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Schreier groups and symmetric neighborhoods with a finite number of open components

Raymond David Phillippi

To the Graduate Council:

I am submitting herewith a thesis written by Raymond David Phillippi entitled "Schreier groups and symmetric neighborhoods with a finite number of open components." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Conrad Plaut, Major Professor

We have read this thesis and recommend its acceptance:

Accepted for the Council:
Dixie L. Thompson

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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Schreier Groups on Symmetric Neighborhoods with a Finite Number of Open Components

A Thesis
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Raymond David Phillippi
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Dedication

This thesis is dedicated to my grandmother Eleanor Jane Norton, whose love of life and knowledge has been a constant source of inspiration.

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Abstract

The purpose of this investigation is to consider the group structure of Schreier groups for both general topological groups and euclidean space in particular where U is taken to have a finite number of components. Theorem 1 exibits a homomorphism from the Schreier group into the direct product of the underlying topological group and a specified finitely presented group with the components of U as generators. Theorem 2 shows that in euclidean space the given homomorphism is an isomorphism. Examples are given which illustrate the process laid out in Theorem 1.

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I. Introduction

Given a topological group G, and a subset U of G one can construct words whose elements are from U. Schreier groups are constructed from equivalence classes of such words when U is symmetric and contains the identity (see Propositions 1 and 2). Schreier groups were first considered by Schreier in 1925 [5]. They have been rediscovered by Tits [6] and in a more general setting of local groups by Mal'tsev[Ma]. Berestovskii and Plaut [1] have used Schreier groups to generalize covering group theory within the setting of topological groups. In these works it is generally assumed that the symmetric neighborhood is also connected. Yelton [7] in an REU project at the University of Tennessee considered Schreier groups in $\mathbb{R}$ arising from a symmetric neighborhood of 0 with a finite number of components. She developed conditions on the components under which the Schreier group becomes the direct product of $\mathbb{R}$ and a finitely generated free group. This paper will again consider Schreier groups on symmetric sets with a finite number of components, but within a more general class of topological groups. Theorem 1 describes a homomorphism from the Schreier group into the direct product of the underlying topological group with a specified finitely presented group. Conditions are specified for the homomorphism to be an epimorphism, or an isomorphism, thus generalizing Yelton's work. In particular, we will show in Theorem 2 that all Schreier groups which arise from a euclidean space $\mathbb{R}^n$ are isomorphic to the direct product of $\mathbb{R}^n$ and a finitely presented group. Propositions 3 and 4 and the examples surrounding them are an attempt to consider some of the finitely presented groups which emerge from Theorem 1.

II. Results

This paper uses the following construction of topological groups (See Plaut [4]). Note that if $(G, \times)$ is a group and $U, V \subset G$ then $UV = \{uv \mid u \in U \text{ and } v \in V\}$ and $U^{-1} = \{u^{-1} \mid u \in U\}$.

Definition 1: Let $(G, \times)$ be a group and $\Gamma = \{F_\alpha\}_{\alpha\in\Lambda}$ a family of subsets of G each containing the identity e. Then G is a (Hausdorff) topological group with fundamental family Γ if the following four conditions hold.
1. $\bigcap_{\alpha\in\Lambda} F_\alpha = \{e\}$
2. For every $F, V \in \Gamma$, there exists a $W \in \Gamma$ such that $WW^{-1} \subset F \cap V$.
3. For all $F \in \Gamma$ and $a \in F$ there exists a $V \in \Gamma$ such that $aV \subset F$.
4. For all $F \in \Gamma$ and $a \in G$ there exists some $V \in \Gamma$ such that $aVa^{-1} \subset F$.

The open sets of a topological group in this sense are defined as those sets V which obey the property that if $x \in V$ then there exists $F \in \Gamma$ such that $xF \subset V$. It will be convenient to show that if V is open we can find $F' \in \Gamma$ such that $F'x \subset V$. To do this, suppose V is open under the given definition. Choose F' such that $xF'x^{-1} \subset F$. Then $F'x = xx^{-1}F'x \subset xF \subset V$. It can be shown [4] that open sets defined in such a way form a Hausdorff topology. If G is a topological group and Γ' is taken to be the family $\{F \subset G \mid e \in F \text{ and } F \text{ is open}\}$ then Γ' satisfies the conditions $1-4$ and the topological group thus obtained is identical to the original one. In other words, we may assume that the fundamental family of a topological group consists of all open sets about the identity. A topological group is called locally generated if $\forall x \in G$ and $F \in \Gamma$ we can write $x = x_1x_2\ldots x_n$ where $x_i \in F$.

Some examples of topological groups used in this paper are:

1. Euclidian Space under the operation of +. Let Γ be the collection of all open balls $B(0,r)$ centered at the origin. In this example we have that $B(0,r) \in \Gamma \Rightarrow B(0,r)^{-1} = B(0,r)$ hence $(F = F^{-1}\ \forall F \in \Gamma)$ and $aB(0,r) = B(a,r)$. Condition 1 is obvious. Condition 2 is satisfied by noting that if $r_1 \leq r_2$ then $B(0,r_1) \cap B(0,r_2) = B(0,r_1)$. If we let $W = B(0,\frac{r_1}{2})$ then $WW^{-1} = W + W = B(0,r_1) \subset B(0,r_1) \cap B(0,r_2)$. If $F = B(0,r)$, and $a \in F$ then condition 3 can be met by choosing $V = B(0, r - \|a\|)$. Condition 4 is trivial since $a + B(0,r) - a = B(0,r)$. We also have that Euclidian Space is locally generated. Let $x \in \mathbb{R}^n$ and $B(0,r) \in \Gamma$. Then choose $k \in \mathbb{N}$ such that $\frac{\|x\|}{k} < r$. Then $\| \frac{1}{k}x \| = \frac{\|x\|}{k} < r$ and $\frac{1}{k}x + \frac{1}{k}x + \ldots + \frac{1}{k}x$ (k times) $= x$.

2. The circle $S^1 = \{e^{i\theta} \in \mathbb{C} \mid \theta \in (-\pi, \pi]\}$ where the group operation is multiplication in $\mathbb{C}$. If Λ is the set $(0,\pi)$ then the collection $\{F_{\alpha\in\Lambda}\}$ where $F_\alpha = \{e^{i\theta} \in \mathbb{C} \mid \theta \in (-\alpha,\alpha)\}$ forms a fundamental family. Since $e^0 = 1 \in F_\alpha\ \forall \alpha \in \Lambda$ we have $1 \in \bigcap_{\alpha\in\Lambda} F_\alpha$. If $e^{i\theta} \neq 1$ then choose $\alpha < \theta$. Then $e^{i\theta} \notin F_\alpha$ and hence $\{1\} = \bigcap_{\alpha\in\Lambda} F_\alpha$. Thus condition 1 is met. Now, let $\alpha_1 \leq \alpha_2$ be arbitrary and $W = F_{\frac{\alpha_1}{2}}$. Then since $F_{\alpha_1} \cap F_{\alpha_2} = F_{\alpha_1}$ and $WW^{-1} = WW = F_{\alpha_1} = F_{\alpha_1} \cap F_{\alpha_2}$ condition 2 is met. Let $a =$

$e^{i\beta} \in F_\alpha$ so that $|\beta| < \alpha$. Condition 3 can be met by choosing $V = F_{\frac{\alpha-|\beta|}{2}}$. Condition 4 is again obvious since S^1 is abelian. The circle can be shown to be locally generated by setting F_α and $e^{i\theta}$ and choosing $k \in \mathbb{N}$ such that $\frac{\theta}{k} < \alpha$. Then $e^{i\theta/k} \in F_\alpha$ and $e^{i\theta/k}e^{i\theta/k}\ldots e^{i\theta/k}$ (k times) $= e^{i\theta}$

3. The set $GL(n,\mathbb{C})$ of n by n matricies with elements in $\mathbb{C}$ whose determinant is non-zero under the operation of matrix multiplication. If $M \in GL(n,\mathbb{C})$ define $|M| = \max_{ij}|m_{ij}|$ where m_{ij} is the element of M in the ith row and jth column. It can be shown [4] that $GL(n,\mathbb{C})$ is a topological group with fundamental family given by the sets $B_r = \{A \in GL(n,\mathbb{C}) \mid |A - I| < r\}$ where $r > 0$ and $I =$ the identity.

The following two definitions and propositions define Schreier groups. Although Schreier groups have been considered by numerous others (see introduction), the following presents such groups in a context useful for our purposes. For the following construction, fix a topological group $(G,+)$ with fundamental family Γ. In a number of ways it will be convenient to use $+$ to represent the operation in G. Please note that commutativity of the group operation is not being assumed and that the identity element will be denoted by e. Let $U(G)$ be the collection of all open sets in G containing the identity e which have a finite number of open components and are symmetric in the sense that $U \in U(G)$ and $x \in U \Rightarrow -x \in U$. We will fix G and $U \in U(G)$ until after Theorem 1.

Definition 2: A U-word with respect to the group G and set U is a finite word $x_1x_2\ldots x_n$ where $x_i \in U \; \forall \; 1 \le i \le n$. The set of all U-words will be denoted by $\hat{U}$. The symbols x,y,z,g and h will be used to represent elements of U and $u,v,$ and w will represent U-words of $\hat{U}$.

Definition 3: If $u = x_1x_2\ldots x_ix_{i+1}\ldots x_n$ where $1 \le i \le n-1$ and if $v = x_1x_2\ldots x_j\ldots x_n$ where $x_j = x_i + x_{i+1}$ in G then v is said to be obtained from u by an expansion, and u is said to be obtained from v by a contraction. Contraction is an inverse operation from expansion in the sense that if u can be obtained from v by a contraction, then v can be obtained from u by an expansion. Define $\sim$ on the set $\hat{U}$ in the following way. If u,v are U-words then $u \sim v$ iff v can be obtained from u by a finite sequence of expansions and contractions; i.e. there exists U-words $u_1, u_2, \ldots, u_k$ such that $u_1 = u$, $u_k = v$ and u_{i+1} can be obtained from u_i by either an expansion or contraction.

Note: If $u,v \in \hat{U}$ and v is obtained from u by an expansion then the sum of the elements of u and v (in G) are unchanged since $x_j = x_i + x_{i+1}$. We have that if $u \sim v$ where $u = x_1x_2\ldots x$ and $v = y_1y_2\ldots y_m$ then the sum of the elements of u and v must be equal, i.e. $\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j$.

Proposition 1: $\sim$ is an equivalence relation on $\hat{U}$.

Proof: Let $u = x_1x_2\ldots x_n$. $v = x_1ex_2\ldots x_n$. Then v is a U-word and can be obtained from u by an expansion since $x_1 + e = x_1$. Similarly u can be obtained again from v by a contraction. Hence $u \sim u$. Now, suppose $u \sim v$. If v can be obtained from u by a single expansion or contraction then it is clear from the definition that u can be obtained from v by a single contraction or expansion respectively, hence $v \sim u$. In general, there exists U-words $u_1, u_2, \ldots u_k$ such that $u = u_1 \sim u_2 \sim \ldots \sim u_k = v$, where u_{i+1} can be obtained from u_i by a single expansion or contraction. But then $v = u_k \sim \ldots \sim u_1 = u$ where u_i can be obtained from u_{i+1} by a single expansion or contraction. Hence $v \sim u$. Finally, suppose $u \sim v$ and $v \sim w$. Then there exist U-words $u_1, u_2 \ldots, u_k$ and U-words $v_1, v_2, \ldots, v_l$ such that $u = u_1 \sim u_2 \sim \ldots \sim u_k = v$ and $v = v_1 \sim v_2 \sim \ldots \sim v_l = w$ where each $\sim$ is a single expansion or contraction. Hence $u = u_1 \sim \ldots \sim u_k \sim v_1 \sim \ldots \sim v_l = w$ and $u \sim w$.

We will denote the quotient $\hat{U} / \sim$ by G_U. Define the following operation on G_U. If $u = x_1 x_2 \ldots x_n$ and $v = y_1 y_2 \ldots y_m$ then $[u][v] = [uv] = [x_1 x_2 \ldots x_n y_1 y_2 \ldots y_m]$. This operation is well-defined since if $u' \in [u]$ and $v' \in [v]$ then there exist $u_1, u_2, \ldots, u_k$ such that $u = u_1 \sim u_2 \sim \ldots \sim u_k = u'$ and $v_1, v_2, \ldots, v_l$ such that $v = v_1 \sim v_2 \sim \ldots \sim v_l = v'$. Then $uv = u_1 v_1 \sim \ldots \sim u_k v_1 \sim \ldots \sim u_k v_l = u'v'$. Hence $[u'][v'] = [u'v'] = [uv] = [u][v]$.

Proposition 2: G_U is a group.
Proof: Let $[u], [v], [w] \in G_U$. $[u]([v][w]) = [u][vw] = [uvw] = [uv][w] = ([u][v])[w]$ and hence the operation is associative. Consider the U-word e. The equivalence class $[e]$ has the property that $[e][u] = [eu] = [u]$ since $u = x_1 x_2 \ldots x_n \sim e x_1 x_2 \ldots x_n = eu$. Similarly $[u][e] = [u]$. Finally, for $[u] \in G_U$ consider the class $[u]^{-1} = [(-x_n)(-x_{n-1}) \ldots (-x_1)]$. Then $[u]$
$[u]^{-1} = [x_1 x_2 \ldots x_n (-x_n)(-x_{n-1}) \ldots (-x_1)]$
$= [x_1 x_2 \ldots x_{n-1} e (-x_{n-1})(-x_{n-2}) \ldots (-x_1)] = [x_1 x_2 \ldots x_{n-1}(-x_{n-1})(-x_{n-2}) \ldots (-x_1)] = \ldots = [x_1(-x$
Similarly $[u]^{-1}[u] = [e]$

One of the components of U must contain the identity of G. In what follows, a significant role is played by those U-words for which all of the elements of the chain belong to this component. We will call such U-words *fine.* The equivalence classes in G_U which have fine representatives are also important, but notice that these classes will also have representatives which are not fine. For instance, if $x_1 x_2 \ldots x_n$ is fine we have $x_1 x_2 \ldots x_n \sim x_1 x_2 \ldots x_n y(-y)$ for any y in U. This leads to the following definition.

Definition 4: Let U_0 be the component of U which contains the identity. A U-word $u = x_1 x_2 \ldots x_n$ will be called *fine* if $x_i \in U_0 \; \forall i$. An equivalence class $[v]$ will be called fine if $v \sim u$ where u is fine.

We wish to evaluate the structure of G_U which will culminate in Theorem 1. The following lemmas prove useful to this end. Recall that G has a fundamental family Γ whose elements can be taken to be all open sets in G containing the identity (see Definition 1). In particular $U_0 \in \Gamma$.

Lemma 1: Let $F \in \Gamma$ and $y,z \in V$ where V is an arbitrary component of U. Then $[z] = [x_1x_2\ldots x_ny]$ where $x_1, x_2, \ldots, x_n \in F$.

Proof: Fix $y \in V$ and $F \in \Gamma$ and let $S = \{z \in V \mid [z] = [x_1x_2\ldots x_ny]$ for some $x_1,x_2,\ldots,x_n \in F\}$. Then $y \in S$ since $[y] = [ey]$ and $e \in F$ by the definition of a fundamental family. We will show that S is both open and closed. To see that S is open, suppose $z \in S$. Then there exists $x_1,x_2,\ldots,x_n \in F$ such that $[z] = [x_1x_2\ldots x_ny]$. Now, since V is open, we can find an $F_1 \in \Gamma$ such that $F_1 + z \subset V$. Since F, F_1 and U_0 are all open and contain e we can define $F_2 = F \cap F_1 \cap U_0$ where $F_2 \in \Gamma$. Then, for all $k \in F_2$ we know that

a) $k \in U_0$ since $F_2 \subset U_0$ and
b) $k + z \in U$ since $F_2 \subset F_1$ and $F_1 + z \subset V$

Thus the following equalities are valid for all $k \in F_2$: $[k+z] = [kz] = [kx_1x_2\ldots x_ny]$ and, since $k \in F$ we have $k+z \in S$. Hence, $F_2 + z \subset S$ and we have that S is open. To show that S is closed, suppose $z \in S^C$. We can find an F_3 such that $F_3 + z \subset V$. Further, we can find an $F_4 \in \Gamma$ such that $F_4 + (-F_4) \subset F \cap F_3 \cap U_0$. Notice in particular that since $e \in F_4$ we have $-F_4 \subset U_0$. As above, we know that if $k \in F_4$ then both k and $k+z$ are elements of U. We wish to show that $F_4 + z \subset S^c$. Suppose not. Then there would be a $k \in F_4$ such that $[k+z] = [x_1x_2\ldots x_ny]$ for some $x_1,x_2,\ldots,x_n \in F$. But since $k \in F_4$, $-k \in U_0$ and $[-kx_1x_2\ldots x_ny] = [-k(k+z)] = [-kkz] = [z]$ which is a contradiction. Thus $F_4 + z \subset S^C$ and we have that S^C is open. Hence S is closed and $S = V$.

Corollary 1: If $x \in U_0$, $F \in \Gamma$ then $[x] = [x_1x_2\ldots x_n]$ where $x_1x_2\ldots,x_n \in F$.

Proof: Since $x,e \in U_0$ we can apply Lemma 1 to get $[x] = [xe] = [x_1x_2\ldots x_ne] = [x_1x_2\ldots x_n]$ where $x_1x_2\ldots,x_n \in F$.

Suppose G is abelian, $x \in U_0$ and g is any element of U. If we choose $F \in \Gamma$ such that $g + F \subset U$ then we can use Lemma 1 to see that $[x][g] = [xg] = [x_1x_2\ldots x_ng]$ (where $x_i \in F$)$= [x_1x_2\ldots(x_n+g)] =$ $[x_1x_2\ldots(g+x_n)] = [x_1x_2\ldots gx_n] =\ldots= [gx_1x_2\ldots x_n] = [gx] = [g][x]$. Thus it follows that the set of all fine U-words is a subset of the center of G_U when G is abelian. We will see, however, that the group operation in G_U is nonabelian in general, even in the case where G is abelian (see for example Proposition 3 below). The following lemma shows that for each element g of U it is possible to find a small neighborhood (dependent on g) of the identity whose elements obey a form of commutativity with g.

Lemma 2: For any $g \in U$ there exists an $F_g \in \Gamma$, $F_g \subset U_0$ such that if $x \in F_g$ then $[xg] = [gy]$ for some $y \in U_0$.

Proof: Let V be the component of U which contains g. We can find an F_1 such that $g+F_1 \subset V$. This guarantees that $\forall x \in F_1$ we have $g+x \in U$. We can also find an F_2 such that $F_2+g \subset V$ which guarantees that $\forall x \in F_2$ the term $x+g$ is in U. We can then define F' such that $F' = F_1 \cap F_2 \cap U_0$. Notice that for each $x \in F'$ we have by definition that $x+g$, $g+x$, and x itself are all in U and can be inserted or deleted as elements in a U-word. By the definition of a fundamental family, we can find an $F \in \Gamma$ so that $-g+F+g \subset F'$. Thus, for all $x \in F$ we can find $y \in F'$ such that $-g+x+g = y$ .This implies that $x+g = g+y$. Since $g,x,y,x+g$, and $g+y \in U$ the following equalities are legal: $[xg] = [(x+g)] = [(g+y)] = [gy]$.

Note that while $x \in F_g$, $y \in U_0$. The following lemma establishes a form of commutativity for all elements of U_0.

Lemma 3: If $x \in U_0$, $h \in U$ then $[xh] = [hx_1x_2\ldots x_n]$ for some $n \in \mathbb{N}, x_1, x_2, \ldots, x_n \in U_0$.

Note: In the abelian case we have $[x] \in Center(G_U)$ $\forall x \in U_0$ from the discussion preceeding Lemma 2.

Proof: Choose F_h so that Lemma 2 holds for h. By the corollary to Lemma 1 we can rewrite $[xh]$ as $[x_1x_2\ldots x_nh]$ where the $x_i \in F_h$. Then applying Lemma 2 (n times) we obtain $[xy] = [hx'_1x'_2\ldots x'_n]$ with the $x'_i \in U_0$.

Lemma 4: Let U_1, U_2, U_3 be components of U. Suppose further that $\exists g_1, g_2, g_3$ such that $g_1 \in U_1, g_2 \in U_2, g_3 \in U_3$ and $g_1+g_2 = -g_3$. Let $x_1 \in U_1, x_2 \in U_2, x_3 \in U_3$ be arbitrary. Then $[x_1x_2x_3]$ is fine, i.e. $x_1x_2x_3 \sim u$ where u is a fine U-word.

Proof: By Lemma 1 above we can write $x_1x_2x_3$ as
$a_1a_2\ldots a_{s_1}g_1b_1b_2\ldots b_{s_2}g_2c_1c_2\ldots c_{s_3}g_3$. where $a_i, b_j, c_k \in U_0$. Then, by Lemma 3,
$x_1x_2x_3 \sim a_1a_2\ldots a_{s_1}g_1g_2g_3b'_1b'_2\ldots b'_{t_2}c'_1c'_2\ldots c'_{t_3}$ (where $s_2 \le t_2$ and $s_3 \le t_3$.)
$\sim a_1a_2\ldots a_{s_1}(-g_3)g_3b'_1b'_2\ldots b'_{t_2}c'_1c'_2\ldots c'_{t_3} \sim a_1a_2\ldots a_{s_1}b'_1b'_2\ldots b'_{t_2}c'_1c'_2\ldots c'_{t_3}$
which is a fine U-word.

In the following theorem we make use of free groups and finitely presented groups. To describe the free group on n elements $\{x_1, x_2, \ldots, x_n\}$, consider the collection of all finite strings of elements of the set $\{e, x_1, x_2, \ldots, x_n, x_1^{-1}, x_2^{-1}\ldots, x_n^{-1}\}$. I will call these strings words. A reduced word is a word in which all e's and all pairs $x_ix_i^{-1}$ and $x_i^{-1}x_i$ are removed. Thus the word $x_1ex_2x_2^{-1}x_3^{-1}x_3x_4^{-1}$ reduces to the word $x_1x_4^{-1}$. It can be shown ([2] pp.64-65) that each word reduces to a unique reduced word and that the operation $x_{i_1}x_{i_2}\ldots x_{i_n} * y_{j_1}y_{j_2}\ldots y_{j_m}$ =(the reduced version of) $x_{i_1}x_{i_2}\ldots x_{i_n}y_{j_1}y_{j_2}\ldots y_{j_m}$ forms a (not necessarily abelian) group on the collection of all reduced words. I will denote this group by $F(n)$. Notice that if a word consists only of elements from the set $\{x_1, x_2, \ldots, x_n\}$ then the word is a reduced word. If R is a collection of words then the finitely presented group $< x_1, x_2, \ldots, x_n \mid R >$ is the quotient group $F(n)/N$ where N is the normal subgroup of $F(n)$ generated by the words in R ([2] p.67).

Lemma 4 shows that conditions on single elements of a component can affect the entire component in profound ways. In fact Theorem 1 will establish that the components themselves have a certain group structure defined by the condition that if there exists $g_1 \in U_1, g_2 \in U_2, g_3 \in U_3$ such that $g_1 + g_2 = g_3$ then $U_1 U_2 = U_3$. In Yelton's work [7] where $G = \mathbb{R}$ the neighborhoods U are composed of intervals and have the form $(-k_n, -k_{n-1}) \cup (-k_{n-2}, -k_{n-3}) \cup \ldots \cup (-k_0, k_0) \cup (k_1, k_2) \ldots \cup (k_{n-1}, k_n)$. She defined a condition for the components of U to be independent which states that if $x \in (k_i, k_{i+1})$ and there exists $y, z \in U$ such that $x + y = z$ then either $y \in (-k_0, k_0)$ and $z \in (k_i, k_{i+1})$ or $y \in (-k_{i+1}, -k_i)$ and $z \in (-k_0, k_0)$. If the intervals in U are all independent then $\mathbb{R}_U \cong \mathbb{R} \times F(\frac{n}{2})$ where $F(\frac{n}{2})$ is a the free group on $\frac{n}{2}$ elements.

Theorem 1: Let $U \in U(G)$. Suppose the components of U are denoted by $U_0, U_1, \ldots, U_k$ where U_0 is the component containing e. Suppose further that:
$R = \{U_i U_j (U_l)^{-1} \mid \exists\, x \in U_i,\ y \in U_j,\ z \in U_l \text{ with } x + y = z\}$.
$K = \{[u] = [x_1 x_2 \ldots x_n] \in G_U \mid [u] \text{ is fine and } \sum_{i=1}^{n} x_i = e\}$

Then there exists a homomorphism $\varphi : G_U \to G \times < U_0, U_1, \ldots, U_k \mid R >$ with kernel K. If G is locally generated (see Definition 1) then φ is surjective.

Note: The condition $\sum_{i=1}^{n} x_i = e$ on one representative of $[u]$ implies the condition on all representatives of $[u]$ by the note preceding Proposition 1. Also, since $e + e = e$ in G we will always have the relation $U_0 U_0 U_0^{-1}$. If N is the normal subgroup of $F(k)$ generated by R then $U_0 U_0 U_0^{-1} \in N$. Since N is normal we have that $U_0^{-1}(U_0 U_0 U_0^{-1}) U_0 \in N$ hence $U_0 \in N$ and $U_0 = e$ in $< U_0, U_1, \ldots, U_k \mid R >$. For a more detailed discussion of the generators in $< U_0, U_1, \ldots, U_k \mid R >$ see the discussion preceding Definition 5.

Proof: Let $[u] = [x_1 x_2 \ldots x_n] \in G_U$ and define $\varphi([u]) = \varphi_1 \times \varphi_2$ where $\varphi_1([u]) = \sum_{i=1}^{n} x_i$ and $\varphi_2([u]) = U_{x_1} U_{x_2} \ldots U_{x_n}$ where U_{x_j} is the component containing x_j. The term $U_{x_1} U_{x_2} \ldots U_{x_n}$ is an element of the free group on the elements $\{U_0, U_1, \ldots, U_k\}$. Since each of the elements U_{x_i} comes from this set, the word is automatically reduced. To see that φ is well-defined, suppose $[u] = [v]$. Notice first that $\varphi_1([u]) = \varphi_1([v])$ by the note preceding Proposition 1. Further, suppose that v can be obtained from u by an expansion and let $u = x_1 x_2 \ldots x_i \ldots x_n$ and $v = x_1 x_2 \ldots y y' \ldots x_n$ where $x_i = y + y'$. Now, let N be the normal subgroup of $F(k)$ generated by R. Since $y + y' = x_i$ we know from the definition of R that $U_y U_{y'} U_{x_i}^{-1} \in N$. Thus $\varphi_2([u]) \varphi_2([v])^{-1} =$
$U_{x_1} U_{x_2} \ldots U_{x_{i-1}} U_y U_{y'} U_{x_{i+1}} \ldots U_{x_n} (U_{x_1} U_{x_2} \ldots U_{x_{i-1}} U_{x_i} U_{x_{i-2}} \ldots U_{x_n})^{-1} = (U_{x_1} \ldots U_{x_{i-1}}) U_y U_{y'} U_{x_i}^{-1} (U$
and hence $U_{x_1} U_{x_2} \ldots U_{x_{i-1}} U_y U_{y'} U_{x_{i+1}} \ldots U_{x_n} = U_{x_1} U_{x_2} \ldots U_{x_{i-1}} U_{x_i} U_{x_{i+1}} \ldots U_{x_n}$ in $< U_0, U_1, \ldots, U_k \mid R >$. We then have $\varphi_2([u]) = \varphi_2([v])$ which implies $\varphi([u]) = \varphi([v])$. Now suppose $[v]$ is any element of G_U such that $[u] = [v]$. Then there exists (see Definition 1) $v_1, v_2, \ldots v_s$ such that $u = v_1$, $v = v_s$ $[v_1] = \ldots = [v_s]$ and v_i can be obtained from v_{i-1} by a single expansion or contraction. Then by above we

have that $\varphi([u]) = \varphi([v_1]) = \ldots = \varphi([v_s]) = \varphi([v])$ and φ is well-defined.

To see that φ is a homomorphism, notice that if $[u] = [x_1x_2\ldots x_n]$ and $[v] = [y_1y_2\ldots y_m]$ then $\varphi([u][v]) = \varphi([x_1x_2\ldots x_ny_1y_2\ldots y_m]) =$
$(x_1 + x_2 + \ldots + x_n + y_1 + y_2 + \ldots + y_m\ ,\ U_{x_1}U_{x_2}\ldots U_{x_n}U_{y_1}U_{y_2}\ldots U_{y_m}) =$
$((x_1 + x_2 + \ldots + x_n) + (y_1 + y_2 + \ldots + y_m)\ ,(U_{x_1}U_{x_2}\ldots U_{x_n})(U_{y_1}U_{y_2}\ldots U_{y_m})) =$
$(x_1 + x_2 + \ldots + x_n\ ,\ U_{x_1}U_{x_2}\ldots U_{x_n}) + (y_1 + y_2 + \ldots + y_m\ ,U_{y_1}U_{y_2}\ldots U_{y_m}) = \varphi([u])\varphi([v])$

To see that the kernel is K, notice first that if $[u] \in K$ then $[u] = [x_1x_2\ldots x_n]$ where $\sum_{i=1}^{n} x_i = e$, and $x_i \in U_0$ for each i. Hence $\varphi([u]) = (\sum_{i=1}^{n} x_i\ ,\ U_0U_0\ldots U_0) = (e,e)$ (by the note preceding this proof) and $[u] \in Ker(\varphi)$. Thus we have $K \subset Ker(\varphi)$. Now, suppose $\varphi([u]) = (e,e)$. To show that $[u] \in K$ it suffices to show that u can be transformed into v where v is a fine U-word. Then we would have $\sum_{i=1}^{n} x_i = e$ by supposition and $[u] = [v]$ where v is a fine U-word and hence $[u] \in K$. To show this, notice that the word $U_{x_1}U_{x_2}\ldots U_{x_n}$ which is the image of $[u]$ under φ_2 must be in the normal subgroup generated by R. Hence $U_{x_1}U_{x_2}\ldots U_{x_n} = w_1R_1w_1^{-1}w_2R_2w_2^{-1}\ldots w_mR_mw_m^{-1}$ as words in the free group $F(k)$ for some $w_i \in F(k)$ and R_i such that $R_i \in R$ or $R_i^{-1} \in R$. Notice that since the right-hand side of the equality may not be reduced, there may not be a one-to-one correspondance between the U_{x_i} on the left side and elements of $w_1R_1w_1^{-1}w_2R_2w_2^{-1}\ldots w_mR_mw_m^{-1}$ on the right. However, since these words must be equal in the free group $F(k)$ we can transform the left side into the right side by inserting e's and pairs of the form $U_iU_i^{-1}$ or $U_i^{-1}U_i$ a finite number of times. We wish to transform $x_1x_2\ldots x_n$ in a similar manner. If e is inserted between U_{x_i} and $U_{x_{i+1}}$ then insert x_ie for x_i in $x_1x_2\ldots x_n$ to get $x_1x_2\ldots x_iex_{i+1}\ldots x_n$. If the pair $U_iU_i^{-1}$ (or $U_i^{-1}U_i$) is inserted between U_{x_i} and $U_{x_{i+1}}$ then fix $a \in U_i$ and insert x_ie for x_i and then $a(-a)$ (or $(-a)a$) for e. This gives $[x_1x_2\ldots x_ia(-a)x_{i+1}\ldots x_n]$ or $[x_1x_2\ldots x_i(-a)ax_{i+1}\ldots x_n]$. In this manner we obtain $[x_1x_2\ldots x_n] = [y_1y_2\ldots y_r]$ where there is a one-to-one correspondance between the y_j and elements of $w_1R_1w_1^{-1}w_2R_2w_2^{-1}\ldots w_mR_mw_m^{-1}$. I will show that $[y_1y_2\ldots y_r]$ can be transformed into a fine U-word.

First, suppose that there is only one term of the form wR_1w^{-1}. Then, let $y_iy_{i+1}y_{i+2}$ correspond to R_1. If $R_1 \in R$, and $R_1 = U_lU_sU_t^{-1}$ then, by the nature of how the y_i were chosen, we have $y_i \in U_l, y_{i+1} \in U_s$, and $y_{i+2} \in -U_t$ (where $-U_t$ is the component symmetric to U_t). Then, by the definition of R there must exist $g_l, g_s, g_t \in G$ such that $g_l \in U_l, g_s \in U_s, -g_t \in -U_t$ and $g_l + g_s = -(-g_t)$. By Lemma 4 above $y_iy_{i+1}y_{i+2}$ relates to a fine U-word. If $R_1^{-1} \in R$, and $R_1 = U_lU_s^{-1}U_t^{-1}$ then $y_i \in U_l, y_{i+1} \in -U_s$, and $y_{i+2} \in -U_t$. By the definition of R there must exist $g_l, g_s, g_t \in G$ such that $g_l \in U_l, g_s \in U_s, g_t \in U_t$ and $g_t + g_s = g_l$. But then $g_l - g_s = -(-g_t)$ with $g_l \in U_l, -g_s \in -U_s$, and $-g_t \in -U_t$. Again by Lemma 4 we have that $y_iy_{i+1}y_{i+2}$ relates to a fine U-word. Now, let $y_1\ldots y_{i-1}$ correspond to w so that $y_{i+3}\ldots y_r$ corresponds to w^{-1}. Then by the nature of how the y_i were chosen we must have $U_{y_{i+j}} = -U_{y_{i-j}}$ for $1 < j < i-1$. We have $[y_{i-1}y_iy_{i+1}y_{i+2}y_{i+3}] = [y_{i-1}b_1b_2\ldots b_cy_{i+3}]$ where $b_i \in U_0$ from the above discussion. By Lemma 1 we may write y_{i+3} as $b_{c+1}b_{c+2}\ldots b_{c'}$. Hence $[y_{i-1}y_iy_{i+1}y_{i+2}y_{i+3}] = [y_{i-1}b_1b_2\ldots b_{c'}(-y_{i-1})]$ where $c \leq c'$. Then by Lemma 3 we may write this as $[b_1b_2\ldots b_{c''}y_{i-1}(-y_{i-1})] = [b_1b_2\ldots b_{c''}]$ which is a fine U-word. By repeated application of this procedure we may write $[y_1y_2\ldots y_r]$ as a fine U-word. Finally, by reducing each block $w_iR_iw_i$ into a fine U-word the entire expression $w_1R_1w_1^{-1}w_2R_2w_2^{-1}\ldots w_mR_mw_m^{-1}$ can be seen to be a fine U-word.

For surjectivity, let G be locally generated and suppose $(g, U_{s_1}U_{s_2}\ldots U_{s_n}) \in G \times < U_0, U_1, \ldots, U_k \mid R >$. Fix an $x_i \in U_{s_i}$ so that $U_{x_i} = U_{s_i}$. Since G is locally generated, we can find $y_1, y_2, \ldots y_l \in U_0$, such that $y_1 + y_2 + \ldots + y_l = g - \sum_{i=1}^{n} x_i$. Then $y_1 + y_2 + \ldots + y_l + x_1 + x_2 + \ldots + x_n = g$. Consider the U-word given by $y_1y_2\ldots y_l\, x_1x_2\ldots x_n$. Then
$\varphi(y_1y_2\ldots y_l\, x_1x_2\ldots x_n) = (y_1 + y_2 + \ldots + y_l + z_1 + z_2 + \ldots + z_n\,,\ U_0U_0\ldots U_0U_{x_1}U_{x_2}\ldots U_{x_n})$
$= (g, U_{s_1}U_{s_2}\ldots U_{s_n})$.

Theorem 1 gives us a tool for evaluating the structure of G_U. The following example illustrates how the above theorem may be used. If U consists of a single component U_0 then $< U_0 \mid R > = < e >$ and φ from Theorem 1 is a homomorphism into G.

Example 1 (The circle): Consider the topological group S^1 considered above. Fix $0 < \alpha < \pi$ and let $U = \{e^{i\theta} : \theta \in (-\alpha, \alpha)\}$. Then

1) If $0 < \alpha \le 2\pi/3$ then $S^1_U / \mathbb{Z} \cong S^1$
2) If $2\pi/3 < \alpha < \pi$ then $S^1_U \cong S^1$

Notice that if $[u] \in K$ where K is the kernel from Theorem 1 and $u = e^{i\theta_1}e^{i\theta_2}\ldots e^{i\theta_n}$ then we have $e^{i\theta_1}e^{i\theta_2}\ldots e^{i\theta_n} = e^0$ since $\sum_{i=1}^{n} x_i = e$ for $[u] \in K$. This implies that $\theta_1 + \theta_2 + \ldots + \theta_n = 2k\pi$ for some $k \in \mathbb{Z}$. I wish to show that

*) If $\theta_1 + \theta_2 + \ldots + \theta_n = 0$ then $u \sim e^0$.

The proof is by induction. First, suppose $n = 1$. Then $\theta_1 = 0$ and $u = e^0$. Now, let $l \in \mathbb{N}$ be arbitrary and suppose that all $[u] \in K$ with $[u] = [e^{i\theta_1}e^{i\theta_2}\ldots e^{i\theta_l}]$ are such that $u \sim e^0$. Consider $[v] = [e^{i\theta_1}e^{i\theta_2}\ldots e^{i\theta_{l+1}}] \in K$ such that $\theta_1 + \theta_2 + \ldots + \theta_{l+1} = 0$. Let j be the first index such that θ_j has opposite sign to θ_1. Then θ_{j-1} and θ_j have opposite sign. This implies that $\theta_{j-1} + \theta_j \in (-\alpha, \alpha)$ and hence $e^{i\theta_1}\ldots e^{i\theta_{j-1}}e^{i\theta_j}\ldots e^{i\theta_{l+1}} \sim e^{i\theta_1}\ldots e^{i(\theta_{j-1}+\theta_j)}\ldots e^{i\theta_{l+1}}$ which has l elements. By the induction hypothesis $v \sim e^0$.

The main difference between the above cases stems from the fact that if $0 < \alpha < 2\pi/3$ and $e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3} \in U$ then $e^{i\theta_1}e^{i\theta_2} = e^{i\theta_3}$ iff $\theta_1 + \theta_2 = \theta_3$. This can be seen by noting that $\theta_1 + \theta_2 < 2\pi/3 + 2\pi/3 = 4\pi/3 = -2\pi/3$. Since $-2\pi/3 < \theta\, \forall\, e^{i\theta} \in U$ the result follows. Hence contractions (or expansions) of U-words formed by elements of U can only occur if the corresponding sums of exponents are equal.

Case 1: Suppose $0 < \alpha \le 2\pi/3$. Let $[u] = [e^{i\theta_1}e^{i\theta_2}\ldots e^{i\theta_n}] \in K$. By the above statement, if $u \sim e^{i\lambda_1}e^{i\lambda_2}\ldots e^{i\lambda_m}$ then $\lambda_1 + \lambda_2 + \ldots + \lambda_m = 2k\pi$. Hence we can define the following map γ from K to $\mathbb{Z}$ by $\gamma([u]) = k = (\theta_1 + \theta_2 + \ldots + \theta_n)/2\pi$. This map is well-defined by the discussion preceding Case 1. Let $v = e^{i\lambda_1}e^{i\lambda_2}\ldots e^{i\lambda_m} \in K$. Then γ is a homomorphism since

$\gamma([u][v]) = \gamma([uv]) = (\theta_1 + \theta_2 + \ldots + \theta_n + \lambda_1 + \lambda_2 + \ldots + \lambda_m)/2\pi = \gamma([u]) + \gamma([v])$. That γ is surjective follows by letting $k \in \mathbb{Z}$ be arbitrary and letting u be the U-word with m terms all of the form $e^{i(2k\pi)/m}$ where m is an integer large enough that $|(2k\pi)/m| < \alpha$. Then clearly $\gamma([u]) = k$. Now, suppose $\gamma([u]) = 0$. Then $\theta_1 + \theta_2 + \ldots + \theta_n = 0$. By (1) we have $u \sim e^0$ and hence $\gamma([u]) = 0 \Rightarrow [u] = [e^0]$ and thus γ is injective. Hence, $K \cong \mathbb{Z}$ and the result follows from Theorem 1, the first isomorphism theorem, and the fact that S^1 is locally generated (see definition 1).

Case 2: Suppose $2\pi/3 < \alpha < \pi$. Let $[u] = [e^{i\theta_1}e^{i\theta_2}\ldots e^{i\theta_n}] \in K$. I will show that $[u] = [e^0]$. We have $\theta_1 + \theta_2 + \ldots + \theta_n = 2k\pi$. If $k = 0$ then the result follows by (*). If $k \neq 0$, let $0 < \varepsilon < \alpha - 2\pi/3$ and notice that $(2\pi/3) + \varepsilon$ and $-(2\pi/3) + \varepsilon/2 \in (-\alpha, \alpha)$. Now, suppose $k > 0$ and $\theta_1 > 0$. Then
$u = e^{i\theta_1}e^{i\theta_2}\ldots e^{i\theta_n} \sim e^{(2\pi/3)+\varepsilon}e^{-[((2\pi/3)+\varepsilon)-\theta_1]}e^{i\theta_2}\ldots e^{i\theta_n} \sim$
$e^{-(2\pi/3)+\varepsilon/2}e^{-(2\pi/3)+\varepsilon/2}e^{-((2\pi/3)+\varepsilon)+\theta_1}e^{i\theta_2}\ldots e^{i\theta_n}$ since $-(2\pi/3) + \varepsilon/2 - (2\pi/3) + \varepsilon/2 = -4\pi/3 + \varepsilon$ and $e^{i(-4\pi/3+\varepsilon)} = e^{i(2\pi/3+\varepsilon)}$ in S^1. Then, since $-(2\pi/3) + \varepsilon/2 - (2\pi/3) + \varepsilon/2 - (2\pi/3) - \varepsilon + \theta_1 + \theta_2 + \ldots + \theta_n = -2\pi + 2k\pi = 2(k-1)\pi$ we have shown that u relates to a U-word whose exponents add to a value one less than that of u. Notice that if θ_1 is negative and k is positive then there must exist an index i with $\theta_i > 0$ and the argument can be applied to θ_i. Applying the above argument k times shows that if $k > 0$, $u \sim e^{i\lambda_1}e^{i\lambda_2}\ldots e^{i\lambda_m}$ with $\lambda_1 + \lambda_2 + \ldots + \lambda_m = 0$. Hence $[u] = [e^0]$ by (*). A symmetric argument with the values $-[(2\pi/3) + \varepsilon]$ and $(2\pi/3) + \varepsilon/2$ can be used to show that if $k < 0$, $[u] = [e^0]$. This shows that K is trivial and hence $S^1_U \cong S^1$ by Theorem 1.

Suppose that a group G is locally generated. An examination of the kernel in Theorem 1 provieds a criterion by which the homomorphism is an isomorphism.

(**) Let $[u]$ be a fine U-word with $u = x_1x_2\ldots x_n$. Then φ is an isomorphism if $\sum_{i=1}^{n} x_i = e \Rightarrow [x_1x_2\ldots x_n] = [e]$.

The following theorem shows that for any euclidean space, $\mathbb{R}^n$ and $V \in U(\mathbb{R}^n)$ the epimorphism in Theorem 1 is an isomorphism.

Theroem 2 ($\mathbb{R}^n$): Consider the topological group $\mathbb{R}^n$. Let U be an arbitrary element of $U(\mathbb{R}^n)$ and let $U_0, U_1, \ldots, U_k$ be the components of U. Then $\forall\, U \in U(\mathbb{R}^n)$ we have $(\mathbb{R}^n)_U \cong \mathbb{R}^n \times < U_0, U_1, \ldots, U_k \mid R >$.

Proof: Suppose $U \in U(\mathbb{R}^n)$. We need to show that if $x_1x_2\ldots x_m$ is a fine U-word with $\Sigma_{i=1}^{m} x_i = 0$ then $x_1x_2\ldots x_m \sim 0$. The proof is by induction on the dimension n.

Suppose $U \in U(\mathbb{R})$ and $x_1x_2\ldots x_m$ is a fine U-word in $\hat{U}$. Let x_i be the first term in the U-word with sign opposite that of x_1. Then x_{i-1} and x_i are elements of U_0 with opposite signs, hence $|x_{i-1} + x_i| < \max\{|x_{i-1}|, |x_i|\} \Rightarrow x_{i-1} + x_i \in U_0$. This gives us $x_1x_2\ldots x_m \sim x_1x_2\ldots x_{i-2}(x_{i-1} + x_i)x_{i+1}\ldots x_m$ which is a fine U-word with $m-1$

terms and with $\Sigma_{i=1}^{m-1} x_i = 0$. Repeating this procedure $m-2$ times leaves us with $x_1x_2\ldots x_m \sim y_1y_2$ where $y_1, y_2 \in U_0$ and $y_1 + y_2 = 0$. Hence $y_1y_2 \sim 0$ and the result follows.

Now, suppose that $\forall U \in \mathcal{U}(\mathbb{R}^n)$ we have $(\mathbb{R}^n)_U \cong \mathbb{R}^n \times < U_0, U_1, \ldots, U_k \mid R >$. Let $U \in \mathcal{U}(\mathbb{R}^{n+1})$ and $x_1x_2\ldots x_m$ be a fine U-word with $\Sigma_{i=1}^{m} x_i = 0$. By Lemma1 above we may suppose that the x_i all lie in a ball $B(0,\varepsilon)$. Let $\hat{\imath}$ denote the unit vector $x_1/\parallel x_1 \parallel$. Further, let $x_i \cdot \hat{\imath}$ represent the projection of x_i onto $\hat{\imath}$ and let $x_i \cdot (x_1)_\perp$ be the projection of x_i onto the n dimensional space perpendicular to $\hat{\imath}$. Then $x_i = x_i \cdot \hat{\imath} + x_i \cdot (x_1)_\perp$. Since $\parallel x_i \cdot \hat{\imath} \parallel \leq \parallel x_i \parallel < \varepsilon$ and $\parallel x_i \cdot (x_1)_\perp \parallel \leq \parallel x_i \parallel < \varepsilon$ we have $x_i \cdot \hat{\imath}$, $x_i \cdot (x_1)_\perp \in U_0$. Hence

$$x_1x_2\ldots x_m \sim x_1(x_2 \cdot \hat{\imath})(x_2 \cdot (x_1)_\perp)\ldots(x_m \cdot \hat{\imath})(x_m \cdot (x_1)_\perp).$$

Now, for each x_j $3 \leq j \leq m$ choose $k_j \in \mathbb{N}$ such that $\|x_j \cdot \hat{\imath}\|/k_j < \min\{\varepsilon - \|x_i \cdot (x_1)_\perp\|\}$. Then we can split $(x_3 \cdot \hat{\imath})$ into k_3 terms all of the form $1/k_3(x_3 \cdot \hat{\imath})$ since $\|r/k_3(x_3 \cdot \hat{\imath})\| = r/k_3\|(x_3 \cdot \hat{\imath})\| < \|(x_3 \cdot \hat{\imath})\| < \varepsilon$ $\forall r \in \mathbb{N}$ with $r \leq k_3$. Then, $\|x_2 \cdot (x_1)_\perp + 1/k_3(x_3 \cdot \hat{\imath})\| \leq \|x_2 \cdot (x_1)_\perp\| + \|1/k_3(x_3 \cdot \hat{\imath})\| < \|x_2 \cdot (x_1)_\perp\| + \varepsilon - \|x_2 \cdot (x_1)_\perp\| = \varepsilon$ implies that $x_2 \cdot (x_1)_\perp + 1/k_3(x_3 \cdot \hat{\imath}) \in U_0$. Since $\mathbb{R}^n$ is abelian and $x_2 \cdot (x_1)_\perp \in U_0$, we have that each term $1/k_3(x_3 \cdot \hat{\imath})$ commutes with $(x_2 \cdot (x_1)_\perp)$ (see the discussion preceding Lemma 2). Thus we have

$$x_1(x_2 \cdot \hat{\imath})(x_2 \cdot (x_1)_\perp)(x_3 \cdot \hat{\imath})(x_3 \cdot (x_1)_\perp)\ldots(x_m \cdot \hat{\imath})(x_m \cdot (x_1)_\perp) \sim$$
$$x_1(x_2 \cdot \hat{\imath})(x_3 \cdot \hat{\imath})(x_2 \cdot (x_1)_\perp)(x_3 \cdot (x_1)_\perp)\ldots(x_m \cdot \hat{\imath})(x_m \cdot (x_1)_\perp)$$

Continuing in order, we see that $(x_j \cdot \hat{\imath})$ can be split into k_j terms (where k_j was chosen above) all of the form $1/k_j(x_j \cdot \hat{\imath})$. Then, for each $2 \leq i < j$ we have $\|x_i \cdot (x_1)_\perp + 1/k_j(x_j \cdot \hat{\imath})\| \leq \|x_i \cdot (x_1)_\perp\| + \|1/k_j(x_j \cdot \hat{\imath})\| < \|x_i \cdot (x_1)_\perp\| + \varepsilon - \|x_i \cdot (x_1)_\perp\| = \varepsilon$, and thus $1/k_j(x_j \cdot \hat{\imath})$ commutes with each $(x_j \cdot \hat{\imath})$. We then have the relation,

$$x_1(x_2 \cdot \hat{\imath})(x_3 \cdot \hat{\imath})(x_2 \cdot (x_1)_\perp)(x_3 \cdot (x_1)_\perp)\ldots(x_m \cdot \hat{\imath})(x_m \cdot (x_1)_\perp) \sim$$
$$x_1(x_2 \cdot \hat{\imath})\ldots(x_m \cdot \hat{\imath})(x_2 \cdot (x_1)_\perp)(x_3 \cdot (x_1)_\perp)\ldots(x_m \cdot (x_1)_\perp)$$

Now, $x_1(x_2 \cdot \hat{\imath})\ldots(x_m \cdot \hat{\imath}) \sim 0$ by an argument identical to the one for $\mathbb{R}$ above. But, $(x_2 \cdot (x_1)_\perp)(x_3 \cdot (x_1)_\perp)\ldots(x_m \cdot (x_1)_\perp) \sim 0$ also by the induction hypothesis since $B(0,\varepsilon) \cap (x_1)_\perp \in \mathcal{U}(\mathbb{R}^n)$. Hence $x_1x_2\ldots x_m \sim 0$ and the result follows by (**).

In Theorem 1, every component of U is listed as a generator in $< U_0, U_1, \ldots, U_k \mid R >$. However, there are many relations which come automatically and which have the effect of reducing the number of generators. In particular, since $e + e = e$ we will alway have the relation $U_0U_0U_0^{-1}$ which implies that $U_0 = e$ in $< U_0, U_1, \ldots, U_k \mid R >$. Also, for each $U_i \subset U$, notice that the set

$-U_i = \{-x \mid x \in U_i\} \subset U$, by definition. Clearly, $U_0 = -U_0$ since $-U_0$ is an open set containing e and hence $-U_0$ is a subset of the component containing e which is U_0. Also, in $\mathbb{R}^2$ with $U_0 = B(0,\varepsilon)$ and $U_1 = \{(x,y) \mid 1 < \sqrt{x^2+y^2} < 2\}$ we have $U_1 = -U_1$. In general, however, it is possible that $U_i \neq -U_i$ as in $\mathbb{R}$ with $U_0 = (-\varepsilon,\varepsilon)$, $U_1 = (1,2)$ and $-U_1 = (-2,-1)$. In the situation $U_i \neq -U_i$ however, we will always have the relation $U_i(-U_i)U_0^{-1}$ since $x + (-x) = e$ which implies that $-U_i = U_i^{-1}$. Thus, even though $-U_i$ may be a distinct component from U_i, it is not really needed as a generator in the finitely presented group $< U_0, U_1, \ldots, U_k \mid R >$. It is also possible that a component $U_i = e$ in $< U_0, U_1, \ldots, U_k \mid R >$ but $U_i \neq U_0$ as in $\mathbb{R}$ with $U = (-5,-2) \cup (-\varepsilon,\varepsilon) \cup (2,5)$ where the component $U_1 = (2,5)$ obeys the relation $U_1U_1(-U_1) = e$ since $2 + 2 = 4 \in U_1$ and hence $U_1 = e$. This discussion leads to the following definition.

Definition 5: Let U_i be a component of U. The following relations are called trivial.
a) $U_iU_iU_i^{-1}$, $U_iU_0U_0^{-1}$, $U_0U_iU_0^{-1}$, $U_0U_0U_i^{-1}$. These establish $U_i = e$ and imply that the component U_i is not a true generator in the group $< U_0, U_1, \ldots, U_k \mid R >$. Since $e + e = e$ in G we always have $U_0U_0U_0^{-1}$ and hence $U_0 = e$.
b) $U_iU_0U_i^{-1}$, $U_0U_iU_i^{-1}$. These establish $U_0 = e$ or $U_i = U_i$. Since $x + e = e + x = x$ in G they are always present for every U_i.
c) $U_i(-U_i)U_0^{-1}$, $(-U_i)U_iU_0^{-1}$. These establish $-U_i = U_i^{-1}$. If $x \in U_i$ then $-x \in -U_i$ and since $x + (-x) = e$ and $-x + x = e$ they are always present for every U_i.

An obvious question at this point is, "Which groups can be achieved as the image of φ_2 in Theorem 1?" The question has proved to be a difficult one, and is an open question. The following propositions and examples demonstrate some of the possibilities especially in the $\mathbb{R}^n$ case.

Proposition 3: If $U \in U(\mathbb{R}^n)$ with k components where $k > 1$, and R consists only of trivial relations then $(\mathbb{R}^n)_U \cong \mathbb{R}^n \times F(m)$ where $F(m)$ is the free group on m generators and $m \leq k$.

Note: The $\mathbb{R}$ case was proved by Yelton in an unpublished paper [7]. She defined a condition of independance (see the discussion following Lemma 4) which can be restated by saying that a component U_i is independent if the only relations it obeys is $U_0U_iU_i^{-1}$ (or $U_iU_0U_i^{-1}$) and $U_i(-U_i)U_0^{-1}$ (or $(-U_i)U_iU_0^{-1}$). If all components are indepentent, then $\mathbb{R}_U \cong \mathbb{R} \times F(m)$ where $m = \frac{k}{2}$. Her paper provided the inspiration for Theorem 1. Proposition 3 represents a slight generalization by considering the possibility of components $U_i \neq U_0$ which obey $U_iU_iU_i^{-1}$. Such components are not independent in the above sense, yet $U_i = e$ in $< U_0, U_1, \ldots, U_k \mid R >$ so that their presence has no effect on the freedom of $< U_0, U_1, \ldots, U_k \mid R >$ (as long as there are no other non trivial relations containing U_i).

Proof: Choose $U_{i_1} \neq U_0$. Then, if $U \neq U_0 \cup U_{i_1} \cup -U_{i_1}$ choose U_{i_2} such that $U_{i_2} \cap (U_0 \cup U_{i_1} \cup -U_{i_1}) = \emptyset$. In general, choose U_{i_s} such that it is distinct from all previous

U_{i_l} and $-U_{i_l}$ i.e $U_{i_s} \cap (U_0 \cup U_{i_1} \cup -U_{i_1} \cup \ldots \cup U_{i_{s-1}} \cup -U_{i_{s-1}}) = \emptyset$. Since there are a finite number of components, the process must end at some U_{i_j} with $j \leq k$. By the choice of U_{i_s} we have $U_{i_s} \cap U_{i_l} = \emptyset$.and $U_{i_s} \neq -U_{i_l}$ for all $0 \leq s, l \leq j$ with $s \neq l$. Remove all $V \in \{U_{i_1}, \ldots U_{i_j}\}$ such that VVV^{-1} is a relation. This gives us a new collection $\{U_{i_1}, \ldots U_{i_m}\}$ where $m \leq j \leq k$. Now, if $V \notin \{U_{i_1}, \ldots U_{i_m}\}$ then either $V = -U_{i_j}$ for some $U_{i_j} \in \{U_{i_1}, \ldots U_{i_m}\}$ or VVV^{-1} is a relation. Hence $V = U_{i_s}^{-1}$ or e in $< U_0, U_1, \ldots, V, \ldots U_k \mid R >$ and hence $< U_0, U_1, \ldots, V, \ldots U_k \mid R > \cong$ $< U_0, U_1, \ldots, U_k \mid R >$. In fact, we have $< U_0, U_1, \ldots, U_k \mid R > \cong < U_{i_1}, \ldots U_{i_m} \mid R >$ Since the only relations are trivial, there are no relations between members of $\{U_{i_1}, \ldots U_{i_m}\}$ and hence $< U_{i_1}, \ldots U_{i_m} \mid R > \cong F(m)$. Thus by Theorem 1 $(\mathbb{R}^n)_U \cong \mathbb{R}^n \times F(m)$.

Example 2: In $\mathbb{R}$ the set $U = (-3,-2) \cup (-1,1) \cup (2,3)$ has $\mathbb{R}_U \cong \mathbb{R} \times \mathbb{Z}$. Set $U_1 = (2,3)$. We must establish that there are no relations between U_0 and U_1 or $-U_1$ by considering all possible $x,y,z \in U$ such that $x + y = z$. Since $(-1,1) + (-1,1) = (-2,2)$ and $(-2,2) \cap (2,3) = \emptyset$ there are no relatons between U_0 and U_1 (or $-U_1$). Further, since $(2,3) + (2,3) = (4,6)$ there are no relations between U_1 and itself (similarly for $-U_1$). Hence, the only relations are trivial, and $\mathbb{R}_U \cong \mathbb{R} \times F(1)$.

If either U_1 is too close to U_0 or if U_1 is too big then the $\mathbb{Z}$ term dissappears and $\mathbb{R}_U \cong \mathbb{R}$. To see this consider the following sets.
1. $U = (-2,-1.5) \cup (-1,1) \cup (1.5,2)$
2. $U = (-5,-2) \cup (-1,1) \cup (2,5)$

Both sets look deceptively similar to the U in Example 2. But in 1 we have $.8 + .8 = 1.6$ which gives the relation $U_0 U_0 U_1^{-1}$ and hence $U_1 = e$ in $< U_0, U_1, -U_1 \mid R >$. Here, U_1 was too close to U_0. On the other hand, in 2 we have $2.1 + 2.1 = 4.2$ which gives the relation $U_1 U_1 U_1^{-1}$ and again $U_1 = e$ in $< U_0, U_1, -U_1 \mid R >$.

Example 3: Let $G = \mathbb{R}^n$. Let $m \in \mathbb{N}$. Let $U_0 = B(0,1)$. In general choose x_i such that $|x_i| > \sup\{|y + z| \mid y,z \in U_0, U_1 \ldots U_{i-1}\} + 2$. This supremum is finite since the sets U_j are bounded and there are only a finite number of them. Choose $U_i = B(x_i, 1)$. Continue this process until U_m is located. Let $U_{m+k} = -U_k$. Then if $U = U_0, U_1, \ldots U_{2m}$ then $(\mathbb{R}^n)_U \cong \mathbb{R}^n \times F(m)$.

Proof: The U_i $1 \leq i \leq m$ were chosen in such a way that there are no relations between them. Since $U_0 = e$ and $U_{m+k} = U_k^{-1}$ $1 \leq k \leq m$ we have by Proposition 3 that $< U_0, U_1, \ldots U_{2m} \mid R > \cong F(m)$ and the result follows.

It is possible in G_U, for a component of U to have finite order in the group $< U_0, U_1, \ldots, U_k \mid R >$ even though all of the group elements of G have infinite order as the following example shows.

Example 4: In $\mathbb{R}^n$, let the set U_0 consist of $2n-1$ parts.
$V_0 = \{(x_1,x_2,\ldots,x_n) \mid |x_1| < 2.01 , |x_2| < .01,\ldots,|x_n| < .01\}$
$V_{1,1} = \{(x_1,x_2,\ldots,x_n) \mid |x_1-2| < .01 , -.01 < x_2 < 2.01, |x_3| < .01,\ldots,|x_n| < .01\}$
$V_{1,2} = \{(x_1,x_2,\ldots,x_n) \mid |x_1+2| < .01, -2.01 < x_2 < .01, |x_3| < .01,\ldots,|x_n| < .01\}$
$V_{2,1} = \{(x_1,x_2,\ldots,x_n) \mid |x_1-2| < .01 , |x_2-2| < .01, -.01 < z < 2.01,$
$|x_4| < .01,\ldots,|x_n| < .01\}$
$V_{2,2} = \{(x_1,x_2,\ldots,x_n) \mid |x_1+2| < .01, |x_2+2| < .01, -2.01 < x_3 < .01,$
$|x_4| < .01,\ldots,|x_n| < .01\}$

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$V_{i,1} = \{(x_1,x_2,\ldots,x_n) \mid |x_1-2| < .01,\ldots,|x_i-2| < .01, -.01 < x_{i+1} < 2.01,$
$|x_{i+2}| < .01,\ldots,|x_n| < .01\}$
$V_{i,2} = \{(x_1,x_2,\ldots,x_n) \mid |x_1+2| < .01,\ldots,|x_i+2| < .01, -2.01 < x_{i+1} < .01,$
$|x_{i+2}| < .01,\ldots,|x_n| < .01\}$

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$V_{n-1,1} = \{(x_1,x_2,\ldots,x_n) \mid |x_1-1| < .01,\ldots,|x_{n-1}-1| < .01, -.01 < x_n < 2.01\}$
$V_{n-1,2} = \{(x_1,x_2,\ldots,x_n) \mid |x_1-1| < .01,\ldots,|x_{n-1}-1| < .01, -2.01 < x_n < .01\}$

Notice that U_0 consists of a "strip" along the x_1 axis from –2 to 2 which has a width of $.02$ in each of the dimensions x_2 through x_n, together with strips of half the length running from the points $(2,2,\ldots,2)$ or $(-2,-2,\ldots,-2)$ in the subspace formed by the first i dimensions to the point $(2,2,\ldots,2)$ or $(-2,-2,\ldots,-2)$ in the subspace formed by the first $i+1$ dimensions which again has a width of $.02$ in each of the dimensions $x_1,\ldots x_i,x_{i+2},\ldots,x_n$. Also notice that the set $V_{i,1}$ intersects the sets $V_{i-1,1}$ and $V_{i+1,1}$ and the set $V_{i,2}$ intersects the sets $V_{i-1,2}$ and $V_{i+1,2}$.

Let $U_1 = B((1,1,\ldots,1),.01)$
$U_2 = B((-1,-1,\ldots,-1),.01)$
Then let $U = U_0 \cup U_1 \cup U_2$ (see Figure 1). We have $\mathbb{R}^n_U \cong \mathbb{R}^n \times \mathbb{Z}_2$ for $n \geq 3$.

To see that U is symmetric, notice that V_0 is symmetric to itself since $|-x_i| = |x_i|$. We also have that $V_{i,2}$ contains the symmetric image of $V_{i,1}$. To see this, let $(z_1,z_2,\ldots,z_n) \in V_1$. Then $(-z_1,-z_2,\ldots,-z_n)$ obeys $|-z_j+1| = |-(z_j-1)| = |z_j-1| < .01$ for $1 \leq j \leq i$ and $|-z_j| = |z_j| < .01$ for $i+2 \leq j$. Also,
$-.01 < z_{i+1} < 2.01 \Rightarrow .01 > -z_{i+1} > -2.01$ so that $(-z_1,-z_2,\ldots,-z_n) \in V_{i,2}$. A similar argument shows that $V_{i,1}$ contains the symmetric image of $V_{i,2}$ hence $V_{i,1}$ and $V_{i,2}$ are symmetric images of each other. Also, we have U_2 as the symmetric image of U_1. Let $(z_1,z_2,\ldots,z_n) \in U_1$. Then $(-z_1,-z_2,\ldots,-z_n)$ obeys
$(-x_1+1)^2 + (-x_2+1)^2 + \ldots + (-x_n+1)^2 =$
$(-(x_1-1))^2 + (-(x_2-1))^2 + \ldots + (-(x_n-1))^2 = (x_1-1)^2 + (x_2-1)^2 + \ldots + (x_n-1)^2 < .01$

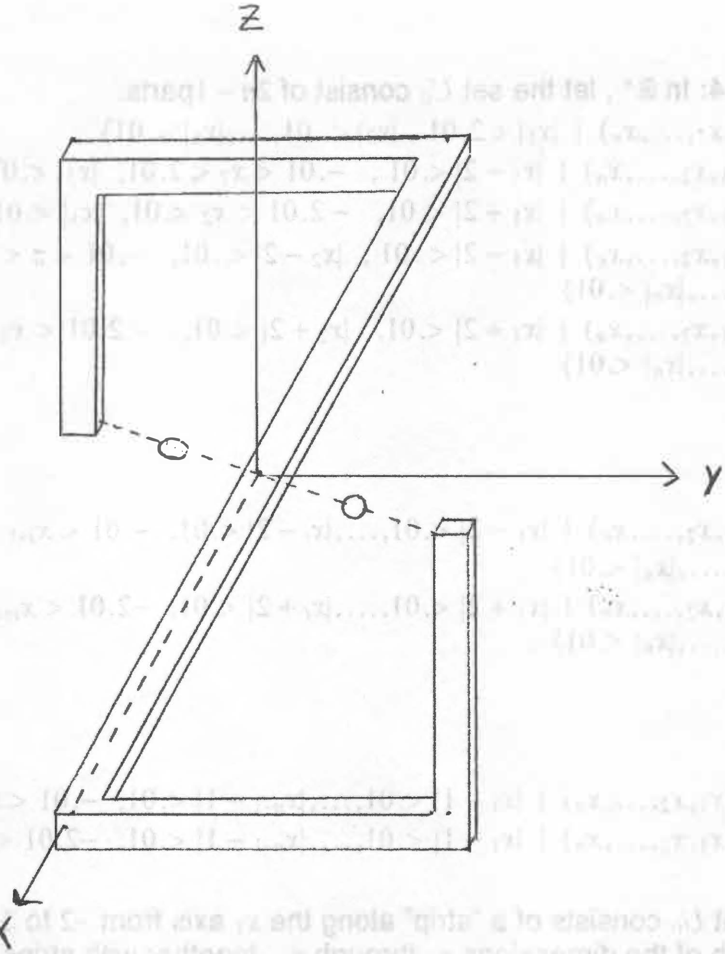

Figure 1. The set U from Example 4 for $n = 3$.

and $(-z_1,-z_2,\ldots,-z_n) \in U_2$. Further, since each V_i is the direct product of n open intervals and both U_1 and U_2 are open balls about the points $(1,1,\ldots,1)$ and $(-1,-1,\ldots,-1)$ respectively, we have that U is open. Finally, notice that each $V_{i,j}$ is connected (as the direct product of connected sets) and open hence pathwise connected. Since each $V_{i,1}$ intersects $V_{i-1,1}$ and each $V_{i,2}$ intersects each $V_{i-1,2}$ we can connect any point pathwise to the origin and hence U_0 is a component. Since U_1 and U_2 are open balls which do not intersect each other or U_0 we have that U consists of three components.

To determine the group $< U_0, U_1, U_2 \mid R >$ we examine all possible relations by considering the sets $U_0 + U_0$, $U_0 + U_1$, $U_0 + U_2$, $U_1 + U_1$, $U_1 + U_2$, $U_2 + U_2$.

1. $U_0 + U_0$. Since $0 + 0 = 0$ we get the trivial relation $U_0 U_0 U_0^{-1}$ which implies that $U_0 = e$. We must establish, however, that $(U_0 + U_0) \cap U_1 = (U_0 + U_0) \cap U_2 = \emptyset$ in order to avoid additional relations which would imply that $U_1 = e$ or $U_2 = e$. First suppose that $w = (w_1, w_2, \ldots, w_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are such that $w, v \in U_0$ and $w + v \in U_1$. Notice that for each $1 \leq i \leq n$ we must have $w_i + v_i \in (.9, 1.1)$. Then we must also have $w \in V_0$ or $v \in V_0$. If not, then we would have $1.99 < w_1 < 2.01$ or $-2.01 < w_1 < -1.99$. and similarly for v_1. Hence $w_1 + v_1 \in (3.98, 4.02)$ or $(-.02, .02)$ or $(-4.02, 3.98)$ which contradicts $w_1 + v_1 \subset (.9, 1.1)$. Hence at least one of v or w must be an element of V_0. Suppose without loss of generality that $w \in V_0$. Then $|w_2| < .01 \Rightarrow v \in V_{1,1}$ or $V_{1,2}$ since otherwise we would have $w_2 + v_2 \in (-.02, .02)$ if $v \in V_0$ or $w_2 + v_2 \in (-2.02, -1.98)$ or $(1.98, 2.02)$ if $v \in V_{i,j}$ for $i \geq 2$. Then $|w_3| < .01$ and $|v_3| < .01 \Rightarrow w_3 + v_3 \in (-.02, .02)$ a contradiction. Hence $(U_0 + U_0) \cap U_1 = \emptyset$. A similar argument shows that $U_0 + U_0 \cap U_2 = \emptyset$. Thus there are no futher relations in $U_0 + U_0$.

2. $U_0 + U_1$. Since $0 + (1,1,\ldots,1) = (1,1,\ldots,1)$ we get the trivial relation $U_0 U_1 U_1^{-1}$ which implies that $U_1 = U_1$ or that $U_0 = e$. Also, since $(-2,-2,\ldots,-2) + (1,1,\ldots,1) = (-1,-1,\ldots,-1)$ we get the relation $U_0 U_1 U_2^{-1}$ which implies that $U_1 = U_2$. We cannot have an $x \in U_0, y \in U_1, z \in U_0$ such that $x + y = z$ because then $x + (-z) = -y$ and we showed in part 1 above that this is not possible.

3. $U_0 + U_2$. As in part 2 we get the trivial relation $U_0 U_2 U_2^{-1}$ and the relation $U_0 U_2 U_1^{-1}$ but not the relation $U_0 U_2 U_0^{-1}$.

4. $U_1 + U_1$. Since $(1,1,\ldots,1) + (1,1,\ldots,1) = (2,2,\ldots,2)$ we have the relation $U_1 U_1 U_0^{-1}$ which implies that $(U_1)^2 = e$. . The $\inf\{x_1 \mid (x_1, x_2, \ldots, x_n) \in U_1\} = .9$ and the $\sup\{x_1 \mid (x_1, x_2, \ldots, x_n) \in U_1\} = 1.1$. Hence the sum of two elements of U_1 must have its x_1 value at least 1.8 so that we do not obtain the relation $U_1 U_1 U_1^{-1}$ or $U_1 U_1 U_2^{-1}$.

5. $U_1 + U_2$. We get the trivial relation $U_1 U_2 U_0^{-1}$ since $x + (-x) = e$ in G. Now let $(x_1, x_2, \ldots, x_n) \in U_1$ and $(y_1, y_2, \ldots, y_n) \in U_2$. Then $-.2 < x_1 + y_1 < .2$ and hence we do not get the relation $U_1 U_2 U_1^{-1}$ or $U_1 U_2 U_2^{-1}$.

6. $U_2 + U_2$. As in part 4 above we get $U_2U_2U_0^{-1}$ but not $U_2U_2U_2^{-1}$ or $U_2U_2U_1^{-1}$.

Since we have $(U_1)^2 = e$ and we do not have $U_1 = e$, $< U_0, U_1, U_2 \mid R > \; \cong \mathbb{Z}_2$ and $\mathbb{R}^n_U \cong \mathbb{R}^n \times \mathbb{Z}_2$

Note: The above example fails in $\mathbb{R}$ or $\mathbb{R}^2$. In $\mathbb{R}$ we have $U_0 = (-2.01, 2.01)$ and $U_1 = (.9, 1.1)$ which gives $U_1 \subset U_0$. In $\mathbb{R}^2$ we have $U_0 = \{(x,y) \mid |x| < 2.01 , |y| < .01\} \cup \{(x,y) \mid |x-2| < .01 , -.01 < y < 2.01\}$ $\cup \{(x,y) \mid |x+2| < .01 , -2.01 < y < .01\}$ and $U_1 = B((1,1),.1)$. Since $(-1,0) + (2,1) = (1,1)$ we obtain the relation $U_0U_0U_1^{-1}$, which implies that $U_1 = e$.

In the attempt to classify all possible groups obtained as the image of φ_2 in Theorem 1, the following lemma shows that every finitely presented group is potentially possible even if the relations don't all have 3 elements to them.

Lemma 5: Any finitely presented group with a finite number of relations is isomorphic to a finitely presented group whose relations have three or fewer elements.

Proof: Let $F_1/N = < x_1, x_2, \ldots, x_n \mid R_1, R_2, \ldots, R_m >$ be an arbitrary finitely presented group with finite number of relations, where F_1 is the free group on the elements $x_1, x_2, \ldots, x_n$ and N represents the normal subgroup generated by the relations $R_1, R_2, \ldots, R_m$. If $|R_j|$ represents the number of elements in relation R_j then the fact that there are a finite number of relations means we can define $k = \max_{1 \le j \le n} |R_j|$. Let R_i be a relation such that $|R_i| = k$ and denote $R_i = x_{i_1}x_{i_2}\ldots x_{i_k}$. Define three new relations $S_1 = a^{-1}x_{i_1}x_{i_2}\ldots x_{i_{k-2}}$,
$S_2 = b^{-1}x_{i_{k-1}}x_{i_k}$, and $S_3 = ab$. Notice that $|S_1|, |S_2|, |S_3| < k$. Now, define $F_2/M = \; < x_1, x_2, \ldots, x_n, a, b \mid R_1, R_2, \ldots R_{i-1}, R_{i+1}, \ldots, R_m, S_1, S_2, S_3 >$ where F_2 is the free group on the elements $x_1, x_2, \ldots, x_n, a, b$, and M is the normal subgroup generated by the relations $R_1, R_2, \ldots, R_{i-1}, R_{i+1}, \ldots, R_m, S_1, S_2, S_3$. I will show that $F_1/N \cong F_2/M$

Define $f : \{x_1, x_2, \ldots, x_n\} \to F_2$ by $f(x_s) = x_s$. Then there is an induced homomorphism $f^* : F_1 \to F_2$ given by $f(z_1z_2\ldots z_l) = z_1z_2\ldots z_l$. In particular, $f(R_j) = R_j \; \forall j \neq i$ hence $f(R_j) \in M$. Also, $f(R_i) = x_{i_1}x_{i_2}\ldots x_{i_k}$. We have $f(R_i) \in M$ since S_3, $b^{-1}S_1b$, and $S_2 \in M$ and $S_3^{-1}b^{-1}S_1bS_2 = x_{i_1}x_{i_2}\ldots x_{i_k}$. Hence $f(N) \subset M$ and we have a homomorphism f° from $F_1/N \to F_2/M$ given by $f^{\circ}(z_1z_2\ldots z_lN) = z_1z_2\ldots z_lM$ (see Hungerford p. 44[2]). Now, define $g : \{x_1, x_2, \ldots, x_n, a, b\} \to F_1$ by $g(x_t) = x_t$, $g(a) = x_{i_1}x_{i_2}\ldots x_{i_{k-2}}$, and $g(b) = x_{i_{k-1}}x_{i_k}$. Then there is an induced homomorphism $g^* : F_2 \to F_1$. In particular, $g^*(R_j) = R_j \forall j \neq i$ hence $g^*(R_j) \in N$. Also, we have $g^*(S_1) = (x_{i_1}x_{i_2}\ldots x_{i_{k-2}})^{-1}(x_{i_1}x_{i_2}\ldots x_{i_{k-2}}) = e$
, $g^*(S_2) = (x_{i_{k-1}}x_{i_k})^{-1}x_{i_{k-1}}x_{i_k} = e$, and $g^*(S_3) = R_i$. Hence $g^*(M) \subset N$. Thus there is an induced homomorphism $g^{\circ} : F_2/M \to F_1/N$. Notice that $(g^{\circ} \circ f^{\circ})(z_1z_2\ldots z_lN) = g^{\circ}(z_1z_2\ldots z_lM) = z_1z_2\ldots z_lN$ since $z_1z_2\ldots z_l$ must contain no a's or

b's. Also we have in particular that
$(f^{\circ} \circ g^{\circ})(aM) = f^{\circ}(x_{i_1}x_{i_2}\ldots x_{i_{k-2}}N) = x_{i_1}x_{i_2}\ldots x_{i_{k-2}}M = aM$ (since $a^{-1}x_{i_1}x_{i_2}\ldots x_{i_{k-2}} \in M$).
Similarly $(f^{\circ} \circ g^{\circ})(bM) = bM$. Hence if $z_1\ldots a\ldots z_lM \in F_2/M$ then
$(f^{\circ} \circ g^{\circ})(z_1\ldots a\ldots z_lM) = f^{\circ}(z_1\ldots x_{i_1}x_{i_2}\ldots x_{i_{k-2}}\ldots z_lN) =$
$z_1\ldots x_{i_1}x_{i_2}\ldots x_{i_{k-2}}\ldots z_lM = z_1\ldots a\ldots z_lM$. Similarly $(f^{\circ} \circ g^{\circ})(z_1\ldots b\ldots z_lM) = z_1\ldots b\ldots z_lM$.
Since each word has only a finite number of a's and b's, we can proceed inductively to show that $(f^{\circ} \circ g^{\circ})(z_1z_2\ldots z_lM)$ is the identity. Hence f° is an isormorphism and the proof is complete.

The following proposition shows that in many cases, if H is a finitely presented subgroup of the topological group G then it is possible to realize the group H as the image of φ_2 in Theorem 1.

Proposition 4: Suppose G is locally connected and that H is a finitely presented subgroup of G. Then there exists $U \in \mathcal{U}(G)$ such that $< U_0, U_1, \ldots, U_k \mid R > \;\cong H$

Proof: By Lemma 5 above we may assume (by choosing more generators if necessary) that the relations all have three elements. Further, if a is a generator, we may throw in the element $-a$ as a generator. We may also assume (by adding them in if necessary) that if $a + b = c$ where a,b,c are generators that the word $ab(-c)$ is in the set of relations.

Let $a_1, a_2, \ldots, a_k, -a_1, -a_2 \ldots, -a_k$ be the generators of H. Since G is Hausdorff, we may find pairwise disjoint open sets around the points $e, a_1, a_2, \ldots, a_k, -a_1, -a_2, \ldots, -a_k$. Since G is locally connected, we may find inside each open set, a connected open set containing each point. Let $U_0, U_1, U_2, \ldots, U_{2k}$ be those sets, so that $e \in U_0, a_i \in U_i$, and $-a_i \in U_{k+i}$. We may further assume that $U_{k+i} = -U_i$ by replacing U_i with $U_i \cap -U_{k+i}$ if necessary and replacing U_{k+i} with $-(U_i \cap -U_{k+i})$. If $U = U_0 \cup U_1 \cup \ldots \cup U_{2k}$ then $U \in \mathcal{U}(G)$.

Suppose $a_i + a_j \neq a_l$. We must exclude the possibility that $U_iU_jU_l^{-1}$ is a word in the set of relations defined in Theorem 1. This may be done by renaming U_i and U_j in the following way. Since G is Hausdorff, we can find an open set V about $a_i + a_j$ such that $a_l \notin V$. Consider the map $\alpha : U_i \times U_j \to G$ given by $\alpha(x,y) = x + y$. Since G is a topological group, the map α must be continuous[4]. Hence $\alpha^{-1}(V)$ is open and contains the point (a_i, a_j). Further, by the definition of the product topology, we can find open sets V_i, V_j such that $a_i \in V_i, a_j \in V_j$ and $V_i \times V_j \subset \alpha^{-1}(V)$. Hence there exist no $x \in V_i, y \in V_j$ such that $x + y = a_l$. If we rename U_i as V_i and U_j as V_j then $U_iU_jU_l^{-1}$ will not be in the set of relations. Finally, if $a_i + a_j = a_l$ then there exists $a_i \in U_i, a_j \in U_j$, and $a_l \in U_l$ such that $a_i + a_j = a_l$ and hence the word $U_iU_jU_l^{-1}$ is a relation for R defined in Theorem 1. Thus there is a one-to-one correspondence between the generators of H and components of U as well as between the relations of H and relations of $< U_0, U_1, \ldots U_{2k}, R >$.

Example 5:(unit quaternions). Consider the following matricies in $GL(2, \mathbb{C})$.

$$i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \; j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $i^2 = j^2 = k^2 = -I$, and $ij = k$, $ji = -k$, $jk = i$, $kj = -i$, $ki = j$, $ik = -j$ the group generated by these matricies is isomorphic to the unit quaternions Q. Hence by Proposition 4, there exists a $U \in U(GL(2,\mathbb{C}))$ such that $< U_0, U_1, \ldots, U_k \mid R > \cong Q$. By Theorem 1 there is a homomorphism from $GL(2,\mathbb{C})_U \to G \times Q$.

III. Conclusion

As has been noted before, an attempt was made to classify all possible groups obtained in the image of φ_2 in Theorem 1. In particluar, it would be interesting to classify the set $\varphi_2((\mathbb{R}^n)_U)$ as U varies over all elements of the set $U(\mathbb{R}^n)$. It seems likely from Example 4 that the answer depends on the dimension n. The theorems and propositions here presented are helpful in establishing that certain groups are in this set. Many attempts, however, were made to establish that certain groups are not in this set. It remains unclear whether $\mathbb{Z}_2$ for instance could be achieved as $\varphi_2((\mathbb{R}^2)_U)$ or $\varphi_2((\mathbb{R})_U)$ for some $U \in U(\mathbb{R}^2)$ or $U(\mathbb{R})$. Another open question is the possibility of achieving the unit quaternions as $\varphi_2((\mathbb{R}^n)_U)$ where n is an arbitrary dimension. The main difficulty lies in the fact that a potentially complicated set of relations could reduce to the given groups in question. Perhaps a deeper knowledge of how the relations in finitely presented groups can combine would be helpful.

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