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Xin Wang

Edwin E. Yaz

Susan C. Schneider

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Coupled State-Dependent Riccati Equation Control for Continuous Time Nonlinear Mechatronics Systems

Xin Wang
Southern Illinois University, Department of Electrical and Computer Engineering, Edwardsville, IL
Edwin E. Yaz
Marquette University, Department of Electrical and Computer Engineering, Milwaukee, WI
Susan C. Schneider
Marquette University, Department of Electrical and Computer Engineering, Milwaukee, WI

Abstract:

This manuscript considers the coupled state-dependent Riccati equation approach for systematically designing nonlinear quadratic regulator and H_∞ control of mechatronics systems. The state-dependent feedback control solutions can be obtained by solving a pair of coupled state-dependent Riccati equations, guaranteeing

nonlinear quadratic optimality with inherent stability property in combination with robust ℓ $_2$ type of disturbance reduction. The derivation of this control strategy is based on Nash's game theory. Both of finite and infinite horizon control problems are discussed. An underactuated robotic system, Furuta rotary pendulum, is used to examine the effectiveness and robustness of this novel nonlinear control approach.

SECTION I. Introduction

Nonlinear H_2 quadratic optimal solutions are traditionally characterized with Hamilton-Jacobi-Issac equations (HJIE), which provide the sufficient conditions for optimal control of nonlinear dynamics. Moreover, the HJIE reduces to algebraic Riccati equation (ARE), when the plant dynamics is linear time invariant (LTI) with linear quadratic regulator (LQR) performance objective. As for nonlinear H_∞ control, only the suboptimal robust control solutions can be obtained, which are equivalent to the solutions to Hamilton-Jacobi-Issac inequalities (HJIIs) [1]. However, there is no efficient algorithm to solve HJIEs and HJIIs for problems with more than a few state variables, due to the imposed numerical problems.

Over the past decades, the mixed H_2/H_∞ control problems have received much attention, with the purpose of deriving the control solutions which enjoy the properties of a quadratic optimal H_2 controllers with the robustness properties of H_∞ controllers. The mixed H_2/H_∞ control problems for linear systems were initially considered in [2] [3] by Doyle, Glover, Khargonekar, Pramod and Francis, where the connection between H_2 and H_∞ optimal control are examined, and state-space solutions are developed to linear H_∞ control problem. In [4], Bernstein and Haddad investigated the linear quadratic Gaussian (LQG) control problem with an H_∞ constraint by solving three cross-coupled algebraic Riccati equations (AREs). And in [5], Zhou, Glover and Doyle introduced an induced norm formulation of a mixed H_2/H_∞ performance criteria. Another contribution to the linear systems H_2/H_∞ control was developed by Mustapha and Glover in [6] [7], in which they proposed entropy minimization approach to obtain an upper bound on the H_2 cost function with an H_∞ constraint. Aiming at simplify the problem of effective computing the controller, Khargonekhar and Rotea in [8], and Scherer et al. in [9], solve more general mixed performance objectives linear control problems by convex optimization involving linear matrix inequalities. And more lately, Limebeer et al. in [10] approach the multi-objectives linear state feedback control problems, based on Nash two-person nonzero-sum differential game theory, which is a theoretical extension to Barsar and Bernhard's minimax approach to H_∞ control [11]. The Nash game approach to output feedback linear control is later studied by Chen and Zhou in [12].

Motivated by the success of linear system control methods, there have been extensive studies in nonlinear system H_2/H_∞ control more recently. As an extension to the results of Lime-beer et. al. in [10], Lin developed cross-coupled Hamilton-Jacobi-Issac's equations as the sufficient conditions for solving the mixed $H_2 - H_\infty$ control problem for continuous and discrete-time nonlinear systems [13]–[14].

Latest development in synthesizing feedback controls for nonlinear H_2/H_∞ control involves solving the state-dependent linear matrix inequality (SDLMI) or the state-dependent Riccati equation (SDRE) techniques. As the further extension to Scherer's results on LMI with mix performance objectives, the purpose behind state-dependent linear matrix inequality (SDLMI), which is also known as nonlinear matrix inequality (NLMI), is to convert a nonlinear system control design into a convex optimization problem involving state-dependent linear matrix inequality solutions. Numerical algorithms for solving convex optimization provides effective means for solving linear matrix inequalities [15]. If a solution can be expressed in an LMI form, then there exist efficient algorithms providing global numerical solutions. As pointed out by Wang and Yaz in [16] [17], SDLMI provides us an effective method to synthesize nonlinear feedback control in achieving nonlinear quadratic regulator (NLQR) and H_∞ control objectives. However, SDLMI method strongly relies on the numerical solutions from linear matrix inequalities, i.e., SDLMI method does not work when solutions to LMI is not strictly feasible.

In the meanwhile, the state dependent Riccati equation (SDRE) control, which is also known as the frozen Riccati equation (FRE) control, has emerged as an alternative nonlinear control design method since the mid-1990s [18]–[19][20]. A survey of the recent development of SDRE method has been summarized by Cimen in [21], Wang and Yaz in [22].

Leveraging our previous work in [22], we focus on applying Nash's game theory approach to design a set of coupled state dependent Riccati equations, which offers a generalized analytical framework in achieving a mixed Nonlinear Quadratic Regulator and H_∞ control of continuous time nonlinear systems. Building on our previous efforts and extending the results of Limebeer, Lin and Cloutier, the main contribution of this paper are the following: i) By utilizing Nash's game theory, the finite and infinite time coupled SDRE (CSDRE) control solutions are derived, which satisfy mixed objectives guaranteeing nonlinear quadratic optimality with inherent stability property in combination with H_∞ type of disturbance reduction. The proposed coupled SDRE control provides a more general SDRE control framework. ii) Instead of using linearizion or energy control, the Furuta rotary pendulum can be effectively controlled/stabilized from pendent to upright positions with the proposed coupled state dependent Riccati equation control method, while achieving the mixed design objectives. iii) Our work unifies Limebeer, Lin and Cloutier's work on Hamilton-Jacobi-Issac equation approach, Nash game theory, and nonlinear quadratic regulator SDRE approach by a more general coupled state-dependent Riccati equation (CSDRE) method for practical nonlinear mechatronics system control applications.

The remainder of this manuscript is organized as follows: In Section II, the finite time nonlinear quadratic regulator/H_∞ SDRE control method is proposed. The infinity time quadratic regulator/H_∞ SDRE control method is presented in Section III. Dynamics model of Furuta rotary pendulum model and coupled state dependent Riccati equation controller implementation details are described in Section IV, and these are followed by concluding remarks in Section V.

SECTION II. Finite-Horizon Nonlinear Quadratic Regulator $\mathcal{H}_\infty$SDRE Control

Consider the following continuous-time input-affine state-space model, which is defined on a smooth n-dimensional manifold $\mathcal{X} \subset \mathcal{R}^n$ containing the origin $x = 0$:

$$\mathcal{P}: \begin{cases} \dot{x} & = f(x) + g_1(x)w + g_2(x)u \\ & = A(x)x + B_1(x)w + B_2(x)u \\ z & = h_1(x) + d_{12}(x)u \\ & = C_1(x)x + D_{12}(x)u \\ y & = x \end{cases}$$

with

$$x(0) = x_0 \quad (1)$$

where $x \in \mathcal{X} \subset \mathcal{R}^n$ denotes the state space variable, $u \in \mathcal{U} \subset \mathcal{R}^p$ denotes the constant input, $w \in \mathcal{W} \subset \mathcal{R}^r$ denotes the disturbance and perturbation. The measurement output $y \in \mathcal{R}^m$ represents the sensor measurements, and the performance output $x \in \mathcal{R}^s$ represents the controlled output.

Consider the notation $\mathcal{M}^{i \times j}$ as the ring of $i \times j$ matrices over $\mathcal{X}$. In system equation of (1), $f(x) = A(x)x$. The state-dependent matrices $A(x): \mathcal{X} \to \mathcal{M}^{n \times n}(\mathcal{X})$, $g_1(x) = B_1(x): \mathcal{X} \to \mathcal{M}^{n \times r}(\mathcal{X})$, $g_2(x) = B_2(x): \mathcal{X} \to \mathcal{M}^{n \times p}(\mathcal{X})$. Meanwhile, in the controlled output equation, $h(x) = C_1(x)x$. The state-dependent matrices $C_1(x): \mathcal{X} \to \mathcal{M}^{s \times n}(\mathcal{X})$, and $d_{12}(x) = D_{12}(x): \mathcal{X} \to \mathcal{M}^{s \times p}(\mathcal{X})$. We assume that $f(x), g_1(x), g_2(x), h(x), d_{12}(x)$ are all real C^∞ functions defined in a neighborhood of the origin with $f(0) = 0$, and $h(0) = 0$.

II. Assumption 1 **Suppose the state-dependent matrices satisfy**

$$d_{12}^T(x)[h_1(x)d_{12}(x)] = [0I] \quad (2)$$

Or the equivalent condition:

$$D_{12}^T(x)[C_1(x)D_{12}(x)] = [0I] \quad (3)$$

The mixed $NLQR/H_\infty$ state-dependent control problem can be formally defined as follows:

Definition 1 Continuous Time Nonlinear $NLQR/H_\infty$ Control Problem with Internal Stability

Find the time-varying state-dependent control feedback law in the form

$$u(x) = K(x)x \quad (4)$$

with $K(0) = 0$, such that, the closed loop system:

$$\mathcal{K} \circ \mathcal{P}: \begin{cases} \dot{x} & = A(x)x + g_1(x)w + g_2(x)K(x)x \\ z & = C_1(x)x + d_{12}(x)K(x)x \\ y & = x \end{cases}$$

with $x(0) = x_0$ satisfies:

1. the suboptimal H_∞ control objective is satisfied.

   $$\int_0^T \|z\|^2 dt \leq \gamma^{*2} \int_0^T \|w\|^2 dt \quad (5)$$

   $\forall t \in [0, T]$, and $\forall w \in \mathcal{W} \subset \mathcal{L}_2[0, T]$.

2. the quadratic energy $x^T Qx + u^T Ru$ is minimized.

3. the closed loop system $\mathcal{K} \circ \mathcal{P}$ defined above with $w = 0$ is locally asymptotically stable in the neighborhood of the origin $x = 0$, starting from the initial state $x_0 = 0$. ◇

As is well-known, the problem mentioned above can be formulated as the two-player Nash game associated with the following H_∞ cost functional and nonlinear quadratic cost functional [10] [13] [23]:

$$\begin{aligned} \min_{u\in\mathcal{U}, w\in\mathcal{W}} J_1(u, w) &= \frac{1}{2}\int_{t_0}^T (\gamma^2\|w(t)\|^2 - \|z(t)\|^2)dt \\ \min_{u\in\mathcal{U}, w\in\mathcal{W}} J_2(u, w) &= \frac{1}{2}\int_{t_0}^T (x^T Q(x)x + u^T R(x)u)dt \end{aligned} \quad (6)(7)$$

The purpose is to seek control strategy u^*, w^*, which satisfy the Nash equilibrium defined by [10] [13] [23]:

$$J_1(u^*, w^*) \leq J_1(u^*, w), \forall w \in \mathcal{W}$$
$$J_2(u^*, w^*) \leq J_2(u, w^*), \forall u \in \mathcal{U} \quad (8)$$

Recall the definition of zero-state detectability from[13] [23].

Definition 2

If there exist $\mathcal{N}$ which is a neighborhood about the origin $x = 0$, S.t. $\forall x \in \mathcal{N}$, we have

$$h(x(t, x_0)) = 0, \forall t > 0 \Rightarrow \lim_{t \to \infty} x(t, x_0) = 0 \quad (9)$$

then the pair $(f(x), h(x))$ is said to be locally zero-state detectable. If $\mathcal{N} = \mathcal{R}^n$, then the pair is said to be (globally) zero-state detectable [13] [14].

Now, we are in the position to describe the main results, which provides sufficient conditions for the solvability of mixed $NLQR - H_\infty$ nonlinear control problems with internal stability.

Theorem 1

Consider the nonlinear plant $\mathcal{P}$ defined by (1) and the finite-horizon continuous time nonlinear quadratic regulator and H_∞ SDRE control problem with cost functionals (6) and (7). Suppose the following.

1. $(f(x), h(x))$ **are locally zero-state detectable.**
2. **there exists a locally negative definite $\mathcal{C}^1$ function $U(x,t) < 0: \mathcal{X} \to \mathcal{R}$, and a locally positive definite $\mathcal{C}^1$ function $V(x,t) > 0: \mathcal{X} \to \mathcal{R}$, such that $U(0,t) = 0$, and $V(0,t) = 0$.**
3. **Assume there exist $P_1 \leq 0$, and $P_2 \geq 0$ solutions of the coupled State Dependent Riccati Equations (CSDRE), which are in the form of ordinary differential equations as:**

$$\begin{aligned}
&-\dot{P}_1 = A^T P_1 + P_1 A - C_1^T C_1 - \\
&[P_1(t) P_2(t)] \begin{pmatrix} \gamma^{-2} B_1 B_1^T & B_2 R^{-1} B_2^T \\ B_2 R^{-1} B_2^T & B_2 R^{-1} B_2^T \end{pmatrix} \begin{bmatrix} P_1(x,t) \\ P_2(x,t) \end{bmatrix}, \\
&\text{with } P_1(x,T) = 0 \\
&-\dot{P}_2 = A^T P_2 + P_2 A + Q - \\
&[P_1(t) P_2(t)] \begin{pmatrix} 0 & \gamma^{-2} B_1 B_1^T \\ \gamma^{-2} B_1 B_1^T & B_2 R^{-1} B_2^T \end{pmatrix} \begin{bmatrix} P_1(x,t) \\ P_2(x,t) \end{bmatrix} \\
&\text{with} P_2(x,T) = 0
\end{aligned} \quad (10)(11)$$

Then, the coupled state dependent Riccati equation control inputs are

$$u^*(x,t) = -R^{-1} B_2^T(x) P_2(x,t) x(x,t)$$
$$w^*(x,t) = -\gamma^{-2} B_1^T(x) P_1(x,t) x(x,t) \quad (12)(13)$$

solve the continuous time finite horizon SDRE problem. Moreover, the optimal costs are given by

$$J_1^*(u^*, w^*) = U(0, x_0) = x^T(0)P_1(0)x(0)$$
$$J_2^*(u^*, w^*) = V(0, x_0) = x^T(0)P_2(0)x(0) \quad (14)(15)$$

Equivalent to Theorem 1, the following theorem provides the coupled Hamilton-Jacobi-Isaacs Equations (HJIEs), which serve as the sufficient conditions for the solvability of finite horizon problem.

Theorem 2

Consider the nonlinear plant $\mathcal{P}$ defined by (1) and the finite horizon continuous time nonlinear quadratic regulator and H_∞ SDRE control problem with cost functionals (6) and (7). Suppose the following conditions hold:

1. $(f(x), h(x))$ **are locally zero-state detectable.**
2. **there exists a locally negative definite $\mathcal{C}^1$ function $U(x,t) < 0: \mathcal{X} \to \mathcal{R}$, and a locally positive definite $\mathcal{C}^1$ function $V(x,t) > 0: \mathcal{X} \to \mathcal{R}$, such that $U(0,t) = 0$, and $V(0,t) = 0$.**
3. **and satisfy the Hamilton-Jacobi-Isaacs Equations (HJIEs):**

$$\begin{aligned} -U_t(x,t) = \quad & U_x(x,t)f(x) - \tfrac{1}{2}V_x(x,t)g_2(x)R^{-2}g_2^T(x)V_x^T(x,t) - \\ & \tfrac{1}{2\gamma^2}U_x(x,t)g_1(x)g_1^T(x)U_x^T(x,t) - \\ & U_x(x,t)g_2(x)R^{-1}g_2^T(x)V_x^T(x,t) - \tfrac{1}{2}h_1^T(x)h_1(x), \\ & \text{with} U(x,T) = 0 \\ -V_t(x,t) = \quad & V_x(x,t)f(x) - \tfrac{1}{2}V_x(x,t)g_2(x)R^{-1}g_2^T(x)V_x^T(x,t) - \\ & \tfrac{1}{\gamma^2}V_x(x,t)g_1(x)g_1^T(x)U_x^T(x,t) + \\ & \tfrac{1}{2}x^T(t)Q(x,t)x(t), \\ & \text{with} V(x,T) = 0 \end{aligned} \quad (16)(17)$$

Then, the state dependent Riccati equation control inputs are

$$u^*(x,t) = -R^{-1}g_2^T(x)V_x^T(x,t)$$
$$w^*(x,t) = -\frac{1}{\gamma^2}g_1^T(x)U_x^T(x,t) \quad (18)(19)$$

solve the continuous time finite horizon SDRE problem. Moreover, the optimal costs are given by

$$J_1^*(u^*, w^*) = U(0, x_0)$$
$$J_2^*(u^*, w^*) = V(0, x_0) \quad (20)(21)$$

SECTION III. Infinite-Horizon Nonlinear Quadratic Regulator/$\mathcal{H}_\infty$ SDRE Control

In this section, we consider the infinite time nonlinear quadratic regulator/$\mathcal{H}_\infty$ SDRE control problem, by letting $T \to \infty$. The following theorem gives the sufficient condition for the solvability of this problem.

Theorem 3

Consider the nonlinear plant $\mathcal{P}$ defined by (1) and the infinite-horizon continuous time nonlinear quadratic regulator and H_∞ SDRE control problem with cost functionals (6) and (7). Suppose the following.

1. $(f(x), h(x))$ **are locally zero-state detectable.**
2. **there exists a locally negative definite $\mathcal{C}^1$ function $U(x,t) < 0: \mathcal{X} \to \mathcal{R}$, and a locally positive definite $\mathcal{C}^1$ function $V(x,t) > 0: \mathcal{X} \to \mathcal{R}$, such that $U(0,t) = 0$, and $V(0,t) = 0$.**
3. **Assume there exist $P_1 \leq 0$, and $P_2 \geq 0$ solutions of the coupled State Dependent Riccati Equations (CSDRE), which are in the form of ordinary differential equations as:**

$$\begin{aligned}&0 = A^T P_1 + P_1 A - C_1^T C_1 - \\ &[P_1(t) P_2(t)] \begin{pmatrix} \gamma^{-2} B_1 B_1^T & B_2 R^{-1} B_2^T \\ B_2 R^{-1} B_2^T & B_2 R^{-1} B_2^T \end{pmatrix} \begin{bmatrix} P_1(x,t) \\ P_2(x,t) \end{bmatrix}, \\ &\text{with} P_1(x,t) \leq 0 \\ &0 = A^T P_2 + P_2 A + Q - \\ &[P_1(t) P_2(t)] \begin{pmatrix} 0 & \gamma^{-2} B_1 B_1^T \\ \gamma^{-2} B_1 B_1^T & B_2 R^{-1} B_2^T \end{pmatrix} \begin{bmatrix} P_1(x,t) \\ P_2(x,t) \end{bmatrix} \\ &\text{with} P_2(x,t) \geq 0\end{aligned} \quad (22)(23)$$

Then, the state dependent Riccati equation control inputs are

$$\begin{aligned} u^*(x,t) &= -R^{-1} B_2^T(x) P_2(x,t) x(x,t) \\ w^*(x,t) &= -\gamma^{-2} B_1^T(x) P_1(x,t) x(x,t) \end{aligned} \quad (24)(25)$$

solve the continuous time infinite horizon SDRE problem. Moreover, the optimal costs are given by

$$\begin{aligned} J_1^*(u^*, w^*) &= U(0, x_0) = x^T(0) P_1(0) x(0) \\ J_2^*(u^*, w^*) &= V(0, x_0) = x^T(0) P_2(0) x(0) \end{aligned} \quad (26)(27)$$

Equivalent to Theorem 3, the following theorem provides the coupled Hamilton-Jacobi-Isaacs equations (HJIEs), which serve as the sufficient conditions for the solvability of infinite horizon problem.

Theorem 4

Consider the nonlinear plant $\mathcal{P}$ defined by (1) and the infinite horizon continuous time nonlinear quadratic regulator and H_∞ SDRE control problem with cost functionals (6) and (7). Suppose the following conditions hold:

1. $(f(x), h(x))$ **are locally zero-state detectable.**

2. **there exists a locally negative definite $\mathcal{C}^1$ function $U(x,t) < 0: \mathcal{X} \to \mathcal{R}$, and a locally positive definite $\mathcal{C}^1$ function $V(x,t) > 0: \mathcal{X} \to \mathcal{R}$, such that $U(0,t) = 0$, and $V(0,t) = 0$.**

3. **and satisfy the Hamilton-Jacobi-Isaacs Equations (HJIEs):**

$$
\begin{aligned}
0 = \quad & U_x(x,t)f(x) - \tfrac{1}{2}V_x(x,t)g_2(x)R^{-2}g_2^T(x)V_x^T(x,t) - \\
& \tfrac{1}{2\gamma^2}U_x(x,t)g_1(x)g_1^T(x)U_x^T(x,t) - \\
& U_x(x,t)g_2(x)R^{-1}g_2^T(x)V_x^T(x,t) - \tfrac{1}{2}h_1^T(x)h_1(x), \\
& \text{with } U(x,t) \le 0 \\
0 = \quad & V_x(x,t)f(x) - \tfrac{1}{2}V_x(x,t)g_2(x)R^{-1}g_2^T(x)V_x^T(x,t) - \\
& \tfrac{1}{\gamma^2}V_x(x,t)g_1(x)g_1^T(x)U_x^T(x,t) + \\
& \tfrac{1}{2}x^T(t)Q(x,t)x(t), \\
& \text{with } V(x,t) \ge 0
\end{aligned}
\quad (28)(29)
$$

Then, the state dependent Riccati equation control inputs are

$$
\begin{aligned}
u^*(x,t) &= -R^{-1}g_2^T(x)V_x^T(x,t) \\
w^*(x,t) &= -\tfrac{1}{\gamma^2}g_1^T(x)U_x^T(x,t)
\end{aligned}
\quad (30)(31)
$$

solve the continuous time infinite horizon SDRE problem. Moreover, the optimal costs are given by

$$
\begin{aligned}
J_1^*(u^*,w^*) &= U(0,x_0) \\
J_2^*(u^*,w^*) &= V(0,x_0)
\end{aligned}
\quad (32)(33)
$$

SECTION IV. Special Case: Nonlinear Regulation State Dependent Riccati Equation Control

To minimize the nonlinear quadratic performance objective J_2 in (7) only

$$
\min_{u \in \mathcal{U}, w \in \mathcal{W}} J(u,w) = \frac{1}{2}\int_0^\infty (x^TQ(x)x + u^TR(x)u)dt \quad (34)
$$

with respect to the state x and control u subject to the nonlinear differential equation, which is a special case of (1) without performance output or external disturbances.

$$
\mathcal{P}: \begin{cases} \dot{x} & = f(x) + g_2(x)u \\ & = A(x)x + B_2(x)u \\ y & = x \end{cases}
$$

The nonlinear quadratic regulator SDRE control approach is to solve the following state dependent Riccati equation:

$$A^T(x)P + PA(x) - PB_2R^{-1}(x)B_2^TP + Q(x) = 0 \quad (35)$$

which is a special case of decoupled equation of (23) by setting $P_2 = P$ and $P_1 = 0$.

The nonlinear feedback control input can be constructed as

$$u = -R^{-1}(x)B_2^TP(x)x \quad (36)$$

which is the decoupled control solution from (24).

SECTION V. Applications to Furuta Pendulum Control

Furuta rotary pendulum is controlled with the proposed coupled state dependent Riccati equation (CSDRE) control. The Furuta pendulum has stable equilibrium point at $\theta_1 = 2\pi K + \pi$ and $\dot{\theta}_1 = 0, \forall K \in \mathcal{N}$; and unstable equilibrium point at $\theta_1 = 2\pi K$ and $\dot{\theta}_1 = 0, \forall K \in \mathcal{N}$. Denote $\mathcal{N}$ as the set of integers. The two links of Furuta pendulum are distinguished using subscripts 0 and 1, respectively.

The following standard Furuta pendulum notations and parameters are used in this manuscript.

Notation	Unit	Explanation
θ_i	$[rad]$	angle of i^{th} link
τ_i	$[Nm]$	torque applied to i^{th} joint
J_i	$[kg \cdot m^2]$	moment of inertia of i^{th} link
m_i	$[kg]$	mass of i^{th} link
d_i	$[kg \cdot m^2/s]$	viscous friction coefficient
l_i	$[m]$	length of i^{th} link
r_i	$[m]$	distance from i^{th} joint to gravity center of i^{th} link

Applying Euler-Lagrange equation, the joint space dynamics of Furuta rotary pendulum is obtained as

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \Gamma + \Delta \quad (37)$$

with the generalized joint coordinate

$$q = [\theta_1, \theta_2]^T \quad (38)$$

The mass matrix

$$M(q) = \begin{pmatrix} j_0 + j_1 s_1^2 & -m_1 l_0 r_1 c_1 \\ -m_1 l_0 r_1 c_1 & j_1 \end{pmatrix} \quad (39)$$

The centrifugal and Coriolis force matrix

$$V(q,\dot{q}) = \begin{pmatrix} m_1 l_0 r_1 s_1 \dot{\theta}_1^2 + 2 j_1 s_1 c_1 \dot{\theta}_0 \dot{\theta}_1 + d_0 \dot{\theta}_0 \\ -i_1 s_1 c_1 \dot{\theta}_0^2 + d_1 \dot{\theta}_1 \end{pmatrix} \quad (40)$$

The gravity force matrix

$$G(q) = \begin{pmatrix} 0 \\ -m_1 g r_1 s_1 \end{pmatrix} \quad (41)$$

The generalized force matrix

$$\Gamma = \begin{pmatrix} \tau_0 \\ 0 \end{pmatrix} (42)$$

where j_i for $i = 0,1$ is the moment of inertia at around i^{th} pivot, i.e. $j_0 = J_0 + m_0 r_0^2 + m_1 l_0^2$ and $j_1 = J_1 + m_1 r_1^2$.

The effect of external disturbance and perturbation w acting on the pendulum beam is included in the disturbance matrix

$$\Delta = \begin{pmatrix} 0 \\ 1 \end{pmatrix} w \quad (43)$$

By choosing the state space variables $x = [\theta_0, \theta_1, \dot{\theta}_0, \dot{\theta}_1]^T$, and control input $u = \tau_0$, Furuta rotary pendulum model can be described in state dependent coefficient (SDC) form as

$$\begin{aligned} \dot{x} &= A(x)x + B_1(x)w + B_2(x)\tau_0 \\ z &= C_1(x)x + D_{12}(x)\tau_0 \\ y &= x \end{aligned} \quad (44)$$

where

$$A(x) = \begin{pmatrix} 0_{2\times 2} & I_{2\times 2} \\ -M^{-1}\Phi & -M^{-1}N \end{pmatrix} \quad (45)$$

with

$$\Phi(x) = \begin{pmatrix} 0 & 0 \\ 0 & -mgr_1 \frac{sin(\theta_1)}{\theta_1} \end{pmatrix}$$

$$N(x) = \begin{pmatrix} j_1 sin(2\theta_1)\dot{\theta}_1 + d_0 & m_1 l_0 r_1 sin(\theta_1)\dot{\theta}_1 \\ -\frac{1}{2} j_1 sin(2\theta_1)\dot{\theta}_0 & d_1 \end{pmatrix} \quad (46)(47)$$

and

$$B_1(x) = \begin{pmatrix} 0_{2\times 1} \\ M^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{pmatrix}$$
$$B_2(x) = \begin{pmatrix} 0_{2\times 1} \\ M^{-1} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{pmatrix} \quad (48)(49)$$

By choosing

$$C_1 = \begin{pmatrix} \sqrt{q1} & 0 & 0 & 0 \\ 0 & \sqrt{q2} & 0 & 0 \\ 0 & 0 & \sqrt{q3} & 0 \\ 0 & 0 & 0 & \sqrt{q4} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (50)$$

and

$$D_{12} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{\rho} \end{pmatrix}$$

The following H_∞ performance can be achieved for any disturbances $w \in \mathcal{L}_2[0,\infty)$

$$\int_0^\infty \{q_1\theta_0^2 + q_2\theta_1^2 + q_3\dot{\theta}_0^2 + q_4\dot{\theta}_1^2 + \rho u^2(t)\}dt < \int_0^\infty w^2(t)dt \quad (52)$$

where $q_1, q_2, q_3, q_4, \rho > 0$ are weighing coefficient.

The proposed nonlinear coupled state dependent Riccati equation control of the Furuta rotary pendulum have been simulated with computer software. The time duration is 3 second, the applied torque input is limited within ±10Nm, the initial state variables are set to be $x(0) = [\theta_0, \theta_1, \omega_0, \omega_1]^T = [\pi/2, \pi - 0.1, 0, 0]^T$, the zero input region is determined by $\theta_\varepsilon = -0.1\ rad$,

Case I: Nonlinear Quadratic Regulator-H_∞ CSDRE Control

The design parameters are set to:

$$Q = diag([10,50,3,3]), R = 5,$$
$$q_1 = 2, q_2 = 1000, q_3 = 1, q_4 = 2000, \rho = 1.2$$

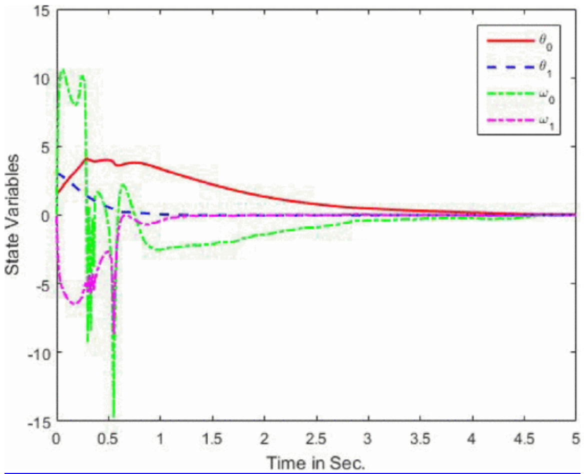

Fig. 1. $NLQR - H_\infty$ SDRE control

Case II: Nonlinear Quadratic Regulator $NLQR$ SDRE Control

The design parameters are set to:

$$Q = diag([100,500,1,1]), R = 20$$

Conclusions

A novel coupled state dependent Riccati equation (CSDRE) approach is proposed to control continuous time nonlinear electromechanical systems. By formulate the system model in state dependent coefficient (SDC) linear structure, optimal control solution can be obtained by solving the coupled state dependent Riccati equation. It is shown that the conventional nonlinear quadratic regulator SDRE is a special case of the CSDRE approach when the nonlinear regulator cost is applied. The Furuta rotary pendulum is used as an illustrative example to demonstrate the efficacy of proposed method.

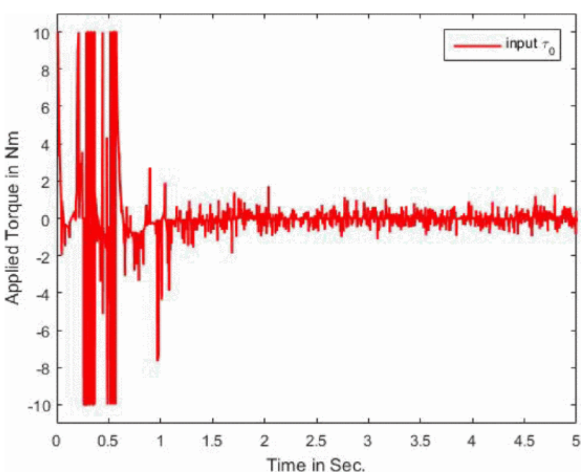

Fig. 2. $NLQR - H_\infty$ SDRE control

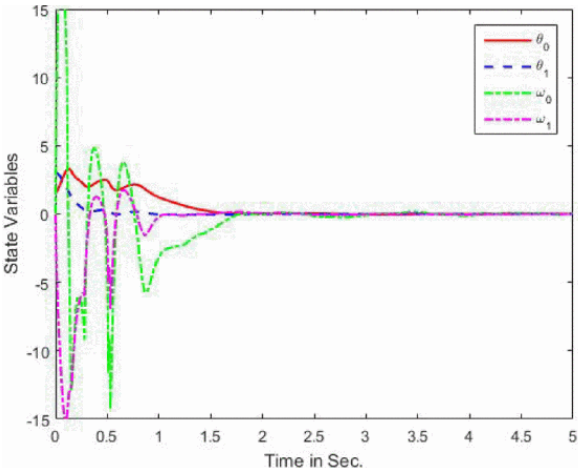

Fig. 3. $NLQR$ SDRE control

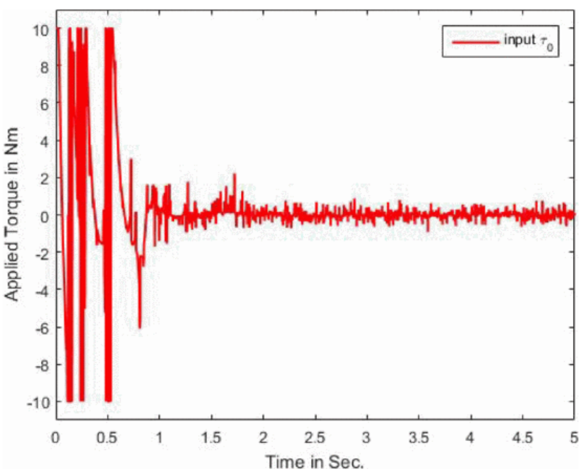

Fig. 4. $NLQR$ SDRE control

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