of a suitable scalar potential.

As last comment, deformations of vector bundle lead to bundle moduli. This deformations are not arbitrary but they should satisfy (2.2.10, 2.2.11), the holomorphicity and stability constraints depend on the complex structure moduli together with the bundle moduli. The naive factorisation of the two different moduli spaces is lost due to these constraints. See [29–32], for a discussion on bundle moduli and their stabilization.

2.3 Beyond Calabi-Yau three-folds

We will now generalize the discussion to a larger class of solution of type IIA/B supergravity. As we have already seen in section 1.2, the supersymmety variations, (1.2.17), are written in terms of two Majorana-Weyl spinors ϵ^1, ϵ^2. We will look for supersymmetric vacuum solutions. We recall that solving the system (1.2.17), together with the Bianchi identities, (1.2.18), implies solving the equation of motions of type II supergravity [33,34]. The goal of this section is to give a sketch of the techniques used to rewrite the general spinor equations in terms of differential equations on forms, which seem to be more manageable.

2.3.1 More general 4-dimensional vacua

We will review now some general aspects of 4d, $\mathcal{N} = 1$, supersymeetric vacua of type II supergravity with fluxes. The gamma matrices are decomposed as (2.2.2), where $\gamma_\mu \to e^A \gamma_\mu$, since we turn on a non-trivial warping and the metric reads (2.1.1) with $d = 4$. The decomposition ansatz for the two ten-dimensional Majorana-Weyl spinors, ϵ^1 and ϵ^2, is

$$\epsilon^1 = \zeta_+ \otimes \eta^1_+ + \zeta_- \otimes \eta^1_- \tag{2.3.1}$$
$$\epsilon^2 = \zeta_+ \otimes \eta^2_\mp + \zeta_- \otimes \eta^2_\pm \tag{2.3.2}$$

where $\zeta_- = (\zeta_+)^*$ and $\eta^{1,2}_- = (\eta^{1,2}_+)^*$ and $\mp$ signs are, respectively, for the cases IIA or IIB supergravity. We plug this decomposition in equations (1.2.17) with the following ansatz:

- For $M_4 = \mathrm{Mink}_4$, $D_\mu \zeta_\pm = 0$, i.e. all spinors are constant with respect to the external coordinates x^μ;
- For $M_4 = \mathrm{AdS}_4$, we have the Killing spinor equation: $D_\mu \zeta_- = \frac{1}{2}\mu\gamma_\mu\zeta_+$, where μ is a parameter related to the cosmological constant: $\Lambda = -\mu^2 < 0$.

Finally, with all previous assumptions decompositions, we get six equations in terms the Killing spinors in the internal space, $\eta^{1,2}$,

$$\left(D_m + \frac{1}{4}H_m\right)\eta^1_+ \pm \frac{e^\phi}{8} f\gamma_m \eta^2_\pm = 0, \tag{2.3.3a}$$

$$\left(D_m + \frac{1}{4}H_m\right)\eta^2_\mp - \frac{e^\phi}{8}\lambda(f)\gamma_m \eta^1_+ = 0, \tag{2.3.3b}$$

$$\mu e^{-A}\eta^1_+ + \partial A \eta^1_+ - \frac{e^\phi}{4} f \eta^2_\pm = 0, \tag{2.3.3c}$$

$$\mu e^{-A}\eta^2_\pm + \partial A \eta^2_\pm - \frac{e^\phi}{4}\lambda(f)\eta^1_+ = 0, \tag{2.3.3d}$$

$$2\mu e^{-A}\eta^1_- + D\eta^1_+ + \left(\partial(2A - \phi) + \frac{1}{4}H\right)\eta^1_+ = 0, \tag{2.3.3e}$$

$$2\mu e^{-A}\eta^2_\pm + D\eta^2_\mp + \left(\partial(2A - \phi) + \frac{1}{4}H\right)\eta^2_\mp = 0, \tag{2.3.3f}$$

where $\pm$ signs are respectively the IIA or IIB case, with $\mu = 0$ it corresponds to the 4d Minkowski case and the contracted three-from flux reads $H_M = \frac{1}{2}H_{MNP}\Gamma^{NP}$. Moreover, the fluxes are written in the following fashion

$$F = f + \mathrm{vol}_4 \wedge *_6 \lambda f, , \tag{2.3.4}$$

where f is a polyform, namely a formal sum of forms of possibly different degrees. All the fields and the operator acting on differential forms, λ, are defined in section 1.2. The new Bianchi identities with the decomposition (2.3.4) are

$$dH = 0; \qquad (d - H\wedge)f = 0; \qquad (d + H\wedge)(e^{4A} *_6 f) = 0, \tag{2.3.5}$$

where A refers to the warp factor of the decomposed metric (2.1.1) with $d = 4$. In order to make the notation simpler and more compact, we redefine the exterior differential operator

$$d_H = d - H\wedge \tag{2.3.6}$$

Without making any further ansatz the goal is to rewrite the equations in terms of forms.

In case $M_4 = \mathrm{Mink}_4$ with constant warping and when all the fluxes vanish, we reobtain the Calabi-Yau solution in type II context. These types of Calabi-Yau solutions in type II have enhanced supersymmetry, $\mathcal{N} = 2$, where the amount of supersymmetry depends on the decomposition ansatz of the spinors, (2.3.1). For instance, a covariantly constant spinor on M_6

results from the system (2.3.3), even when we have a decomposition of the type, $\epsilon^1 = \zeta^1 \otimes \eta$, and $\epsilon^2 = \zeta^2 \otimes \eta$, with vanishing fluxes and constant warping. On the other hand, we can have more general solutions with non-trivial fluxes. The spinors are no longer covariantly constant with respect to the Levi-Civita connection but rather they satisfy an equation of the kind, $D\eta = f(F, H)$. Saying it differently, the spinors would be covariantly constant with respect to another connection with non-trivial torsion, which is related to the fluxes, and they define a manifold with reduced structure group of the tangent bundle, i.e a G-structure manifold.

In equations (2.3.3) we have two Killing spinors, so we have to understand which G-structure they define. The G-structure depends on how the spinors are related to each other. For instance, if η_1, η_2 are everywhere parallel we have an SU(3)-structure (sometimes called strict SU(3)-structure), if they are everywhere non-parallel they lead to an SU(2)-structure in six dimensions. However, the natural framework to discuss the general case of two spinors that are neither everywhere parallel nor non-parallel is the generalized tangent bundle, $TM_6 \oplus T^*M_6$. This extended bundle is at the heart of generalized complex geometry developed by Hitchin [35] and Gualtieri [36].

The idea is that $\{dx^m \wedge, \iota_m\} \in TM_6 \oplus T^*M_6$ (with x_m, $m = 1, \ldots, 6$, coordinates of M_6) have a natural action on differential forms as operators, where $\iota_m \equiv \frac{\partial}{\partial x_m} \llcorner$ is the contraction operator of indices of a differential form, see definition in A. The algebra generated by these operators satisfies the cliff(6) $\oplus$ cliff(6) algebra relations[8]. In this sense differential forms are represented as spinors of cliff(6) $\oplus$ cliff(6). Eventually, every polyform can be written as bispinor of cliff(6), under the Clifford map (1.2.16). In six dimensions the following bispinors of cliff(6),

$$\Phi_- = \eta^1_+ \otimes \eta^{2\dagger}_- = \sum_{k\,(\text{even})=1}^{6} \frac{1}{4k!} \eta^{2\dagger}_- \gamma_{i_k \ldots i_1} \eta^1_+ \gamma^{i_1 \ldots i_k}, \qquad (\text{IIA}) \qquad (2.3.7)$$

$$\Phi_+ = \eta^1_+ \otimes \eta^{2\dagger}_+ = \sum_{k\,(\text{odd})=1}^{6} \frac{1}{4k!} \eta^{2\dagger}_+ \gamma_{i_k \ldots i_1} \eta^1_+ \gamma^{i_1 \ldots i_k}, \qquad (\text{IIB}) \qquad (2.3.8)$$

are also called "pure spinors" in type IIB/A supergravity respectively, where "pure" means that $\Phi_\pm$ are annihilated by half of the generators of cliff(6) $\oplus$ cliff(6). Since Φ is a nowhere vanishing spinor of cliff(6) $\oplus$ cliff(6), one can define a spinor bundle associated to the extended tangent bundle, $TM_6 \oplus T^*M_6$. The structure group of this extended tangent bundle would be the stabilizer of the spin group associated to the clifford algebra cliff(6) $\oplus$ cliff(6) acting on its spinor Φ, i.e. Spin(6) $\times$ Spin(6). This implies that the structure group of $TM_6 \oplus T^*M_6$ reduces to SU(3) $\times$ SU(3) and, hence, $\Phi_\pm$ define a SU(3) $\times$ SU(3) generalized structure everywhere on M_6. Using these powerful techniques, one can prove that the supersymmetry equations (2.3.3) are equivalent to the system:

$$d_H \Phi_+ = 0, \qquad d_H(e^{-A}\mathrm{Re}\,\Phi_-) = 0; \qquad (2.3.9)$$

$$d_H(e^{A}\mathrm{Im}\,\Phi_-) = e^{4A} * \lambda f, \qquad (2.3.10)$$

[8]Each copy of the algebra, cliff(6), is given by the anticommutators $\{\gamma^m, \gamma^n\} = 2g^{mn}$, where g^{mn} is the metric on M_6 and γ^m are matrices acting on elements of the Spin(6) bundle on M_6.

for $M_4 = \text{Mink}_4$, with the cosmological constant $\Lambda = 0$ and to the system

$$d_H\Phi_+ = -2\mu \text{Re}\,\Phi_-; \tag{2.3.11}$$
$$d_H(e^A \text{Im}\,\Phi_-) + 3\mu \text{Im}\,\Phi_+ = e^{4A} * \lambda f, \tag{2.3.12}$$

for $M_4 = \text{AdS}_4$ and $\Lambda = -3\mu^2 < 0$. For more details see lecture notes [37].

2.3.2 Further generalization: the ten-dimensional supersymmetric system

The final step is to generalize this procedure to any supersymmetric ten-dimensional solution of type II supergravity. The idea is the same as in the previous section, however, the story here is much more complicated. First of all, let us define some of the main objects that will appear in the system of equations. The bispinor Φ is a bilinear in the two ten-dimensional spinors, ϵ_1, ϵ_2,

$$\Phi = \epsilon_1 \otimes \bar{\epsilon}_2 = \sum_p \frac{1}{32\,p!} \bar{\epsilon}_2 \Gamma_{M_1 \dots M_p} \epsilon_1 \Gamma^{M_p \dots M_1}, \tag{2.3.13}$$

where $\bar{\epsilon}_2 = (\epsilon^2)^\dagger \Gamma^0$ and p is even in IIB and odd in IIA supergravity, and

$$K_1 = \frac{1}{32} \bar{\epsilon}_1 \Gamma_M \epsilon_1 \Gamma^M; \tag{2.3.14}$$
$$K_2 = \frac{1}{32} \bar{\epsilon}_2 \Gamma_M \epsilon_2 \Gamma^M, \quad K_{1,2}^2 = 0, \tag{2.3.15}$$
$$K = \frac{1}{2}(K_1 + K_2), \tag{2.3.16}$$
$$\tilde{K} = \frac{1}{2}(K_1 - K_2), \tag{2.3.17}$$

can be all associated to one-forms by (1.2.16) as well as Φ can be related to a polyform by the same map . As it was shown in [38] with an extensive usage of generalized geometry on $TM_{10} \oplus T^*M_{10}$, the following system of equation

$$d_H(e^{-\phi}\Phi) = -(\tilde{K} + \iota_K)F \tag{2.3.18}$$
$$L_K g = 0 \quad d\tilde{K} = \iota_K H, \tag{2.3.19}$$

is equivalent to (1.2.17), where L_K is the Lie derivative with respect to K acting on the ten-dimensional metric, g, $d_H \equiv d - H$, F is the total RR field strength, $F = \sum F_k$ (k is odd in IIB and even in IIA supergravity), and ι_K is the contraction of K on the three-form flux H. The equation, (2.3.18), is called "*master equation*" and (2.3.19) are the symmetry constraints. The system of equations is supplemented by two more complicated constraints called "*constraint or pairing equations*" ,

$$\langle \text{e}_{+_1} \cdot \Phi \cdot \text{e}_{+_2} \,,\, \gamma^{MN} \left[\pm d_H(e^{-\phi}\Phi \cdot \text{e}_{+_2}) + e^{\phi} d^\dagger (e^{-2\phi} \text{e}_{+_2})\Phi - F \right] \rangle = 0, \tag{2.3.20a}$$
$$\langle \text{e}_{+_1} \cdot \Phi \cdot \text{e}_{+_2} \,,\, \left[d_H(e^{-\phi} \text{e}_{+_2} \cdot \Phi) - e^{\phi} d^\dagger (e^{-2\phi} \text{e}_{+_2})\Phi - F \right] \gamma^{MN} \rangle = 0, \tag{2.3.20b}$$

which are written in terms of Mukai pairing[9] and

$$\begin{aligned} \mathrm{e}_{+_i} \cdot \omega &= \mathrm{e}_{+_i} \wedge \omega + (-)^{\deg(\omega)} \mathrm{e}_{+_i} \llcorner \omega, & \text{(2.3.22a)} \\ \omega \cdot \mathrm{e}_{+_i} &= \mathrm{e}_{+_i} \wedge \omega - (-)^{\deg(\omega)} \mathrm{e}_{+_i} \llcorner \omega, & \text{(2.3.22b)} \end{aligned}$$

on any p-form ω. D^M is the covariant derivative on M_{10} (with respect to the Levi-Civita connection) and $d^\dagger = D^M \iota_M$, the vector $e_{\pm_i}$ are defined by taking the contraction[10] of them on the one-forms K_i.

$$e_{\pm_i} \llcorner e_{\pm_i} = 0, \quad e_{+_i} \llcorner K_i = \frac{1}{2}, \quad i = 1, 2. \tag{2.3.23}$$

Finally to have a necessary and sufficient system of equation for any supersymmetric ten-dimensional solution of type II supergravity, we need to include the Bianchi identities

$$dH = 0, \qquad d_H F = 0, \tag{2.3.24}$$

which are valid away from the sources. In the presence of branes one should include source terms. For instance, in case of localized Dp-brane or Op-plane sources, the Bianchi identities will be modified such that

$$d_H F \sim \delta_{\mathrm{Dp/Op}}(y). \tag{2.3.25}$$

In this thesis Dp-branes and Op-planes will only appear as sources in the Bianchi equations, which is enough to describe their low-energy effects, whereas their other microscopic behaviour will be neglected.

[9]The Mukai pairing is defined as follows:

$$\langle A, B \rangle \equiv (A \wedge \lambda(B))_{top}, \tag{2.3.21}$$

for any two forms A and B, where the label "$_{top}$" means that we keep only the maximal degree form (top-form).

[10]In this thesis, the symbols ι and $\llcorner$ are both used for contractions of forms on forms, as defined in appendix A.

Chapter 3

Wilson lines and Chern-Simons flux in explicit heterotic Calabi-Yau compactifications

Heterotic string compactifications on Calabi-Yau (CY) manifolds with Wilson lines have had considerable success in string model building [39–45], with abundant explicit examples containing only a supersymmetric standard model, a hidden sector and a few geometric and vector bundle moduli. There are also several ideas on how to address the moduli stabilization problem, although their realization in explicit constructions has proven more challenging. An important observation is that the holomorphicity and stability conditions on vector bundles could lift many of the flat directions already at tree-level [24, 29–32]. Another mechanism proposed by [46] is to stabilize moduli with fractional H-flux sourced by Wilson lines in conjunction with gaugino condensation. In ref. [47] it was argued that this mechanism would generically lead to GUT scale supersymmetry breaking.

Wilson lines were first introduced in order to break GUT gauge groups without breaking supersymmetry. However, any concomitant H-flux might also unintentionally affect the self-consistency of the compactification background. Indeed, it is well-known that the backreaction of H-flux deforms away from supersymmetric Calabi-Yau compactifications of the leading order 10D heterotic supergravity theory, either by breaking supersymmetry or by leading to non-Kähler internal spaces [48]. Moreover, it has also long been known that the Wilson lines' contribution to H-flux may be associated with global worldsheet anomalies and could thus be inconsistent as string backgrounds [49].

Since for a given choice of Wilson lines and background manifold, the fractional H-flux is completely determined and not a matter of choice, it is important to develop techniques that allow one to compute it in concrete examples to address the above issues. In this work we focus on complete intersection Calabi-Yau (CICY) manifolds (or rather quotients thereof by a freely acting discrete symmetry group) as these provide a well understood class of potentially realistic particle physics models [39–45]. In order to compute the induced H-flux from given Wilson lines we use a class of special Lagrangian submanifolds (sLags) as representatives of the three-cycles of the CICYs. One reason for this is that these sLags are easily explicitly constructed as fixed point loci of certain anti-holomorphic involutions that are completely classified [50]. Furthermore, the intersection theory of sLags is particularly simple. We then show that the projection of the Wilson line and its induced Chern-Simons term on these sLags can be systematically determined. Hence, if the above sLags span a basis for the third homology group (i.e. if the rank of their intersection matrix matches the dimension of the third homology group), the superpotential can be expressed as a linear combination of explicitly computable Chern-Simons invariants on these sLags. Our procedure can then be summarized as follows:

1. Identify sLags in the CICY under consideration, as fixed point sets of isometric anti-holomorphic involutions classified in [50]. We do this in section 3.2.2. Within this classification, we also show how the Wilson lines project onto the sLags in section 3.2.3.

2. Calculate the intersection matrix of the sLags and compare its rank with the dimension of the third homology group. We provide details and further references on how this computation can be done systematically in sections 3.3.1 and 3.3.2.

3. Compute the Chern-Simons invariants on the sLags. To this end we review some results from the mathematics literature on Chern-Simons invariants on three-manifolds in section 3.2.4. In order to apply these results one has to determine the topology of the relevant sLags, and a central role will be played by Seifert fibered manifolds or compositions thereof.

We discuss the consistency of non-trivial H-flux, be it fundamental or induced by Wilson lines, in supersymmetric CY compactifications, recalling subtleties associated with the inclusion of gaugino condensation. On one hand, a dimensional reduction of the 10D effective theory including non-trivial H-flux and possibly fermionic bilinears does not allow for a supersymmetric vacuum on CY internal spaces [11, 12, 51]. On the other hand, including non-perturbative effects together with threshold corrections directly in the 4D effective theory, one can restore supersymmetry [46,47] in an anti-de Sitter vacuum. The 10D description of this 4D solution is not yet understood [11, 12]. We discuss the Chern-Simons contributions to H-flux from both non-standard embeddings and Wilson lines. Chern-Simons fluxes from non-standard embeddings correspond to higher derivative corrections. They preserve the leading order supersymmetric CY compactification, and the would-be α'-corrections to the 4D superpotential vanish for the massless modes due to non-renormalization theorems [8, 24, 52]. Wilson lines, in contrast, can contribute both to leading order H-flux and the superpotential and are therefore potentially dangerous for the consistency of the 10D solution. On a similar note, we also mention the relation between H-flux due to Wilson lines and 2D global worldsheet anomalies [49].

3.1 The heterotic 3-form flux

In this section we will use the notation of the preliminary section 1.3. We will discuss two seemingly contradictory results that are important to bear in mind when considering H-flux in heterotic string compactifications. Whether and how these results are concordant has not been worked out in detail.

- Compactifying leading order heterotic supergravity on CY 3-folds to a supersymmetric 4D maximally symmetric vacuum forces the 3-form flux H to be zero, as we described in section 2.2. This is true even when vacuum expectation values of fermionic bilinears are taken into account in the 10D action [11, 12, 51].
- By including the non-perturbative effects of fermionic condensates and threshold corrections directly in the effective 4D theory of a CY compactification, one can in principle turn on H-flux while simultaneously preserving supersymmetry [46].

This section is therefore largely a review of the literature on various subtleties associated with H-flux and gaugino condensation on CY internal spaces. We will consider in particular the effects of non-trivial Chern-Simons terms in this context. We will also briefly discuss the 4D superpotential from Chern-Simons flux, considering the well known non-renormalization theorem. Finally, we will mention the relation between Chern-Simons flux and global anomalies in the associated 2D sigma model.

H-flux in heterotic compactifications was discussed soon after the foundational work on CY compactifications [53]. The seminal paper by Strominger [48] showed that, for supersymmetric Minkowski solutions, H-flux generates torsion and deforms away from Kählerity[1]. Indeed, the supersymmetry conditions imply $H = *\mathrm{d}J$, so that the (3,0) and (0,3) contributions to H must

[1]CY compactifications with H-flux are, however, possible if we relax the condition of a maximally symmetric 4D external space and consider 4D domain wall solutions [54].

vanish, and the (1,2) and (2,1) contributions induce non-Kählerity. One question that has been considered is then what is the effect of gaugino condensation on these statements, especially as the H-flux and the fermion bilinear, Σ, corresponding to the 4D gaugino condensate, appear in a related way in the 10D theory.

H-flux and gaugino condensation were first considered in [55, 56]. For CY compactifications, the vanishing of the gravitino variation together with the equations of motion requires Σ to vanish [11, 55]. The gaugino condensate in 4D is expected to descend from a non-vanishing expectation value of Σ. This would then imply that gaugino condensation is not compatible with the supersymmetry conditions on CY internal spaces. However, H-flux and gaugino condensation are compatible with a Minkowski $\times$ CY compactification, if we allow supersymmetry to be broken spontaneously [55]. In detail, the condition for 4D Minkowski space fixes $T = 0$, which then leads to non-vanishing supersymmetry transformations for the dilatino and part of the gravitino. Note that satisfying the Minkowski condition $T = 0$ requires balancing the quantized H-flux against non-perturbative effects, which are exponentially small at weak coupling [57]. Dine et al. [55] compared the scalar potential obtained from dimensional reduction with the scalar potential obtained via a superpotential, $W \sim c + Ae^{-aS}$, directly in 4D field theory. The results matched up to power law corrections, which had been neglected in the 10D analysis.

Gukov et al. in [46] later argued from a 4D perspective that a supersymmetric AdS solution is also possible with H-flux and gaugino condensation, provided we include one-loop threshold corrections. A non-vanishing H-flux leads to the well known superpotential [58, 59]

$$W_{\text{flux}} = \int_{Y_3} H \wedge \Omega, \tag{3.1.1}$$

where the internal space Y_3 is assumed to be a CY 3-fold with a holomorphic 3-form Ω. When gaugino condensates are taken into account we also have to include a corresponding term in the superpotential [60]

$$W_{\text{gaugino}} \sim -\mathrm{e}^{-8\pi^2 f/C}, \tag{3.1.2}$$

where f is the holomorphic gauge kinetic function of the gauge group from which the gauginos condense and C is the dual Coxeter number of the gauge group. Gukov et al. [46] showed that an AdS supersymmetric solution is possible in the resulting 4D effective field theory provided that threshold corrections are taken into account so that the gauge coupling function takes the form

$$f = S + \beta T, \tag{3.1.3}$$

where S and T are the dilaton and volume moduli, and βT is the one-loop correction term. From this point of view, however, it is not completely clear if the internal space can remain a CY 3-fold, as we lack a 10D description of the 4D threshold corrections. Also, as was mentioned, the H-flux is generically quantized to integers [57] which would imply that the dilaton is stabilized at strong coupling. However, as was discussed in [46], this problem is ameliorated by using the Chern-Simons contribution to H, which is only fractionally quantized.

An attempt to capture the 4D physics described above within the 10D theory was made by Frey and Lippert in [11], by solving the 10D supersymmetry conditions. However, as we have already seen, it is clear from the leading order 10D equations that the internal manifold cannot be CY, rather, the solutions they found were a product of 4D AdS spacetime and non-complex internal spaces. The treatment of higher order corrections in the 10D theory which correspond to the gaugino condensates with threshold effects in the 4D theory is still missing. In fact it is unclear how to derive the full 4D superpotential from 10D in the presence of H-flux and — in particular — gaugino condensation. Usually, the 4D superpotential can be derived from the gravitino supersymmetry variation. But Frey and Lippert [11] showed that the contributions from the fermion bilinears Σ (the gaugino condensate in 4D) cancel here, so that the 10D theory does not seem to catch the 4D non-perturbative effect (see also [12, 61]).

To summarize, if we have non-trivial H-flux together with a 10D fermion bilinear, both non-supersymmetric Minkowski $\times$ CY compactifications [55] and supersymmetric AdS $\times$ non-CY compactifications [11, 12, 15, 16, 61] are possible. Matching these solutions to a corresponding solution obtained directly in 4D (with gaugino condensates) is non-trivial and not fully understood. As for supersymmetric CY compactifications with non-trivial H-flux and gaugino condensation, a 4D construction that also relies on threshold effects was given in [46] (see also [47]). A 10D construction of these solutions has so far not been obtained, as – at leading order – H-flux and fermion bilinears in the equations of motion are not compatible with vanishing supersymmetry transformations.

3.1.1 The Chern-Simons flux

For the heterotic string, the 3-form H, i.e. the gauge invariant field strength for the Kalb-Ramond 2-form B, is given not simply by $\mathrm{d}B$, but rather as:

$$H = \mathrm{d}B - \frac{\alpha'}{4}\left(\omega_{3\mathrm{Y}} - \omega_{3\mathrm{L}}\right), \tag{3.1.4}$$

where the 3-form $\omega_{3\mathrm{Y}}$ is the Chern-Simons form

$$\omega_{3\mathrm{Y}} = \mathrm{tr}\left(A \wedge F - \frac{1}{3} A \wedge A \wedge A\right), \tag{3.1.5}$$

which locally satisfies $\mathrm{d}\omega_{3\mathrm{Y}} = \mathrm{tr}F \wedge F$, and similar expressions can be written down for the Lorentz Chern-Simons form $\omega_{3\mathrm{L}}$. The Bianchi identity for H therefore has a non-trivial contribution on the right hand side:

$$\mathrm{d}H = \frac{\alpha'}{4}\left(\mathrm{tr}R \wedge R - \mathrm{tr}F \wedge F\right), \tag{3.1.6}$$

which requires $P_1(V;\mathbb{R}) = P_1(T;\mathbb{R})$, that is, the first Pontryagin classes over real numbers for the tangent bundle and vector bundle should be equal. It is important to note that – despite the appearance of α' in both Chern-Simons contributions – the Yang-Mills contribution is actually leading order in the derivative expansion [2].

[2] For a nice and exhaustive treatment of the α' vs. higher derivative-expansion in heterotic M-theory and its relation to 10d heterotic supergravity is given in [62, 63].

As a result of the Chern-Simons contributions to H, we can have a non-zero H-flux, even if we choose $\mathrm{d}B = 0$ globally. The full expression for the H-flux superpotential is:

$$W = \int \left[\mathrm{d}B - \frac{\alpha'}{4}\omega_{3\mathrm{Y}}\right] \wedge \Omega\,. \tag{3.1.7}$$

Note that the Lorentz Chern-Simons term in H does not contribute to W because it appears at higher order in the derivative expansion, whilst the superpotential does not receive any perturbative corrections beyond the leading order term [8, 51]. We now consider the Yang-Mills Chern-Simons contribution to W. The Yang-Mills Chern-Simons term in H can give rise to a background H-field via both the non-standard embedding and Wilson lines. These, however, affect the background solution and W in different ways.

A non-standard embedding solves the leading order supersymmetry conditions using a holomorphic connection on a holomorphic stable vector bundle. However, imposing also the leading order Bianchi identity, $\mathrm{d}H = -\frac{\alpha'}{4}\mathrm{tr}F \wedge F$, implies $F = 0$ and vanishing background gauge field [8]. The non-trivial gauge field and the torsion due to H-flux is induced only when balancing with the higher derivative effects, from the Lorentz Chern-Simons contribution, in the integrated Bianchi identity. The non-renormalization theorem then implies that H-flux due to the non-standard embedding does not contribute to[3] W. Moreover, the non-renormalization theorem can then be used to argue that the non-standard embedding is a consistent solution to all finite orders in perturbation theory [8]. Indeed, as $W = \mathrm{d}W = 0$ in the background at leading order, this must remain true to all finite orders, and there exists a supersymmetric 4D Minkowski solution. The internal geometry is Calabi-Yau at leading order, and receives corrections at higher order. In contrast to the non-standard embedding, we will see next that Wilson lines are a wholly leading order effect. A non-trivial H-flux induced by Wilson lines may thus contribute to the background W, and spoil the consistency of the leading order supersymmetric Calabi-Yau compactification. Whether or not consistency can be restored by higher loop effects is an open question.

3.1.2 Wilson lines

Wilson lines are flat vector bundle connections, that is, non-trivial gauge configurations with $F = 0$ but a global restriction to setting $A = 0$ everywhere. In particular, when the fundamental group of the CY is non-trivial, we can define the gauge invariant Wilson line operator, which is an embedding of $\pi_1(Y_3)$ into the gauge group G:

$$\mathrm{WL}_\gamma = \mathrm{P}\exp\left(i\int_\gamma A^j T_j\right)\,, \tag{3.1.8}$$

[3]Note that the non-renormalization theorem only applies to the light modes in the low energy effective field theory. A non-standard choice of holomorphic stable vector bundle in general fixes some of the would-be CY-moduli by obstructing the corresponding geometric deformations. Formal inclusion of these massive fluctuations in the low energy theory then does lead to a non-trivial W for those modes and reproduces their expected stabilization from a 4D point of view [24, 29–31].

where γ is a non-trivial homotopy cycle on the CY space, and $\mathrm{P}\exp$ denotes the path ordered exponential. As $b_1(Y_3) = 0$, there are no Wilson line moduli or corresponding *continuous* Wilson lines in CY compactifications. Instead we can have at most *discrete* Wilson lines corresponding to a finite fundamental group on a CY.

Discrete Wilson lines were introduced into CY compactifications as a way to break the gauge symmetry without breaking supersymmetry [8, 64]. Indeed, since $F = 0$, they do not contribute to the Yang-Mills supersymmetry equations. However, they may still contribute non-trivially to the other supersymmetry conditions and equations of motion via the Chern-Simons term in H, eq. (3.1.4). Moreover, any H-flux and torsion induced by Wilson lines is leading order, as A is non-trivial although $F = 0$ exactly and is vanishing in the Bianchi identity. Therefore, Wilson lines can contribute to the background superpotential. Notice that only the (0,3) and harmonic part of $\omega_{3\mathrm{Y}}$ contributes [47].

3.1.3 Chern-Simons invariants and global worldsheet anomalies

The Chern-Simons contribution to the superpotential can be expressed in terms of a Chern-Simons invariant. Indeed, we can write

$$W = -\frac{\alpha'}{4}\int_{\mathrm{Y}_3} \omega_{3\mathrm{Y}} \wedge \Omega = -\frac{\alpha'}{4}\int_{\Lambda} \omega_{3\mathrm{Y}} \ , \tag{3.1.9}$$

where Λ is the 3-cycle Poincaré dual to the holomorphic 3-form Ω. In general, the Chern-Simons invariant cannot be computed directly, as an expression for the gauge field is not known. Indeed, the gauge field A is neither uniquely nor globally defined.

Chern-Simons invariants for flat vector bundles have been well-studied in the mathematics literature. In particular the Chern-Simons invariant

$$CS(A, Q) = \int_Q \omega_{3\mathrm{Y}} \ , \tag{3.1.10}$$

has been computed explicitly for several real 3-dimensional manifolds, denoted here by Q. In section 3.2.4, we summarize the known results on Chern-Simons invariants for a large class of real 3-manifolds. Among the simplest examples that give a non-trivial Chern-Simons invariant are the Lens spaces $S^3/\mathbb{Z}_p$ for which one obtains [49, 65–67]:

$$CS(A, S^3/\mathbb{Z}_p) = -\sum_a \frac{k_a^2}{2p} \ \mathrm{mod}\ \mathbb{Z} \tag{3.1.11}$$

for a gauge connection A with the Wilson line fitting into SU(N) as specified by the integers k_a,

$$U = \mathrm{diag}(\mathrm{e}^{2\pi \mathrm{i} k_1/p}, \ldots, \mathrm{e}^{2\pi \mathrm{i} k_N/p}) \ . \tag{3.1.12}$$

It is obvious from this example that the Chern-Simons invariant can take fractional values; in fact it is only defined modulo integers, as large gauge transformations shift $CS(A, Q)$ by integer values. This is precisely the reason why [46] suggested to use Chern-Simons flux instead of the integer quantized $\mathrm{d}B$-flux for moduli stabilization as it facilitates the balance between flux and non-perturbative effects at weak coupling. This proposal has recently been discussed in a wider context in ref. [47], where it was found that even the fractional Chern-Simons flux would generically lead to GUT scale supersymmetry breaking. From a phenomenological point of view, it is thus very important to know whether a non-trivial Chern-Simons invariant is induced by a given set of Wilson lines. This is also true for ensuring the mathematical self-consistency of such a scenario, as the mutual consistency of unbroken supersymmetry, internal CY geometry, and non-trivial 3-form flux could so far not be rigorously established from a purely 10D or even a worldsheet point of view. Regarding the consistency of the 2D theory the situation may be even more demanding due to worldsheet anomalies that cannot be cancelled with any known methods[4]. More specifically this case occurs when $CS(A, Q)$ is fractional for a 3-manifold Q that corresponds to a torsion class of $H_3(Y_3, \mathbb{Z})$ [46, 49]. Motivated by all this, it is the purpose of the present work to explicitly compute Chern-Simons invariants induced by Wilson lines on a class of phenomenologically realistic CY spaces.

3.2 Computing Chern-Simons flux in explicit models

We will now proceed to develop a strategy to compute the Chern-Simons flux and its superpotential for Calabi-Yau compactifications with Wilson lines, and apply this strategy to some explicit models with promising phenomenology. More concretely our focus is on complete intersection Calabi-Yau (CICY) 3-folds, which are common setups for model building in [39–42, 44, 45].

3.2.1 Special Lagrangian submanifolds

In order to compute the Chern-Simons fluxes in CY compactifications, we will need to construct explicit 3-cycles, which the fluxes thread. We will therefore consider special Lagrangian submanifolds (sLags), which provide explicit representatives of 3-cycles in a CICY space (quickly introduced in 2.2.3) and moreover have a particularly simple intersection theory. Slags in a CY space are real 3D submanifolds defined by the conditions:

$$J\,\Big|_Q = 0 \quad \text{and} \quad \mathrm{Im}(\mathrm{e}^{i\frac{\theta}{2}}\Omega)\Big|_Q = 0\,, \tag{3.2.1}$$

with J the Kähler 2-form, and θ is the so-called calibration angle associated with the sLag (see [68, 69] for some introductory lectures on these geometries). They are volume minimizing in their homology class, with the volume form given by

$$\mathrm{Re}(\mathrm{e}^{i\frac{\theta}{2}}\Omega)\Big|_Q = \mathrm{dVol}_Q\,. \tag{3.2.2}$$

[4]In fact, the relationship between Wilson lines and global worldsheet anomalies was used in [49] to indirectly compute the Chern-Simons invariant on the Lens space.

Although general sLag submanifolds are difficult to construct explicitly, there is one well-known method to obtain examples. An isometric anti-holomorphic involution[5] σ acts on the CY manifold as

$$\sigma(J) = -J \quad \sigma(\Omega) = \overline{e^{i\theta}\Omega}\,. \tag{3.2.3}$$

Therefore, the fixed locus of σ is a sLag submanifold; we will write this as

$$Q_\sigma = \mathrm{Fix}(\sigma)\,, \tag{3.2.4}$$

where Q_σ is the sLag and $\mathrm{Fix}(\sigma)$ denotes the fixed point set of the involution σ. Given a CICY with defining polynomials P_a, an isometric anti-holomorphic involution σ on the ambient space descends to the CICY if it satisfies

$$P_i \circ \sigma = \bar{P}_i. \tag{3.2.5}$$

The sLag submanifolds in a CICY are therefore 3D submanifolds and give rise to 3-cycles, which we can construct and analyze explicitly using the defining polynomials. As we will see in sections 3.3.1, 3.3.2, their intersection theory is also simple, so that it is straightforward to check whether a given set of sLags generates the full third homology group of the CICY. Furthermore, all the information required can be obtained by going to a simple point in moduli space, that is, choosing a particularly symmetric form of the defining polynomials, for which we can find many homologically distinct sLags. Let Q_σ be one such sLag. As mentioned, different polynomials corresponding to the same configuration matrix determine manifolds that are diffeomorphic, so if $\tilde{Y}_3$ is another CICY corresponding to the same configuration matrix as Y_3, then there exists a diffeomorphism f between Y_3 and $\tilde{Y}_3$. The restriction of f to Q_σ defines a submanifold $f(Q_\sigma)$ in $\tilde{Y}_3$, which may or may not be a sLag (in fact, sLags turn out to be surprisingly stable under deformations of the CY structure [68]). As we are interested in topological properties of the sLags as representatives of their homology class, namely their Chern-Simons invariants, our final results will be independent of these choices.

3.2.2 A classification of sLags in CICYs

We will now provide a classification of the sLags in CICYs, which correspond to the fixed point sets of isometric anti-holomorphic involutions. We will start with relevant involutions on the ambient space; these will descend to the CICY when the condition (3.2.5) is satisfied. Isometric anti-holomorphic involutions on $\mathbb{C}P^n$ can be classified into two different types, A and B which act on the coordinates in the following way [50]

$$\begin{aligned} \sigma_A : (z_1, z_2, \ldots, z_n, z_{n+1}) &\mapsto (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n, \bar{z}_{n+1}), && (3.2.6)\\ \sigma_B : (z_1, z_2, \ldots, z_n, z_{n+1}) &\mapsto (-\bar{z}_2, \bar{z}_1, \ldots, -\bar{z}_{n+1}, \bar{z}_n)\,. && (3.2.7)\end{aligned}$$

Note that σ_B applies only for projective spaces $\mathbb{C}P^n$ with n odd. All other involutions of $\mathbb{C}P^n$ can be constructed by a projective $GL(n+1,\mathbb{C})$ transformation acting on either σ_A or σ_B [50],

$$\sigma^U_{A,B} = U^{-1} \circ \sigma_{A,B} \circ U\,. \tag{3.2.8}$$

[5] The isometricity property is $\sigma(g) = g$, whereas anti-holomorphicity is $\sigma(I) = -I$ for I the complex structure. Also $J = Ig$, and g, J only define Ω up to a phase, $J \wedge J \wedge J = \frac{3}{4} i\Omega \wedge \bar{\Omega}$.

We will use the terminology $A(B)$-type involution for an involution that is constructed by the action of $GL(n+1, \mathbb{C})$ on $\sigma_A(\sigma_B)$. Note that B-type involutions act freely on $\mathbb{C}P^n$ and therefore Fix(σ_B^U) is empty for all $GL(n+1, \mathbb{C})$ transformations U. For the A-type involutions, Fix(σ_a^U) is non-empty and furthermore

$$\begin{aligned} \text{Fix}(\sigma_A^U) &= \{z \in \mathbb{C}P^n \,|\, \sigma_A^U(z) = z\} \\ &= \{z \in \mathbb{C}P^n \,|\, U^{-1}\overline{Uz} = z\} \\ &= U^{-1}\{(z' = Uz) \in \mathbb{C}P^n \,|\, \overline{z'} = z'\} \\ &= U^{-1}\text{Fix}(\sigma_A) \,. \end{aligned} \tag{3.2.9}$$

Applying this to a CY hypersurface in $\mathbb{C}P^n$, we see that if σ_A is an involution on the CY, then all matrices U that are symmetries of the defining polynomial will give involutions σ_A^U on the CICY, and the corresponding sLags are

$$Q_{\sigma_A^U} = U^{-1}(Q_{\sigma_A}) \,. \tag{3.2.10}$$

This is an important result that, in particular, shows that all A-type sLags are homeomorphic $Q_{\sigma_A^U} \sim Q_{\sigma_A}$.

In the following, it will sometimes be useful to write the A-type involutions in terms of the matrices $M \equiv U^{-1}\overline{U}$,

$$\sigma_A^U = M \circ \sigma_A, \tag{3.2.11}$$

where M is a symmetry of the polynomial equations.

The A-type involutions on $\mathbb{C}P^n$ generalize to products of projective spaces, for which the basic A-type involutions act individually on each factor with complex conjugation

$$(\sigma_A, \sigma_A, \ldots, \sigma_A) : \mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \cdots \times \mathbb{C}P^{n_k} \to \mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \cdots \times \mathbb{C}P^{n_k} \,. \tag{3.2.12}$$

The fixed point set is given by

$$\text{Fix}(\sigma_A, \sigma_A, \ldots, \sigma_A) = \mathbb{R}P^{n_1} \times \mathbb{R}P^{n_2} \times \cdots \times \mathbb{R}P^{n_k} \,. \tag{3.2.13}$$

A general A-type involution is now given by the map

$$(M_1 \circ \sigma_A, \ldots, M_k \circ \sigma_A) : \mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_k} \to \mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_k} \,. \tag{3.2.14}$$

where the matrices $M_1, \ldots, M_k$ are given in terms of $GL(n_i + 1, \mathbb{C})$ transformations $M_i = U_i^{-1}\overline{U}_i$. The fixed point set in this case is given by

$$(U_1^{-1}, U_2^{-1}, \cdots, U_k^{-1})\text{Fix}(\sigma_A, \sigma_A, \ldots, \sigma_A) \,. \tag{3.2.15}$$

In this chapter we will only make use of diagonal matrices U to generate sLags, and the condition (3.2.5) will then often force the diagonal elements to be roots of unity.

When we have a product space of two identical projective spaces $\mathbb{C}P^n \times \mathbb{C}P^n$ there is another type of involution, which we will call C [50]:

$$\sigma_C : (z_a, w_a) \mapsto (\bar{w}_{\bar{a}}, \bar{z}_{\bar{a}}) \,. \tag{3.2.16}$$

It is easy to see that the fixed point set of σ_C is the diagonal in $\mathbb{C}P^n \times \mathbb{C}P^n$,

$$\text{Fix}(\sigma_C) = \{(z, \bar{z}) \in \mathbb{C}P^n \times \mathbb{C}P^n\} . \tag{3.2.17}$$

All C-type involutions can be constructed by a pair of $GL(n+1, \mathbb{C})$ transformations U_1 and U_2

$$\Big(M,\ \overline{M}^{-1}\Big) \circ \sigma_C : \mathbb{C}P^n \times \mathbb{C}P^n \to \mathbb{C}P^n \times \mathbb{C}P^n , \tag{3.2.18}$$

where $M = U_1^{-1}\overline{U}_2$ and the fixed point set is found to be

$$\big(U_1^{-1},\ U_2^{-1}\big)\ \text{Fix}(\sigma_C) . \tag{3.2.19}$$

Therefore, assuming that (U_1, U_2) is a symmetry of the defining polynomials of the CICY, it gives rise to a sLag $Q_{\sigma_C^{(U_1,U_2)}}$, which is homeomorphic to the basic C-type sLag Q_{σ_C}. Here, as for the A-type sLags, we will restrict our attention to diagonal matrices U_1 and U_2 which by (3.2.5) usually forces the elements to be roots of unity.

Having identified sLags via the isometric anti-holomorphic involutions of the CICY, an important question will be how the quotient symmetry Γ, which is freely acting on the CICY, acts on the sLags. We will now turn to this and related questions.

3.2.3 Wilson lines on sLags

Our objective is to compute the contribution from discrete Wilson lines to the Chern-Simons invariant on a given sLag. Consider a field ϕ on a quotient CY, Y_3/Γ, transforming in some non-trivial representation of the GUT gauge group. Each element, g, of the fundamental group, Γ, of Y_3/Γ defines an action of Γ on ϕ by parallel transport with respect to the gauge connection,

$$\mathrm{g} : \phi \mapsto \mathrm{WL}_{\mathrm{g}} \cdot \phi , \tag{3.2.20}$$

where

$$\mathrm{WL}_{\mathrm{g}} = \mathrm{P}\exp\left(i \int_{\gamma_{\mathrm{g}}} A^j T_j\right) \tag{3.2.21}$$

is the Wilson line operator with a homotopy loop γ_{g} corresponding to g, and the dot refers to the action on ϕ induced by its gauge group representation. Without Wilson lines, this action is of course trivial. As the fundamental group Γ is discrete and the Wilson line operators define a group homorphism, it is sufficient to specify the Wilson line operators, WL_g, corresponding to the *generators*, g, of Γ.

Now consider the field $\phi|_Q$ restricted to a sLag, Q, of Y_3. Since Γ acts freely on Y_3, we encounter two possibilities for the action of each generator g of Γ on the sLag $Q \subset Y_3$ (see figure 3.1):

- g maps Q pointwise to *another* sLag $Q' \subset Y_3$ so that Q and Q' are identified in Y_3/Γ. In this case, any Wilson line WL_g on Q on the quotient space Y_3/Γ would have to be already present on Q in the covering space Y_3. On Y_3, however, the homotopy loop γ_g would be contractible so the projection of the Wilson line on Q must vanish. If this is true for all generators g of Γ, it means that all Wilson line operators project to the identity on the sLag Q, and hence they can never give rise to a non-trivial Chern-Simons invariant on Q in the quotient space Y_3/Γ.

- If instead g acts freely *within* Q, then the corresponding sLag Q/Γ in the quotient space Y_3/Γ may acquire a new homotopy loop on which the Wilson line on Y_3/Γ projects non-trivially. In this case, there is the possibility to have a non-trivial Chern-Simons invariant on the sLag Q/Γ.

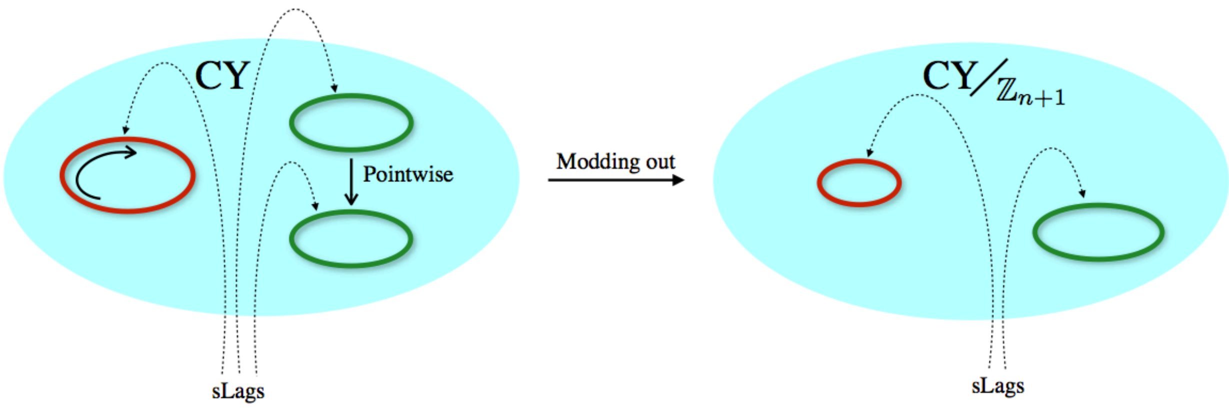


Figure 3.1: A cartoon of the two possibilities for the free action of a generator of Γ on sLags. The black arrows depict the action of the generator. On the left hand side we show the action on the covering CY. It can act freely within the sLag (red), or it can identify two (or more) distinct ones (green). On the right hand side we can see what happens in the quotient CY. The red sLag can have modified topology, because of the free action of the generator. The green sLags are simply identified, and the resulting sLags have the same topology as before. Only the red sLags can possibly inherit a non-trivial CS invariant from a Wilson line on the quotient Calabi-Yau.

Having classified a large set of sLags in the CICY as in subsection 3.2.2, our next task is then to determine how the discrete symmetry Γ, by which we quotient, acts on them. Only sLags Q that are mapped to themselves by at least one generator g of Γ, can have non-trivial Wilson lines and hence possible Chern-Simons invariants on their quotients Q/Γ. As we will now see, this is a model independent question. Whether or not a non-trivial Chern-Simons terms on such a quotient sLag is then really induced, depends also on its topology and the details of the Wilson line in Y_3/Γ and will be discussed further below.

The discrete symmetry groups usually encountered and considered in following are rotations and permutations. We take $\Gamma = \mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1}$, where the first $\mathbb{Z}_{n+1}$ factor refers to rotations, R, and the second to cyclic permutations, S, of the coordinates of $\mathbb{C}P^n$. When we specify how the discrete symmetry group Γ acts on the coordinates of $\mathbb{C}P^n$, we implicitly fix some or all of the coordinate freedom of this ambient space. We give the action of these symmetries in terms of their respective generators, g_R and g_S. The rotations are generated by

$$g_R : z_a \mapsto \omega^a z_a, \qquad a = 1, \dots, n+1, \tag{3.2.22}$$

where ω is the primitive $(n+1)$-th root of unity. The generator of the cyclic permutations acts as

$$g_S : z_a \mapsto z_{a+1}, \qquad a = 0, \dots, n, \quad z_0 := z_{n+1} \, . \tag{3.2.23}$$

Note that R and S have fixed points on $\mathbb{C}P^n$, but the CICY under consideration will not contain these fixed points.

A-type sLags

We begin by discussing the action of the generators $g \in \Gamma$ on the basic A-type sLags, i.e. the fixed point loci of σ_A or, more generally, σ_A^U. As these involutions do not mix different ambient $\mathbb{C}P^n$'s, it is sufficient to restrict our discussion to a single $\mathbb{C}P^n$ factor.

Rotations R**:** We first consider the action of the rotations generated by g_R on the A-type sLags. We can treat the basic A-type sLag based on the involution σ_A as a special case of the more general case corresponding to σ_A^U. The original sLag $Q_{\sigma_A^U}$ is associated with the fixed point set $\mathrm{Fix}(\sigma_A^U) = \{z \in \mathbb{C}P^n \mid U^{-1}\overline{Uz} = z\}$. The rotation g_R maps this to the fixed point set

$$g_R \mathrm{Fix}(\sigma_A^U) = \mathrm{Fix}(\sigma_A^{Ug_R^{-1}}) \, . \tag{3.2.24}$$

Note that due to the projective identification, $z \sim \lambda z$, this is the same as the original fixed point set if

$$g_R U^{-1} \overline{U} g_R = \lambda U^{-1} \overline{U}, \tag{3.2.25}$$

where we used $g_R^{-1} = \overline{g_R}$ and λ is a phase factor. Because $g_R^{n+1} = 1$, re-iterating this equation implies $\lambda^{n+1} = 1$, i.e. λ is an integer power of the primitive $(n+1)$th root of unity, $\lambda = \omega^l$, $l \in \mathbb{Z}$. For diagonal U, the condition (3.2.25) becomes

$$g_R^2 = \lambda \mathbf{1}, \tag{3.2.26}$$

which is only satisfied if n, the dimension of the ambient space, equals one. We therefore see that if $n > 1$ the rotational symmetry g_R always maps the sLag based on σ_A^U to a different sLag, so that there can be no Wilson lines or Chern-Simons invariant induced by rotational identifications on any A-type sLag.

If on the other hand, the ambient space is $\mathbb{C}P^1$, the rotational symmetry is $R \cong \mathbb{Z}_2$ and the generator g_R automatically satisfies (3.2.26) with $\lambda = 1$. In this case, the generator g_R maps the original sLag (non-trivially) to itself, and a Chern-Simons invariant might in principle be induced on any A-type sLag by a Wilson line associated with the generator g_R.

In our examples in section 3.3, only the first case with $n > 1$ will occur so that we do not have to worry about rotational identifications and their associated Wilson lines on Y_3/Γ.

Cyclic permutations S: Next we consider the $(n+1) \times (n+1)$ matrices g_S corresponding to the cyclic permutations (3.2.23). As they are real, the condition for g_S to map a sLag based on the involution σ_A^U to itself, and hence to induce possible Wilson lines and Chern-Simons invariants, is not of the form (3.2.25), but rather:

$$g_S U^{-1} \overline{U} g_S^{-1} = \lambda U^{-1} \overline{U}. \tag{3.2.27}$$

Let us now give the most general solution of (3.2.27) for a diagonal matrix $U = \text{diag}(u_1, \ldots, u_{n+1})$ that is assumed to be a symmetry of the defining polynomial of the CY-space Y_3. Obviously, $U^{-1}\overline{U} = \text{diag}(\mu_1, \ldots, \mu_{n+1})$ with $\mu_i \equiv \overline{u}_i / u_i$, and the left hand side of (3.2.27) becomes

$$g_S U^{-1} \overline{U} g_S^{-1} = \text{diag}(\mu_2, \mu_3, \ldots, \mu_{n+1}, \mu_1). \tag{3.2.28}$$

It is then easily seen that the general solution of (3.2.27) is given by

$$U^{-1}\overline{U} = \mu_{n+1}\, \text{diag}(\lambda, \lambda^2, \ldots, \lambda^n, 1), \quad \lambda = \omega^l, \quad l \in \mathbb{Z}. \tag{3.2.29}$$

Any A-type sLag on Y_3 based on a matrix U that satisfies this equation for some $l \in \mathbb{Z}$ is then mapped to itself by g_S and possibly gives rise to a non-trivial Wilson line and Chern-Simons invariant on the corresponding quotient sLag.

We now show, however, that in many cases (and in particular in all cases we study in this chapter) this apparent multitude of sLags with potential Chern-Simons terms actually collapses to just the basic A-type sLag corresponding to the simple involution σ_A when also the rotational symmetries R are modded out. More precisely, we show that for nl even, any A-type sLag that satisfies (3.2.29) is identified with the basic A-type sLag by modding out the rotation $g_R^{nl/2}$.

In order to prove this, one needs to find an integer k such that $g_R^k\, \text{Fix}(\sigma_A^U) = \text{Fix}(\sigma_A)$, i.e.

$$g_R^k U^{-1} \overline{U} g_R^k \propto \mathbf{1}\ . \tag{3.2.30}$$

Using (3.2.29), the left hand side of (3.2.30) becomes

$$\begin{aligned} g_R^k U^{-1} \overline{U} g_R^k &= \mu_{n+1}\, \text{diag}(\lambda\omega^{2k}, \lambda^2\omega^{4k}, \ldots, \lambda^n \omega^{2nk}, 1) \\ &= \mu_{n+1}\, \text{diag}(\omega^{l+2k}, \omega^{2(l+2k)}, \omega^{3(l+2k)}, \ldots, \omega^{n(l+2k)}, 1)\ , \end{aligned} \tag{3.2.31}$$

which is proportional to the identity for $2k = -l \bmod n+1 = nl \bmod n+1$. This then implies:

- **n even:** Every A-type sLag that satisfies (3.2.29) is mapped to the basic A-type sLag corresponding to σ_A by the rotation $g_R^{nl/2}$. Thus, for $\mathbb{Z}_{\text{odd}}$, one only has to check whether this *basic* A-type sLag inherits a Chern-Simons invariant from the Wilson line associated with the permutation g_S.

- **n odd:** In this case, all A-type sLags that satisfy (3.2.29) with l *even* are also identified with the basic A-type sLag upon modding out by $g_R^{-l/2}$ and hence don't have to be studied separately. On the other hand, the sLags that satisfy (3.2.29) with l *odd* are not mapped to the basic A-type sLag, but rather the one corresponding to $\sigma_A^{\sqrt{g_R}}$. This is because if we choose k such that $l + 2k = -1 \bmod n+1$, we see that eq. (3.2.31) implies

$$g_R^k U^{-1} \overline{U} g_R^k \propto g_R^{-1} = g_R^{-1/2} \cdot \overline{g_R^{1/2}} \,. \tag{3.2.32}$$

It should be noted that for n odd, $g_R^{1/2}$ is in general not a symmetry of the polynomial, but still satisfies (3.2.5) because $\sigma_A^{\sqrt{g_R}} = g_R^{-1/2} \circ \sigma_A \circ g_R^{1/2} = g_R^{-1} \circ \sigma_A$ and g_R is by assumption a symmetry of the polynomials. Hence Fix($\sigma_A^{\sqrt{g_R}}$) is still a sLag, but it is not necessarily homoeomorphic to the basic A-type sLag.

For n odd, we therefore may have possible non-trivial Chern-Simons invariants on the basic A-type sLag and *one* other A-type sLag corresponding to $\sigma_A^{\sqrt{g_R}}$, which have to be studied separately. In our examples, however, n is always even and this case does not occur.

To summarize: If one mods out by the group $\Gamma = R \times S \cong \mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1}$ of rotations *and* cyclic permutations, and if $(n+1)$ is *odd*, the only A-type sLag one has to check for a possible Chern-Simons invariant is the basic one based on the simple involution σ_A, and one only has to consider Wilson lines due to g_S. This will be the case for all the examples discussed in section 3.3. If $(n+1)$ is *even*, by contrast, one further A-type sLag might carry non-trivial Chern-Simons invariants on the quotient space Y_3/Γ due to modding out cyclic permutations S. For the special case $(n+1) = 2$, Chern-Simons invariants might also occur from modding out certain rotations R (see table 3.1).

C-type sLags

The C-type sLags are fixed point sets of involutions (3.2.16) or (3.2.18) that involve the exchange of the coordinates of two $\mathbb{C}P^n$-factors in an ambient space $\mathbb{C}P^n \times \mathbb{C}P^n$. This leaves some freedom in defining the action of the symmetries R and S on each factor. We will consider transformations generated by (g_R, g_R^{-1}) and (g_S, g_S), as these are precisely of the form we will encounter in our explicit examples in section 3.3.

To begin with, let us recall the fixed point sets of the involutions σ_C and $\sigma_C^{(U_1,U_2)}$:

$$\begin{aligned} \text{Fix}(\sigma_C) &= \{(z,w) \in \mathbb{C}P^n \times \mathbb{C}P^n \mid \overline{z} = w\} && (3.2.33) \\ \text{Fix}(\sigma_C^{(U_1,U_2)}) &= \{(z,w) \in \mathbb{C}P^n \times \mathbb{C}P^n \mid U_2^{-1}\overline{U_1 z} = w\}, && (3.2.34) \end{aligned}$$

where U_1 and U_2 are independent elements of $GL(n+1,\mathbb{C})$. Due to the projective identifications, two sLags associated to $\sigma_C^{(U_1,U_2)}$ and $\sigma_C^{(U_1',U_2')}$ are equivalent whenever $U_2^{-1}\overline{U}_1 = \lambda U'^{-1}_2 \overline{U'}_1$ for some $\lambda \in \mathbb{C}$.

We now consider the action of the discrete symmetries R and S on C-type sLags.

Rotations: From (3.2.16), (3.2.17) and (3.2.18) one can see that the generator of the rotation, (g_R, g_R^{-1}), acts on the sLag associated to $\sigma_C^{(U_1,U_2)}$ in the following way:

$$(g_R, g_R^{-1})\mathrm{Fix}(\sigma_C^{(U_1,U_2)}) = \{(z,w) \in \mathbb{C}P^n \times \mathbb{C}P^n \mid g_R^{-1}U_2^{-1}\overline{U_1}g_R\overline{z} = w\}. \tag{3.2.35}$$

We are interested in the case when this action maps a given sLag non-trivially to itself. This is the case when

$$g_R^{-1}U_2^{-1}\overline{U}_1 g_R = \lambda U_2^{-1}\overline{U}_1, \qquad \lambda \in \mathbb{C}. \tag{3.2.36}$$

Note that this equation differs from (3.2.25) in an important way because the first g_R is inverted. When U_1 and U_2 are diagonal matrices and commute with g_R, eq. (3.2.36) is always satisfied for *any* U_1, U_2. Hence, (g_R, g_R^{-1}) acts freely within each C-type sLag associated to $\mathrm{Fix}(\sigma_C^{(U_1,U_2)})$. A Wilson line can thus project non-trivially to any of them, and hence all C-type sLags could a priori inherit a Chern-Simons invariant from a Wilson line associated with modding out a rotation.

Cyclic permutations: The generator, (g_S, g_S), of a cyclic permutation maps the fixed point set of a C-type involution $\sigma_C^{(U_1,U_2)}$ to itself whenever the following equation is satisfied:

$$g_S U_2^{-1}\overline{U}_1 g_S^{-1} = \lambda U_2^{-1}\overline{U}_1, \qquad \lambda = \omega^l. \tag{3.2.37}$$

In contrast to g_R, g_S is not diagonal, and hence it does not in general commute with $U_2^{-1}\overline{U}_1$. In analogy with the A-type involutions, we have $U_2^{-1}\overline{U}_1 = \mathrm{diag}(\mu_1, \ldots, \mu_{n+1})$ with $\mu_i \equiv \overline{u}_i^{(2)}/u_i^{(1)}$, where $u_i^{(j)}$ is the ith diagonal element of U_j, so that the left hand side of (3.2.37) becomes

$$g_S U_2^{-1}\overline{U}_1 g_S^{-1} = \mathrm{diag}(\mu_2, \mu_3, \ldots, \mu_{n+1}, \mu_1). \tag{3.2.38}$$

It is then easily seen that the general solution of (3.2.37) is given by

$$U_2^{-1}\overline{U}_1 = \mu_{n+1}\mathrm{diag}(\lambda, \lambda^2, \ldots, \lambda^n, 1), \quad \lambda = \omega^l. \tag{3.2.39}$$

Any sLag on Y_3 based on matrices (U_1, U_2) that satisfies this equation for some $l \in \mathbb{Z}$ is thus mapped to itself by g_S and could possibly give rise to a non-trivial Chern-Simons invariant on the corresponding quotient sLag.

As we did for A-type involutions, we can try to see if we can rotate the sLag corresponding to such a $\sigma_C^{(U_1,U_2)}$ to the basic one. However, this is not possible here, since, as seen above, any rotation (g_R^m, g_R^{-m}) ($\forall m \in \mathbb{Z}_{n+1}$) only maps a C-type sLag to itself.

To summarize: Wilson lines associated with permutations S and rotations R may project non-trivially to the basic C-type sLag, which could thus inherit a non-trivial Chern-Simons invariant from both these Wilson lines. The more general C-type sLags associated to $\sigma_C^{(U_1,U_2)}$, on the other hand, are likewise sensitive to any Wilson lines associated to R, but carry Wilson line projections corresponding to permutations S only when (3.2.39) is satisfied. Thus these general C-type sLags have to be checked for corresponding Chern-Simons invariants as well (see table 3.1).

$\Gamma = \mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1}$	Fix(σ_A)	Fix(σ_A^U)	Fix(σ_C)	Fix($\sigma_C^{(U_1,U_2)}$)
$n = 1$	g_R, g_S	g_R, g_S♦	g_R, g_S	g_R, $g_S^♣$
n even	g_S	g_S♢	g_R, g_S	g_R, $g_S^♣$
$n > 1$ odd	g_S	g_S♦	g_R, g_S	g_R, $g_S^♣$

Table 3.1: In the table we summarize the cases encountered for the action of the generators g_R and g_S of the symmetry group $\Gamma = R \times S \cong \mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1}$ on the A- and C-type sLags . In the first row we label the sLags associated to their involutions. The entries indicate which generators map the sLags non-trivially into themselves and hence could potentially induce non-trivial Chern-Simons invariants. The symbol ♢ means that the corresponding sLag is mapped into Fix(σ_A) by the action of R if (3.2.30) is satisfied so that one does not have to study it separately for the Wilson lines of g_S. The symbol ♦ indicates that the corresponding sLag is either mapped to Fix(σ_A) (and hence does not have to be studied separately) or to Fix($\sigma_A^{\sqrt{g_R}}$) by the action of R (if (3.2.30) is satisfied). Which of these two possibilities is realized depends on whether l in (3.2.30) is even or odd, respectively. The superscript ♣, finally, means that the generator g_S maps the sLag into itself only if (3.2.39) is satisfied.

3.2.4 Chern-Simons invariants on Seifert fibered 3-manifolds

In the previous subsections, we have provided a classification of particular 3D submanifolds, sLags, that can be explicitly constructed in CICYs. We have also considered how Wilson lines in a CICY project onto these sLags. The next step in computing the flux superpotential due to Wilson lines is to compute the Chern-Simons invariants on the sLags on which the Wilson lines project non-trivially. Therefore, in this subsection, we will give some general mathematical results relevant to computing Chern-Simons invariants on a large class of closed, compact, orientable 3D (sub)manifolds. As we will see, a class of 3D manifolds very widely encountered are so-called Seifert fibred manifolds, or compositions thereof.

We will apply the results presented here to treat our explicit examples in the next section, and indeed expect them to be useful more generally. This section is a somewhat technical summary of the mathematical literature, and the reader may wish to skip it on the first read.

Decomposition theorems We begin by discussing two important ways to simplify the description of a 3-manifold, by decomposing it into more basic pieces [70].

The first is called a *prime decomposition*; every compact orientable 3-manifold M has a unique decomposition along 2-spheres as a connected sum[6] $M = P_1 \sharp \ldots \sharp P_n$, where each P_i is a prime manifold (i.e., the only way that P_i splits as a connected sum is the trivial one $P_i = P_i \sharp S^3$). Note that a prime manifold is either irreducible (every 2-sphere bounds a ball) or diffeomorphic to $S^2 \times S^1$.

[6] The connected sum of two 3-manifolds is formed by deleting a 3-ball from each, and gluing together the resulting boundary 2-spheres.

The second is called a *torus decomposition*; every irreducible compact orientable 3-manifold M can be decomposed by cutting along incompressible 2-tori T_i (i.e., a torus T_i such that the induced map $\pi_1(T_i) \to \pi_1(M)$ is injective), to give the union $M = X_1 \cup \cdots \cup X_n$, where each X_i is either Seifert fibered or atoroidal (i.e., every incompressible torus in X_i is isotopic to a torus component of ∂X_i). Note that atoroidal 3-manifolds are hyperbolic.

The sLags we encounter in our concrete CICY examples indeed simply turn out to be Seifert fibered manifolds, or can be decomposed into Seifert fibered manifolds using a torus decomposition.

Seifert fibered manifolds Seifert fibered manifolds are among the best understood 3D manifolds, and their Chern-Simons invariants can be explicitly calculated using the results of [67,71]. Let us start with a definition of Seifert fibered manifolds (see e.g. [70, 72–74] for some lectures on these spaces): A Seifert fibered manifold, Q_{Sf}, is a 3D manifold that is a union of pairwise disjoint circles (the fibers) such that the neighborhood of each circle fiber is diffeomorphic to a, possibly fibered, solid torus.[7] Equivalently, a Seifert fibered manifold can be described as an S^1 fibration over a 2-dimensional orbifold base called the orbit surface. The fibered solid torus and orbifold surface and the relation between them are explained in figure 3.2.

A Seifert fibration is characterized by a so-called Seifert invariant, which is the collection of relevant topological data,

$$Q_{Sf} = \{O, o, g; b, (\alpha_1, \beta_1), \ldots, (\alpha_s, \beta_s)\} \ . \tag{3.2.40}$$

Here, the symbol O denotes that the Seifert fibered manifold is orientable and the symbol o denotes that the orbit surface is orientable[8], g is the genus of the orbit surface, b is called the section obstruction of the Seifert fibration[9] which vanishes for manifolds with non-empty boundary, s is the number of exceptional fibers, i.e. the number of orbifold points in the base, and the pairs (α_j, β_j) (with $j = 1, \ldots, s$) describe the exceptional fibers. For each exceptional fiber, the invariant (α_j, β_j) is given in terms of the invariant (p_j, q_j), which describes the associated fibered solid torus as in figure 3.2, by $\alpha_j = p_j$ and

$$0 < \beta_j < \alpha_j, \quad \beta_j q_j \equiv 1 \text{ mod } \alpha_j \ . \tag{3.2.41}$$

Note that one and the same Seifert fibered manifold might be describable in terms of different Seifert invariants in case it admits several ways of splitting it into base and fibers.

Finally, in order to describe Wilson lines and Chern-Simons invariants on Seifert fibered manifolds, one needs to know their fundamental groups. A presentation of the fundamental group of a Seifert fibration can be read off directly from the Seifert invariant, with the generators

[7]In case Q_{Sf} has boundaries, the boundary fibers are located on the boundary of a suitable fibered solid torus.

[8]We only consider orientable Seifert fibered manifolds and orbit surfaces in this chapter, but this restriction can easily be lifted.

[9]More precisely the section obstruction refers to the circle bundle with no exceptional fibers, which is obtained by *drilling out* the fibered solid tori of the Seifert fibered manifold and *filling in* with standard solid tori; the resulting smooth fibration has global section iff $b = 0$. We refer to [73, 74] for more details.

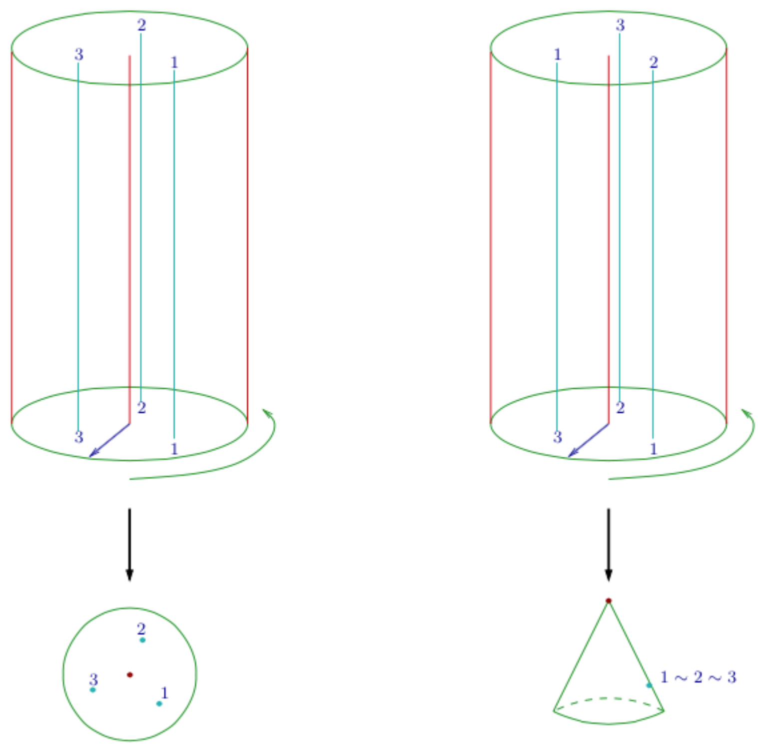


Figure 3.2: Fibered solid tori in a Seifert fibration. On the left we show an ordinary solid torus, and on the right a fibered solid torus. They are $D \times I$ with D being the unit disk in $\mathbb{C}$ and the ends of the interval I identified, and fibered by the intervals $\{x\} \times I$ with $x \in D$. Defining the homeomorphism $\rho : D \to D$ by $\rho(x) = xe^{2\pi \mathrm{i} q/p}$, we construct the fibered solid torus by identifying $(x, 0)$ with $(\rho(x), 1)$. The integers p, q are co-prime, and are chosen to satisfy $0 \leq q < p$. The ordinary solid torus has $(p, q) = (1, 0)$ and the fibered solid torus depicted has $(p, q) = (3, 1)$. The central fiber of the fibered solid torus $\{0\} \times I$ is called the exceptional fiber. It covers the interval I, and intersects the disk D, once. The other fibers are regular fibers. They cover the interval – and intersect the disk D – a multiple p times before closing. Taking the quotient space of a Seifert fibered manifold by identifying all circular fibers to a point results in a 2-dimensional orbifold B, with orbifold points at the location of the exceptional fibers, as illustrated at the bottom of the figure.

and relations given by [73]:

$$\begin{aligned}\pi_1(Q_{Sf}) = \quad & \langle h, a_1, b_1, \ldots, a_g, b_g,\ c_0, c_1, \ldots, c_s,\ d_1, \ldots d_m, | h \text{ is central} \\ & c_0 h^b = c_j^{\alpha_j} h^{\beta_j} = \prod [a_i, b_i] \prod c_j \prod d_k = 1 \rangle \,, \end{aligned} \tag{3.2.42}$$

where $i = 1, \ldots, g$, $j = 1, \ldots, s$ and $k = 1, \ldots, m$, and m is the number of boundary components of the 3-manifold. As a simple illustration, consider e.g. the 3-torus as a trivial S^1-bundle over the orbit surface T^2, so that $g = 1$, $b = 0$ and $s = 0$, and hence eq. (3.2.42) gives $c_0 = 1$ and three commuting non-trivial generators h, a_1, b_1, i.e. the expected result $\pi_1(T^3) = \mathbb{Z}^3$.

Chern-Simons invariants on Seifert fibered manifolds and their compositions We now summarize some known results for Chern-Simons invariants on closed Seifert fibered manifolds, and closed manifolds that decompose into Seifert fibered manifolds with boundary under a torus decomposition. The Chern-Simons invariant for all flat SU(2) connections on all closed Seifert fibered spaces was computed in [71]. The Chern-Simons invariant for a general class of flat SU(N) bundles on any closed Seifert fibered 3-manifold was computed in [67]. These results are stated in terms of irreducible and reducible flat connections (a reducible flat connection is one for which the subgroup H commuting with the image of the homomorphism $\rho:\ \pi_1(Q_{sf}) \to$ G has continuous parameters, otherwise it is irreducible[10]). Notice that the Wilson lines of interest to us are always reducible flat connections, because H should always contain the gauge group of the Standard Model. Moreover, our Wilson lines always lie in a maximal torus of the gauge group G. The Chern-Simons invariant for Abelian reducible SU(N) connections with $\rho:\ \pi_1(Q_{sf}) \to$ SU(N) given by $\rho(h) = \exp 2\pi\mathrm{i}Y$, $\rho(c_j) = 1$, on Seifert fibered 3-manifolds without boundaries is[11] [67]

$$CS(A, Q_{Sf}) = \frac{1}{2} b\,\mathrm{tr}Y^2 + \frac{1}{2}\sum_{j=1}^{s} \beta_j\,\delta_j\,\mathrm{tr}Y^2 \ \mathrm{mod}\ \mathbb{Z} \tag{3.2.43}$$

where Y is in the Lie algebra of a maximal torus of SU(N) and $\delta_j \in \mathbb{Z}$ is such that $\alpha_j\delta_j - \beta_j\gamma_j = 1$ for some integers γ_j. It is immediate that for the 3-torus with $b = 0 = s$ this Chern-Simons invariant is zero (modulo integers), as we will use later.

In our examples, we will also encounter sLags that are not Seifert fibered manifolds, but reduce to Seifert fibered manifolds with boundary under a torus decomposition. For such more general manifolds, we may use the results of [75], where it was shown how to compute Chern-Simons invariants on 3-manifolds decomposed along tori[12]. Indeed, for a 3-manifold M that decomposes into a union of Seifert fibered spaces, X_i, the Chern-Simons invariant on M may be obtained by first computing the Chern-Simons invariants on the pieces X_i, and then computing the effect of gluing the pieces together. Some extra care is required because Chern-Simons invariants on manifolds with boundary are not gauge invariant, even up to integers.

For example, consider M a closed 3-manifold decomposed along a torus T as $M = X_1 \cup_T X_2$, and an SU(2) flat connection over it. The toroidal boundaries $\partial X_i = T_i$ have fundamental group $\pi_1(T_i) = \langle \mu_i, \lambda_i \rangle$. The gluing together of X_1 and X_2 along their boundaries is described by a map between these generators: $\mu_1 \to p\mu_2 + q\lambda_2,\ \lambda_1 \to r\mu_2 + s\lambda_2$, with $ps - qr = 1$. Meanwhile, the restriction of the Wilson lines on X_i, $\rho:\ \pi_1(X_i) \to$ SU(2), to T_i is given by:

$$\rho(\mu_i) = \begin{pmatrix} e^{2\pi\mathrm{i}a_i} & 0 \\ 0 & e^{-2\pi\mathrm{i}a_i} \end{pmatrix} \qquad \rho(\lambda_i) = \begin{pmatrix} e^{2\pi\mathrm{i}b_i} & 0 \\ 0 & e^{-2\pi\mathrm{i}b_i} \end{pmatrix}. \tag{3.2.44}$$

[10] A sufficient, but not necessary, condition for a connection $\rho:\ \pi_1(Q_{Sf}) \to$ G to be reducible is that $\rho(h)$ lies outside the center of G. In these cases, all elements of $\pi_1(Q_{Sf})$ must map to the Cartan subalgebra, and H is at least U(1)^r with r the rank of G.

[11] This result follows from the expression given above Lemma 3.3 in [67]. Indeed, we need to relax the condition applied in Lemma 3.3 that $\rho(h)$ be a scalar matrix, as the Wilson lines encountered are typically not scalar matrices.

[12] Ref. [75] also considers cases when some components of the torus decomposition are not Seifert fibered but hyperbolic manifolds.

We then define equivalence classes of Chern-Simons invariants on each X_i:

$$\left[\left\{a_i, b_i;\ e^{2\pi \mathrm{i} CS(A,X_i)}\right\}\right] , \tag{3.2.45}$$

where the square brackets indicate the orbit of SU(2), with the equivalence relation:

$$\left\{a_i, b_i; e^{2\pi \mathrm{i} CS(A,X_i)}\right\} = \left\{a_i + m, b_i + n; e^{2\pi \mathrm{i}(mb_i - na_i)} e^{2\pi \mathrm{i} CS(A,X_i)}\right\} \tag{3.2.46}$$

for $m, n \in \mathbb{Z}$. Finally, the Chern-Simons invariant on M is defined as the inner product:

$$CS(A, M) = \langle CS(A, X_1), CS(A, X_2)\rangle , \tag{3.2.47}$$

which is simply given by the sum $CS(A, X_1) + CS(A, X_2)$ after choosing gauge fixings that are compatible with the gluing map, $a_1 = pa_2 + qb_2,\ b_1 = ra_2 + sb_2$.

3.2.5 The superpotential from Chern-Simons invariants

Before considering some explicit examples, let us here outline the full procedure for computing the superpotential due to Chern-Simons fluxes from Wilson lines.

1. Identify sLags in a given quotient CICY via its isometric anti-holomorphic involutions of type A and C. If the discrete group is $\Gamma = R \times S$ with R and S cyclic groups of odd order, then only the basic A-type sLag could inherit a Wilson line associated only with S. For the C-type sLags, on the other hand, all can inherit Wilson lines associated with R, and sometimes also associated with S. The case of even order cyclic groups does not occur in our examples but a complete discussion on which sLags are relevant or not is given in section 3.2.3.

2. Compute the intersection matrix for sLags on the quotient CICY. If the rank of the intersection matrix equals the dimension of the third homology group, then the sLags constitute a basis for the 3-cycles in the quotient CICY. In this case, we can write the 3-cycle, Λ, Poincaré dual to the holomorphic 3-form, as

$$\Lambda = \frac{4}{\alpha'} \sum_K c^K Q_K , \tag{3.2.48}$$

in homology, where Q_K are the sLags, satisfying the specialness condition with various calibration angles (so Λ is in general not sLag), and c^K are constant coefficients that depend on the complex structure moduli. Therefore, the background superpotential is given by[13],

$$W = -\frac{\alpha'}{4} \int_{Y_3} \omega_{3Y} \wedge \Omega = -\frac{\alpha'}{4} \int_{\Lambda} \omega_{3Y} = -\sum_K c_K \int_{Q_K} \omega_{3\mathrm{Y}} = -\sum_K c_K\, CS(A, Q_K) , \tag{3.2.49}$$

[13] $\int_C \phi = \int_{C'} \phi$ for C and C' in the same homology class and ϕ closed. In the vacuum, $\mathrm{d}\omega_{3\mathrm{Y}} = 0$.

3. Study the topology of the A-type and C-type sLags of the modded out CICY. For the sLags on which the Wilson lines project, one then has to compute the Chern-Simons invariants, and finally write down the explicit superpotential. For example, suppose the Chern-Simons invariant is non-trivial only on the basic A-type sLag, and that the A-type sLags are Lens spaces $L(p,1)$ (we will see below that this is the case for the $\mathbb{Z}_5 \times \mathbb{Z}_5$ quotient of the Fermat quintic). Then, using (3.1.11), we have for the superpotential in the vacuum,

$$W = -c\, CS(A, Q_{\sigma_A}) = c \left(\sum_a \frac{k_a^2}{2p} \bmod \mathbb{Z} \right) . \tag{3.2.50}$$

Should we wish $W = 0$ in the vacuum, due to any of the reasons mentioned in section 3.1, we require the Chern-Simons flux on Q_{σ_A} to be vanishing (assuming a non-vanishing value c), and this provides a constraint on the Wilson lines that can be introduced in any explicit model. In the example above, the necessary and sufficient condition is that the Wilson lines satisfy

$$\sum_a \frac{k_a^2}{2p} = 0 \bmod \mathbb{Z} . \tag{3.2.51}$$

The same result would be a necessary condition for setting $H = 0$, even if the third homology group were not spanned by sLags.

Note that although the Chern-Simons invariants are (fractionally) quantized, the coefficients c_K may take on more general values. In principle, the vacuum expectation value of W might thus be accidentally small leading to additional suppression of the gravitino mass in the scenario discussed in [46]. It is not clear whether this is actually possible; it was argued in [47] that moduli stabilization from Chern-Simons flux and gaugino condensation generically leads to high-scale supersymmetry breaking.

3.3 Concrete examples

In this section we will apply our strategy to compute the Chern-Simons flux superpotential in explicit compactifications. Several of the steps are model dependent, in particular the computations of the sLag intersection matrix and the sLag Chern-Simons invariants. We therefore begin this programme by treating two concrete examples. Although not realistic, the four generation quintic quotient provides a simple first example to illustrate our arguments. We will then progress to the three generation split-bicubic quotient, which has a potentially realistic particle spectrum, corresponding to the MSSM, a hidden sector and moduli.

3.3.1 The four generation quintic quotient

The Fermat quintic, $X^{1,101}$, is defined by the following hypersurface in $\mathbb{C}P^4$:

$$X^{1,101} = \left\{ z \in \mathbb{C}P^4 \;\middle|\; \sum_{a=1}^{5} z_a^5 = 0 \right\} . \tag{3.3.1}$$

The notation $X^{1,101}$ refers to the two non-trivial Hodge numbers $(h^{1,1}, h^{1,2}) = (1, 101)$. The quintic has two freely acting order five symmetries, each isomorphic to $\mathbb{Z}_5$, generated respectively by:

$$\begin{aligned} g_R : (z_1, z_2, z_3, z_4, z_5) &\rightarrow (\omega z_1, \omega^2 z_2, \omega^3 z_3, \omega^4 z_4, z_5) \\ g_S : (z_1, z_2, z_3, z_4, z_5) &\rightarrow (z_5, z_1, z_2, z_3, z_4) \end{aligned} \tag{3.3.2}$$

with $\omega = \mathrm{e}^{2\pi\mathrm{i}/5}$. These are precisely the symmetry groups R and S discussed in section 3.2.3.

A four-generation model [8] can be constructed by compactifying on the quintic quotiented by $\Gamma = R \times S$, to give non-trivial fundamental group $\pi_1(Y_3) = R \times S \cong \mathbb{Z}_5 \times \mathbb{Z}_5$. The choice of vector bundle corresponding to the standard embedding breaks the $\mathrm{E}_8\times\mathrm{E}_8$ gauge group to $\mathrm{E}_6\times\mathrm{E}_8$. Depending on the choice of Wilson lines, the E_6 is broken further to some extension of the Standard Model gauge group with chiral matter representations. We will take just one of the two possible Wilson lines, associated with either R or S, to be non-trivial. Using E_6's maximal subgroup $\mathrm{SU}(3)_c\times \mathrm{SU}(3)_L\times \mathrm{SU}(3)_R$, we can write the Wilson line as the 27×27-matrix,

$$\mathrm{WL}_\gamma = (\mathbf{1}_3)_c \otimes \mathrm{diag}(\alpha, \alpha, \alpha^{-2})_L \otimes \mathrm{diag}(\beta, \rho, \delta)_R\,, \tag{3.3.3}$$

with $\alpha^5 = \beta^5 = \rho^5 = \delta^5 = 1$ and $\beta\rho\delta = 1$, which is the most general WL_γ that commutes with the SM gauge group. E.g. for $\beta = \rho = \alpha$ and $\delta = \alpha^{-2}$, the unbroken gauge group is $\mathrm{SU}(3)_c\times \mathrm{SU}(2)_L\times \mathrm{SU}(2)_R\times \mathrm{U}(1)^2$. The Hodge numbers of the quintic quotient, $X^{1,5}$, are $(h^{1,1}, h^{1,2}) = (1, 5)$.

The intersection matrix for the quintic sLags

The Fermat quintic has a number of isometric anti-holomorphic involutions, whose actions are not free, and whose fixed points correspond to special Lagrangian submanifolds [76]. The involution $\sigma_A : z_a \mapsto \bar{z}_a$ has as fixed points the real quintic

$$Q_{\sigma_A} = \mathrm{Fix}(\sigma_A) \cap X^{1,101} = \mathbb{R}P^4 \cap X^{1,101} = \left\{ x \in \mathbb{R}P^4 \;\middle|\; \sum_{a=1}^{5} x_a^5 = 0 \right\}\,. \tag{3.3.4}$$

One of the coordinates, say x_5, can always be expressed uniquely in terms of the other coordinates which are completely unrestricted but just subject to the projective rescaling. This means that Q_{σ_A} is topologically $\mathbb{R}P^3 \cong S^3/\mathbb{Z}_2$ (note that this is a Lens space and hence also a Seifert fibered manifold). As discussed above we can construct many more A-type involutions by considering $\sigma_A^U = M \circ \sigma_A$ where $M = U^{-1}\overline{U}$, and U is a symmetry of the defining polynomial of the quintic. Taking only diagonal matrices U, we get $5^4 = 625$ non-trivial and distinct involutions of this type. The fixed point loci of these involutions are given by

$$Q_{\sigma_A^U} = \mathrm{Fix}(\sigma_A^U) \cap X^{1,101} = U^{-1}(Q_{\sigma_A}) \cong \mathbb{R}P^3. \tag{3.3.5}$$

We will give here some details on how the intersection matrices of sLags are calculated. A more detailed discussion is presented in [76–78]. For the quintic we use the simplest polynomial

(3.3.1)

$$P = \sum_{a=0}^{5} z_a^5 = 0, \tag{3.3.6}$$

where $z_a \in \mathbb{C}P^4$. From the definitions of involutions presented in section 3.2.2 we notice that the only possible involutions we can consider are of A-type. We will limit ourselves to A-type sLags defined as the simultaneous solutions of (3.3.6) and

$$z_a = \omega^{l_a} \bar{z}_a, \tag{3.3.7}$$

where $\omega = \mathrm{e}^{2\pi \mathrm{i}/5}$ and $l_a \in \mathbb{Z}^5$. The intersection number of two sLags is given by the Euler number of the intersection subspace [76, 77]. For instance in the quintic, the A-type sLags are given by

$$Q_1 = \{z_a \subset \mathbb{C}P^4 \mid P = 0, z_a = \omega^{l_z} \bar{z}_z\}, \qquad Q_2 = \{z_a \subset \mathbb{C}P^4 \mid P = 0, z_a = \omega^{k_a} \bar{z}_a\}. \tag{3.3.8}$$

The dimension of the intersection is

$$3 - n\,, \tag{3.3.9}$$

where n is the number of $l_a \neq k_a$. For example if $l_a = k_a$ for all a, the intersection is simply the sLag itself which is three dimensional. If $k_5 = 1$ and all other k_a's and l_a's are zero then z_1 simultaneously has to satisfy $z_5 = \bar{z}_5$ and $z_5 = \omega \bar{z}_5$ which implies that $z_5 = 0$. We therefore lose one degree of freedom and the intersection is a surface, as is consistent with $n = 1$. The Euler number of the surface is 1, because generally (real) surface intersections of a pair of manifolds, each diffeomorphic to $\mathbb{R}P^3$, is topologically an $\mathbb{R}P^2$. This can also be noted from the fact that the intersection is a single solution of a real equation in $\mathbb{R}P^3$. The intersection number in this case is -1, where the sign is due to a relative orientation[14] between the sLags. To compute the intersection matrix is important to keep track of the orientation of cycles. In the quintic as well as in the split-bicubic in section 3.3.2, we actually do not need to compute the orientation of each sLag explicitly, but rather the orientation just for the intersection of two sLags Q_1 and Q_2. We compute the orientation of the intersection by evaluating the sign of the following, [77],

$$p(Q_1 \cap Q_2) = \mathrm{sgn}\left(\frac{\mathrm{Re}(\omega^{l_a}\Omega)|_{Q_1} \wedge \mathrm{Re}(\omega^{k_a}\Omega)|_{Q_2}}{i\Omega \wedge \bar{\Omega}}\right), \tag{3.3.10}$$

where $\mathrm{Re}(\omega^{l_a}\Omega)|_Q$ is the volume form on the sLag Q and $\omega^{l_a}\Omega$ is not a product, but the action of ω^{l_a} on Ω descends from the definition of the holomorphic three-form in CICYs, (2.2.22), and from (3.3.8). The orientation of the intersection is then given by the non-intersecting part, namely $l_a \neq k_a$, as it was done in [76, 77],

$$p(Q_1 \cap Q_2) = \mathrm{sgn} \prod_a \mathrm{Im}(\omega^{l_a}\omega^{-k_a}) = \mathrm{sgn} \prod_a \sin \frac{2\pi(l_a - k_a)}{5}, \qquad l_a \neq k_a \tag{3.3.11}$$

[14] This does not refer to the geometrical orientations of intersection between the sLags, but it is a relative orientation between them, due to the definition of anti-holomorphic involutions and to the intersection theory of the sLags, [77].

where again only non-trivial terms are included in the product[15] [76–78]. In summary, if n is odd, the intersection number is equal to ± 1, where the sign is determined by the orientation, p. If n is even, then either the intersection is the sLag itself or a real curve, topologically a circle. In both cases the intersection number vanishes.

It is convenient to introduce the notation

$$\langle k_1 k_2 k_3 k_4 k_5 | l_1 l_2 l_3 l_4 l_5 \rangle, \tag{3.3.12}$$

to denote the intersection matrix. From the above example we see that

$$\langle 00001|00000 \rangle = -1. \tag{3.3.13}$$

The orientation formula, together with the fact that intersection numbers with n even vanish, ensures that the intersection matrix is anti-symmetric.

The sLags defined by the rotation angles l_a are not all independent. By employing the scaling symmetry $z_a \mapsto \mathrm{e}^{\pi i \lambda/5} z_a$ we effectively transform the l_a's by the formula $l_a \mapsto l_a + \lambda$ for $\lambda \in \mathbb{Z}_5$. We have only used the scaling symmetry to make this transformation and so the two sLags have to be the same. We therefore define an equivalence class

$$[l_a] \equiv \{ l_a \sim l_a + \lambda, \ \forall \lambda \in \mathbb{Z}_5 \}. \tag{3.3.14}$$

We calculate the intersection number of two equivalence classes simply by summing the intersection numbers of all elements in the classes

$$\langle [k_a] | [l_a] \rangle \equiv \sum_{k_a \in [k_a], l_a \in [l_a]} \langle k_1 k_2 k_3 k_4 k_5 | l_1 l_2 l_3 l_4 l_5 \rangle. \tag{3.3.15}$$

This does not give the actual numerical value for the intersection number, but the whole intersection matrix is scaled by a common factor which of course does not affect its rank. We also want to compute the intersection matrix of a CICY which is modded out by a discrete group. This modding out is taken care of in the same way as for the scaling symmetries. The equivalence classes of sLags are enlarged by the discrete symmetry. For example in the quintic we mod out by $\mathbb{Z}_5$ generated by the cyclic permutation $z_a \mapsto z_{a+1}$, which translates to a permutation of the l_a's, $p : l_a \mapsto l_{a-1}$. We then define a new equivalence class

$$[l_a]_{\mathbb{Z}_5} \equiv \{ l_a \sim l_a + \lambda, \ l_a \sim p^\kappa(l)_a = l_{a-\kappa}, \ \forall \lambda, \kappa \in \mathbb{Z}_5 \}, \tag{3.3.16}$$

and again the intersection number of equivalence classes is defined by the sum

$$\langle [k_a]_{\mathbb{Z}_5} | [l_a]_{\mathbb{Z}_5} \rangle \equiv \sum_{k_a \in [k_a]_{\mathbb{Z}_5}, l_a \in [l_a]_{\mathbb{Z}_5}} \langle k_1 k_2 k_3 k_4 k_5 | l_1 l_2 l_3 l_4 l_5 \rangle. \tag{3.3.17}$$

[15] Definitions (3.3.10) is valid for all complete intersection CYs, however it needs to be modified a bit when the ambient space is given by product of projective spaces of odd degree. In these cases, due to the appearance of square roots in M of (3.2.11), we would need to consider orientation of the single sLags in the definition of the intersection orientation, see [77] for some examples.

Using this procedure we find that the rank of the intersection matrix precisely matches the dimension of the third homology group for both the quintic and the modded out quintic. In fact, computing the intersection matrix, one can show that only 204 of the 625 sLags $Q_{\sigma_A^U}$ are distinct in homology, and that they span the homology group of the quintic $X^{1,101}$, [76].

For the four-generation quintic quotient $X^{1,5}$, the number of distinct A-type sLags on the quotient $X^{1,5}$ can be computed to be 129, and the rank of the 129×129 dimensional intersection matrix is reduced to 12. This matches the dimension of the third homology group for the quintic quotient, so that the sLags continue to provide a basis for the 3-cycles, as expected. We have seen in subsection 3.2.3 that the only A-type sLag one has to check for a non-trivial Wilson line, is the basic one (3.3.4). Moreover, this basic A-type sLag can at most inherit Wilson lines, and hence Chern-Simons invariants, from the permutation group S.

Wilson lines and Chern-Simons flux on sLags

We can immediately write down the full Chern-Simons flux superpotential. Choosing to embed the Wilson line only in R, all the Chern-Simons invariants are trivial, and therefore, the superpotential is also trivial. Embedding instead the Wilson line in S, the only non-trivial Chern-Simons invariant is on the sLag Q_{σ_A}, which on the quotient is the Lens space $\mathbb{R}P^3/\mathbb{Z}_5 = S^3/\mathbb{Z}_{10}$. Writing $\alpha = e^{2\pi \mathrm{i} 2k_1/10}$, $\beta = e^{2\pi \mathrm{i} 2k_2/10}$, $\rho = e^{2\pi \mathrm{i} 2k_3/10}$ and $\delta = e^{2\pi \mathrm{i} 2k_4/10}$ ($k_{1,2,3,4} = 0, \ldots, 4$) in (3.3.3), and using (3.1.11), the Chern-Simons invariant is immediately given by

$$CS(A, Q_{\sigma_A}) = -\frac{9}{5}\left(6k_1^2 + k_2^2 + k_3^2 + k_4^2\right) \mod \mathbb{Z}\,, \tag{3.3.18}$$

which reduces to $CS(A, Q_{\sigma_A}) = -\frac{108}{5}k_1^2 \bmod \mathbb{Z}$ for the SU(3)$_c\times$ SU(2)$_L\times$ SU(2)$_R\times$ U(1)^2 model. The full superpotential from the visible sector Wilson lines in the vacuum is then simply:

$$W = c\left(\frac{108}{5}k_1^2 \mod \mathbb{Z}\right) = c\left(\frac{3}{5}k_1^2 \mod \mathbb{Z}\right) \tag{3.3.19}$$

for c a (possibly) non-vanishing constant, depending on the choice of complex structure. The mod $\mathbb{Z}$ can be interpreted as a possible integer H-flux contribution. There may also be non-trivial contributions from hidden sector Wilson lines, which could e.g. be chosen to ensure two or more condensing gauge sectors to help stabilize moduli. Of course, the hidden Wilson lines project in the same way as the visible ones on each sLag, and they only differ in their explicit values.

3.3.2 The three generation split-bicubic quotient

We now turn to a potentially realistic compactification, based on a quotient of the split-bicubic CY threefold [79, 80]. After introducing the CICY and its quotient we will follow the same procedure as above, which is here somewhat more involved. We identify the A-type and C-type sLags, and study their topology, particularly in the quotient CICY. Then we can compute the relevant Chern-Simons invariants by using the torus decomposition into Seifert fibered manifolds, discussed in section 3.2.4. Finally, we compute the intersection matrix for the sLags and show that we can generate the full third homology group. In this way, we obtain the full Chern-Simons flux superpotential.

The split-bicubic CICY It will be useful to have several pictures of the split-bicubic in mind. The first is as a Schoen manifold, which is a fiber product of two rational elliptic surfaces, B and B', with a common base $\mathbb{C}P^1$,

$$X^{19,19} = B \times_{\mathbb{C}P^1} B' = \{(b, b') \in B \times B' \mid \beta(b) = \beta'(b')\}, \tag{3.3.20}$$

where

$$\beta : B \to \mathbb{C}P^1, \qquad \beta' : B' \to \mathbb{C}P^1, \tag{3.3.21}$$

are the projections of B and B' on the common $\mathbb{C}P^1$-base. This can be represented by the following pull back diagram

$$\begin{tikzcd} & X^{19,19} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ B \arrow[dr, "\beta"'] & & B' \arrow[dl, "\beta'"] \\ & \mathbb{C}P^1 & \end{tikzcd}$$

so that the CY admits a fibration over $\mathbb{C}P^1$ with generic fiber the product of two elliptic curves. The rational elliptic surfaces B, B' are known as dP_9, due to their similarity to the del Pezzo surfaces. Indeed, dP_9 is a blow up[16] of $\mathbb{C}P^2$ at nine points to $\mathbb{C}P^1$ and may be represented by the configuration matrix

$$\left[\begin{array}{c|c} \mathbb{C}P^1 & 1 \\ \mathbb{C}P^2 & 3 \end{array}\right] . \tag{3.3.22}$$

In other words, it can be written as the hypersurface

$$B = \left\{(t, \zeta) \in \mathbb{C}P^1 \times \mathbb{C}P^2 \,\middle|\, t_1 f(\zeta) - t_2 g(\zeta) = 0\right\}, \tag{3.3.23}$$

where t_i ($i = 1, 2$) are homogeneous coordinates of $\mathbb{C}P^1$, ζ_j ($j = 1, 2, 3$) are homogeneous coordinates of $\mathbb{C}P^2$, and $f(\zeta)$ and $g(\zeta)$ are cubic polynomials. The equation $t_1 f(\zeta) - t_2 g(\zeta) = 0$ can be solved uniquely for t_i in terms of ζ_j, except for those nine points of $\mathbb{C}P^2$ where $f(\zeta) = 0 = g(\zeta)$. At those nine points of $\mathbb{C}P^2$ the t_i are unrestricted and hence parameterize an entire $\mathbb{C}P^1$.

[16] A blow up of an n-dimensional complex manifold, M, at m points to $\mathbb{C}P^1$ is diffeomorphic to the connected sum $M \sharp_m \overline{\mathbb{C}P^2}$, where $\overline{\mathbb{C}P^2}$ has opposite orientation to M [81]. So dP_9 may also be considered as the connected sum $\mathbb{C}P^2 \sharp_9 \overline{\mathbb{C}P^2}$.

As there is a similar description for B', the elliptically fibered Calabi-Yau can also be described as a CICY with the configuration matrix:

$$\left[\begin{array}{c|cc} \mathbb{C}P^1 & 1 & 1 \\ \mathbb{C}P^2 & 3 & 0 \\ \mathbb{C}P^2 & 0 & 3 \end{array}\right]. \tag{3.3.24}$$

In other words,

$$X^{19,19} = \left\{(t,\zeta,\eta) \in \mathbb{C}P^1 \times \mathbb{C}P^2 \times \mathbb{C}P^2 \,\middle|\, P_1(t,\zeta) = P_2(t,\eta) = 0\right\} , \tag{3.3.25}$$

where

$$\begin{aligned} P_1(t,\zeta) &= t_1 f(\zeta) - t_2 g(\zeta), \\ P_2(t,\eta) &= t_1 \hat{g}(\eta) - t_2 \hat{f}(\eta), \end{aligned} \tag{3.3.26}$$

η_j ($j = 1,2,3$) are homogeneous coordinates for the second $\mathbb{C}P^2$ factor, and $f, g, \hat{f}, \hat{g}$ are cubic polynomials. When specifying the polynomials, we have 19 degrees of freedom as the Hodge number $h^{1,2} = 19$ indicates. Here we will make the same choice as in [82],

$$\begin{aligned} &f(\zeta) = \zeta_1^3 + \zeta_2^3 + \zeta_3^3 - a\,\zeta_1\zeta_2\zeta_3, \qquad g(\zeta) = -c\,\zeta_1\zeta_2\zeta_3\,, \\ &\hat{g}(\eta) = c\,\eta_1\eta_2\eta_3, \qquad \hat{f}(\eta) = -\eta_1^3 - \eta_2^3 - \eta_3^3 + b\,\eta_1\eta_2\eta_3\,. \end{aligned} \tag{3.3.27}$$

This turns out to be the most general choice of polynomials for which the split-bicubic has a freely acting discrete symmetry $\Gamma = R \times S$ with R, S both isomorphic to $\mathbb{Z}_3$, with the following generators[17] [82]:

$$\begin{aligned} g_R: &\quad \zeta_j \to \omega^j \zeta_j, \quad \eta_j \to \omega^{-j}\eta_j, \quad t_i \to t_i\,, \\ g_S: &\quad \zeta_j \to \zeta_{j+1}, \quad \eta_j \to \eta_{j+1}, \quad t_i \to t_i\,, \end{aligned} \tag{3.3.28}$$

where $\omega = e^{2\pi \mathrm{i}/3}$. The Hodge numbers of the quotient split-bicubic, $X^{3,3} = X^{19,19}/\Gamma$, are $(h^{1,1}, h^{2,1}) = (3,3)$. The coefficients a, b, c in (3.3.27) correspond, roughly speaking, to the three complex structure moduli of $X^{3,3}$. In order to analyze the equations explicitly, we will take $a = b = 0$ and leave $c = 1$. The polynomials then satisfy $f = -\hat{f}$, $g = -\hat{g}$ and

$$\begin{aligned} P_1(t,\zeta) &= t_1 f(\zeta) - t_2 g(\zeta) = t_1\left(\zeta_1^3 + \zeta_2^3 + \zeta_3^3\right) + t_2\zeta_1\zeta_2\zeta_3, \\ P_2(t,\eta) &= t_2 f(\eta) - t_1 g(\eta) = t_2\left(\eta_1^3 + \eta_2^3 + \eta_3^3\right) + t_1\eta_1\eta_2\eta_3. \end{aligned} \tag{3.3.29}$$

Note that since $\mathrm{d}P_1 \wedge \mathrm{d}P_2$ does not vanish in this case, the resulting manifold is diffeomorphic to all smooth split-bicubic CICYs. Putting all three parameters to zero would also be an attractive choice, but corresponds to a singular limit of $X^{3,3}$. A heterotic MSSM with no exotics (beyond hidden sectors and moduli) can be obtained from a compactification on $X^{3,3}$. To this end, one introduces an SU(4) holomorphic stable vector bundle, and the following Wilson lines,

[17] Another, equivalent, choice is made in [80, 83].

which embed the $\mathbb{Z}_3 \times \mathbb{Z}_3$ fundamental group into the SO(10) GUT gauge group using the $\mathbf{10}$ representation of SO(10) [40, 45, 80]:

$$\mathrm{WL}_{\gamma_1} = \begin{pmatrix} e^{4\pi \mathrm{i}/3}\mathbf{1}_5 & \\ & e^{2\pi \mathrm{i}/3}\mathbf{1}_5 \end{pmatrix} \quad \text{and} \quad \mathrm{WL}_{\gamma_2} = \begin{pmatrix} \mathbf{1}_2 & & & \\ & e^{4\pi \mathrm{i}/3}\mathbf{1}_3 & & \\ & & \mathbf{1}_2 & \\ & & & e^{2\pi \mathrm{i}/3}\mathbf{1}_3 \end{pmatrix} \tag{3.3.30}$$

As the results on Chern-Simons invariants are usually given in terms of SU(N) flat connections, it is useful to note that the above Wilson lines embed into an SU(5) $\subset$ U(5) $\subset$ SO(10) subgroup of the SO(10) GUT group.

Having set up the compactification, we are ready to compute the Wilson line contribution to the superpotential. The split-bicubic $X^{3,3}$ has both A-type and C-type sLags. We now turn our attention to studying these sLags in the smooth split-bicubic quotient and computing their Chern-Simons invariants.

The C-type sLags Let us first consider the C-type sLags. The basic C-type sLag is obtained from the isometric anti-holomorphic involution:

$$\sigma_C : \quad \zeta_j \to \bar{\eta}_j, \quad \eta_j \to \bar{\zeta}_j, \quad t_1 \to \bar{t}_2, \quad t_2 \to \bar{t}_1\,. \tag{3.3.31}$$

Further C-type sLags can be identified by considering involutions $(M, \overline{M}^{-1}) \circ \sigma_C$, and those we will consider are:

$$(M, \overline{M}^{-1}) \circ \sigma_C : \quad \zeta_j \to \omega^{l_j}\bar{\eta}_j, \quad \eta_j \to \omega^{l_j}\bar{\zeta}_j, \quad t_1 \to \bar{t}_2, \quad t_2 \to \bar{t}_1, \tag{3.3.32}$$

where $l_1 + l_2 + l_3 = 0 \mod 3$. Together, these give three distinct C-type sLags on $X^{19,19}$.

In order to understand the topology of the C-type sLags, it is enough to consider the basic one. The sLag Q_{σ_C} can be described by the equations

$$0 = t_1 f(\zeta) - \bar{t}_1\, g(\zeta) \quad \text{and} \quad t_1 = \bar{t}_2 \tag{3.3.33}$$

in $\mathbb{C}P^1 \times \mathbb{C}P^2$. Notice that on the sLag $t_1 = \bar{t}_2 \neq 0$, so this equation reduces as a hypersurface in $\mathbb{C}P^2$ to:

$$0 = f(\zeta) - \frac{\bar{t}_1}{t_1}\, g(\zeta)\,, \tag{3.3.34}$$

which corresponds to the configuration matrix $[\mathbb{C}P^2\,|\,3]$ describing a smooth CY 1-fold, that is, a 2-torus. The total sLag is then a fibration over $\mathbb{R}P^1$ ($t_1 = \bar{t}_2$ in $\mathbb{C}P^1$), with smooth fibers $\mathbb{T}^2$. As the monodromy of this torus bundle is clearly trivial, the resulting 3-manifold is simply a 3-torus. All C-type sLags are diffeomorphic to the basic C-type sLag and hence they are also all 3-tori.

The free action of a cyclic group on a 3-torus corresponds to trivial or free actions along each of the S^1 factors, so that the quotient is again a 3-torus. As explained in section 3.2.4, the Chern-Simons contributions from discrete Wilson lines on a 3-torus vanish. Hence the C-type sLags do not contribute to the superpotential for $X^{3,3}$.

A-type sLags on the covering CICY Next we consider the A-type sLags, whose basic isometric anti-holomorphic involution is:

$$\sigma_A: \quad \zeta_j \to \bar{\zeta}_j, \quad \eta_j \to \bar{\eta}_j, \quad t_i \to \bar{t}_i\,. \tag{3.3.35}$$

Further sLags can be identified from the involutions $M \circ \sigma_A$, which we take to be:

$$M \circ \sigma_A: \; \zeta_j \to \omega^{l_j}\bar{\zeta}_j, \quad \eta_j \to \omega^{m_j}\bar{\eta}_j, \quad t_i \to \bar{t}_i, \tag{3.3.36}$$

where $l_j, m_j \in \{0,1,2\}$, and $l_1+l_2+l_3 = m_1+m_2+m_3 = 0$ mod 3. This gives only nine A-type sLags in total.

The basic A-type sLag can be described as the complete intersection,

$$\begin{aligned} 0 &= r_1 f(x) - r_2 g(x) = r_1\left(x_1^3+x_2^3+x_3^3\right) + r_2 x_1 x_2 x_3 \\ 0 &= r_2 f(y) - r_1 g(y) = r_2\left(y_1^3+y_2^3+y_3^3\right) + r_1 y_1 y_2 y_3 \end{aligned} \tag{3.3.37}$$

in $\mathbb{R}P^1 \times \mathbb{R}P^2 \times \mathbb{R}P^2$, with r_i, x_j and y_j being the homogeneous coordinates on $\mathbb{R}P^1$, $\mathbb{R}P^2$ and $\mathbb{R}P^2$ respectively. In analogy with the split-bicubic itself, our real 3-manifold can then be described as a fiber product,

$$\begin{tikzcd} & Q_{\sigma_A} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ N \arrow[dr, "\beta"'] & & N' \arrow[dl, "\beta'"] \\ & \mathbb{R}P^1 & \end{tikzcd} \tag{3.3.38}$$

where the map π (π') forgets the y_i (x_i) coordinates, and the map β (β') forgets the x_i (y_i) coordinates.

In order to understand the topology of Q_{σ_A}, we start by characterizing the topology of the 2-manifolds N and N', in analogy to the rational elliptic surface dP_9. N is described as the hypersurface

$$N = \left\{(r,x) \in \mathbb{R}P^1 \times \mathbb{R}P^2 \,\middle|\, r_1 f(x) - r_2 g(x) = 0\right\}, \tag{3.3.39}$$

and similarly for N'. The smooth surface N can be viewed[18] as a singular fibration over $\mathbb{R}P^1$ (parameterized by r_i) where the fibers are given by the following cubic equation in $\mathbb{R}P^2$:

$$r_1(x_1^3+x_2^3+x_3^3) + r_2 x_1 x_2 x_3 = 0\,. \tag{3.3.40}$$

This well-known plane cubic curve can immediately be understood with some plots, see figure 3.3. The generic smooth fibers are a single $\mathbb{R}P^1$ for $r_1/r_2 > 0$ and $r_1/r_2 < -1/3$, or a disjoint

[18] Just as for the complex case (see the discussion below eq. (3.3.24)), the manifold N can also be viewed as the blowup of $\mathbb{R}P^2$ at three points (where $f(x) = g(x) = 0$) to $\mathbb{R}P^1$. This is topologically equivalent to the connected sum of four $\mathbb{R}P^2$'s, i.e. a 2-sphere with four crosscaps. The Euler characteristic for this blowup is given by $\chi(N) = \chi(\mathbb{R}P^2) - 3\,\chi(\text{point}) + 3\,\chi(\mathbb{R}P^1) = 1 - 3 + 0 = -2$.

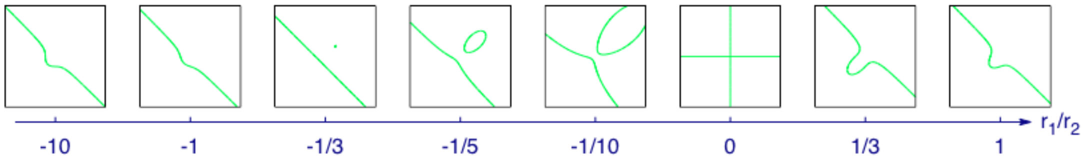


Figure 3.3: Solutions to the cubic equation (3.3.40) in $\mathbb{R}P^2$, treating r_1/r_2 as a parameter. In the figure, we have used affine coordinates with x_3 scaled to unity and plotted x_2 against x_1. The complement, $x_3 = 0$, defines an $\mathbb{R}P^1$ which, in the chosen affine coordinates, sits at infinity. In this way we find apparantly non-compact curves, but the curves that seem noncompact are connected at infinity due to the antipodal identification on the $\mathbb{R}P^1$ defined by $x_3 = 0$. We see that for all $r_1/r_2 \neq 0$ and $r_1/r_2 \neq -1/3$ we find either a single curve which is topologically $\mathbb{R}P^1 \cong S^1$ or a disjoint union of two such curves. For $r_1/r_2 = 0$, the eq. (3.3.40) reduces to $x_1x_2x_3 = 0$, whose solution is three intersecting $\mathbb{R}P^1$'s. In this case the plot is not complete since the entire $\mathbb{R}P^1$ at infinity, corresponding to $x_3 = 0$, is also a solution but not shown. Finally, for $r_1/r_2 = -1/3$, the solution is a disjoint union of $\mathbb{R}P^1$ and a single point.

union of two $\mathbb{R}P^1$'s for $-1/3 < r_1/r_2 < 0$.[19] There are also, however, two singular fibers: For $r_1/r_2 = -1/3$, the equation for the fiber is solved both by the $\mathbb{R}P^1$ described by $x_1 = -x_2 - x_3$, and the point $x_1 = x_2 = x_3$; for $r_1 = 0$ it gives a connected union of three $\mathbb{R}P^1$'s with three singular points. It is then straightforward to verify that the surface N has Euler characteristic (see figure 3.4)

$$\chi(N) = \chi(\text{point}) \times \chi(\text{point}) + \chi(\text{point}) \times \chi(3 \text{ intersecting } \mathbb{R}P^1\text{'s}) = 1 - 3 = -2\,,$$

and similarly for N'.

Building on these results, we can describe the A-type sLag. First of all, we have just seen from (3.3.38) that it is the fiber product $N \times_{\mathbb{R}P^1} N'$, i.e. a singular fibration over $\mathbb{R}P^1$, where the fibers are products of two plane cubic curves described above (see eq. (3.3.40)). In fact, for any ratio r_1/r_2 at least one of the two plane cubic curve fibers is always a single smooth $\mathbb{R}P^1$ (see figure 3.5). By cutting up Q_{σ_A} at two places in the $\mathbb{R}P^1$ base where *both* fibers are locally smooth $\mathbb{R}P^1$'s, say at $r_1 = \pm r_2$, the manifold Q_{σ_A} can be decomposed into two diffeomorphic pieces (see figure 3.5). We denote the piece corresponding to $r := r_1/r_2 \in [-1, 1]$ by $\tilde{Q}_{\sigma_A}$, i.e.

$$\tilde{Q}_{\sigma_A} = \left\{(r, x, y) \in [-1, 1] \times \mathbb{R}P^2 \times \mathbb{R}P^2 \middle| rf(x) - g(x) = 0 = f(y) - rg(y)\right\}. \qquad (3.3.41)$$

Since the fibers above $r = \pm 1$ are 2-tori, the above cutting operation is an example of a torus decomposition, which we discussed in section 3.2.4. The map $\tilde{\pi} : \tilde{Q}_{\sigma_A} \to \tilde{N}$, where $\tilde{\pi}(r, x, y) = (r, x)$ and

$$\tilde{N} = \left\{(r, x) \in [-1, 1] \times \mathbb{R}P^2 \middle| rf(x) - g(x) = 0\right\}, \qquad (3.3.42)$$

[19] Indeed, it follows from a classic theorem due to Harnack [84] that a smooth cubic in $\mathbb{R}P^2$ has up to two connected components, each circles, exactly one of which must correspond to the non-zero element of $H_1(\mathbb{R}P^2, \mathbb{Z}) \cong \mathbb{Z}_2$.

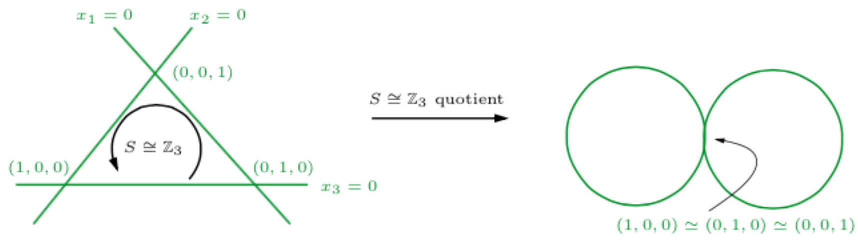


Figure 3.4: A singular fiber in the A-type sLag and its quotient, solution to the plane cubic curve (3.3.40) at $r_1 = 0$. Before modding out by $S \cong \mathbb{Z}_3$, it is a connected union of three $\mathbb{R}P^1$'s, each two of which intersect at a point. The Euler characteristic of this curve is then given by $\chi(3 \text{ intersecting } \mathbb{R}P^1\text{'s}) = 3\,\chi(\mathbb{R}P^1) - 3\,\chi(\text{point}) = -3$ or $\chi(3 \text{ intersecting } \mathbb{R}P^1\text{'s}) = b_0 - b_1 = 1 - 4 = -3$. Modding out by the permutation symmetry S, leads to a figure of eight, with Euler characteristic $\chi(\text{figure of eight}) = 2\chi(\mathbb{R}P^1) - \chi(\text{point}) = -1$.

defines an S^1-bundle over $\tilde{N}$ since $\tilde{\pi}$ projects out *smooth* S^1 fibers (see figure 3.5),

$$\tilde{\pi}^{-1}(r, x) = \left\{ y \in \mathbb{R}P^2 \middle| f(y) - rg(y) = 0 \right\} \cong \mathbb{R}P^1 \cong S^1. \tag{3.3.43}$$

This is a trivial Seifert fibration (i.e. S^1-bundle over a smooth surface, $\tilde{N}$), where the base $\tilde{N}$ has two circular boundaries.

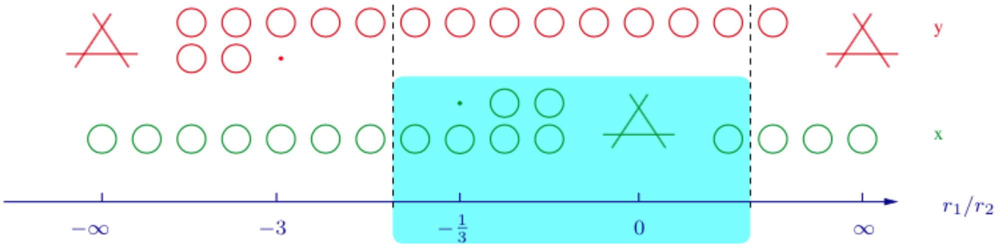


Figure 3.5: The A-type sLag Q_{σ_A} as fiber product. The cubic curves in the $\mathbb{R}P^2$ factor parameterized by x_j's fibered over $\mathbb{R}P^1$ parameterized by r_i's give a smooth surface, $N \cong \sharp_4 \mathbb{R}P^2$. The same is true of the cubic curves in $\mathbb{R}P^2$ parameterized by y_j's fibered over $\mathbb{R}P^1$. Alternatively, by cutting up the manifold into two pieces at $r_1 = \pm r_2$, we obtain two diffeomorphic S^1-bundles over the bounded base $\tilde{N}$ indicated by the shaded area in the figure.

.

A-type sLags on the quotient Up to now, we have identified the A-type sLags in the simply connected split-bicubic, $X^{19,19}$, together with their topological structure. Next we have to understand how the sLags are modified when we mod out $X^{19,19}$ by the discrete symmetry $\Gamma = S \times R$ to obtain $X^{3,3}$. The only A-type sLag on $X^{3,3}$ that can inherit a Wilson line is the basic one, which may only inherit a Wilson line associated with S. In the covering space $X^{19,19}$, the permutation group does not act on the base $\mathbb{R}P^1$ of the sLag Q_{σ_A}. Therefore, the quotient sLag $Q_{\sigma_A}/S \cong Q_{\sigma_A}/\mathbb{Z}_3$ can still be described as a fibration over $\mathbb{R}P^1$ with the fibers being a product of two plane cubic curves (3.3.40) subject to identifications. Let us consider the action of S on these plane cubic curves. We first note that S is a symmetry of the defining polynomial (3.3.40) so that for a fixed $r = r_1/r_2$ each plane cubic curve is mapped to itself by S. Moreover the only fixed point of S in the ambient $\mathbb{R}P^2$ is $x_1 = x_2 = x_3$. We now examine how the permutation group S acts on the four topologically different types of plane cubic curve (see figure 3.3). Referring to (3.3.40):

- For $r = -1/3$, the plane cubic curve is topologically a disjoint union of a circle and the point $x_1 = x_2 = x_3$. The permutation S acts freely on the circle component which thus stays topologically a circle after modding out by S and the point component is a fixed point.
- For $r \notin [-1/3, 0]$, the plane cubic curve is topologically a single circle which is mapped freely to itself by S. Again, the quotient curve remains a circle.
- For $r = 0$, the plane cubic curve consists of three intersecting circles as depicted in figure 3.4. Each circle is given by the vanishing of one of the coordinates, and hence the permutation action maps the circles onto one another. Moreover on each circle there are two distinguished points that map into each other, namely the intersection points of that circle with the other two. The quotient topology is then easily verified to be the so-called figure of eight.
- For $r \in (-1/3, 0)$, the plane cubic curve consists of two disjoint circles. The permutation group S acts freely within each circle component. This can be seen as follows, one of the two circles has all x_j with the same sign (the smaller circle in the corresponding diagrams of figure 3.3) while the x_j in the other circle do not have the same sign.

As S acts trivially on the base $\mathbb{R}P^1$ parameterized by r_i, we can now perform essentially the same torus decomposition as for the unquotiented sLag, namely cut $Q_{\sigma_A}/\mathbb{Z}_3$ along toroidal boundaries located at $r_1 = \pm r_2$. Each of the two resulting components is now diffeomorphic to $\tilde{Q}_{\sigma_A}/\mathbb{Z}_3$. Before we mod out by S, $\tilde{Q}_{\sigma_A}$ is a S^1-bundle over the smooth base $\tilde{N}$. The permutation group $S \cong \mathbb{Z}_3$ acts freely within each S^1-fiber so that the quotient $\tilde{Q}_{\sigma_A}/\mathbb{Z}_3$ is also an S^1-bundle, but over the base manifold $\tilde{N}/\mathbb{Z}_3$. As explained above, $\tilde{N}$ has precisely one fixed point located at $(r, x_1, x_2, x_3) = (-1/3, 1, 1, 1)$. Increasing r from $r = -1/3$ to $r = -1/3 + \epsilon$, the isolated fixed point grows into a circle (see figure 3.5) so that the coordinates r and x locally parameterize a disk neighbourhood of the fixed point. The permutation group $S \cong \mathbb{Z}_3$ acts on this disk neighbourhood by rotating the disk about the fixed point in its center. It is therefore clear that $\tilde{N}/\mathbb{Z}_3$ has an orbifold singularity of order three at the center of the disk whereas everywhere else the quotient $\tilde{N}/\mathbb{Z}_3$ is smooth. Thus the space $\tilde{Q}_{\sigma_A}/\mathbb{Z}_3$ is now a non-trivial Seifert fibration with one exceptional fiber, see figure 3.7. The manifold has Seifert invariant (c.f. (3.2.40)):

$$\tilde{Q}_{\sigma_A}/\mathbb{Z}_3 = (O, o, 0; 0, (3, 1)) , \tag{3.3.44}$$

where we have used that the underlying topology of the orbit surface is a cylinder (see figure 3.6) and recalled that the section obstruction b is trivial on manifolds with boundary.

Intersection	A · A	A · C	C · C
point	1	1	0
curve	0	0	0
surface	-2	0	0

Table 3.2: The intersection numbers for intersections of A- and C-type sLags in the split bicubic, given by the Euler characteristic of the intersection loci.

A basis for the third homology group and the flux superpotential

Finally, we should check whether or not we span the basis for the third homology group, as required to obtain all the Wilson line contributions to the Chern-Simons flux superpotential. For the split-bicubic a similar procedure to that used for the quintic holds. We identify sLags using isometric antiholomorphic involutions of the CICY. Then, using the description of these sLags as complete intersections, we can easily compute their intersection loci, the corresponding Euler characteristics and hence the intersection numbers. Taking care of the orientations and the scaling symmetry as done for the quintic, we can then compute the rank of the intersection matrix. We will, however, encounter one additional complication, which is that we must pass through a singular limit of the split-bicubic in order to find sufficient 3-cycles to span a basis of the third homology group.

Ensuring first a choice of complex structure parameters that give a smooth CY ($a = b = 0,\ c \neq 0$), we take:

$$\begin{aligned} P_1(t,\zeta) &= t_1(\zeta_1^3+\zeta_2^3+\zeta_3^3)+c\,t_2\zeta_1\zeta_2\zeta_3\,, \\ P_2(t,\eta) &= t_2(\eta_1^3+\eta_2^3+\eta_3^3)+c\,t_1\eta_1\eta_2\eta_3\,. \end{aligned} \tag{3.3.45}$$

As discussed in the main text, this smooth split bicubic has 9 A-type sLags and 3 C-type sLags, described respectively by $(k_1,k_2,k_3,l_1,l_2,l_3)$ with $k_1+k_2+k_3 = 0 \bmod 3$, $l_1+l_2+l_3 = 0 \bmod 3$, and $(k_1,k_2,k_3,l_1,l_2,l_3)$ with $k_1+k_2+k_3 = l_1+l_2+l_3 \bmod 3 = 0 \bmod 3$, where we have taken $c = \epsilon$ real. Notice that, as we will discuss further below, more sLags could be obtained by taking the singular CY with $a = b = c = 0$, indeed it is then easy to identify 81 A-type sLags and 9 C-type sLag. Also, different sets of 9 A-type and 3 C-type sLags can be obtained by choosing different smooth choices for c, $c = \epsilon\omega^n$ with $\omega = e^{2\pi i/3}$ and $n = 0,1,2$. The A-type are labelled by $(k_1,k_2,k_3,l_1,l_2,l_3)$ with $k_1+k_2+k_3 = 2n \bmod 3$, $l_1+l_2+l_3 = 2n \bmod 3$. The C-type are labelled by $(k_1,k_2,k_3,l_1,l_2,l_3)$ with $k_1+k_2+k_3 = l_1+l_2+l_3 \bmod 3 = 2n \bmod 3$. The equations describing these sLags as complete intersections in $\mathbb{R}P^1\times\mathbb{R}P^2\times\mathbb{R}P^2$ are identical for all A-type sLags and all C-type sLags. We compute the orientation of the intersections using formula (3.3.10) as in 3.3.1.

In 3.2 we present the intersection numbers for all A- and C-type sLags in the unmodded smooth split-bicubic, given by the Euler characteristic of the intersection loci. The only non-trivial entry in table 3.2 is the surface intersection of two A-type sLags, so let us explain how this can be obtained. An A-type sLag is given by the solution of

$$\zeta_j = \omega^{l_j}\bar{\zeta}_j, \qquad \eta_j = \omega^{k_j}\bar{\eta}_j, \qquad t_i = \bar{t}_i, \tag{3.3.46}$$

together with the defining polynomials (3.3.45). For two such sLags, a simultaneous solution is a surface when only one of the angles k_j and l_j are different. Let us assume that in the smooth split-bicubic the missing sLags, which are present in the singular limit, are deformed cycles (actually the sLags in the singular point are all present as sLags in different smooth choice of the complex structure moduli space), eventually, deformed sLags. We are allowed then to translate the intersection computation of the singular manifold to smooth points in moduli space, considering all the intersection loci smooth. Then we consider the basic A-type sLag with $k_j = l_j = 0$ intersecting with the sLag defined by $k_1 = 1$ and other k's and l's vanishing. We find that the intersection locus is defined by $\zeta_1 = 0$ and ζ_2, ζ_3, η_j and t_i real. We can denote $\zeta_j = x_j, \eta_j = y_j$ and $t_i = r_i$ to distinguish from the complex coordinates on the ambient space. The intersection surface satisfies the equations

$$0 = r_1(x_2^3 + x_3^3) = r_2(y_1^3 + y_2^3 + y_3^3) + r_1 y_1 y_2 y_3, \tag{3.3.47}$$

where $r, (x_2, x_3) \in \mathbb{R}P^1$ and $y \in \mathbb{R}P^2$. As indicated in the table 3.2, this surface has Euler characteristic -2. We can see this by the fact that for $r_1 \neq 0$ the first equation simply has a point solution $x_2 = -x_3$, the second equation, has a solution space which is topologically a $\mathbb{R}P^1 \cong S^1$ except for $r_2 = 0$ and $r_1 = -3r_2$. For $r_2 = 0$ the solution space is three intersecting $\mathbb{R}P^1$'s and for $r_1 = -3r_2$ the solution space is a point and a $\mathbb{R}P^1$. The total Euler characteristic of the surface is determined only by these contributions, i.e. $\chi = -3 + 1 = -2$ where -3 is the Euler characteristic of the three intersecting $\mathbb{R}P^1$'s.

If we use the sLag allowed just in the smooth split-bicubic, the intersection matrix turns out to be the zero matrix. A similar computation can be carried out for the modded out split bicubic but of course the rank of the intersection matrix in all cases turns out to vanish. Note that this does not imply that all the A-type and C-type sLags are homologically equivalent, but only that the number of linearly independent homology elements covered by the cycles is *at least* zero.

We can, however, identify a set of deformed sLags which do span a basis for the third homology group of the smooth split-bicubic. We do so by considering first the singular split-bicubic, taking $a = b = c = 0$:

$$\begin{aligned} P_1(t,\zeta) &= t_1(\zeta_1^3 + \zeta_2^3 + \zeta_3^3)\,, \\ P_2(t,\eta) &= t_2(\eta_1^3 + \eta_2^3 + \eta_3^3)\,. \end{aligned} \tag{3.3.48}$$

Assuming this CICY has a well-defined intersection theory, we can fill out its intersection matrix. First note that it is easy to write down equations describing all 81 A-type sLags and 9 C-type sLags, as well as identify point, curve and surface intersections as described above. Note that each intersection of a given dimension is described by the same equation. Next, observe that 9 out of the 81 A-type sLags and 3 out of the 9 C-type sLags persist as sLags when we deform to a smooth CICY, taking c from 0 to ϵ. Assuming that the intersection numbers do not change in going back to the singular limit, they are given by table 3.2. Moreover, these are the intersection numbers for all point, curve and surface intersections, given that they are described by the same equations. Having filled out the intersection matrix, we can compute its rank, finding 16 and 8, respectively, for $X^{7,7} = X^{19,19}/S$ and $X^{3,3} = X^{19,19}/S \times R$. That is, the A-type and C-type sLags span the basis for the third homology group of the singular CICY.

Finally, we know that all these sLags survive as 3-cycles when we deform to a smooth CICY[20], even though they are not all fixed point sets of any isometric antiholomorphic involution (and thus likely not all sLags). In this way, we obtain a set of deformed sLags that generate the full third homology group of the smooth (quotient) split bicubic.

Wilson lines on the sLags and their Chern-Simons invariants

Given the Seifert invariant, one can immediately write down a presentation of the fundamental group (c.f. (3.2.42)):

$$\pi_1(\tilde{Q}_{\sigma_A}/\mathbb{Z}_3) = \langle h, c_0, c_1, d_1, d_2 \,|\, h \text{ is central},\ c_0 = c_1^3 h = c_0 c_1 d_1 d_2 = 1 \rangle \,. \tag{3.3.49}$$

This fundamental group is infinite and non-Abelian.

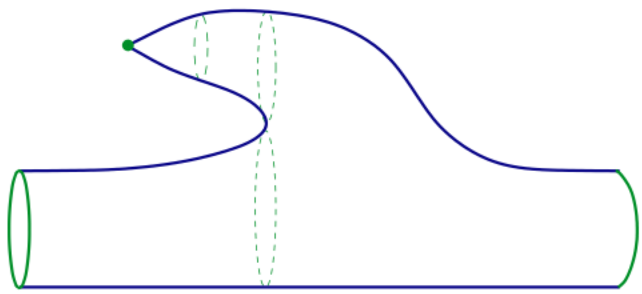


Figure 3.6: The base $\tilde{N}/\mathbb{Z}_3$ of the quotient sLag $Q_{\sigma_A}/\mathbb{Z}_3$ after torus decomposition.

We compute now the Chern-Simons invariant of the sLag $Q_{\sigma_A}/S \cong Q_{\sigma_A}/\mathbb{Z}_3$ in the quotient split-bicubic, $X^{3,3}$. To do so, we first have to understand how the Wilson line associated with the symmetry group $S \cong \mathbb{Z}_3$, which is a homomorphism $\rho: \ \pi_1(X^{3,3}) \to$ SO(10), is compatible with the fundamental group $\pi_1(Q_{\sigma_A}/\mathbb{Z}_3)$ of the sLag. In fact, we will show that the Wilson line associated with S on $X^{3,3}$ cannot project to a Wilson line on the sLag $Q_{\sigma_A}/\mathbb{Z}_3$.

The strategy is to check whether the fundamental group of the manifold $Q_{\sigma_A}/\mathbb{Z}_3$ admits a homomorphism $\rho: \ \pi_1(Q_{\sigma_A}/\mathbb{Z}_3) \to$ SO(10) whose image can be written as (3.3.30). We start by recalling that the sLag has been cut into two pieces, $\tilde{Q}^{(I)}_{\sigma_A}/\mathbb{Z}_3$ with $I = 1, 2$, as in figure 3.5. Each piece is a Seifert fibered manifold with boundary and their fundamental group is given by (3.3.49). In order to understand the generators of the fundamental group, we look at the fibration structure of the manifold described in section 3.3.2, and list the non-contractible loops present:

- $h^{(I)}$ is associated with the S^1 fiber;

[20] Indeed, for many kinds of singularities, the map between third homology groups $H_3(X_{smth}) \to H_3(X_{sing})$ is surjective, so that cycles can disappear when going to the singular limit, but no new cycles can appear. One way to see this in our case is to notice that we can define the holomorphic 3-form and the periods in the singular limit, and deform them away from the singular limit. Therefore, the cycles also exist in the smooth limit.

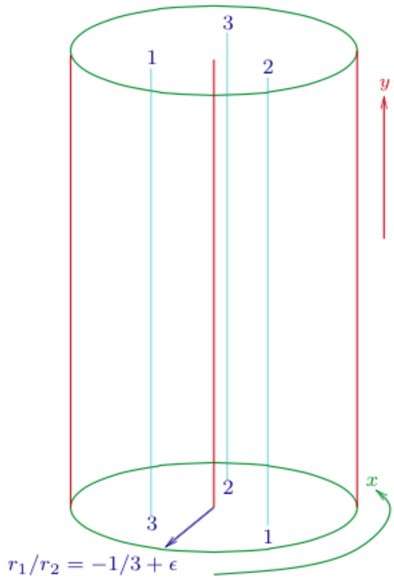


Figure 3.7: The exceptional fiber in the Seifert fibration of the quotient $\tilde{Q}_{\sigma_A}/\mathbb{Z}_3$. The exceptional fiber lies above the fixed point of the $\mathbb{Z}_3$ action in the orbit surface $\tilde{B}$. The figure shows the structure close to an exceptional fiber as follows. We consider a disk neighbourhood of the orbifold point $(\tilde{r}, \tilde{x}_1/\tilde{x}_3, \tilde{x}_2/\tilde{x}_3) = (-1/3, 1, 1)$ in the base surface. The disk forms the base of a fibered solid torus, which is the product $D^2_{\tilde{r},\tilde{x}} \times I_{\tilde{y}}$ with the ends of the interval $I_{\tilde{y}}$ identified after twisting by an angle of $2\pi/3$. The center of the disc $\{0\}$ lifts to the core circle of the solid torus, and points in $D^2_{r,x} - \{0\}$ lift to fibers that wrap 3 times around the core in the longitudinal direction and 1 times in the meridianal direction. An example of a fiber is shown in blue, the three line segments are joint together as indicated when the endcaps of the cylinder are glued together. Thus the data describing the exceptional fiber is $(p, q) = (3, 1)$ or $(\alpha, \beta) = (3, 1)$.

- $c_0^{(I)}$ is associated with an eventual twisting of the base $\tilde{N}^I$;
- $c_1^{(I)}$ corresponds to the non-contractible loop around the orbifold point in $\tilde{N}^I$;
- $d_1^{(I)}, d_2^{(I)}$, are the two boundaries of the cylinder $\tilde{N}^I$, see figure 3.6.

The next step is to glue the two manifolds $\tilde{Q}^{(1)}_{\sigma_A}/\mathbb{Z}_3$ and $\tilde{Q}^{(2)}_{\sigma_A}/\mathbb{Z}_3$ along the two boundaries given by the plane cubic curves at the points $r = r_1/r_2 = \pm 1$. As we have already seen, the boundaries are 2-tori, and the gluing condition is an automorphism of the torus, namely an $SL(2, \mathbb{Z})$ transformation, that maps the two circular boundaries of $\tilde{Q}^{(1)}_{\sigma_A}/\mathbb{Z}_3$ to the ones of $\tilde{Q}^{(2)}_{\sigma_A}/\mathbb{Z}_3$ (and the reverse for the other boundary). Note that the symmetry group $S \cong \mathbb{Z}_3$ acts such that there is no twisting of the two fibers in the neighbourhood of $r = r_1/r_2 = \pm 1$ on the original uncut manifold, where we recall that the fibers are given by the two plane cubic curves (see figures 3.5 and 3.3). Therefore, we can write the gluing conditions as follows. Along the boundary $r = 1$ we have

$$h^{(1)} = d_1^{(2)}, \tag{3.3.50}$$
$$d_2^{(1)} = h^{(2)}, \tag{3.3.51}$$

and along the boundary $r = -1$ we have

$$h^{(2)} = d_1^{(1)}, \tag{3.3.52}$$
$$d_2^{(2)} = h^{(1)}. \tag{3.3.53}$$

So far, together with the relations in (3.3.49), we have listed all the topological ingredients of our sLag $Q_{\sigma_A}/\mathbb{Z}_3$. Wilson lines on the sLag would correspond to the homomorphism $\pi_1(Q_{\sigma_A}/\mathbb{Z}_3) \to$ SO(10), given by:

$$\rho : h^{(I)} \mapsto e^{2\pi i Y^{(I)}}; \qquad \rho : c_k^{(I)} \mapsto e^{2\pi i X_k^{(I)}},\ \ k = 0, 1; \qquad \rho : d_l^{(I)} \mapsto e^{2\pi i D_l^{(I)}},\ \ l = 1, 2; \tag{3.3.54}$$

where at least one of the generators of the fundamental group should generate a $\mathbb{Z}_3$ subgroup, in order to be mapped to the matrices in (3.3.30). To check if this is possible we start from the relations (in (3.3.49)) given by

$$\begin{aligned} \left(c_1^{(I)}\right)^3 h &= 1 \Rightarrow (3X_1^{(I)} + Y^I) \in \mathrm{diag}(\mathbb{Z}), &(3.3.55)\\ c_0^{(I)} \left(h^{(I)}\right)^b &= 1 \Rightarrow (X_0^{(I)} + bY^I) \in \mathrm{diag}(\mathbb{Z}), &(3.3.56)\\ c_0^{(I)} c_1^{(I)} d_1^{(I)} d_2^{(I)} &= 1 \Rightarrow (X_0^{(I)} + X_1^{(I)} + D_1^{(I)} + X_2^{(I)}) \in \mathrm{diag}(\mathbb{Z}), &(3.3.57) \end{aligned}$$

where we have used (3.3.54) and $\mathrm{diag}(\mathbb{Z})$ is the set of integer valued 10×10 diagonal matrices. Again using the map ρ in (3.3.54), the boundary gluing conditions (3.3.50-3.3.51) become

$$Y^{(1)} = D_1^{(2)} \mod \mathrm{diag}(\mathbb{Z}), \tag{3.3.58}$$
$$D_2^{(1)} = Y^{(2)} \mod \mathrm{diag}(\mathbb{Z}). \tag{3.3.59}$$

and (3.3.52–3.3.53) become

$$Y^{(2)} = D_1^{(1)} \mod \mathrm{diag}(\mathbb{Z}), \tag{3.3.60}$$
$$D_2^{(2)} = Y^{(1)} \mod \mathrm{diag}(\mathbb{Z}). \tag{3.3.61}$$

Since we want a Wilson line that is a homomorphism ρ of $\mathbb{Z}_3$ into SO(10), suppose that every generator g fulfils the following relation

$$g^3 = 1. \tag{3.3.62}$$

This implies that $3X_1^{(I)} \in \mathrm{diag}(\mathbb{Z})$, which together with (3.3.55) gives also $Y^{(I)} \in \mathrm{diag}(\mathbb{Z})$. Plugging these results into (3.3.56), we find that also $X_0^{(I)} \in \mathrm{diag}(\mathbb{Z})$. Using now the boundary gluing conditions (3.3.58-3.3.61) and, plugging them into (3.3.57), we obtain that also $X_1^{(I)} \in \mathrm{diag}(\mathbb{Z})$.

To sum up, we have obtained a completely trivial representation, and therefore $\mathbb{Z}_3$ Wilson lines do not project onto the sLag $Q_{\sigma_A}/\mathbb{Z}_3$. Therefore, the fundamental group together with the appropriate gluing condition to compose $Q_{\sigma_A}/\mathbb{Z}_3 = \tilde{Q}_{\sigma_A}/\mathbb{Z}_3 \cup \tilde{Q}_{\sigma_A}/\mathbb{Z}_3$, does not allow one to define a $\mathbb{Z}_3$ Wilson line consistently on the entire sLag $Q_{\sigma_A}/\mathbb{Z}_3$. We can therefore conclude that the corresponding Chern-Simons invariant vanishes

$$CS(A, Q_{\sigma_A}/\mathbb{Z}_3) = 0\,, \tag{3.3.63}$$

Summary and superpotential on the split-bicubic: The topology of 27 out of the 81 deformed A-type sLags and all deformed C-type sLags are the same as that of the basic A-type and C-type sLags, as can be seen by considering the different smooth limits, $c = \epsilon, \epsilon\omega, \epsilon\omega^2$ which are diffeomorphic to each other. The rank of the A- and C-type intersection matrix can be computed to be zero for the smooth split-bicubic[21]. However, the singular split-bicubic, with complex structure parameters $a = b = c = 0$ has additional A-type and C-type sLags, due to its larger set of isometric anti-holomorphic involutions. Starting from this singular limit, we can obtain a set of deformed sLags, which do complete a basis for the third homology group of the smooth quotient split-bicubic. We have to consider the Wilson lines and Chern-Simons invariants for these deformed sLags which complete the basis. Whether or not Wilson lines wrap the cycles can be inferred from the singular limit, where it is clear from section 3.2.3 that Wilson lines can project non-trivially on the basic A-type sLag and C-type sLags. All the C-type sLags in the singular limit of the split-bicubic are smooth, and they are topologically 3-tori. Hence, like the basic C-type sLag, their Chern-Simons invariants are zero. Recalling that the basic A-type sLag also has a vanishing Chern-Simons invariant, we therefore conclude that all the Chern-Simons invariants vanish and we can write down the full Wilson line contribution to the Chern-Simons flux superpotential,

$$W_{\text{CS}} = 0\,. \tag{3.3.64}$$

In contrast to the quintic, one therefore cannot introduce fractional terms in the flux superpotential coming from the visible or hidden sector Wilson lines. On one hand the consistency of the leading order 10D supersymmetric CY compactification is clear, and on the other hand Chern-Simons fluxes from Wilson lines cannot help with moduli stabilization.

3.4 Summary of the results and outlook

Discrete Wilson lines are a key ingredient in heterotic Standard Model constructions based on Calabi-Yau compactifications.[22]

[21] Note that this does not imply that all the A-type and C-type sLags are homologically equivalent, but only that the number of linearly independent homology elements covered by the cycles is *at least* zero.

[22] See [85] for some Standard Model like constructions without Wilson lines on simply-connected Calabi-Yau spaces.

They are introduced to break grand unified gauge groups down to the standard model whilst maintaining supersymmetry and the control that this provides. However, they can sometimes induce a non-trivial fractional H-flux via their Chern-Simons contributions, which may affect the internal self-consistency of the assumed string background and could lead to possibly unintended phenomenological consequences such as high-scale supersymmetry breaking. Since, for a given Wilson line, the presence or absence of fractional H-flux is not a choice, it is important to develop methods for its computation.

We analyzed this problem for complete intersection Calabi-Yau manifolds that admit freely acting symmetry groups of discrete rotations, R, and cyclic permutations, S. We used the well understood special Lagrangian submanifolds based on isometric anti-holomorphic involutions as explicit representatives for the 3-cycles of the third homology group. If they span a basis for the third homology group, the full background superpotenial from Chern-Simons flux can be expressed in terms of Chern-Simons invariants on these submanifolds. The special Lagrangian submanifolds come in two types, the A-type associated with complex conjugation of the coordinates in the ambient projective spaces, and the C-type associated with complex conjugation and exchange of coordinates between any two of the ambient projective spaces of equal dimension. In a systematic analysis we determined which sLags could potentially inherit non-trivial Wilson lines from the Calabi-Yau space. This first step is model independent.

The actual value of the Chern-Simons invariant depends both on the topology of the submanifold and the choice of Wilson line, but it is computable on a model-by-model basis. As an illustration we carried out this computation for two explicit complete intersection Calabi-Yau manifolds, namely for the quintic and the split-bicubic. The 3-dimensional spaces we encountered in these models are Seifert fibered 3-manifolds or composition thereof. For Wilson lines in such spaces we can compute the Chern-Simons invariants by applying results from the mathematics literature.

For the quintic modded out by $\mathbb{Z}_5 \times \mathbb{Z}_5$, we were able to obtain an expression for the full superpotential induced by Wilson lines. The result depends on whether we choose to embed the Wilson line in the R or S factor of the Calabi-Yau fundamental group. Notice that the low energy particle spectrum and couplings are independent of this choice. Choosing an R-type Wilson line, all Chern-Simons invariants and the superpotential are vanishing in this model. In this way, we can ensure a consistent leading order supersymmetric Calabi-Yau 10D compactification. Choosing an S-type Wilson line, by contrast, there is a non-vanishing Chern-Simons invariant and superpotential, which might be used for moduli stabilization, but may also introduces subtleties regarding the self-consistency of the string background.

We then progressed to the potentially realistic three generation quotient split-bicubic with two discrete Wilson lines. The special Lagrangian submanifolds we found for the smooth quotient split-bicubic do not generate the full third homology group, but by starting from a more symmetric singular limit, we were able to identify deformed sLags that do span a basis. Contrary to the quintic case we found that the Wilson lines do not generate any H-flux and therefore do not contribute to the flux superpotential. This is completely independent of the choice of Wilson lines and is due solely to the topological properties of the three dimensional submanifolds in the split-bicubic. Therefore, we have new evidences that this three generation split-bicubic preserves its consistency as supersymmetric ten-dimensional solution. This is a very interesting new result, since it supports the self-consistency of the models constructed on the split-bicubic, but it also means that moduli stabilization must be achieved by some mechanism different to the one proposed in [46], see e.g. [29, 31, 47, 86, 87].

Our work leaves several important open questions. The consistency of incorporating Chern-Simons flux into supersymmetric Calabi-Yau compactifications with gaugino condensation has not yet been established in a definitive way. In any case, it would be necessary to compute the Chern-Simons flux (and its superpotential) from Wilson lines in any explicit non-simply connected Calabi-Yau compactification. Our procedure should be applicable to a wide range of models, but there are also some model dependent steps. It would be relevant to extend our computation for a broader class of CICYs and to develop methods to implement these within computerized scans like [88].

Chapter 4

Supersymmetric AdS_6 solutions of type II supergravity

One of the interesting theoretical results of string theory is that it helps defining several nontrivial quantum field theories in dimensions higher than four, which are hard to study with traditional methods. For example, several five-dimensional superconformal field theories (SCFT_5's) have been defined, using D4-branes in type I' [89, 90], M-theory on Calabi–Yau manifolds with shrinking cycles [90, 91], (p, q)-fivebrane webs [92] (sometimes also including (p, q)-sevenbranes [93]). These various realizations are dual to each other [93, 94]; some of these theories are also related by compactification [95] to the four-dimensional "class S" theories [96].

However, not too many AdS_6 duals are known to these SCFT_5's. Essentially the reason is that there is no D-brane stack whose near-horizon limit gives AdS_6. Indeed the string realizations quoted above originate from intersecting branes, whose localized metrics are notoriously difficult to find, as illustrated for example in [97]; even were they known, the relevant near-horizon limit would probably be far from obvious. One exception is when one of the branes is completely inside the other; in such cases some partially delocalized solutions [98] become actually localized. This was used by Brandhuber and Oz [99] to obtain the first AdS_6 solution in string theory. (It was also anticipated to exist [100] as a lift of a vacuum in the six-dimensional supergravity of [101].) It is in massive IIA, and it represents the near-horizon limit of a stack of D4's near an O8–D8 wall; thus it is dual to the theories in [89]. The internal space is half an S^4; the warping function A and the dilaton ϕ go to infinity at its boundary. This is just a consequence of the presence of the O8–D8 system there, and it is a reflection of the peculiar physics of the corresponding SCFT_5's. The fact that the dilaton diverges at the wall roughly corresponds to a Yang–Mills kinetic term of the type $\phi F_{\mu\nu} F^{\mu\nu}$; the scalar ϕ plays the role of $\frac{1}{g^2_{\mathrm{YM}}}$, and at the origin $\phi \to 0$ one finds a strongly coupled fixed point.

One can also study a few variations on the Brandhuber–Oz solution, such as orbifolding it [102] and performing T-duality [103, 104] or even the more recently developed [105, 106] nonabelian T-duality [104, 107]. The latter is not thought to be an actual duality, but rather a solution-generating duality; thus the solution should represent some new physics, although its global features are puzzling [107].

We attack the problem systematically, using the "pure spinor" techniques. In general, the procedure reformulates the equations for preserved supersymmetry ($\mathcal{N} = 1$, namely 8 real supercharges in 6 dimensions) in terms of certain differential forms defining G-structures on the "generalized tangent bundle" $T \oplus T^*$. It originates from generalized complex geometry [35,36] and its first application was to $\mathrm{Minkowski}_4$ or $\mathrm{AdS}_4 \times M_6$ solutions of type II supergravity [108], in which case the relevant G was $\mathrm{SU}(3) \times \mathrm{SU}(3)$. In [38] the method was extended (still in type II supergravity) to any ten-dimensional geometry; in this chapter we apply to $\mathrm{AdS}_6 \times M_4$ the general system obtained there. We work in IIB, since in massive IIA the Brandhuber–Oz solution is unique [109], and in eleven-dimensional supergravity there are no solutions, as we show in appendix B.1. We will classify $\mathcal{N} = 1$ AdS_6 solutions of type IIB supergravity that preserve eight real supercharges in six dimensions.

The relevant structure on $T \oplus T^*$ is an "identity" structure (in other words, G is the trivial group). Such a structure is defined by a choice of two vielbeine $e^a_\pm$ (roughly associated with left- and right-movers in string theory). We actually prefer working with a single "average" vielbein e^a and with some functions on M_4 encoding the map between the two vielbeine $e^a_\pm$. We then use these data to parameterize the forms appearing in the supersymmetry system. The supersymmetry equations then determine e^a in terms of the functions on M_4, thus also determining completely the local form of the metric. As usual for this kind of formalism, the fluxes also come out as an output; less commonly, but the Bianchi identities are automatically satisfied.

When the dust settles, it turns out that we have completely reduced the problem to a system of two PDEs (see (4.3.12b), (4.3.13) below) on a two-dimensional space Σ. The metric is that of an S^2-fiberation over Σ. This should not come as a surprise: a SCFT$_5$ has an SU(2) R-symmetry, which manifests itself in the gravity dual as the isometry group of the S^2. In 5, for similar reasons the internal space M_3 will be an S^2-fiberation over an interval.

The problem is reduced to PDEs, which are harder to study even numerically. Using EDS techniques (see for example [110, Chap. III] or [111, Sec. 10.4.1]) we have checked that the system is "well-formed": the general solution is expected to depend on two functions of one variable, which can be thought of as the values of the warping function A and the dilaton ϕ at the boundary of Σ. (We expect regularity of the metric to fix those degrees of freedom as well, up to discrete choices.) We do recover two explicit solutions to the PDEs, corresponding to the abelian and nonabelian T-duals of the Brandhuber–Oz solution mentioned above.

Even though we do not present any new solutions, it seems likely that our PDEs will describe (p, q)-fivebrane webs. The common directions of the (p, q)-fivebranes would be $x^0, \ldots, x^4$, and which would be stretched along a line in the x^5–x^6 plane (such that $\frac{x^5}{x^6} = \frac{p}{q}$). It is natural to conjecture that the solutions to our PDEs would correspond to near-horizon limits of such configurations, with the x^5–x^6 plane somehow corresponding to our Σ; the remaining directions x^7, x^8, x^9 would provide our S^2 (as well as the radial direction of AdS$_6$). For such cases we would expect Σ to have a boundary, at which the S^2 shrinks; the (p, q)-fivebranes would then be pointlike sources at this boundary.

4.1 Supersymmetry and pure spinor equations for AdS$_6$

We will start by presenting the system of pure spinor equations that we need to solve. Although this is similar to systems in other dimensions, there are some crucial differences, which we will try to highlight.

The original example of the pure spinor approach to supersymmetry was found for Mink$_4 \times M_6$ or AdS$_4 \times M_6$ solutions in type II supergravity [108], where the BPS conditions were reformulated in terms of certain differential equations on an $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure on the "generalized tangent bundle" $TM_6 \oplus T^*M_6$. Other examples followed over the years; for instance, [112] applied the strategy to Mink$_d \times M_{10-d}$ for even d (for $d = 2$ the situation was improved in [113–115]); the case $\mathbb{R} \times M_9$ was considered in [116].

Partially motivated by the need of generating quickly pure-spinor-like equations for different setups, [38] formulated a system directly in ten dimensions, using the geometry of the generalized tangent bundle of M_{10}. We will adopt the procedure described in 2.3.

The ten-dimensional system contains two "symmetry" equations (2.3.19) that usually simply fix the normalizations of the pure spinors; two "pairing" equations (2.3.20) that often end up being redundant (although not always, see [114, 117]); and one "exterior" equation (2.3.18) that usually generates the pure spinor equations one is most interested in. This pattern is repeated for our case. One important difference is that the spinor decomposition we have to start with is clumsier than the one in other dimensions. Usually, the ten-dimensional spinors ϵ_a are the sum of two (or sometimes even one) tensor products. For AdS$_4 \times M_6$ in IIB, for example, we simply have $\epsilon_a = \zeta_{4+} \otimes \eta^a_{6+} + \text{c.c.}$. The analogue of this for Mink$_6 \times M_4$ in IIB would be

$$\begin{aligned} \epsilon_1 &= \zeta_{6+} \otimes \eta^1_{4+} + \zeta^c_{6+} \otimes \eta^{1\,c}_{4+} \\ \epsilon_2 &= \zeta_{6+} \otimes \eta^2_{4\mp} + \zeta^c_{6+} \otimes \eta^{2\,c}_{4\pm} \end{aligned} \qquad (\text{Mink}_6 \times M_4;\ \text{IIA/IIB}) \, , \tag{4.1.1}$$

where $(\ )^c \equiv C(\ )^*$ denotes Majorana conjugation. For AdS$_6 \times M_4$, however, such an Ansatz cannot work: compatibility with the negative cosmological constant of AdS$_6$ demands that the ζ_6 obey the Killing spinor equation on AdS$_6$,

$$D_\mu \zeta_6 = \frac{1}{2} \gamma^{(6)}_\mu \zeta_6 \, , \tag{4.1.2}$$

and solutions to this equation cannot be chiral, while the ζ_{6+} in (4.1.1) are chiral. This issue does not arise in AdS$_4$ because in that case $(\zeta_{4+})^c$ has negative chirality; here $(\zeta_{6+})^c$ has positive chirality. This forces us to add "by hand" to (4.1.1) a second set of spinors with negative chirality, ending up with the unpromising-looking

$$\begin{aligned} \epsilon_1 &= \zeta_+ \eta^1_+ + \zeta^c_+ \eta^{1\,c}_+ + \zeta_- \eta^1_- + \zeta^c_- \eta^{1\,c}_- \\ \epsilon_2 &= \zeta_+ \eta^2_\mp + \zeta^c_+ \eta^{2\,c}_\mp + \zeta_- \eta^2_\pm + \zeta^c_- \eta^{2\,c}_\pm \end{aligned} \qquad (\text{AdS}_6 \times M_4;\ \text{IIA/IIB}) \tag{4.1.3}$$

where we have dropped the $_6$ and $_4$ labels (and the $\otimes$ sign), as we will do elsewhere. Attractive or not, (4.1.3) will turn out to be the correct one for our classification.

In the main text from now on we will consider the IIB case (unless otherwise stated). This is because AdS$_6 \times M_4$ solutions in massive IIA were already analyzed in [109], where it was found that the only solution is the one in [99]. We did find it useful to check our methods on that solution as well; we sketch how that works in appendix B.2. As for the massless case, we found it more easily attacked by direct analysis in eleven-dimensional supergravity, which we present in appendix B.1, given that it is methodologically a bit outside the stream of our pure spinor analysis in IIB.

4.1.1 Derivation of the system

With the spinor Ansatz (4.1.3) in hand, we can apply the system of differential-forms equations (2.3.18, 2.3.19):

$$d_H(e^{-\phi}\Phi) = -(\tilde{K} \wedge + \iota_K) F_{(10)} \, ; \tag{4.1.4a}$$

$$L_K g = 0 \, , \quad d\tilde{K} = \iota_K H \, . \tag{4.1.4b}$$

$\Phi = \epsilon_1 \otimes \overline{\epsilon_2}$ is the key ten-dimensional polyform,[1] which is adapted to our background; g is the ten-dimensional metric while K and $\tilde{K}$ are ten-dimensional one-forms which will be defined

[1] It should not be confused with the SU(2)-covariant internal even forms $\Phi_\pm$.

momentarily. The decomposition of the ten-dimensional spinors ϵ_a suggests we decompose accordingly the ten-dimensional gamma matrices:

$$\gamma_\mu^{(6+4)} = e^A \gamma_\mu^{(6)} \otimes 1 \, , \qquad \gamma_{m+5}^{(6+4)} = \gamma^{(6)} \otimes \gamma_m^{(4)} \, . \tag{4.1.5}$$

Here $\gamma_\mu^{(6)}$, $\mu = 0, \ldots, 5$, are a basis of six-dimensional gamma matrices ($\gamma^{(6)}$ is the chiral gamma), while $\gamma_m^{(4)}$, $m = 1, \ldots, 4$ are a basis of four-dimensional gamma matrices. We can now expand via Fierz identities (see formula (A.12) in [38]) the bilinear $\epsilon_1 \otimes \overline{\epsilon_2}$, by plugging in the decomposition (4.1.3) and (C.1.1). We get a sum of terms such as the following:

$$\sum_{k=0}^{6} \frac{1}{8k!} \left(\overline{\zeta_+} \gamma_{(6)}^{j} \gamma^{(6)}_{\mu_k \ldots \mu_1} \zeta_+ \right) \gamma_{(6)}^{\mu_1 \ldots \mu_k} \sum_{j=0}^{4} \frac{1}{4j!} \left(\eta_\mp^{2\dagger} \gamma^{(4)}_{m_j \ldots m_1} \eta_+^1 \right) \gamma_{(4)}^{m_1 \ldots m_j} = \mp \zeta_+ \overline{\zeta_+} \wedge \eta_+^1 \eta_\mp^{2\dagger} \, . \tag{4.1.6}$$

What we mean by e.g. $\zeta_+ \overline{\zeta_+}$ is the six-dimensional polyform corresponding to this bilinear via the Clifford map (see footnote 2). All in all we get:

$$\begin{aligned} \Phi = & \mp \zeta_+ \overline{\zeta_+} \wedge \eta_+^1 \eta_\mp^{2\dagger} \mp \zeta_+ \overline{\zeta_+^c} \wedge \eta_+^1 \overline{\eta_\mp^2} + \zeta_- \overline{\zeta_-} \wedge \eta_-^1 \eta_\pm^{2\dagger} + \zeta_- \overline{\zeta_-^c} \wedge \eta_-^1 \overline{\eta_\pm^2} + \\ & + \zeta_+ \overline{\zeta_-} \wedge \eta_+^1 \eta_\pm^{2\dagger} + \zeta_+ \overline{\zeta_-^c} \wedge \eta_+^1 \overline{\eta_\mp^2} \pm \zeta_- \overline{\zeta_+} \wedge \eta_-^1 \eta_\mp^{2\dagger} \pm \zeta_- \overline{\zeta_+^c} \wedge \eta_-^1 \overline{\eta_\mp^2} + \text{c.c.} \, . \end{aligned} \tag{4.1.7}$$

The presence of the complex conjugates (of all summands) is due to relations such as $\zeta_\pm^c \overline{\zeta_\pm} = -(\zeta_\pm \overline{\zeta_\pm^c})^*$ and $\eta_\pm^{1\,c} \eta_\pm^{2\,\dagger} = -(\eta_\pm^1 \overline{\eta_\pm^2})^*$. If we were interested in the Minkowski case, the system would only contain the bispinors $\eta_+^1 \otimes \eta_+^{2\,\dagger}$ and $\eta_+^1 \otimes (\eta_+^{2\,c})^\dagger$.[2] (As usual in the pure spinor approach, we need not consider spinors of the type e.g. $\eta_+^1 \otimes \eta_+^{1\,\dagger}$ to formulate a system which is necessary and sufficient.) Mathematically, this would describe an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure on $TM_4 \oplus T^*M_4$. Since in (4.1.3) we also have the negative chirality spinors η_-^1 and $\eta_-^{1\,c}$, there are many more forms we can build. We have the even forms:[3]

$$\phi_\pm^1 = e^{-A} \eta_\pm^1 \otimes \eta_\pm^{2\,\dagger} \, , \qquad \phi_\pm^2 = e^{-A} \eta_\pm^1 \otimes (\eta_\pm^{2\,c})^\dagger \equiv e^{-A} \eta_\pm^1 \otimes \overline{\eta_\pm^2} \, ; \tag{4.1.8a}$$

and the odd forms:

$$\psi_\pm^1 = e^{-A} \eta_\pm^1 \otimes \eta_\mp^{2\,\dagger} \, , \qquad \psi_\pm^2 = e^{-A} \eta_\pm^1 \otimes (\eta_\mp^{2\,c})^\dagger \equiv e^{-A} \eta_\pm^1 \otimes \overline{\eta_\mp^2} \, . \tag{4.1.8b}$$

The factors e^{-A} are inserted so that the bispinors have unit norm, in a sense to be clarified shortly; A is the warping function, defined as usual by

$$ds_{10}^2 = e^{2A} ds^2_{\mathrm{AdS}_6} + ds^2_{M_4} \, . \tag{4.1.9}$$

Already by looking at (4.1.8a), we see that we have *two* $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structures on $TM_4 \oplus T^*M_4$. If both of these structures come for example from $\mathrm{SU}(2)$ structures on TM_4, we see

[2] As usual, we will identify forms with bispinors via the Clifford map $dx^{m_1} \wedge \ldots \wedge dx^{m_k} \mapsto \gamma^{m_1 \ldots m_k}$.

[3] Notice that the ^{1} or ^{2} on ϕ has nothing to do with the ^{1} or ^{2} on the η's; rather, it has to do with whether the second spinor is Majorana conjugated (^{2}) or not (^{1}). Another caveat is that the $_\pm$ does not indicate the degree of the form, as it is often the case in similar contexts; all the ϕ's in (4.1.8a) are even forms. One can think of the $_\pm$ as indicating whether these forms are self-dual or anti-self-dual.

that we get an identity structure on TM_4, i.e. a vielbein. In fact, this is true in general: (4.1.8a) always defines a vielbein on M_4. We will see in section 4.2 how to parameterize both (4.1.8a) and (4.1.8b) in terms of the vielbein they define.

Since we already know from (4.1.8a) and (4.1.8b) the forms defined by the bispinors along the internal space M_4, we just need to compute the bispinors along AdS$_6$, as $\zeta_+\overline{\zeta_+}$. The structure of these bispinors actually depends on how ζ_+ is chosen. One way to see this is to notice that some of the algebraic relations depend on whether the bilinear $\overline{\zeta_+}\zeta_-$ vanishes or not. A more invariant way to describe the situation is to notice that a pair $\zeta_\pm$ of chiral spinors has the same properties as another pair $\zeta'_\pm$ if they can be related via a Lorentz transformation, $\zeta'_\pm = \Lambda\zeta_\pm$; or in other words if they lie in the same *orbit*. The orbits for SO$(1,5)$ have been studied in [118, Sec. 2.4.5.2]. Two orbits correspond to the case where either ζ_+ or ζ_- is zero; these are not compatible with the Killing spinor equation (4.1.2), and are therefore not interesting to us. There is then a one-parameter family of orbits whose stabilizer (i.e. the little group under the SO$(1,5)$ action) is the abelian group $\mathbb{R}^4$; each of these orbits has dimension 11. Finally, there is a four-parameter family of orbits whose stabilizer is SU(2); each of these orbits has dimension 12.

The properties of the forms that one can define from spinor bilinears depend on whether we consider an orbit with stabilizer $\mathbb{R}^4$ or SU(2). The system in [38] will give systems of equations which are superficially different for these two types of orbits. However, the original system for supersymmetry is linear in the supersymmetry parameters ϵ_a. So its solution space should be a linear space, which must in fact have dimension 8 (since this is the smallest number of supercharges for a superalgebra in this dimension). Even if two choices of spinor pairs on this linear space might give superficially different systems of equations, eventually these two different systems must agree. So we can choose the spinor pair in such a way as to get the most convenient system of equations. It turns out that this is one of the orbits with $\mathbb{R}^4$ stabilizer.

To get more concrete, let us decompose the external spinors splitting the external index μ into a “lightcone” part, $a = +, -$, and a four-dimensional Euclidean part, $m = 1, \dots, 4$:

$$\gamma^a_{(6)} = \sigma_a \otimes 1_{(4)} = \frac{1}{2}(\gamma^0_{(6)} \pm \gamma^1_{(6)}) \ , \quad \gamma^m_{(6)} = \sigma_3 \otimes \gamma^m_{(4)} \ , \tag{4.1.10}$$

with $\sigma_\pm = \frac{1}{2}(\pm\sigma_1 + i\sigma_2)$. The matrices $\gamma^\mu_{(6)}$ satisfy the algebra $\mathrm{Cl}(1,5)$ with lightcone metric $\tilde{\eta}^{\mu\nu} = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \oplus \delta^{mn}_{(4)}$, so that $\gamma^{(6)}_\pm = -2\gamma^\mp_{(6)}$ and $\gamma^{(6)}_m = \gamma^m_{(6)}$.

Using this decomposition, we choose now a spinor pair of the form

$$\zeta_\pm \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \chi_\pm \ , \tag{4.1.11}$$

with $\chi_\pm$ a chiral spinor in four dimensions. This corresponds to an orbit with $\mathbb{R}^4$ stabilizer. (Orbits with SU(2) stabilizer would correspond to taking $\zeta_+ = \binom{1}{0} \otimes \chi_+$, $\zeta_- = \binom{0}{1} \otimes \chi_-$.) One consequence of this (which would not be true for the SU(2) orbit) is that the one-form part of the bilinears $\zeta_+\overline{\zeta_+}$ and $\zeta_-\overline{\zeta_-}$ coincide; we will call it z. It is light-like, and it only has components in the two-dimensional part of the decomposition (4.1.10). As for the bilinears in the four dimensions $1, \ldots, 4$, they can be evaluated in the same way as those along M_4, in terms of two one-forms that we will call V and W and which satisfy exactly the same properties as the forms v and w introduced in (4.2.8).

z and the real and imaginary parts of V and W are independent, and in fact orthogonal. They are not quite a vielbein: if we think of z as of the element of a vielbein in the null direction $-$, we are missing another element in direction $+$. As stressed in [38], this cannot be obtained as a bilinear of the supersymmetry parameters; we will see in section 4.1.3 that the remaining equations in the ten-dimensional system of [38] require picking such a null vector as an auxiliary piece of data. In conclusion,

$$\{z = e_-, e_+, \mathrm{Re}V, \mathrm{Im}V, \mathrm{Re}W, \mathrm{Im}W\} \tag{4.1.12}$$

is a vielbein in AdS_6.

We will also define $\Omega_+ = -V \wedge W$, $\Omega_- = \bar{V} \wedge W$, $J_\pm = \pm\frac{i}{2}(V \wedge \bar{V} \pm W \wedge \bar{W})$, just as in (4.2.17), (4.2.17b) for M_4. With all these definitions, we can evaluate

$$\zeta_\pm\overline{\zeta_\pm} = z \wedge e^{-iJ_\pm} \ , \tag{4.1.13a}$$
$$\zeta_+\overline{\zeta_-} = -z \wedge (V + *_4 V) \ , \tag{4.1.13b}$$
$$\zeta_-\overline{\zeta_+} = -z \wedge (\overline{V} - *_4 \overline{V}) \ , \tag{4.1.13c}$$
$$\zeta_\pm\overline{\zeta^c_\pm} = z \wedge \Omega_\pm \ , \tag{4.1.13d}$$
$$\zeta_\pm\overline{\zeta^c_\mp} = \mp z \wedge (W \pm *_4 W) \ . \tag{4.1.13e}$$

Specializing to IIB from now on, we can now plug (4.1.13) into (4.1.7); we have:

$$\begin{aligned}\Phi_{\mathrm{IIB}} = e^A \big[& (z \wedge e^{-iJ_+}) \wedge \phi^1_+ + (z \wedge e^{-iJ_-}) \wedge \phi^1_- \\ & + z \wedge \Omega_+ \wedge \phi^2_+ + z \wedge \Omega_- \wedge \phi^2_- \\ & - z \wedge (V + *_4 V) \wedge \psi^1_+ + z \wedge (\overline{V} - *_4\overline{V}) \wedge \psi^1_- \\ & - z \wedge (W + *_4 W) \wedge \psi^2_+ - z \wedge (W - *_4 W) \wedge \psi^2_- + \text{c.c.}\big] \ .\end{aligned} \tag{4.1.14}$$

This is an odd form, as should be the case for IIB.

To evaluate (4.1.4a), we need to compute the ten-dimensional exterior derivative of $e^{-\phi}\Phi$; schematically, it takes the form:

$$d_H(e^{-\phi}\Phi) = d_H\left(\sum \text{ext} \wedge e^{A-\phi}\,\text{int}\right) = \sum d_6\text{ext} \wedge e^{A-\phi}\,\text{int} + (-)^{\deg(\text{ext})}\text{ext} \wedge d_H(e^{A-\phi}\,\text{int}) \ . \tag{4.1.15}$$

d_6 is the differential along the AdS_6 coordinates, while $d_H = d_4 - H\wedge$ in the last identity is a combination of the exterior differential d_4 along M_4 and of the NS three-form H (which only has components along M_4). Since we are looking for vacuum solutions to (4.1.4a) which are compatible with supersymmetry on AdS_6, we need to take the external spinors $\zeta_\pm$ to be the chiral components of a Killing spinor ζ on this spacetime, i.e. $D_\mu\zeta = \frac{1}{2}\mu\gamma_\mu\zeta$. The norm of the complex constant μ (which is proportional to $\sqrt{-\Lambda}$) can be reabsorbed in the warping function

A; its phase can be reabsorbed by multiplying $\eta^a_\pm$ by $e^{\pm i\theta}$. Hence in what follows we will set $\mu = 1$, resulting in the equation (4.1.2) that we already quoted in the main text.

Exploiting (4.1.2) we can now compute the derivatives of the external forms (4.1.13):

$$d_6(\zeta_\pm\overline{\zeta_\pm}) = -2z \wedge (\mathrm{Re}V + 2i *_4 \mathrm{Im}V) \,, \tag{4.1.16a}$$
$$d_6(\zeta_\pm\overline{\zeta_\mp}) = \pm 3iz \wedge \mathrm{Re}V \wedge \mathrm{Im}V \pm 5z \wedge \mathrm{Re}v \wedge \mathrm{Im}V \wedge \mathrm{Re}W \wedge \mathrm{Im}W \,, \tag{4.1.16b}$$
$$d_6(\zeta_\pm\overline{\zeta^c_\pm}) = -4z \wedge *_4 W \,, \tag{4.1.16c}$$
$$d_6(\zeta_\pm\overline{\zeta^c_\mp}) = \pm 3z \wedge \mathrm{Re}V \wedge W \,. \tag{4.1.16d}$$

As an illustration, (4.1.16a) is computed as follows:

$$\begin{aligned} d_6(\zeta_+\overline{\zeta_+}) &= \frac{1}{2}\left[\gamma^\mu_{(6)}, D_\mu(\zeta_+\overline{\zeta_+})\right] = \frac{1}{4}(\gamma^\mu\gamma_\mu\zeta_-\overline{\zeta_+} - \gamma^\mu\zeta_+\overline{\zeta_-}\gamma_\mu - \gamma_\mu\zeta_-\overline{\zeta_+}\gamma^\mu + \zeta_+\overline{\zeta_-}\gamma_\mu\gamma^\mu) \\ &= \frac{1}{2}(-3z \wedge (\overline{V} - *_4\overline{V}) - 3z \wedge (V + *_4 V) + z \wedge (V - *_4 V) + z \wedge (\overline{V} + *_4\overline{V})) \\ &= -2z \wedge (\mathrm{Re}V + 2i *_4 \mathrm{Im}V) \,, \end{aligned} \tag{4.1.17}$$

having used the formula $\gamma^\mu\omega_k\gamma_\mu = (-)^k(D-2k)\omega_k$ for a k-form ω_k in D dimensions.

The left-hand side $d_H(e^{-\phi}\Phi)$ of (4.1.4a) then contains only unknown derivatives of the internal forms, since those of the external forms have been traded for the right-hand sides of (4.1.16). Once we compute its right-hand side, the complete equation will only involve internal forms and will be valid for any of the sixteen independent components of $\zeta = \zeta_+ + \zeta_-$, as appropriate for an $\mathcal{N} = 1$ vacuum in six dimensions.

Before computing the right-hand side of (4.1.4a), namely $-(\tilde{K} \wedge + \iota_K)F$, we will look at the simpler (4.1.4b): as it happens in other dimensions, they imply that the norms of the internal spinors are related to the warping function A. Let us see how. First, recall the definitions of K and $\tilde{K}$ [38]:

$$K = \frac{1}{64}(\bar{\epsilon}_1\gamma^{(10)}_M\epsilon_1 + \bar{\epsilon}_2\gamma^{(10)}_M\epsilon_2)\, dx^M \,, \quad \tilde{K} = \frac{1}{64}(\bar{\epsilon}_1\gamma^{(10)}_M\epsilon_1 - \bar{\epsilon}_2\gamma^{(10)}_M\epsilon_2)\, dx^M \,. \tag{4.1.18}$$

Plugging in these formulas the decomposition (4.1.3), we obtain:

$$K = \frac{e^{-A}}{4}z\,(\|\eta^1\|^2 + \|\eta^2\|^2) \,, \quad \tilde{K} = \frac{e^{-A}}{4}z\,(\|\eta^1\|^2 - \|\eta^2\|^2) \,. \tag{4.1.19}$$

The external part of the second equation in (4.1.4b) gives $e^{-A}d_6z\,(\|\eta^1\|^2 - \|\eta^2\|^2) = 0$ (the right-hand side vanishes since H is purely internal). One can explicitly compute $d_6 z$, recalling that z is the one-form part of $\zeta_\pm\overline{\zeta_\pm}$; using (4.1.2), one can show that it is nonvanishing. Thus we get:

$$\|\eta^1\|^2 = \|\eta^2\|^2 \,. \tag{4.1.20}$$

Hence $K = \frac{e^{-A}}{2}z\,\|\eta^1\|^2$ and $\tilde{K} = 0$. On the other hand, the first equation in (4.1.4b) says that K is a Killing vector with respect to the ten-dimensional metric g: its external part says that z

is Killing with respect to g_{AdS_6} (this is obvious, since z is a bilinear constructed out of Killing spinors), while its internal part implies $\partial_m\left(\frac{e^{-A}}{2}\|\eta^1\|^2\right)=0$, which upon integration gives

$$\|\eta^1\|^2 = e^A \,, \tag{4.1.21}$$

where without loss of generality we have set to one a possible integration constant. Putting (4.1.20) and (4.1.21) together we get (4.1.25f). Moreover $K = z/2$. Recalling (4.1.26) we now have:

$$\begin{aligned} -(\tilde{K}\wedge + \iota_K)F_{(10)} &= -\iota_K(e^{6A}\mathrm{vol}_6 \wedge *_4\lambda F) = -\frac{e^{6A}}{2} *_6 z \wedge *_4\lambda F \\ &= \frac{e^{6A}}{2}(z\wedge \mathrm{Re}V \wedge \mathrm{Im}V \wedge \mathrm{Re}W \wedge \mathrm{Im}W)\wedge *_4\lambda F \,. \end{aligned} \tag{4.1.22}$$

Putting everything together, we can now separate the various terms in (4.1.4a) that multiply different wedge products of the one-forms in (4.1.12); since those forms are a vielbein in AdS$_6$, they are linearly independent, and each term has to be set to zero separately. In particular, we see from (4.1.22) that the RR flux only contributes to one equation. This gives rise to many equations that can then be arranged in SU(2)$_R$ representations defining SU(2)-covariant forms $\Phi_\pm$ and $\Psi_\pm$. We will rearrange the bispinors (4.1.8) in a very nice way and we will state the result in the nex sub-section.

4.1.2 SU(2)-covariant system of pure spinors

In the meantime, we can already now notice that the (4.1.8a) and (4.1.8b) can be assembled more conveniently using the SU(2) R-symmetry. This is the group that rotates $\binom{\zeta}{\zeta^c}$ and each of $\binom{\eta^a_\pm}{\eta^{a\,c}_\pm}$ as a doublet. One can check that (4.1.3) is then left invariant, so it is a symmetry; since it acts on the external spinors, we call it an R-symmetry. It is the manifestation of the R-symmetry of a five-dimensional SCFT. The SU(2) covariant formalist will be used from the very beginning to yield more manageable results. Let us define

$$\begin{aligned} \Phi_\pm \equiv \begin{pmatrix}\eta^1_\pm \\ \eta^{1\,c}_\pm\end{pmatrix} \otimes \begin{pmatrix}\eta^{2\dagger}_\pm & \overline{\eta^2_\pm}\end{pmatrix} &= \begin{pmatrix}\phi^1_\pm & \phi^2_\pm \\ -(\phi^2_\pm)^* & (\phi^1_\pm)^*\end{pmatrix} \\ &= \mathrm{Re}\phi^1_\pm \mathrm{Id}_2 + i(\mathrm{Im}\phi^2_\pm\sigma_1 + \mathrm{Re}\phi^2_\pm\sigma_2 + \mathrm{Im}\phi^1_\pm\sigma_3) \equiv \Phi^0_\pm \mathrm{Id}_2 + i\Phi^\alpha_\pm\sigma_\alpha \,, \end{aligned} \tag{4.1.23a}$$

$$\begin{aligned} \Psi_\pm \equiv \begin{pmatrix}\eta^1_\pm \\ \eta^{1\,c}_\pm\end{pmatrix} \otimes \begin{pmatrix}\eta^{2\dagger}_\mp & \overline{\eta^2_\mp}\end{pmatrix} &= \begin{pmatrix}\psi^1_\pm & \psi^2_\pm \\ -(\psi^2_\pm)^* & (\psi^1_\pm)^*\end{pmatrix} \\ &= \mathrm{Re}\psi^1_\pm \mathrm{Id}_2 + i(\mathrm{Im}\psi^2_\pm\sigma_1 + \mathrm{Re}\psi^2_\pm\sigma_2 + \mathrm{Im}\psi^1_\pm\sigma_3) \equiv \Psi^0_\pm \mathrm{Id}_2 + i\Psi^\alpha_\pm\sigma_\alpha \,. \end{aligned} \tag{4.1.23b}$$

σ_α, $\alpha = 1,2,3$, are the Pauli matrices. Here and in what follows, the superscript ^{0} denotes an SU(2) singlet, and not the zero-form part; the superscript $^\alpha$ denotes an SU(2) triplet, not a one-form. We hope this will not create confusion.

As we already mentioned, the forms $\Phi_\pm$, $\Psi_\pm$ will define an identity structure on M_4. However, not any random forms $\Phi_\pm$, $\Psi_\pm$ may be written as bispinors as in (4.1.23). In other cases, such as for $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structures in six dimensions [108], it is useful to formulate a set of constraints on the forms that guarantee that they come from spinors; this allows to completely forget about the original spinors, and formulate supersymmetry completely in terms of some forms satisfying some constraints. In the present case, it would be possible to set up such a fancy approach, by saying that $\Phi_\pm$ and $\Psi_\pm$ should satisfy a condition on their inner products. For example we could impose that the Φ's and Ψ's be pure spinors on M_4 obeying the compatibility conditions[4]

$$(\Phi^\alpha_\pm, \Phi^\beta_\pm) = (\Psi^\alpha_\pm, \Psi^\beta_\pm) = \delta^{\alpha\beta}(\Phi^0_\pm, \Phi^0_\pm) = \delta^{\alpha\beta}(\Psi^0_\pm, \Psi^0_\pm) \ . \tag{4.1.24}$$

This would however be an overkill, since in section 4.2 we will directly parameterize $\Phi_\pm$ and $\Psi_\pm$ in terms of a vielbein and some functions on M_4. This will achieve the end of forgetting about the spinors $\eta^a_\pm$ by different means.

We can finally give the system of equations equivalent to preserved supersymmetry:

$$d_H \left[e^{3A-\phi}(\Psi_- - \Psi_+)^0\right] - 2e^{2A-\phi}(\Phi_- + \Phi_+)^0 = 0 \ , \tag{4.1.25a}$$
$$d_H \left[e^{4A-\phi}(\Phi_- - \Phi_+)^\alpha\right] - 3e^{3A-\phi}(\Psi_- + \Psi_+)^\alpha = 0 \ , \tag{4.1.25b}$$
$$d_H \left[e^{5A-\phi}(\Psi_- - \Psi_+)^\alpha\right] - 4e^{4A-\phi}(\Phi_- + \Phi_+)^\alpha = 0 \ , \tag{4.1.25c}$$
$$d_H \left[e^{6A-\phi}(\Phi_- - \Phi_+)^0\right] - 5e^{5A-\phi}(\Psi_- + \Psi_+)^0 = -\frac{1}{4}e^{6A} *_4 \lambda F \ , \tag{4.1.25d}$$
$$d_H \left[e^{5A-\phi}(\Psi_- + \Psi_+)^0\right] = 0 \ ; \tag{4.1.25e}$$
$$||\eta^1||^2 = ||\eta^2||^2 = e^A \ . \tag{4.1.25f}$$

As usual, ϕ here is the dilaton; $d_H = d - H\wedge$; A was defined in (4.1.9); λ is a sign operator defined in footnote 4; $F = F_1 + F_3$ is the "total" allowed internal RR flux, which also determines the external flux via

$$F_{(10)} = F + e^{6A}\mathrm{vol}_6 \wedge *_4\lambda F \ . \tag{4.1.26}$$

Again, we remind the reader that the superscript ^{0} denotes a singlet part, and $^\alpha$ a triplet part, as in (4.1.23).

The last equation, (4.1.25f), can be reformulated in terms of Φ and Ψ. Since $\|\eta^a\|^2 \equiv \|\eta^a_+\|^2 + \|\eta^a_-\|^2$, we can define $||\eta^1_+|| = e^{A/2}\cos(\alpha/2)$, $||\eta^1_-|| = e^{A/2}\sin(\alpha/2)$, $||\eta^2_+|| = e^{A/2}\cos(\tilde\alpha/2)$, $||\eta^2_-|| = e^{A/2}\sin(\tilde\alpha/2)$, where $\alpha, \tilde\alpha \in [0, \pi]$; we then get

$$\begin{aligned}
&(\Phi^0_+, \Phi^0_+) = \frac{1}{8}\cos^2(\alpha/2)\cos^2(\tilde\alpha/2) \ , \qquad (\Phi^0_-, \Phi^0_-) = -\frac{1}{8}\sin^2(\alpha/2)\sin^2(\tilde\alpha/2) \ ; \\
&(\Psi^0_+, \Psi^0_-) = \frac{1}{8}\cos^2(\alpha/2)\sin^2(\tilde\alpha/2) \ , \qquad (\Psi^0_-, \Psi^0_+) = -\frac{1}{8}\sin^2(\alpha/2)\cos^2(\tilde\alpha/2) \ .
\end{aligned} \tag{4.1.27}$$

Just as (4.1.24), however, such a fancy formulation will be ultimately made redundant by our parameterization of Φ and Ψ in section 4.2, which will satisfy (4.1.24) automatically, and where we will take care to implement (4.1.25f), so that (4.1.27) will be satisfied too.

[4]The Chevalley–Mukai pairing is defined as $(\alpha, \beta) = (\alpha \wedge \lambda(\beta))_4$, where on a k-form $\lambda\omega_k = (-)^{\lfloor \frac{k}{2} \rfloor}\omega_k$.

We can check immediately that (4.1.25) imply the equations of motion for the flux, by acting on (4.1.25d) with d_H and using (4.1.25e). The equations of motion for the metric and dilaton are then satisfied (as shown in general in [34] for IIA, and in [119] for IIB); the equations of motion for H are also implied, since they are [33] for Minkowski$_4$ compactifications (which include Minkowski$_5$ as a particular case, and hence also AdS$_6$ by a conical construction). We will see later that the Bianchi identities for F and H are also automatically satisfied for this case.

It is also interesting to compare the system (4.1.25) with the above-mentioned system for Minkowski$_6$ in [112]. First of all the second summands in the left-hand side of (4.1.25a)–(4.1.25d) implicitly come with a factor proportional to $\sqrt{-\Lambda}$ that we have set to one (since it can be reabsorbed in the warping factor A). To take the Mink$_6$ limit, we can imagine to restore those factors, and then take $\Lambda \to 0$. Hence all the second summands in the left-hand side of (4.1.25a)–(4.1.25d) will be set to zero. This is not completely correct, actually, because implicit in (4.1.25a)–(4.1.25c) there are more equations, that one can get by acting on them with d_H (before taking the $\Lambda \to 0$ limit); we have to keep these equations as well. So far the limit works in the same way as for taking the $\Lambda \to 0$ limit from AdS$_4$ to Minkowski$_4$ in [108]. In the present case, however, there is one more thing to take into account. As we have seen, in the Minkowski$_6$ case the spinor Ansatz can be taken to be (4.1.1) rather than the more complicated (4.1.3) we had to use for AdS$_6$. To go from (4.1.3) to (4.1.1), we can simply set $\eta^1_- = 0$ and $\eta^2_\pm = 0$. This sets to zero some of our bispinors; for the IIB case on which we are focusing, it sets to zero everything but Φ_+. This makes some of the equations disappear; some others become redundant. All in all, we are left with

$$d_H(e^{2A-\phi}\Phi^0_+) = 0 \,, \qquad d_H(e^{4A-\phi}\Phi^\alpha_+) = 0 \,, \qquad d_H(e^{6A-\phi}\Phi^0_+) = -\frac{1}{4}e^{6A} *_4 \lambda F \,, \quad (4.1.28)$$

which is [112, Eq. (4.11)] in our SU(2)-covariant language.

4.1.3 Redundancy of pairing equations

We are now left with equations (2.3.20) of the general ten-dimensional system specialized for IIB backgrounds of the type AdS$_6 \times M_4$. We will now show that they are redundant, as we expected.

It is convenient to rewrite here equations: (2.3.20),[5]

$$\left(e_{+_1} \cdot \Phi \cdot e_{+_2}, \gamma^{MN}_{(10)} \left[\pm d_H(e^{-\phi}\Phi \cdot e_{+_2}) + \frac{1}{2}e^{\phi} d^\dagger(e^{-2\phi}e_{+_2})\Phi - F_{(10)}\right]\right) = 0 \,, \qquad (4.1.29a)$$

$$\left(e_{+_1} \cdot \Phi \cdot e_{+_2}, \left[d_H(e^{-\phi}e_{+_1} \cdot \Phi) - \frac{1}{2}e^{\phi} d^\dagger(e^{-2\phi}e_{+_2})\Phi - F_{(10)}\right] \gamma^{MN}_{(10)}\right) = 0 \,, \qquad (4.1.29b)$$

[5]The Clifford action from the left (right) of a ten-dimensional gamma matrix on a k-form ω_k is given by [38]:

$$\gamma^M_{(10)}\,\omega_k = (dx^M \wedge + g^{MN}\iota_N)\omega_k \,, \qquad \omega_k\,\gamma^M_{(10)} = (-)^k(dx^M \wedge - g^{MN}\iota_N)\omega_k \,.$$

are completely redundant when specialized to $\mathrm{AdS}_6 \times M_4$ solutions in IIB, i.e. they are automatically satisfied by the expressions for bispinors and fluxes we found in section 4.3. Since the analysis of the case at hand is similar to the ones presented in [38] and [114] (for four- and two-dimensional Minkowski vacua respectively), we will briefly describe the main computations and point out the novelties arising for an AdS vacuum.

Firstly, we need to choose the vectors e_{+_a}. Intuitively, these auxiliary vectors are needed because the form Φ is not enough by itself to specify a vielbein; for more details, see [38]. The e_{+_a} can be chosen quite freely, provided they satisfy the constraints

$$e_{+_a}^2 = 0 \ , \quad e_{+_a} \llcorner K_a = \frac{1}{2} \ . \tag{4.1.30}$$

Since $K_1 = K_2 = K = \frac{1}{2}z$ has only external indices, we will set

$$e_{+_1} = e_{+_2} \equiv e_+ \ , \tag{4.1.31}$$

and we will consider e_+ to be purely external as well. This is just the one-form that in (4.1.12) we had to leave undetermined; as we anticipated there, it is an auxiliary piece of data and cannot be determined as a bilinear of $\zeta_\pm$. For Minkowski vacua, K is a constant vector, and one can then simply take e_+ to be constant too. In AdS, however, the requirement that K be a Killing vector does not imply that it is constant, and hence there is no reason to have e_+ constant either. However, we will argue that e_+ can be chosen in such a way to at least make the $d_6^\dagger e_+$ terms in (4.1.29) vanish. To this end, let us first define the spinors $\tilde{\zeta}_\pm$ along the lines of (4.1.11):

$$\tilde{\zeta}_\pm \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \chi_\pm \ , \tag{4.1.32}$$

and the one-form

$$e_+ \equiv (\tilde{\zeta}_+ \overline{\tilde{\zeta}_+})_{\text{one-form}} \propto \overline{\tilde{\zeta}_+} \gamma_\mu^{(6)} \tilde{\zeta}_+ \, dx^\mu \ , \tag{4.1.33}$$

which satisfies $e_+^2 = 0$, $e_+ \cdot K \neq 0$; thus, by appropriate rescaling, taking (4.1.31) and (4.1.33) will indeed satisfy (4.1.30). Since (4.1.32) now also satisfies the Killing spinor equations (4.1.2), $d_6^\dagger e_+$ vanishes.

Another difference with respect to the Minkowski case comes from the term $d_H(e^{-\phi}\Phi \cdot e_+)$. Using the formula $\left\{d, \cdot\, e_+(-)^{\deg}\right\} = e^{-A}\partial_+ + dA \wedge e_+ \cdot$, we can write it as

$$d_H(e^{-\phi}\Phi \cdot e_+) = (d_H(e^{-\phi}\Phi)) \cdot e_+ - e^{-\phi} dA \wedge e_+ \cdot \Phi - e^{-(A+\phi)}\partial_+\Phi \ . \tag{4.1.34}$$

As usual, the first term on the right hand side vanishes inside a pairing,[6] while the last one does not (contrary to the Minkowski case), and we must evaluate it. Since $\partial_+\Phi = \delta^{+\mu}D_\mu\Phi = \delta^{+\mu}D_\mu(\epsilon_1\overline{\epsilon_2})$, we can use the decomposition (4.1.3) and the equations (4.1.2) to conclude that

$$\partial_+\Phi = \frac{1}{2} e_+ \cdot \hat{\Phi} + \ldots \ , \tag{4.1.35}$$

[6] This is because $e_+^2 = 0$. Just replace C with $(d_H(e^{-\phi}\Phi)) \cdot e_+$ in the formula [38, Sec. B.4]

$$(e_+ \cdot \Phi \cdot e_+, C) = -\frac{(-)^{\deg(\Phi)}}{32} \overline{\epsilon_1} e_+ C e_+ \epsilon_2 \ .$$

where the dots denote terms that vanish in the pairing in (4.1.29a), and where we defined

$$\hat{\Phi} \equiv (\hat{\epsilon}_1 \overline{\epsilon_2}) \, , \qquad \hat{\epsilon}_1 \equiv \zeta_- \eta^1_+ + \zeta^c_- \eta^{1\,c}_+ + \zeta_+ \eta^1_- + \zeta^c_+ \eta^{1\,c}_- \, . \tag{4.1.36}$$

To sum up, for type IIB $\mathrm{AdS}_6 \times M_4$ vacua we can rewrite (4.1.29a) as

$$\left(e_+ \cdot \Phi \cdot e_+ , \gamma^{MN}_{(10)} \left[e^{-\phi} dA \wedge (e_+ \cdot \Phi) + \frac{e^{-(A+\phi)}}{2} e_+ \cdot \hat{\Phi} - 2F \right] \right) = 0 \, ; \tag{4.1.37}$$

to rewrite the flux term we have made use of the formula

$$\big(e_+ \cdot \Phi \cdot e_+ , F_{(10)} \big) = 2 \left(e_+ \cdot \Phi \cdot e_+ , F \right) \, . \tag{4.1.38}$$

From now on the analysis parallels the one for Minkowski vacua, and we will not repeat it here. Specializing (4.1.29a), (4.1.29b) to the case $M = m$, $N = n$ does not give any equations; specializing them to the cases $M = \mu$, $N = \nu$ and $M = m$, $N = \nu$ gives[7]

$$\big(\Psi^0_+ + \Psi^0_- , F \big) = e^{-\phi} \, , \tag{4.1.40a}$$
$$\big(\Psi^\alpha_+ - \Psi^\alpha_- , F \big) = 0 \, , \tag{4.1.40b}$$
$$\big(dx_m \wedge (\Phi^0_+ - \Phi^0_-) , F \big) = -e^{A-\phi} \partial_m A \, , \tag{4.1.40c}$$
$$\big(\iota_m (\Phi^0_+ - \Phi^0_-) , F \big) = 0 \, . \tag{4.1.40d}$$

It can be shown that these equations transform into identities upon plugging in the expressions for the solutions to the system (4.1.25). This completes the proof of the redundancy of (4.1.29a) and (4.1.29b) for $\mathrm{AdS}_6 \times M_4$ vacua in type IIB.

In summary, in this section we have presented the system (4.1.25), which is equivalent to preserved supersymmetry for backgrounds of the form $\mathrm{AdS}_6 \times M_4$. The forms Φ and Ψ are not arbitrary: they obey certain algebraic constraints expressing their origin as spinor bilinears in (4.1.23), (4.1.8). We will now give the general solution to those constraints, and then proceed in section 4.3 to analyze the system.

4.2 Parameterization of the pure spinors

We have introduced in section 4.1 the even forms $\Phi_\pm$ and the odd forms $\Psi_\pm$ (see (4.1.23), (4.1.8a), (4.1.8b)). These are the main characters in the system (4.1.25), which is equivalent to preserved supersymmetry. Before we start using the system, however, we need to characterize what sorts of forms $\Phi_\pm$ and $\Psi_\pm$ can be: this is what we will do in this section.

[7] As a curiosity, notice that (4.1.40c) can also be written as

$$\sqrt{g} * \big((\Phi^0_+ - \Phi^0_-) \wedge \lambda(F) \big) = -e^{A-\phi} dA \, . \tag{4.1.39}$$

4.2.1 Even forms

We will first deal with $\Phi_\pm$. We will actually first focus on Φ_+, and then quote the results for Φ_-. The computations in this subsection are actually pretty standard, and we will be brief.

Let us start with the case $\eta^1_+ = \eta^2_+ \equiv \eta_+$. Assume also for simplicity that $||\eta_+||^2 = 1$. In this case the bilinears define an SU(2) structure:

$$\eta_+\eta_+^\dagger = \frac{1}{4}e^{-ij_+} , \qquad \eta_+\overline{\eta_+} = \frac{1}{4}\omega_+ , \tag{4.2.1}$$

where the two-forms j_+, ω_+ satisfy

$$j_+ \wedge \omega_+ = 0 , \qquad \omega_+^2 = 0 , \qquad \omega_+ \wedge \overline{\omega_+} = 2j_+^2 = -\mathrm{vol}_4 . \tag{4.2.2}$$

We can also compute

$$\eta^c_+\eta^{c\,\dagger}_+ = \frac{1}{4}e^{ij_+} , \qquad \eta^c_+\eta^\dagger_+ = -\frac{1}{4}\overline{\omega_+} . \tag{4.2.3}$$

Let us now consider the case with two different spinors, $\eta^1_+ \neq \eta^2_+$; let us again assume that they have unit norm. We can define (in a similar way as in [120])

$$\eta_{0+} = \frac{1}{2}(\eta^1_+ - i\eta^2_+) , \qquad \tilde\eta_{0+} = \frac{1}{2}(\eta^1_+ + i\eta^2_+) . \tag{4.2.4}$$

Consider now $a_+ = \eta^{2\,\dagger}_+\eta^1_+$, $b_+ = \overline{\eta^2_+}\eta^1_+$. $\{\eta^2_+, \eta^{2\,c}_+\}$ is a basis for spinors on M_4; a_+, b_+ are then the coefficients of η^1_+ along this basis. Since η^a_+ have both unit norm, we have $|a_+|^2 + |b_+|^2 = 1$. By multiplying η^a_+ by phases, we can assume that a_+ and b_+ are for example purely imaginary, and we can then parameterize them as $a_+ = -i\cos(\theta_+)$, $b_+ = i\sin(\theta_+)$. Going back to (4.2.4), we can now compute their inner products:

$$\eta^\dagger_{0+}\eta_{0+} = \cos^2\left(\frac{\theta_+}{2}\right) , \qquad \eta^\dagger_{0+}\tilde\eta_{0+} = 0 , \qquad \overline{\eta_{0+}}\tilde\eta_{0+} = \frac{1}{2}\sin(\theta_+) . \tag{4.2.5}$$

From this we can in particular read off the coefficients of the expansion of $\tilde\eta_{0+}$ along the basis $\{\eta_{0+}, \eta^c_{0+}\}$. This gives $\tilde\eta_{0+} = \frac{1}{||\eta_{0+}||^2}(\eta^\dagger_{0+}\tilde\eta_{0+}\eta_{0+} + \overline{\eta_{0+}}\tilde\eta_{0+}\eta^c_{0+}) = \tan\left(\frac{\theta_+}{2}\right)\eta^c_{0+}$. Recalling (4.2.4), and defining now $\eta_{0+} = \cos\left(\frac{\theta_+}{2}\right)\eta_+$, we get

$$\eta^1_+ = \cos\left(\frac{\theta_+}{2}\right)\eta_+ + \sin\left(\frac{\theta_+}{2}\right)\eta^c_+ , \qquad \eta^2_+ = i\left(\cos\left(\frac{\theta_+}{2}\right)\eta_+ - \sin\left(\frac{\theta_+}{2}\right)\eta^c_+\right) . \tag{4.2.6}$$

From this it is now easy to compute $\eta^1_+\eta^{2\,\dagger}_+$ and $\eta^1_+\overline{\eta^2_+}$. Recall, however, that in the course of our computation we have first fixed the norms and then the phases of η^a_+. The norms of the spinors we need in this chapter are not one; they were actually already parameterized before (4.1.27), so as to satisfy (4.1.25f). The factor e^A, however, simplifies with the e^{-A} in the definition (4.1.8a).

Let us also restore the phases we earlier fixed, by rescaling $\eta^1_\pm \to e^{iu_\pm}\eta^1_\pm$, $\eta^2_\pm \to e^{it_\pm}\eta^2_\pm$. All in all we get

$$\phi^1_+ = \frac{1}{4}\cos(\alpha/2)\cos(\tilde{\alpha}/2)e^{i(u_+-t_+)}\cos(\theta_+)\exp\left[-\frac{1}{\cos(\theta_+)}(ij_+ + \sin(\theta_+)\mathrm{Re}\omega_+)\right] , \tag{4.2.7a}$$

$$\phi^2_+ = \frac{1}{4}\cos(\alpha/2)\cos(\tilde{\alpha}/2)e^{i(u_++t_+)}\sin(\theta_+)\exp\left[\frac{1}{\sin(\theta_+)}(\cos(\theta_+)\mathrm{Re}\omega_+ + i\mathrm{Im}\omega_+)\right] . \tag{4.2.7b}$$

The formulas for $\phi^{1,2}_-$ can be simply obtained by changing $\cos(\alpha/2) \to \sin(\alpha/2)$, $\cos(\tilde{\alpha}/2) \to \sin(\tilde{\alpha}/2)$, and $_+ \to {}_-$ everywhere. The only difference to keep in mind is that the last equation in (4.2.2) is now replaced with $\omega_- \wedge \overline{\omega_-} = 2j_-^2 = \mathrm{vol}_4$.

4.2.2 Odd forms

We now turn to the bilinears of “mixed type”, i.e. the $\psi^{1,2}_\pm$ we defined in (4.1.8b), which result in odd forms. We will again start from the case where $\eta^1_\pm = \eta^2_\pm \equiv \eta_\pm$.

There are two vectors we can define:

$$v_m = \eta^{2\,\dagger}_- \gamma_m \eta^1_+ , \qquad w_m = \overline{\eta^2_-}\gamma_m \eta^1_+ . \tag{4.2.8}$$

In bispinor language, we can compute

$$\eta_+\eta_-^\dagger = \frac{1}{4}(1+\gamma)v , \qquad \eta^c_+\eta^{c\,\dagger}_- = \frac{1}{4}(1+\gamma)\overline{v} , \tag{4.2.9a}$$

$$\eta_-\eta_+^\dagger = \frac{1}{4}(1-\gamma)\overline{v} , \qquad \eta^c_-\eta^{c\,\dagger}_+ = \frac{1}{4}(1-\gamma)v , \tag{4.2.9b}$$

and

$$\eta_+\eta^{c\,\dagger}_- = \frac{1}{4}(1+\gamma)w , \qquad \eta^c_+\eta^\dagger_- = -\frac{1}{4}(1+\gamma)\overline{w} , \tag{4.2.9c}$$

$$\eta_-\eta^{c\,\dagger}_+ = -\frac{1}{4}(1-\gamma)w , \qquad \eta^c_-\eta^\dagger_+ = \frac{1}{4}(1-\gamma)\overline{w} . \tag{4.2.9d}$$

(In four Euclidean dimensions, the chiral $\gamma = *_4\lambda$, so that $(1+\gamma)v = v + *_4 v$, and so on. See [38, App. A] for more details.) For the more general case where $\eta^1_\pm \neq \eta^2_\pm$, we can simply refer back to (4.2.6). For example we get

$$\psi^1_+ = \frac{e^{i(u_+-t_-)}}{4}\cos(\alpha/2)\sin(\tilde{\alpha}/2)(1+\gamma)\left[\cos\left(\frac{\theta_++\theta_-}{2}\right)\mathrm{Re}v + i\cos\left(\frac{\theta_+-\theta_-}{2}\right)\mathrm{Im}v + \right.$$
$$\left. -\sin\left(\frac{\theta_++\theta_-}{2}\right)\mathrm{Re}w + i\sin\left(\frac{\theta_+-\theta_-}{2}\right)\mathrm{Im}w\right] . \tag{4.2.10}$$

For the time being we do not show the lengthy expressions for the other odd bispinors ψ^2_+ and $\psi^{1,2}_-$, because they will all turn out to simplify quite a bit as soon as we impose the zero-form equations in (4.1.25).

The v and w we just introduced are a complex vielbein; let us see why. First, a standard Fierz computation gives

$$v \cdot \eta_+ = 0 \, , \qquad \overline{v} \cdot \eta_+ = 2\eta_- \, , \tag{4.2.11}$$

where $\cdot$ denotes Clifford product. Multiplying from the left by $\eta_-^\dagger$, we obtain

$$v^2 = 0 \, , \qquad v \llcorner \overline{v} = v^m \overline{v}_m = 2 \, . \tag{4.2.12}$$

Similarly to (4.2.11), we can compute the action of w:

$$w \cdot \eta_\pm = 0 \, , \qquad \overline{w} \cdot \eta_\pm = \pm 2\eta^c_\mp \, . \tag{4.2.13}$$

Multiplying by $\overline{\eta_\mp}$, we get

$$w^2 = 0 \, , \qquad w \llcorner \overline{w} = 2 \, . \tag{4.2.14}$$

From (4.2.11) we can also get $v \cdot \eta_+ \overline{\eta_-} = 0$, $\overline{v} \cdot \eta_+ \overline{\eta_-} = 2\eta_- \overline{\eta_-}$, whose zero-form parts read

$$v \llcorner w = 0 = \overline{v} \llcorner w \, . \tag{4.2.15}$$

Together, (4.2.12), (4.2.14), (4.2.15) say that

$$\{\mathrm{Re} v, \ \mathrm{Re} w, \ \mathrm{Im} v, \ \mathrm{Im} w\} \tag{4.2.16}$$

are a vielbein.

We can also now try to relate the even forms of section 4.2.1 to this vielbein. From (4.2.11) we also see $v \cdot \eta_+ \overline{\eta_+} = 0$, which says $v \wedge \omega_+ = 0$; similarly one gets $\overline{v} \wedge \omega_- = 0$. Also, (4.2.13) implies that $w \cdot \eta_+ \overline{\eta_+} = w \cdot \omega_+ = 0$, and thus that $w \wedge \omega_\pm = 0$. So we have $\omega_+ \propto v \wedge w$, $\omega_- \propto \overline{v} \wedge w$. One can fix the proportionality constant by a little more work:

$$\omega_+ = -v \wedge w \, , \qquad \omega_- = \overline{v} \wedge w \, . \tag{4.2.17a}$$

Similar considerations also determine the real two-forms:

$$j_\pm = \pm \frac{i}{2} (v \wedge \overline{v} \pm w \wedge \overline{w}) \, . \tag{4.2.17b}$$

So far we have managed to parameterize all the pure spinors $\Phi_\pm$, $\Psi_\pm$ in terms of a vielbein given by (4.2.16). The expressions for Φ_+ are given in (4.2.7); Φ_- is given by changing $(\cos(\alpha/2), \cos(\tilde{\alpha}/2)) \to (\sin(\alpha/2), \sin(\tilde{\alpha}/2))$, and $_+ \to {}_-$ everywhere. The forms $j_\pm$, $\omega_\pm$ are given in (4.2.17) in terms of the vielbein. Among the odd forms of $\Psi_\pm$, we have only quoted one example, (4.2.10); similar expressions exist for ψ^2_+ and for $\psi^{1,2}_-$. We will summarize all this again after the simplest supersymmetry equations will allow us to simplify the parameterization quite a bit.

4.3 General analysis

We will now use the parameterization obtained for Φ and Ψ in section 4.2 in the system (4.1.25). As anticipated in the introduction, we will reduce the system to the two PDEs (4.3.12a), (4.3.13), and we will determine the local form of the metric and of the fluxes in terms of a solution to those equations.

4.3.1 Zero-form equations

The only equations in (4.1.25) that have a zero-form part are (4.1.25a) and (4.1.25c):

$$(\Phi_+ + \Phi_-)^0_0 = 0 \, , \qquad (\Phi_+ + \Phi_-)^\alpha_0 = 0 \, . \tag{4.3.1}$$

The subscript $_0$ here denotes the zero-form part. (Recall that the superscripts ^{0} and $^\alpha$ denote SU(2) singlets and triplets respectively.) To simplify the analysis, it is useful to change variables so as to make the SU(2) R-symmetry more manifest.

In (4.2.7), apart for the overall factor $\cos(\alpha/2)\cos(\tilde\alpha/2)/4$, we have $\phi^1_{+\,0} \propto e^{i(u_+ - t_+)}\cos(\theta_+)$, $\phi^2_{+\,0} \propto e^{i(u_+ + t_+)}\sin(\theta_+)$. The singlet is $\mathrm{Re}\phi^1_{+\,0} \propto \cos(\theta_+)\cos(u_+ - t_+)$, and it is a good idea to give it a name, say x_+. On the other hand, the triplet is $\{\mathrm{Im}\phi^2_+, \mathrm{Re}\phi^2_+, \mathrm{Im}\phi^1_+\} \propto \{\sin(\theta_+)\sin(u_+ + t_+), \sin(\theta_+)\cos(u_+ + t_+), \cos(\theta_+)\sin(u_+ - t_+)\}$. If we sum their squares, we obtain:

$$\sin^2(\theta_+) + \cos(\theta_+)^2 \sin^2(u_+ - t_+) = x_+^2 \tan^2(u_+ - t_+) + \sin^2(\theta_+) = 1 - x_+^2 \, . \tag{4.3.2}$$

This suggests that we parameterize the triplet using the combination $\sqrt{1 - x_+^2}\, y^\alpha$, where y^α should obey $y_\alpha y^\alpha = 1$ and can be chosen to be the $\ell = 1$ spherical harmonics on S^2. What we are doing is essentially changing variables on an S^3, going from coordinates that exhibit it as an $S^1 \times S^1$ fiberation over an interval to coordinates that exhibit it as an S^2 fiberation over an interval:

$$\left\{\cos(\theta_+)e^{i(u_+ - t_+)}, \sin(\theta_+)e^{i(u_+ + t_+)}\right\} \to \left\{x_+, \sqrt{1 - x_+^2} y^\alpha\right\} \, . \tag{4.3.3}$$

An identical discussion can of course be given for $\phi^{1,2}_-$. Summing up, we are led to the following definitions:

$$x_\pm \equiv \cos(\theta_\pm)\cos(u_\pm - t_\pm) \, , \quad \sin\beta_\pm \equiv \frac{\sin(\theta_+)}{\sqrt{1 - x_+^2}} \, , \quad \gamma_\pm \equiv \frac{\pi}{2} - u_\pm - t_\pm \, , \tag{4.3.4}$$

and

$$y^\alpha_\pm \equiv \Big(\sin(\beta_\pm)\cos(\gamma_\pm), \; \sin(\beta_\pm)\sin(\gamma_\pm), \; \cos(\beta_\pm)\Big) \, , \tag{4.3.5}$$

in terms of which

$$\begin{aligned} \Phi_{+\,0} &= \cos(\alpha/2)\cos(\tilde\alpha/2)\left(x_+ + i y^\alpha_+ \sqrt{1 - x_+^2}\sigma_\alpha\right) \, , \\ \Phi_{-\,0} &= \sin(\alpha/2)\sin(\tilde\alpha/2)\left(x_- + i y^\alpha_- \sqrt{1 - x_-^2}\sigma_\alpha\right) \, . \end{aligned} \tag{4.3.6}$$

Going back to (4.3.1), summing the squares of all four equations we get $\cos^2(\alpha/2)\cos^2(\tilde{\alpha}/2) = \sin^2(\alpha/2)\sin^2(\tilde{\alpha}/2)$. Given that α and $\tilde{\alpha} \in [0,\pi]$, this is uniquely solved by

$$\tilde{\alpha} = \pi - \alpha \, . \tag{4.3.7}$$

Now (4.3.1) reduces to

$$-x_- = x_+ \equiv x \, , \qquad -y^\alpha_- = y^\alpha_+ \equiv y^\alpha \, . \tag{4.3.8}$$

In terms of the original parameters, this means $\theta_+ = \theta_-$, $u_- = u_+$, $t_- = t_+ + \pi$.

The parameterization obtained in section 4.2 now simplifies considerably:

$$\phi^1_\pm = \pm\frac{1}{8}\sin\alpha\cos\theta\, e^{i(u-t)}\exp\left[-\frac{1}{\cos\theta}(ij_\pm + \sin\theta\mathrm{Re}\omega_\pm)\right] \, , \tag{4.3.9a}$$

$$\phi^2_\pm = \pm\frac{1}{8}\sin\alpha\sin\theta\, e^{i(u+t)}\exp\left[\frac{1}{\sin\theta}(\cos\theta\mathrm{Re}\omega_+ + i\mathrm{Im}\omega_+)\right] \, ; \tag{4.3.9b}$$

$$\psi^1_\pm = \mp\frac{1}{8}(1\pm\cos\alpha)e^{i(u-t)}(1\pm\gamma)\left[\cos\theta\mathrm{Re}v \pm i\mathrm{Im}v \mp \sin\theta\mathrm{Re}w\right] \, , \tag{4.3.9c}$$

$$\psi^2_\pm = \mp\frac{1}{8}(1\pm\cos\alpha)e^{i(u+t)}(1\pm\gamma)\left[\sin\theta\mathrm{Re}v \pm i\mathrm{Im}w \pm \cos\theta\mathrm{Re}w\right] \, . \tag{4.3.9d}$$

We temporarily reverted here to a formulation where $\mathrm{SU}(2)_\mathrm{R}$ is not manifest; however, in what follows we will almost always use the $\mathrm{SU}(2)$-covariant variables x and y^α introduced above.

4.3.2 Geometry

We will now describe how we analyzed the higher-form parts of (4.1.25), although not in such detail as in section 4.3.1.

The only equations that have a one-form part are (4.1.25b). From (4.3.9c), (4.3.9d), we see that the second summand $(\Psi_+ + \Psi_-)^\alpha_1$ is a linear combination of the forms in the vielbein (4.2.16). The first summand consists of derivatives of the parameters we have previously introduced. This gives three constraints on the four elements of the vielbein. We used it to express $\mathrm{Im}v$, $\mathrm{Re}w$, $\mathrm{Im}w$ in terms of $\mathrm{Re}v$;[8] the resulting expressions are at this point still not particularly illuminating, and we will not give them here. These expressions are not even manifestly $\mathrm{SU}(2)$-covariant at this point; however, once one uses them into $\Phi_\pm$ and $\Psi_\pm$, one does find $\mathrm{SU}(2)$-covariant forms. Just by way of example, we have

$$\begin{aligned}
(\Phi_+ + \Phi_-)^\alpha_2 &= -\frac{1}{3}e^{-3A+\phi}\sin\alpha\,\mathrm{Re}v\wedge d\left(y^\alpha\sin\alpha\, e^{4A-\phi}\sqrt{1-x^2}\right) \, , \\
(\Psi_- - \Psi_+)^\alpha_1 &= y^\alpha\sqrt{1-x^2}\sin^2(\alpha)\mathrm{Re}v + \frac{1}{3}e^{-3A+\phi}\cos\alpha\, d\left(y^\alpha\sin\alpha\, e^{4A-\phi}\sqrt{1-x^2}\right) \, .
\end{aligned} \tag{4.3.10}$$

[8]Doing so requires $x \neq 0$; the case $x = 0$ will be analyzed separately in section 4.3.4.

We chose these particular 2-form and 1-form triplet combinations because they are involved in the 2-form part of (4.1.25c). The result is a triplet of equations of the form $y^\alpha\, E_2 + dy^\alpha \wedge E_1 = 0$, where E_i are i-forms and $\mathrm{SU}(2)_\mathrm{R}$ singlets. If we multiply this by y_α, we obtain $E_2 = 0$ (since $y_\alpha dy^\alpha = 0$); then also $E_1 = 0$ necessarily. The latter gives a simple expression for $\mathrm{Re}v$, the one-form among the vielbein (4.2.16) that we had not determined yet:

$$\mathrm{Re}v = -\frac{e^{-A}}{\sin\alpha} d(e^{2A}\cos\alpha)\ . \tag{4.3.11}$$

Once this is used, the two-form equation $E_2 = 0$ is automatically satisfied.

There are some more two-form equations from (4.1.25). The easiest is (4.1.25e), which gives

$$d\left(\frac{e^{4A-\phi}}{x}\cot\alpha\, d(e^{2A}\cos\alpha) + \frac{1}{3x}e^{2A}\sqrt{1-x^2}\,d\left(e^{4A-\phi}\sqrt{1-x^2}\sin\alpha\right)\right) = 0\ . \tag{4.3.12a}$$

Locally, this can be solved by saying

$$\boxed{x dz = e^{4A-\phi}\cot\alpha\, d(e^{2A}\cos\alpha) + \frac{1}{3}e^{2A}\sqrt{1-x^2}\,d\left(e^{4A-\phi}\sqrt{1-x^2}\sin\alpha\right)} \tag{4.3.12b}$$

for some function z. The two-form part of (4.1.25a) reads, on the other hand,

$$\boxed{e^{-8A}d(e^{6A}\cos\alpha)\wedge dz = d(xe^{2A-\phi}\sin\alpha)\wedge d(e^{2A}\cos\alpha)}\ . \tag{4.3.13}$$

If one prefers, dz can be eliminated, giving

$$3\sin(2\alpha)dA\wedge d\phi = d\alpha\wedge\left(6dA + \sin^2(\alpha)\left(-dx^2 - 2(x^2+5)dA + (1+2x^2)d\phi\right)\right)\ . \tag{4.3.14}$$

We will devote the whole section 4.4 to analyze the PDEs (4.3.12a), (4.3.13) and we will also exhibit two explicit solutions.

Taking the exterior derivative of (4.3.13) one sees that $d\alpha\wedge dA\wedge dz = 0$. Wedging (4.3.12a) with an appropriate one-form, one also sees $d\alpha\wedge dA\wedge dx = 0$. Taken together, these mean that only two among the remaining variables (α, x, A, ϕ) are really independent. For example we can take α and x to be independent, and

$$A = A(\alpha, x)\ , \qquad \phi = \phi(\alpha, x)\ . \tag{4.3.15}$$

We are not done with the analysis of (4.1.25), but there will be no longer any purely geometrical equations: the remaining content of (4.1.25) determines the fluxes, as we will see in the next subsection. Let us then pause to notice that at this point we have already determined the metric: three of the elements of the vielbein (4.2.16) were determined already at the beginning of this section in terms of $\mathrm{Re}v$, and the latter was determined in (4.3.11). This gives the metric

$$ds^2 = \frac{\cos\alpha}{\sin^2(\alpha)}\frac{dq^2}{q} + \frac{1}{9}q(1-x^2)\frac{\sin^2(\alpha)}{\cos\alpha}\left(\frac{1}{x^2}\left(\frac{dp}{p} + 3\cot^2(\alpha)\frac{dq}{q}\right)^2 + ds^2_{S^2}\right)\ , \tag{4.3.16}$$

where the S^2 is spanned by the functions β and γ introduced in (4.3.5) (namely, $ds^2_{S^2} = d\beta^2 + \sin^2(\beta)d\gamma^2$), and we have eliminated A and ϕ in favor of

$$q \equiv e^{2A}\cos\alpha\ , \qquad p \equiv e^{4A-\phi}\sin\alpha\sqrt{1-x^2}\ . \tag{4.3.17}$$

These variables could also be used in the equations (4.3.12a), (4.3.13) above, with marginal simplification. Notice that positivity of (4.3.16) requires $|x| \leq 1$.

Thus we have found in this section that the internal space M_4 is an S^2 fiberation over a two-dimensional space Σ, which we can think of as spanned by the coordinates (α, x).

4.3.3 Fluxes

We now turn to the three-form part of (4.1.25b). This is an SU(2)$_\mathrm{R}$ triplet. It can be written as $y^\alpha H = \epsilon^{\alpha\beta\gamma}y^\beta dy^\gamma \wedge \tilde{E}_2 + y^\alpha \mathrm{vol}_{S^2} \wedge \tilde{E}_1$, where $\tilde{E}_i$ are i-forms and SU(2)$_\mathrm{R}$ singlets. Actually, from (4.3.12a) and (4.3.13) it follows that $\tilde{E}_2 = 0$; we are then left with a single equation setting $H = \mathrm{vol}_{S^2} \wedge \tilde{E}_1$:

$$H = -\frac{1}{9x}e^{2A}\sqrt{1-x^2}\sin\alpha\left[-\frac{6dA}{\sin\alpha} + 2e^{-A}(1+x^2)d(e^A\sin\alpha) + \sin\alpha\, d(\phi + x^2)\right] \wedge \mathrm{vol}_{S^2}\ . \tag{4.3.18}$$

As expected, H is a singlet under SU(2)$_\mathrm{R}$.

All the four-form equations in (4.1.25e), (4.1.25a), (4.1.25c) turn out to be automatically satisfied. We can then finally turn our attention to (4.1.25d), which we have ignored so far. It gives the following expressions for the fluxes:

$$F_1 = \frac{e^{-\phi}}{6x\cos\alpha}\left[\frac{12dA}{\sin\alpha} + 4e^{-A}(x^2-1)d(e^A\sin\alpha) + e^{2\phi}\sin\alpha\, d(e^{-2\phi}(1+2x^2))\right]\ ; \tag{4.3.19a}$$

$$\begin{aligned} F_3 = \frac{e^{2A-\phi}}{54}\sqrt{1-x^2}\frac{\sin^2(\alpha)}{\cos\alpha}&\left[\frac{36dA}{\sin\alpha} + 4e^{-A}(x^2-7)d(e^A\sin\alpha) + \right.\\ &\left. +\ e^{2\phi}\sin\alpha\, d(e^{-2\phi}(1+2x^2))\right] \wedge \mathrm{vol}_{S^2}\ . \end{aligned} \tag{4.3.19b}$$

The Bianchi identities

$$dH = 0\ , \quad dF_1 = 0\ , \quad dF_3 + H \wedge F_1 = 0\ , \tag{4.3.20}$$

are all automatically satisfied, using of course the PDEs (4.3.12a), (4.3.13). As usual, this statement is actually true only if one assumes that the various functions appearing in those equations are smooth. One can introduce sources by relaxing this condition.

4.3.4 The case $x = 0$

In section 4.3.2, we used the three-form part of (4.1.25b) to express $\mathrm{Im}v$, $\mathrm{Re}w$, $\mathrm{Im}w$ in terms of $\mathrm{Re}v$. This actually can only be done for $x \neq 0$: the expressions we get contain x in the denominator, as can be seen for example in (4.3.12a). This left out the case $x = 0$; we will analyze it in this section, showing that it leads to a single solution, discussed in [103, 104] — namely, to a T-dual of the AdS_6 solution found in [99] and reviewed in our language in appendix B.2.

Keeping in mind that $-x_- = x_+ = x$ (from (4.3.8)), from (4.3.4) we have $x = \cos(\theta)\cos(u-t)$. Imposing $x = 0$ then means either $\theta = \frac{\pi}{2}$ or $u - t = \frac{\pi}{2}$. Of these two possibilities, the first does not look promising, because on the S^3 parameterized by $(\cos(\theta)e^{i(u-t)}, \sin(\theta)e^{i(u+t)})$ it effectively restricts us to an S^1: only the function $u + t$ is left in the game, and indeed going further in the analysis one finds that the metric becomes degenerate.[9] The second possibility, $u - t = \frac{\pi}{2}$, restricts us instead to an $S^2 \subset S^3$; we will now see that this possibility survives. It gives

$$\beta = \theta\ , \qquad t = -\frac{1}{2}\gamma\ , \qquad u = \frac{\pi}{2} - \frac{1}{2}\gamma\ . \tag{4.3.21}$$

This leads to a dramatic simplification in the whole system. The one-form equations from (4.1.25b) do not involve $\mathrm{Im}v$ any more; we can now use them to solve for $\mathrm{Re}v$, $\mathrm{Re}w$, $\mathrm{Im}w$ (rather than for $\mathrm{Im}v$, $\mathrm{Re}w$, $\mathrm{Im}w$ as we did in previous subsections, for $x \neq 0$). This strategy would actually have been possible for $x \neq 0$ too, but it would have led to far more involved expressions; for this reason we decided to isolate the $x = 0$ case and to treat it separately in this subsection. We get

$$\mathrm{Re}v = \frac{e^{-3A+\phi}}{3\cos\alpha}d(\sin\alpha e^{4A-\phi})\ , \quad \mathrm{Re}w = \frac{e^A}{3}\sin\alpha\, d\beta\ , \quad \mathrm{Im}w = -\frac{e^A}{3}\sin\alpha\,\sin\beta d\gamma\ . \tag{4.3.22}$$

We now turn to the 2-form equation in (4.1.25c). As in the previous subsections of this section, this can be separated into a 2-form multiplying y^α and a 1-form multiplying dy^α, which have to vanish separately:

$$d(e^{5A-\phi}\mathrm{Re}v) = 0\ , \qquad e^{5A-\phi}(3 - 4\sin^2(\alpha))\mathrm{Re}v = d(e^{6A-\phi}\sin\alpha\cos\alpha)\ . \tag{4.3.23}$$

Hitting the second equation with d and using the first, we find $\sin\alpha\cos\alpha\, d\alpha \wedge \mathrm{Re}v = 0$, and hence, recalling (4.3.22), to $\sin\alpha d\alpha \wedge d(4A - \phi) = 0$. Now, $\sin\alpha$ is not allowed to vanish because of (4.3.22) (recall that $\mathrm{Re}v$, $\mathrm{Re}w$, $\mathrm{Im}w$ are part of a vielbein); hence $d\alpha \wedge d(4A-\phi) = 0$. This can be interpreted as saying that $4A-\phi$ is a function of α. On the other hand, using (4.3.22) in the first in (4.3.23), we get $d(\frac{e^{2A}}{\cos\alpha}) \wedge d(\sin\alpha e^{4A-\phi}) = 0$, which shows that $A = A(\alpha)$, and hence also that $\phi = \phi(\alpha)$. Going back to the second in (4.3.23), it now reads

$$2(\cos^2(\alpha) + 2)\partial_\alpha A + \sin^2(\alpha)\partial_\alpha\phi = \sin(2\alpha)\ . \tag{4.3.24}$$

Turning to (4.1.25e), its 2-form part reads

$$d(e^{5A-\phi}\mathrm{Im}v) = 0 \qquad \Rightarrow \qquad \mathrm{Im}v = e^{-(5A-\phi)}dz \tag{4.3.25}$$

[9] At the stage of (4.3.22) below, one would find $\mathrm{Re}w \propto \mathrm{Im}v$.

for some function z. This completes (4.3.22).

Finally, (4.1.25a) gives

$$\left(d(e^{-2A}\cos\alpha) + 2e^{-3A}\sin\alpha \mathrm{Re}v\right) \wedge \mathrm{Im}v = 0 \,. \tag{4.3.26}$$

In view of (4.3.25), the parenthesis has to vanish by itself; this leads to

$$4(7\cos^2(\alpha) - 4)\partial_\alpha A + 4\sin^2(\alpha)\partial_\alpha \phi = -\sin(2\alpha) \,. \tag{4.3.27}$$

Notice that now (4.3.24) and (4.3.27) are two *ordinary* (as opposed to partial) differential equations, which can be solved explicitly:

$$e^A = \frac{c_1}{\cos^{1/6}(\alpha)} \,, \qquad e^\phi = \frac{c_2}{\sin\alpha\cos^{2/3}(\alpha)} \,, \tag{4.3.28}$$

where c_i are two integration constants. These are exactly the warping and dilaton presented in [107, (A.1)], for $c_1 = \frac{3}{2}Lm^{-1/6}$, $c_2 = 4/(3L^2m^{2/3})$. It is now possible to derive the fluxes, as we did in subsection 4.3.3 for $x \neq 0$, and check that they coincide with those in [107, (A.1)].

The metric can now be computed too, using the vielbein (4.3.23), (4.3.25); it also agrees with the one given in [107, (A.1)]. It inherits the singularity at $\alpha = \frac{\pi}{2}$ from the Brandhuber–Oz solution [99]; moreover, it now has a singularity at $\alpha = 0$. The latter is actually the singularity one always gets when one T-dualizes along a Hopf direction in a S^3 that shrinks somewhere. It represents an NS5 smeared along the T-dual S^1; one expects worldsheet instantons to modify the metric so that the NS5 singularity gets localized along that direction, as in [121]. As for the singularity at $\alpha = \frac{\pi}{2}$, it now cannot be associated with an O8–D8 system as it was in IIA, since we are in IIB. It probably now represents a smeared O7–D7 system; it is indeed always the case that T-dualizing a brane along a parallel direction in supergravity gives a smeared version of the correct D-brane solution on the other side, as we just saw for the NS5-brane. It is possible that again instanton effects localize the singularity, this time to an O7–D7 system. (Even more correctly, we should expect the O7 to split into an $(1,1)$-sevenbrane and an $(1,-1)$-sevenbranes, as pointed out in [93] following [122].)

Notice finally that, although we have found it convenient to treat the $x = 0$ case separately from the rest, it is in fact a particular case of the general treatment (although a slightly degenerate one). Indeed one can check that (4.3.12b) is satisfied by (4.3.28); in contrast to the general case, this does not determine a function z, but we can use (4.3.14), where z has been eliminated, instead of (4.3.13), which contains z. Thus the solution presented in this subsection is already an example of our general formalism. In section 4.4.2 we will see another, more elaborate example.

4.4 The PDEs

In section 4.3, we reduced the problem of finding AdS$_6$ $\times$ M_4 solutions to the two PDEs (4.3.12a), (4.3.13). As anticipated in the introduction, we will not try to find the most general solution to these equations in this chapter. In this section we will make some general remarks about the PDEs, and we will recover via a simple Ansatz the known solution [104], originally obtained via nonabelian T-duality. (As we mentioned in that section, one can also see the $x = 0$ case as a particular solution to the PDEs.)

4.4.1 General considerations

We derived in section 4.3.2 the two equations (4.3.12a), (4.3.13). Recall that z is an auxiliary variable, defined by (4.3.12b). As we already remarked, among the four remaining variables (α, x, A, ϕ), only two (for example α and x) are independent. The other two, A and ϕ, can be taken to be dependent as in (4.3.15). The equations (4.3.12a) and (4.3.13) can then be reexpressed as two scalar PDEs in the two dimensions spanned by α and x:

$$3\sin(2\alpha)(A_\alpha\phi_x - A_x\phi_\alpha) = 6A_x + \sin^2\alpha\left(-2x - 2(x^2+5)A_x + (1+2x^2)\phi_x\right) \ , \quad (4.4.1a)$$

$$\cos\alpha(2+3x\phi_x) + \sin\alpha\phi_\alpha = 2x\left(\frac{3}{\sin\alpha} + (x^2-4)\sin\alpha\right)(A_\alpha\phi_x - A_x\phi_\alpha) + \quad (4.4.1b)$$
$$- 2x\cos\alpha\left(\frac{3}{\sin^2\alpha} - (5+x^2)\right)A_x + 2\left(\frac{3}{\sin\alpha} - (1+x^2)\sin\alpha\right)A_\alpha \ ,$$

where $A_\alpha \equiv \partial_\alpha A$ etc. As we will see, they are actually easier to study in their original form manifestations (4.3.12a) and (4.3.13).

These equations are nonlinear, and as such they are rather hard to study. Even so, there are quite a few techniques that have been developed over the years to tackle such systems. Perhaps the first natural question is how many solutions one should expect. For a first-order system of ODEs, it is roughly enough to compare the number of equations to the number of functions. If there are n equations and n functions, the system is neither over- nor under-constrained: geometrically, the system gives a vector field in an open set in $\mathbb{R}^{n+1}$ (including time), and solving the system means finding integral curves to this vector field. (When the system is "autonomous", i.e. it does not depend explicitly on time, one can more simply consider a vector field on $\mathbb{R}^n$).

The picture is more complicated for a system of PDEs. In general, if we have k "times" and m functions, the system will define a distribution of dimension k (namely, a choice of subspaces $V_x \subset T_x\mathbb{R}^{k+m}$ of dimension k for every point $x \in \mathbb{R}^{k+m}$); solving the system then means finding "integral submanifolds" for the distribution, namely submanifolds $S \subset \mathbb{R}^{k+m}$ such that V_x is tangent to S for every $x \in S$. This distribution is in general however not guaranteed to admit integral submanifolds. (A famous example is given by Frobenius theorem: a distribution defined by the span of vector fields v_i will only be integrable if all the Lie brackets $[v_i, v_j]$ are linear combinations of the v_i themselves.) Fortunately, the machinery of "exterior differential systems" (EDS) has been developed to deal with these issues, culminating in the Cartan–Kähler theorem (see for example [110, Chap.III], or [111, Sec. 10.4.1] in slightly more informal language).

Describing and applying such methods in detail is beyond the scope of this thesis, but here is a sketch. First one defines a "differential ideal", namely a vector space of the equations in the system and their exterior derivatives. In our case, denote by E_i the two two-forms that have to vanish in (4.3.12a) and (4.3.13); the ideal is then the linear span $I = \langle E_1, E_2, dE_2 \rangle$ (since $dE_1 = 0$ automatically). We then want to construct the distribution V on which the forms in I vanish, in the sense that each multi-vector built from vectors in the distribution has zero pairing with the forms in I. One proceeds iteratively. We first consider a single vector field e_1 on which the forms vanish (in our case this is trivial, since there are no one-forms in I; we can take for example $e_1 = \partial_\alpha$). We then add a second vector: this is done by solving the "polar equations" $H(E_1) \equiv \{v \llcorner e_1 \llcorner E_i = 0\}$. The rank of this system is denoted by c_1. (In general there might be a c_0 too, but in our case the first choice of a vector was free because there are no one-forms in I; c_0 is then considered to be 0.) For us it turns out that $c_1 = 2$. In general one would go on by choosing a solution e_2 to the polar equations above, and would consider new polar equations $H(E_2) \equiv \{v \llcorner e_1 \llcorner E_i = v \llcorner e_2 \llcorner E_i = 0, v \llcorner e_1 \llcorner e_2 \llcorner dE_2 = 0\}$; the rank of this new system would be denoted by c_2, which in our case also happens to be 2. However, solving our PDEs means finding a two-dimensional integral manifold, and hence we can stop at the second step and disregard the higher polar equations $H(E_2)$. (The general theory would also show that for our system there is actually no three-dimensional integral manifold.) We can then apply the so-called "Cartan test" and a corollary to the Cartan–Kähler theorem (respectively Thm. 1.11 and Cor. 2.3 in [110]) to infer that an integral submanifold of dimension 2 actually does exist. The proof of the theorem also says that the general solution depends on $s_1 = c_1 - c_0 = 2$ functions of one variable. ($s_i = c_i - c_{i-1}$ are called "Cartan characters".) These two functions can be thought of as functions at the boundary of the two-dimensional domain in α and x on which the solution exists.

Having determined the structure of the solutions, it would be nice to find as many as possible of them. A strategy which is common in this context is to impose some extra symmetry. This is less obvious than usual to implement. We cannot for example just assume that A and ϕ do not depend on one of the coordinates α and x: the metric (4.3.16) would become degenerate. Another perhaps more promising idea is to use the so-called "method of characteristics" to reduce the problem to a system of ODEs. We plan to return on this in the future.

Finally, let us point out that two solutions to our PDEs are already known. One is the case $x = 0$, which we studied in section 4.3.4. Although we had to treat it separately, we also mentioned that it is a solution of the general system of PDEs (once we eliminate dz from (4.3.13), obtaining (4.3.14)).

We will now see another particular solution. Although the global properties of the resulting M_4 are even more puzzling than those of the solution in section 4.3.4, it might be possible to generalize it to new solutions which are better-behaved; for example, one might start by studying perturbations around it.

4.4.2 A local solution: nonabelian T-duality

Many PDEs are reduced to ODEs by a separation of variables Ansatz. For our nonlinear PDEs, this does not work. However, we will now see that a particular case does lead to a solution, namely:

$$\phi = f(\alpha) + \log(x) , \qquad A = A(\alpha) . \tag{4.4.2}$$

Notice that this Ansatz restricts x to be in $(0, 1]$. (We already observed after (4.3.16) that $|x| \leq 1$ in general.)

We begin by considering (4.3.12b). With (4.4.2), after a few manipulations it reduces to

$$\begin{aligned} dz = d\left(e^{6A-f}\frac{\sin\alpha}{6x^2}\right) - \frac{1}{3}e^{2A}d(e^{4A-f}\sin\alpha) + \\ + \frac{1}{x^2}\left[-\frac{1}{6}e^{4A}d(e^{2A-f}\sin\alpha) + e^{4A-f}\cot\alpha\, d(e^{2A}\cos\alpha)\right] . \end{aligned} \tag{4.4.3}$$

The first line in (4.4.3) is manifestly exact, since everything is a function of α alone. The second line is of the form $\frac{1}{x^2}d(\text{function}(\alpha))$, and cannot be exact unless it vanishes, which leads to

$$d(e^{2A-f}\sin\alpha) = 6e^{-f}\cot\alpha\, d(e^{2A}\cos\alpha) . \tag{4.4.4}$$

The first line in (4.4.3) then determines dz (and can be integrated to produce z). We can now use this expression for dz in (4.3.13). Most terms in (4.3.13) actually vanish because they involve wedges of forms proportional to $d\alpha$; the only one surviving is of the form $d(e^{6A}\cos\alpha) \wedge dx$. In other words, we are forced to take

$$e^A = c_1(\cos\alpha)^{-1/6} , \tag{4.4.5}$$

with c_1 an integration constant. Plugging this back into (4.4.4) we get

$$e^f = c_2\frac{(\cos\alpha)^{-1/3}}{\sin^3\alpha} \tag{4.4.6}$$

for c_2 another integration constant.

This is actually the solution found in [104]. To see this, one needs to identify

$$\alpha = \theta , \qquad x = \frac{e^{2\hat{A}}}{\sqrt{r^2 + e^{4\hat{A}}}} , \tag{4.4.7}$$

where $\hat{A}$ is the function denoted by A in [104]. One can check that indeed the fluxes (4.3.18), (4.3.19) and metric (4.3.16) give the expressions in [104]. The metric one gets has a singularity at $\alpha = \pi/2$, just like the solution [99], and a new singularity at $\alpha = 0$ [107]. More worryingly, it is noncompact; it might be possible to find a suitable analytic continuation, with the help of the PDEs (4.3.12a), (4.3.13).

4.5 Summary of the results and outlook

In this chapter, we analyzed the $\mathrm{AdS}_6 \times M_4$ supersymmetric system of equations of type II supergravity, given by the vanishing supersymmetry variations with all the fluxes turned on. We were able to rewrite this system in terms of pure-spinors for the unique solution of massive type IIA, namely the Brandhuber and Oz solution [99, 109], see B.2. We also show that there are no AdS_6 solutions in eleven-dimensional supergravity with eight preserved supercharges, B.1.

The main and new result of this chapter is a classification of this type of $\mathrm{AdS}_6 \times M_4$ backgrounds in type IIB supergravity. In this case, we reduced the pure-spinors system to two PDEs in terms of the dilaton and the warp factor, which depends on the local coordinates of a two dimensional space Σ. M_4 is an S^2-fiberation over Σ. The metric, the dilaton and the fluxes are completely determined by the two PDEs, and the classification is given in the sense that each AdS_6 solution of type IIB must be a solution of these two PDEs. As a consistency check and as a confirmation of our classification result, we recover also two already known solutions in type IIB by plugging in two different ansatz in the PDEs. One is the T-dual of the Brandhuber and Oz solution, see 4.3.4, the other one was obtained using non-abelian T-duality [104], and see section 4.4.2. However, this last solution has some divergences and it is non-compact. It will be interesting to solve analytically the two PDEs with a clever ansatz (or even numerically) and/or with a change of coordinates on Σ, in order to find an analytic continuation of the solution of [104], which makes it compact. In general, it would be nice to find also new compact solutions, which we believe are dual to five-dimensional superconformal field theories coming from (p, q)-webs of five and seven-branes.

Chapter 5

All supersymmetric AdS_7 solutions of type II supergravity

Interacting quantum field theories generally become hard to define in more than four dimensions. A Yang–Mills theory, for example, becomes strongly coupled in the UV. In six dimensions, a possible alternative would be to use a two-form gauge field. Its nonabelian formulation is still unclear, but string theory predicts that a $(2,0)$-superconformal completion of such a field actually exists on the worldvolume of M5-branes. Understanding these branes is still one of string theory's most interesting challenges.

This prompts the question of whether other non-trivial six-dimensional theories exist. There are in fact several other string theory constructions [123–126] that would engineer such theories. Progress has also been made (see for example [127, 128]) in writing explicitly their classical actions.

A way to establish the existence of superconformal theories in six dimensions is to look for supersymmetric AdS$_7$ solutions in string theory. In this chapter, we classify such solutions. As we will review later, in M-theory, one only has AdS$_7 \times S^4$ (which is holographically dual to the $(2,0)$ theory) or an orbifold thereof. That leaves us with AdS$_7 \times M_3$ in IIA with non-zero Romans mass $F_0 \neq 0$ (which cannot be lifted to M-theory) or in IIB.

Here we will show that, while there are no such solutions in IIB, many do exist in IIA with non-zero Romans mass F_0.

Our methods are reminiscent of the generalized complex approach for Mink$_4 \times M_6$ or AdS$_4 \times M_6$ solutions [108]. We start with a similar system [112] for Mink$_6 \times M_4$, and we then use the often-used trick of viewing AdS$_7$ as a warped product of Mink$_6$ with a line. This allows us to obtain a system valid for AdS$_7 \times M_3$. A similar procedure was applied in [129] to derive a system for AdS$_5 \times M_5$ from Mink$_4 \times M_6$. The system we derive is written in terms of differential forms satisfying some algebraic constraints; mathematically, these constraints mean that the forms define a generalized identity$\times$identity structure on $T_{M_3} \oplus T^*_{M_3}$. This fancy language, however, will not be needed here; we will give a parameterization of such structures in terms of a vielbein $\{e_a\}$ and some angles, and boil the system down to one written in terms of those quantities.

When one writes supersymmetry as a set of PDEs in terms of forms, they may have some interesting geometrical interpretation (such as the one in terms of generalized complex geometry in [108]); but, to obtain solutions, one usually needs to make some Ansatz, such as that the space is homogeneous or that it has cohomogeneity one. One then reduces the differential equations to algebraic equations or to ODEs, respectively.

The AdS$_7 \times M_3$ case is different with $\mathcal{N} = 1$, or 16 real supercharges preserved in 7 dimensions. As we will see, the equations actually determine explicitly the vielbein $\{e_a\}$ in terms of derivatives of our parameterization function. This gives a local, explicit form for the metric, without any Ansatz. By a suitable redefinition we find that the metric describes an S^2 fibration over a one-dimensional space.

This is actually to be expected holographically. A $(1,0)$ superconformal theory has an Sp(1)≅SU(2) R-symmetry group, which should appear as the isometry group of the internal space M_3. With a little more work, all the fluxes can also be determined, and they are also left invariant by the SU(2) isometry group of our S^2 fiber. All the Bianchi identities and equations of motion are automatically satisfied, and existence of a solution is then reduced to a system of two coupled ODEs.[1] From this point on, our analysis is pretty standard: in order for M_3 to be compact, the coordinate r on which everything depends should in fact parameterize an interval $[r_{\rm N}, r_{\rm S}]$, and the S^2 should shrink at the two endpoints of the interval, which we from now on will call "poles". This requirement translates into certain boundary conditions for the system of ODEs.

We have studied the system numerically. We can obtain regular[2] solutions if we insert brane sources. We exhibit solutions with D6's, and solutions with one or two D8 stacks, appropriately stabilized by flux. For example, in the solution with two D8 stacks, they have opposite D6 charge, and their mutual electric attraction is balanced against their gravitational tendency to shrink. (For D8-branes, there is no problem with the total D-brane charge in a compact space; usually such problems are found by integrating the flux sourced by the brane over a sphere surrounding the brane, whereas for a D8 such a transverse sphere is simply an S^0.) We think that there should exist generalizations with an arbitrary number of stacks.

It is natural to think that our regular solutions with D8-branes might be related to D-brane configurations in [124, 125], which should indeed engineer six-dimensional $(1,0)$ superconformal theories. Supersymmetric solutions for configurations of that type have actually been found [130] (see also [131]); non-trivially, they are fully localized. It is in principle possible that their results are related to ours by some limit. Such a relationship is not obvious, however, in part because of the SU(2) symmetry, that forces our sources to be only parallel to the S^2-fiber. It would be interesting to explore this possibility further.

5.1 Supersymmetry and pure spinor equations in three dimensions

In this section, we will derive a system of differential equations on forms in three dimensions that is equivalent to preserved supersymmetry for solutions of the type $\mathrm{AdS}_7 \times M_3$. We will derive it by a commonly-used trick: namely, by considering AdS_{d+1} as a warped product of Mink_d and $\mathbb{R}$. We will begin in section 5.1.1 by reviewing a system equivalent to supersymmetry for $\mathrm{Mink}_6 \times M_4$. In section 5.1.2 we will then translate it to a system for $\mathrm{AdS}_7 \times M_3$.

[1] This is morally a hyper-analogue to the reduction performed in [129] along the generalized Reeb vector, although in our case the situation is so simple that we need not introduce that reduction formalism.

[2] On the loci where branes are present, the metric is of course not regular, but such singularities are as usual excused by the fact that we know that D-branes have an alternative definition as boundary conditions for open strings, and are thought to be objects in the full theory. The singularity is particularly mild for D8's, which manifest themselves as jumps in the derivatives of the metric and other fields — which are themselves continuous.

5.1.1 $\mathrm{Mink}_6 \times M_4$

Preserved supersymmetry for $\mathrm{Mink}_4 \times M_6$ was found [108] to be equivalent to the existence on M_6 of an $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure satisfying certain differential equations reminiscent of generalized complex geometry [35, 36].

Similar methods can be useful in other dimensions. For $\mathrm{Mink}_6 \times M_4$ solutions, [112] found a system in terms of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure on M_4, described by a pair of pure spinors $\phi^{1,2}$. Similarly to the $\mathrm{Mink}_4 \times M_6$ case, they can be characterized in two ways. One is as bilinears of the internal parts $\eta^{1,2}$ of the supersymmetry parameters in (C.1.2):[3]

$$\phi^1_\mp = e^{-A_4}\eta^1_+ \otimes \eta^{2\,\dagger}_\mp \ , \qquad \phi^2_\mp = e^{-A_4}\eta^1_+ \otimes \eta^{2c\,\dagger}_\mp \ , \tag{5.1.1}$$

where the warping function A_4 is defined by

$$ds^2_{10} = e^{2A_4} ds^2_{\mathrm{Mink}_6} + ds^2_{M_4} \ . \tag{5.1.2}$$

The upper index in (5.1.1) is relevant to IIA, the lower index to IIB; so in IIA we have that $\phi^{1,2}$ are both odd forms, and in IIB that they are both even. One can also give an alternative characterization of $\phi^{1,2}$, as a pair of pure spinors which are *compatible*. This stems directly from their definition as an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ structure, and it means that the corresponding generalized almost complex structures commute. This latter constraint can also be formulated purely in terms of pure spinors as $(\phi^1, \phi^2) = (\bar{\phi}^1, \phi^2)$.[4] This can be shown similarly to an analogous statement in six dimensions; see [132, App. A].

The system equivalent to supersymmetry now reads [112] [5]

$$d_H\big(e^{2A_4-\phi}\mathrm{Re}\phi^1_\mp\big) = 0 \ , \tag{5.1.3a}$$
$$d_H\big(e^{4A_4-\phi}\mathrm{Im}\phi^1_\mp\big) = 0 \ , \tag{5.1.3b}$$
$$d_H\big(e^{4A_4-\phi}\phi^2_\mp\big) = 0 \ , \tag{5.1.3c}$$
$$e^\phi F = \mp 16 *_4 \lambda(dA_4 \wedge \mathrm{Re}\phi^1_\mp) \ , \tag{5.1.3d}$$
$$(\overline{\phi^1_\pm}, \phi^1_\pm) = (\overline{\phi^2_\pm}, \phi^2_\pm) = \frac{1}{4} \ . \tag{5.1.3e}$$

Here, ϕ is the dilaton; $d_H = d - H\wedge$ is the twisted exterior derivative; A_4 was defined in (5.1.2); F is the internal RR flux, which, as usual, determines the external flux via self-duality:

$$F_{(10)} \equiv F + e^{6A_4}\mathrm{vol}_6 \wedge *_4\lambda F \ . \tag{5.1.4}$$

Actually, (5.1.3) contains an assumption: that the norms of the η^i are equal. For a noncompact M_4, it might be possible to have different norms; (5.1.3) would then have to be slightly

[3] As usual, we are identifying forms with bispinors via the Clifford map $dx^{m_1} \wedge \ldots \wedge dx^{m_k} \mapsto \gamma^{m_1\ldots m_k}$. $\mp$ denotes chirality, and $\eta^c \equiv B_4\eta^*$ denotes Majorana conjugation; for more details see appendix C.1. The factors e^{-A_4} are included for later convenience.

[4] As usual, the Chevalley pairing in this equation is defined as $(\alpha, \beta) = (\alpha \wedge \lambda(\beta))_{\mathrm{top}}$; λ is the sign operator defined on k-forms as $\lambda\omega_k \equiv (-)^{\lfloor \frac{k}{2} \rfloor}\omega_k$.

[5] We have massaged a bit the original system in [112], by eliminating $\mathrm{Re}\phi^1_\mp$ from the first equation of their (4.11).

changed. (See [133, Sec. A.3] for comments on this in the $\mathrm{Mink}_4 \times M_6$ case.) As shown in appendix C.1, however, for our purposes such a generalization is not relevant.

With this caveat, the system (5.1.3) is equivalent to supersymmetry for $\mathrm{Mink}_6 \times M_4$. It can be found by direct computation, or also as a consequence of the system for $\mathrm{Mink}_4 \times M_6$ in [108]: one takes $M_6 = \mathbb{R}^2 \times M_4$, with warping $A = A_4$, internal metric $ds^2_{M_6} = e^{2A_4}((dx^4)^2 + (dx^5)^2) + ds^2_{M_4}$, and, in the language of [133],

$$\Phi_1 = e^{A_4}(dx^4 + idx^5) \wedge \phi^2_\mp \, , \qquad \Phi_2 = (1 + ie^{2A_4} dx^4 \wedge dx^5) \wedge \phi^1_\mp \, . \tag{5.1.5}$$

Furthermore, (5.1.3) can also be found as a consequence of the ten-dimensional system in [38]. [112] also give an interpretation of the system in terms of calibrations, along the lines of [134].

5.1.2 $\mathrm{AdS}_7 \times M_3$

As we anticipated, we will now use the fact that AdS can be used as a warped product of Minkowski space with a line. We would like to classify solutions of the type $\mathrm{AdS}_7 \times M_3$. These in general will have a metric

$$ds^2_{10} = e^{2A_3} ds^2_{\mathrm{AdS}_7} + ds^2_{M_3} \tag{5.1.6}$$

where A_3 is a new warping function (different from the A_4 in (5.1.2)). Since

$$ds^2_{\mathrm{AdS}_7} = \frac{d\rho^2}{\rho^2} + \rho^2 ds^2_{\mathrm{Mink}_6} \, , \tag{5.1.7}$$

(5.1.6) can be put in the form (5.1.2) if we take

$$e^{A_4} = \rho e^{A_3} \, , \qquad ds^2_{M_4} = \frac{e^{2A_3}}{\rho^2} d\rho^2 + ds^2_{M_3} \, . \tag{5.1.8}$$

A genuine AdS_7 solution is one where not only the metric is of the form (5.1.7), but where there are also no fields that break its SO(6,2) invariance. This can be easily achieved by additional assumptions: for example, A_3 should be a function of M_3. The fluxes F and H, which in section 5.1.1 were arbitrary forms on M_4, should now be forms on M_3. For IIA, $F = F_0 + F_2 + F_4$: in order not to break $\mathrm{SO}(6,2)$, we impose $F_4 = 0$, since it would necessarily have a leg along AdS_7; for IIB, $F = F_1 + F_3$.

Following this logic, solutions to type II equations of motion of the form $\mathrm{AdS}_7 \times M_3$ are a subclass of solutions of the form $\mathrm{Mink}_6 \times M_4$. In appendix C.1, we also show how the $\mathrm{AdS}_7 \times M_3$ supercharges get translated in the $\mathrm{Mink}_6 \times M_4$ framework, and that the internal spinors have equal norm, as we anticipated in section 5.1.1. Using (C.1.10), we also learn how to express the $\phi^{1,2}$ in (5.1.1) in terms of bilinears of spinors $\chi_{1,2}$ on M_3:

$$\phi^1_\mp = \frac{1}{2}\left(\psi^1_\mp + ie^{A_3}\frac{d\rho}{\rho} \wedge \psi^1_\pm\right) \, , \qquad \phi^2_\mp = \mp\frac{1}{2}\left(\psi^2_\mp + ie^{A_3}\frac{d\rho}{\rho} \wedge \psi^2_\pm\right) \, , \tag{5.1.9}$$

with

$$\psi^1 = \chi_1 \otimes \chi_2^\dagger \, , \qquad \psi^2 = \chi_1 \otimes \chi_2^{c\dagger} \, . \tag{5.1.10}$$

As in section 5.1.1, we have implicitly mapped forms to bispinors via the Clifford map, and in (5.1.9) the subscripts $_\pm$ refer to taking the even or odd form part. (Recall also that $\phi_-^{1,2}$ is relevant to IIA, and $\phi_+^{1,2}$ to IIB; see (5.1.3).) The spinors $\chi_{1,2}$ have been taken to have unit norm.

$\psi^{1,2}$ are differential forms on M_3, but not just any forms. (5.1.10) imply that they should obey some algebraic constraints. Those constraints could be interpreted in a fancy way as saying that they define an identity$\times$identity structure on $T_{M_3} \oplus T^*_{M_3}$. However, three-dimensional spinorial geometry is simple enough that we can avoid such language: rather, in section 5.2 we will give a parameterization that will allow us to solve all the algebraic constraints resulting from (5.1.10).

We can now use (5.1.9) in (5.1.3). Each of those equations can now be decomposed in a part that contains $d\rho$ and one that does not. Thus, the number of equations would double. However, for (5.1.3a), (5.1.3b) and (5.1.3c), the part that does not contain $d\rho$ actually follows from the part that does. The "norm" equation, (5.1.3e), simply reduces to a similar equation for a three-dimensional norm. Summing up:

$$d_H \mathrm{Im}(e^{3A_3-\phi}\psi^1_\pm) = -2e^{2A_3-\phi}\mathrm{Re}\psi^1_\mp \, , \tag{5.1.11a}$$
$$d_H \mathrm{Re}(e^{5A_3-\phi}\psi^1_\pm) = 4e^{4A_3-\phi}\mathrm{Im}\psi^1_\mp \, , \tag{5.1.11b}$$
$$d_H (e^{5A_3-\phi}\psi^2_\pm) = -4ie^{4A_3-\phi}\psi^2_\mp \, , \tag{5.1.11c}$$
$$\pm \frac{1}{8}e^{\phi} *_3 \lambda F = dA_3 \wedge \mathrm{Im}\psi^1_\pm + e^{-A_3}\mathrm{Re}\psi^1_\mp \, , \tag{5.1.11d}$$
$$dA_3 \wedge \mathrm{Re}\psi^1_\mp = 0 \, , \tag{5.1.11e}$$
$$(\psi^{1,2}_+, \overline{\psi^{1,2}_-}) = -\frac{i}{2} \, ; \tag{5.1.11f}$$

again with the upper sign for IIA, and the lower for IIB.

The system (5.1.11) is equivalent to supersymmetry for AdS$_7 \times M_3$. As we show in appendix C.1, a supersymmetric AdS$_7 \times M_3$ solution can be viewed as a supersymmetric Mink$_6 \times M_4$ solution, and for this the system (5.1.3) is equivalent to supersymmetry. (5.1.11) can also be obtained directly from the ten-dimensional system in subsection 2.3.2, but other equations also appear, and extra work is needed to show that those extra equations are redundant. The derivation of the supersymmetry conditions by using the ten-dimensional system of 2.3.2 was work done in the master thesis [135].

In (5.1.11) the cosmological constant of AdS$_7$ does not appear directly, since we have taken its radius to be one in (5.1.7). We did so because a non-unit radius can be reabsorbed in the factor e^{2A_3} in (5.1.6).

Before we can solve (5.1.11), we have to solve the algebraic constraints that follow from the definition of $\psi^{1,2}$ in (5.1.10); we will now turn to this problem.

5.2 Parameterization of the pure spinors

In section 5.1.2 we obtained a system of differential equations, (5.1.11), that is equivalent to supersymmetry for an $\mathrm{AdS}_7 \times M_3$ solution. The $\psi^{1,2}$ appearing in that system are not arbitrary forms; they should have the property that they can be rewritten as bispinors (via the Clifford map $dx^{i_1} \wedge \ldots \wedge dx^{i_k} \mapsto \gamma^{i_1 \ldots i_k}$) as in (5.1.10). In this section, we will obtain a parameterization for the most general set of $\psi^{1,2}$ that has this property. This will allow us to analyze (5.1.11) more explicitly in section 5.3.

We will begin in section 5.2.1 with a quick review of the case $\chi_1 = \chi_2$, and then show in section 5.2.2 how to attack the more general situation where $\chi_1 \neq \chi_2$.

5.2.1 One spinor

We will use the Pauli matrices σ_i as gamma matrices, and use $B_3 = \sigma_2$ as a conjugation matrix (so that $B_3 \sigma_i = -\sigma_i^t B_3 = -\sigma_i^* B_3$). We will define

$$\chi^c \equiv B_3 \chi^* \, , \qquad \overline{\chi} \equiv \chi^t B_3 \, ; \tag{5.2.1}$$

notice that $\chi^{c\,\dagger} = \chi^t B_3^\dagger = \overline{\chi}$.

We will now evaluate $\psi^{1,2}$ in (5.1.10) when $\chi_1 = \chi_2 \equiv \chi$; as we noted in section 5.1.2, χ is normalized to one. Notice first a general point about the Clifford map $\alpha_k = \frac{1}{k!} \alpha_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \mapsto \slashed{\alpha}_k \equiv \frac{1}{k!} \alpha_{i_1 \ldots i_k} \gamma^{i_1 \ldots i_k}$ in three dimensions (and, more generally, in any odd dimension). Unlike what happens in even dimensions, the antisymmetrized gamma matrices $\gamma^{i_1 \ldots i_k}$ are a redundant basis for bispinors. For example, we see that the slash of the volume form is a number: $\slashed{\mathrm{vol}}_3 = \sigma^1 \sigma^2 \sigma^3 = i$. More generally we have

$$\slashed{\alpha} = -i \, \slashed{* \lambda \alpha}. \tag{5.2.2}$$

In other words, when we identify a form with its image under the Clifford map, we lose some information: we effectively have an equivalence $\alpha \cong -i * \lambda \alpha$. When evaluating $\psi^{1,2}$, we can give the corresponding forms as an even form, or as an odd form, or as a mix of the two.

Let us first consider $\chi \otimes \chi^\dagger$. We can choose to express it as an odd form. In its Fierz expansion, both its one-form part and its three-form part are a priori non-zero; we can parameterize them as

$$\chi \otimes \chi^\dagger = \frac{1}{2}(e_3 - i \mathrm{vol}_3) \, . \tag{5.2.3}$$

(We can also write this in a mixed even/odd form as $\chi \otimes \chi^\dagger = \frac{1}{2}(1 + e_3)$; recall that the right hand sides have to be understood with a Clifford map applied to them.) e_3 is clearly a real vector, whose name has been chosen for later convenience. The fact that the three-form part is simply $-\frac{i}{2}\mathrm{vol}_3$ follows from $||\chi|| = 1$. Notice also that

$$e_3 \chi = \sigma_i \chi e_3^i = \sigma_i \chi \chi^\dagger \sigma^i \chi = \frac{1}{2}(-e_3 - 3i \mathrm{vol}_3) \chi \quad \Rightarrow \quad e_3 \chi = \chi \tag{5.2.4}$$

where we have used (5.2.3), and that $\sigma_i \alpha_k \sigma^i = (-)^k(3-2k)\alpha_k$ on a k-form. (5.2.4) also implies that e_3 has norm one.[6]

Coming now to $\chi \otimes \overline{\chi}$, we notice that the three-form part in its Fierz expansion is zero, since $\overline{\chi}\chi = \chi^t B_3 \chi = 0$. The one-form part is now a priori no longer real; so we write

$$\chi \otimes \overline{\chi} = \frac{1}{2}(e_1 + ie_2) \ . \tag{5.2.5}$$

Similar manipulations as in (5.2.4) show that $(e_1 + ie_2)\chi = 0$; using this, we get that

$$e_i \cdot e_j = \delta_{ij} \ . \tag{5.2.6}$$

In other words, $\{e_i\}$ is a vielbein, as notation would suggest.

5.2.2 Two spinors

We will now analyze the case with two spinors $\chi_1 \neq \chi_2$ (again both with norm one). We will proceed in a similar fashion as in [120, Sec. 3.1].

Our aim is to parameterize the bispinors $\psi^{1,2}$ in (5.1.10). Let us first consider their zero-form parts, $\chi_2^\dagger \chi_1$ and $\chi_2^{c\dagger}\chi_1$. The parameterization (5.2.4) can be applied to both χ_1 and χ_2, resulting in two one-forms e_3^i. (This notation is a bit inconvenient, but these two one-forms will cease to be useful very soon.) Using then (5.2.3) twice, we see that

$$|\chi_2^\dagger\chi_1|^2 = \chi_2^\dagger\chi_1\chi_1^\dagger\chi_2 = \mathrm{Tr}(\chi_1\chi_1^\dagger\chi_2\chi_2^\dagger) = \frac{1}{4}\mathrm{Tr}\left((1+e_3^1)(1+e_3^2)\right) = \frac{1}{2}(1+e_3^1\cdot e_3^2) \ . \tag{5.2.7}$$

Similarly we have

$$|\chi_2^{c\dagger}\chi_1|^2 = \mathrm{Tr}(\chi_1\chi_1^{c\dagger}\chi_2\chi_2^{c\dagger}) = \frac{1}{4}\mathrm{Tr}\left((1+e_3^1)(1-e_3^2)\right) = \frac{1}{2}(1-e_3^1\cdot e_3^2) = 1-|\chi_2^\dagger\chi_1|^2 \ . \tag{5.2.8}$$

Both $|\chi_2^\dagger\chi_1|^2$ and $|\chi_2^{c\dagger}\chi_1|^2$ are positive and ≤ 1. Thus we can parameterize $\chi_2^\dagger\chi_1 = e^{ia}\cos(\psi)$, $\chi_2^{c\dagger}\chi_1 = e^{ib}\sin(\psi)$. (The name of this angle should not be confused with the forms $\psi^{1,2}$.) By suitably multiplying χ_1 and χ_2 by two phases, we can assume $a = -\frac{\pi}{2}$ and $b = \frac{\pi}{2}$; we will reinstate generic values of these phases at the very end. Thus we have

$$\chi_2^\dagger\chi_1 = -i\cos(\psi) \ , \qquad \chi_2^{c\dagger}\chi_1 = i\sin(\psi) \ . \tag{5.2.9}$$

Just as in [120, Sec. 3.1], we can now introduce

$$\chi_0 = \frac{1}{2}(\chi_1 - i\chi_2) \ , \qquad \tilde{\chi}_0 = \frac{1}{2}(\chi_1 + i\chi_2) \ . \tag{5.2.10}$$

[6] An alternative, perhaps more amusing, way of seeing this is to consider $\chi \otimes \chi^\dagger$ as a two-by-two spinorial matrix. It has rank one, which will be true if and only if its determinant is one. Using that $\det(A) = \frac{1}{2}(\mathrm{Tr}(A)^2 - \mathrm{Tr}(A^2))$ for 2×2 matrices, one gets easily that e_3 has norm one.

In three Euclidean dimensions, a spinor and its conjugate form a (pointwise) basis of the space of spinors. For example, χ_0 and χ_0^c are a basis. We can then expand $\tilde{\chi}_0$ on this basis. Actually, its projection on χ_0 vanishes, due to (5.2.9): $\chi_0^\dagger \tilde{\chi}_0 = \frac{i}{4}(\chi_1^\dagger \chi_2 + \chi_2^\dagger \chi_1) = 0$. With a few more steps we get

$$\tilde{\chi}_0 = \frac{{\chi_0^c}^\dagger \tilde{\chi}_0}{||\chi_0||^2} \chi_0^c = \tan\left(\frac{\psi}{2}\right) \chi_0^c \,. \tag{5.2.11}$$

We can now invert (5.2.10) for χ_1 and χ_2, and use (5.2.11). It is actually more symmetric-looking to define $\chi_0 \equiv \cos\left(\frac{\psi}{2}\right)\chi$, to get

$$\chi_1 = \cos\left(\frac{\psi}{2}\right)\chi + \sin\left(\frac{\psi}{2}\right)\chi^c \,, \qquad \chi_2 = i\left(\cos\left(\frac{\psi}{2}\right)\chi - \sin\left(\frac{\psi}{2}\right)\chi^c\right) \,. \tag{5.2.12}$$

We have thus obtained a parameterization of two spinors χ_1 and χ_2 in terms of a single spinor χ and of an angle ψ. Let us count our parameters, to see if our result makes sense. A spinor χ of norm 1 accounts for 3 real parameters; ψ is one more. We should also recall we have rotated both $\chi_{1,2}$ by a phase at the beginning of our computation, to make things easier. We have a grand total of 6 real parameters, which is correct for two spinors of norm 1 in three dimensions.

We can now use the parameterization (5.2.12), and the bilinears (5.2.3), (5.2.5) obtained in section 5.2.1:

$$\begin{aligned}\chi_1 \otimes \chi_2^\dagger &= -i\left[\cos^2\left(\frac{\psi}{2}\right)\chi\chi^\dagger - \sin^2\left(\frac{\psi}{2}\right)\chi^c\chi^{c\dagger} + \cos\left(\frac{\psi}{2}\right)\sin\left(\frac{\psi}{2}\right)(\chi^c\chi^\dagger - \chi\chi^{c\dagger})\right] \\ &= -\frac{i}{2}\left[e_3 - i\sin(\psi)e_2 - i\cos(\psi)\mathrm{vol}_3\right] \,.\end{aligned} \tag{5.2.13}$$

A computation along these lines allows us to evaluate $\chi_1 \otimes \overline{\chi_2}$ as well. We can also reinstate at this point the phases of χ_1 and χ_2, absorbing the overall factor $-i$. The bilinear in (5.2.13) is expressed as an odd form, but we will also need its even-form expression; this can be obtained by using (5.2.2). Recalling the definition (5.1.10), we get:

$$\psi_+^1 = \frac{e^{i\theta_1}}{2}\left[\cos(\psi) + e_1 \wedge (-ie_2 + \sin(\psi)e_3)\right] \,, \quad \psi_-^1 = \frac{e^{i\theta_1}}{2}\left[e_3 - i\sin(\psi)e_2 - i\cos(\psi)\mathrm{vol}_3\right] \,; \tag{5.2.14a}$$

$$\psi_+^2 = \frac{e^{i\theta_2}}{2}\left[\sin(\psi) - (ie_2 + \cos(\psi)e_1) \wedge e_3\right] \,, \quad \psi_-^2 = \frac{e^{i\theta_2}}{2}\left[e_1 + i\cos(\psi)e_2 - i\sin(\psi)\mathrm{vol}_3\right] \,. \tag{5.2.14b}$$

Notice that these satisfy automatically (5.1.11f).

Armed with this parameterization, we will now attack the system (5.1.11) for $\mathrm{AdS}_7 \times M_3$ solutions.

5.3 General results

In section 5.1.2, we have obtained the system (5.1.11), equivalent to supersymmetry for $\mathrm{AdS}_7 \times M_3$ solutions. The $\psi_\pm^{1,2}$ appearing in that system are not just any forms; they should have the property that they can be written as bispinors as in (5.1.10). In section 5.2.2, we have obtained a parameterization for the most general set of $\psi_{\pm}^{1,2}$ that fulfills that constraint; it is (5.2.14), where $\{e_i\}$ is a vielbein.

Thus we can now use (5.2.14) into the differential system (5.1.11), and explore its consequences.

5.3.1 Purely geometrical equations

We will start by looking at the equations in (5.1.11) that do not involve any fluxes. These are (5.1.11e), and the lowest-component form part of (5.1.11a), (5.1.11b) and (5.1.11c).

First of all, we can observe quite quickly that the IIB case cannot possibly work. (5.1.11a), (5.1.11b) and (5.1.11c) all have a zero-form part coming from their right-hand side, which, using (5.2.14), read respectively

$$\cos(\psi)\cos(\theta_1) = 0 \, , \qquad \cos(\psi)\sin(\theta_1) = 0 \, , \qquad \sin(\psi)e^{i\theta_2} = 0 \, . \tag{5.3.1}$$

These cannot be satisfied for any choice of ψ, θ_1 and θ_2. Thus we can already exclude the IIB case.[7]

Having disposed of IIB so quickly, we will devote the rest of this chapter to IIA. Actually, we already know that we can get something new only with non-zero Romans mass, $F_0 \neq 0$. This is because for $F_0 = 0$ we can lift to an eleven-dimensional supergravity solution AdS$_7 \times N_4$. There, we only have a four-form flux G_4 at our disposal, and the only way not to break the SO(6,2) invariance of AdS$_7$ is to switch it on along the internal four-manifold N_4. This is the Freund–Rubin Ansatz, which requires N_4 to admit a Killing spinor. This means that the cone $C(N_4)$ over N_4 admits a covariantly constant spinor; but in five dimensions the only manifold with restricted holonomy is $\mathbb{R}^5$ (or one of its orbifolds, of the form $\mathbb{R}^4/\Gamma \times \mathbb{R}$). Thus we know already that all solutions with $F_0 = 0$ lift to AdS$_7 \times S^4$ (or AdS$_7 \times S^4/\Gamma$) in eleven dimensions. (In fact we will see later how AdS$_7 \times S^4$ reduces to ten dimensions.) We will thus focus on $F_0 \neq 0$, and use the case $F_0 = 0$ as a control.

In IIA, the lowest-degree equations of (5.1.11a), (5.1.11b) and (5.1.11c) are one-forms; they are less dramatic than (5.3.1), but still rather interesting. Using (5.2.14), after some manipulations we get

$$\begin{aligned} e_1 &= -\frac{1}{4}e^A \sin(\psi) d\theta_2 \, , \qquad e_2 = \frac{1}{4}e^A (d\psi + \tan(\psi) d(5A - \phi)) \, , \\ e_3 &= \frac{1}{4}e^A \left(-\cos(\psi) d\theta_1 + \frac{\cot(\theta_1)}{\cos(\psi)} d(5A - \phi) \right) \, , \end{aligned} \tag{5.3.2}$$

and

$$x dx = (1 + x^2) d\phi - (5 + x^2) dA \, , \tag{5.3.3}$$

[7] This quick death is reminiscent of the fate of AdS$_4 \times M_6$ with SU(3) structure in IIB. The system in [108] has a zero-form equation and two-form equation coming from the right-hand side of its fluxless equation, which look like $\cos(\theta) = 0 = \sin(\theta) J$, where θ is an angle similar to ψ in (5.2.14). This is consistent with a no-go found with lengthier computations in [136].

where

$$x \equiv \cos(\psi)\sin(\theta_1) \ , \tag{5.3.4}$$

and we have dropped the subscript $_3$ on the warping function: $A \equiv A_3$ from now on. Notice that (5.3.2) determine the vielbein. Usually (i.e. in other dimensions), the geometrical part of the differential system coming from supersymmetry gives the *derivative* of the forms defining the metric. In this case, the forms themselves are determined in terms of derivatives of the angles appearing in our parameterizations. This will allow us to give a more complete and concrete classification than is usually possible.

We still have (5.1.11e). Notice that (5.1.11a) allows to write it as $dA \wedge d(e^{3A-\phi}x) = 0$. Using also (5.3.3), we get

$$dA \wedge d\phi = 0 \ . \tag{5.3.5}$$

This means that ϕ is functionally dependent on A:[8]

$$\phi = \phi(A) \ . \tag{5.3.6}$$

(5.3.3) then means that x too is functionally dependent on A: $x = x(A)$.

5.3.2 Fluxes

So far, we have analyzed (5.1.11e), and the one-form part of (5.1.11a), (5.1.11b) and (5.1.11c). Before we look at their three-form part too, it is convenient to look at (5.1.11d), which gives us the RR flux, for reasons that will become apparent.

First we compute F_0 from (5.1.11d):

$$F_0 = 4xe^{-A-\phi}\frac{3-\partial_A\phi}{5-2x^2-\partial_A\phi} \ . \tag{5.3.7}$$

The Bianchi identity for F_0 says that it should be (piecewise) constant. It will thus be convenient to use (5.3.7) to eliminate $\partial_A\phi$ from our equations.

Before we go on to analyze our equations, let us also introduce the new angle β by

$$\sin^2(\beta) = \frac{\sin^2(\psi)}{1-x^2} \ . \tag{5.3.8}$$

We can now use x as defined in (5.3.4) to eliminate θ_1, and β to eliminate ψ. This turns out to be very convenient in the following, especially in our analysis of the metric in section 5.3.4 below (which was our original motivation to introduce β).

After these preliminaries, let us give the expression for F_2 as one obtains it from (5.1.11d):

$$F_2 = \frac{1}{16}\sqrt{1-x^2}e^{A-\phi}(xe^{A+\phi}F_0 - 4)\mathrm{vol}_{S^2} \ , \tag{5.3.9}$$

[8](5.3.6) excludes the case where A is constant in a region. However, it is easy to see that this case cannot work. Indeed, in this case (5.3.3) can be integrated as $e^\phi \propto \sqrt{1-x^2}$, which is incompatible with (5.3.7) below.

where

$$\mathrm{vol}_{S^2} = \sin(\beta) d\beta \wedge d\theta_2 \tag{5.3.10}$$

is formally identical to the volume form for a round S^2 with coordinates $\{\beta, \theta_2\}$. We will see later that this is no coincidence.

Finally, let us look at the three-form part of (5.1.11a), (5.1.11b) and (5.1.11c). One of them can be used to determine H:

$$H = \frac{1}{8} e^{2A} \sqrt{1-x^2}\, \frac{6 + xF_0 e^{A+\phi}}{4 + xF_0 e^{A+\phi}}\, dx \wedge \mathrm{vol}_{S^2} \ , \tag{5.3.11}$$

while the other two turn out to be identically satisfied.

Our analysis is not over: we should of course now impose the equation of motion, and the Bianchi identities for our fluxes. The equation of motion for F_2, $d * F_2 + H * F_0 = 0$, follows automatically from (5.1.11d), much as it happens in the pure spinor system for AdS$_4 \times M_6$ solutions [108]. We should then impose the Bianchi identity for F_2, which reads $dF_2 - HF_0 = 0$ (away from sources). This does not follow manifestly from (5.1.11d), but in fact it is a consequence of the explicit expressions (5.3.7), (5.3.9) and (5.3.11) above. When $F_0 \neq 0$, it also implies that the B field such that $H = dB$ can be locally written as

$$B_2 = \frac{F_2}{F_0} + b \tag{5.3.12}$$

for a closed two-form b. Using a gauge transformation, it can be assumed to be proportional (by a constant) to vol_{S^2}; we then have that it is a constant, $\partial_A b = 0$.

The equation of motion for H, which reads for us $d(e^{7A-2\phi} *_3 H) = e^{7A} F_0 *_3 F_2$ (again away from sources), is also automatically satisfied, as shown in general in [33]. Finally, since we have checked all the conditions for preserved supersymmetry, the Bianchi identities and the equations of motion for the fluxes, the equations of motion for the dilaton and for the metric will now follow [34].

5.3.3 The system of ODEs

Let us now sum up the results of our analysis of (5.1.11). Most of our equations determine some fields: (5.3.2) give the vielbein, and (5.3.7), (5.3.9), (5.3.11) give the fluxes. The only genuine differential equations we have are (5.3.3), and the condition that F_0 should be constant. Recalling that ϕ is functionally dependent on A, (5.3.6), these two equations can be written as

$$\partial_A \phi = 5 - 2x^2 + \frac{8x(x^2-1)}{4x - F_0 e^{A+\phi}} \ , \tag{5.3.13a}$$

$$\partial_A x = 2(x^2-1)\, \frac{x e^{A+\phi} F_0 + 4}{4x - F_0 e^{A+\phi}} \ . \tag{5.3.13b}$$

We thus have reduced the existence of a supersymmetric solution of the form AdS$_7 \times M_3$ in IIA to solving the system of ODEs (5.3.13). It might look slightly unsettling that we are essentially using at this point A as a coordinate, which might not always be a wise choice (since A might not be monotonic). For that matter, our analysis has so far been completely local; we will start looking at global issues in section 5.3.4, and especially 5.3.6.

Unfortunately we have not been able to find analytic solutions to (5.3.13), other than in the $F_0 = 0$ case (which we will see in section 5.4.1). For the more interesting $F_0 \neq 0$ case, we can gain some intuition by noticing that the system becomes autonomous (i.e. it no longer has explicit dependence on the "time" variable A) if one defines $\tilde{\phi} \equiv \phi + A$. The system for $\{\partial_A \tilde{\phi}, \partial_A x\}$ can now be thought of as a vector field in two dimensions; we plot it in figure 5.1.

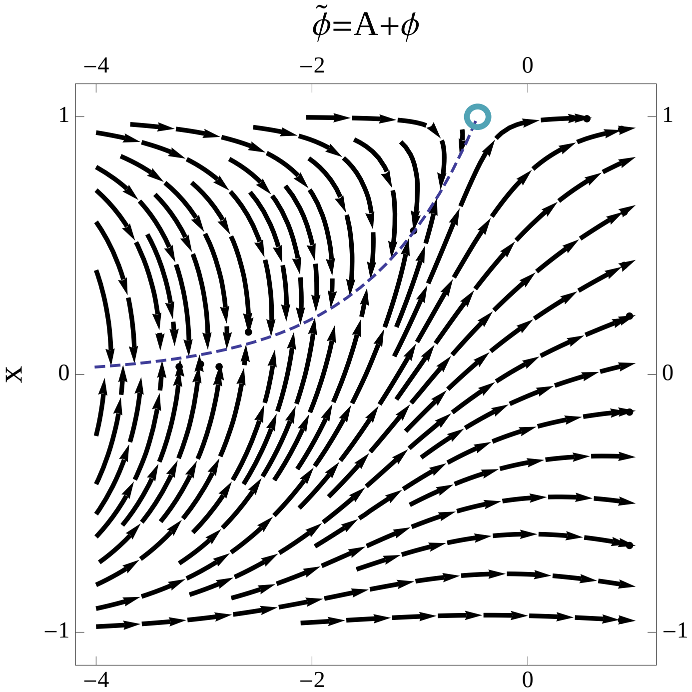


Figure 5.1: A plot of the vector field induced by (5.3.13) on $\{\tilde{\phi} \equiv \phi + A, x\}$, for $F_0 = 40/2\pi$ (in agreement with flux quantization, (5.3.37) below). The green circle represents the point $\{\phi + A = \log(4/F_0), x = 1\}$, whose role will become apparent in section 5.3.7. The dashed line represents the locus along which the denominators in (5.3.13) vanish.

We will study the system (5.3.13) numerically in section 5.4. Before we do that, we should understand what boundary conditions we should impose. We will achieve this by analyzing global issues about our set-up, that we have so far ignored.

5.3.4 Metric

The metric

$$ds^2_{M_3} = e_a e_a \tag{5.3.14}$$

following from (5.3.2) looks quite complicated. However, it simplifies enormously if we rewrite it in terms of β in (5.3.8):[9]

$$ds^2_{M_3} = e^{2A}(1-x^2)\left[\frac{16}{(4x-e^{A+\phi}F_0)^2}\,dA^2 + \frac{1}{16}\,ds^2_{S^2}\right] , \qquad ds^2_{S^2} = d\beta^2 + \sin^2(\beta)d\theta_2^2 \ . \tag{5.3.15}$$

Without any Ansatz, the metric has taken the form of a fibration of a round S^2, with coordinates $\{\beta, \theta_2\}$, over an interval with coordinate A. Notice that none of the scalars appearing in (5.3.15) (and indeed in the fluxes (5.3.7), (5.3.9), (5.3.11)) were originally intended as coordinates, but rather as functions in the parameterization of the pure spinors $\psi^{1,2}$. Usually, one would then need to introduce coordinates independently, and to make an Ansatz about how all functions should depend on those coordinates, sometimes imposing the presence of some particular isometry group in the process.

Here, on the other hand, the functions we have introduced are suggesting themselves as coordinates to us rather automatically. Since so far our expressions for the metric and fluxes were local, we are free to take their suggestion. We will take β to be in the range $[0,\pi]$, and θ_2 to be periodic with period 2π, so that together they describe an S^2 as suggested by (5.3.15), and also by the two-form (5.3.10) that appeared in (5.3.9), (5.3.11).[10]

It is not hard to understand why this S^2 has emerged. The holographic dual of any solutions we might find is a $(1,0)$ CFT in six dimensions. Such a theory would have SU(2) R-symmetry; an SU(2) isometry group should then appear naturally on the gravity side as well. This is what we are seeing in (5.3.15).

The fact that the S^2 in (5.3.15) is rotated by R-symmetry also helps to explain a possible puzzle about IIB. Often, given a IIA solution, one can produce a IIB one via T-duality along an isometry. All the Killing vectors of the S^2 in (5.3.15) vanish in two points; T-dualizing along any such direction would produce a non-compact solution in IIB, but still a valid one. But the IIB case died very quickly in section 5.3.1; there are no solutions, not even non-compact or singular ones. Here is how this puzzle is resolved. Since the SU(2) isometry group of the S^2 is an R-symmetry, supercharges transform as a doublet under it (we will see this more explicitly in section 5.3.5). Thus even the strange IIB geometry produced by T-duality along a U(1) isometry of S^2 would not be supersymmetric.

Even though we have promoted β and θ_2 to coordinates, it is hard to do the same for A, which actually enters in the seven-dimensional metric (see (5.1.6)). We would like to be able to cover cases where A is non-monotonic. One possibility would be to use A as a coordinate piecewise. We find it clearer, however, to introduce a coordinate r defined by $dr = 4e^A\frac{\sqrt{1-x^2}}{4x-e^{A+\phi}F_0}dA$, so that the metric now reads

$$ds^2_{M_3} = dr^2 + \frac{1}{16}e^{2A}(1-x^2)ds^2_{S^2} \ . \tag{5.3.16}$$

[9] In fact, the definition of β was originally found by trying to understand the global properties of the metric (5.3.14). Looking at a slice x =const, one finds that the metric in $\{\theta_1, \theta_2\}$ has constant positive curvature; the definition of β becomes then natural. Nontrivially, this definition also gets rid of non-diagonal terms of the type $dAd\theta_1$ that would arise from (5.3.2).

[10] A slight variation is to take $\mathbb{RP}^2 = S^2/\mathbb{Z}_2$ instead of S^2; this will not play much of a role in what follows, except for some solutions with O6-planes that we will mention in sections 5.4.1 and 5.4.2.

In other words, r measures the distance along the base of the S^2 fibration. Now A, x and ϕ have become functions of r. From (5.3.13) and the definition of r we have

$$\begin{aligned} \partial_r\phi &= \frac{1}{4}\frac{e^{-A}}{\sqrt{1-x^2}}(12x+(2x^2-5)F_0e^{A+\phi}) \ , \\ \partial_r x &= -\frac{1}{2}e^{-A}\sqrt{1-x^2}(4+xF_0e^{A+\phi}) \ , \\ \partial_r A &= \frac{1}{4}\frac{e^{-A}}{\sqrt{1-x^2}}(4x-F_0e^{A+\phi}) \ . \end{aligned} \tag{5.3.17}$$

We have introduced a square root in the system, but notice that $-1 \leq x \leq 1$ already follows from requiring that $ds^2_{M_3}$ in (5.3.15) has positive signature. (We choose the positive branch of the square root.)

Let us also record here that the NS three-form also simplifies in the coordinates introduced in this section:

$$H = -(6e^{-A} + xF_0e^{\phi})\mathrm{vol}_3 \ , \tag{5.3.18}$$

where vol_3 is the volume form of the metric $ds^2_{M_3}$ in (5.3.16) or (5.3.15).

We have obtained so far that the metric is the fibration of an S^2 (with coordinates (β, θ_2)) over a one-dimensional space. The SU(2) isometry group of the S^2 is to be identified holographically with the R-symmetry group of the $(1, 0)$-superconformal dual theory. For holographic applications, we would actually like to know whether the total space of the S^2-fibration can be made compact. We will look at this issue in section 5.3.6. Right now, however, we would like to take a small detour and see a little more clearly how the R-symmetry SU(2) emerges in the pure spinors $\psi^{1,2}$.

5.3.5 $\mathrm{SU}(2)$-covariance

We have just seen that the metric takes the particularly simple form (5.3.16) in coordinates (r, β, θ_2); the appearance of the S^2 is related to the SU(2) R-symmetry group of the $(1, 0)$ holographic dual.

Since these coordinates are so successful with the metric, let us see whether they also simplify the pure spinors $\psi^{1,2}$. We can start by the zero-form parts of (5.2.14), which read

$$\psi^1_0 = ix + \sqrt{1-x^2}\,\cos(\beta) \ , \qquad \psi^2_0 = \sqrt{1-x^2}\,\sin(\beta)e^{i\theta_2} \ . \tag{5.3.19}$$

Recalling that (β, θ_2) are the polar coordinates on the S^2 (see the expression of $ds^2_{S^2}$ in (5.3.15)), we recognize in (5.3.19) the appearance of the $\ell = 1$ spherical harmonics

$$y^\alpha = \{\sin(\beta)\cos(\theta_2), \sin(\beta)\sin(\theta_2), \cos(\beta)\} \ . \tag{5.3.20}$$

Notice that y^3 appears in $\psi^1 = \chi_1 \otimes \chi_2^\dagger$, while $y^1 + iy^2$ appears in $\psi^2 = \chi_1 \otimes \chi_2^{c\,\dagger}$. This suggests that we introduce a 2×2 matrix of bispinors. From (C.1.4) we see that for IIA $\binom{\chi_1}{\chi_1^c}$ and $\binom{\chi_2}{-\chi_2^c}$ are both SU(2) doublets, so that it is natural to define

$$\Psi = \begin{pmatrix} \chi_1 \\ \chi_1^c \end{pmatrix} \otimes (\chi_2^\dagger\, , -\chi_2^{c\,\dagger}) = \begin{pmatrix} \psi^1 & \psi^2 \\ (-)^{\deg}(\psi^2)^* & -(-)^{\deg}(\psi^1)^* \end{pmatrix} \ , \tag{5.3.21}$$

where $(-)^{\rm deg}$ acts as $\pm$ on a even (odd) form. The even-form part can then be written as

$$\Psi_+^{ab} = i{\rm Im}\psi_+^1\,{\rm Id}_2 + \left({\rm Re}\psi_+^2\sigma_1 - {\rm Im}\psi_+^2\sigma_2 + {\rm Re}\psi_+^1\sigma_3\right)\ , \tag{5.3.22a}$$

where σ_α are the Pauli matrices while the odd-form part is

$$\Psi_-^{ab} = {\rm Re}\psi_-^1\,{\rm Id}_2 + i\left({\rm Im}\psi_-^2\sigma_1 + {\rm Re}\psi_-^2\sigma_2 + {\rm Im}\psi_-^1\sigma_3\right)\ . \tag{5.3.22b}$$

(5.3.22) shows more explicitly how the R-symmetry SU(2) acts on the bispinors Ψ^{ab}, which split between a singlet and a triplet. If we go back to our original system (5.1.11), we see that (5.1.11a), (5.1.11d), (5.1.11e) each behave as a singlet, while (5.1.11b), (5.1.11c) behave as a triplet — thanks also to the fact that the factor $e^{5A-\phi}$ appears in both those equations.

More concretely, (5.3.19) can now be written as

$$\Psi_0^{ab} = ix\,{\rm Id}_2 + \sqrt{1-x^2}\,y^\alpha\sigma_\alpha\ ; \tag{5.3.23a}$$

the one-form part reads

$$\Psi_1^{ab} = \sqrt{1-x^2}dr\,{\rm Id}_2 + i\left[xy^\alpha dr + \frac{1}{4}e^A\sqrt{1-x^2}\,dy^\alpha\right]\sigma_\alpha\ . \tag{5.3.23b}$$

The rest of Ψ^{ab} can be determined by (5.2.2): $\Psi_3^{ab} = -i *_3 \Psi_0^{ab} = -i\Psi_0^{ab}\,{\rm vol}_3$, $\Psi_2^{ab} = -i *_3 \Psi_1^{ab}$. (The three-dimensional Hodge star can be easily computed from (5.3.16).)

We will now turn to the global analysis of the metric (5.3.16).

5.3.6 Topology

We now wonder whether the S^2 fibration in (5.3.15) can be made compact.

One possible strategy would be for r to be periodically identified, so that the topology of M_3 would become $S^1 \times S^2$. This is actually impossible: from (5.3.17) we have

$$\partial_r(xe^{3A-\phi}) = -2\sqrt{1-x^2}e^{2A-\phi} \le 0\ . \tag{5.3.24}$$

This can also be derived quickly from (5.1.11a) using the singlet part of (5.3.23). Now, $xe^{3A-\phi}$ is continuous;[11] for r to be periodically identified, $xe^{3A-\phi}$ should be a periodic function. However, thanks to (5.3.24), it is nowhere-increasing. It also cannot be constant, since x would be ± 1 for all r, which makes the metric in (5.3.15) vanish. Thus r cannot be periodically identified.

We then have to look for another way to make M_3 compact. The only other possibility is in fact to shrink the S^2 at two values of r, which we will call $r_{\rm N}$ and $r_{\rm S}$; the topology of M_3 would then be S^3. The subscripts stand for "north" and "south"; we can visualize these two points as the two poles of the S^3, and the other, non-shrunk copies of S^2 over any $r \in (r_{\rm N}, r_{\rm S})$ to be the "parallels" of the S^3. Of course, since (5.3.17) does not depend on r, we can assume without any loss of generality that $r_{\rm N} = 0$.

We will now analyze this latter possibility in detail.

[11]This might not be fully obvious in presence of D8-branes, but we will see later that it is true even in that case, basically because ϕ is a physical field, and A and x appear as coefficients in the metric.

5.3.7 Local analysis around poles

We have just suggested to make M_3 compact by having the S^2 fiber over an interval $[r_{\rm N}, r_{\rm S}]$, and by shrinking it at the two extrema. In this case M_3 would be homeomorphic to S^3.

To realize this idea, from (5.3.16) we see that x should go to 1 or -1 at the two poles $r_{\rm N}$ and $r_{\rm S}$. To make up for the vanishing of the $\sqrt{1-x^2}$'s in the denominators in (5.3.17), we should also make the numerators vanish. This is accomplished by having $e^{A+\phi} = \pm 4/F_0$ at those two poles (which is obviously only possible when $F_0 \neq 0$). We can now also see that $\partial_r x \sim -4e^{-A}\sqrt{1-x^2} \leq 0$ around the poles. Since, as we noticed earlier, $-1 \leq x \leq 1$, x should actually be 1 at $r_{\rm N}$, and -1 at $r_{\rm S}$. Summing up:

$$\left\{ x = 1,\ e^{A+\phi} = \frac{4}{F_0} \right\} \ \text{at } r = r_{\rm N} \ , \qquad \left\{ x = -1,\ e^{A+\phi} = -\frac{4}{F_0} \right\} \ \text{at } r = r_{\rm S} \ . \tag{5.3.25}$$

Since we made both numerators and denominators in (5.3.17) vanish at the poles, we should be careful about what happens in the vicinity of those points. We want to study the system around the boundary conditions (5.3.25) in a power-series approach. (The same could also be done directly with (5.3.13).) Let us first expand around $r_{\rm N}$. As mentioned earlier, thanks to translational invariance in r we can assume $r_{\rm N} = 0$ without any loss of generality. We get

$$\begin{aligned} \phi &= -A_0^+ + \log\left(\frac{4}{F_0}\right) - 5e^{-2A_0^+}r^2 + \frac{172}{9}e^{-4A_0^+}r^4 + O(r)^6 \ , \\ x &= 1 - 8e^{-2A_0^+}r^2 + \frac{400}{9}e^{-4A_0^+}r^4 + O(r)^6 \ , \\ A &= A_0^+ - \frac{1}{3}e^{-2A_0^+}r^2 - \frac{4}{27}e^{-4A_0^+}r^4 + O(r)^6 \ . \end{aligned} \tag{5.3.26}$$

A_0^+ here is a free parameter. The way it appears in (5.3.26) is explained by noticing that (5.3.17) is symmetric under

$$A \to A + \Delta A \ , \qquad \phi \to \phi - \Delta A \ , \qquad x \to x \ , \qquad r \to e^{\Delta A} r \ . \tag{5.3.27}$$

Applying (5.3.26) to (5.3.16), and setting for a moment $r_{\rm N} = 0$, we find that the metric has the leading behavior

$$ds^2_{M_3} = dr^2 + r^2 ds^2_{S^2} + O(r)^4 = ds^2_{\mathbb{R}^3} + O(r)^4 \ . \tag{5.3.28}$$

This means that the metric is regular around $r = r_{\rm N}$. The expansion of the fluxes (5.3.9), (5.3.11) is

$$F_2 = -\frac{10}{3}F_0 e^{-A_0^+} r^3 \text{vol}_{S^2} + O(r)^5 \ , \qquad H = -10e^{-A_0^+} r^2 dr \wedge \text{vol}_{S^2} + O(r)^3 \ . \tag{5.3.29}$$

As for the B field, recall that it can be written as in (5.3.12). (5.3.29) shows that around $r = r_{\rm N} = 0$, the term F_2/F_0 is regular as it is, without the addition of b; this suggests that one should set $b = 0$. To make this more precise, consider the limit

$$\lim_{r\to 0} \int_{\Delta_r} H = \lim_{r \to 0} \int_{S^2_r} B_2 \tag{5.3.30}$$

where Δ_r is a three-dimensional ball such that $\partial\Delta_r = S^2_r$. In (5.3.12), the first term goes to zero because $x \to 1$; so the limit is equal to $\int_{S^2} b$, which is constant. This constant signals a delta in H. So we are forced to conclude that

$$b = 0 \tag{5.3.31}$$

near the pole. (However, we will see in section 5.3.8 that b can become non-zero if one crosses a D8 while going away from the pole.)

To be more precise, (5.3.31) should be understood up to gauge transformations. B is not a two-form, but a 'connection on a gerbe', in the sense that it transforms non-trivially on chart intersections: on $U \cap U'$, $B_U - B_{U'}$ can be a 'small' gauge transformation $d\lambda$, for λ a 1-form, or more generally a 'large' gauge transformation, namely a two-form whose periods are integer multiples of $4\pi^2$. In our case, if we cover S^3 with two patches $U_{\rm N}$ and $U_{\rm S}$, around the equator we can have $B_{\rm N} - B_{\rm S} = N\pi \mathrm{vol}_{S^2}$. In this case $\int_{S^3} H = B_{\rm N} - B_{\rm S} = N\pi\mathrm{vol}_{S^2} = (4\pi^2)N$, in agreement with flux quantization for H. Thus $b = 0$ is also gauge equivalent to any integer multiple of $\pi\mathrm{vol}_{S^2}$. In practice, however, we will prefer to work with $b = 0$ around the poles, and perform a gauge transformation whenever

$$\hat{b}(r) \equiv \frac{1}{4\pi}\int_{S^2_r} B_2 \tag{5.3.32}$$

gets outside the "fundamental region" $[0, \pi]$. In other words, we will consider $\hat{b}$ to be a variable with values in $[0, \pi]$, and let it begin and end at 0 at the two poles. $\hat{b}$ will then wind an integer number N of times around $[0, \pi]$, and this will make sure that $\int_{S^3} H = (4\pi^2)N$, thus taking care of flux quantization for H.

So far we have discussed the expansion around the north pole; a similar discussion holds for the expansion around the south pole $r_{\rm S}$. The expressions that replace (5.3.26), (5.3.28), (5.3.29) can be obtained by using the symmetry of (5.3.17) under

$$x \to -x\,, \qquad F_0 \to -F_0\,, \qquad r \to -r\,. \tag{5.3.33}$$

The free parameter A_0^+ can now be changed to a possibly different free parameter A_0^-.

We have hence checked that the boundary conditions (5.3.25) are compatible with our system (5.3.17), and that they give rise to a regular metric at the poles.

5.3.8 D8

There is one more ingredient that we will need in section 5.4 to exhibit compact solutions: brane sources. In presence of branes the metric cannot be called regular: their gravitational backreaction will give rise to a singularity. A random singularity would call into question the validity of a solution, since the curvature and possibly the dilaton[12] would diverge there,

[12] In presence of Romans mass, the string coupling is bounded by the inverse radius of curvature in string units: $e^\phi \lesssim \frac{l_s}{R_{\rm curv}}$, and is actually generically of the order of the bound [137].

making the supergravity approximation untrustworthy. We are however sure of the existence of D-branes, in spite of the singularities in their geometry, because we have an open string realization for them.

D8-branes in particular are even more benign, in a way, because the singularity manifests itself simply as a discontinuity in the derivatives of the coefficients in the metric. In general relativity, such a discontinuity would be subject to the so-called Israel junction conditions [138], which are a consequence of the Einstein equations. As we mentioned earlier, in our case, however, supersymmetry guarantees that the equations of motion for the dilaton and metric are automatically satisfied [34]. Hence, the conditions on the first derivatives will follow from imposing continuity of the fields and supersymmetry.

Let us be more concrete. We will suppose we have a stack of $n_{\rm D8}$ D8-branes, possibly with a worldvolume gauge field-strength f_2 (not to be confused with the RR field-strength F_2), which induces a D6-brane charge distribution on it. The Bianchi identity for such an object reads

$$d_H F = \frac{1}{2\pi} n_{\rm D8} e^{\mathcal{F}} \delta \qquad \Rightarrow \qquad d\tilde{F} = \frac{1}{2\pi} n_{\rm D8} e^{2\pi f_2} \delta \qquad (\delta \equiv dr\delta(r)) \ . \tag{5.3.34}$$

As usual $\mathcal{F} = B_2 + 2\pi f_2$; recall from section 5.1.2 that $F = F_0 + F_2$; and likewise we have defined

$$\tilde{F} \equiv e^{-B_2} F = F_0 + (F_2 - B_2 F_0) \ . \tag{5.3.35}$$

In other words, $\tilde{F} = F_0 + \tilde{F}_2$, with $\tilde{F}_2 = F_2 - B_2 F_0$. Since $\tilde{F}_2$ is closed away from sources, it makes sense to define

$$n_2 \equiv \frac{1}{2\pi} \int_{S^2} \tilde{F}_2 \ . \tag{5.3.36}$$

Flux quantization then requires n_2 to be an integer, and that

$$F_0 = \frac{n_0}{2\pi} \ , \tag{5.3.37}$$

with n_0 an integer. (We are working in string units where $l_s = 1$.) Integrating now (5.3.34) across the magnetized stack of D8's gives

$$\Delta n_0 = n_{\rm D8} \ , \qquad \Delta \tilde{F}_2 = f_2 \Delta n_0 \ . \tag{5.3.38}$$

All physical fields should be continuous across the D8 stack. For example, $\Delta\phi = 0$. Also, the coefficients of the metric should not jump; in particular, from (5.1.6), we see that $\Delta A = 0$. Also, since x appears in front of $ds^2_{S^2}$ in (5.3.16), we should have $\Delta x = 0$.

Imposing that the B field does not jump is trickier. A first caveat is that B would actually be allowed to jump by a gauge transformation, as discussed in section 5.3.7. However, we find it less confusing to put the intersection between the charts $U_{\rm N}$ and $U_{\rm S}$ away from the D8's, and to treat $\int_{S^2} B_2$ as a periodic variable as described in section 5.3.7.

Thus we will simply impose that B does not jump. First, recall that it can be written as in (5.3.12), when $F_0 \neq 0$. The b term was shown in (5.3.31) to be vanishing near the pole, but we will soon see that this conclusion is not valid between D8's. In fact, it is connected to the flux integer n_2 defined in (5.3.36): from (5.3.12) we have

$$\tilde{F}_2 = -F_0 b \, ; \tag{5.3.39}$$

integrating this on S^2, we get $2\pi n_2 = -F_0 \int_{S^2} b$, or in other words

$$b = -\frac{n_2}{2F_0}\mathrm{vol}_{S^2} \, . \tag{5.3.40}$$

We can use our result (5.3.9) for F_2; for this section, it will be convenient to define

$$p \equiv \frac{1}{16}x\sqrt{1-x^2}e^{2A} \, , \qquad q \equiv \frac{1}{4}\sqrt{1-x^2}e^{A-\phi} \, , \tag{5.3.41}$$

so that

$$F_2 = (pF_0 - q)\mathrm{vol}_{S^2} \, . \tag{5.3.42}$$

From this and (5.3.40) we now have

$$B_2 = \left(p - \frac{q}{F_0} - \frac{n_2}{2F_0}\right)\mathrm{vol}_{S^2} \, . \tag{5.3.43}$$

Let us call n_0, n_2 the flux integers on one side of the D8 stack, and n_0', n_2' the fluxes on the other side. Let us at first assume that both n_0 and n_0' are non-zero. Then, equating B on the two sides, we see that p cancels out, and we get

$$\frac{1}{n_0}\left(q + \frac{1}{2}n_2\right) = \frac{1}{n_0'}\left(q + \frac{1}{2}n_2'\right) \, , \tag{5.3.44}$$

or in other words

$$q|_{r=r_{\mathrm{D8}}} = \frac{n_2' n_0 - n_2 n_0'}{2(n_0' - n_0)} \, , \tag{5.3.45}$$

with q as defined in (5.3.41). Notice that, in (5.3.12), the term F_2/F_0 and b can both separately jump, while the whole B_2 is staying continuous. For this reason, as we anticipated in section 5.3.7, the conclusion $b = 0$ (which implies $n_2 = 0$ by (5.3.40)) will hold near the poles, but can cease to hold after one crosses a D8. (5.3.45) is also satisfying in that it is symmetric under exchange $\{n_0, n_2\} \leftrightarrow \{n_0', n_2'\}$. Notice also that, under a gauge transformation for the B field, $n_2 \to n_2 + n_0\Delta B$, $n_2' \to n_2' + n_0'\Delta B$, and (5.3.45) remains unchanged.

A constraint on the discontinuity should also come from the F_2 Bianchi identity (5.3.34). Using (5.3.42), we see that the only discontinuities are coming from the jump in F_0, so that we get

$$d_H F = \Delta F_0 (1 + p\mathrm{vol}_{S^2})\delta = \Delta F_0 \, e^{p\mathrm{vol}_{S^2}}\delta \, . \tag{5.3.46}$$

Comparing this with (5.3.34) we see that $\mathcal{F} = p\mathrm{vol}_{S^2}$. It also follows that

$$d\tilde{F}_2 = \Delta F_0(-B_2 + p\mathrm{vol}_{S^2})\delta = \frac{\Delta F_0}{F_0}\left(q + \frac{1}{2}n_2\right)\mathrm{vol}_{S^2} \, . \tag{5.3.47}$$

The expression on the right-hand side is not ambiguous thanks to (5.3.42). Comparing (5.3.47) with (5.3.34) again, we see that $f_2 = \frac{1}{F_0}\left(q + \frac{n_2}{2}\right)$. Going back to (5.3.38), we learn that

$$\frac{\Delta n_2}{\Delta n_0} = \frac{1}{n_0}\left(q + \frac{1}{2}n_2\right) . \tag{5.3.48}$$

This is actually nothing but (5.3.45) again.

(5.3.47) shows that our D8 is actually also charged under F_2, and thus that it is actually a D8/D6 bound state.

In fact, we should mention that it also acts as a source for H. This should not come as a surprise: it comes from the fact that B appears in the DBI brane action. The simplest way to see this phenomenon for us is to notice that H in (5.3.18) contains F_0. Since F_0 jumps across the D8, so does H, and its equation of motion now gets corrected to

$$d(e^{7A-2\phi} * H) - e^{7A} F_0 * F_2 = -x e^{7A-\phi} \Delta F_0 \delta . \tag{5.3.49}$$

The localized term on the right hand side is exactly what one obtains by varying the DBI action $-\int_{S^2} e^{7A-\phi}\sqrt{\det(g+\mathcal{F})}$: the variation for a single D8 is $-e^{7A-\phi}\frac{\mathcal{F}}{\det(g+\mathcal{F})}\delta = -xe^{7A-\phi}\delta$. This was guaranteed to work: the equation of motion for H was shown in [33] to follow in general from supersymmetry even in presence of sources. (The CS term $\int Ce^{\mathcal{F}}$ does not contribute, as remarked below [33, (B.7)].)

Yet another check one could perform is whether the D8 source is now BPS — namely, whether the supersymmetry variation induced on its worldvolume theory can be canceled by an appropriate κ-symmetry transformation. This check is made simpler by the fact that brane calibrations are actually the same forms that appear in the bulk supersymmetry conditions (as first noticed in [134] for compactifications to four dimensions). In our case, we see from [112, Table 1] that the appropriate calibration for a space-filling brane is $e^{6A_4-\phi}\mathrm{Re}\phi^1_-$; for our AdS$_7$ case, we should pick in (5.1.9) its part along $d\rho$. So our brane calibration is

$$e^{7A-\phi}\mathrm{Im}\psi^1_+ . \tag{5.3.50}$$

The condition that a single brane should be BPS boils down to demanding that the pull-back of the form $e^{\mathcal{F}}\mathrm{Im}\psi^1_+$ equal the generalized volume form $\sqrt{\det(g+\mathcal{F})}$ on the brane. Alternatively, this is equivalent to demanding that the pullback on the brane of

$$e^{\mathcal{F}}\mathrm{Re}\psi^1_+ , \qquad e^{\mathcal{F}}\psi^2_+ \tag{5.3.51}$$

vanish. We checked explicitly that this condition holds precisely if (5.3.45) does.

We should be a bit more careful, however, about what happens for multiple branes. In that case, (5.3.51) become non-abelian, because they both contain the worldsheet field f_2. Satisfying this condition now requires $\mathcal{F}$ to be proportional to the identity, and this in turn requires that the D6-brane charge Δn_2 should be an integer multiple of $n_{\mathrm{D8}} = \Delta n_0$. In other words, a bunch of D8-branes should be made of magnetized branes which all have the same induced D6-brane charge.

Finally, in our analysis so far we have left out the case where F_0 is zero on one of the sides of the D8 stack, say the right side, so that $n_0' = 0$. This time we cannot apply (5.3.43) on the right side of the D8. An expression for B in this case will be given in (5.4.8) below. Imposing continuity of B this time does not lead to (5.3.45), but to a different condition in terms of the integration constants appearing in (5.4.8). However, the Bianchi identity for F_2 can still be applied on the left side of the D8, where $F_0 \neq 0$; this still leads to (5.3.45). In other words, in this case we have (5.3.45) plus an extra condition imposing continuity of B. This will be important in our example with two D8's in section 5.4.3.

Let us summarize the results of this section. We have obtained that one can insert D8's in our set-up, provided their position r_{D8} is such that the condition (5.3.45) is satisfied. When F_0 is non-zero on both sides of the D8, this ensures that the Bianchi for F_2 is satisfied, and that B is continuous. In the special case where $F_0 = 0$ on one side, continuity of B has to be imposed independently.

5.3.9 Summary of this section

Supersymmetric solutions of the form $\mathrm{AdS}_7 \times M_3$ cannot exist in IIB. In IIA we have reduced the problem to solving the system of ODEs (5.3.13) (or (5.3.17)). Given a solution to this system, the flux is given by (5.3.7), (5.3.9) and (5.3.11), and the metric is given by (5.3.15) (or (5.3.16)). This describes an S^2 fibration over a segment; the space is compact if the S^2 fiber shrinks at the endpoints of the segment, giving a topology $M_3 = S^3$. This imposes the boundary conditions (5.3.25) on the system (5.3.17). D8-branes can be inserted along the S^2, at values $r = r_{\mathrm{D8}}$ that satisfy (5.3.45).

We now turn to a numerical study of the system, which will show that nontrivial solutions do indeed exist.

5.4 Explicit solutions

We will now show some explicit $\mathrm{AdS}_7 \times M_3$ solutions, by solving the system (5.3.17). We will start in section 5.4.1 by looking briefly at the massless solution, which is in a sense unique; it has a D6-brane and an anti-D6 at the two poles. In section 5.4.2 we will switch on Romans mass, and we will obtain a solution with a D6 at one pole only. In section 5.4.3 we will then obtain regular solutions with D8-branes.

5.4.1 Warm-up: review of the $F_0 = 0$ solution

We will warm up by reviewing the solution one can get for $F_0 = 0$.

As we remarked in section 5.3.1, in the massless case one can always lift to eleven-dimensional supergravity, and there we can only have $\mathrm{AdS}_7 \times S^4$ (or an orbifold thereof). The metric simply reads

$$ds^2_{11} = R^2 \left(ds^2_{\mathrm{AdS}_7} + \frac{1}{4} ds^2_{S^4} \right) , \tag{5.4.1}$$

being R an overall radius. Let us now have a look at how this reduces to IIA. It is not obvious whether the reduction will preserve any supersymmetry; but, as we will now see, this can be arranged.

To reduce, we have to choose an isometry. Since S^4 has Euler characteristic $\chi = 2$, like any even-dimensional sphere, any vector field has at least two zeros, and so our reduction will have at least two points where the dilaton goes to zero; we expect some other strange feature at those two points, and as we will see this expectation is borne out.

How should we choose the isometry? We can think about U(1) isometries on S^d as rotations in $\mathbb{R}^{d+1}$. The infinitesimal generator v is an element of the Lie algebra $\mathfrak{so}(d+1)$, namely an antisymmetric $(d+1) \times (d+1)$ matrix v. Moreover, two such elements v_i that can be related by conjugation, $v_1 = O v_2 O^t$, for $O \in \mathrm{SO}(d+1)$, can be thought of as equivalent. Any antisymmetric matrix can be put in a canonical block-diagonal form where every block is of the form $\left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix} \right)$, with a an angle. For even d, this implies that there is at least one zero eigenvalue, which corresponds to the fact that there is no vector field without zeros on the sphere. For $d = 4$, we have two angles a_1 and a_2. Our solution can be reduced along any of these vector fields, but we also want the reduction to preserve some supersymmetry. The infinitesimal spinorial action of the vector field we just described is proportional to $a_1 \gamma_{12} + a_2 \gamma_{34}$. If we demand that this matrix annihilates at least one spinor χ (so that, at the finite level, χ is kept invariant), we get either $a_1 = a_2$ or $a_1 = -a_2$.

To make things more concrete, let us introduce a coordinate system on S^4 adapted to the isometry we just found:

$$ds^2_{S^4} = d\alpha^2 + \sin^2(\alpha) ds^2_{S^3} = d\alpha^2 + \sin^2(\alpha) \left(\frac{1}{4} ds^2_{S^2} + (dy + C_1)^2 \right) , \quad dC_1 = \frac{1}{2} \mathrm{vol}_{S^2} \tag{5.4.2}$$

with $\alpha \in [0, \pi]$. We have written the S^3 metric as a Hopf fibration over S^2; the $1/4$ is introduced so that all spheres have unitary radius. The reduction will now proceed along the vector

$$\partial_y . \tag{5.4.3}$$

We can actually generalize this a bit by considering the orbifold $S^4/\mathbb{Z}_k$, where $\mathbb{Z}_k$ is taken to be a subgroup of the U(1) generated by ∂_y. This is equivalent to multiplying the $(dy + C_1)^2$ term in (5.4.2) by $\frac{1}{k^2}$.

We can now reduce the eleven-dimensional metric (5.4.1), quotiented by the $\mathbb{Z}_k$ we just mentioned, using the string-frame reduction $ds^2_{11} = e^{-\frac{2}{3}\phi} ds^2_{10} + e^{\frac{4}{3}\phi} (dy + C_1)^2$. We obtain a metric of the form (5.1.6), with

$$e^{2A} = R^2 e^{\frac{2}{3}\phi} = \frac{R^3}{2k} \sin(\alpha) , \qquad ds^2_{M_3} = \frac{R^3}{8k} \sin(\alpha) \left(d\alpha^2 + \frac{1}{4} \sin^2(\alpha) ds^2_{S^2} \right) . \tag{5.4.4}$$

We could now also reduce the Killing spinors on S^4, which are given in appendix C.2 in our coordinates. There are indeed two of them which can be reduced, confirming our earlier arguments. This would allow us to compute directly the $\psi^{1,2}$. We will instead proceed by using the equations we derived in section 5.3. It is actually more convenient, in this case, to work directly with the system (5.3.13), that can be more easily solved explicitly:

$$x = \sqrt{1 - e^{4(A-A_0)}} \;, \qquad \phi = 3A - \phi_0 \tag{5.4.5}$$

where A_0 and ϕ_0 are two integration constants. This can be seen to be the same as (5.4.4) by taking

$$x = \cos(\alpha) \;, \qquad A_0 = \frac{1}{2}\log\left(\frac{R^3}{2k}\right) \;, \qquad \phi_0 = 3\log R \;. \tag{5.4.6}$$

The fluxes can now be computed from (5.3.9) and (5.3.11):

$$F_2 = -\frac{1}{2}k\mathrm{vol}_{S^2} \;, \qquad H = -\frac{3}{32}\frac{R^3}{k}\sin^3(\alpha)d\alpha \wedge \mathrm{vol}_{S^2} \;; \tag{5.4.7}$$

the B field then can be written as

$$B_2 = \frac{3}{32}\frac{R^3}{k}\left(x - \frac{x^3}{3}\right)\mathrm{vol}_{S^2} + b \tag{5.4.8}$$

where again b is a closed two-form. The simple result for F_2 in (5.4.7) could be expected from the fact that the metric (5.4.2) is an S^1 fibration over S^2 with Chern class $c_1 = -k$.

However, (5.4.4) might appear problematic for two reasons. First of all, the warping function goes to zero at the two poles $\alpha = 0$, $\alpha = \pi$.[13] Second, $ds^2_{M_3}$ would be singular at the poles even if it were not multiplied by an overall factor $e^{2A} = \frac{R^3}{2k}\sin(\alpha)$, because of the $1/4$ in front of $ds^2_{S^2}$. Indeed, when we expand it around, say, $\alpha = 0$, we find $d\alpha^2 + \frac{\alpha^2}{4}ds^2_{S^2}$; this would be regular without the $1/4$, but as it stands it has a conical singularity.

However, these singularities at the poles have the behavior one expects near a D6. Near the north pole $\alpha = 0$, $ds^2_{M_3}$ in (5.4.4) looks like $ds^2_{M_3} \sim \alpha\left(d\alpha^2 + \frac{1}{4}\alpha^2 ds^2_{S^2}\right)$. In terms of the r variable we used in (5.3.16), this looks like

$$ds^2_{M_3} \sim dr^2 + \left(\frac{3}{4}r\right)^2 ds^2_{S^2} \;. \tag{5.4.9}$$

Near the ordinary flat-space D6-brane metric, $ds^2_{M_3} \sim \rho^{-1/2}(d\rho^2 + \rho^2 ds^2_{S^2})$, which also looks like (5.4.9) with $r = \frac{4}{3}\rho^{3/4}$.

The presence of D6's could actually be inferred more directly. First of all, we know that D6-branes result from loci where the size of the eleventh dimension goes to zero; this indeed happens at the two poles. Moreover, from the expression of F_2 in (5.4.7), the integral of F_2 over the S^2 is constant and equal to $-2\pi k$. We can take the S^2 close to the north or the south pole, where it signals the presence of D6-brane charge. More precisely, there are k anti-D6-branes at the north pole and k D6-branes at the south pole.

[13] The warping function also goes to zero at the equator of the AdS$_6 \times S^4$ solution [99], recently shown [109] to be the only AdS$_6$ solution in massive IIA. This solution can also be T-dualized, without breaking supersymmetry, both using its non-abelian and the more usual abelian isometries [104], differently from what we saw for AdS$_7$ in section 5.3.4.

One crucial difference with the usual D6 behavior, however, is the presence of the NS three-form H. From (5.4.7) we see that it does not vanish near the D6. Rather, it diverges: near the anti-D6 at $r = r_{\rm N} = 0$,[14]

$$H \sim r^{-1/3}{\rm vol}_3 \ . \tag{5.4.10}$$

This can also be inferred directly from eleven-dimensional supergravity, using the reduction formula $G_4 = e^{\phi/3} H \wedge e^{11}$. Since $\phi \sim r$, the three-form energy density diverges as $e^{-2\phi}H^2 \sim (r_{\rm N} - r)^{-8/3}$. We should remember, in any case, that this solution is non-singular in eleven dimensions; the diverging behavior in (5.4.10) is cured by M-theory, just like the divergence of the curvature of (5.4.9) is.

The simultaneous presence of D6's and anti-D6's in a BPS solution might look unsettling at first, since in flat space they cannot be BPS together. It is true that the conditions imposed on the supersymmetry parameters ϵ_i by a D6 and by an anti-D6 brane are incompatible. But in flat space the ϵ_i are constant, while in our present case they are not. The condition changes from the north pole to the south pole; so much so that an anti-D6 is BPS at the north pole, and a D6 is BPS at the south pole. Although we have not been working explicitly with spinors, but rather with forms, we can see this by performing a brane probe analysis in the language of calibrations, as we did for D8-branes at the end of section 5.3.8. The relevant polyform is again (5.3.50); for a D6 we should use its zero-form part, which from (5.2.14) is simply $\cos(\theta_1)\sin(\psi) = x$. For a D6 or anti-D6, this should be equal to plus or minus the internal volume form of the D6, which is ± 1; this happens precisely at the north and south pole.

In figure 5.2 we show some parameters for the solution as a function of the r defined in (5.3.16), for uniformity with latter cases. We also show the radius of the transverse sphere, which near the poles has the angular coefficient $3/4$ of (5.4.9).

We have obtained this massless IIA solution by reducing the M-theory solution ${\rm AdS}_7 \times S^4/\mathbb{Z}_k$, but other orbifolds would be possible as well. One could for example have quotiented by the $\hat{D}_{k-2}$ groups, which would have resulted in IIA in an orientifold by the action of the antipodal map on the S^2. The transverse S^2 would have been replaced by an $\mathbb{RP}^2$; at the poles we would have had O6's together with the k D6's/anti-D6's of the A_k case.

We will see in section 5.4.3 solutions with $F_0 \neq 0$ and without any D6-branes. But we will at first try in the next subsection to introduce F_0 without any D8-branes.

5.4.2 Massive solution without D8-branes

In section 5.4.1 we reviewed the only solution for $F_0 = 0$, related to ${\rm AdS}_7 \times S^4$ by dimensional reduction; it has a D6 and an anti-D6 at the poles of $M_3 \cong S^3$.

We now start looking at what happens in presence of a non-zero Romans mass, $F_0 \neq 0$. We saw in section 5.3.7 that in this case it is possible for the poles to be regular points. It remains to be seen whether those boundary conditions can be joined by a solution of the system (5.3.17).

[14] It is interesting to ask what happens in the Minkowski limit. From (5.3.18) we see that $H = -6e^{-A}{\rm vol}_3$; taking $R \to \infty$, e^{-A} tends to zero except than in a region $\alpha \ll R^{-1/3}$, which gets smaller and smaller in the limit.

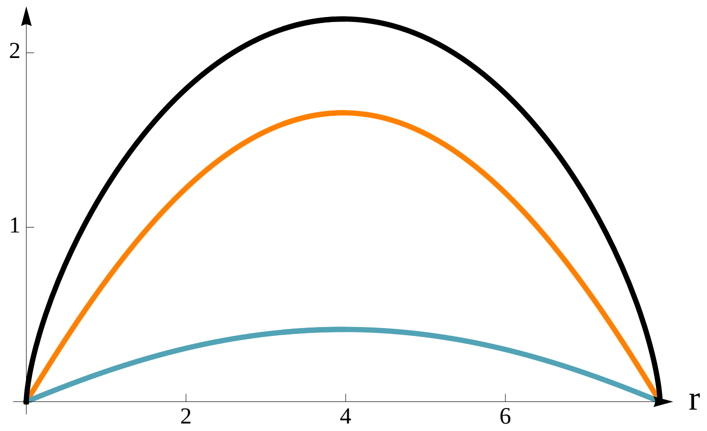


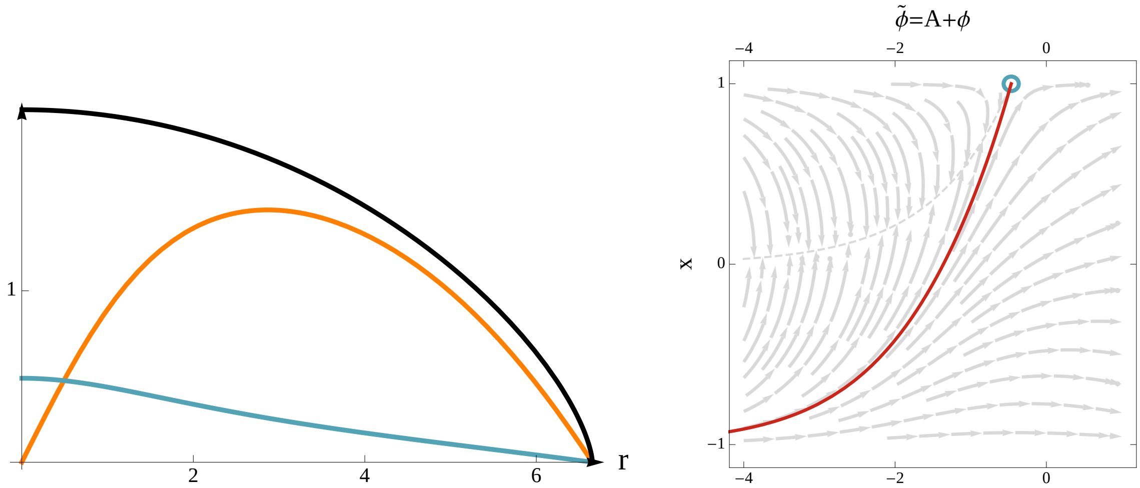
Figure 5.2: Massless solution in IIA. We show here the radius of the S^2 (orange), the warping factor e^{2A} (black; multiplied by a factor $1/20$), and the string coupling e^{ϕ} (green; multiplied by a factor 5). We see that the warping goes to zero at the two poles. The angular coefficient of the orange line can be seen to be $3/4$ as in (5.4.9). The two singularities are due to k D6 and k anti-D6 (in this picture, $k = 20$).

We can for example impose the boundary condition (5.3.25) at $r = r_{\rm N}$, and evolve numerically towards positive r using (5.3.17). The procedure is standard: we use the approximate power-series solution (5.3.26) from $r = r_{\rm N} = 0$ to a very small r, and then use the values of A, ϕ, x thus found as boundary conditions for a numerical evolution of (5.3.17). One example of solution is shown in figure 5.3(a). It stops at a finite value of r, where it resembles there the south pole behavior of the massless case in figure 5.2; for example, e^A goes to zero at the right extremum.

This is actually easy to understand already from the system, both in (5.3.13) and in (5.3.17). As A and ϕ get negative, they suppress the terms containing F_0, and the system tends to the one for the massless case.

An alternative, and perhaps more intuitive, understanding can be found using the form (5.3.13) of the system, which we drew in figure 5.1 as a vector field flow on the space $\{A+\phi, x\}$. The green circle in that figure represents the point $\{A + \phi = \log(4/F_0), x = 1\}$, which is the appropriate boundary condition for the north pole in (5.3.25). In that figure the 'time' variable is A. From (5.3.26), we see that A has a local maximum at $r = r_{\rm N}$. So the stream in figure 5.1 has to be followed backwards, starting from the green circle at the top. We can see that the integral curve asymptotically approaches $x = -1$, but does not get there in finite 'time'; in other words, $A \to -\infty$. The flow corresponding to the solution in figure 5.3(a) is shown in figure 5.3(b).

In the massless case, we saw in section 5.4.1 that the singularities at the poles are actually D6-branes. In this case too we have D6's at the south pole. This is confirmed by considering the integral of F_2 along a sphere S^2 in the limit where it reaches the south pole: it gives a non-zero number. By tuning A_0^+, this can be arranged to be 2π times an integer k, where k is the



(a) (b)

Figure 5.3: Solution for $F_0 = 40/2\pi$. We imposed regularity at the north pole, and evolved towards positive r. In (a) we again plot the radius of the S^2 (orange), the warping factor e^{2A} (black; multiplied by a factor $1/20$), and the string coupling e^{ϕ} (green; multiplied by a factor 5). With increasing r, the plot gets more and more similar to the one for the massless case in figure 5.2. There is a stack of D6's at the south pole (in this picture, $k = 112$ of them), as in the massless case, although this time it also has a diverging NS three-form H. Notice that the size of the S^2 goes linearly near both poles, but with angular coefficients 1 near the north pole (appropriate for a regular point) and $3/4$ for the south pole (appropriate for a D6, as seen in (5.4.9)). In (b), we see the path described by the solution in the $\{A + \phi, x\}$ plane, overlaid to the vector field shown in figure 5.1.

number of D6-branes at the south pole. The presence of these D6-branes without any anti-D6 is not incompatible with the Bianchi identity $dF_2 - HF_0 = k\delta_{D6}$, because integrating it gives $-F_0 \int H = k$. In other words, the flux lines of the D6's are absorbed by H-flux, as is often the case for flux compactifications. Notice also that these D6's are calibrated; the computation runs along similar lines as the one we presented for the massless solution in section 5.4.1.

To be more precise, the singularity is not the usual D6 singularity, in that there is also a NS three-form H diverging as in (5.4.10). This is consistent with the prediction in [139, Eq. (4.15)] (given there in Einstein frame), and in general with the analysis of [140–142], which found that it is problematic to have ordinary D6-brane behavior in a massive $\mathrm{AdS}_7 \times S^3$ set-up precisely like the one we are considering here. (In the language of [140], the parameter α of our solution goes to a negative constant; this enables the solution to exist and to evade the global no-go they found, but at the cost of the diverging H in (5.4.10), [139, Eq. (4.15)].) More precisely, the asymptotic behavior we find is the one discovered in [141, Eq. (3.4)].

Thus the singularity at the south pole in figure 5.3 is the same we found in the massless case we saw in section 5.4.1. In that case, the singularity is cured by M-theory. In the present case, the non-vanishing Romans mass prevents us from doing that. However, we still think it should be interpreted as the appropriate response to a D6; for this reason we think it is a physical solution.

So far we have examined what happens when we impose that the north pole is regular. It is also possible to have a D6 and anti-D6 singularity at both poles, as in the previous section, or an O6 at one of the poles (keeping D6's at the other pole). Roughly speaking, this corresponds to a trajectory similar to the one in figure 5.3(b), in which one "misses" the green circle to the left or to the right, respectively. As we have seen, the D6 solution is very similar to the massless one. The O6 solutions also turn out to be very similar to their massless counterpart:[15] near the pole, their asymptotics is $e^A \sim r^{-1/5}$, $e^\phi \sim r^{-3/5}$, $x \sim 1 - r^{4/5}$. This leads to the same asymptotics for the metric as in the massless O6 solution near the critical radius $\rho_0 = g_s l_s$. Once again, however, in the massive case we have a diverging NS three-form; this time $H \sim r^{-3/5}\mathrm{vol}_3$. Finally, in such a case the S^2 is replaced by an $\mathbb{RP}^2$ because of the orientifold action.

5.4.3 Regular massive solution with D8-branes

We will now examine what happens in presence of D8-branes.

The first possibility that comes to mind is to put all of them together in a single stack. The idea is the following. We once again use the power-series expansion (5.3.26) from $r = r_\mathrm{N} = 0$ to a small r, and use the resulting values of A, ϕ and x as boundary conditions for a numerical evolution of (5.3.17). This time, however, we should stop the evolution at a value of r where (5.3.45) is satisfied. At this point F_0 will change, and (5.3.17) will change as well. Generically, the evolution on the other side of the D8 will lead to a D6 or an O6 singularity, as discussed in section 5.4.2. However, if F_0 is negative, according to (5.3.25), the point $\{x = -1,\, e^{A+\phi} = -\frac{4}{F_0}\}$ leads to a regular South Pole. Fortunately, our solution still has a free parameter, namely $A_0^+ = A(r_\mathrm{N})$. By fine-tuning this parameter, we can try to reach $\{x = -1,\, e^{A+\phi} = -\frac{4}{F_0}\}$ and obtain a regular solution.

Alternatively, after stopping the evolution from the North Pole to the D8, one can look for a similar solution starting from the South Pole, and then match the two — in the sense that one should make sure that A, ϕ, and x are continuous. One combination of them, namely q, will already match by construction. It is then enough to match two variables, say A and x; this can be done by adjusting A_0^+ and A_0^-.

Naively, however, we face a problem when we try to choose the flux parameters on the two sides of the D8's. We concluded in (5.3.31) that near the poles we should have $b = 0$; this seems to imply, via (5.3.40), that $n_2 = 0$ on both sides of the D8. (5.3.45) would then lead to $q = 0$ on the D8, which can only be true at the poles $x = \pm 1$.

This confusion is easily cleared once we remember that B can undergo a large gauge transformation that shifts it by $k\pi\mathrm{vol}_{S^2}$, as we explained towards the end of section 5.3.7. We saw there that we can keep track of this by introducing the variable $\hat{b}$ in (5.3.32). We now simply have to make sure that $\hat{b}$ winds an integer amount of times N around the fundamental domain

[15] In the different set-up of [143], an O6 in presence of F_0 gets modified in such a way that its singularity disappears. This does not happen here.

$[0, \pi]$; this can be interpreted as the presence of N large gauge transformations, or as the presence of a non-zero quantized flux $N = \frac{1}{4\pi^2} \int H$.

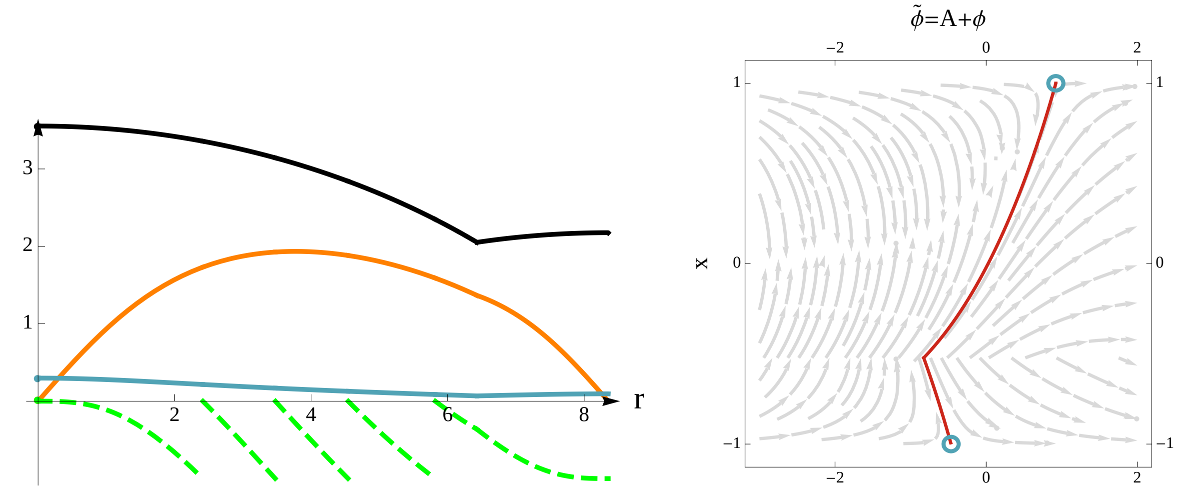
We still face one last apparent problem. It might seem that making sure that $\hat{b}$ winds an integer amount of times requires a further fine-tuning on the solution; this we cannot afford, since we have already used both our free parameters $A_0^{\pm}$ to make sure all the variables are continuous, and that the poles are regular.

Fortunately, such an extra fine-tuning is in fact not necessary. Let us call (n_0, n_2) the flux parameters before the D8, and (n_0', n_2') after it. For simplicity let us also assume $n_2' = 0$, so that no large gauge transformations are needed on that side. As we remarked at the end of section 5.3.8, $\Delta n_2 = n_2' - n_2 = -n_2$ should be an integer multiple of $\Delta n_0 = n_0' - n_0 = n_{\mathrm{D8}}$: $\Delta n_2 = \mu \Delta n_0$, $\mu \in \mathbb{Z}$. To take care of flux quantization, it is enough to also demand that $n_2 = N n_0$ for N integer. Indeed, from (5.3.37), (5.3.40), (5.3.32), we see that in that case at the North Pole we get $\hat{b} = -\pi N$; since this is an integer multiple of π, it can be brought to zero by using large gauge transformations. Together, the conditions we have imposed determine $n_0' = n_0 \left(1 - \frac{N}{\mu}\right)$.

All this gives a strategy to obtain solutions with one D8 stack. We show one concrete example in figure 5.4. One might find it intuitively strange that the D8-branes are not "slipping" towards the South Pole. The branes back-react on the geometry, bending the S^3, much as a rubber band on a balloon. This by itself, however, would not be enough to prevent them from slipping. Rather, we also have to take into account the Wess–Zumino term in the brane action. This term, which takes into account the interaction of the branes with the flux, balances with the gravitational DBI term to stabilize the D8's. The formal check of this is that the branes are calibrated, something we have already seen in section 5.3.8 (see discussion around (5.3.50), (5.3.51)). The D8 stack is made of $n_{\mathrm{D8}} = 50$ D8-branes; each of these D8's has worldsheet flux f_2 such that $\int_{S^2} f_2 = -2\pi$, which means that it has an effective D6-brane charge equal to -1. A single D8/D6 bound state probe with this charge is calibrated exactly at $r = r_{\mathrm{D8}}$, and thus will not slip to the South Pole. The solution can perhaps be thought of as arising from the one in figure 5.3 via some version of Myers' effect. These solutions are consistent with the analysis in [142, 144], or more recently in [145, 146], and they might give some evidence on the polarization of D6-branes solutions into regular solution with D8-branes. See the list of references in [140–142, 144–146] for polarization mechanisms in other set-ups like M-theory or IIB.

We can also look for a configuration with two stacks of D8-branes, again with regular poles. The easiest thing to attempt is a symmetric configuration where the two stacks have the same number of D8's, with opposite D6 charge. As for the solution with one D8, (5.3.25) implies F_0 at the north pole and negative F_0 at the south pole. For our symmetric configuration, these two values will be opposite, and there will be a central region between the two D8 stacks where $F_0 = 0$.

We show one such solution in figure 5.5. As for the previous solution with one D8, we have started from the North Pole and South Pole; now, however, we did not try to match these two solutions directly, but we inserted a massless region in between. From the northern solutions,



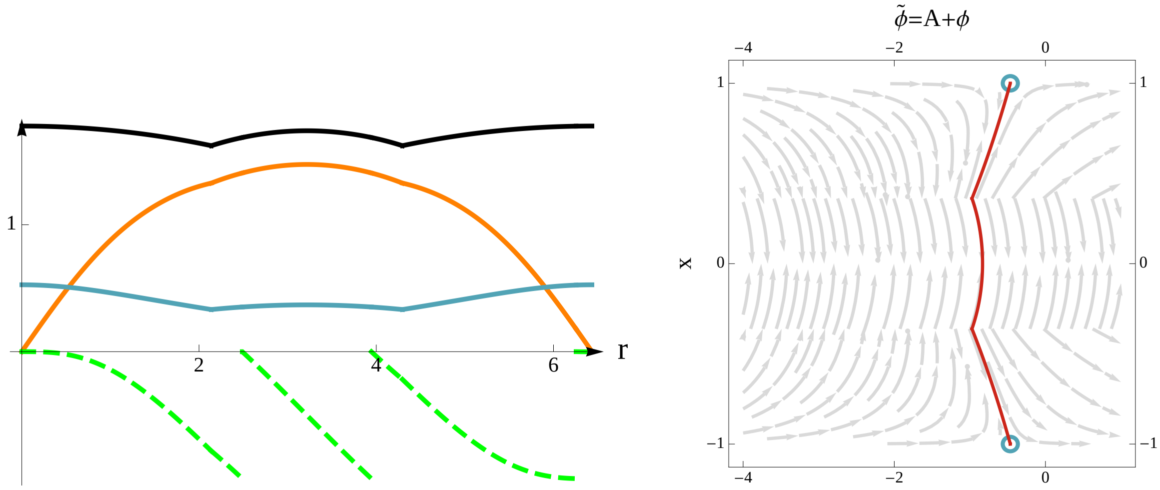
(a) (b)

Figure 5.4: Regular solution with one D8 stack. Its position can be seen in the graph as the value of r where the derivatives of the functions jump; it is fixed by (5.3.45). In (a) we again plot the radius of the S^2 (orange), the warping factor e^{2A} (black; rescaled by a factor $1/20$), and the string coupling e^{ϕ}. We also plot $\frac{1}{\pi}\hat{b}(r) = \frac{1}{4\pi^2}\int_{S^2_r} B_2$ (dashed, light green); to guide the eye, we have periodically identified it as described in section 5.3.7. (The apparent discontinuities are an artifact of the identification.) The fact that it starts and ends at $\hat{b} = 0$ is in compliance with flux quantization for H; we have $\frac{1}{4\pi^2}\int H = -5$. The flux parameters are $\{n_0, n_2\} = \{10, -50\}$ on the left (namely, near the north pole), $\{-40, 0\}$ on the right (near the south pole). In (b), we see the path described by the solution in the $\{A + \phi, x\}$ plane, overlaid to the relevant vector field, that this time changes with n_0.

again we found at which value of $r = r_{\text{D8}}$ it satisfies (5.3.45). We then stopped the evolution of the system there, evaluated A, ϕ, x at r_{D8}, and used them as a boundary condition for the evolution of (5.3.17), now with $F_0 = 0$. Now we matched this solution to the southern one; namely, we found at which values of $r = r_{\text{D8}'}$ their A, ϕ and x matched. This requires translating the southern solution in r by an appropriate amount, and picking $A_0^- = A_0^+$. Given the symmetry of our configuration, this is not surprising: the southern solution is related to the northern one under (5.3.33). Moreover, matching a region with $F_0 \neq 0$ to the massless one means imposing an extra condition, namely the continuity of B in r_{D8}, as we mentioned at the end of 5.3.8.

The parameter $A_0^+ = A_0^- = A_0$ would at this point be still free. However, one still has to impose flux quantization for H. As we recalled above, this is equivalent to requiring that the periodic variable $\hat{b}$ starts and ends at zero. Unlike the case with one D8 above, this time we do need a fine-tuning to achieve this, since the expression for B is not simply controlled by the massive expression (5.3.43). Fortunately we can use the parameter A_0 for this purpose. The solution in the end has no moduli.

As for the solution with one D8 stack we saw earlier, in this case too the D8-branes are not "slipping" towards the North and South Pole because of their interaction with the RR flux: each



(a) (b)

Figure 5.5: Regular solution with two D8 stacks. As in figure 5.4, their positions are the two values of r where the derivatives of the functions jump. In (a) we again plot the radius of the S^2 (orange), the warping factor e^{2A} (black; rescaled by a factor $1/20$), and the string coupling e^{ϕ} (green; rescaled by a factor 5), and $\hat{b}$ (as in figure 5.4; this time $\frac{1}{4\pi^2}\int H = -3$). The flux parameters are: $\{n_0, n_2\} = \{40, 0\}$ on the left (namely, near the north pole); $\{0, -40\}$ in the middle; $\{-40, 0\}$ on the right (near the south pole). The region in the middle thus has $F_0 = 0$; it is indeed very similar to the massless case of figure 5.2. In (b), we see the path described by the solution in the $\{A + \phi, x\}$ plane, overlaid to the relevant vector field, that again changes with n_0.

of the two stacks is calibrated. In this case, intuitively this interaction can be understood as the mutual electric attraction between the two D8 stacks, which indeed have opposite charge under F_2; the balance between this attraction and the "elastic" DBI term is what stabilizes the branes.

Let us also remark that for both solutions (the one with one D8 stack, and the one with two) it is easy to make sure, by taking the flux integers to be large enough, that the curvature and the string coupling e^{ϕ} are as small as one wishes, so that we remain in the supergravity regime of string theory. In figures 5.4 and 5.5 they are already rather small (moreover, in the figure we use some rescalings for visualization purposes).

Thus we have found regular solutions, with one or two stacks of D8-branes. It is now in principle possible to go on, and to add more D8's. We have found examples with four D8 stacks, which we are not showing here. We expect that generalizations with an arbitrary number of stacks should exist, especially if there is a link with the brane configurations in [124, 125]. Another possibility that might also be realized is having an O8-plane at the equator of the S^3.

5.5 Summary of the results and outlook

As in the previous chapter, we reduced the supersymmetric system of equation, given by the vanishing of the supersymmetry variation, to a system of ODEs, by applying the same techniques described in 4. All the fluxes are turned on and they backreact on the internal geometry. We also show that in type IIB there are no solutions, whereas in massless type IIA we got that the solution is the reduction of the known Freund-Rubin AdS$_7 \times S^4$ (and its orbifolds) background of eleven-dimensinal supergravity.

The main and new result is of this chapter is the complete and explicit classification of all the AdS$_7 \times M_3$ solutions in type IIA. Especially the solutions in massive type IIA are new and infinitely many and preserve 16 supercharges in seven dimensions. We reached this goal by solving numerically the ODEs. We determined the fluxes and the metric. The topology of M_3 is the same as a 3-sphere, which is explicitly given by a S^2-fiberation over an interval. All the functions, namely warping, dilaton and radius of the S^2 depend on the interval coordinate r. The S^3 is not exactly round, but is distorted by space-time filling D6-branes/O6-planes at the poles and/or by space-time filling D8-branes/D8-planes wrapping the S^2 inside M_3. Among all the solutions, the subclass with only D8-branes is regular. All these backgrounds are gravity dual to the 6d (1,0) sCFT, described in [125, 147, 148]. One subclass of these backgrounds were previously locally studied in [139–142] in the context of stability of solutions with D6-branes, which manifest a divergent near-brane behaviour of the energy density for the H-flux. The global geometry, together with completely regular new solutions with D8-branes, suggest that the singularity tend to be resolved by a polarization mechanisms of the D6-branes into the D8's [142, 144–146]. Our new results clarify several aspects of the backreaction of branes and fluxes on the geometry of the vacuum solutions and the stability thereof.

It would be very interesting also to rigorously show the supersymmetry breaking of the numerical solutions found in [144], which manifest a modulus that seems to break supersymmetry, and to generalize them to regular non-supersymmetric solution with D8-branes. We would also like to understand physically the supersymmetry breaking mechanism. Finally, starting from our explicit solutions, it would be also nice to study the operator content of the 6d sCFT theories using the AdS/CFT dictionary.

Chapter 6

Supersymmetric AdS_5 solutions of type IIA supergravity

The study of supersymmetric conformal field theories (CFT) in four dimensions using holography is by now a venerable subject. Their holographic duals are AdS$_5$ solutions in either IIB supergravity or M-theory. A comprehensive analysis of supersymmetric AdS$_5$ solutions of IIB supergravity was carried out in [119]; these include the Freund-Rubin compactifications and the Pilch-Warner solution [149], see also [129, 150]. Analogous studies were performed for $\mathcal{N} = 1$ [155] and $\mathcal{N} = 2$ [151] supersymmetric AdS$_5$ backgrounds of M-theory, where new analytic solutions were found. AdS$_5$ solutions in M-theory usually have a higher-dimensional origin: they are compactifications ("twisted" in a certain way) of CFT's in six dimensions. Actually this latter CFT is essentially always the $(2, 0)$ theory living on the world-volume of M5-branes, as in [152] (and in the more recent examples [153–155]).

In chapter 5 we have classified AdS$_7$ solutions in type II supergravity. A new infinite class of solutions was found in massive IIA: the internal space M_3 is always topologically an S^3, but its shape is not round — rather, it is a fiberation of a round S^2 over an interval.[1] Both D6's and D8's can be present (and, a bit more exotically, O6's and O8's). The CFT duals of these solutions are $(1, 0)$-supersymmetric theories, which were argued in [147] to be the ones obtained in [124, 125] from NS5-D6-D8 configurations (see also [156, 157] for earlier related theories). A similar class of $(1, 0)$ theories can be found in F-theory [148, 158].

This prompts the question of whether these $(1, 0)$ theories, when compactified on a Riemann surface, can also give rise to CFTs in four dimensions. If so, their duals should be AdS$_5$ solutions in massive IIA.

In this chapter we classify AdS$_5$ solutions of massive IIA, and we find many analytic examples. The new (and physically sensible) ones are in bijective correspondence with the AdS$_7$ solutions and they have 8 real supercharges preserved in 5 dimensions; this strongly suggests that their dual CFT$_4$ are indeed twisted compactifications of the $(1, 0)$ CFT$_6$. The correspondence is via a simple universal map, which was directly inspired by the map in [159] from AdS$_4$

[1]This Ansatz was also considered in [139–141, 144], also in a non-supersymmetric setting.

to AdS_7 solutions. At the level of the metric it reads

$$\begin{aligned} &e^{2A}(ds^2_{\mathrm{AdS}_5}+ds^2_{\Sigma_g})+dr^2+e^{2A}v^2ds^2_{S^2} \;\to \\ &\qquad\qquad \sqrt{\frac{4}{3}}\left(\frac{4}{3}e^{2A}ds^2_{\mathrm{AdS}_7}+dr^2+\frac{v^2}{1+3v^2}e^{2A}ds^2_{S^2}\right)\,, \end{aligned} \tag{6.0.1}$$

where A, v are functions of r and Σ_g is a Riemann surface of genus $g\geq 2$. This map is so simple that it also allows us to find analytic expressions for the AdS_7 solutions. For example, the simplest massive AdS_5 solution has metric

$$ds^2=\sqrt{\frac{3}{4}}\frac{n_2}{F_0}\left(\sqrt{\tilde y+2}\,(ds^2_{\mathrm{AdS}_5}+ds^2_{\Sigma_g})+\frac{d\tilde y^2}{4(1-\tilde y)\sqrt{\tilde y+2}}+\frac{1}{9}\frac{(1-\tilde y)(\tilde y+2)^{3/2}}{2-\tilde y}ds^2_{S^2}\right) \tag{6.0.2}$$

with $\tilde y\in[-2,1]$. Its AdS_7 "mother", obtained via the map (6.0.1), reads on the other hand

$$ds^2=\frac{n_2}{F_0}\left(\frac{4}{3}\sqrt{\tilde y+2}\,ds^2_{\mathrm{AdS}_7}+\frac{d\tilde y^2}{4(1-\tilde y)\sqrt{\tilde y+2}}+\frac{1}{3}\frac{(1-\tilde y)(\tilde y+2)^{3/2}}{8-4\tilde y-\tilde y^2}ds^2_{S^2}\right)\,. \tag{6.0.3}$$

Both these solutions have a stack of n_2 D6-branes at $\tilde y=2$, and are regular elsewhere. The D6's can also partially or totally be replaced by several D8-branes, much like in a Myers effect [160]. (In a way, these solutions realize the vision of [161].) Such more complicated solutions are obtained by gluing together copies of (6.0.3), or sometimes also of a more complicated metric that we will see later on.

We start our analysis in complete generality. We use the time-honored trick of reducing the study of AdS_5 solutions to that of $\mathrm{Minkowski}_4$ solutions whose internal space M_6 has a conical isometry. One can then use the general classification of [108], which uses generalized complex geometry on M_6. Due to the conical structure of M_6, the "pure spinor equations" of [108] become a certain new set of equations on M_5. (The idea of applying the pure spinor equations to AdS_5 solutions in this way goes back to [129, 150], where it was applied to IIB solutions.) It is immediately seen that the only possibility that leads to solutions is that of an SU(2) structure on M_6 (where the pure spinors are of so-called type 1 and type 2), which means in turn that there is an identity structure on M_5.

The practical consequence of this is that we can determine the metric on M_5 in full generality. It is a fiberation of a three-dimensional fiber M_3 over a two-dimensional space $\mathcal{C}$. The three-dimensional fiber also has a Killing vector, which is holographically dual to R-symmetry on the field theory side. The fluxes are also fully determined. The independent functions (one function a_2 in the metric, the warping A, and the dilaton ϕ) have to satisfy a total of six PDEs.

The problem simplifies dramatically once we impose what we will call the "compactification Ansatz". This consists in imposing that: 1) The metric of $\mathcal{C}$ is conformally related to that of a surface Σ, which does not depend on the coordinates of the three-dimensional space orthogonal to $\mathcal{C}$ inside M_5. The conformal factor is equal to the warping function e^{2A} in front of the AdS_5 metric; 2) neither A, nor the dilaton ϕ, nor the function a_2 entering the metric and fluxes, depend on the coordinates of Σ. Under this Ansatz, Σ has constant curvature[2] (and we can compactify

[2]For compactifications of $(2,0)$ theories, the fact that Σ has constant curvature was explained in [162].

it to produce a compact Riemann surface Σ_g); the PDEs reduce to only three. Moreover, these PDEs are all polynomial in one of the local coordinates on M_3. Thus they can be in fact reduced to a set of ODEs. At this point the analysis branches out in several possibilities; for each of those, only one ODE survives. In the massless case, there is a "generic case", which is the reduction to IIA of the BBBW solution [154, 163], and two special cases being the reduction of the $\mathcal{N} = 1$ Maldacena–Núñez solution [152] and the INST solution [164]. In the massive case, we get new solutions. Again there is a generic case and two special cases. In the generic case, we solve the ODE explicitly, but the solution appears not to be physically sensible: it has singularities which we cannot interpret. The first special case, with positive curvature on Σ_g, again has singularities. The second, with constant negative curvature[3] on Σ_g, leads to physically sensible solutions.

Solving the ODE produces several solutions, of which (6.0.2) is the simplest. Without D8's, the most general solution has either two D6 stacks (unlike (6.0.2), which has one), or one D6 stack and one O6. As we already mentioned, there is also the possibility of introducing D8's, which can be done by gluing together copies of (6.0.2), of the Maldacena–Núñez solution, and possibly also of the more complicated solution we just mentioned. As we also anticipated, the map (6.0.1) can then be used to produce analytical expressions for all the AdS$_7$ solutions in 5 and [147].

All these new explicit solutions are begging further investigation, particularly regarding their field theory interpretation. This might be the beginning of a correspondence between CFT$_6$ and CFT$_4$ similar to the celebrated class S theories [96] (although notice that we do not discuss Riemann surfaces with punctures here, as was done in [153]). A feature that those theories also had is that (at the supergravity level) the ratio of the number of degrees of freedom in four and six dimensions is proportional to $g - 1$, just like for [153] (and for [154]); this is a simple consequence of the map (6.0.1). We compute the central charges for the CFT$_6$ in a couple of simple cases; for example, for a symmetric solution with two D8's. Along with the NSNS flux integer N, there is also another flux integer μ, which is basically the D6 charge of the D8's; the number of degrees of freedom is a simple cubic polynomial in N and μ, and agrees with an earlier approximate computation in [147]. It would be interesting to also compute contributions from stringy corrections, which we have not done here.

6.1 The conditions for supersymmetry

In this section, we will derive a system of differential equations on forms in five dimensions that is equivalent to preserved supersymmetry for solutions of the type $\mathrm{AdS}_5 \times M_5$. We will derive it by considering AdS_5 as a warped product of Mink_4 and $\mathbb{R}$. We will begin in section 6.1.1 by reviewing a system equivalent to supersymmetry for $\mathrm{Mink}_4 \times M_6$. In section 6.1.2 we will then translate it to a system for $\mathrm{AdS}_5 \times M_5$.

[3]Compactifying on T^2 the NS5–D6–D8 configurations of [124, 125] and T-dualizing twice should lead to the NS5–D4–D6 system of [165]; the holographic dual to those solutions was found in [166].

6.1.1 $\mathrm{Mink}_4 \times M_6$

Preserved supersymmetry for $\mathrm{Mink}_4 \times M_6$ was found [108] to be equivalent to the existence on M_6 of an $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure satisfying a set of differential equations. The system is described by a pair of pure spinors

$$\phi_- \equiv e^{-A_6} \chi_1^+ \otimes \chi_2^{-\,\dagger} \,, \qquad \phi_+ \equiv e^{-A_6} \chi_1^+ \otimes \chi_2^{+\,\dagger} \,, \tag{6.1.1}$$

where the warping function A_6 is defined by

$$ds_{10}^2 = e^{2A_6} ds^2_{\mathrm{Mink}_4} + ds^2_{M_6} \,, \tag{6.1.2}$$

and the $\pm$ superscripts indicate the chirality of χ_1 and χ_2. The pure spinors ϕ_- and ϕ_+ can be expressed as a sum of odd and even forms respectively, via application of the Fierz expansion and the Clifford map

$$dx^{m_1} \wedge \cdots \wedge dx^{m_k} \to \gamma^{m_1 \ldots m_k} \,. \tag{6.1.3}$$

The system of differential equations equivalent to supersymmetry for type IIA supergravity reads:

$$d_H\big(e^{2A_6-\phi}\mathrm{Re}\phi_-\big) = -\frac{c_-}{16} F \,, \tag{6.1.4a}$$

$$d_H\big(e^{3A_6-\phi}\phi_+\big) = 0 \,, \tag{6.1.4b}$$

$$d_H\big(e^{4A_6-\phi}\mathrm{Im}\phi_-\big) = -\frac{c_+ e^{4A_6}}{16} *_6 \lambda F \,. \tag{6.1.4c}$$

Here, ϕ is the dilaton, $d_H = d - H\wedge$ is the twisted exterior derivative and $c_\pm$ are constants such that

$$\|\chi_1\|^2 \pm \|\chi_2\|^2 = c_\pm e^{\pm A_6} \,. \tag{6.1.5}$$

F is the internal Ramond-Ramond flux which determines the external flux via self-duality:

$$F_{(10)} \equiv F + e^{6A_6} \mathrm{vol}_4 \wedge *_6 \lambda F \,. \tag{6.1.6}$$

λ is an operator acting on a p-form F_p as $\lambda F_p = (-1)^{\left[\frac{p}{2}\right]} F_p$, where square brackets denote the integer part.

6.1.2 $\mathrm{AdS}_5 \times M_5$

As we anticipated, we will now use the fact that anti-de Sitter space can be treated as a warped product of Minkowski space with a line. We would like to classify solutions of the type $\mathrm{AdS}_5 \times M_5$. These in general will have a metric[4]

$$ds_{10}^2 = e^{2A} ds^2_{\mathrm{AdS}_5} + ds^2_{M_5} \,. \tag{6.1.7}$$

[4]Here $ds^2_{\mathrm{AdS}_5}$ is the unit radius metric on AdS_5.

Since

$$ds^2_{\mathrm{AdS}_5} = \frac{d\rho^2}{\rho^2} + \rho^2 ds^2_{\mathrm{Mink}_4} , \tag{6.1.8}$$

ds^2_{10} in equation (6.1.7) can be put in the form of equation (6.1.2) if we take

$$e^{A_6} = \rho e^A , \qquad ds^2_{M_6} = \frac{e^{2A}}{\rho^2} d\rho^2 + ds^2_{M_5} . \tag{6.1.9}$$

In order to preserve the $\mathrm{SO}(4,2)$ invariance of AdS_5, A should be a function of M_5. In addition, the fluxes F and H, which in subsection 6.1.1 were arbitrary forms on M_6, should now be forms on M_5. For IIA, $F = F_0 + F_2 + F_4 + F_6$; in order not to break $\mathrm{SO}(4,2)$, we impose $F_6 = 0$.

Following the decomposition of the geometry of M_6 we wish to decompose the system of equations (6.1.4) so as to obtain the system equivalent to preserved supersymmetry for $\mathrm{AdS}_5 \times M_5$. We start by decomposing the generators of $\mathrm{Cliff}(6)$ as

$$\gamma^{(6)}_\rho = \frac{e^A}{\rho} 1 \otimes \sigma_1 , \qquad \gamma^{(6)}_m = \gamma_m \otimes \sigma_2 , \qquad m = 1, \ldots, 5 \tag{6.1.10}$$

where σ_1, σ_2 are the Pauli matrices and γ_m generate $\mathrm{Cliff}(5)$. Accordingly, the chirality matrix $\gamma^{(6)}_7 = 1 \otimes \sigma_3$ and the chiral spinors χ^+_1, χ^-_2 are decomposed in terms of $\mathrm{Spin}(5)$ spinors η_1, η_2 as

$$\chi^+_1 = \sqrt{\frac{\rho}{2}} \eta_1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad \chi^-_2 = \sqrt{\frac{\rho}{2}} \eta_2 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{6.1.11}$$

ϕ_- and ϕ_+ now read

$$\phi_- = \frac{1}{2}\left(\frac{e^A}{\rho} d\rho \wedge \psi^1_+ + i\psi^1_-\right) , \qquad \phi_+ = \frac{1}{2}\left(-i\frac{e^A}{\rho} d\rho \wedge \psi^2_- + \psi^2_+\right) , \tag{6.1.12}$$

where

$$\psi^1 \equiv e^{-A} \eta_1 \otimes \eta_2^\dagger , \qquad \psi^2 \equiv e^{-A} \eta_1 \otimes \overline{\eta_2} . \tag{6.1.13}$$

The bar is defined as $\overline{\eta} \equiv (\eta^c)^\dagger \equiv (B\eta^*)^\dagger = -\eta^t B$, where B is a conjugation matrix that in five Euclidean dimensions can be taken to satisfy $B^* = B$, $B^t = -B$, $B^2 = BB^* = -1$. The subscripts plus and minus on ψ^1, ψ^2 refer to taking the even and odd form part respectively, in their expansion as forms. One should keep in mind here a comment about odd dimensions: the Clifford map (6.1.3) is not injective. Rather, a form ω and its cousin $*\lambda\omega$ are mapped to the same bispinor (recall the definition of λ right after (6.1.6)). Thus a bispinor can always be expressed both as an even and as an odd form, and in particular we have

$$\psi^{1,2}_- = *\lambda \psi^{1,2}_+ . \tag{6.1.14}$$

Applying the decomposition (6.1.12) to equations (6.1.4) we obtain a *necessary and sufficient* system of equations for supersymmetric $\mathrm{AdS}_5 \times M_5$ solutions:

$$d_H\big(e^{3A-\phi}\mathrm{Re}\psi^1_+\big) + 2e^{2A-\phi}\mathrm{Im}\psi^1_- \;=\; 0\,, \tag{6.1.15a}$$

$$d_H\big(e^{4A-\phi}\psi^2_-\big) - 3ie^{3A-\phi}\psi^2_+ \;=\; 0\,, \tag{6.1.15b}$$

$$d_H\big(e^{4A-\phi}\mathrm{Re}\psi^1_-\big) \;=\; 0\,, \tag{6.1.15c}$$

$$d_H\big(e^{5A-\phi}\mathrm{Im}\psi^1_+\big) - 4e^{4A-\phi}\mathrm{Re}\psi^1_- \;=\; \frac{c_+}{8}e^{5A} * \lambda F\,. \tag{6.1.15d}$$

We also obtain the condition $c_- = 0$; it follows that the relation $\|\chi_1\|^2 \pm \|\chi_2\|^2 = c_\pm e^{\pm A_6}$ becomes

$$\|\eta_1\|^2 = \|\eta_2\|^2 = \frac{1}{2}c_+ e^A\,. \tag{6.1.16}$$

Henceforth, without loss of generality, we set $c_+ = 2$.

The stabilizer group $\mathcal{G} \in \mathrm{Spin}(5)$ of η_1 and η_2 can be either SU(2) or the identity group. In the next section we parametrize ψ^1, ψ^2 in terms of these structures. We will see however that only the identity case leads to supersymmetric solutions. An identity structure is actually a choice of vielbein; so we will end up parameterizing the ψ^1 and ψ^2 in terms of a vielbein.

6.1.3 Parametrization of ψ^1, ψ^2 and the identity structure

We first consider the case where there is only one spinor, $\eta_1 = \eta_2$ of norm $e^{\frac{A}{2}}$. In five dimensions it defines an SU(2) structure. This can be read off from the Fierz expansions of $\eta_1 \otimes \eta_1^\dagger$ and $\eta_1 \otimes \overline{\eta_1}$, which as remarked in (6.1.14) can be written both as even and as odd forms:

$$\begin{aligned} \psi^1_+ &= \frac{1}{4}e^{-ij}\,, \qquad & \psi^2_+ &= \frac{1}{4}\omega\,, \\ \psi^1_- &= \frac{1}{4}v \wedge e^{-ij}\,, \qquad & \psi^2_- &= v \wedge \omega\,. \end{aligned} \tag{6.1.17}$$

Application of Fierz identities yields

$$v\eta_1 = \eta_1 \tag{6.1.18}$$

and the following set of algebraic constraints on the 1-form v and 2-forms j and ω:

$$\begin{gathered} \iota_v v = 1\,, \qquad \iota_v j = \iota_v \omega = 0 \\ j \wedge \omega = 0\,, \qquad \omega \wedge \omega = 0\,, \qquad \omega \wedge \overline{\omega} = 2j \wedge j = \mathrm{vol}_4, \end{gathered} \tag{6.1.19}$$

where vol_4 is the volume form on the four-dimensional subspace orthogonal to v. This set of forms and constraints define precisely an SU(2) structure in five dimensions.

In this case, however, the two-form part of (6.1.15b) tells us $\psi^2 = 0$, which is only possible for $\eta_1 = 0$. Hence, there are no supersymmetric $\mathrm{AdS}_5 \times M_5$ solutions in type IIA supergravity with an $SU(2)$ structure on M_5.

Let us then consider the case of two spinors η_1 and η_2, which as mentioned earlier define an identity structure. We can expand η_2 in terms of η_1 as

$$\eta_2 = a\eta_1 + a_0\eta_1^c + \frac{1}{2}b\overline{w}\,\eta_1 \ , \tag{6.1.20}$$

where $a,\ a_0 \in \mathbb{C}$, $b \in \mathbb{R}$ and w is a complex vector that we normalize such that $w\cdot\overline{w} = 2$ (so that $\mathrm{Re}w$ and $\mathrm{Im}w$ are orthogonal and have norm 1). Also, by redefining if necessary $a \to a+\frac{b}{2}\overline{w}\cdot v$, $w \to w - (w\cdot v)v$ (which leaves (6.1.20) invariant, upon using (6.1.18)), we can assume

$$w \cdot v = 0 \ . \tag{6.1.21}$$

Now (6.1.16) implies

$$|a|^2 + |a_0|^2 + b^2 = 1 \ . \tag{6.1.22}$$

The identity structure is then spanned by v, w and

$$u \equiv \frac{1}{2}\iota_{\overline{w}}\,\omega \ , \tag{6.1.23}$$

in terms of which

$$\omega = w \wedge u \ , \qquad -ij = \frac{1}{2}(w\wedge\overline{w} + u\wedge\overline{u}) \ . \tag{6.1.24}$$

From (6.1.19) we now see that u is also orthogonal to v, as well as to w and $\overline{w}$; moreover, it satisfies $u\cdot\overline{u} = 2$. In other words,

$$\{v, \mathrm{Re}w, \mathrm{Im}w, \mathrm{Re}u, \mathrm{Im}u\} \tag{6.1.25}$$

are a vielbein.

We can now expand ψ^1 and ψ^2 in terms of this vielbein. We separate out their even and odd parts:

$$\begin{aligned}
\psi^1_+ &= \frac{1}{4}\overline{a}\exp\left[-ij + \frac{b}{\overline{a}}v\wedge w\right] \ , \qquad \psi^2_+ = -\frac{a_0}{4}\exp\left[-ij + \frac{u}{a_0}\wedge(a_+ w - bv)\right] \ , \\
\psi^1_- &= \frac{1}{4}(\overline{a}v + bw)\wedge e^{-ij} \ , \qquad \psi^2_- = -\frac{1}{4}(a_0 v + bu)\wedge\exp\left[-ij + \frac{u}{a_0}\wedge(a_+ w - bv)\right] \ .
\end{aligned} \tag{6.1.26}$$

6.2 Analysis of the conditions for supersymmetry

Having obtained the expansions (6.1.26) of ψ^1, ψ^2 in terms of the identity structure on M_5, we can proceed with the study of the system (6.1.15). In section 6.2.1 we study the constraints imposed on the geometry of M_5 while in section 6.2.2 we obtain the expressions of the fluxes in terms of the geometry. The analysis in 6.2.1 is *local*.

6.2.1 Geometry

The equations of the system (6.1.15) which constrain the geometry of M_5 are (6.1.15a), (6.1.15b) and (6.1.15c) with the exception of the three-form part of (6.1.15a) which determines H. In the following study of these constraints, it is convenient to introduce the notation

$$a \equiv a_1 + ia_2 \ , \qquad k_1 \equiv \bar{a}v + bw \ , \qquad k_2 \equiv -bv + aw \ . \tag{6.2.1}$$

The zero form part of (6.1.15b), the one-form part of (6.1.15a), the two-form part of (6.1.15c) and the two-form part (6.1.15b) yield the following set of equations:

$$\begin{aligned} a_0 &= 0 \ , &(6.2.2a)\\ d\big(e^{3A-\phi}a_1\big) + 2e^{2A-\phi}\mathrm{Im}k_1 &= 0 \ , &(6.2.2b)\\ d\big(e^{4A-\phi}\mathrm{Re}k_1\big) &= 0 \ , &(6.2.2c)\\ d\big(e^{4A-\phi}bu\big) - 3ie^{3A-\phi}u\wedge k_2 &= 0 \ . &(6.2.2d) \end{aligned}$$

It can then be shown that the higher-form parts of (6.1.15a), (6.1.15b) and (6.1.15c) follow from the above equations.

(6.2.2a) simplifies quite a bit (6.1.26), which now becomes

$$\begin{aligned} &\psi_+^1 = \frac{1}{4}\bar{a}\exp\left[-ij + \frac{b}{\bar{a}}v\wedge w\right] \ , \qquad \psi_+^2 = \frac{1}{4}(aw - bv)\wedge u\wedge e^{-ij} \ , \\ &\psi_-^1 = \frac{1}{4}(\bar{a}v + bw)\wedge e^{-ij} \ , \qquad \psi_-^2 = -\frac{1}{4}bu\wedge\exp\left[-ij - \frac{a}{b}v\wedge w\right] \ . \end{aligned} \tag{6.2.3}$$

It is also interesting to see what the pure spinors $\phi_\pm$ on M_6 look like:

$$\phi_+ = \frac{1}{4}E_1\wedge E_2\wedge\exp\left[\frac{1}{2}E_3\wedge\overline{E_3}\right] \ , \qquad \phi_- = E_3\wedge\exp\left[\frac{1}{2}(E_1\wedge\overline{E_1} + E_2\wedge\overline{E_2})\right] \ , \tag{6.2.4}$$

where

$$E_1 \equiv ie^A b\frac{d\rho}{\rho} + aw - bv \ , \qquad E_2 \equiv u \ , \qquad E_3 \equiv e^A\bar{a}\frac{d\rho}{\rho} + i(\bar{a}v + bw) \ . \tag{6.2.5}$$

(6.2.4) are the canonical forms of a type 1 – type 2 pure spinor pair (where the “type” of a pure spinor is the lowest form appearing in it); or, in other words, of a pure spinor pair associated with an SU(2) structure on M_6 (although remember that the structure on M_5 is the identity). It would be interesting to push this further, and to start an analysis similar to the one in [129]: in that paper, the language of generalized complex geometry is used to set up a generalized reduction procedure, which eventually leads to a set of four-dimensional equations.

Let us now go back to (6.2.2). Given (6.2.2a), equation (6.1.22) becomes

$$a_1^2 + a_2^2 + b^2 = 1 \ . \tag{6.2.6}$$

Equations (6.2.2b) and (6.2.2c) can be integrated by introducing local coordinates y,

$$y = -\frac{1}{2} e^{3A-\phi} a_1 \; , \tag{6.2.7}$$

and x such that

$$\mathrm{Im} k_1 = e^{-2A+\phi} dy \; , \qquad \mathrm{Re} k_1 = e^{-4A+\phi} dx \; . \tag{6.2.8}$$

M_5 possesses an abelian isometry generated by the Killing vector

$$\xi \equiv \frac{1}{2} (\eta_1^\dagger \gamma^m \eta_2 - \eta_2^\dagger \gamma^m \eta_2) \partial_m = -e^A b (\mathrm{Re} k_2)^\sharp \tag{6.2.9}$$

where $m = 1, \ldots, 5$ and the $\sharp$ superscript denotes the vector dual to the one-form it acts on. A straightforward way to show that ξ is a Killing vector is to work directly with the supersymmetry variations (see appendix D.1) which yield $D_{(m} \xi_{\nu)} = 0$ and $\mathcal{L}_\xi \phi = \mathcal{L}_\xi A = 0$, where D is the spin connection associated to the Levi-Civita connection and $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ. It would be interesting to show this directly using the language of generalized complex geometry, and to make contact with the analysis in [129, 167].

Expressing w, v in terms of $\mathrm{Re} k_2$, $\mathrm{Re} k_1$, and $\mathrm{Im} k_1$ we can write the metric on M_5 as

$$ds^2_{M_5} = ds^2_{\mathcal{C}} + (\mathrm{Re} k_2)^2 + \frac{e^{-4A+2\phi}}{b^2} \left[(b^2 + a_2^2) e^{-4A} dx^2 + (b^2 + a_1^2) dy^2 + 2 a_1 a_2 e^{-2A} dx dy \right] \; , \tag{6.2.10}$$

where $ds^2_{\mathcal{C}} = u \bar{u}$, and $\mathcal{C}$ denotes the two-dimensional subspace spanned by u.

Let us introduce local coordinates x^I, $I = 1, 2, 3$ such that

$$ds^2_{\mathcal{C}} + (\mathrm{Re} k_2)^2 = g_{IJ}(x^I, x, y) dx^I dx^J \; . \tag{6.2.11}$$

ϕ, A and a_2 are in principle functions of x^I, x and y. Given the fact that $\mathcal{L}_\xi \mathrm{Re} k_1 = \mathcal{L}_\xi \mathrm{Im} k_1 = 0$[5], we can further introduce a coordinate $x^3 \equiv \psi$ adapted to the the Killing vector

$$\xi = 3 \partial_\psi \; , \tag{6.2.12}$$

in terms of which

$$\mathrm{Re} k_2 = -\frac{1}{3} e^A b D\psi \; , \qquad D\psi \equiv d\psi + \rho \; , \qquad \rho = \rho_i(x^i, x, y) dx^i \; . \tag{6.2.13}$$

where x^i, $i = 1, 2$ are local coordinates on $\mathcal{C}$. Thus

$$g_{IJ}(x^I, x, y) dx^I dx^J = (g_{\mathcal{C}})_{ij}(x^i, x, y) dx^i dx^j + \frac{1}{9} e^{2A} b^2 D\psi^2 . \tag{6.2.14}$$

In addition, since ξ is a Killing vector and $\mathcal{L}_\xi \phi = \mathcal{L}_\xi A = 0$, A, ϕ and a_2 are independent of ψ.

The exterior derivative on M_5 can be decomposed as

$$d = d_2 + d\psi \wedge \partial_\psi + dx \wedge \partial_x + dy \wedge \partial_y \; , \tag{6.2.15}$$

[5] Deduced from $\iota_\xi \mathrm{Re} k_1 = \iota_\xi \mathrm{Im} k_1 = 0$ and equation (6.2.8)

where d_2 is the exterior derivative on $\mathcal{C}$. We can thus further refine equation (6.2.2d) as follows:

$$d_2 u = i\rho_0 \wedge u \ , \tag{6.2.16a}$$
$$\partial_\psi u = iu \ , \tag{6.2.16b}$$
$$\partial_x u = f_1 u \ , \tag{6.2.16c}$$
$$\partial_y u = f_2 u \ , \tag{6.2.16d}$$

where

$$\rho_0 \equiv \rho + *_2 d_2 \log\big(be^{4A-\phi}\big) \ , \tag{6.2.17a}$$
$$f_1(x^i, x, y) \equiv -\partial_x \log\big(e^{4A-\phi} b\big) + \frac{3e^{-5A+\phi} a_2}{b^2} \ , \tag{6.2.17b}$$
$$f_2(x^i, x, y) \equiv -\partial_y \log\big(e^{4A-\phi} b\big) + \frac{3e^{-3A+\phi} a_1}{b^2} \ . \tag{6.2.17c}$$

$*_2$ is the Hodge star defined by $g_\mathcal{C}$, such that $*_2 u = -iu$. Integrability of equations (6.2.16) yields the constraints

$$\partial_y f_1 = \partial_x f_2 \tag{6.2.18}$$

and

$$\partial_x \rho_0 = - *_2 d_2 f_1 \ , \tag{6.2.19a}$$
$$\partial_y \rho_0 = - *_2 d_2 f_2 \ . \tag{6.2.19b}$$

We can write $ds^2_\mathcal{C}$ as

$$ds^2_\mathcal{C} = e^{2\varphi(x^i,x,y)}(dx_1^2 + dx_2^2) \ . \tag{6.2.20}$$

The Gaussian curvature or one-half the scalar curvature of $\mathcal{C}$, $\ell(x^i, x, y)$, is

$$\ell(x^i, x, y) = -e^{-2\varphi}(\partial^2_{x_1} + \partial^2_{x_2})\varphi \ . \tag{6.2.21}$$

Equations (6.2.16b) and (6.2.16c), (6.2.16d) are solved by

$$u = e^{\varphi + i\psi}(dx_1 + idx_2) \ , \qquad \partial_x \varphi = f_1 \ , \qquad \partial_y \varphi = f_2 \ . \tag{6.2.22}$$

Equation (6.2.16a) then yields

$$\rho_0 = \partial_{x_2}\varphi dx_1 - \partial_{x_1}\varphi dx_2 \ , \tag{6.2.23}$$

and thus

$$d_2 \rho_0 = \ell(x^i, x, y)\text{vol}_\mathcal{C} \ . \tag{6.2.24}$$

Compatibility of (6.2.24) with (6.2.19a), (6.2.19b) requires that ℓ obey the equations

$$\partial_x \ell + 2f_1 \ell = \Delta_2 f_1 \ , \tag{6.2.25a}$$
$$\partial_y \ell + 2f_2 \ell = \Delta_2 f_2 \ , \tag{6.2.25b}$$

where $\Delta_2 \equiv {d_2}^\dagger d_2 + d_2 {d_2}^\dagger$. The last two equations also follow from (6.2.21), bearing in mind that $\Delta_2 \varphi = -e^{-2\varphi}(\partial^2_{x_1} + \partial^2_{x_2})\varphi$.

6.2.2 Fluxes

In this section we give the expressions for the fluxes in terms of the geometry of M_5. In the following expressions we employ the notation

$$\begin{aligned}\zeta_1 &\equiv \mathrm{Re}(ak_1)^\sharp = -2ye^A\partial_x - a_2e^{2A-\phi}\partial_y\ ,\\ \zeta_2 &\equiv \frac{1}{b^2}\mathrm{Im}(ak_1)^\sharp = a_2e^{4A-\phi}\partial_x - 2ye^{-A}\partial_y\ .\end{aligned} \tag{6.2.26}$$

The NSNS three-form flux H is given by the three-form part of equation (6.1.15a):

$$\begin{aligned}H\ &= d\left(\frac{1}{6y}dx\wedge D\psi + \frac{1}{3}e^A\mathrm{Re}(ak_1)\wedge D\psi + \frac{e^{3A-\phi}a_2}{2y}\mathrm{vol}_\mathcal{C}\right)\\ &\quad - \frac{1}{6y^2}dx\wedge dy\wedge D\psi + \frac{e^{-2A}}{y}dx\wedge\mathrm{vol}_\mathcal{C} + \frac{e^{3A-\phi}a_2}{2y^2}dy\wedge\mathrm{vol}_\mathcal{C}\ ,\end{aligned} \tag{6.2.27a}$$

where $\mathrm{Re}(ak_1) = -2ye^{-7A+2\phi}dx - a_2e^{-2A+\phi}dy$.

The RR fluxes can be computed from equation (6.1.15d):

$$F_0\ =\ -\,4e^{2A-2\phi}b^2\partial_yA - e^{-A}\iota_{\zeta_1}d\big(e^{A-\phi}a_2\big)\ , \tag{6.2.28a}$$

$$\begin{aligned}F_2\ =\ &\big[-4e^{-A-\phi}a_2 + 4e^{4A-2\phi}\partial_xA - e^{-5A}\iota_{\zeta_2}d\big(e^{5A-\phi}a_2\big)\big]\ \mathrm{vol}_\mathcal{C}\\ &+\frac{1}{3}d\big(e^{A-\phi}a_2\big)\wedge D\psi + F_0\frac{1}{3}e^A\mathrm{Re}(ak_1)\wedge D\psi\\ &-\frac{e^{-A}}{b^2}*_2 d_2\big(e^{A-\phi}a_2\big)\wedge\mathrm{Im}(ak_1) + 4e^{-4A}*_2 d_2A\wedge dx\ ,\end{aligned} \tag{6.2.28b}$$

$$\begin{aligned}F_4\ &= \frac{1}{3}\left[e^{-6A}\partial_y\big(e^{5A-\phi}a_2\big)dx - e^{-2A}\partial_x\big(e^{5A-\phi}a_2\big)dy - 4e^{-2A}dy\right]\wedge d\psi\wedge\mathrm{vol}_\mathcal{C}\\ &\quad - \frac{1}{3}\left[4e^{-\phi}a_2 + e^{-4A}\iota_{\zeta_2}d\big(e^{5A-\phi}a_2\big)\right]\mathrm{Re}(ak_1)\wedge d\psi\wedge\mathrm{vol}_\mathcal{C}\\ &\quad - \frac{1}{3}e^{-10A+2\phi}*_2\left[d_2\big(e^{5A-\phi}a_2\big)\right]\wedge dx\wedge dy\wedge D\psi\ ,\end{aligned} \tag{6.2.28c}$$

where $\mathrm{Im}(ak_1) = a_2e^{-4A+\phi}dx - 2ye^{-5A+2\phi}dy$.

The fluxes can also be computed from the expression

$$F = \mathcal{J}_+\cdot d_H(e^{-\phi}\mathrm{Im}\phi_-) \tag{6.2.29}$$

on M_6 [132]. The operator $\mathcal{J}_+\cdot$ is associated with the pure spinor ϕ_+, which can be found in (6.2.4):

$$\mathcal{J}_+\cdot = \frac{i}{2}\sum_{i=1}^{2}(E_i\wedge\overline{E_i}\llcorner - \overline{E_i}\wedge E_i\llcorner) + \frac{i}{2}(E_3\llcorner\overline{E_3}\llcorner + E_3\wedge\overline{E_3}\wedge)\ . \tag{6.2.30}$$

The degree of difficulty of computing the fluxes from (6.2.29) is proportional to the degree of the flux. The opposite is true for computing the fluxes from (6.1.15d).

6.2.3 Bianchi identities

In order to have a complete supersymmetric $\mathrm{AdS}_5 \times M_5$ solution, apart from the conditions for supersymmetry (which imply the equations of motion [34]) the Bianchi identities of the fluxes need to be imposed. In this section we study the latter and the extra constraints that follow from their application.

We start with the Bianchi identity of H i.e. $dH = 0$. We find that it determines

$$d_2\rho = e^{-2A}\big[6 + 12y(\partial_y A - f_2) - 6e^{5A-\phi}a_2(\partial_x A - f_1) + 3\partial_x\big(e^{5A-\phi}a_2\big)\big]\mathrm{vol}_{\mathcal{C}} \, . \tag{6.2.31}$$

Next, we turn to the Bianchi identities of the RR fluxes. The Bianchi identity of F_0 just says that it is a constant. The Bianchi identity of F_2 is

$$dF_2 - F_0\, H = 0 \, . \tag{6.2.32}$$

The non-zero components on the left-hand side are the $dx \wedge \mathrm{vol}_{\mathcal{C}}$ and $dy \wedge \mathrm{vol}_{\mathcal{C}}$ components and imposing that they vanish yields the equations:

$$\begin{aligned}\partial_x\mathcal{Q} + 2f_1\mathcal{Q} - \left[\frac{1}{3}\partial_x\big(e^{A-\phi}a_2\big) - \frac{F_0}{6y}\right] *_2 d_2\rho - F_0\frac{e^{-2A}}{y} \\ + \Delta_2\big(e^{A-\phi}a_2\big)\frac{e^{-5A+\phi}a_2}{b^2} - \Delta_2(e^{-4A}) - d_2\big(e^{A-\phi}a_2\big)\cdot d_2\left(\frac{2e^{-5A+\phi}a_2}{b^2}\right) = 0 \, ,\end{aligned} \tag{6.2.33a}$$

$$\begin{aligned}\partial_y\mathcal{Q} + 2f_2\mathcal{Q} - \frac{1}{3}\partial_y\big(e^{A-\phi}a_2\big) *_2 d_2\rho - F_0\frac{e^{3A-\phi}a_2}{2y^2} \\ - \Delta_2\big(e^{A-\phi}a_2\big)\frac{2e^{-6A+2\phi}y}{b^2} + d_2\big(e^{A-\phi}a_2\big)\cdot d_2\left(\frac{4e^{-6A+2\phi}y}{b^2}\right) = 0 \, .\end{aligned} \tag{6.2.33b}$$

where

$$\mathcal{Q}(x^i, x, y) \equiv -4e^{-A-\phi}a_2 + 4e^{4A-2\phi}\partial_x A - e^{-5A}\iota_{\zeta_2}d\big(e^{5A-\phi}a_2\big) - F_0\frac{e^{3A-\phi}a_2}{2y} \, . \tag{6.2.34}$$

Finally, the Bianchi identity of F_4

$$dF_4 - H \wedge F_2 = 0 \, , \tag{6.2.35}$$

is automatically satisfied.

6.2.4 Summary so far

So far, we have analyzed the constraints imposed by supersymmetry and the Bianchi identities without any Ansatz; let us summarize what we have obtained.

First of all, we have already determined the local form of the metric: (6.2.10), (6.2.13). Most notably, we see the emergence of a Killing vector ξ generating a U(1) isometry, and of a two-dimensional space $\mathcal{C}$. The geometry of $\mathcal{C}$ is constrained by (6.2.16). The S^1 upon which the U(1) acts is fibered over $\mathcal{C}$ with ρ being the connection of the fiberation. The curvature of the connection is given by (6.2.31).

In fact the U(1) isometry is a symmetry of the full solution as it also leaves invariant the fluxes; the latter can be verified by computing the Lie derivative with respect to ξ of the fluxes' expressions as presented in section 6.2.2. This symmetry was to be expected: it is a U(1) R-symmetry corresponding to the R-symmetry of the dual $\mathcal{N} = 1$ field theory. The surface $\mathcal{C}$ is of less immediate interpretation, but already at this stage it seems to suggest that the field theory should be a compactification on $\mathcal{C}$ of a six-dimensional field theory. We will see later that this expectation is indeed borne out for the explicit solutions we will find.

We have also reduced the task of finding solutions to a set of partial differential equations on three functions: a_2, the dilaton ϕ, and the warp factor A, which in general depend on four variables i.e. the coordinates x^i, x, y. Supersymmetry equations alone give us (6.2.18), (6.2.25a), (6.2.25b). Moreover, the fluxes should satisfy the relevant Bianchi identities, which away from sources give the further equations (6.2.28a), (6.2.33a), (6.2.33b). Thus we have a total of six partial differential equations. Solving all of them might seem a daunting task, but we will see in the next section that they simplify dramatically with a simple Ansatz. This will allow us to find many explicit solutions.

6.3 A simple Ansatz

We assume that ϕ and A are functions of y only and that $g_{\mathcal{C}}$ is independent of x i.e. $f_1 = 0$. From equation (6.2.17b) it follows that $a_2 = 0$. The metric becomes

$$ds^2_{M_5} = ds^2_{\mathcal{C}} + \frac{1}{9} e^{2A} b^2 D\psi^2 + e^{-8A+2\phi} dx^2 + \frac{e^{-4A+2\phi}}{b^2} dy^2 \ , \qquad (6.3.1)$$

where now $b^2 = 1 - a_1^2$. Equation (6.2.18) is satisfied trivially while equations (6.2.19a) and (6.2.19b) yield $\partial_x \rho = \partial_y \rho = 0$ (in the present Ansatz $\rho_0 = \rho$). $A(y)$ and $\phi(y)$ are subject to the differential equations coming from the Bianchi identities of F_0 and F_2, and equation (6.2.25b),

$$\partial_y \ell + 2 f_2 \ell = 0 \ . \qquad (6.3.2)$$

ℓ is determined by (6.2.24) and (6.2.31) to be $\ell = 6e^{-2A} + 12 e^{-2A} y(\partial_y A - f_2)$.

We first look at the Bianchi identity of F_2; it yields:

$$F_0(\partial_y A - f_2) = 0 \ , \qquad (6.3.3)$$

so either $F_0 = 0$ or $f_2 = \partial_y A$. We consider the two cases $F_0 = 0$ and $F_0 \neq 0$ separately.

6.3.1 $F_0 = 0$

In this case, from the expression (6.2.28a) for F_0 we conclude that $\partial_y A = 0$ i.e. A is constant which without loss of generality we set to zero. F_2 is zero, as can be seen from its expression (6.2.28c). We thus need to solve equation (6.3.2). This yields the ODE

$$\partial_y f_2 + 2f_2^2 = 0 \, , \tag{6.3.4}$$

which is solved by

$$f_2 = \frac{1}{2}\frac{c}{cy-k} \, , \qquad c, \, k = \text{const.} \, . \tag{6.3.5}$$

Recalling the definition (6.2.17c) of f_2, equation (6.3.5) is in turn solved for

$$e^{2\phi} = \frac{k-cy}{2(c_1 - ky^2)} \, , \qquad c_1 = \text{const.} \, . \tag{6.3.6}$$

Equations (6.2.16) are then solved by

$$u = e^{i\psi}\sqrt{2(k-cy)}\, \widehat{u}(x_1, x_2) \, . \tag{6.3.7}$$

Substituting (6.3.5) and (6.3.7) in (6.2.31) yields

$$d_2\rho = 12k \, \text{vol}_\Sigma \, , \tag{6.3.8}$$

where Σ is the surface spanned by $\widehat{u}$. Its Gaussian curvature is thus $12k$.

This solution was first discovered by Gauntlett, Martelli, Sparks and Waldram [155] (see section 6.5.1), and it is the T-dual of the $\text{AdS}_5 \times Y^{p,q}$ solution in type IIB supergravity.

6.3.2 $F_0 \neq 0$

In this case $f_2 = \partial_y A$; equations (6.2.16) are solved by

$$u = e^{i\psi} e^{A}\, \widehat{u}(x_1, x_2) \, . \tag{6.3.9}$$

$\ell = 6e^{-2A}$ obeys equation (6.3.2) automatically and

$$d_2\rho = 6\text{vol}_\Sigma \, . \tag{6.3.10}$$

Substituting f_2 in (6.2.17b) gives

$$e^{4A} = -\frac{1}{12y}\partial_y \beta \, , \tag{6.3.11}$$

where $\beta(y) \equiv e^{10A-2\phi}b^2$. The Bianchi identity of F_0 becomes then an ODE for ϵ:

$$e^{12A}F_0 = -\beta\, \partial_y e^{4A} \, . \tag{6.3.12}$$

The situation appears promising: we have reduced the problem to the ODE (6.3.12). However, as we will now see, one cannot obtain physical compact solutions to this system.

Let us introduce the coordinate $\tilde{y}$ by $d\tilde{y} = \frac{e^{-2A+\phi}}{b} dy$, so that the metric (6.3.1) contains $d\tilde{y}^2$. (6.3.11) now reads

$$F_0 = 16 e^{-\phi} b^2 \partial_{\tilde{y}} A \ . \tag{6.3.13}$$

In order to obtain a compact solution, we should have the factor in front of the S^1 in (6.3.1), namely $e^A b$, go to zero for some $y = y_0$. For a regular point, this is impossible: since A and ϕ should go to constant at y_0, we should have b go to zero; but from (6.3.13) we see that this is in contradiction with $F_0 \neq 0$. We might think of having a singularity corresponding to a brane, but since only an S^1 would shrink at $y = y_0$, such a brane would be codimension-2; there are no such objects in IIA supergravity.

6.4 A compactification Ansatz

We have reduced the general classification problem to a set of six PDEs. To simplify the problem, we will now make an Ansatz.

We assume that A, ϕ and a_2 are functions of x and y only, and that

$$ds^2_C = e^{2A} ds^2_\Sigma(x_1, x_2) \ . \tag{6.4.1}$$

In other words, the ten-dimensional metric becomes $ds^2_{10} = e^{2A}(ds^2_{\mathrm{AdS}_5} + ds^2_\Sigma) + ds^2_{M_3}$. It will soon follow that Σ has constant curvature; from now on we will assume it to be a compact Riemann surface Σ_g. For $g \geq 1$ this involves a quotient by a discrete subgroup, but since no functions depend on its coordinates, this presents no difficulty.

This Ansatz is motivated by the fact that most known solutions in eleven-dimensional supergravity (and hence in massless IIA) are of this type. We also have in mind our original motivation: finding solutions dual to twisted compactifications of CFT$_6$. If one wants to study a CFT$_6$ on $\mathbb{R}^4 \times \Sigma_g$ rather than on $\mathbb{R}^6$, one needs to replace $ds^2_{\mathrm{AdS}_7} = \frac{d\rho^2}{\rho^2} + \rho^2 ds^2_{\mathbb{R}^6}$ with $\frac{d\rho^2}{\rho^2} + \rho^2(ds^2_{\mathbb{R}^4} + ds^2_\Sigma)$ in the UV, and then look for a solution that represents the flow to the IR. Our Ansatz is basically that in the IR fixed point this metric is only modified in the ρ^2 term multiplying ds^2_Σ, which drops out and becomes a constant.

Whatever its origin, we will now see that this Ansatz is remarkably effective at simplifying the system of PDEs: we will be able to completely classify the resulting solutions. One particular case will be source of many solutions, which will be analyzed in section 6.6.

6.4.1 Simplifying the PDEs

(6.4.1) implies

$$f_1 = \partial_x A \ , \qquad f_2 = \partial_y A \ . \tag{6.4.2}$$

The integrability condition (6.2.18) is then satisfied trivially, while equations (6.2.19a) and (6.2.19b) yield $\partial_x \rho = \partial_y \rho = 0$ (in the present Ansatz $\rho_0 = \rho$).

Equations (6.2.24), (6.2.31) yield $\ell = e^{-2A}[6 + 3\partial_x(e^{5A-\phi}a_2)]$. (6.2.25a), (6.2.25b) are then solved by

$$e^{5A-\phi}a_2 = cx + \epsilon \qquad c = \text{const.}\ , \tag{6.4.3}$$

where $\epsilon = \epsilon(y)$ is a function of y only. It follows that

$$\ell = e^{-2A}(6 + 3c) \tag{6.4.4}$$

i.e. the Gaussian curvature of Σ_g is equal to $6 + 3c$.

Given the definitions (6.2.17b), (6.2.17c), the equations $f_1 = \partial_x A$ and $f_2 = \partial_y A$ become

$$\partial_x\left(e^{10A-2\phi}b^2\right) = 6e^{5A-\phi}a_2\ , \tag{6.4.5a}$$

$$\partial_y\left(e^{10A-2\phi}b^2\right) = 6e^{7A-\phi}a_1\ . \tag{6.4.5b}$$

Recall that $a_1 = -2ye^{-3A+\phi}$ and $b^2 = 1 - a_1^2 - a_2^2$. Using (6.4.3) we can solve these for

$$e^{10A-2\phi} - 4y^2e^{4A} = c(c+3)x^2 + 2(c+3)\epsilon x + \beta\ , \tag{6.4.6a}$$

$$e^{4A} = -\frac{\epsilon'}{2y}x - \frac{1}{12y}\left(\beta' - 2\epsilon\epsilon'\right)\ , \tag{6.4.6b}$$

where $\beta = \beta(y)$ is a function of y only, and a prime denotes differentiation with respect to y.

So far we have solved the differential equations imposed by supersymmetry; we now need to impose the Bianchi identities. First, the expression for F_0, (6.2.28a), becomes

$$e^{12A}F_0 = -[c(c+3)x^2 + 2(c+3)\epsilon x + \beta](e^{4A})' + e^{4A}\partial_y(cx+\epsilon)^2 + 2e^{8A}cy\ . \tag{6.4.7}$$

Recalling (6.4.6b), we see that this equation is polynomial in x, of degree 3. In other words, we can view it as a set of four ODEs in y.

The Bianchi identities for F_2, (6.2.33), become

$$\partial_x^2(e^{6A-2\phi}) = 0\ , \tag{6.4.8a}$$

$$\partial_y\partial_x(e^{6A-2\phi}) + F_0\frac{\epsilon'}{2y} = 0\ . \tag{6.4.8b}$$

Substituting equations (6.4.3) and (6.4.6) in (6.4.8a) yields the differential equation

$$36(\epsilon')^2\beta = -(c+3)(\beta' - 2\epsilon\epsilon')\left[c\beta' - 2(c+6)\epsilon\epsilon'\right]\ . \tag{6.4.9}$$

Notice that the x dependence has dropped out from this equation. Concerning (6.4.8b), just as for (6.4.7), it can be written as a polynomial in x of degree 3, and viewed as four ODEs in y.

So we appear to have reduced the problem to four ODEs from (6.4.7), one from (6.4.8a) (which becomes (6.4.9)), and four from (6.4.8b), for a total of nine ODEs in y. However, many of these ODEs actually happen not to be independent from each other. For example, the x^3 component of both (6.4.7) and (6.4.8b) gives

$$4c(c+3)\left(\frac{\epsilon'}{y}\right)' + F_0\left(\frac{\epsilon'}{y}\right)^3 = 0\ , \tag{6.4.10}$$

as well as the x^2 component of (6.4.8b).

To analyze the remaining ODEs, as a warm-up we will first look at the case $F_0 = 0$, where we will reproduce several known solutions. We will then look at the case $F_0 \neq 0$, which we will further split into a generic case where $\epsilon' \neq 0$, and a special case where $\epsilon' = 0$; both will give rise to new solutions.

6.4.2 $F_0 = 0$

For $F_0 = 0$, (6.4.10) becomes $c(c+3)(\epsilon' - y\epsilon'') = 0$. We can then have either $\epsilon' = y\epsilon''$, $c = -3$, or $c = 0$. In the $c = 0$ case, actually the x^2 coefficient of (6.4.7) gives again $\epsilon' = y\epsilon''$. So this case becomes a subcase of the $\epsilon' = y\epsilon''$ case.

- **Case 1:** $\epsilon' = y\epsilon''$. In this case we have

$$\epsilon = \frac{1}{2}c_1 y^2 + c_2 \, , \qquad c_1,\, c_2 = \text{const.} \, . \tag{6.4.11}$$

The x^3 component of (6.4.7) is (6.4.10), which we just looked at. The x^2 and x^1 components both require

$$\left(\frac{\beta'}{y}\right)' = 2\frac{c+3}{c+6}c_1 y \, . \tag{6.4.12}$$

The solution to this ODE is

$$\beta = \frac{c+6}{c+3}\frac{1}{4}c_1^2 y^4 + \frac{1}{2}c_3 y^2 + c_4 \qquad c_3,\, c_4 = \text{const.} \, . \tag{6.4.13}$$

The x^0 component of (6.4.7) then gives

$$(2c_1c_2 - c_3)(2(c+6)c_1c_2 - cc_3) + \frac{36}{(c+3)}c_1^2 c_4 = 0 \, . \tag{6.4.14}$$

Generically this can be solved for c_4. In this case, the transformation

$$x \to x + \frac{\delta}{c} \, , \qquad c_2 \to c_2 - \delta \, , \qquad \beta \to \beta + \frac{(3+c)(\delta^2 - 2\delta\epsilon)}{c} \tag{6.4.15}$$

leaves the solution invariant and δ can be chosen such that

$$\beta = \frac{c+6}{c+3}\epsilon^2. \tag{6.4.16}$$

This branch reproduces the solution obtained from reduction to ten dimensions of the BBBW AdS$_5$ solution of M-theory [154], as described in section 6.5.4.

This however does not cover the case $c_1 = 0$. Treating this separately, we find that (6.4.14) leads to $c = 0$. This branch reproduces the INST solution [164], discussed in section 6.5.2.

- **Case 2:** $c = -3$. In this case, the x^2 component of (6.4.7) gives $\epsilon' = 0$. With this, the whole of (6.4.7) gives

$$2\beta(\beta' - y\beta'') + y\beta'^2 = 0 \ . \qquad (6.4.17)$$

This equation is nonlinear, but if one defines $z = y^2/2$ it becomes $2\beta\partial_z^2\beta = (\partial_z\beta)^2$, which is easily solved by the square of a linear function; in other words, by

$$\beta = c_2(y^2 + 4c_1)^2 \ , \qquad c_1, \, c_2 = \text{const.} \ . \qquad (6.4.18)$$

This case reproduces the solution obtained from reduction to ten dimensions of the Maldacena–Núñez AdS_5 solution of M-theory [152], described in section 6.5.3.

6.4.3 $F_0 \neq 0$

We will divide the analysis in the generic case, where $c \neq 0$ and -3, and two special cases $c = 0$ or -3. Let us note that from (6.4.9), we see that $\epsilon' = 0$ implies either $c = 0$ or -3; in other words, if $c \neq 0$ and -3, then $\epsilon' \neq 0$. On the other hand, from (6.4.10), we see that $\epsilon' \neq 0$ implies $c \neq 0$ and -3; in other words, if $c = 0$ or -3, then $\epsilon' = 0$.

Generic case

We begin by analyzing (6.4.7) with the aid of (6.4.10) and (6.4.9). In particular, combining the last two we derive

$$(e^{4A})' = \frac{(\epsilon')^2}{8c(c+3)y^3}\left[F_0\epsilon' x + \frac{1}{6}F_0(\beta' - 2\epsilon\epsilon') - 4cy^2\right] \ . \qquad (6.4.19)$$

Substituting $(e^{4A})'$ as expressed in the above equation, and β as expressed in (6.4.9), in (6.4.7), the whole of the latter gives

$$\beta' = \frac{c+3}{c}2\epsilon\epsilon' \ . \qquad (6.4.20)$$

(6.4.9) then actually fixes

$$\beta = \frac{c+3}{c}\epsilon^2 \ . \qquad (6.4.21)$$

Finally, (6.4.8b) follows from (6.4.9) and (6.4.10), which can be solved by quadrature. The solution is

$$\epsilon = -\frac{2\sqrt{2c(c+3)}}{3F_0^2}(F_0 y - 2c_1)\sqrt{F_0 y + c_1} + c_2 \ , \qquad c_1, \, c_2 = \text{const.} \ . \qquad (6.4.22)$$

This yields an AdS_5 solution which was not known before; we will analyze its features here. Let us define

$$\tilde{x} \equiv cx + \epsilon \ , \qquad (6.4.23)$$

since this quantity will appear several times. We know already, from (6.4.3), that $e^{5A-\phi}a_2 = \tilde{x}$. By substituting β in (6.4.6a), we also find $e^{10A-2\phi}(1-a_1^2) = \frac{c+3}{c}\tilde{x}^2$. Recalling (6.2.6), we also find $e^{5A-\phi}b = \sqrt{\frac{3}{c}}\tilde{x}$. Again from the expression of β and from (6.4.6b) we find A:

$$e^{4A} = -\frac{\epsilon'}{2cy}\tilde{x}\ , \tag{6.4.24}$$

while (6.4.6a) determines ϕ:

$$e^{\phi} = \sqrt{c}\,\tilde{x}^{3/4}\left(\frac{-2cy}{\epsilon'}\right)^{-5/4}(-2y\epsilon' + (c+3)\tilde{x})^{-1/2}\ . \tag{6.4.25}$$

Finally, collecting everything and recalling the expression of the metric (6.2.10), we can write the metric for this solution as

$$\begin{aligned} ds^2 &= e^{2A}ds^2_{\Sigma_g} + e^{-8A+2\phi}\left(\frac{\tilde{x}^2}{3c}D\psi^2 + \frac{c+3}{c}dQ^2 + \frac{2y\epsilon' - (c+3)\tilde{x}}{2(c+3)y}\frac{\epsilon'}{c}dy^2\right) \\ &= e^{2A}ds^2_{\Sigma_g} + \sqrt{-\frac{c\epsilon'}{2y\tilde{x}}}\left(\frac{1}{-2y\epsilon' + (c+3)\tilde{x}}\left(\frac{\tilde{x}^2}{3c}D\psi^2 + \frac{c+3}{c}dQ^2\right) - \frac{\epsilon'}{2c(c+3)y}dy^2\right)\ , \end{aligned} \tag{6.4.26}$$

where $Q \equiv x + \frac{1}{c+3}\epsilon$.

Unfortunately, as it stands the metric (6.4.26) appears to be unphysical. To make M_5 compact, we should be able to find some locus where the coefficient of $D\psi^2$ vanishes. One way this could happen is if $\tilde{x} = 0$. However, this also leads to $e^A = 0$. Hence this cannot be a regular point. One might think about the presence of a brane (where e^A might legitimately go to zero), but the locus $\tilde{x} = 0$ appears to be codimension 2, and there are no such branes in IIA supergravity.

So we look for other loci where the coefficient of $D\psi^2$ might vanish. Notice that the coefficient $\sqrt{-\frac{c\epsilon'}{2y\tilde{x}}}$ cannot go to zero, since $\epsilon' = -y\frac{\sqrt{2c(c+3)}}{\sqrt{F_0y+c_1}}$. Nonetheless, we have the combination

$$-2y\epsilon' + 3\epsilon = \frac{c_1[2c(c+3)]^{1/2}}{F_0^2}\frac{F_0y + 2c_1}{\sqrt{F_0y + c_1}} + 3c_2\ . \tag{6.4.27}$$

So the denominator $-2y\epsilon' + (c+3)\tilde{x}$ can go to infinity where $F_0y + c_1 = 0$. However, on this locus $e^{\phi} \to 0$, and this locus cannot be regular. It also cannot correspond to the presence of branes, for the same reason noted above for the locus $\tilde{x} = 0$.

One last possibility, which we will not analyze here, would be to try to glue this solution to other solutions (perhaps ones with $F_0 = 0$) along a D8, much as we will do in the next section. As we will see, such gluing can happen along loci where $\sqrt{1-a_1^2}e^{A-\phi}$ is constant; in our case this happens to be proportional to $\frac{y}{\epsilon'}$, which is a function of y. We leave the study of such a possibility for the future.

Special cases

- For $c = 0$, equation (6.4.10) is trivially satisfied, while the x^1 component of (6.4.7) yields $\epsilon = 0$. Then $a_2 = 0$ and this leads to the (unphysical) massive solution of section 6.3.

- For $c = -3$, (6.4.10) is again trivially satisfied, while (6.4.7) yields the following ODE for β:
$$e^{12A} F_0 = -\beta\,(e^{4A})' - 6e^{8A} y\ . \tag{6.4.28}$$
Using (6.4.6b) we see $e^{4A} = -\frac{\beta'}{12y}$. This ODE is nonlinear, and a little tougher than the ones we saw so far in this subsection. Hence we defer its further analysis to the next section. We will see there that it leads to many new AdS_5 solutions.

6.5 Recovered solutions

In this section we discuss a set of known, supersymmetric $\mathrm{AdS}_5 \times M_5$ solutions of type IIA supergravity with zero Romans mass, which we recovered in our analysis. Two of them descend from AdS_5 solutions of M-theory, whose reduction to ten dimensions we present. We focus on the geometry of the solutions, as the fluxes are determined by it. We aim to adhere to the notation of the original papers; whenever there is overlap with notation used in the main body of this chapter, we add a hat $\hat{}$.

There are more supersymmetric AdS_5 solutions in IIA [105, 166, 168, 169] that should be particular cases of our general classification of section 6.2. These are outside the compactification Ansatz of section 6.4.

6.5.1 The Gauntlett–Martelli–Sparks–Waldram (GMSW) solution

The metric on M_5 reads

$$ds^2_{M_5} = \frac{k - cy}{6m^2} ds^2_{C_k} + e^{-6\lambda} \sec^2\zeta + \frac{1}{9m^2} \cos^2\zeta D\psi^2 + e^{-6\lambda} dx_3^2\ , \tag{6.5.1}$$

where

$$e^{6\lambda} = \frac{2m^2(\hat{a} - ky^2)}{k - cy}\ , \qquad \cos^2\zeta = \frac{\hat{a} - 3ky^2 + 2cy^2}{\hat{a} - ky^2}\ . \tag{6.5.2}$$

The dilaton is given by $e^{-2\phi} = e^{6\lambda}$.

$\hat{a}$, c are constants, $k = 0, \pm 1$ and m^{-1} is the radius of AdS_5. C_k is a Riemann surface of unit radius; it is a sphere S^2, a torus T^2 or a hyperbolic space H^2 for $k = 1, 0$ or -1 respectively. The GMSW solution is the reduction to ten dimensions of an $\mathrm{AdS}_5 \times M_6$ solution of M-theory, where M_6 is a fiberation of S^2 over $C_k \times T^2$ and the reduction is along an $S^1 \in T^2$.

The solution is the one recovered in subsection 6.3.1. The constants c and k are identified with the corresponding of 6.3.1, while $\hat{a} = c_1$. The coordinate x_3 is related to x via $x_3 = -x$; a minus is introduced for matching the expressions of the fluxes. Finally, in 6.3.1 $m = 1$.

6.5.2 The Itsios–Núñez–Sfetsos–Thompson (INST) solution

The INST solution [164] was discovered by nonabelian T-dualizing the AdS$_5 \times T^{1,1}$ solution in type IIB supergravity. The metric on M_5 reads

$$ds^2_{M_5} = \lambda_1^2 ds^2_{S^2} + \frac{\lambda_2^2\lambda^2}{\Delta} x_1^2 D\psi^2 + \frac{1}{\Delta}\Big[(x_1^2 + \lambda^2\lambda_2^2)dx_1^2 + (x_2^2 + \lambda_2^4)dx_2^2 + 2x_1x_2dx_1dx_2\Big] , \tag{6.5.3}$$

where

$$\Delta = \lambda_2^2 x_1^2 + \lambda^2(x_2^2 + \lambda_2^4) , \qquad \lambda_1^2 = \lambda_2^2 = \frac{1}{6} , \qquad \lambda^2 = \frac{1}{9} , \tag{6.5.4}$$

and

$$ds^2_{S^2} = d\theta_1^2 + \sin^2\theta_1\phi_1^2 , \qquad \rho = \cos\theta_1 d\phi_1 . \tag{6.5.5}$$

The dilaton is given by $e^{-2\phi} = \Delta$.

The INST solution fits into the $c_1 = 0$ branch of the first case of subsection 6.4.2 for $c_3 = -12$ (achieved by setting the constant warp factor to zero) and $\epsilon = c_2 = \lambda\lambda_2^2$. Σ_g is S^2 of radius $\frac{1}{\sqrt{6}}$. The coordinate transformation relating x_1, x_2 to x, y is:

$$x_1^2 = -36y^2 + 36\epsilon x + 6c_4 - 6\epsilon^2 , \qquad x_2 = 6y . \tag{6.5.6}$$

6.5.3 The Maldacena–Núñez solution

We write the metric of the $\mathcal{N} = 1$ Maldacena–Núñez solution [152] in the form presented in [155]:

$$e^{-2\lambda} ds^2_{11} = ds^2_{\mathrm{AdS}_5} + \frac{1}{3} ds^2_{H^2} + e^{-6\lambda}\sec^2\zeta dy^2 + \frac{1}{9m^2}\cos^2\zeta\left((d\psi + \tilde{P})^2 + ds^2_{S^2}\right) , \tag{6.5.7}$$

where

$$e^{6\lambda} = \hat{a} + y^2 , \qquad \cos^2\zeta = \frac{\hat{a} - 3y^2}{\hat{a} + y^2} , \tag{6.5.8}$$

and m^{-1} is the radius of AdS$_5$. The metrics on H^2 and S^2 are

$$ds^2_{H^2} = \frac{dX^2 + dY^2}{Y^2} , \qquad ds^2_{S^2} = d\theta^2 + \cos^2\theta d\nu^2 , \tag{6.5.9}$$

while the connection of the fiberation of ψ is

$$\tilde{P} = -\cos\theta d\nu - \frac{dX}{Y} . \tag{6.5.10}$$

Reduction to ten dimensions

We reduce the Maldacena–Núñez solution to ten dimensions, along ν. In order to do so, we rewrite the part of $ds^2_{M_6}$ involving $d\psi$ or $d\nu$ as

$$\frac{1}{9m^2}\cos^2\zeta\left[(d\nu+A_1)^2+\sin^2\theta D\psi^2\right] \ , \tag{6.5.11}$$

where

$$A_1=-\cos\theta D\psi\ , \qquad \rho=-\frac{dX}{Y}\ . \tag{6.5.12}$$

Reducing along $d\nu$ yields then

$$ds^2_{10}=e^{2A}ds^2_{\mathrm{AdS}_5}+ds^2_{M_5}\ , \tag{6.5.13}$$

where

$$e^{-2A}ds^2_{M_5}=\frac{1}{3}ds^2_{H^2}+e^{-6A+2\phi}\sec^2\zeta dy^2+\frac{1}{9m^2}\cos^2\zeta\left(d\theta^2+\sin^2\theta D\psi^2\right)\ . \tag{6.5.14}$$

Furthermore,

$$\phi=\frac{3}{4}\log\left(\frac{1}{9m^2}e^{2\lambda}\cos^2\zeta\right)\ , \qquad A=\lambda+\frac{1}{3}\phi\ . \tag{6.5.15}$$

The reduced Maldacena–Núñez solution fits into the second case of section 6.4.2, for $\epsilon=0$ (achieved by by a $x\to x+\frac{\epsilon}{3}$ shift), $c_1=-\frac{\hat{a}}{12}$ and $c_2=1$. Σ_g is H^2 of radius $\frac{1}{\sqrt{3}}$. In our conventions $m=1$. The coordinate transformation relating x,y to θ is:

$$x=-\frac{1}{9}(\hat{a}-3y^2)\cos\theta\ . \tag{6.5.16}$$

AdS$_7$ variables

For our discussion in the main text, it is useful to also include two parameters R and k which are usually set to one. If we use the slightly awkward-looking

$$\beta=\frac{4}{k^2}\left(y^2-\frac{3^4}{2^{10}}R^6\right)^2 \tag{6.5.17}$$

the corresponding solution, using (6.6.4) and (6.6.7), is

$$ds^2_{M_5}=e^{2A}ds^2_{\sigma_g}+\frac{1}{3^{3/2}k}\frac{64dy^2}{\sqrt{9^2R^6-32^2y^2}}+\frac{(9^2R^6-32^2y^2)^{3/2}}{16(3^5R^6+32^2y^2)}\ , \tag{6.5.18}$$

$$e^{4A}=\frac{9^2R^6-32^2y^2}{3\cdot 2^8k^2}\ , \qquad e^{4\phi}=\frac{(9^2R^6-32^2y^2)^3}{2\cdot 6^3k^6(3^5R^6+32^2y^2)^2}\ . \tag{6.5.19}$$

These again look messy, but upon using the map (6.6.16) and defining an angle α via

$$\cos\alpha\equiv\frac{32}{9R^3}y \tag{6.5.20}$$

turn into the expressions for the metric, A and ϕ of the massless AdS$_7$ solution, obtained by reducing AdS$_7 \times S^4/\mathbb{Z}_k$ to IIA supergravity: see section 5.4.1.

In the main text we will need an expression for the B field of the AdS$_5$ solution. We give it directly in terms of x_7, which is related to (6.6.14) via (6.6.16):

$$B = \frac{R^3}{48k} x_7 \frac{(5 - x_7^2)}{1 + \frac{1}{3}x_7^2} \mathrm{vol}_{S^2} + \frac{1}{\sqrt{3}} \frac{x_7}{\sqrt{1 - x_7^2}} \cos\theta \mathrm{vol}_{\Sigma_g} \ . \tag{6.5.21}$$

This is similar to the one given for the AdS$_7$ solution in (5.4.8).

6.5.4 The Bah–Beem–Bobev–Wecht (BBBW) solution

The metric of the BBBW [154] solution is

$$ds^2_{11} = e^{2\lambda} \left[ds^2_{\mathrm{AdS}_5} + e^{2\nu + 2\hat{A}(x_1, x_1)} (dx_1^2 + dx_2^2) \right] + e^{-4\lambda} ds^2_{M_4} \ , \tag{6.5.22}$$

where $ds^2_{\mathrm{AdS}_5}$ is the unit radius metric on AdS$_5$, and $\hat{A}(x_1, x_2)$ is the conformal factor of the constant curvature metric on the Riemann surface $\widehat{\Sigma}_g$ of genus g, obeying

$$(\partial^2_{x_1} + \partial^2_{x_2}) \hat{A} + \kappa e^{2\hat{A}} = 0 \ . \tag{6.5.23}$$

The constant κ is the Gaussian curvature of the Riemann surface which is set to 1, 0 or -1 for the sphere S^2, the torus T^2 or a hyperbolic surface respectively. ν is a real constant. The metric $ds^2_{M_4}$ is

$$ds^2_{M_4} = \left(1 + \frac{4y^2}{qf}\right) dy^2 + \frac{qf}{k} \left(dq + \frac{12yk}{qf} dy\right)^2 + \frac{\hat{a}_1^2}{4} \frac{fk}{q} (d\chi + V)^2 + \frac{qf}{9} (d\psi + \hat{\rho})^2 \ . \tag{6.5.24}$$

The metric functions are

$$e^{6\lambda} = qf + 4y^2 \ , \qquad f(y) \equiv 1 + 6\frac{\hat{a}_2}{\hat{a}_1} y^2 \ , \qquad k(q) \equiv \frac{\hat{a}_2}{\hat{a}_1} q^2 + q - \frac{1}{36} \ , \tag{6.5.25}$$

while the one-forms which determine the fiberation of the ψ and χ directions are given by

$$\hat{\rho} = (2 - 2g)V - \frac{1}{2} \left(\hat{a}_2 + \frac{\hat{a}_1}{2q}\right) (d\chi + V) \ , \qquad dV = \frac{\kappa}{2 - 2g} e^{2\hat{A}} dx_1 \wedge dx_2 \ . \tag{6.5.26}$$

The constants $\hat{a}_1$, $\hat{a}_2$ are fixed as

$$\hat{a}_1 \equiv \frac{2(2 - 2g)e^{2\nu}}{\kappa} \ , \qquad \hat{a}_2 \equiv 2(2 - 2g) \left(1 - \frac{6e^{2\nu}}{\kappa}\right) \ . \tag{6.5.27}$$

Reduction to ten dimensions

We reduce the BBBW solution to ten dimensions, along χ. In order to do so, we rewrite the part of $ds^2_{M_4}$ involving $d\psi$ or $d\chi$ as

$$e^{2\alpha_1}(d\chi + A_1)^2 + e^{2\alpha_2} D\psi^2 \, , \tag{6.5.28}$$

where

$$e^{2\alpha_1(y,q)} \equiv \frac{\hat{a}_1^2}{4}\frac{fk}{q} + \frac{1}{4}\frac{qf}{9}\left(\hat{a}_2 + \frac{\hat{a}_1}{2q}\right)^2 \, , \tag{6.5.29a}$$

$$e^{2\alpha_2(y,q)} \equiv \frac{qf}{9} - \frac{1}{4}\left(\frac{qf}{9}\right)^2\left(\hat{a}_2 + \frac{\hat{a}_1}{2q}\right)^2 \, , \tag{6.5.29b}$$

and

$$\rho = (2-2g)V \, , \qquad A_1 = V - \frac{qf}{9}\frac{1}{2}\left(\hat{a}_2 + \frac{\hat{a}_1}{2q}\right) e^{-\alpha_1} D\psi \, . \tag{6.5.30}$$

Reducing along $d\chi$ yields then

$$ds^2_{10} = e^{2A}\left[ds^2_{\mathrm{AdS}_5} + e^{2\nu + 2\hat{A}}(dx_1^2 + dx_2^2)\right] + ds^2_{M_3} \, , \tag{6.5.31}$$

where

$$e^{4A-2\phi} ds^2_{M_3} = \left(1 + \frac{4y^2}{qf}\right) dy^2 + \frac{qf}{k}\left(dq + \frac{12yk}{qf}dy\right)^2 + e^{2\alpha_2} D\psi^2 \, . \tag{6.5.32}$$

Furthermore,

$$\phi = \frac{3}{2}(\alpha_1 - 2\lambda) \, , \qquad A = \frac{1}{2}\alpha_1 \, . \tag{6.5.33}$$

The reduced BBBW solution fits into the generic branch of the first case of subsection 6.4.2 for $c_1 = \frac{9\hat{a}_1 + \hat{a}_2}{108}$, $c_2 = \frac{(9\hat{a}_1 + \hat{a}_2)\hat{a}_2}{18\hat{a}_1}$ and $c = \frac{\hat{a}_2}{3\hat{a}_1}$. The coordinate transformation relating x, y to q is:

$$x = -\frac{\hat{a}_1(18\hat{a}_1 + \hat{a}_2 + 18\hat{a}_2 q)}{36\hat{a}_2}\left(1 + 6\frac{\hat{a}_2}{\hat{a}_1}y^2\right) \, . \tag{6.5.34}$$

Certain generalizations of the BBBW class of solutions have also appeared [170, 171]. It would be interesting to reduce these to solutions of IIA supergravity and verify that they fit in our classification of section 6.2.

6.6 Compactification solutions

We will now analyze further the case we started considering in section 6.4.3. We will see that it corresponds to a compactification of the AdS_7 solutions considered in chapter 5. Moreover, we will be able to find the most general explicit solution, thus providing a new infinite class of AdS_5 solutions.

6.6.1 Metric and fluxes

In section 6.4.3, we found that there are AdS$_5$ solutions associated with solutions of the ODE (6.4.28). Replacing the expression of A given there, we have

$$\beta\,(y\beta'' - \beta') = \frac{1}{2}y(\beta')^2 - \frac{F_0}{144y}(\beta')^3 \,. \tag{6.6.1}$$

This equation is non-linear; however, it can be rewritten as

$$(q_5^2)' = \frac{2}{9}F_0 \,, \qquad q_5 \equiv -\frac{4y\sqrt{\beta}}{\beta'} \,. \tag{6.6.2}$$

We will see later that q_5 has actually a useful physical interpretation (similar to the q of 5): it will turn out to be related to D8-brane positions. In any case, the trick (6.6.2) allows us to solve the ODE (6.6.1): indeed we can write $16y^2\frac{\beta}{(\beta')^2} = \frac{2}{9}F_0(y-\hat{y}_0)$, which can now be integrated by quadrature.

We will postpone the detailed analysis of the solutions of (6.6.1) to sections 6.6.5 and 6.6.6. For the time being, in this subsection we will collect various features of the resulting AdS$_5$ solutions.

The internal metric for the class we are considering can be extracted from the general expression (6.2.10). However, at first its global meaning is not transparent. It proves useful to trade the coordinate x for a new coordinate θ, defined by

$$\cos\theta = \frac{-3x+\epsilon}{\sqrt{\beta}} \,. \tag{6.6.3}$$

The metric then becomes

$$ds^2_{M_5} = e^{2A}ds^2_{\Sigma_g} + ds^2_{M_3} \,, \qquad ds^2_{M_3} = dr^2 + \frac{1}{9}e^{2A}(1-a_1^2)ds^2_{S^2} \,. \tag{6.6.4}$$

Here

$$ds^2_{S^2} = d\theta^2 + \sin^2\theta D\psi^2 \tag{6.6.5}$$

is the metric of the round S^2, fibered over Σ_g, which is a Riemann surface of Gaussian curvature -3 (recalling (6.4.4), and $c=-3$) and hence $g \geq 2$; The new coordinate r is defined by

$$dr = \frac{e^{3A}}{\sqrt{\beta}}dy \,. \tag{6.6.6}$$

Moreover, from (6.2.7) and (6.4.6) we have

$$1-a_1^2 = \frac{3\beta}{3\beta - y\beta'} \,, \qquad e^{4A} = -\frac{\beta'}{12y} \,, \qquad e^{\phi} = \frac{\sqrt{3}e^{5A}}{\sqrt{3\beta - y\beta'}} \,. \tag{6.6.7}$$

We can now remark that the q_5 defined in (6.6.2) is

$$q_5 \equiv e^{-\phi}R_{S^2} \equiv \frac{1}{3}e^{A-\phi}\sqrt{1-a_1^2} = -\frac{4y\sqrt{\beta}}{\beta'} \,. \tag{6.6.8}$$

$R_{S^2} = \frac{1}{3}e^A\sqrt{1-a_1^2}$ is the radius of the round S^2, as inferred from (6.6.4). The role of this particular combination of the radius and dilaton will become clearer in section 6.6.4.

From (6.6.8) and (6.6.7) we see that for the solution to make sense we must require

$$\beta \geq 0\ , \qquad -\frac{\beta'}{y} \geq 0\ . \tag{6.6.9}$$

We can now also obtain the fluxes, from the formulas in section 6.2.2. We have

$$F_2 = q_5\left[-(\mathrm{vol}_{S^2} + 3\cos\theta\mathrm{vol}_{\Sigma_g}) + \frac{1}{3}F_0 a_1 e^{A+\phi}\mathrm{vol}_{S^2}\right]\ , \tag{6.6.10}$$

where $\mathrm{vol}_{S^2} \equiv \sin\theta d\theta \wedge D\psi$. The four-form flux reads

$$F_4 = \frac{1}{3}\mathrm{vol}_{\Sigma_g} \wedge \left[\frac{2y\beta}{3\beta - y\beta'}\cos\theta\mathrm{vol}_{S^2} + \sin^2\theta D\psi \wedge dy\right]\ . \tag{6.6.11}$$

When $F_0 \neq 0$, we need not give an expression for H: as usual for massive IIA, it can be written as $H = dB$, where

$$B = \frac{F_2}{F_0} + b\ , \tag{6.6.12}$$

where b is a closed two-form. When $F_0 = 0$, the only solution in the class we are considering in this section is the Maldacena–Núñez solution; an expression for B is presented for that case in (6.5.21).

We can observe already now that the metric (6.6.4) and the flux (6.6.10) look related to those for the AdS_7 solutions in 5; see (5.3.16) and (5.3.9). The expressions are very similar; one obvious difference is that the three-dimensional metric in (6.6.4) is fibered over Σ_g, and that the flux (6.6.10) has extra legs along Σ_g. Except for a few numerical factors, everything seems to correspond nicely; the role of x in chapter 5 seems to be played here by a_1:

$$x \text{ in } \mathrm{AdS}_7 \to a_1 \text{ here.} \tag{6.6.13}$$

Actually this correspondence can be justified a little better. In 5, x is the zero-form part of $\mathrm{Im}\psi^1_+$ introduced in 5.3.1, which is the calibration for a D6-brane extended along AdS_7. The analogue of this in our case would be a D6-brane extended along $\mathrm{AdS}_5 \times \Sigma_g$; the relevant calibration is the part along $u \wedge \bar{u}$ of $\mathrm{Im}\psi^1_+$ of the present work. Looking at (6.1.26), we see that that is indeed $\mathrm{Re}a = a_1$.

Motivated by this, in this section we will also use the name

$$x_5 \equiv a_1\ . \tag{6.6.14}$$

This x_5 is meant to evoke the x in 5.3.1, and is not to be confused with the coordinate x we temporarily used in sections 6.2 and 6.4.

6.6.2 Correspondence with AdS$_7$

We will now show that solutions of the type considered in section 6.4.3 are in one-to-one correspondence with the AdS$_7$ solutions in 5. The map we will find is directly inspired from a similar map from AdS$_4$ to AdS$_7$ found in [159]. It would be possible to present our new AdS$_5$ solutions perfectly independently from the map to AdS$_7$; in fact, in finding the analytic solutions the map does not help at all. However, the existence of the map tells us right away that infinitely many regular solutions do exist, and what data they depend on.

Let us start from (6.6.1). Using the definition (6.6.6), the expressions (6.6.7) and the expression $x_5 = a_1 = -2ye^{-3A+\phi}$ from (6.6.14), (6.2.7), we can see that

$$\begin{aligned} \partial_r \phi &= \frac{1}{4}\frac{e^{-A}}{\sqrt{1-x_5^2}}(11x_5 - 2x_5^3 + (2x_5^2-5)F_0 e^{A+\phi}) \, , \\ \partial_r x_5 &= -\frac{1}{2}e^{-A}\sqrt{1-x_5^2}(4 - x_5^2 + x_5 F_0 e^{A+\phi}) \, , \\ \partial_r A &= \frac{1}{4}\frac{e^{-A}}{\sqrt{1-x_5^2}}(3x_5 - F_0 e^{A+\phi}) \, . \end{aligned} \tag{6.6.15}$$

Conversely, given a solution to this system, one may define $\beta = e^{10A-2\phi}(1-x_5^2)$, $y = -\frac{1}{2}x_5 e^{3A-\phi}$ (with an eye to (6.4.6), (6.2.7), which correspond to (6.6.7)); if one then eliminates r from (6.6.15), the resulting equations imply $\beta' = -12ye^{4A}$ (the second in (6.6.7)), and (6.6.1). So the system (6.6.15) is in fact an equivalent way to characterize our solutions. It looks much more complicated than the original ODE (6.6.1). We write it because it bears an uncanny resemblance with the system in (5.3.17): a few numerical factors have changed, and two new terms have appeared. This suggests that there might be a close relationship between solutions of one system and solutions of the other. This is in fact the case: to any solution (ϕ_5, x_5, A_5) of (6.6.15) one can associate a solution (ϕ_7, x_7, A_7) of (5.3.17) given by

$$\begin{aligned} e^{\phi_7} &= \left(\frac{3}{4}\right)^{1/4}\frac{e^{\phi_5}}{\sqrt{1-\frac{1}{4}x_5^2}} \, , \qquad & e^{A_7} &= \left(\frac{4}{3}\right)^{3/4} e^{A_5} \, , \\ x_7 &= \left(\frac{3}{4}\right)^{1/2}\frac{x_5}{\sqrt{1-\frac{1}{4}x_5^2}} \, , \qquad & r_7 &= \left(\frac{4}{3}\right)^{1/4} r_5 \, . \end{aligned} \tag{6.6.16}$$

Comparing (6.6.4) with (5.3.16), we find that the map acts on the metrics as

$$\begin{aligned} &e^{2A_5}(ds^2_{\mathrm{AdS}_5} + ds^2_{\Sigma_g}) + dr_5^2 + \frac{1-x_5^2}{9}e^{2A_5}ds^2_{S^2} \to \\ &\qquad \sqrt{\frac{4}{3}}\left(\frac{4}{3}e^{2A_5}ds^2_{\mathrm{AdS}_7} + dr_5^2 + \frac{e^{2A_5}}{12}\frac{1-x_5^2}{1-\frac{1}{4}x_5^2}ds^2_{S^2}\right) \, . \end{aligned} \tag{6.6.17}$$

Conversely, to any solution (ϕ_7, x_7, A_7) of (5.3.17), one can associate a solution (ϕ_5, x_5, A_5) of

(6.6.15) given by

$$e^{\phi_5} = \left(\frac{4}{3}\right)^{1/4} \frac{e^{\phi_7}}{\sqrt{1+\frac{1}{3}x_7^2}} \,, \qquad e^{A_5} = \left(\frac{3}{4}\right)^{3/4} e^{A_7} \,,$$
$$x_5 = \left(\frac{4}{3}\right)^{1/2} \frac{x_7}{\sqrt{1+\frac{1}{3}x_7^2}} \,, \qquad r_5 = \left(\frac{3}{4}\right)^{1/4} r_7 \,. \tag{6.6.18}$$

This inverse map now acts on the metrics as

$$e^{2A_7} ds^2_{\mathrm{AdS}_7} + dr_7^2 + \frac{1-x_7^2}{16} e^{2A_7} ds^2_{S^2} \;\to$$
$$\sqrt{\frac{3}{4}} \left(\frac{3}{4} e^{2A_7} (ds^2_{\mathrm{AdS}_5} + ds^2_{\Sigma_g}) + dr_7^2 + \frac{1}{12} \frac{1-x_7^2}{1+\frac{1}{3}x_7^2} e^{2A_7} ds^2_{S^2} \right) \,. \tag{6.6.19}$$

The simplicity of this map is basically a generalization of the simple Maldacena–Núñez solution [152], with the $1+\frac{1}{3}x_7^2$ factor ultimately playing the role of the $\Delta = 1+\sin^2\theta$ factor in [152].

One can also apply (6.6.16) directly to (6.6.7), and infer the expressions for the variables of the seven-dimensional solution:

$$e^{A_7} = \frac{2}{3}\left(-\frac{\beta'}{y}\right)^{1/4} \,, \qquad x_7 = \sqrt{\frac{-y\beta'}{4\beta - y\beta'}} \,, \qquad e^{\phi_7} = \frac{(-\beta'/y)^{5/4}}{12\sqrt{4\beta - y\beta'}} \,. \tag{6.6.20}$$

Moreover, $dr_7 = \left(\frac{3}{4}\right)^2 \frac{e^{3A_7}}{\sqrt{\beta}} dy$.

In 5, solving the system of ODEs in (5.3.17) was only part of the problem. First, one had to take care of flux quantization; second, most solutions include D8's, and one must take care that supersymmetry be preserved also on top of them. We will see in section 6.6.4 that the relevant conditions also map nicely under (6.6.16); that will lead us to conclude that there are infinitely many AdS_5 solutions, each one of them corresponding to the AdS_7 solutions in 5 and [147]. Moreover, the map is quite simple: for example, it acts on the metrics as in (6.6.19).

6.6.3 Regularity analysis

We showed that solutions of (6.6.1) are in one-to-one correspondence with solutions of the system of ODEs relevant for AdS_7 solutions. However, (6.6.1) looks much simpler than that system; hence one may hope to learn more about both the AdS_5 and the AdS_7 solutions by studying it.

In this subsection we will see what boundary conditions on (6.6.1) have to be imposed in order to obtain compact and regular solutions.

We saw in (6.6.4) that the internal metric consists of an M_3 fibered over a Riemann surface Σ_g; M_3 is itself a fiberation of S^2 over a one-dimensional space with coordinate r.

To make M_3 compact, we can use the same logic as for the AdS$_7$ solutions of 5. One might think of making it compact by periodically identifying r, but this doesn't work for the same reason as in (5.3.24): the quantity $y = -\frac{1}{2}e^{3A-\phi}x_5$ is monotonic — from (6.6.15) we see $\partial_r y = e^{2A-\phi}\sqrt{1-x_5^2}$, which is always positive; or also, directly from (6.6.6) we see $\frac{\partial y}{\partial r} = e^{-3A}\sqrt{\beta}$. So periodically identifying r is not an option. The other way to make M_3 compact is to make the S^2 shrink for two values of r, just like in chapter 5. This is what we will now devote ourselves to.

To make the S^2 shrink, we should make the coefficient $(1-a_1^2)$ in (6.6.4) go to zero, which, recalling (6.6.7), can be accomplished by making β vanish. If β has a single zero,

$$\beta = \beta_1(y-y_0) + O(y-y_0)^2 \ , \tag{6.6.21}$$

the metric (6.6.4) near y_0 is proportional to

$$\frac{dy^2}{4(y-y_0)} + (y-y_0)ds^2_{S^2} \ , \tag{6.6.22}$$

which in fact upon defining $r = \sqrt{y-y_0}$ turns into

$$dr^2 + r^2 ds^2_{S^2} \ , \tag{6.6.23}$$

which is the flat metric on $\mathbb{R}^3$. Hence if β has a single zero at $y_0 \neq 0$ the metric is regular.

One might wonder what happens if β has a double zero:

$$\beta = \beta_2(y-y_0)^2 + O(y-y_0)^3 \ . \tag{6.6.24}$$

In this case, (6.6.4) is proportional to $\frac{dy^2}{\sqrt{y-y_0}} + (y-y_0)^{3/2}ds^2_{S^2}$, which upon defining $\rho = y - y_0$ turns into

$$\frac{1}{\sqrt{\rho}}(d\rho^2 + \rho^2 ds^2_{S^2}) \ ; \tag{6.6.25}$$

we also have $e^A \sim \rho^{1/4}$, $e^\phi \sim \rho^{3/4}$. This is obviously not a regular point, but it is the local behavior appropriate for a D6 stack whose transverse directions are ρ and the S^2.

Higher-order zeros do not lead to anything of physical relevance, and in fact they would not lead to solutions, as we will see later. However, given that we have obtained boundary conditions corresponding to a regular point and to presence of a D6 stack, it is natural to wonder whether we can find boundary conditions corresponding to presence of an O6. This is realized when

$$\beta = \beta_0 + \beta_{1/2}\sqrt{y-y_0} + O(y-y_0) \ ; \tag{6.6.26}$$

in this case the metric is proportional to $(y-y_0)^{1/4}\left(\frac{dy^2}{y-y_0} + 16\alpha_0^2 ds^2_{S^2}\right)$, with $\alpha_0 \equiv \frac{\beta_{1/2}}{\beta_0}$. With the definition $\rho = \sqrt{y-y_0}$, this turns into

$$\sqrt{\rho}\left(d\rho^2 + 4\alpha_0^2 ds^2_{S^2}\right) \ ; \tag{6.6.27}$$

moreover, $e^A \sim \rho^{-1/4}$, $e^\phi \sim \rho^{-3/4}$. These are the appropriate behaviors for fields near the beginning of an O6 hole: to see this, one can start from the flat space O6 metric, given by $ds^2_\perp = H^{1/2}(d\rho^2 + \rho^2 ds^2_{S^2})$, $e^A = H^{1/4}$, $e^\phi \propto H^{3/4}$, $H = 1 - \frac{\rho_0}{\rho}$, and expand around $\rho = \rho_0$, which is indeed the boundary of the O6 hole.

This concludes our study of the physically relevant boundary conditions for the ODE (6.6.1); as it will turn out, these are the only ones which are actually realized in its solutions. Later in this section we will turn to the task of finding such solutions.

6.6.4 Flux quantization, D8 branes

Before we look at explicit solutions, we will discuss flux quantization. We will also introduce D8-branes in our construction, as we have done in 5. This subsection is in many ways similar to 5.3.8.

We will start with some preliminary comments about the B field. In (6.6.12) we expressed it in terms of a closed two-form b. We will need this second term because the term $\frac{F_2}{F_0}$ in (6.6.12) will jump as we cross a D8 (since F_0 will jump there, by definition). More precisely, looking at F_2 we see that only the term proportional to $\mathrm{vol}_{S^2} + 3\cos\theta \mathrm{vol}_{\Sigma_g}$ jumps (since in the other term an F_0 cancels out). Thus we can limit ourselves to considering b of the form

$$b = b_0(\mathrm{vol}_{S^2} + 3\cos\theta \mathrm{vol}_{\Sigma_g}) \ , \tag{6.6.28}$$

which is indeed closed (while $\mathrm{vol}_{S^2} = \sin\theta d\theta \wedge D\psi$ would not be, because of the presence of ρ). (6.6.12) now becomes

$$B = \left(b_0 - \frac{q_5}{F_0}\right)(\mathrm{vol}_{S^2} + 3\cos\theta \mathrm{vol}_{\Sigma_g}) + \frac{q_5}{3} x_5 e^{A+\phi}\mathrm{vol}_{S^2} \ . \tag{6.6.29}$$

At the poles, for regularity we should have that what multiplies vol_{S^2} should go to zero.

However, more precisely B should be understood as a “connection on a gerbe”. Concretely, this means that it is not necessarily a globally well-defined two-form. On a chart intersection $U \cap U'$, $B_U - B_{U'}$ can be any closed two-form whose periods are integer multiples of $4\pi^2$ (known as a “large gauge transformation”). This translates into the requirement that the coefficient of vol_{S^2} in (6.6.29) should wind $\pi\times$ an integer number of times in going from the north to the south pole. Alternatively, using Stokes’ theorem, we see that the integral of H between r_N and r_S (the positions of the two poles) is

$$\int_{M_3} H = \int_{S^2}\int_{r_\mathrm{N}}^{r_\mathrm{S}} dr H = \int_{S^2}(B(r_\mathrm{N}) - B(r_\mathrm{S})) \ ; \tag{6.6.30}$$

thus $\int_{M_3} H$ will be an integer multiple of $4\pi^2$, in agreement with flux quantization.

After these comments on the NSNS flux H, let us now consider the RR fluxes. First of all, the zero-form should satisfy $F_0 = \frac{n_0}{2\pi}$, $n_0 \in \mathbb{Z}$. For the higher forms, we should consider

$$\tilde{F}_2 \equiv F_2 - BF_0 \ , \qquad \tilde{F}_4 \equiv F_4 - B \wedge F_2 + \frac{1}{2} B \wedge BF_0 \ , \tag{6.6.31}$$

which are d-closed (unlike the original F_2 and F_4, which in our notation are $(d - H\wedge)$-closed). Flux quantization imposes that those should have integer periods. For the two-form we simply have

$$\tilde{F}_2 = -bF_0 = -b_0 F_0(\mathrm{vol}_{S^2} + 3\cos\theta \mathrm{vol}_{\Sigma_g}) \ . \tag{6.6.32}$$

Integrating this on the fiber S^2 and imposing that it is of the form $2\pi n_2$, $n_2 \in \mathbb{Z}$, we find

$$b_0 = -\frac{n_2}{2F_0} = -\pi\frac{n_2}{n_0} \ , \tag{6.6.33}$$

just like in 5. A gauge transformation will change $b_0 \to b_0 + k\pi$, and simultaneously $n_2 \to n_2 - k$, so that (6.6.33) remains satisfied.

Near the north and south pole it is convenient to work in a gauge where B is regular; then $\int \tilde{F}_2 \to \int F_2$, and n_2 should be equal to the limit near the pole of $\left(b_0 - \frac{q_5}{F_0} + \frac{q_5}{3}x_5 e^{A+\phi}\right)$, the coefficient of vol_{S^2} in (6.6.29). For a regular point, n_2 near the pole is zero, and both $q_5 \to 0$ and $q_5 x_5 e^{A+\phi} \to 0$. For a stack of n_2 D6-branes, $q_5 x_5 e^{A+\phi} \to 0$, and $q_5 \to -\frac{n_2}{2}$. In section 6.6.3, we saw that presence of a D6 corresponds to a double zero in β, (6.6.24). The condition we just saw will then discretize the parameter β_2, giving

$$\beta_2 = \left(\frac{4y_0}{n_2}\right)^2 \ . \tag{6.6.34}$$

An O6 point is different: $n_2 = \pm 1$ (depending on whether we are considering the north or south pole), $q_5 \to 0$, but $\frac{q_5}{3}x_5 e^{A+\phi}$ is non zero, and will have to tend to $-\frac{n_2}{2F_0}$. Again in section 6.6.3 we saw that an O6 corresponds in our class of solutions to the presence of a square root, (6.6.26). Flux quantization will then fix

$$\beta_0 = (18y_0)^2 \ . \tag{6.6.35}$$

The four-form $\tilde{F}_4$ can now be written, after some manipulations, as

$$\tilde{F}_4 = \left(\frac{3}{F_0}\left(-q_5^2 + \frac{n_2^2}{4}\right)\cos\theta \mathrm{vol}_{S^2} + \frac{1}{3}\sin^2\theta D\psi \wedge dy\right) \wedge \mathrm{vol}_{\Sigma_g} \ . \tag{6.6.36}$$

Using (6.6.2) we can also write $\tilde{F}_4 = d\tilde{C}_3$, where

$$\tilde{C}_3 = \frac{3}{2F_0}\left(-q_5^2 + \frac{n_2^2}{4}\right)\sin^2\theta D\psi \wedge \mathrm{vol}_{\Sigma_g} \ . \tag{6.6.37}$$

If both poles are regular points, $\tilde{C}_3$ is a regular form. Indeed, as we saw, at such a pole we should have $n_2 = 0$ and $q_5 \to 0$. So the coefficient $\left(-q_5^2 + \frac{n_2^2}{4}\right)$ will actually go to zero at the pole. Now, using the fact that β has a single zero (6.6.21), from (6.6.6) and (6.6.8) we see that q_5 starts with a linear power in r. Hence we have

$$\tilde{C}_3 \sim r^2 \sin^2\theta D\psi \wedge \mathrm{vol}_{\Sigma_g} \ . \tag{6.6.38}$$

Now, $r^2 \sin^2 \theta d\psi$, going from spherical to cartesian coordinates x^i, $i = 1, 2, 3$, is proportional to $x^1 dx^2 - x^2 dx^1$, and hence is regular. All in all, we conclude that $\tilde{F}_4$ does not have any non-zero periods, since it is exact. In presence of a D6 or O6 point, it is best to go back to (6.6.36). The space is topologically an S^3 fiberation over Σ_g; standard topological arguments tell us that its cohomology is just the product of that of S^3 and that of Σ_g. As such it would have no four-cycles. Thus so far flux quantization for $\tilde{F}_4$ is not an issue.

We will now introduce D8-branes. We will consider them to be extended along all directions except r. Their treatment is very similar to chapter 5 and we will be brief. The defining feature of a D8 stack is that the Romans mass F_0 jumps as we go across them. Let us call n_0 and n_0' the flux integers on the two sides. Moreover, we will allow the D8's to have non-zero worldsheet flux, which can also be thought of as a smeared D6 charge. This will make the flux integer for $\tilde{F}_2$ jump as well; we will call n_2 and n_2' its value on the two sides. The "slope" $\mu \equiv \frac{\Delta n_2}{\Delta n_0} \equiv \frac{n_2' - n_2}{n_0' - n_0}$ needs to be an integer. With this notation, imposing that (6.6.29) be continuous we find the condition

$$q_5|_{\mathrm{D8}} = \frac{1}{2}\frac{n_2' n_0 - n_2 n_0'}{n_0' - n_0} = \frac{1}{2}(-n_2 + \mu n_0) = \frac{1}{2}(-n_2' + \mu n_0') \; . \tag{6.6.39}$$

This is to be read as a condition fixing the D8's position.

One might now also wonder whether the flux of $\tilde{F}_4$ along $\Sigma_g \times S^2$ might jump between D8's, as does the integral of $\tilde{F}_2$. But actually $\int_{S^2} \cos\theta \mathrm{vol}_{\mathrm{S}^2} = 0$. So even in presence of D8's we need not worry about flux quantization for $\tilde{F}_4$.

Crucially, (6.6.39) is exactly the same condition that was found for D8-branes in (5.3.45). The function called q in chapter 5, which we will call q_7 here, is not exactly the same as our q_5 defined in (6.6.8): indeed $q_7 \equiv \frac{1}{4} e^{A_7 - \phi_7} \sqrt{1 - x_7^2}$. However, using the map (6.6.16), we see that the different overall factor is reabsorbed:[6]

$$q_5 = q_7 \; . \tag{6.6.40}$$

So (6.6.39) fixes the D8's at exactly the same position in an AdS$_5$ solution and in its AdS$_7$ solution.

Since (6.6.39) was found by imposing that B should be continuous, it looks easy to impose the condition on flux quantization. As remarked earlier, by Stokes' theorem we can relate the integrality of H to the periodicity of the coefficient of vol_{S^2} in B. (This periodicity was expressed visually in several figures in 5 and [147], as a dashed green line.) However, in presence of D8's one might encounter a region where $F_0 = 0$; generically such a region will exist (although there are also "limiting cases" where it does not exist; see [147, Sec. 4.2]). In such a region, (6.6.12) (and hence (6.6.29)) cannot be used; we have to resort to (6.5.21). This allows to write a general expression for the integral of H, as shown in [147, Eq.(4.7)].

Since we are going to simplify that formula for AdS$_7$ solutions, let us review it quickly here. To simplify things a bit, one derives first an expression for the integral in the "northern

[6] Actually, the condition that the system (6.6.15) be mapped to the similar system (5.3.17) for AdS$_7$ solutions only fixed the map (6.6.16) up to a constant. We fixed the constant so that (6.6.39) would look exactly equal to (5.3.45).

hemisphere", between $x_7 = 1$ and $x_7 = 0$; it can be shown that $x_7 = 0$ is in the massless region, where $F_0 = 0$. There might be many D8's; let D8$_n$ be the one right before the massless region, $\{n_{2,n}, n_{0,n} = 0\}$ the flux parameters right after it, and $\{n_{2,n-1}, n_{0,n-1}\}$ the ones right before it. Then we can divide the integral into a contribution from the massive region and one from the massless region:

$$
\begin{aligned}
\int_{\text{north}} H &= \int_{r_{\text{N}}}^{\text{D8}_n} H + \int_{\text{D8}_n}^{x=0} H \\
&= 4\pi \left[q_7 \left(\frac{x_7}{4} e^{A_7+\phi_7} - \frac{1}{F_{0,n-1}} \right) - \frac{n_{2,n-1}}{2F_{0,n-1}} + \frac{3}{32} \frac{R^3}{n_{2,n}} \left(x_7 - \frac{x_7^3}{3} \right) \right]_{\text{D8}_n} \\
&= 4\pi \left[-\pi\mu_n + \frac{1}{4} q x_7 e^{A_7+\phi_7} - \frac{1}{4} q_7 x_7 e^{A_7+\phi_7} \frac{3 - x_7^2}{1 - x_7^2} \right]_{\text{D8}_n} \\
&= 4\pi \left[-\pi\mu_n + \frac{R^3}{16 n_{2,n}} x_7 \right]_{\text{D8}_n} .
\end{aligned}
\tag{6.6.41}
$$

We have used that for the massless solution $-8q_7 \frac{e^{A_7+\phi_7}}{1-x_7^2} = -2\frac{e^{2A_7}}{\sqrt{1-x_7^2}} = \frac{R^3}{n_2}$, where R is a constant. After this simplification, and putting together the contribution from $\int_{\text{south}} H$ from the "southern hemisphere", we can write

$$
N \equiv -\frac{1}{4\pi^2} \int H = (|\mu_n| + |\mu_{n+1}|) + \frac{1}{4\pi} e^{2A(x=0)} (|x_n| + |x_{n+1}|) \; , \tag{6.6.42}
$$

where x_n and x_{n+1} are the values of x_7 at the branes D8$_n$ and D8$_{n+1}$.[7]

To derive a similar expression for AdS$_5$ solutions, we follow a similar logic. It proves convenient to use from the very beginning (A_7, x_7, ϕ_7) variables, which are related to (A_5, x_5, ϕ_5) variables via (6.6.16). We can use (6.6.29) and (6.5.21), the latter of which is already expressed in terms of x_7. Some factors in the computation change, but remarkably the result turns out to be exactly the same as in (6.6.42). As a consequence, if the H flux quantization is satisfied for an AdS$_7$ solution, it is also satisfied for an AdS$_5$ solution, and viceversa.

So the conclusion of this section is that the flux quantization conditions and the constraints fixing the D8-brane positions are all precisely mapped by (6.6.16), in such a way that if they are satisfied for an AdS$_7$ solution they are also automatically satisfied for an AdS$_5$ solution. This proves that the map (6.6.16) produces infinitely many AdS$_5$ solutions.

6.6.5 The simplest massive solution

We will now start studying solutions to (6.6.1), and their associated physics. We have already indicated in (6.6.2) how to solve it analytically. However, in this section we will warm up by a perturbative study, which we find instructive and which will allow us to isolate a particularly nice and useful solution.

[7] The μ_i and x_i before the massless region are positive, while those after the massless region are negative.

In section 6.6.3 we studied the boundary conditions for the ODE (6.6.1). We can now proceed to study it in the neighborhood of such a solution. We will do so by assuming analytic behavior around y_0: $\beta = \sum_{k=1}^{\infty} \beta_k (y - y_0)^k$, by plugging this Taylor expansion in (6.6.1), and solving order by order.

Already at order zero we find

$$\left(\beta_1 - \frac{72 y_0^2}{F_0}\right) \beta_1^2 = 0 \, . \tag{6.6.43}$$

The first branch, $\beta_1 = \frac{72y_0^2}{F_0}$, lets β have a single zero, which as we saw after (6.6.21) corresponds to a regular point. The second branch, $\beta_1 = 0$, makes β have a double zero, which as we saw after (6.6.24) corresponds to a D6. In this section we will use the first branch, leaving the second for section 6.6.6.

Continuing to solve (6.6.1) perturbatively after having set $\beta_1 = \frac{72y_0^2}{F_0}$, we find a nice surprise: the perturbative expansion stops after three iterations. This leads to a very simple solution to (6.6.1):

$$\beta = \frac{8}{F_0}(y - y_0)(y + 2y_0)^2 \, . \tag{6.6.44}$$

This has the desired single zero at $y = y_0$, and it also has a double zero at $y = -2y_0$, signaling that M_3 has a D6 stack there. These are the qualitative features one expects from the solution in 5.4.2; in that chapter, that solution was argued to exist (along with many others, which we shall discuss in due course) on numerical grounds — see in particular 5.3. It would also be possible to find (6.6.44) by finding the general solution, and imposing the presence of a simple zero; we will see this in section 6.6.6.

(6.6.44) looks superficially very similar to (6.4.18). Taking $c_1 = -y_0^2/4$, we see that (6.4.18) has *two* double zeros, at $y = \pm y_0$, corresponding to two D6 stacks. This is indeed correct for that massless solution: the two D6 stacks are generated by the reduction from eleven dimensions, in a similar way as in section 5.4.1. Notice also that the massless limit of (6.6.44), on the other hand, does not exist, since F_0 appears there in the denominator.

Now that we have obtained one solution of (6.6.44), we can pause to explore what the resulting AdS$_5$ solution looks like; moreover, using the map (6.6.16), we can also produce an AdS$_7$ solution which will indeed be the one found numerically in section 5.4.2.

The conditions (6.6.9) give us two possibilities:

$$\{y_0 < 0,\ F_0 > 0,\ y \in [y_0, -2y_0]\} \ \text{ or } \ \{y_0 > 0,\ F_0 < 0,\ y \in [-2y_0, y_0]\} \, . \tag{6.6.45}$$

We will assume the first possibility. One can then write the metric and fields most conveniently in terms of

$$\tilde{y} \equiv \frac{y}{y_0} \, , \tag{6.6.46}$$

which then has to belong to $[-2, 1]$. We have

$$ds^2_{M_5} = e^{2A} ds^2_{\Sigma_g} + \sqrt{-\frac{y_0}{8F_0}} \left(\frac{d\tilde{y}^2}{(1-\tilde{y})\sqrt{\tilde{y}+2}} + \frac{4}{9} \frac{(1-\tilde{y})(\tilde{y}+2)^{3/2}}{2-\tilde{y}} ds^2_{S^2} \right) \, , \tag{6.6.47}$$

and

$$e^{4A} = -2\frac{y_0}{F_0}(2+\tilde{y}) , \qquad e^{2\phi} = \sqrt{-\frac{1}{2y_0F_0^3}\frac{(\tilde{y}+2)^{3/2}}{2-\tilde{y}}} . \tag{6.6.48}$$

We also need to implement flux quantization, which in this case is the statement that the D6 stack at the $\tilde{y} = -2$ point has an integer number n_2 of D6-branes. This constraint was discussed right below (6.6.33). From (6.6.8) and (6.6.44) we find $q_5 = \frac{1}{3}\sqrt{2F_0(y-y_0)}$, which implies

$$y_0 = -\frac{3}{8}\frac{n_2^2}{F_0} . \tag{6.6.49}$$

We did not replace this constraint in (6.6.47), as we did in (6.0.2), because later we will glue pieces of it together with other metrics and with itself, and in that context the parameter y_0 will be fixed by flux quantization a bit differently.

The AdS$_7$ solutions can now be found easily by applying the map (6.6.16), and in particular its action on the metric, (6.6.17). The internal metric on M_3 is

$$ds^2_{M_3} = \sqrt{-\frac{y_0}{6F_0}}\left(\frac{d\tilde{y}^2}{(1-\tilde{y})\sqrt{\tilde{y}+2}} + \frac{4}{3}\frac{(1-\tilde{y})(\tilde{y}+2)^{3/2}}{8-4\tilde{y}-\tilde{y}^2}ds^2_{S^2}\right) , \tag{6.6.50}$$

and

$$e^{4A} = -\left(\frac{4}{3}\right)^3 2\frac{y_0}{F_0}(\tilde{y}+2) , \qquad e^{2\phi} = \sqrt{-\frac{6}{y_0F_0^3}\frac{(\tilde{y}+2)^{3/2}}{8-4\tilde{y}-\tilde{y}^2}} . \tag{6.6.51}$$

(6.6.50) and (6.6.51) give analytically the solution found numerically in section 5.4.2. The flux F_2 can be read off from the expression $F_2 = q(\frac{x_7}{4}F_0e^{A+\phi} - 1)\mathrm{vol}_{S^2}$ in (5.3.42):

$$F_2 = \frac{k}{\sqrt{3}}\frac{(1-\tilde{y})^{3/2}(\tilde{y}+4)}{8-4\tilde{y}-\tilde{y}^2}\mathrm{vol}_2 . \tag{6.6.52}$$

For both the AdS$_5$ and AdS$_7$ solutions, from (6.6.49) we can see that, making n_2 large, curvature and string coupling become as small as one wishes. This guarantees that the supergravity approximation is applicable. Similar limits can be taken for the solutions that we will present later. (This was shown in general in [147, Sec. 4.1].)

6.6.6 General massive solution

Let us now go back to (6.6.43) and see what happens if we use the branch $\beta_1 = 0$. This means that β has a double zero, which corresponds to presence of a D6 stack at $y = -2y_0$.

The perturbative expansion for (6.6.1) now does not truncate anymore. It is possible to go to higher order, guess an expression for the k-th term β_k in the Taylor expansion $\beta = \sum_{k=1}^{\infty}\beta_k(y-y_0)^k$, and resum this guess. (This is in fact the way we originally proceeded.) At this point it is of course much easier to use the trick explained below (6.6.2), and find the general solution directly. Assuming $y_0 > 0$, it reads

$$\beta = \frac{y_0^3}{b_2^3F_0}\left(\sqrt{\hat{y}}-6\right)^2\left(\hat{y}+6\sqrt{\hat{y}}+6b_2-72\right)^2 , \tag{6.6.53}$$

where

$$\hat{y} \equiv 2b_2\left(\frac{y}{y_0}-1\right)+36\ , \qquad b_2 \equiv \frac{F_0}{y_0}\beta_2\ . \tag{6.6.54}$$

This solution now depends on the two parameters y_0 and b_2, rather than just one as (6.6.44), and we expect it to be the most general solution to (6.6.1). To see whether this is true, let us analyze its features and compare them to what we expect from the qualitative study in section 5.4.2.

(6.6.53) has zeros in $\hat{y} = 36$ (which corresponds to $y = y_0$) and for $b_2 < 12$ also in $\hat{y} = \sqrt{-3+\sqrt{81-6b_2}}$. Also, at $\hat{y} = 0$ it has a point where it behaves as $\beta \sim \beta_0 + \sqrt{\hat{y}} + O(\hat{y})$, which up to translation is the same as in (6.6.26), which corresponds to an O6 point. Taking also into account the constraints in (6.6.9), we find two possibilities, and one special case between them.

- If $b_2 < 12$, the solution is defined in the interval $\hat{y} \in [\sqrt{-3+\sqrt{81-6b_2}}, 36]$; there are two double zeros at both extrema. This represents a solution with two D6 stacks at both ends, but where the numbers of D6s are not the same on the two sides (unlike for (6.4.18)). Under the map (6.6.16) to AdS_7, it becomes a solution that was briefly mentioned at the end of 5.4.2; in terms of the graph in 5.3 in chapter 5, its path would come from below and miss the green dot on the top side from the left, so as to end up in a D6 asymptotics on the top side as well.

- If $b_2 > 12$, the solution is defined for $\hat{y} \in [0, 36]$; there is a double zero at $\hat{y} = 36$, and an O6 singularity (see (6.6.26)) at $\hat{y} = 0$. This represents a solution with one D6 stack at one end, and one O6 at the other extremum. Under the map to AdS_7, it becomes another solution that was briefly mentioned in section 5.4.2; in terms of the graph in 5.3 in chapter 5, its path would come from below and miss the green dot on the top side from the right, so as to end up in an O6 asymptotics on the top side.

- In the limiting case, $b_2 = 12$, the solution is again defined for $\hat{y} \in [0, 36]$; under the map to AdS_7 we expect to find the case where (again referring to 5.3) we hit the green dot at the top, which should correspond to having a regular point. Indeed in this case (6.6.53) reduces to

$$\beta = \frac{y_0^3}{1728F_0}\hat{y}(\hat{y}-36)^2\ , \tag{6.6.55}$$

  which has a double zero in $\hat{y} = 36$ and a single zero in $\hat{y} = 0$; it is essentially (6.6.44). It would have been possible to obtain (6.6.44) this way, but we chose to highlight it in a subsection by itself because of its simplicity.

So the solution (6.6.53) has the features we expected from the qualitative analysis in 5.4.2.

We record also here some data of the corresponding solutions. For the AdS_5 solution, the metric, warping and dilaton read

$$
\begin{aligned}
ds^2_{M_5} &= e^{2A} ds^2_{\Sigma_g} + \frac{y_0^{5/4} d\hat{y}^2}{4(b_2^5 F_0^3 \hat{y}^3 \beta)^{1/4}} + \frac{(b_2^7 F_0 \hat{y})^{1/4}}{18 y_0^{7/4}} \frac{\beta^{3/4} ds^2_{S^2}}{2(b_2-18)^2 + 18(b_2-12)\sqrt{\hat{y}} - (b_2-18)\hat{y}} \,, \\
e^{8A} &= \frac{b_2 \beta}{F_0 y_0 \hat{y}} \,, \qquad e^{8\phi} = \frac{b_2^{11}\beta^3}{16 F_0^3 y_0^{11} \hat{y}^3 \left(2(b_2-18)^2 + 18(b_2-12)\sqrt{\hat{y}} - (b_2-18)\hat{y}\right)^4} \,.
\end{aligned} \tag{6.6.56}
$$

The AdS$_7$ solution reads

$$
\begin{aligned}
ds^2_{M_3} &= \frac{y_0^{5/4} d\hat{y}^2}{4(b_2^5 F_0^3 \hat{y}^3 \beta)^{1/4}} + \frac{(b_2^7 F_0 \hat{y})^{1/4}}{3 y_0^{7/4}} \frac{\beta^{3/4} ds^2_{S^2}}{12(b_2-18)^2 + 144(b_2-12)\sqrt{\hat{y}} - 12(b_2-18)\hat{y} - \hat{y}^2} \,, \\
e^{8A} &= \frac{2^{12} b_2 \beta}{3^6 F_0 y_0 \hat{y}} \,, \qquad e^{8\phi} = \frac{144 b_2^{11}\beta^3}{F_0^3 y_0^{11} \hat{y}^3 \left(-12(b_2-18)^2 - 144(b_2-12)\sqrt{\hat{y}} + 12(b_2-18)\hat{y} + \hat{y}^2\right)^4} \,.
\end{aligned} \tag{6.6.57}
$$

Finally, flux quantization can be taken into account by using (6.6.33), (6.6.34) and the expansion of β around its zeros (or around its zero and its square root point, for the O6–D6 case). We obtain two equations, which discretize the two parameters b_2 and y_0. The expressions are not particularly inspiring (especially in the D6–D6 case) and we will not give them here.

6.6.7 Some solutions with D8's

We will now show two simple examples of solutions with D8-branes. These will be the ones studied numerically in 5.4.3; here we will give their analytic expressions. We will simply have to piece together solutions we have already studied; all we will have to work out is the position of the D8's.

The first example is a solution with only one D8 stack. This can be obtained by gluing two metrics of the type (6.6.47). We will assume

$$
y_0 < 0 \,, \qquad F_0 > 0 \,; \qquad y_0' > 0 \,, \qquad F_0' < 0 \,. \tag{6.6.58}
$$

Following the logic in section 5.4.3, the flux quantization conditions can be satisfied by taking for example the two-form flux integer after the D8 stack to vanish, $n_2' = 0$, $n_2 = \mu(n_0' - n_0)$, $\mu \in \mathbb{Z}$, and

$$
n_0' = n_0 \left(1 - \frac{N}{\mu}\right) \,, \tag{6.6.59}
$$

where $N = \frac{1}{4\pi^2}\int H$ is the NSNS flux integer. (Recall that $F_0 = \frac{n_0}{2\pi}$, and similarly for F_0'.) As usual the metric can be written as $ds^2_{M_5} = e^{2A} ds^2_{\Sigma_g} + ds^2_{M_3}$, and putting together two copies of

(6.6.47) we can write[8]

$$ds^2_{M_3} = \begin{cases} \frac{1}{\sqrt{8F_0}}\left(\frac{dy^2}{(y-y_0)\sqrt{-2y_0-y}} + \frac{4}{9}\frac{(y-y_0)(-2y_0-y)^{3/2}}{-y_0(y-2y_0)}ds^2_{S^2}\right), & y_0 < y < y_{\rm D8}\,; \\ \frac{1}{\sqrt{-8F_0'}}\left(\frac{dy^2}{(y_0'-y)\sqrt{2y_0'+y}} + \frac{4}{9}\frac{(y_0'-y)(2y_0'+y)^{3/2}}{y_0'(2y_0'-y)}ds^2_{S^2}\right), & y_{\rm D8} < y < y_0'\,. \end{cases} \tag{6.6.60}$$

We reverted to using y rather than $\hat{y}$, so as to be able to use the same coordinate before and after the D8 stack. Imposing that A and ϕ (or, equivalently, that β and β') be continuous across the D8 stack, we get

$$y_0 = \frac{1}{2}\frac{2F_0 - F_0'}{F_0 + F_0'}y_{\rm D8}\,, \qquad y_0' = \frac{1}{2}\frac{2F_0' - F_0}{F_0 + F_0'}y_{\rm D8}\,. \tag{6.6.61}$$

We also have to impose (6.6.39), which fixes

$$y_{\rm D8} = y_0 + \frac{9(F_0')^2 n_2^2}{8F_0(F_0 - F_0')^2}\,, \tag{6.6.62}$$

which together with (6.6.61) and (6.6.59) gives

$$y_0 = -\frac{3}{2}F_0\pi^2(N^2-\mu^2)\,, \qquad y_0' = \frac{3}{2}F_0\pi^2(N-\mu)(2N-\mu)\,, \qquad y_{\rm D8} = 3F_0\pi^2(N-2\mu)(N-\mu)\,. \tag{6.6.63}$$

One can also obtain the corresponding AdS$_7$ solution. This can be done using the map (6.6.19) on (6.6.60). Alternatively, we can just write one copy of (6.6.50) for $y_0 < y < y_{\rm D8}$, and a second copy of (6.6.50), formally obtained by $y \to -y$, $y_0 \to -y_0'$, $F_0 \to -F_0'$. This provides the analytic expression of the solution in figure 5.4.

We can also consider a configuration with two D8 stacks. We will take it to by symmetric, in the sense that the flux integers before the first D8 stack will be $(n_0, 0)$, between the two stacks $(0, n_2 = -k < 0)$, and after the second stack $(-n_0, 0)$. This corresponds to figure 5.5. Again we will assume $y_0 < 0$; the positions of the two D8 stacks will be $y_{\rm D8} < 0$ and $y_{\rm D8'} = -y_{\rm D8} > 0$. We will give only the AdS$_7$ internal metric:

$$ds^2_{M_3} = \begin{cases} \frac{1}{\sqrt{6F_0}}\left(\frac{dy^2}{(y-y_0)\sqrt{-2y_0-y}} + \frac{4}{3}\frac{(y-y_0)(-2y_0-y)^{3/2}}{8y_0^2 - 4yy_0 - y^2}ds^2_{S^2}\right), & y_0 < y < y_{\rm D8}\,; \\ \frac{24^4 R^6 dy^2 + (9^2R^6 - 32^2y^2)^2 ds^2_{S^2}}{3\cdot 6^5(9^2R^6 - 32^2y^2)^{1/2}}\,, & y_{\rm D8} < y < -y_{\rm D8}\,; \\ \frac{1}{\sqrt{6F_0}}\left(\frac{dy^2}{(-y_0-y)\sqrt{-2y_0+y}} + \frac{4}{3}\frac{(-y_0-y)(-2y_0+y)^{3/2}}{8y_0 + 4yy_0 - y^2}ds^2_{S^2}\right) & -y_{\rm D8} < y < -y_0\,; \end{cases} \tag{6.6.64}$$

[8] The sign differences between the expression before and after the D8 have to do with the simplification of factors involving $\sqrt{F_0^2} = |F_0|$ from applying (6.6.7) to (6.6.44).

the metric in the middle region is the known massless metric in (5.4.4), with the change of coordinate (6.5.20).

We now have three unknowns: R, y_0, $y_{\rm D8}$. Continuity of β and β' this time only imposes one condition; we then have (6.6.39) and the condition (6.6.42). We get

$$\begin{gathered} y_0 = -\frac{9}{4}k\pi(N-\mu)\ , \qquad y_{\rm D8} = -\frac{9}{4}k\pi(N-2\mu)\ , \\ R^6 = \frac{64}{3}k^2\pi^2(3N^2-4\mu^2)\ , \end{gathered} \tag{6.6.65}$$

where in this case $\mu = \frac{k}{n_0}$. Notice that the in this case the bound in [147, Eq.(4.10)] (which can also be found by (6.6.42)) implies $N > 2\mu$.

It would now be possible to produce solutions with a larger number of D8's. It is in fact possible to introduce an arbitrary number of them, although there are certain constraints on their numbers and their D6 charges [147, Sec. 4]. The most general solution can be labeled by the choice of two Young diagrams; there is also a one-to-one correspondence with the brane configurations in [124, 125]. One can in fact think of the AdS$_7$ solutions as a particular near-horizon limit of the brane configurations. For more details, see [147]. For these more general solutions, we expect to have to glue together not only pieces of the solution in subsection 6.6.5 and of the massless solution, but also pieces of the more complicated solution in 6.6.6.

6.6.8 Field theory interpretation

In this section we have found infinitely many new AdS$_5$ solutions in massive IIA, and we have established that they are in one-to-one correspondence with the AdS$_7$ solutions of 5 and [147].

It is easy to guess the field theory interpretation of this correspondence. Recall first the Maldacena–Núñez $\mathcal{N} = 2$ solutions [152]. The original AdS$_7 \times S^4$ solution of M-theory has an SO(5) R-symmetry; when one compactifies on a Riemann surface Σ_g, one "mixes" the SO(2) of local transformations on Σ_g with an SO(2) $\subset$ SO(5) subgroup; the commutant SO(2)$\times$SO(3)$\cong$U(2) remains as the R-symmetry of the resulting $\mathcal{N} = 2$ CFT$_4$. This is reflected in the form of the metric of the S^4, that gets distorted (except for the directions protected by the R-symmetry).

In similar $\mathcal{N} = 1$ solutions [152, 154], the SO(2) is embedded in SO(5) in a more intricate way, so that its commutant is a U(1), which is indeed the R-symmetry of an $\mathcal{N} = 1$ theory.

For us, the CFT$_6$ has only $(1, 0)$ supersymmetry, and thus its R-symmetry is already only SU(2). The twisting is very similar to the usual one in [152]: it is signaled by the fact that the ψ coordinate is fibered over the Riemann surface Σ_g.

When we mix this with the SO(2) of local transformations on Σ_g, the commutant is only a U(1). So in principle there is no symmetry protecting the shape of the internal S^2 in the AdS$_7$ solutions; indeed the metric (6.6.4) does not have SO(3) isometry, because the ψ direction is fibered over Σ_g. What is a bit surprising is that the breaking is not more severe: (6.6.5) might have become considerably more complicated, with $\sin\theta$ for example being replaced by a different function. Likewise, in the fluxes, one can see that there is no SO(3) symmetry: the

$\cos\theta$ in front of vol_{Σ_g}, for example, breaks it. Still, there are various nice vol_{S^2} terms which were not guaranteed to appear.

In any case, we interpret our solutions as the twisted compactification of the CFT$_6$ dual to the AdS$_7$ solutions in 5 and [147]. Recently, there has been a lot of progress in understanding such compactifications for the $(2,0)$ theories [96, 153, 154], and it would be very interesting to extend those results to our AdS$_5$ solutions. Here, we will limit ourselves to pointing out a couple of preliminary results about the number of degrees of freedom.

A common way of estimating the number of degrees of freedom using holography in any dimension is to introduce a cut-off in AdS, and estimate the Bekenstein–Hawking entropy (see for example [172, Sec. 3.1.3]). This leads to $\frac{R^5_{\mathrm{AdS}_7}}{G_{\mathrm{N},7}}$ in AdS$_7$, and to $\frac{R^3_{\mathrm{AdS}_5}}{G_{\mathrm{N},5}}$ in AdS$_5$, where $G_{\mathrm{N},d}$ is Newton's constant in d dimensions. The latter can be computed as $\frac{1}{g_s^2}\mathrm{vol}_{10-d}$. In a warped compactification with non-constant dilaton, both R_{AdS} and g_s are non-constant, and should be integrated over the internal space. In our case, for AdS$_7$ this leads to

$$\mathcal{F}_{0,6} \equiv \int_{M_3} e^{5A_7-2\phi_7}\mathrm{vol}_3 \tag{6.6.66}$$

and for AdS$_5$ to $\mathcal{F}_{0,4} \equiv \int_{M_5} e^{5A_5-2\phi_5}\mathrm{vol}_5$. These can be thought of as the coefficient in the thermal partition function, $\mathcal{F} = \mathcal{F}_{0,d}VT^d$, where V is the volume of space and T is temperature. These computations however are basically the same for the coefficients in the Weyl anomaly, at least at leading order (i.e. in the supergravity approximation).

As a consequence of our map (6.6.16), $\mathcal{F}_{0,6}$ and $\mathcal{F}_{0,4}$ are related. Taking into account the transformation of the volume form according to (6.6.17), we find

$$\mathcal{F}_{0,4} = \left(\frac{3}{4}\right)^4 \mathcal{F}_{0,6}\mathrm{Vol}(\Sigma_g)\ . \tag{6.6.67}$$

The volume of Σ_g can be easily computed using Gauss–Bonnet and the fact that its scalar curvature equals -6: we get

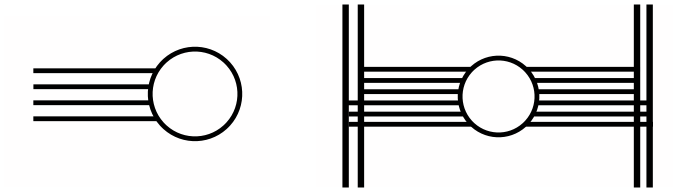
$$\mathrm{Vol}(\Sigma_g) = \frac{4}{3}\pi(g-1)\ . \tag{6.6.68}$$

So the ratio of degrees of fredom in four and six dimensions is universal, in that it depends only on g and not on the precise $(1,0)$ theory we are considering in our class. This is reminiscent of what happens for compactifications of the $(2,0)$ theory; see e.g. [153, Eq.(2.8)], or [154, Eq. (2.22)].

We have not computed $\mathcal{F}_{0,6}$ in full generality for the $(1,0)$ theories. This would now be possible in principle, since the analytic expressions are now known. One first example is the solution in section 6.6.5. The corresponding brane configuration according to the identification in [147] consists in k D6's ending on $N = \frac{k}{n_0}$ NS5-branes; see figure 6.1(a). We get

$$\mathcal{F}_{0,6} = \frac{512}{45}k^2\pi^4 N^3\ , \tag{6.6.69}$$

which reassuringly goes like N^3. (By way of comparison, for the massless case one gets $\mathcal{F}_{0,6} = \frac{128}{3}k^2\pi^4 N^3$.)



(a) (b)

Figure 6.1: Brane configurations for two sample theories. The circles represent stacks of N NS5-branes; the horizontal lines represent D6-branes; the vertical lines represent D8-branes. In the second case, on each side we have $n_0 = 2$ D8-branes; $|\mu| = 3$ D6-branes end on each, for a total of $k = n_0|\mu| = 6$.

We also computed $\mathcal{F}_{0,6}$ for the solution (6.6.64), which has two D8's and a massless region between them. The corresponding brane configuration would be N NS5-branes in the middle with $k = \mu n_0$ D6's sticking out of them, ending on n_0 D8-branes both on the left and on the right; see figure 6.1(b). This case was considered in [147, Sec. 5], where approximate expressions for $\mathcal{F}_{0,6}$ were computed, using perturbation theory around the massless limit. Using (6.6.64) we can now obtain the exact result:

$$\mathcal{F}_{0,6} = \frac{128}{3} k^2 \pi^4 \left(N^3 - 4N\mu^2 + \frac{16}{5}\mu^3 \right) . \tag{6.6.70}$$

This agrees with [147, Sec. 5], but is now exact. Recall that $\mu = \frac{k}{n_0}$; since this number can be large, the second and third term are also large, and are not competing with stringy corrections. Using (6.6.67), and comparing with the $(2, 0)$ theory to fix the proportionality factors, we get that for the CFT$_4$ theory $a = c = \frac{1}{3}(g-1)\left(N^3 - 4N\mu^2 + \frac{16}{5}\mu^3\right)$. Stringy corrections will modify this result with terms linear in N and probably in μ.

6.7 Summary of the results and outlook

The main achieved goal of this chapter is the classification of AdS$_5 \times M_5$ solutions of massive type IIA supergravity. As in 4 and 5, we reduced the supersymmetry equations in terms of six PDEs. All the fields, such us the dilaton, the metric and the fluxes, would be completely determined by the solution of those PDE's. The geometry of M_5 is given by a fiberation of a three-dimensional manifold, M_3, over a two-dimensional space $\mathcal{C}$. We were able to recover many known solutions, mainly arising from compactification of eleven-dimensional supergravity or obtained before by T-dualitizing type IIB solutions.

The most relevant new result is that we found a subclass of infinitely many new solutions which are in one-to-one correspondence with all the AdS$_7 \times S^3$ backgrounds classified in 5, moreover, they preserve eight supercharges in five dimensions. We were able to explicitly find the map between this subclass of AdS$_5 \times M_5$ backgrounds and all the AdS$_7 \times S^3$, where the geometry of M_3 inside M_5 is the same as the distorted S^3 in the AdS$_7$ compactification. $\mathcal{C}$ has to be a Riemann surface with constant negative curvature and genus, $g > 1$. We also needed to twist the S^2 over $\mathcal{C}$, breaking the SU(2)-isometry of the S^3 to U(1), as we expected from the dual, $\mathcal{N} = 1$, CFT picture in four dimensions. For this subclass of AdS_5 backgrounds of massive type IIA supergravity, the PDEs simplify to some algebraic conditions together with a single ODE. A second surprising new result was that we were able to analytically solve this equation. We have, then, a nice analytical version of all the AdS_7 solutions in 5, by means of the map that relates them to the new subclass of the AdS_5 backgrounds found in this chapter.

Due to the analytic expressions for the fields on the gravity side, we can explicitly compute the free energy for some example of 4d, $\mathcal{N} = 1$, and 6d, $(1, 0)$, CFTs with large N, by using the AdS/CFT dictionary, perhaps along the lines of [96]. It would be very interesting to probe further some general aspect, using these gravity duals. Another interesting development would be to study the RG-flows by looking at seven-dimensional gauged supergravity and how it is related to our solutions, in the spirit of [173–176]. It would be nice to find a gauged supergravity, which is a consistent truncation and includes all the solutions we found, perhaps, by also extending the computation in [177] turning on all the components of the embedding tensor in the 7d gauged supergravity. It would be also nice to make contact of all our AdS compactifications with the extended generalized geometry formalism developed in [178–180] and how we can recduce to a consistent truncation using the extended generalized geometry approach along the lines of [181]. We would also like to apply the same pure spinor techinques used in this chapter to study $\mathcal{N} = 2$ solutions of massive type IIA supergravity.

Appendix A

Useful Definitions and Notations

In this thesis capital letter indices, $(M, N, \ldots)$, are 10-dimensional, whereas Greek letter, $(\mu, \nu, \ldots)$, are indices of the external Minkowski or anti-de-Sitter spaces. $(n, m, \ldots, r, s)$ are indices for internal compact spaces and $(a, b, \ldots)$ are usually their holomorphic version. $(i, j, \ldots)$ are sometimes used as indices which transform in some representation of the gauge groups, other times they indicate some general coordinates system. This distinction will be clear from the context. At this point we can introduce some useful definitions, which we have omitted in the previous chapter.

A.1 Operators on Differential Forms

The *Hodge dual* is an operator, which, for a d-dimensional manifold M_d, acts on k-forms in the following way:

$$*_d : \Lambda^k T^* M \to \Lambda^{d-k} T^* M \tag{A.1.1}$$

where $\Lambda^k T^* M$ is the space of k-forms. In components we have that for a k-form ω_k

$$(*\omega_k)_{n_1 \ldots i_k} = \frac{1}{(d-k)!} \sqrt{|g|} \epsilon_{n_1 \ldots n_k m_{k+1} \ldots m_d} \omega^{m_{k+1} \ldots m_d} \tag{A.1.2}$$

where we need a metric g_{mn} in order to rise and to lower the indices. where the form ϵ is the levi-civita form in d dimensions:

$$\epsilon_{n_1 \ldots n_n} = \begin{cases} sign \begin{pmatrix} 1 & 2 & \ldots & d \\ n_1 & n_2 & \ldots & n_d \end{pmatrix}, & \text{all indices } n_1, n_2, \ldots, n_d \text{ are distinct;} \\ \\ 0, \quad \text{otherwise.} \end{cases} \tag{A.1.3}$$

The Symbol, $T^* M$, means the cotangent bundle for any manifold M. Another important operator is the wedge product between forms.

The *wedge product* or external product between two forms is a map, such that

$$(\ \wedge \ ) : (\Lambda^k T^* M, \Lambda^l T^* M) \to \Lambda^{k+l} T^* M \tag{A.1.4}$$

for every k-form and l-form and we have that the components

$$(\alpha \wedge \beta)_{n_1 \dots n_{k+l}} = \sum_{\sigma \in S_{k+l}} sign\ \sigma\ \alpha_{n_{\sigma_1} \dots n_{\sigma_k}} \beta_{n_{\sigma_{k+1}} \dots n_{\sigma_{k+l}}}. \tag{A.1.5}$$

where σ is an element in the permutation group S_{k+l}. The fundamental property of the wedge product is that

$$\alpha_k \wedge \beta_l = (-1)^{kl} \alpha_l \wedge \beta_k. \tag{A.1.6}$$

for this property when k and l are odd the wedge product vanishes. We have another useful operator

The *contraction* is a map, such that

$$\iota_n \equiv \iota_{\partial_n} : \Lambda^k T^* M \to \Lambda^{k-1} T^* M \tag{A.1.7}$$

such that the action on the basis of differentials is

$$\iota_n(dx^{n_1} \wedge \dots \wedge dx^{n_p}) = p \delta_n^{[n_1} dx^{n_2} \wedge \dots \wedge dx^{n_p]}; \tag{A.1.8}$$

sometimes one can use an equivalent notation for contraction of a vector v on a one-form w,

$$v \llcorner w. \tag{A.1.9}$$

Finally, the *exterior differential* on a d-dimensional manifold is a map, such that $d^2 = 0$

$$d : \Lambda^k T^* M \to \Lambda^{k+1} T^* M \tag{A.1.10}$$

and, in coordinates $\{x^n\}$ with $n = 1, \dots, d$, it is given by

$$d = \partial_n dx^n. \tag{A.1.11}$$

The action on a k-form ω_k is given by (in coordinates)

$$d\omega = \partial_{[n} \omega_{n_1 \dots n_k]} dx^n \wedge dx^{n_1} \wedge \dots \wedge dx^{n_k}. \tag{A.1.12}$$

In an almost complex manifold, M, one can define the *Dolbeault operators* ∂ and $\overline{\partial}$ such that

$$\partial \omega^{(p,q)} = \omega^{(p+1.q)}, \qquad \overline{\partial} \omega^{(p,q)} = \omega^{(p,q+1)}, \tag{A.1.13}$$

where the ω's are all three-form as real form, but different (p, q) forms with respect to an ACS I defined on the manifold M. Moreover the exterior differential on a (p, q)-form reads

$$d\omega^{(p,q)} = \omega^{(p-1,q+2)} + \omega^{(p,q+1)} + \omega^{(p+1,q)} + \omega^{(p+2,q-1)}. \tag{A.1.14}$$

When M is a complex manifold the $\omega^{(p-1,q+2)} = 0$ and $\omega^{(p+2,q-1)} = 0$, and the exterior differential can be decomposed into $d = \partial + \overline{\partial}$.

The *Lie derivative* of a differential form, ω_p, along a vector field, v, is (Cartan's formula):

$$\mathcal{L}_v \omega_p = \{d, \iota_v\} \omega_p = \iota_v d\omega_p + d(\iota_v \omega_p). \tag{A.1.15}$$

Appendix B

More AdS$_6$ systems

B.1 AdS$_6$ solutions in eleven-dimensional supergravity

We will show here that there are no AdS$_6 \times M_5$ solutions in eleven-dimensional supergravity.[1] This case is easy enough that we will deal with it by using the original fermionic form of the supersymmetry equations, without trying to reformulate them in terms of bilinears as we did in the main text for IIB.

The bosonic fields of eleven-dimensional supergravity consist of a metric g_{11} and a three-form potential C with four-form field strength $G = dC$. The action is

$$S = \frac{1}{(2\pi)^8 \ell_p^9} \int R *_{11} 1 - \frac{1}{2} G \wedge *_{11} G - \frac{1}{6} C \wedge G \wedge G \, , \tag{B.1.1}$$

with ℓ_p the eleven-dimensional Planck length.

We take the eleven-dimensional metric to have the warped product form

$$ds^2_{11} = e^{2A} ds^2_{\mathrm{AdS}_6} + ds^2_{M_5} \, . \tag{B.1.2}$$

In order to preserve the SO$(2,5)$ invariance of AdS$_6$ we take the warping factor to be a function of M_5, and G to be a four-form on M_5. Preserved supersymmetry is equivalent to the existence of a Majorana spinor ϵ satisfying the equation

$$D_M \epsilon + \frac{1}{288} \left(\gamma_M^{(11)NPQR} - 8 \delta_M^N \gamma_{(11)}^{PQR} \right) G_{NPQR} \, \epsilon = 0 \, . \tag{B.1.3}$$

[1]This conclusion was also reached independently by F. Canoura and D. Martelli and later reported also in [182].

We may decompose the eleven-dimensional gamma matrices via

$$\gamma_\mu^{(6+5)} = e^A \gamma_\mu^{(6)} \otimes 1 \,, \qquad \gamma_{m+5}^{(6+5)} = \gamma^{(6)} \otimes \gamma_m^{(5)} \,. \tag{B.1.4}$$

Here $\gamma_\mu^{(6)}$, $\mu = 0, \ldots, 5$ are a basis of six-dimensional gamma matrices ($\gamma^{(6)}$ is the chiral gamma), while $\gamma_m^{(5)}$, $m = 1, \ldots, 5$ are a basis of five-dimensional gamma matrices. The spinor Anzatz preserving $\mathcal{N} = 1$ supersymmetry in AdS$_6$ is

$$\epsilon = \zeta_+ \eta_+ + \zeta_- \eta_- + \text{c.c.} \tag{B.1.5}$$

where $\zeta_\pm$ are the chiral components of a Killing spinor on AdS$_6$ satisfying

$$D_\mu \zeta_\pm = \frac{1}{2} \gamma_\mu^{(6)} \zeta_\mp \,, \tag{B.1.6}$$

while $\eta_\pm$ are Dirac spinors on M_5.

Substituting (B.1.5) in (B.1.3) leads to the following equations for the spinors $\eta_\pm$:

$$\frac{1}{2} e^{-A} \eta_\mp \pm \frac{1}{2} \gamma^m_{(5)} \partial_m A \, \eta_\pm + \frac{1}{12} *_5 G_m \gamma^m_{(5)} \eta_\pm = 0 \,, \tag{B.1.7a}$$

$$D_m \eta_\pm \pm \frac{1}{4} *_5 G_m \eta_\pm \mp \frac{1}{6} *_5 G_n \gamma_m^{(5)} \gamma^n_{(5)} \eta_\pm = 0 \,. \tag{B.1.7b}$$

Using (B.1.7) it is possible to derive the following differential conditions on the norms $\eta_\pm^\dagger \eta_\pm \equiv e^{B_\pm}$ of the internal spinors:

$$*_5 G = \mp 6 \, d_5 B_\pm \,, \tag{B.1.8}$$

$$B_+ = -B_- + \text{const.} \,. \tag{B.1.9}$$

We can absorb the constant in a redefinition of η_- so that $B_+ = -B_- \equiv B$; thus

$$*_5 \, G = -6 \, d_5 B \,. \tag{B.1.10}$$

The equation of motion for G is then automatically satisfied; in absence of sources, the Bianchi identity reads $d_5 G = 0$, resulting in $*_5 G$ being harmonic. This is in contradiction with $*_5 G$ being exact. This still leaves open the possibility of adding M5-branes extended along AdS$_6$, which would modify the Bianchi identity to $d_5 G = \delta_{\text{M5}}$. However, we will now show that even that possibility is not realized.

Defining $\tilde\eta_\pm \equiv e^{-B/2} \eta_\pm$ we can rewrite (B.1.7b) as

$$D_m \tilde\eta_\pm \pm \partial_n B \, \gamma_m^n \tilde\eta_\pm = 0 \,. \tag{B.1.11}$$

Upon rescaling the metric $ds^2_{M_5} \to e^{-4B} ds^2_{M_5'}$ the equation for $\tilde\eta_+$ becomes

$$D'_m \tilde\eta_+ = 0 \,. \tag{B.1.12}$$

In five dimensions the only compact manifold admitting parallel spinors is the torus T^5, so we are forced to set $ds^2_{M'_5} = ds^2_{T^5}$. Similarly if we rescale the metric $ds^2_{M_5} \to e^{4B} ds^2_{M''_5}$ the equation for $\tilde{\eta}_-$ becomes

$$D''_m \tilde{\eta}_- = 0 \,, \tag{B.1.13}$$

so that $ds^2_{M''_5} = ds^2_{T^5}$.[2] We are thus led to the relation

$$e^{-4B} ds^2_{M'_5} = e^{4B} ds^2_{M''_5} \,. \tag{B.1.14}$$

Since $ds^2_{M'_5} = ds^2_{M''_5} = ds^2_{T^5}$, this implies $B = 0$, and hence $G = 0$ (from (B.1.10)). This makes the whole system collapse to flat space.

B.2 The massive IIA solution

We have shown in appendix B.1 that there are no AdS$_6$ solutions in eleven-dimensional supergravity — and hence in massless IIA. As for massive IIA, it was shown in [109] that the only solution is the one in [99]. In this section, we show how that solution fits in the IIA version of the formalism presented in the main text.

For the bispinors Φ and Ψ, we will keep using the definitions given in section 4.1 and the parameterizations given in section 5.2. The main difference is the system for supersymmetry, which in IIB was (5.1.3), and in IIA reads instead

$$d_H \left[e^{3A-\phi} (\Phi_- + \Phi_+)^0 \right] + 2 e^{2A-\phi} (\Psi_- - \Psi_+)^0 = 0 \,, \tag{B.2.1a}$$
$$d_H \left[e^{4A-\phi} (\Psi_- + \Psi_+)^\alpha \right] + 3 e^{3A-\phi} (\Phi_- - \Phi_+)^\alpha = 0 \,, \tag{B.2.1b}$$
$$d_H \left[e^{5A-\phi} (\Phi_- + \Phi_+)^\alpha \right] + 4 e^{4A-\phi} (\Psi_- - \Psi_+)^\alpha = 0 \,, \tag{B.2.1c}$$
$$d_H \left[e^{6A-\phi} (\Psi_- + \Psi_+)^0 \right] + 5 e^{5A-\phi} (\Phi_- - \Phi_+)^0 = -\frac{1}{4} e^{6A} *_4 \lambda F \,, \tag{B.2.1d}$$
$$d_H \left[e^{5A-\phi} (\Phi_- - \Phi_+)^0 \right] = 0 \,; \tag{B.2.1e}$$
$$||\eta^1||^2 = ||\eta^2||^2 = e^A \,. \tag{B.2.1f}$$

The bispinors Φ and Ψ can be easily extracted from the supersymmetry parameters: in terms of the vielbein $\{e^\alpha, e^4\}$,

$$e^\alpha = -w^{-1/6} \frac{1}{2} \sin\alpha \, \hat{e}^\alpha \,, \qquad e^4 = -w^{-1/6} d\alpha \,, \qquad w \equiv \frac{3}{2} F_0 \cos\alpha \,, \tag{B.2.2}$$

where $\hat{e}^\alpha$ are the left-invariant one-forms on S^3, satisfying

$$d\hat{e}^\alpha = \frac{1}{2} \epsilon^\alpha{}_{\beta\gamma} \hat{e}^\beta \wedge \hat{e}^\gamma \,, \tag{B.2.3}$$

[2]One might try to avoid this conclusion by setting $\tilde{\eta}_-$ to zero. However, (B.1.7a) would then also set $\tilde{\eta}_+$ to zero.

we have

$$\Phi_\pm = \frac{1}{8}(\pm 1 - \cos\alpha)\left((1 \pm \mathrm{vol}_4)\mathrm{Id}_2 + i\left(\frac{1}{2}\epsilon^{\alpha}{}_{\beta\gamma}e^{\beta}\wedge e^{\gamma} \mp e^{\alpha}\wedge e^{4}\right)\sigma_\alpha\right) \ ; \tag{B.2.4a}$$

$$\Psi_\pm = \frac{1}{8}\sin\alpha\,(1 \pm *_4)\Big(\mp e^4\mathrm{Id}_2 + ie^{\alpha}\sigma_\alpha\Big)\ , \tag{B.2.4b}$$

being σ_α the Pauli matrices.

The physical fields then read:

$$e^{\phi} = w^{-5/6}\ , \qquad e^{A} = \frac{3}{2}w^{-1/6}\ , \qquad ds^2_{M_4} = e^{\alpha}e^{\alpha} + e^4e^4\ , \qquad F_4 = \frac{10}{3}w\,\mathrm{vol}_4\ . \tag{B.2.5}$$

Appendix C

More on the AdS$_7$ system of type II supergravity

In these appendices there is some overlap with the master thesis [135], however we decided to include the following material for convenience.

C.1 Supercharges

At the beginning of section 5.1.2 we reviewed an old argument that shows how a solution of the form AdS$_7 \times M_3$ can also be viewed as a solution of the type Mink$_6 \times M_4$. In this appendix we show how the AdS$_7 \times M_3$ supercharges get translated in the Mink$_6 \times M_4$ framework.

A decomposition of gamma matrices appropriate to six-dimensional compactifications reads

$$\gamma^{(6+4)}_\mu = e^{A_4}\gamma^{(6)}_\mu \otimes 1 \ , \qquad \gamma^{(6+4)}_{m+5} = \gamma^{(6)} \otimes \gamma^{(4)}_m \ . \tag{C.1.1}$$

Here $\gamma^{(6)}_\mu$, $\mu = 0, \ldots, 5$, are a basis of six-dimensional gamma matrices, while $\gamma^{(4)}_m$, $m = 1, \ldots, 4$ are a basis of four-dimensional gamma matrices. For a supersymmetric Mink$_6 \times M_4$ solution, the supersymmetry parameters can be taken to be

$$\begin{aligned} \epsilon^{(6+4)}_1 &= \zeta^0_+ \otimes \eta^1_+ + \zeta^{0\,c}_+ \otimes \eta^{1\,c}_+ \ , \\ \epsilon^{(6+4)}_2 &= \zeta^0_+ \otimes \eta^2_\mp + \zeta^{0\,c}_+ \otimes \eta^{2\,c}_\mp \ , \end{aligned} \tag{C.1.2}$$

where ζ_+ is a constant spinor; $_\mp$ denotes the chirality, and ^{c} Majorana conjugation both in six and four dimensions. Supersymmetry implies that the norms of the internal spinors satisfy $||\eta^1||^2 \pm ||\eta^2||^2 = c_\pm e^{\pm A_4}$, where $c_\pm$ are constant.

On the other hand, for seven-dimensional compactifications a possible gamma matrix decomposition reads

$$\begin{aligned} \gamma_\mu^{(7+3)} &= e^{A_3}\gamma_\mu^{(7)} \otimes 1 \otimes \sigma_2 \ , \\ \gamma_{i+6}^{(7+3)} &= 1 \otimes \sigma_i \otimes \sigma_1 \ . \end{aligned} \tag{C.1.3}$$

This time $\gamma_\mu^{(7)}$, $\mu = 0, \ldots, 6$, are a basis of seven-dimensional gamma matrices, and σ_i, $i = 1, 2, 3$, are a basis of gamma matrices in three dimensions (which in flat indices can be taken to be the Pauli matrices). For a supersymmetric solution of the form AdS$_7 \times M_3$, the supersymmetry parameters are now of the form

$$\begin{aligned} \epsilon_1^{(7+3)} &= (\zeta \otimes \chi_1 + \zeta^c \otimes \chi_1^c) \otimes v_+ \ , \\ \epsilon_2^{(7+3)} &= (\zeta \otimes \chi_2 \mp \zeta^c \otimes \chi_2^c) \otimes v_\mp \ . \end{aligned} \tag{C.1.4}$$

Here, $\chi_{1,2}$ are spinors on M_3, with $\chi_{1,2}^c \equiv B_3\chi_{1,2}^*$ their Majorana conjugates; a possible choice of B_3 is $B_3 = \sigma_2$. ζ is a spinor on AdS$_7$, and $\zeta^c \equiv B_7\zeta^*$ is its Majorana conjugate; there exists a choice of B_7 which is real and satisfies $B_7\gamma_\mu = \gamma_\mu^* B_7$. (It also obeys $B_7B_7^* = -1$, which is the famous statement that one cannot impose the Majorana condition in seven Lorentzian dimensions.) The ten-dimensional conjugation matrix can then be taken to be $B_{10} = B_7 \otimes B_3 \otimes \sigma_3$; the last factor in (C.1.4), $v_\pm$, are then spinors chosen in such a way as to give the $\epsilon_i^{(7+3)}$ the correct chirality, and to make them Majorana; with the above choice of B_{10}, $v_+ = \frac{1}{\sqrt{2}}\binom{1}{-1}$, $v_- = \frac{1}{\sqrt{2}}\binom{1}{1}$. The minus sign (for the IIA case) in front of the term $\zeta^c \otimes \chi_2^c$ in (C.1.4) is due to the fact that, both in seven Lorentzian and three Euclidean dimensions, conjugation does not square to one: $(\zeta^c)^c = -\zeta$, $(\chi^c)^c = -\chi$.

The presence of the cosmological constant in seven dimensions means that ζ is not constant, but rather that it satisfies the so-called Killing spinor equation, which for $R_{\mathrm{AdS}} = 1$ reads

$$D_\mu\zeta = \frac{1}{2}\gamma_\mu^{(7)}\zeta \ . \tag{C.1.5}$$

One class of solutions to this equation [183, 184] is simply of the form

$$\zeta_+ = \rho^{1/2}\zeta_+^0 \ . \tag{C.1.6}$$

The coordinate ρ appears in (5.1.7), which expresses AdS$_7$ as a warped product of Mink$_6$ and $\mathbb{R}$. ζ_+^0 is a spinor constant along Mink$_6$ and such that $\gamma_{\hat\rho}\zeta_+^0 = \zeta_+^0$ (the hat denoting a flat index).

Just like for Mink$_6 \times M_4$, supersymmetry again implies that the norms of the internal spinors $\chi^{1,2}$ should be related to the warping function: $||\chi_1||^2 \pm ||\chi_2||^2 = c_\pm e^{\pm A_3}$, where $c_\pm$ are constant. We will now see, however, that for AdS$_7 \times M_3$ actually $c_- = 0$. We use the ten-dimensional system in [38, Eq. (3.1)]. As we mentioned in section 5.1, it can be used to derive quickly the system 5.1.1, while applying it directly to AdS$_7 \times M_3$ to derive (5.1.11) is more lengthy. For our purposes, however, it will be enough to apply one equation of that system to the AdS$_7 \times M_3$ setup, namely

$$d\tilde{K} = \iota_K H \tag{C.1.7}$$

This is equation (3.1b) in [38], but it appeared previously in [116, 185, 186]. K and $\tilde{K}$ are the ten-dimensional vector and one-form defined by $K = \frac{1}{64}(\bar{\epsilon}_1\gamma_M^{(10)}\epsilon_1 + \bar{\epsilon}_2\gamma_M^{(10)}\epsilon_2)dx^M$ and $\tilde{K} = \frac{1}{64}(\bar{\epsilon}_1\gamma_M^{(10)}\epsilon_1 - \bar{\epsilon}_2\gamma_M^{(10)}\epsilon_2)dx^M$. Plugging the decomposed spinors (C.1.4) in these definitions and calling $\beta_1 = e^{A_3}(\frac{1}{8}\bar{\zeta}\gamma_\mu^{(7)}\zeta)dx^\mu$, the part of (C.1.7) along AdS$_7$ leads to $e^{A_3}d_7\beta_1(||\chi_1||^2 - ||\chi_2||^2) = (d_7\beta_1)c_- = 0$, where d_7 is the exterior derivative along AdS$_7$. (The right hand side does not contribute, because H has only internal components.) On the other hand, using the Killing spinor equation (C.1.5) in AdS$_7$, we have that $d_7\beta_1 = e^{2A_3}(\bar{\zeta}\gamma_{\mu\nu}^{(7)}\zeta)dx^{\mu\nu} \equiv \beta_2$. A spinor in seven dimensions can be in different orbits (defining an SU(3) or an SU(2)$\ltimes\mathbb{R}^5$ structure [187, 188]), but for none of them the bilinear β_2 is identically zero. Consequently, the norms of the two Killing spinors have to be equal, namely $c_- = 0$.

Let us now see how to translate the spinors ϵ_i for an AdS$_7 \times M_3$ solution into a language relevant for Mink$_6 \times M_4$. First, we split the seven-dimensional gamma matrices $\gamma_\mu^{(7)}$; the first six give a basis of gamma matrices in six dimensions, $\tilde{\gamma}_\mu^{(6)} = \rho\gamma_\mu^{(7)}$, $\mu = 0, \ldots, 5$, while the radial direction, $\gamma_{\hat{\rho}}^{(7)} = \gamma^{(6)}$ becomes the chiral gamma in six dimensions. (The hat denotes a flat index.) This split is by itself not enough to turn (C.1.3) into (C.1.1), because the three-dimensional gamma's in (C.1.3) have no $\gamma^{(6)}$ in front. This can be cured by applying a change of basis:

$$\gamma_M^{(6+4)} = O\gamma_M^{(7+3)}O^{-1}\,, \qquad O = \frac{1}{\sqrt{2}}(1 - i\gamma_{\hat{\rho}}^{(7+3)})\,, \tag{C.1.8}$$

with, however, a change of basis in six dimensions: $\gamma_\mu^{(6)} \to -i\gamma^{(6)}\gamma_\mu^{(6)}$. Likewise, the spinors (C.1.4) are related to (C.1.2) by

$$\epsilon_i^{(6+4)} = O\epsilon_i^{(7+3)}\,, \tag{C.1.9}$$

if we take

$$\eta_1 = \rho^{1/2}\chi_1 \otimes v_+ = \frac{1}{\sqrt{2}}\rho^{1/2}\chi_1 \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \eta_2 = \rho^{1/2}\chi_2 \otimes v_\mp = \frac{1}{\sqrt{2}}\rho^{1/2}\chi_2 \otimes \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}. \tag{C.1.10}$$

Notice that the two η^i have equal norm, because the χ^i have equal norm, as shown earlier. Moreover, since the norm of the χ^i is $e^{A_3/2}$, and because of the factor $\rho^{1/2}$ in (C.1.10), the η^i have norm equal to $\rho^{1/2}e^{A_3/2}$; recalling (6.1.9), this is equal to $e^{A_4/2}$, as it should.

Besides (C.1.6), there is also a second class of solution to the Killing spinor equation $D_\mu\zeta = \frac{1}{2}\gamma_\mu^{(7)}\zeta$ on AdS$_7$: it reads $\zeta = (\rho^{-1/2} + \rho^{1/2}x^\mu\gamma_\mu^{(7)})\zeta_-^0$, where now $\gamma_{\hat{\rho}}\zeta_-^0 = -\zeta_-^0$. If we plug this into (C.1.4) and use the above procedure (C.1.9) to translate it in the Mink$_6 \times M_4$ language, we find a generalization of (C.1.2) where both a positive and negative chirality six-dimensional spinor appear (namely, $x^\mu\gamma_\mu\zeta_-^0$ and ζ_-^0) instead of just a positive chirality spinor ζ_+^0. Because of the $x^\mu\gamma_\mu$ factor, this spinor Ansatz would break Poincaré invariance if used by itself; if four supercharges of the form (C.1.2) are preserved, Poincaré invariance is present, and these additional supercharges simply signal that an AdS$_7 \times M_3$ solution is $\mathcal{N} = 2$ in terms of Mink$_6 \times M_4$.

C.2 Killing spinors on S^4

The AdS$_7 \times S^4$ is a familiar Freund–Rubin solution; the flux is taken to be proportional to the internal volume form, $G_4 = g\mathrm{vol}_{S^4}$. The eleven-dimensional supersymmetry transformation reads $\left(\nabla_M + \frac{1}{144}G_{NPQR}(\gamma^{NPQR}{}_M - 8\gamma^{NPQ}\delta^R_M)\right)\epsilon_{11} = 0$; decomposing $\epsilon_{11} = \sum_{a=1}^4 \zeta_a \otimes \eta_a + \text{c.c.}$, and using (C.1.5), one reduces the requirement of supersymmetry (for $R_{\text{AdS}} = 1$) to taking $g = 3/4$, and to the equation

$$(\nabla_m - \frac{1}{2}\gamma\gamma^m)\eta = 0 \tag{C.2.1}$$

on S^4. This is an alternative form of the Killing spinor equation; it was solved in [189] in any dimension. However, we are using different coordinates, adapted to the S^1 reduction used in section 5.4.1; we will here solve (C.2.1) again, using more or less the same method.

The idea is to start from the easiest components of the equation, and to work one's way to the more complicated ones. Our coordinates in section 5.4.1 are α, β, γ, y, the latter being the reduction coordinate. Our vielbein reads $e^1 = d\alpha$, $e^2 = \frac{1}{2}\sin(\alpha)d\beta$, $e^3 = \frac{1}{2}\sin(\alpha)\sin(\beta)d\gamma$, $e^4 = \frac{1}{2}\sin(\alpha)(dy + \cos(\beta)d\gamma)$. We begin with the α component of (C.2.1):

$$\partial_\alpha\eta = \frac{1}{2}\gamma\gamma_1\eta \qquad \Rightarrow \qquad \eta = e^{\frac{1}{2}\alpha\gamma\gamma_1}\eta_1\ . \tag{C.2.2}$$

The next component we use is

$$\left(\partial_\beta - \frac{1}{4}\cos(\alpha)\right)\eta = \frac{1}{4}\sin(\alpha)\gamma\gamma_2\eta\ . \tag{C.2.3}$$

This can be manipulated as follows:

$$0 = \left(\partial_\beta - \frac{1}{4}e^{\alpha\gamma\gamma_1}\gamma_{12}\right)\eta = e^{\frac{1}{2}\alpha\gamma\gamma_1}\left(\partial_\beta - \frac{1}{4}\gamma_{12}\right)\eta_1 \qquad \Rightarrow \qquad \eta_1 = e^{\frac{1}{4}\beta\gamma_{12}}\eta_2\ . \tag{C.2.4}$$

We proceed in a similar way for the two remaining coordinates; the details are complicated, and we omit them here. The final result is

$$\eta = \exp\left[\frac{\alpha}{2}\gamma\gamma_1\right]\exp\left[\frac{\beta}{4}\gamma_{12} + \frac{\beta-\pi}{4}\gamma_{34}\right]\exp\left[\frac{y+\gamma}{4}\gamma_{13} + \frac{y-\gamma}{4}\gamma_{24}\right]\eta_0 \tag{C.2.5}$$

where η_0 is a constant spinor. When we reduce, we demand that $\partial_y\eta = 0$, which becomes $(\gamma_{13} + \gamma_{24})\eta_0 = 0$; this condition indeed keeps two out of four spinors, as anticipated in our discussion in section 5.4.1.

C.3 Sufficiency of the system (5.1.11)

In section 5.1.2 we obtained the system of equations (5.1.11) starting from (5.1.3) and using the fact that AdS$_7$ can be considered as a warped product of Mink$_6$ and $\mathbb{R}$. In this section we will explain how one can show that (5.1.11) is completely equivalent to supersymmetry for $\mathrm{AdS}_7 \times M_3$ with a direct computation. Our strategy will be very similar to the one in [133, Sec. A.4], with some relevant differences that we will promptly point out.

To begin with, we write the system of equations resulting from setting to zero the type II supersymmetry variations (of gravitinos and dilatinos) using the spinorial decomposition (C.1.4):[1]

$$\left(D_m - \frac{1}{4}H_m\right)\chi_1 - \frac{e^\phi}{8}F\sigma_m\chi_2 = 0\ , \tag{C.3.1a}$$

$$\left(D_m + \frac{1}{4}H_m\right)\chi_2 - \frac{e^\phi}{8}\lambda(F)\sigma_m\chi_1 = 0\ , \tag{C.3.1b}$$

$$\frac{1}{2}e^{-A}\chi_1 - \frac{i}{2}\partial A\chi_1 + i\frac{e^\phi}{8}F\chi_2 = 0\ , \tag{C.3.1c}$$

$$\frac{1}{2}e^{-A}\chi_2 + \frac{i}{2}\partial A\chi_2 - i\frac{e^\phi}{8}\lambda(F)\chi_1 = 0\ , \tag{C.3.1d}$$

$$\left(D - \frac{1}{4}H\right)\chi_1 + i\frac{7}{2}e^{-A}\chi_1 + \left(\frac{7}{2}\partial A - \partial\phi\right)\chi_1 = 0\ , \tag{C.3.1e}$$

$$\left(D + \frac{1}{4}H\right)\chi_2 - i\frac{7}{2}e^{-A}\chi_2 + \left(\frac{7}{2}\partial A - \partial\phi\right)\chi_2 = 0\ . \tag{C.3.1f}$$

As in [133, Sec. A.4], we introduce a set of intrinsic torsions p^a_m, q^a_m, and T^a, $\hat{T}^a$, with $a = 1, 2$:

$$\left(D_m - \frac{1}{4}H_m\right)\chi_1 \equiv p^1_m\chi_1 + q^1_m\chi^c_1\ , \qquad \left(D_m + \frac{1}{4}H_m\right)\chi_2 \equiv p^2_m\chi_2 + q^2_m\chi^c_2\ , \tag{C.3.2a}$$

$$\left(D - \frac{1}{4}H\right)\chi_1 \equiv T^1\chi_1 + \hat{T}^1\chi^c_1\ , \qquad \left(D + \frac{1}{4}H\right)\chi_2 \equiv T^2\chi_2 + \hat{T}^2\chi^c_2\ , \tag{C.3.2b}$$

where $D = \gamma^m_{(7)}D_m$, $H_m \equiv \frac{1}{2}H_{mnp}\gamma^{np}_{(7)}$, $H \equiv \frac{1}{6}H_{mnp}\gamma^{mnp}_{(7)}$ as usual. We used the fact that χ_1 and χ^c_1 (or χ_2 and χ^c_2) constitute a basis for the three-dimensional spinors. Taking tensor products of these two bases, we also obtain a basis for bispinors, on which we can now expand F:

$$F \equiv R_{00}\,\chi_1\otimes\chi_2^\dagger + R_{10}\,\chi^c_1\otimes\chi^\dagger_2 + R_{01}\,\chi_1\otimes\chi^{c\dagger}_2 + R_{11}\,\chi^c_1\otimes\chi^{c\dagger}_2\ . \tag{C.3.3}$$

Using (C.3.2) and (C.3.3) in (C.3.1), we can rewrite the conditions for unbroken supersymmetry as a set of equations relating the intrinsic torsions to the coefficients R_{ij}. Let us call this system of equations the "spinorial system". Using instead (C.3.2) and (C.3.3) in (5.1.11), we obtain a second set of equations, again in terms of the intrinsic torsions and R_{ij}; let us call this system the "form system". Our aim is to show the equivalence between the spinorial and the form systems.

Although we are using the same technique appearing in [133, Sec. A.4] (there applied to four-dimensional vacua), proving this equivalence in the case at hand is more involved. Relying

[1] We choose to show the equivalence in the IIA case, hence we pick $\epsilon_1^{(7+3)}$ and $\epsilon_2^{(7+3)}$ with opposite chirality.

on a superficial counting, it would seem that the form system contains fewer equations than the spinorial one. To see why this happens, we first notice that the definitions (C.3.2) are redundant. Indeed the torsions T^a and $\hat{T}^a$ can be rewritten in terms of the torsions p^a, q^a and H; however, in three dimensions, $\gamma^{(7)}_{mnp}$, hence H, is proportional to the identity (use (5.2.2) with $\alpha = H$). Thus in (C.3.2b) four complex numbers (T's and $\hat{T}$'s) are used to describe a single real number H. This suggests that some of the equations in the spinorial system are redundant and could be dropped. However, this redundancy is not manifest.

To make it manifest, we could use the following strategy. On the one hand (C.3.1a) and (C.3.1b) give a natural expansion of the torsions p^a and q^a in terms of the vielbein e^b, with $a \neq b$, defined by the spinor χ_b (see (5.2.3) and (5.2.5)); that is, they transform into equations for the components $q^1 \cdot e^2_3$, $q^1 \cdot e^2_1$ and so forth. On the other hand the intrinsic torsions T^a and $\hat{T}^a$ give expressions like $q^1 \cdot e^1_3$, $q^1 \cdot e^1_1$. Therefore, we would need a formula relating the vielbein e^1 defined by χ_1 to the vielbein e^2 defined by χ_2.

Actually, there exists a simpler method. Indeed we can use the following equations,

$$\begin{aligned} &d_H(e^{2A_3-\phi}\mathrm{Re}\psi^1_-) = 0 \,, \\ &d_H(e^{4A_3-\phi}\mathrm{Im}\psi^1_-) = 0 \,, \\ &d_H(e^{4A_3-\phi}\psi^2_-) = 0 \,, \end{aligned} \tag{C.3.4}$$

obtained by simply applying d_H to the equations (5.1.11a), (5.1.11b) and (5.1.11c) respectively (in other words, they are redundant with respect to the original system (5.1.11)). If we now express (C.3.4) in terms of (C.3.2), and add the resulting equations to the form system we obtained earlier, we obtain a new, equivalent expression for the form system. With some linear manipulations, it can now be shown that it is equivalent to the spinorial system. This concludes our alternative proof that (5.1.11) is completely equivalent to the requirement of unbroken supersymmetry.

Appendix D

More on AdS$_5$

D.1 Supersymmetry variations and the Killing vector

Setting to zero the type IIA supersymmetry variations (of gravitinos and dilatinos) yields the following set of equations[1]

$$0 = \left(D_M + \frac{1}{4}H_M\right)\epsilon_1 + \frac{e^\phi}{16}\lambda(F)\Gamma_M\epsilon_2\ , \tag{D.1.1a}$$

$$0 = \left(D_M - \frac{1}{4}H_M\right)\epsilon_2 + \frac{e^\phi}{16}F\,\Gamma_M\epsilon_1\ , \tag{D.1.1b}$$

$$0 = \left(D - \partial\phi + \frac{1}{4}H\right)\epsilon_1\ , \tag{D.1.1c}$$

$$0 = \left(D - \partial\phi - \frac{1}{4}H\right)\epsilon_1\ , \tag{D.1.1d}$$

where suppressed indices are contracted with antisymmetric products of gamma matrices and ϵ_1, ϵ_2 are Spin$(1,9)$ Majorana–Weyl spinors of opposite chirality.

We wish to obtain a set of differential and algebraic equations for the Spin(5) spinors η_1, η_2 and so we decompose the the generators of Cliff$(1,9)$ as

$$\Gamma_\mu = e^A\gamma_\mu^{(1,4)} \otimes 1 \otimes \sigma_3 \qquad \Gamma_i = 1 \otimes \gamma_m^{(5)} \otimes \sigma_1\ , \tag{D.1.2}$$

where $\mu = 0,\dots,4$, $m = 1,\dots,5$ and σ_1 and σ_3 are the Pauli matrices; $\gamma_\mu^{(1,4)}$ generate Cliff$(1,4)$ and $\gamma_m^{(5)}$ Cliff(5). Accordingly, the chirality matrix Γ_{11} and the intertwiner B_{10} relating Γ_M and Γ_M^*, are decomposed as

$$\Gamma_{11} = 1 \otimes 1 \otimes \sigma_2\ , \qquad B_{10} = B_{1,4} \otimes B_5 \otimes \sigma_1\ . \tag{D.1.3}$$

[1]The first two equations follow from setting the gravitino variation $\delta\psi_M$ to zero, while the last two equations follow from $\Gamma^M\delta\psi_M - \delta\lambda = 0$ where λ is the dilatino.

Furthermore, the supersymmetry parameters ϵ_1, ϵ_2 split as

$$\epsilon_1 = (\zeta \otimes \eta_1 + \zeta^c \otimes \eta_1^c) \otimes \theta \ , \tag{D.1.4a}$$
$$\epsilon_2 = (\zeta \otimes \eta_2 + \zeta^c \otimes \eta_2^c) \otimes \theta^* \ , \tag{D.1.4b}$$

where $\eta^c_{1,2} = B_5 \eta^*_{1,2}$ and $\zeta^c = B_{1,4}\zeta^*$. ζ is a $\mathrm{Spin}(1,4)$ spinor obeying the AdS$_5$ Killing spinor equation

$$D_\mu \zeta = \frac{1}{2}\gamma_\mu \zeta \ , \tag{D.1.5}$$

while θ obeys $\sigma_2\theta = \theta$ and $\sigma_1\theta = \theta^*$.

Applying the above decomposition, the equations (D.1.1) become

$$0 = \left(D_i + \frac{1}{4}H_i\right)\eta_1 + \frac{e^\phi}{16}\lambda(F)\gamma_m^{(5)}\eta_2 \ , \tag{D.1.6a}$$
$$0 = \left(D_i - \frac{1}{4}H_i\right)\eta_2 + \frac{e^\phi}{16}F\,\gamma_m^{(5)}\eta_1 \ , \tag{D.1.6b}$$
$$0 = \left(\frac{i}{2}e^{-A} - \frac{1}{2}\partial A\right)\eta_1 - \frac{e^\phi}{16}\lambda(F)\eta_2 \ , \tag{D.1.6c}$$
$$0 = \left(\frac{i}{2}e^{-A} + \frac{1}{2}\partial A\right)\eta_2 + \frac{e^\phi}{16}F\eta_1 \ , \tag{D.1.6d}$$
$$0 = \left(\frac{5i}{2}e^{-A} - D - \frac{5}{2}\partial A + \partial\phi - \frac{1}{4}H\right)\eta_1 \ , \tag{D.1.6e}$$
$$0 = \left(\frac{5i}{2}e^{-A} + D + \frac{5}{2}\partial A - \partial\phi - \frac{1}{4}H\right)\eta_2 \ . \tag{D.1.6f}$$

Using equations (D.1.6a) and (D.1.6b) it is straightforward to show that $\xi \equiv \frac{1}{2}(\eta_1^\dagger\gamma^m\eta_2 - \eta_2^\dagger\gamma^m\eta_2)\partial_m$ satisfies

$$D_{(m}\,\xi_{n)} = 0 \ , \tag{D.1.7}$$

i.e. that ξ is a Killing vector, while equations (D.1.6c) and (D.1.6d) yield $\mathcal{L}_\xi A = 0$. That $\mathcal{L}_\xi \phi = 0$ follows from the algebraic equations obtained from (D.1.6e) and (D.1.6f) after eliminating D, using (D.1.6a) and (D.1.6b).[2]

[2]These conditions also follow directly from setting the dilatino variation $\delta\lambda$ to zero.

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List of Publications

1. F. Apruzzi, M. Fazzi, D. Rosa, and A. Tomasiello "*All AdS_7 solutions of type II supergravity*," *JHEP* **1404** (2014) 064, `hep-th/1309.2949`.

2. F. Apruzzi, M. Fazzi, A. Passias, D. Rosa, and A. Tomasiello "*AdS_6 solutions of type II supergravity*," *JHEP* **1505** (2015) 012, `hep-th/1406.0852`.

3. F. Apruzzi, F. F. Gautason, S. Parameswaran, and M. Zagermann "*Wilson lines and Chern-Simons flux in explicit heterotic Calabi-Yau compactifications*," *JHEP* **1502** (2015) 183, `hep-th/1410.2603`.

4. F. Apruzzi, M. Fazzi, A. Passias, A. Rota, A. Tomasiello, *"Holographic compactifications of (1,0) theories from massive IIA supergravity ", accepted on PRL, to appear*, `hep-th/1502.06616`.

5. F. Apruzzi, M. Fazzi, A. Passias, A. Tomasiello, *"Supersymmetric AdS_5 solutions of massive IIA supergravity ", JHEP* **1506** (2015) 195, `hep-th/1502.06620`

6. L. B. Anderson, F. Apruzzi, X. Gao, J. Gray, S.-J. Lee, *"A New Construction of Calabi-Yau Manifolds: Generalized CICYs "* `hep-th/1507.03235`

Curriculum Vitæ

Fabio Apruzzi

Personal data **Fabio Apruzzi**

Institut für Theoretische Physik
Graduiertenkolleg 1463, Analysis, Geometry and String Theory
Leibniz Universität Hannover
Welfengarten 1
D–30167 Hannover
Germany

Campus E-mail address: `fabio.apruzzi@itp.uni-hannover.de`
Personal E-mail address: `fabio.apruzzi@gmail.com`

Born in Monza (Italy) on **May 12th, 1988**
Nationality: Italian

Home address:
Nelkenstrasse, 18
D-30167, Hannover,
Germany

Education

October 2007 – September 2010, *Università degli Studi di Milano Bicocca: Laurea di primo livello (***Bachelor***) in* **Physics***; mark: 107/110; thesis on "Geometria Simplettica e Classificazione di Dirac dei Vincoli" ("Symplectic Geometry and Dirac Classification of Constraints, in Italian); supervisor: Prof. Alessandro Tomasiello.*

November 2010 – September 2012, *Università degli Studi di Milano Bicocca: Laurea di secondo livello (***Master***) in* **Physics***; mark: 110/110 cum laude; thesis on "String Compactification and Generalized Geometry of Supersymmetric Vacua in Type II Supergravity" (in English); supervisor: Prof. Alessandro Tomasiello.*

October 2012 – August 2015, *Leibniz Universität Hannover, Fakultät für Mathematik und Physik and RTG GRK 1463 "Analysis, Geometry and String Theory":* ***PhD*** *in Theoretical Physics. Thesis on "Flux Backreaction in Supergravity: Heterotic Calabi-Yau Compactifications and New Type II Anti-de-Sitter Vacua" (in English); Supervisor: Prof. Marco Zagermann. Degree of Doctor of Natural Science with grade: Summa cum laude, obtained on July,27th (2015).*

September 2014 – December 2014, and March – April 2015, *Particle theory group, Virginia Tech, Blacksburg, 24061, Virginia U.S.A: **Visiting scholar** hosted by Prof. Lara Anderson and Prof. James Gray.*

September 2015 – ..., *University of North Carolina, Chapel-hill (Durham), North Carolina U.S.A: **Post-Doc**.*

September 2015 – August 2016, *Columbia University and CUNY, New York: **Visiting scholar**.*

Research Interests **Physics:** *String theory and Supergravity (M-theory, F-theory, Heterotic, Type II), Supersymmetry, String Phenomenology, Gauge Theory, Conformal Field Theory, Holography and AdS/CFT.*

Mathematics: *Differential Geometry, Generalized Geometry and Structures, Algebraic Geometry and Topology.*

Teaching **Tutorial class:** *Analytical Mechanics. Main lecturer: Prof. Manfred Lein. Winter semester 2013/2014, Leibniz Universität Hannover.*

Supervision: *Student Seminar on "classical meechanics from non-linear systems to chaos". Main lecturer: Prof. Michael Flohr. and Prof. Luis Santos, Summer semester 2015, Leibniz Universität Hannover.*

Other Activities: **Short-Course Organizer:**

- Short-course in F-theory, *July 2014. Lecturer: Prof. Thomas Grimm (MPI Munich). Leibniz Universität Hannover.*

Talks and Presentations:

- "All AdS_7 solutions of type II supergravity", *Nordic string meeting, February (2014), Potsdam (Germany);*
- "Chern-Simons flux in Heterotic CY compactification", *XXVII Workshop Beyond the Standard Model, March (2014), Bad Honnef Bohn (Germany);*
- "Wilson lines and Chern-Simons Fluxes in Standard Model Heterotic CY Compactifications", *Stringpheno, July (2014), Trieste (Italy).*
- "Holographic dual for higher-dimensional sCFT's", *October 2014, Virginia Tech, Blacksburg, Virginia (USA).*

Conference Participations:

• "CERN Winter School on Supergravity, Strings and Gauge Theory ", *February (2013), Geneva (Switzerland);*

• "Spring School on Superstring Theory and Related Topics", *ICTP, March (2013), Trieste (Italy);*

• "Summer School on Toric Geometry, Dimers and String Theory ", *May (2013), Hannover (Germany).*

• "StringPheno 2013", *July (2013), DESY, Hamburg (Germany).*

• "Nordic String Meeting 2014", *February (2014), Potsdam (Germany).*

• "Beyond the standard model", *March (2014), Bad Honnef (Germany).*

• "StringPheno 2014", *July (2014), ICTP, Trieste (Italy).*

• "Regional Meeting", *October (2014), Duke University, Duhram, North Carolina (USA).*

• "Physics and Geometry of F-Theory", *February (2015), MPI, Munich (Germany).*

• "Regional Meeting", *April (2015), Duke University, Duhram, North Carolina (USA).*

• "StringPheno 2015", *June (2015), IFT, Madrid (Spain).*

Journal Club:

• "Supersymmetric theories on curved spaces", *January 2013, Leibniz Universität Hannover;*

• "All AdS_7 solutions of type II supergravity", *December 2014, Leibniz Universität Hannover;*

• "New Developments in Holographic Methods for higher-dimensional sCFT's", *June 2014, Leibniz Universität Hannover;*

• "On the Classification of 6D SCFTs and Generalized ADE Orbifolds ", *November 2014, Virginia Tech.*

Publication list:

1. *F. Apruzzi, M. Fazzi, D. Rosa, A. Tomasiello*, "All AdS_7 solutions of type II supergravity", ***JHEP 1404 (2014) 064***, *arXiv:1309.2949 [hep-th];*

2. *F. Apruzzi, M. Fazzi, A. Passias, D. Rosa, A. Tomasiello*, "AdS_6 solutions of type II supergravity", ***JHEP 1411 (2014) 099***, *arXiv:1406.0852 [hep-th].*

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6. *L. B. Anderson, F. Apruzzi, X. Gao, J. Gray, S.-J. Lee,* "A New Construction of Calabi-Yau Manifolds: Generalized CICYs ", *arXiv:1507.03235 [hep-th].*

IT knowledges **Operating systems:** *Linux/Unix and Mac OS X, Windows.*
Programming languages: *C/C++, Mathematica, SAGE, Palp.*

I am mainly experienced with classical programming (C/C++) and a bit on object-oriented programming and I learned to use Mathematica, for algebraic and differential calculus and SAGE, Palp for algebraic computations.

Languages *Italian, mother tongue*
English (Good)
German (basic, A1)

Reference persons

Prof. Marco Zagermann
Leibniz Universität Hannover
Institut für Theoretische Physik
Appelstrasse, 2
D– 30167 Hannover
Germany
E-mail address: `zagermann@itp.uni-hannover.de`

Prof. Alessandro Tomasiello
Università di Milano-Bicocca
Dipartimento di Fisica G. Occhialini
Piazza della Scienza 3 - Edificio U2
I-20126 Milano
Italy

E-mail address: `alessandro.tomasiello@mib.infn.it`
Second E-mail address: `alessandro.tomasiello@unimib.it`

Prof. Lara Anderson
Virginia Tech
Department of physics (Robeson Hall)
850 West Campus Drive
24061 Blacksburg, Virginia
U.S.A.
Phone number: +1 540-231-0317 (office)
E-mail address: `lara.anderson@vt.edu`

Prof. James Gray
Virginia Tech
Department of physics (Robeson Hall)
850 West Campus Drive
24061 Blacksburg, Virginia
U.S.A.
Phone number: +1 540-231-0618 (office)
E-mail address: `jamesgray@vt.edu`