η-EINSTEIN SASAKIAN IMMERSIONS IN NON-COMPACT SASAKIAN SPACE FORMS

GIANLUCA BANDE, BENIAMINO CAPPELLETTI–MONTANO, AND ANDREA LOI

Abstract. The aim of this paper is to study Sasakian immersions of (non-compact) complete regular Sasakian manifolds into the Heisenberg group and into $\mathbb{B}^N \times \mathbb{R}$ equipped with their standard Sasakian structures. We obtain a complete classification of such manifolds in the η-Einstein case.

1. Introduction

Sasakian geometry is considered as the odd-dimensional counterpart of Kähler geometry. Despite the Kähler case, where the study of Kähler immersions is well developed, due to the seminal work of Calabi [6] (see also [15] for a modern treatment and an account on the subject), in the Sasakian setting there are few results. Most of the Sasakian results are concerned with finding conditions which ensure that a Sasakian submanifold is totally geodesic or similar geometric properties (see, for instance, [12, 13, 14]).

In [7] the second and the third authors studied Sasakian immersions into spheres. In particular they proved the following classification result:

Theorem ([7]). *Let S be a $(2n+1)$-dimensional compact η-Einstein Sasakian manifold. Assume that there exists a Sasakian immersion of S into $\mathbb{S}^{2N+1}$. If $N = n+2$ then S is Sasaki equivalent to $\mathbb{S}^{2n+1}$ or to the Boothby-Wang fibration over Q_n, where $Q_n \subset \mathbb{C}P^{n+1}$ is the complex quadric equipped with the restriction of the Fubini–Study form of $\mathbb{C}P^{n+1}$.*

Since the (Sasakian) sphere is one of the three "models" of Sasakian space forms, it is quite natural to study as a second step the immersions into Sasakian space forms.

In this paper we give a complete characterisation of Sasakian immersions of complete, regular, η-Einstein Sasakian manifolds into a non-compact Sasakian space form $M(N,c)$, proving the following:

Theorem 1. *Let S be a $(2n+1)$-dimensional connected, complete, regular η-Einstein Sasakian manifold. Suppose that there exists $p \in S$, an open neighborhood U_p of p and a Sasakian immersion $\phi : U_p \to M(N,c)$, where $c \leq -3$. Then S is Sasaki equivalent to $M(n,c)/\Gamma$ where Γ is some discrete subgroup of the Sasakian-isometry group of $M(n,c)$. Moreover, if $U_p = S$ then $\Gamma = \{1\}$ and ϕ is, up to a Sasakian transformation of $M(N,c)$, given by*

$$\phi(z,t) = (z, 0, t+c)$$

Date: February 17, 2020.

2010 *Mathematics Subject Classification.* 53C25; 53C55.

Key words and phrases. Sasakian; Sasaki–Einstein; η-Einstein; Sasakian immersion; Kähler manifolds; Kähler immersions, Sasakian space form.

The authors were supported by Prin 2015 – Real and Complex Manifolds: Geometry, Topology and Harmonic Analysis – Italy and by Fondazione di Sardegna (Project STAGE) and Regione Autonoma della Sardegna (Project KASBA). All the authors are members of INdAM-GNSAGA - Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni.

Theorem 1 is a strong generalisation of [12, Theorem 3.2] which asserts that a complete, ϕ-invariant, η-Einstein submanifold of codimension 2 of the $(2N+1)$-dimensional Heisenberg group is necessarily a totally geodesic submanifold Sasaki-equivalent to a copy of a $(2N-1)$-dimensional Heisenberg group and similarly for totally geodesic submanifolds of $\mathbb{B}^N \times \mathbb{R}$, where $\mathbb{B}^N$ denotes the unit disc of $\mathbb{C}^N$ equipped with the hyperbolic metric. In fact in our result there is no restriction on the codimension and we assume that we have a Sasakian immersion instead of a ϕ-invariant submanifold. Moreover the immersion is not necessarily injective and is not assumed to be from the whole space but from an open neighbourhood of a point.

The general philosophy in [7] and in this paper is to take into account the transversal Kähler geometry of the Reeb foliation. When a regular Sasakian manifold is compact as in [7], one can use the so-called Boothby-Wang construction [3], which realises the space of leaves as a Kähler manifold which is the base of a principal $\mathbb{S}^1$-fibration. Then one translates the immersion problem into a Kähler immersion problem of the base spaces.

Trying the same trick in the non-compact case is more complicated because the Boothby-Wang construction fails in general, even if the Sasakian manifold is regular. Nevertheless, the Reeb foliation has the strong property to be both a totally geodesic and a Riemannian foliation. Assuming the Sasakian manifold complete, one can appeal to the result of Reinhart [19] which says that the space of leaves is the base space of a fibration, and once again translate the problem into one on Kähler immersions.

The paper contains two other sections. In Section 2 we recall the main definitions and some foliation theory needed in the proof of Theorem 1 to whom Section 3 is dedicated.

2. Preliminaries

A *contact metric manifold* is a contact manifold (S, η) admitting a Riemannian metric g compatible with the contact structure, in the sense that, defined the $(1,1)$-tensor ϕ by $d\eta = 2g(\cdot, \phi\cdot)$, the following conditions are fulfilled

$$\phi^2 = -Id + \eta \otimes \xi, \quad g(\phi\cdot, \phi\cdot) = g - \eta \otimes \eta, \tag{1}$$

where ξ denotes the *Reeb vector field* of the contact structure, that is the unique vector field on S such that

$$i_\xi \eta = 1, \quad i_\xi d\eta = 0.$$

A contact metric manifold is said to be *Sasakian* if the following integrability condition is satisfied

$$N_\phi(X,Y) := [\phi X, \phi Y] + \phi^2[X,Y] - \phi[X, \phi Y] - \phi[\phi X, Y] = -d\eta(X,Y)\xi, \tag{2}$$

for any vector fields X and Y on S.

Two Sasakian manifolds (S_1, η_1, g_1) and (S_2, η_2, g_2) are said to be *equivalent* if there exists a contactomorphism $F : S_1 \longrightarrow S_2$ between them which is also an isometry, i.e.

$$F^*\eta_2 = \eta_1, \quad F^*g_2 = g_1. \tag{3}$$

One can prove that if (3) holds then F satisfies also

$$F_{*_x} \circ \phi_1 = \phi_2 \circ F_{*_x}, \quad F_{*_x}\xi_1 = \xi_2$$

for any $x \in S_1$. An isometric contactomorphism $F : S \longrightarrow S$ from a Sasakian manifold (S, η, g) to itself will be called a *Sasakian transformation* of (S, η, g).

It is a well-known fact [4] that the foliation defined by the Reeb vector field of a Sasakian manifold S has a transversal Kähler structure. Using the theory of Riemannian submersions one can prove that the transverse geometry is Kähler-Einstein if and only if the Ricci tensor of S satisfies the following equality

$$\mathrm{Ric} = \lambda g + \nu \eta \otimes \eta \tag{4}$$

for some constants λ and ν. Any Sasakian manifold satisfying (4) is said to be *η-Einstein* (see [5] for more details).

A remarkable property of η-Einstein Sasakian manifolds is that, contrary to Sasaki-Einstein ones, they are preserved by $\mathcal{D}_a$-homothetic deformations, that is the change of structure tensors of the form

$$\phi_a := \phi, \quad \xi_a := \frac{1}{a}\xi, \quad \eta_a := a\eta, \quad g_a := ag + a(a-1)\eta \otimes \eta \tag{5}$$

where $a > 0$.

By a *Sasakian immersion* (often called invariant submanifolds or Sasakian submanifolds in the literature) of a Sasakian manifold (S_1, η_1, g_1) into the Sasakian manifold (S_2, η_2, g_2) we mean an isometric immersion $\varphi : (S_1, g_1) \longrightarrow (S_2, g_2)$ that preserves the Sasakian structures, i.e. such that

$$\varphi^* g_2 = g_1, \quad \varphi^* \eta_2 = \eta_1, \tag{6}$$

$$\varphi_* \xi_1 = \xi_2, \quad \varphi_* \circ \phi_1 = \phi_2 \circ \varphi_*. \tag{7}$$

We refer the reader to the standard references [2, 4] for a more detailed account of Riemannian contact geometry and Sasakian manifolds.

Sasakian space forms. Recall that the curvature tensor of a Sasakian manifold is completely determined [2] by its *ϕ-sectional* curvature, that is the sectional curvature of plane sections of the type $(X, \phi X)$, for X a unit vector field orthogonal to the Reeb vector field.

A *Sasakian space form* is a connected, complete Sasakian manifold with constant ϕ-sectional curvature. According to Tanno [20] there are exactly three simply connected Sasakian space forms depending on the value c of the ϕ-sectional curvature: the standard Sasakian sphere up $\mathcal{D}_a$-homothetic deformation if $c > -3$, the Heisenberg group $\mathbb{C}^n \times \mathbb{R}$ if $c = -3$ and the hyperbolic Sasakian space form $\mathbb{B}^n \times \mathbb{R}$ if $c < -3$. Notice that each simply connected space form admits a fibration over a Kähler manifold and in the non-compact cases the fibration is trivial.

We denote by $M(n, c)$ the simply connected $(2n+1)$-dimensional Sasakian space form with ϕ-sectional curvature equal to c. Every connected, complete Sasakian space form is Sasakian equivalent to $M(n, c)/\Gamma$, where Γ is a discrete subgroup of the Sasakian transformation group of $M(n, c)$.

Immersions and regular foliations. We recall some basic concepts from foliation theory (see e.g. [16, 18]). Let M be a smooth manifold of dimension n. A foliation can be defined as a maximal foliation atlas on M, where a foliation atlas of codimension q on M is an atlas

$$\{\varphi_i : U_i \longrightarrow \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q\}_{i \in I}$$

of M such that the change of charts diffeomorphisms φ_{ij} locally takes the form

$$\varphi_{ij}(x, y) = (g_{ij}(x, y), h_{ij}(y)).$$

Each foliated chart is divided into *plaques*, the connected components of

$$\varphi_i^{-1}(\mathbb{R}^p \times \{y\}),$$

where $y \in \mathbb{R}^q$, and the changes of chart diffeomorphism preserve this division.

Definition 2. A *foliated map* is a map $f:(M,\mathcal{F})\longrightarrow(M',\mathcal{F}')$ between foliated manifolds which preserves the foliation structure, i.e. which maps leaves of $\mathcal{F}$ into leaves of $\mathcal{F}'$.

Now, let $(M,\mathcal{F})$ and $(M',\mathcal{F}')$ be foliated manifolds and $f:M\longrightarrow M'$ be an immersion. Moreover, assume that f is a foliated map. Thus

$$f_{*x}(L(x))\subset L'(f(x))$$

for each $x\in M$, where $L=T(\mathcal{F})$ and $L'=T(\mathcal{F}')$. In particular, it follows that $\dim(\mathcal{F})\leq\dim(\mathcal{F}')$. The proof of the following proposition is quite standard and will be omitted:

Proposition 3. *$(M,\mathcal{F})$ and $(M',\mathcal{F}')$ be foliated manifolds of dimension n and n', respectively, and $f:M\longrightarrow M'$ be a foliated immersion. Suppose that $\dim(\mathcal{F})=\dim(\mathcal{F}')$. Then for each $x\in M$ there are charts $\varphi:U\longrightarrow\mathbb{R}^p\times\mathbb{R}^q$ for M about x and $\varphi':U'\longrightarrow\mathbb{R}^p\times\mathbb{R}^{q'}$ for M' about $f(x)$ such that*

(i) $\varphi(x)=(0,\dots,0)\in\mathbb{R}^n$
(ii) $\varphi'(f(x))=(0,\dots,0)\in\mathbb{R}^{n'}$
(iii) $F(x_1,\dots,x_n)=(x_1,\dots,x_n,0,\dots,0)$, *where* $F:=\varphi'\circ f\circ\varphi^{-1}$
(iv) $L(x)=\operatorname{span}\left\{\frac{\partial}{\partial x_1}(x),\dots,\frac{\partial}{\partial x_p}(x)\right\}$
(v) $L'(f(x))=\operatorname{span}\left\{\frac{\partial}{\partial x_1}(f(x)),\dots,\frac{\partial}{\partial x_p}(f(x))\right\}$

where $p=\dim(\mathcal{F})=\dim(\mathcal{F}')$, $q=n-p$, $q'=n'-p$.

Let $\mathcal{F}$ be a foliation on a manifold M and let L be a leaf of $\mathcal{F}$. It is well known that L intersects at most a countable number of plaques in a foliated chart U. Now we give the following definition.

Definition 4 ([18]). A foliation $\mathcal{F}$ is said to be *regular* if for any $x\in M$ there exists a foliated chart U containing x such that every leaf of $\mathcal{F}$ intersects at most one plaque of U.

The following proposition is a generalisation to the non-compact case and to immersions of [11, Proposition 3.1]:

Proposition 5. *Let $(M,\mathcal{F})$ and $(M',\mathcal{F}')$ be foliated manifolds such that $\dim(\mathcal{F})=\dim(\mathcal{F}')$. If there exists a foliated immersion $f:(M,\mathcal{F})\longrightarrow(M',\mathcal{F}')$ and $\mathcal{F}'$ is regular, then $\mathcal{F}$ is also regular.*

Proof. Assume that $\mathcal{F}$ is not regular. Then there exists a point $x\in M$ and a leaf L of $\mathcal{F}$ such that, for any foliated chart U containing x, L intersects more then one plaque in U. Let us consider the foliated charts U and U', respectively about x and $f(x)$, satisfying the properties stated in Proposition 3. Then there exist at least two plaques, say $P_1=\varphi^{-1}(\mathbb{R}^p\times\{\mathbf{y}_1\})$ and $P_1=\varphi^{-1}(\mathbb{R}^p\times\{\mathbf{y}_2\})$, such that

$$L\cap P_1\neq\emptyset,\quad L\cap P_2\neq\emptyset, \tag{8}$$

where $\mathbf{y}_1,\mathbf{y}_2\in\mathbb{R}^q$. Notice that, for each $i\in\{1,2\}$, $f(P_i)$ is a plaque of $\mathcal{F}'$ in $U':=f(U)$. Indeed, using Proposition 3, we have $f(P_i)=f(\varphi^{-1}(\mathbb{R}^p\times\{\mathbf{y}_i\}))=\varphi'^{-1}(F(\mathbb{R}^p\times\{\mathbf{y}_i\}))=\varphi'^{-1}(\mathbb{R}^p\times\{(\mathbf{y}_i,0,\dots,0)\})$. Now, since f is a foliated map, $L'=f(L)$ is a leaf of $\mathcal{F}'$ and from (8) it follows that $L'\cap f(P_1)\neq\emptyset$ and $L'\cap f(P_2)\neq\emptyset$. But this contradicts the regularity of $\mathcal{F}'$. □

3. Classification

In this Section we prove the main result of this paper, that is the classification of connected, regular η-Einstein Sasakian manifolds immersed into Sasakian space forms.

Proof of Theorem 1. Let $M(N,c)$ be one of the non-compact simply connected Sasakian space forms and $\pi' : M(N,c) \to K'$ the (trivial) fibration over its Kähler quotient. Recall that K' is either $\mathbb{C}^N$ with its flat Kähler metric or the hyperbolic Kähler space form $\mathbb{B}^N$.

Since S is complete and regular, by [19] there exists a fibration $\pi : S \to K$, whose fibers are the leaves of the Reeb foliation of S. By assumption S is Sasakian η-Einstein and then K is necessarily Kähler-Einstein (see [9]). The fibration $\pi : S \to K$ is a Riemannian submersion and since S is complete then K also is (see [17, 9]).

By assumption there exists an open neighbourhood U_p of $p \in S$ and a Sasakian immersion $\varphi : U_p \to M(N,c)$. By Proposition 5 the submanifold U_p is still regular. The restriction of π to U_p gives then a projection of the Sasakian η-Einstein manifold U_p to the Kähler manifold $\pi(U_p) \subset K$.

The Sasakian immersion $\varphi : U_p \to M(N,c)$ covers a Kähler immersion $i(\varphi)$ (see [7, 10]) making the following diagram commutative:

$$\begin{array}{ccc} U_p & \xrightarrow{\ \varphi\ } & M(N,c) \\ {\scriptstyle \pi}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi'} \\ \pi(U_p) \subset K & \xrightarrow[i(\varphi)]{} & K' \end{array}$$

We have proved that there exists $q = \pi(p) \in K$, an open neighbourhood $V_q = \pi(U_p)$ of q and a Kähler immersion of V_q in K'. By Umehara [21] V_q is flat or complex hyperbolic.

On the other end, by [8] (see also [1, Theorem 5.26]), the Kähler-Einstein manifold K is real analytic and by [6, Theorem 4 and Theorem 10], for every $q \in K$ there exists an open neighbourhood V'_q and a Kähler immersion of V'_q in K'. Then K is locally flat and [9, Formula 1.31] implies that the ϕ-sectional curvature of S is less or equal to -3.

By Tanno [20] there exists a discrete group Γ of the Sasakian transformations of S such that $S = M(n,c)/\Gamma$ and this proves the first part of the theorem.

For the second part of the theorem, let us suppose $U_p = S$. Reasoning as before, by completeness of K and by Calabi's Rigidity Theorem [6] for Kähler immersions into Kähler space forms (see also [15]) one obtains the stronger result that either $K = \mathbb{C}^n$ or $K = \mathbb{B}^n$ and the projection is just the trivial fibration because in both cases K contractible. Then, since S is complete, the fibres of the fibration are diffeomorphic either to $\mathbb{R}$ or to $\mathbb{S}^1$.

The second case cannot occur because φ is a Sasakian immersion and then it restricts to immersions on the leaves of the Reeb foliations of S and $M(N,c)$. But the leaves of $M(N,c)$ are diffeomorphic to $\mathbb{R}$ and if the leaves of S are circles we would obtain immersions of the circle in $\mathbb{R}$ which is not possible.

Now it remains to prove that φ is the standard embedding up to Sasakian transformations.

First observe that, again by Calabi's Rigidity Theorem, the immersion $i(\varphi) : K \to K'$ has (up to unitary transformation) the following form:

$$i(\varphi)(z_1, \dots, z_n) = (z_1, \dots, z_n, 0, \dots, 0).$$

Because the fibrations are trivial φ must have the following expression:

$$\varphi(z_1, \dots, z_n, t) = (z_1, \dots, z_n, 0, \dots, 0, f_N((z_1, \dots, z_n, t))).$$

Since ϕ is a Sasakian immersion, in particular we have $\varphi^*(\eta_N) = \eta_n$, where η_N and η_n are the standard contact forms of $M(N, c)$ and $M(n, c)$ respectively. Then a direct calculation of $\varphi^*(\eta_N) = \eta_n$ yields $\frac{\partial f_N}{\partial t} = 1$ and $\frac{\partial f_N}{\partial x_i} = \frac{\partial f_N}{\partial y_i} = 0$ for $i = 1, \dots n$, where we put $z_j = x_j + iy_j$. □

Remark 6. In Theorem 1 the case $U_p = S$ cannot occur if S is compact because if S is compact, a Sasakian immersion cannot exist for otherwise, from the regularity of a compact Sasakian manifold, we would obtain a (compact) Kähler quotient immersed either in $\mathbb{C}^N$ or in $\mathbb{B}^N$, which is impossible by the Maximum Principle.

The following result is a variation of Theorem 1:

Theorem 7. *Let S be a $(2n+1)$-dimensional connected, complete, η-Einstein Sasakian manifold. Suppose that for every $p \in S$ there exists an open neighbourhood U_p of p and a Sasakian immersion $\phi : U_p \to M(N, c)$, where $c \leq -3$. Then S is Sasaki-equivalent to $M(n, c)/\Gamma$ where Γ is some discrete subgroup of the Sasakian-isometry group of $M(n, c)$. Moreover, if $U_p = S$ then $\Gamma = \{1\}$ and ϕ is, up to a Sasakian transformation of $M(N, c)$, given by*

$$\phi(z, t) = (z, 0, t + c)$$

Proof. For every point $p \in S$ we have an immersion of some U_p . After possibly shrinking the open set U_p we obtain an open set where the Reeb foliation is given by a fibration over a Kähler base. Then we proceed exactly as in the proof of Theorem 1 and we obtain that U_p (and then S) has constant ϕ-sectional curvature at every point. Then S is Sasaki-equivalent to $M(n, c)/\Gamma$ where Γ is some discrete subgroup of the Sasakian-isometry group of $M(n, c)$.

If $U_p = S$ we cannot directly conclude as in Theorem 1 because a priori we don't know if $M(n, c)/\Gamma$ is regular. On the other end we are assuming the existence of an immersion of $U_p = S$ into the regular Sasakian space form $M(N, c)$ and then S is regular by Proposition 5. We can now apply Theorem 1 to conclude. □

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Gianluca Bande, Dipartimento di Matematica e Informatica, Università di Cagliari, Italy.
Email address: gbande@unica.it

Beniamino Cappelletti–Montano, Dipartimento di Matematica e Informatica, Università di Cagliari, Italy.
Email address: b.cappellettimontano@unica.it

Andrea Loi, Dipartimento di Matematica e Informatica, Università di Cagliari, Italy.
Email address: loi@unica.it