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Third Order Opial Inequalities

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To the Graduate Council:

I am submitting herewith a thesis written by Brandi Michelle Gierhart entitled "Third Order Opial Inequalities." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Don Hinton, Major Professor

We have read this thesis and recommend its acceptance:

Suzanne Lenhart, G. Samuel Jordan

Accepted for the Council:
Dixie L. Thompson

Vice Provost and Dean of the Graduate School

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Dr. Anne Mayhew

Vice Provost and Dean of Graduate Studies

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THIRD ORDER OPIAL INEQUALITIES

A Thesis Presented for the Master of Science Degree

The University of Tennessee, Knoxville

Brandi Michelle Gierhart

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Abstract

The purpose of the thesis was to find extremals y and constants K for a basic problem and for several variations of the problem. The basic problem was the third order opial inequality. We proved the existence of an extremal of this problem and other related problems. We conjectured that the extremal was a quintic spline with at most one knot, then found the quintic splines and constants for the basic problem and several variations. Maple was an important tool in finding these extremals and constants due to the complexity of the equations. We also proved that these extremals can be approximated by polynomials. We discussed an application involving a bound for the least eigenvalue for $y^{(vi)} = -\lambda\rho(x)y$ with certain boundary conditions. Finally, we discussed the related problem in L^p space.

Contents

1 Introduction

We begin with the third order opial inequality

$$K\int_a^b |yy'|dx \leq \int_a^b |y'''|^2 dx.$$

This inequality leads to a problem to solve:

$$K = \inf_{y\epsilon D\setminus\{0\}} \frac{\int_a^b |y'''|^2 dx}{\int_a^b |yy'|dx}$$

Given the set D of admissible functions, we seek K and an extremal function y that achieves the infimum. Using Maple, we will solve this equation for various sets D. We will also examine a few more general equations of a similar form. The extremals for which we are searching are quintic splines. Later, we will discuss in greater detail why we can approximate these extremals with polynomials. We will also discuss applications to a related problem, a bound for the least eigenvalue of

$$y^{(vi)} = -\lambda\rho(x)y$$

with certain boundary conditions. Finally, we will discuss the related equation

$$K_{0,p} = \inf_{y\epsilon D_0\setminus\{0\}} \frac{(\int_a^b |y'''|^p dx)^{2/p}}{\int_a^b |yy'|dx}$$

and solve for K using a specific set D_0 and its extremal y. We define

$$D_0 := \{y\epsilon A : y(0) = y'(0) = y''(0) = y(1) = y'(1) = y''(1) = 0\}$$

and

$$A := \{\mathrm{y : y\,is\,real\,on}\,[0,1],\ \mathrm{y,y',y''\,continuous,\ y'''\,piecewise\,continuous}\}.$$

All of this work is to explore generalizations of

$$K\int_a^b |yy'|dx \le \int_a^b (y''')^2 dx,$$

the third order Opial inequality.

We begin in Chapter 2 with our equation

$$K_0 = \inf_{y \epsilon D_0 \setminus \{0\}} \frac{\int_a^b |y'''|^2 dx}{\int_a^b |yy'|dx}$$

involving a specific admissible set D_0 and the constant K_0, still to be found. Also to be found is the extremal y_0. Using Maple, we will solve for K_0 and y_0 on the class of functions where $y_0 y_0'$ has one change of sign. We will also discuss the necessary conditions for the extremal y_0 to exist.

In Chapter 3 we shall discuss two functionals that are more general and the existence of solutions to the infimum problem. By solution we mean the constant K and the extremal y. First we will examine the problem

$$K = \inf_{y \epsilon D_0 \setminus \{0\}} J_{p,w}(y)$$

where

$$J_{p,w} = \frac{\int_0^1 p(x)(y'''(x))^2 dx}{\int_0^1 w(x)|y(x)y'(x)|dx}$$

and where $p(x) > 0$ and $w(x) > 0$ are continuous functions. We will show that the solution exists and that $K_{p,w}$ is positive. The other problem we will discuss is

$$K_G = \inf_{y \epsilon D_0 \setminus \{0\}} J_G(y)$$

where

$$J_G = \frac{\int_a^b [p(x)(y'''(x))^2 + r(x)(y''(x))^2 + m(x)(y'(x))^2 + q(x)(y(x))^2] dx}{\int_a^b w(x)|y(x)y'(x)| dx}.$$

We will find the necessary conditions for the extremal y.

Chapter 4 discusses other classes of admissible functions. Instead of using the set D_0, we will examine six other sets D_i for $i = 1, ..., 6$. We will examine in detail how to solve the equation

$$K_i = \inf_{y \epsilon D_i \setminus \{0\}} \frac{\int_0^1 (y''')^2 dx}{\int_0^1 |yy'| dx}$$

for the $i = 2$ case. The other five cases are similar; therefore, the solutions will be provided, omitting the details..

Chapter 5 will deal with one of the other classes of admissible functions. We will examine a special aspect of the $i = 4$ case. Because the extremal spline of the D_4 case, found in Chapter 4, is a quintic polynomial on $[a, b]$ and not a quintic spline, we can examine it differently. We will show that there always exists an extremal of the D_4 case which is monotone. This will involve working with Maple and the extremal spline that we will calculate for Chapter 4.

In Chapter 6 we will discuss applications to $y^{(vi)} = -\lambda\rho(x)y$ with boundary conditions. We will show that there is a lower bound on the smallest eigenvalue. Examples of sixth order eigenvalue problems may be found in [2].

Chapter 7 will cover the approximation of extremals by polynomials. We will state and prove two lemmas concerning this approximation. Lemma 1 deals with extremals in the class of admissible functions D_0 which has boundary conditions

$$y(0) = y'(0) = y''(0) = y(1) = y'(1) = y''(1) = 0.$$

Lemma 2 covers the D_1 case. D_1 is the class of admissible functions with boundary conditions

$$y(0) = y''(0) = y(1) = y''(1) = 0$$

and natural boundary conditions

$$y^{(iv)}(0) = y^{(iv)}(1) = 0.$$

Finally, in Chapter 8 we examine what happens if we alter our original equation from

$$K_0 = \inf_{y \epsilon D_0 \setminus \{0\}} J(y_0)$$

where

$$J(y_0) = \frac{\int_0^1 |y'''|^2 dx}{\int_0^1 |yy'| dx}$$

to

$$K_{0,p} = \inf_{y \epsilon D_0 \setminus \{0\}} J_p(y)$$

where

$$J_p(y) = \frac{(\int_0^1 |y'''(x)|^p dx)^{2/p}}{\int_0^1 |y(x)y'(x)| dx}.$$

Both problems have the same class of admissible functions D_0.

We will discuss all of these topics related to the third order Opial inequality. The discussion will begin with the basic problem involving D_0, the first class of admissible functionals. Also discussed will be necessary conditions, other general functions, six other classes of admissible functions, applications involving the smallest eigenvalue, and approximation of extremals by polynomials. Finally we will examine an altered version of the D_0 equation. All of these topics elaborate on various aspects of the third order Opial inequality.

2 Basic Problem

The basic problem is to find extremals $y \epsilon D_0(a,b)$ and constant

$$K_0(a,b) := \inf_{y \epsilon D_0(a,b) \setminus \{0\}} J(y) \tag{1}$$

where

$$J(y) := \frac{\int_a^b |y'''|^2 dx}{\int_a^b |yy'| dx}, \tag{2}$$

$$A := \{\mathrm{y : y\,is\,real\,on\,[a,b],\ y, y', y''\,continuous,\ y'''\,piecewise\,continuous}\},$$

and

$$D_0(a,b) := \{y \epsilon A : y(a) = y'(a) = y''(a) = y(b) = y'(b) = y''(b) = 0\}.$$

By an extremal we mean a function $y \epsilon D_0(a,b)$ such that $K_0(a,b) = J(y)$. In this case we are looking at a general interval [a,b]. However, we will show that it is only necessary to examine the interval [0,1]. For $Y \epsilon D_0(a,b)$, define $y(x) = Y(a + x(b-a))$. Then $y \epsilon D_0(0,1)$ and

$$J(y) := \frac{\int_0^1 (y''')^2 dx}{\int_0^1 |y(x)y'(x)| dx} = \frac{\int_a^b (Y'''(u))^2 (b-a)^6 \frac{du}{(b-a)}}{\int_a^b |Y(u)Y'(u)|(b-a)\frac{du}{(b-a)}} = (b-a)^5 J(Y).$$

Then

$$K_0(a,b) = \frac{K_0(0,1)}{(b-a)^5}$$

Let $K_0 := K_0(0,1)$. Therefore, without loss of generality, it is sufficient to work with the interval [0,1] instead of a general interval [a,b]. We now define

$$D_0 := D_0(0,1).$$

We begin with a preliminary discussion of some useful mathematics information important to the later explanations and work. Before determining necessary conditions for an extremal, we need to investigate the differentiation of $|f(x)|$ when f is a continuously differentiable function. Define

$$g(x) = |f(x)|.$$

Then we claim

$$g'(x_0) = \begin{cases} f'(x_0) & \text{if } f(x_0) > 0 \\ -f'(x_0) & \text{if } f(x_0) < 0. \end{cases}$$

Therefore, for $|f(x_0)| \neq 0$,

$$g'(x_0) = [sgn(f(x_0))] \cdot [f'(x_0)],$$

where $sgn(\cdot)$ is the signum function, giving the sign of the function as either $+1$ or -1.

Now examine the case in which $f(x_0) = 0$. If $f'(x_0) = 0$, then

$$g'(x_0) = \lim_{h \longrightarrow 0} \frac{g(x_0 + h) - g(x_0)}{h} = \lim_{h \longrightarrow 0} \frac{|f(x_0 + h)| - 0}{h}$$

$$= \lim_{h \longrightarrow 0} \frac{|f(x_0 + h) - f(x_0)|}{h} = 0.$$

Thus, $g'(x_0) = 0$. If $f'(x_0) \neq 0$, say $f'(x_0) > 0$, then

$$g'(x_0^+) = f'(x_0)$$

and

$$g'(x_0^-) = -f'(x_0).$$

The function g has one-sided derivatives at x_0, but they are unequal.

From real analysis we know that f is absolutely continuous on [a,b] if for $\epsilon > 0$ there exists $\delta > 0$ such that if $I_1 = [a_1, b_1], ..., I_n = [a_n, b_n]$ are nonoverlapping subintervals of [a,b] with

$$\sum_{i=1}^{n}(b_i - a_i) < \delta,$$

then

$$\sum_{i=1}^{n}|f(b_i) - f(a_i)| < \epsilon.$$

Now,

$$|\,|f(b_i)| - |f(a_i)|\,| \leq |f(b_i) - f(a_i)|$$

tells us that $|f(x)|$ is absolutely continuous if $f(x)$ is. Real analysis tells us that absolutely continuous functions are differentiable except possibly on a set of (Lebesgue) measure 0.

If f is continuously differentiable on [a,b], then $f(x)$ is absolutely continuous (from real analysis, but easy since $|f(b_i) - f(a_i)| = |f'(c_i)(b_i - a_i)| \leq M(b_i - a_i)$ if $|f'(x)| \leq M$). Therefore, $g(x) = |f(x)|$ is absolutely continuous, and the set $S = \{x_0 : f(x_0) = 0 \text{ and } f'(x_0) \neq 0\}$ where g is not differentiable has measure 0. Thus, $g'(x) = [sgn(f(x))]f'(x)$ is well-defined except in a set involving g of measure 0 which does not affect the value of an integral.

We need to see what necessary conditions that an extremal $y_0 \epsilon D_0$ of (2) must satisfy. We are only interested in intervals where $y_0 y_0'$ is of constant sign. Later we will explain specifically what kind of extremals for which we are searching. Let $y_0, h \epsilon D_0$. We calculate

$$\frac{d}{d\epsilon} J(y_0 + \epsilon h)|_{\epsilon=0} = 0.$$

First of all,

$$J(y_0 + \epsilon h) = \frac{\int_0^1 (y_0''' + \epsilon h''')^2 dx}{\int_0^1 |y_0 y_0' + \epsilon h y_0' + \epsilon h' y_0 + \epsilon^2 h h'| dx}$$

For simplicity, we write this as

$$J(y_0 + \epsilon h) = \frac{T}{B}.$$

Then

$$\frac{d}{d\epsilon} J(y_0 + \epsilon h) = \frac{BT_\epsilon - TB_\epsilon}{B^2}$$

where

$$T_\epsilon = \int_0^1 2(y_0''' + \epsilon h''') h''' dx$$

and

$$B_\epsilon = \int_0^1 (sgn(y_0 y_0' + \epsilon h y_0' + \epsilon h' y_0 + \epsilon^2 h h'))(h y_0' + h' y_0 + 2\epsilon h h') dx$$

Then

$$0 = \frac{d}{d\epsilon} J(y_0 + \epsilon h)|_{\epsilon=0}$$

$$= \frac{\int_0^1 |y_0 y_0'| dx \int_0^1 2 y_0''' h''' dx - \int_0^1 (y_0''')^2 dx \int_0^1 (sgn(y_0 y_0'))(h y_0)' dx}{(\int_0^1 |y_0 y_0'| dx)^2}$$

If the fraction is 0, then the numerator must be 0. This gives

$$\int_0^1 2y_0'''h'''dx \int_0^1 |y_0y_0'|dx - \int_0^1 (y_0''')^2dx \int_0^1 (sgn(y_0y_0'))(y_0h)'dx = 0 \qquad (3)$$

Since we are only interested in necessary conditions on intervals where y_0y_0' is of constant sign, we do not concern ourselves with intervals where $y_0y_0' = 0$. Rearranging (2) gives

$$\int_0^1 (y_0''')^2dx = K_0 \int_0^1 |y_0y_0'|dx. \qquad (4)$$

We substitute (4) into (3) to get

$$\int_0^1 2y_0'''h'''dx - K_0 \int_0^1 (sgn(y_0y_0'))(y_0h)'dx = 0. \qquad (5)$$

On an interval (c, d) on which y_0y_0' is of constant sign, suppose $h(x)$ is a smooth test function with support $[c, d]$. This implies that

$$h(c) = h'(c) = h''(c) = h(d) = h'(d) = h''(d) = 0.$$

Then

$$\int_c^d 2y_0'''h'''dx - K_0 \int_c^d (sgn(y_0y_0'))(y_0h)'dx = 0$$

The second term is zero on this interval, leaving

$$\int_c^d y_0'''h'''dx = 0.$$

Integrating by parts three times (standard theory of calculus of variations justifies this, i.e. [5], pg. 56) and using the boundary conditions for h gives

$$\int_c^d y_0'''h'''dx = \int_c^d y_0^{(vi)}hdx = 0.$$

From this we get $y_0^{(vi)}(x) = 0$ on (c, d). Thus, $y_0(x)$ is a quintic polynomial on (c, d).

Now, let us look for natural boundary conditions for y_0. Suppose that $h(x)$ is a smooth test function with $h(x) = 0$ for $x > \delta$, and that $y_0 y_0'$ has constant sign on $[0, \delta]$. Integrating (5) by parts three times and using $y_0^{(vi)}(x) = 0$ gives

$$-y_0'''(0)h''(0) + y_0^{(iv)}(0)h'(0) - y_0^{(v)}(0)h(0) + \frac{K_0}{2}(sgn(y_0 y_0'))(0)(y_0 h)(0) = 0. \tag{6}$$

Similarly, for smooth test function $h(x)$ with $h(x) = 0$ for $x < 1 - \delta$ and $y_0 y_0'$ with constant sign on $[1 - \delta, 1]$,

$$y_0'''(1)h''(1) - y_0^{(iv)}(1)h'(1) + y_0^{(v)}(1)h(1) - \frac{K_0}{2}(sgn(y_0 y_0'))(1)(y_0 h)(1) = 0 \tag{7}$$

We prove in Chapter 3 that an extremal exists for the basic problem as well as for more general problems. The necessary conditions (6) and (7) together with $y_0^{(vi)}(x) = 0$ on intervals where $y_0 y_0'$ is of constant sign will be used to determine a candidate for y_0. The proof that we have the value of K_0 and an extremal y_0 remains incomplete however. We assume that there is an extremal which is a quintic spline with a single knot. A knot in a spline is a point, denoted here by (c, K), where two polynomial pieces meet. Thus, we prove in general an extremal exists, but we find K_0 and extremals over a smaller class of functions, i.e., on those y where $y_0 y_0'$ has one change of sign. In the case of the functional $J_2(y) = \frac{\int_0^1 (y'')^2 dx}{\int_0^1 |yy'| dx}$, R. Brown, V. Burenkov, S.

Clark, and D. Hinton [4] have proved that extremals exist which are cubic splines with a single knot by using polynomial approximations. We show in Chapter 7 that polynomial approximate extremals exist, but have not been able to use them to give a proof that our problem has an extremal which is a quintic spline with a single knot. C. Fitzgerald [3] claimed to have a proof, but as noted by R. Brown, V. Burenkov, S. Clark, and D. Hinton, the proof is incorrect. Thus it is an open problem for functionals of the form, $n > 2$,

$$J_n(y) = \frac{\int_a^b (y^{(n)})^2 dx}{\int_a^b |yy'| dx}$$

if there are extremals which are splines with a single knot. In case $n = 1$, this is the Opial inequality

$$\int_a^b |yy'| dx \leq \frac{b-a}{4} \int_a^b (y')^2 dx \tag{8}$$

for piecewise smooth functions y such that $y(a) = y(b) = 0$. Further, equality holds in (8) only if $y = cy_0$, where y_0 is a nontrivial linear spline with knot at $\frac{a+b}{2}$.

We now compute this quintic spline for (1) on [0,1]. From the definition of D_0, we start with the conditions y_0 must satisfy:

$$y_0(0) = y_0'(0) = y_0''(0) = y_0(1) = y_0'(1) = y_0''(1) = 0. \tag{9}$$

Suppose that y_0 is a quintic spline with one knot, which is at $x = c$. We use Maple to solve this problem. We start by supposing that

$$s = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 \tag{10}$$

and

$$z = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5, \tag{11}$$

where s is the piece to the left of the knot and z is the piece to the right of the knot. Since together s and z make up a quintic spline, $s'(c) = z'(c), s''(c) = z''(c), s'''(c) = z'''(c)$, and $s^{(iv)}(c) = z^{(iv)}(c)$. Also, we normalize to get $s(c) = 1$ and $z(c) = 1$. This gives us twelve equations in thirteen unknowns.

$$a_0 = 0$$
$$a_1 = 0$$
$$2a_2 = 0$$
$$b_0 + b_1 + b_2 + b_3 + b_4 + b_5 = 0$$
$$b_1 + 2b_2 + 3b_3 + 4b_4 + 5b_5 = 0$$
$$2b_2 + 6b_3 + 12b_4 + 20b_5 = 0$$
$$a_0 + a_1c + a_2c^2 + a_3c^3 + a_4c^4 + a_5c^5 = 1$$
$$b_0 + b_1c + b_2c^2 + b_3c^3 + b_4c^4 + b_5c^5 = 1$$
$$a_1 + 2a_2c + 3a_3c^2 + 4a_4c^3 + 5a_5c^4 = b_1 + 2b_2c + 3b_3c^2 + 4b_4c^3 + 5b_5c^4$$
$$2a_2 + 6a_3c + 12a_4c^2 + 20a_5c^3 = 2b_2 + 6b_3c + 12b_4c^2 + 20b_5c^3$$
$$6a_3 + 24a_4c + 60a_5c^2 = 6b_3 + 24b_4c + 60b_5c^2$$
$$24a_4 + 120a_5c = 24b_4 + 120b_5c$$

Solving these twelve equations in terms of c gives us

$$s = \frac{5x^3}{3c^3(c^2 - 2c + 1)} + \frac{5(3c + 1)x^4}{c^4(c^2 - 2c + 1} + \frac{(6c^2 + 3c + 1)x^5}{6c^5(c^2 - 2c + 1)}$$

and

$$
\begin{aligned}
z = {} & \frac{-1}{6(-10c^2+5c-1-5c^4+10c^3+c^5)} + \\
& \frac{5x}{6c(-10c^2+5c-1-5c^4+10c^3+c^5)} \\
& - \frac{5x^2}{3c^2(-10c^2+5c-1-5c^4+10c^3+c^5)} \\
& + \frac{5(c^2-3c+3)x^3}{3c^2(-10c^2+5c-1-5c^4+10c^3+c^5)} \\
& - \frac{(3c^2-8c+6)x^4}{6c^2(-10c_5^2c-1-5c^4+10c^3+c^5)} \\
& + \frac{(10-15c+6c^2)x^5}{6c^2(-10c^2+5c-1-5c^4+10c^3+c^5)}.
\end{aligned}
$$

The next step is to evaluate

$$J(y_0) = \frac{\int_0^c (s''')^2 dx + \int_c^1 (z''')^2 dx}{\int_0^c |ss'| dx + \int_c^1 |zz'| dx}.$$

We get

$$J(y_0) = \frac{-20}{c^5(-10c^2+5c-1-5c^4+10c^3+c^5)}.$$

We graph $J(y_0)$ with respect to c and discover that there does indeed exist a minimum. Minimizing $J(y_0)$ gives $c = 1/2$ and $K_0 = 20,480$. A knot at $x = 1/2$ gives extremal

$$y_0 = \begin{cases} \frac{160x^3}{3} - \frac{400x^4}{3} + \frac{256x^5}{3}, & 0 \le x \le 1/2 \\ \frac{16}{3} - \frac{160x}{3} + \frac{640x^2}{3} - \frac{1120x^3}{3} + \frac{880x^4}{3} - \frac{256x^5}{3}, & 1/2 \le x \le 1 \end{cases}$$

Note that $y_0(1-x) = y_0(x)$ for $0 \le x \le 1$ as verified by Maple.

3 More General Functionals and Existence

We examine a more general problem

$$K := \inf_{y \epsilon D_0 \setminus \{0\}} J_{p,w}(y) \tag{12}$$

for

$$J_{p,w} = \frac{\int_0^1 p(x) y'''(x)^2 dx}{\int_0^1 w(x) |y(x) y'(x)| dx}$$

and where $p(x) > 0$ and $w(x) > 0$ are continuous functions. We conjectured in Chapter 2 that for $p(x) = w(x) = 1$, the basic problem (2) for $y \epsilon D_0 \setminus \{0\}$ has an extremal which is a quintic spline with at most one knot. We use the notation $L^2(0,1)$ for the Lebesgue measurable functions f that satisfy $\int_0^1 |f(x)|^2 dx < \infty$ and $\|f\|_2 = (\int_0^1 |f(x)|^2 dx)^{1/2}$.

In order to prove an extremal exists, we must enlarge the class D_0 to a larger class $\tilde{D_0}$. This is done in Theorem 1 below. Similar extensions are made for the other boundary conditions considered in Chapter 4.

The quintic spline computed earlier in this chpater is actually the extremals over the class of functions where y' has at most one sign change. It is our conjecture, but there is an open problem, that these splines are actual extremals for this large class of functions in which it is known an extremal exists.

Theorem 1: The infimum K of (12) is positive and an extremal y_0 exists in $\tilde{D}_0$, the larger class of functions where $y'''\epsilon L^2(0,1)$ replaces y''' piecewise continuous.

<u>Proof:</u> Let y_n be a sequence in $\tilde{D}_0$ such that $J_{p,w}(y_n) \longrightarrow K$ as $n \longrightarrow \infty$, where K is the infimum in (12) over $\tilde{D}_0 \setminus \{0\}$. Without loss of generality, we may assume $\int_0^1 w|y_n y_n'|dx = 1$. Then as $n \longrightarrow \infty$,

$$\int_0^1 p(x)|y_n'''|^2 dx = \int_0^1 p(x)(y_n''')^2 dx \longrightarrow K.$$

And then as $n \longrightarrow \infty$,

$$\left(\int_0^1 p(y_n''')^2 dx\right)^{1/2} = \|y_n'''\sqrt{p}\|_2 \longrightarrow K^{1/2}.$$

Then $\{\|y_n'''\sqrt{p}\|_2\}$ is a bounded sequence. Therefore, [1] $\{y_n'''\sqrt{p}\}$ has a weakly convergent subsequence in $L^2(0,1)$, say $y_n'''\sqrt{p} \rightharpoonup m$ as $n \longrightarrow \infty$. Then $y_n''' \rightharpoonup \frac{m}{\sqrt{p}}$ as $n \longrightarrow \infty$. From $y_n''(t) = \int_0^t y_n''' dx$, we have, since $\{\|y_n'''\|_2\}$ is also bounded,

$$|y_n''(t) - y_n''(s)|^2 = \left|\int_s^t y_n''' dx\right|^2 \leq |t-s| \int_0^1 (y_n''')^2 dx. \qquad (13)$$

Equation (13) implies that $\{y_n''\}$ is equicontinuous and uniformly bounded (set $s = 0$). Then the Ascoli-Arzela Theorem tells us that $\{y_n''\}$ has a uniformly convergent subsequence. Without loss of generality, assume $y_n'' \longrightarrow g$ uniformly as $n \longrightarrow \infty$. Letting $n \longrightarrow \infty$ in $y_n''(t) = \int_0^t y_n''' dx$, we have that $g(t) = \int_0^t \frac{m}{\sqrt{p}} dx$. Since $y_n'' \longrightarrow g$ uniformly as $n \longrightarrow \infty$, and $y_n'(x) = \int_0^x y_n''(t)dt$, we have that $y_n' \longrightarrow h$ uniformly as $n \longrightarrow \infty$ where $h(x) = \int_0^x g(t)dt$. Set $y(t) = \int_0^t (t-s)g(s)ds$. Then $y(0) = 0$. And $y'(t) = \int_0^t g(s)ds =$

$h(t)$ implies $y'(0) = h(0) = 0$. We have

$$y'(1) = h(1) = \int_0^1 g(t)dt = \lim_{n\longrightarrow\infty} \int_0^1 y_n''(t)dt = \lim_{n\longrightarrow\infty} [y_n'(1) - y_n'(0)] = 0$$

since $y_n \epsilon \tilde{D_0}$. Similarly,

$$y(1) = \int_0^1 hdx = \lim_{n\longrightarrow\infty} \int_0^1 y_n' dx = \lim_{n\longrightarrow\infty} [y_n(1) - y_n(0)] = 0$$

because $y_n \epsilon \tilde{D_0}$. Also, $y'(t) = \int_0^t g(x)dx$ gives $y''(t) = \frac{d}{dt}\int_0^t g(x)dx = g(t)$. Then $y''(0) = g(0) = \lim_{n\longrightarrow\infty} y_n''(0) = 0$ since $y_n \epsilon \tilde{D_0}$. Similarly,

$$y''(1) = g(1) = \lim_{n\longrightarrow\infty} y_n''(1) = 0.$$

Thus, $y \epsilon \tilde{D_0}$ with $y''' = h'' = g' = \frac{m}{\sqrt{p}}$. The uniform convergence of y_n' to h implies $\{y_n\}$ converges uniformly to $y(x)$; hence,

$$\int_0^1 w|yy'|dx = \lim_{n\longrightarrow\infty} \int_0^1 w|y_n y_n'|dx = 1.$$

The convergence $y_n''' \rightharpoonup \frac{m}{\sqrt{p}} = y'''$ implies $y_n'''\sqrt{p} \rightharpoonup m = y'''\sqrt{p}$. Therefore, since $y_n'''\sqrt{p} \rightharpoonup m$ [1], we obtain

$$\|m\|_2^2 = \|y'''\sqrt{p}\|_2^2 = \int_0^1 p(y''')^2 dx \leq \liminf_{n\longrightarrow\infty} \int_0^1 p(y_n''')^2 dx = K.$$

On the other hand,

$$\|y'''\sqrt{p}\|_2^2 = \int_0^1 p(y''')^2 dx \geq K$$

since $y \epsilon \tilde{D_0} \setminus \{0\}$. Therefore, $\int_0^1 p(y''')^2 dx = K$ and y is an extremal for (12). Finally, since $\int_0^1 w|yy'|dx = 1$, y is nonzero on a set of positive measure. The initial conditions are $y(0) = y'(0) = y''(0) = 0$. To show $K > 0$, suppose that $y''' \equiv 0$ so that y is a nontrivial quadratic function, say $y = mx^2 + bx + c$. Then $y' = 2mx + b$ and $y'' = 2m$. But $y(0) = 0 = c$, $y'(0) = 0 = b$, and $y''(0) = 0 = 2m$ implies that $y \equiv 0$, the trivial quadratic solution. This contradiction shows $y''' \not\equiv 0$. Thus, $\int_0^1 p(y''')^2 dx \neq 0$. Therefore, $K > 0$, and we are done.

The polynomial approximations in Chapter 7 show that

$$\inf_{y \epsilon D_0 \setminus \{0\}} J(y) = \inf_{y \epsilon \tilde{D_0} \setminus \{0\}} J(y)$$

so that the infimum is the same over the smaller set of functions $D_0 \setminus \{0\}$.

Now, let us examine a very general case. Let

$$J_G(y) = \frac{\int_a^b [p(x)y'''(x)^2 + r(x)y''(x)^2 + m(x)y'(x)^2 + q(x)y(x)^2] dx}{\int_a^b w(x)|y(x)y'(x)| dx} \tag{14}$$

and look for $K_G = \inf_{y \epsilon D_0 \setminus \{0\}} J_G(y)$. What are the necessary conditions? Suppose y_0 is an extremal and $h \epsilon D_0$. With calculations similar to those used to obtain (13), $\frac{d}{d\epsilon} J_G(y_0 + \epsilon h)|_{\epsilon=0} = 0$ reduces to

$$2\int_0^1 w|y_0 y_0'| dx \int_0^1 [p y_0''' h''' + r y_0'' h'' + m y_0' h' + q y_0 h] dx$$

$$- \int_0^1 [p(y_0''')^2 + r(y_0'')^2 + m(y_0')^2 + q y_0^2] dx \int_0^1 w(sgn(y_0 y_0'))(y_0 h)' dx = 0. \tag{15}$$

Rearranging (14) and (15) gives

$$\int_0^1 [p(y_0''')^2 + r(y_0'')^2 + m(y_0')^2 + q y_0^2] dx = K_G \int_0^1 w|y_0 y_0'| dx, \tag{16}$$

where $K_G = \inf_{y \epsilon D_0 \backslash \{0\}} J_G(y)$. Substituting (16) into (15) gives

$$2\int_0^1 [py_0'''h''' + ry_o''h'' + my_0'h' + qy_0h]dx - K_G \int_0^1 w(sgn(y_0y_0'))(y_0h)'dx = 0. \tag{17}$$

Suppose y_0y_0' is of constant sign on (c, d) and h is a smooth test function with support $[c, d]$. Thus,

$$h(c) = h'(c) = h''(c) = h(d) = h'(d) = h''(d) = 0.$$

Then the second term of (17) is zero. This leaves

$$\int_0^1 [py_0'''h''' + ry_0''h'' + my_0'h' + qy_0h]dx = 0.$$

Then after integration by parts,

$$\int_c^d [py_0'''h''' + ry_0''h'' + my_0'h' + qy_0h]dx =$$

$$\int_c^d [(py_0''')''' - (ry_0'')'' + (my_0')' - qy_0]hdx = 0.$$

Since this holds for all smooth h, we conclude that

$$0 = (py_0''')''' - (ry_0'')'' + (my_0')' - qy_0$$

almost everywhere on (c, d). Thus, $y_0(x)$ is a quintic polynomial on (c, d) if p is a real number, and $r = 0$, $m = 0$, and $q = 0$. Suppose y_0y_0' is of constant sign on $(0, \delta)$ and $h(x)$ is a smooth test function with $h(0) = 0$ for $x > \delta$. Then integrating by parts three times, (17) becomes

$$-p(0)y_0'''(0)h''(0) + (py_0''')'h'(0) - (py_0''')''(0)h(0) - r(0)y_0''(0)h'(0)$$

$$+(ry_0'')'(0)h(0) - m(0)y_0'(0)h(0) - \frac{K_G}{2}\int_0^\delta w(sgn(y_0y_0'))(y_0h)'dx = 0. \quad (18)$$

Similarly, if y_0y_0' is of constant sign on some $(1-\delta, 1)$ and h is a smooth test function with $h(x) = 0$ for $x < 1-\delta$, then

$$p(1)y_0'''(1)h''(1) - (py_0''')'(1)h'(1) + (py_0''')''(1)h(1) + r(1)y_0''(1)h'(1)$$

$$-(ry_0'')(1)h(1) + m(1)y_0'(1)h(1) - \frac{K_G}{2}\int_{1-\delta}^1 w(sgn(y_0y_0'))(y_0h)'dx = 0.$$

4 Other Classes of Admissible Functions

Up until now, we have only allowed one set of boundary conditions. Recall that

$$D_0 = \{y \epsilon A : y(0) = y'(0) = y''(0) = y(1) = y'(1) = y''(1) = 0\}$$

and

$$A := \{y : y \,\text{is real}, y, y', y''\, \text{continuous}, y'''\, \text{piecewise continuous}\}.$$

Let us now consider other classes of admissible functions.

$$\begin{aligned}
D_1 &:= \{y \epsilon A : y(0) = y''(0) = y(1) = y''(1) = 0\}, \\
D_2 &:= \{y \epsilon A : y(0) = y'(0) = y''(0) = y(1) = y''(1) = 0\}, \\
D_3 &:= \{y \epsilon A : y(0) = y'(0) = y''(0) = y'(1) = y''(1) = 0\}, \\
D_4 &:= \{y \epsilon A : y(0) = y'(0) = y''(0) = y''(1) = 0\}, \\
D_5 &:= \{y \epsilon A : y(0) = y'(0) = y(1) = y''(1) = 0\}, \\
D_6 &:= \{y \epsilon A : y(0) = y'(0) = y(1) = y'(1) = 0\}.
\end{aligned}$$

For $i = 1, ..., 6$, let

$$K_i := \inf_{y \epsilon D_i \setminus \{0\}} J(y). \tag{19}$$

where J is as in (2).

Theorem 2: For $i = 1, ..., 6$, the infimum K_i of (19) is positive and an extremal y_0 exists in the larger class of functions with $y''' \epsilon L^2(0,1)$ replacing y''' piecewise continuous.

The proof is similar to that of Theorem 1. We include one case, $i = 2$, for illustration.

First we need to consider the boundary conditions:

$$y(0) = y'(0) = y''(0) = y(1) = y''(1) = 0.$$

We want to find any natural boundary conditions for this case. Suppose that $h(x)$ is a smooth test function with $h(x) = 0$ for $x > \delta$ and that $y_0 y_0'$ has constant sign on $[1-\delta, 1]$. We begin with

$$\frac{d}{d\epsilon} J(y_0 + \epsilon h)|_{\epsilon=0} = 0$$

where

$$J(y_0 + \epsilon h) = \frac{\int_0^1 (y_0''' + \epsilon h''')^2 dx}{\int_0^1 |y_0 y_0' + \epsilon h y_0' + \epsilon h' y_0 + \epsilon^2 h h'| dx}.$$

In Chapter 2 we evaluated this same expression and got the equation

$$\int_0^1 2y_0''' h''' dx \int_0^1 |y_0 y_0'| dx - \int_0^1 (y_0''')^2 dx \int_0^1 [sgn(y_0 y_0')](y_0 h)' dx = 0.$$

Integration by parts three times gives us

$$-y_0'''(1)h''(1) + y_0^{(iv)}(1)h'(1) - y_0^{(v)}(1)h(1) +$$

$$\frac{K_2}{2}[sgn(y_0 y_0')(1)](y_0 h)(1) = 0.$$

But $h''(1) = 0$ and $h(1) = 0$ since h is a smooth test function. Thus we are left with

$$y_0^{(iv)}(1)h'(1) = 0.$$

Because $h'(1)$ is not restricted to be 0 and h is a smooth test function, we are left with the natural boundary condition $y_0^{(iv)}(1) = 0$.

We want to find extremal y_2 and constant K_2 for $y \epsilon D_2$. We suppose that y_2 is a quintic spline with at most one knot. We let s represent the quintic polynomial to the left of the knot and z the quintic polynomial to the right of the knot. Because y_2 is a quintic spline, $s'(c) = z'(c), s''(c) = z''(c), s'''(c) = z'''(c)$, and $s^{(iv)}(c) = z^{(iv)}(c)$. We normalize $s(c) = 1$ and $z(c) = 1$. The boundary conditions for this case are $y(0) = y'(0) = y''(0) = y(1) = y''(1) = 0$. There is also the natural boundary condition $y^{(iv)}(1) = 0$. Altogether this gives us twelve equations in thirteen unknowns.

$$a_0 = 0$$

$$a_1 = 0$$

$$2a_2 = 0$$

$$b_0 + b_1 + b_2 + b + 3 + b_4 + b_5 = 0$$

$$2b_2 + 6b_3 + 12b_4 + 20b_5 = 0$$

$$24b_4 + 120b_5 = 0$$

$$a_0 + a_1c + a_2c^2 + a_3c^3 + a_4c^4 + a_5c^5 = 1$$

$$b_0 + b_1c + b_2c^2 + b_3c^3 + b_4c^4 + b_5c^5 = 1$$

$$a_1 + 2a_2c + 3a_3c^2 + 4a_4c^3 + 5a_5c^4 = b_1 + 2b_2c + 3b_3c^2 + 4b_4c^3 + 5b_5c^4$$

$$2a_2 + 6a_3c + 12a_4c^2 + 20a_5c^3 = 2b_2 + 6b_3c + 12b_4c^2 + 20b_5c^3$$

$$6a_3 + 24a_4c + 60a_5c^2 = 6b_3 + 24b_4c + 60b_5c^2$$

$$24a_4 + 120a_5c = 24b_4 + 120b_5c$$

Solving these twelve equations in terms of c gives us

$$\begin{aligned} s = & \frac{20(c^2 - 4c + 4)x^3}{c^3(3c^4 - 27c^3 + 88c^2 - 112c + 48} \\ & - \frac{5(3c^3 - 12c^2 + 8c + 8)x^4}{c^4(3c^4 - 27c^3 + 88c^2 - 112c + 48)} \\ & + \frac{3c^4 - 12c^3 + 8c^2 + 8c + 8)x^5}{c^5(3c^4 - 27c^3 + 88c^2 - 112c + 48)} \end{aligned}$$

and

$$\begin{aligned} z = & \frac{-8}{115c^3 - 30c^4 + 160c - 200c^2 + 3c^5 - 48} \\ & + \frac{40x}{c(115c^3 - 30c^4 + 160c - 200c^2 + 3c^5 - 48)} \\ & - \frac{80x^2}{c^2(115c^3 - 30c^4 + 160c - 200c^2 + 3c^5 - 48)} \\ & + \frac{20(c^2 - 5c + 8)x^3}{c^2(115c^3 - 30c^4 + 160c - 200c^2 + 3c^5 - 48)} \\ & - \frac{5(20 - 15c + 3c^2)x^4}{c^2(115c^3 - 30c^4 + 160c - 200c^2 + 3c^5 - 48)} \\ & + \frac{(20 - 15c + 3c^2)x^5}{c^2(115c^3 - 30c^4 + 160c - 200c^2 + 3c^5 - 48)}. \end{aligned}$$

Note that in this case, we do not have symmetric boundary conditions at the two endpoints; thus, we do not expect to have $z(1-x) = s(x)$ nor $c = 1/2$. It can be verified with Maple that $z(1-x) \neq s(x)$.

Next we evaluate

$$J(y_2) = \frac{\int_0^c (s''')^2 dx + \int_c^1 (z''')^2 dx}{\int_0^c |ss'| dx + \int_c^1 |zz'| dx}.$$

Evaluating these integrals gives

$$J(y_2) = \frac{-960}{c^5(115c^3 - 30c^4 + 160c - 200c^2 + 3c^5 - 48)}.$$

Graphing $J(y_2)$ as a function of c shows that the function has a minimum. Minimizing $J(y_2)$ gives constant $K_2 = 4289.04905$ and $c = 0.6386479177$. A knot at $x = 0.6386479177$ gives quintic spline

$$y_2 = \begin{cases} 24.4069919708x^3 - 46.3778220168x^4 & \\ +22.1910377838x^5, & 0 \le x \le c \\ 3.79740807422 - 29.7300591529x & \\ +93.1031271815x^2 - 121.374626666x^3 & \\ +67.7551882043x^4 - 13.5510376409x^5, & c \le x \le 1. \end{cases}$$

Assuming again that an extremal exists which is a quintic spline with at most one knot, we have computed the constants K_i together with the corresponding spline y_i and all boundary conditions, including natural boundary conditions, for problems D_i; these are given below.

D_1 : $K_1 = 480, y_1(0) = y_1''(0) = y_1(1) = y_1''(1) = 0, y_1(c) = 1, y_1^{(iv)}(0) = y_1^{(iv)}(1) = 0, c = 1/2.$

$$y_1 = \begin{cases} 3.125x - 5x^3 + 2x^5, & 0 \le x \le 1/2 \\ 0.125 + 1.875x + 5x^2 - 15x^3 + 10x^4 - 2x^5, & 1/2 \le x \le 1. \end{cases}$$

D_2 : $K_2 = 4289.04905, y_2(0) = y_2'(0) = y_2''(0) = y_2(1) = y_2''(1) = 0, y_2(c) = 1, y_2^{(iv)}(1) = 0, c = 0.6386479177.$

$$y_2 = \begin{cases} 24.4069919708x^3 - 46.3778220168x^4 & \\ +22.1910377838x^5, & 0 \le x \le c \\ 3.79740807422 - 29.7300591529x & \\ +93.1031271815x^2 - 121.374626666x^3 & \\ +67.7551882043x^4 - 13.5510376409x^5, & c \le x \le 1. \end{cases}$$

D_3 : $K_3 = 20750.4, y_3(0) = y_3'(0) = y_3''(0) = y_3(1) = y_3''(1) = 0, y_3(c) = 1, y_3^{(v)}(1) = 0, c = 0.5427288488.$

$$y_3 = \begin{cases} 34.41731142x^3 - 73.57528469x^4 + 39.95681333x^5, & 0 \le x \le c \\ 1.881505015 - 17.33374797x + 63.87627266x^2 & \\ -83.27732149x^3 + 34.85329178x^4, & c \le x \le 1. \end{cases}$$

D_4 : $K_4 = 90, y_4(0) = y_4'(0) = y_4''(0) = y_4''(1) = 0, y_4(c) = 1, y_4^{(iv)}(1) = y_4^{(v)}(1) = 0, c = 1.$

$$y_4 = \frac{5x^3}{2} - \frac{15x^4}{8} + \frac{3x^5}{8}.$$

D_5 : $K_5 = 2296.483317, y_5(0) = y_5'(0) = y_5(1) = y_5''(1) = 0, y_5(c) = 1, y_5'''(0) = y_5^{(iv)}(1) = 0, c = 0.6013056920.$

$$y_5 = \begin{cases} 6.746341828x^2 - 17.72938718x^4 & \\ +11.17583433x^5, & 0 \le x \le c \\ 1.504383705 - 12.50930870x + 48.35349392x^2 & \\ -69.19467531x^3 + 39.80763301x^4 - 7.961526601x^5, & c \le x \le 1. \end{cases}$$

D_6 : $K_6 = 7680, y_6(0) = y_6'(0) = y_6(1) = y_6'(1) = 0, y_6(c) = 1, y_6'''(0) = y_6'''(1) = 0, c = 1/2$.

$$y_6 = \begin{cases} 10x^2 - 40x^4 + 32x^5, & 0 \le x \le 1/2, \\ 2 - 20x + 90x^2 - 160x^3 + 120x^4 - 32x^5, & 1/2 \le x \le 1. \end{cases}$$

5 The $\mathbf{i = 4}$ Case

The $i = 4$ case is special because the knot is at $x = 1$. This means that y_4 is not a quintic spline on the interval (0,1). The extremal y_4 is actually a quintic polynomial with minimum at $x = 1$ on interval (0,1).

We show that if y is an extremal for the $i = 4$ case, then there is an extremal $\tilde{y}$ which is nondecreasing on [0,1]. Since then $\tilde{y}'$ has no change of sign on [0,1], from Chapter 2 we have $\tilde{y}^{(vi)} \equiv 0$ and thus $\tilde{y}$ is a quintic polynomial. Set

$$\tilde{y}'(x) = \int_0^x |y''(s)|ds,\ \tilde{y}(0) = \tilde{y}'(0) = 0$$

so that

$$\tilde{y}(x) = \int_0^x \tilde{y}'(s)ds.$$

Then $\tilde{y}''(x) = |y''(x)|$ so that $\tilde{y}'''(x) = [sgn(y'''(x))]y'''(x)$ almost everywhere, and almost everywhere,

$$|\tilde{y}'''(x)| = |y'''(x)|. \tag{20}$$

On the interval (0,1),

$$|y'(x)| = \left|\int_0^x y''(x)dx\right| \leq \int_0^x |y''(x)|dx = \tilde{y}'(x)$$

so that

$$\int_0^1 |\tilde{y}(x)\tilde{y}'(x)|dx \geq \int_0^1 |y(x)y'(x)|dx \tag{21}$$

and

$$\int_0^1 (\tilde{y}'''(x))^2 dx = \int_0^1 (y'''(x))^2 dx. \tag{22}$$

This gives $J(\tilde{y}) \leq J(y)$. Since y is an extremal, we have $J(\tilde{y}) = J(y)$. Thus we have $\tilde{y}$ is a multiple of

$$y_4 = \frac{5x^3}{2} - \frac{15x^4}{8} + \frac{3x^5}{8}$$

which we found in Chapter 4.

6 Applications to $y^{(vi)} = -\lambda\rho(x)y$ With Boundary Conditions

Let us examine the sixth order ordinary differential equation

$$y^{(vi)} = -\lambda\rho(x)y,\ 0 \leq x \leq 1, \tag{23}$$

with the boundary conditions of D_0 imposed, i.e.,

$$y(0) = y'(0) = y''(0) = y(1) = y'(1) = y''(1) = 0. \tag{24}$$

Suppose that the smallest eigenvalue of (23) and (24) is λ_0. We further assume that the first derivative of the eigenfunction has only one change of sign (of its first derivative) in (0,1) so that the inequalities derived earlier apply. Then multiplying by y gives

$$yy^{(vi)} = -\lambda_0\rho(x)y^2. \tag{25}$$

Rearranging, we get

$$yy^{(vi)} + \lambda_0\rho(x)y^2 = 0. \tag{26}$$

Now we integrate. This gives us

$$\int_0^1 (yy^{(vi)} + \lambda_0\rho(x)y^2)dx = 0$$

and

$$\int_0^1 yy^{(vi)}dx + \lambda_0\int_0^1 \rho(x)y^2dx = 0.$$

Integrating by parts three times and using the boundary conditions listed in (24) gives us

$$\int_0^1 (y''')^2 dx = \lambda_0 \int_0^1 \rho(x) y^2 dx. \tag{27}$$

Set $P(x) = \int_0^x \rho(u)du - \frac{1}{2}\int_0^1 \rho(u)du$. Then we obtain

$$\int_0^1 (y''')^2 dx = \lambda_0 \int_0^1 P'(x) y^2 dx. \tag{28}$$

Integrating by parts gives

$$\int_0^1 (y''')^2 dx = -2\lambda_0 \int_0^1 P(x) y y' dx$$

$$\leq 2\lambda_0 \int_0^1 |P(x) y y'| dx$$

$$\leq 2\lambda_0 \int_0^1 |P(x)||y y'| dx.$$

Note that $|P(x)| \leq \frac{1}{2}\int_0^1 \rho(x)dx$ as $\rho(x) > 0$. Then this implies

$$2\lambda_0 \int_0^1 |P(x)||yy'|dx \leq 2\lambda_0 \int_0^1 \left(\frac{1}{2}\int_0^1 \rho(x)dx\right)|yy'|dx.$$

Thus, combining the above two inequalities yields

$$\int_0^1 (y''')^2 dx \leq 2\lambda_0 \int_0^1 \left(\frac{1}{2}\int_0^1 \rho(x)dx\right)|yy'|dx$$

$$= \lambda_0 \int_0^1 \rho(x)dx \int_0^1 |yy'|dx.$$

Then we have

$$\int_0^1 (y''')^2 dx \leq \lambda_0 \int_0^1 \rho(x)dx \int_0^1 |yy'|dx.$$

But (over the functions in D_0 such that y' has at most one sign change)

$$K_0 = 20,480 = \inf_{D_0\setminus\{0\}} \frac{\int_0^1 (y''')^2 dx}{\int_0^1 |yy'| dx}$$

or

$$\int_0^1 (y''')^2 dx \geq 20,480 \int_0^1 |yy'| dx. \tag{29}$$

Then (28) and (29) tell us that

$$\lambda_0 \geq \frac{\int_0^1 (y''')^2 dx}{\int_0^1 \rho(x)dx \int_0^1 |yy'|dx} \geq \frac{20,480}{\int_0^1 \rho(x)dx}. \tag{30}$$

Now we want to show why (30) is the best possible. Let $M \equiv \int_0^1 \rho(x)dx$ and for ϵ small, set

$$\rho_\epsilon(x) = \begin{cases} \epsilon, & x \notin [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon] \\ \frac{M}{2\epsilon} - (\frac{1}{2} - \epsilon), & x\epsilon[\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]. \end{cases}$$

Then $\int_0^1 \rho_\epsilon(x)dx = M$. Let $\lambda_{0,\epsilon}$ be the smallest eigenvalue of

$$y^{(vi)} = -\lambda_{0,\epsilon}\rho(\epsilon)y,$$

$$y(0) = y'(0) = y''(0) = y(1) = y'(1) = y''(1) = 0.$$

By the Rayleigh quotient characterization of $\lambda_{0,\epsilon}$,

$$\lambda_{0,\epsilon} = \inf_{y\epsilon D_0\setminus\{0\}} \frac{\int_0^1 (y''')^2 dx}{\int_0^1 \rho_\epsilon(x)y^2 dx}.$$

In particular, for $y(x)$ given by the quintic spline

$$y(x) = \begin{cases} \frac{160x^3}{3} - \frac{400x^4}{3} + \frac{256x^5}{3}, & 0 \leq x \leq 1/2 \\ \frac{16}{3} - \frac{160x}{3} + \frac{640x^2}{3} - \frac{1120x^3}{3} + \frac{880x^4}{3} - \frac{256x^5}{3}, & 1/2 \leq x \leq 1, \end{cases}$$

$$\lambda_{0,\epsilon} \leq \frac{\int_0^1 (y''')^2 dx}{\int_0^1 \rho_\epsilon(x) y^2 dx}.$$

Therefore, we conclude

$$\lim_{\epsilon \longrightarrow 0} \sup \lambda_{0,\epsilon} \leq \frac{\int_0^1 (y''')^2 dx}{M(y(1/2))^2} = \frac{20,480}{M},$$

which shows the constant 20,480 in (30) cannot be increased. Similarly, it can be shown that for each set D_i, there is a lower bound on the smallest eigenvalue λ_i, where λ_i is the smallest eigenvalue of (23) with the boundary conditions of D_i imposed. Again, we further suppose the first derivative of the corresponding eigenfunction has at most one change of sign (of its first derivative). The lower bounds are listed here: $D_1 : \lambda_1 \geq \frac{480}{\int_0^1 \rho(x)dx}$, $D_2 : \lambda_2 \geq \frac{4289.04905}{\int_0^1 \rho(x)dx}$, $D_3 : \lambda_3 \geq \frac{20,750.4}{\int_0^1 \rho(x)dx}$, $D_4 : \lambda_4 \geq \frac{90}{\int_0^1 \rho(x)dx}$, $D_5 : \lambda_5 \geq \frac{2296.483317}{\int_0^1 \rho(x)dx}$, and $D_6 : \lambda_6 \geq \frac{7680}{\int_0^1 \rho(x)dx}$.

7 Approximation of Extremals by Polynomials

It is possible to use polynomials to approximate extremals. Let us examine two cases: D_0 and D_1. We recall that the polynomials are dense in $L^2(0,1)$ [1].

Lemma 1: Let $y \epsilon D_0$ (or $\tilde{D_0}$) and let $\{d_n\}$ be a sequence of polynomials such that $\|y''' - d_n\|_2 \to 0$ as $n \longrightarrow \infty$. Define

$$q_n(x) := \frac{1}{2}\int_0^x (x-s)^2 d_n(s)ds + a_n x^3 + b_n x^4 + c_n x^5 \tag{31}$$

where a_n, b_n, and c_n are determined by the requirement

$$q_n(1) = q_n'(1) = q_n''(1) = 0.$$

Then $q_n \epsilon D_0$, $q_n \to y$, $q_n' \to y'$, and $q_n'' \to y''$ uniformly as $n \to \infty$, and $\|y''' - q_n\|_2 \to 0$ as $n \to \infty$.

Proof: Since $y \epsilon D_0$, we know that

$$y(x) = \frac{1}{2}\int_0^x (x-s)^2 y'''(s)ds.$$

Then we can calculate $y(1)$, $y'(1)$, and $y''(1)$. First we find that

$$y'(x) = \int_0^x (x-s)y'''(s)ds$$

and

$$y''(x) = \int_0^x y'''(s)ds.$$

Then we obtain

$$y(1) = \frac{1}{2}\int_0^1 (1-s)^2 y'''(s)ds = 0,$$

$$y'(1) = \int_0^1 (1-s)y'''(s)ds = 0,$$

and

$$y''(1) = \int_0^1 y'''(s)ds = 0.$$

Therefore, a_n, b_n, and c_n satisfy these equations:

$$q_n(1) = \frac{1}{2}\int_0^1 (1-s)^2[d_n(s) - y'''(s)]ds + a_n + b_n + c_n = 0, \qquad (32)$$

$$q_n'(1) = \int_0^1 (1-s)[d_n(s) - y'''(s)]ds + 3a_n + 4b_n + 5c_n = 0, \qquad (33)$$

$$q_n''(1) = \int_0^1 [d_n(s) - y'''(s)]ds + 6a_n + 12b_n + 20c_n = 0. \qquad (34)$$

Note that the determinant of the coefficients a_n, b_n, and c_n is

$$\begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 6 & 12 & 20 \end{vmatrix} = 2 \neq 0.$$

Because $\|d_n - y'''\|_2 \to 0$ as $n \to \infty$, equations (32), (33), and (34) and Kramer's rule imply that $a_n \to 0$, $b_n \to 0$, and $c_n \to 0$ as $n \to \infty$. The uniform convergence of q_n to y, q_n' to y', and q_n'' to y'' follow from the equations,

$$q_n(x) - y(x) = \frac{1}{2}\int_0^x (x-s)^2[d_n(s) - y'''(s)]ds + a_n x^3 + b_n x^4 + c_n x^5,$$

$$q_n'(x) - y'(x) = \int_0^x (x-s)[d_n(s) - y'''(s)]ds + 3a_n x^2 + 4b_n x^3 + 5c_n x^4,$$

$$q_n''(x) - y''(x) = \int_0^x [d_n(s) - y'''(s)]ds + 6a_n x + 12b_n x^2 + 20c_n x^3.$$

We are now done.

Now we need to examine D_1. Recall that

$$D_1 = \{y\epsilon A : \; y(0) = y''(0) = y(1) = y''(1) = 0\}.$$

The natural boundary conditions for D_1 are $y^{(iv)}(0) = y^{(iv)}(1) = 0$.

Lemma 2: Let $y\epsilon D_1$ and $\{d_n\}$ be a sequence of polynomials such that $\|y''' - d_n\|_2 \to 0$ as $n \to \infty$. Define

$$q_n(x) := \frac{1}{2}\int_0^x (x-s)^2 d_n(s)ds + a_n x + b_n x^3 + c_n x^5, \tag{35}$$

where a_n, b_n, and c_n are determined by the requirement that $q_n(1) = q_n''(1) = q_n^{(iv)}(1) = 0$. Then $q_n \epsilon D_1$, $q_n \to y$, $q_n' \to y'$, and $q_n'' \to y''$ uniformly as $n \to \infty$, and $\|y''' - q_n'''\|_2 \to 0$ as $n \to \infty$.

Proof: Since $y\epsilon D_1$, we have that

$$y(x) = \frac{1}{2}\int_0^x (x-s)^2 y'''(s)ds,$$

$$y(1) = \frac{1}{2}\int_0^1 (1-s)^2 y'''(s)ds = 0,$$

$$y'(1) = \int_0^1 (1-s)^2 y'''(s)ds = 0,$$

and

$$y''(1) = \int_0^1 y'''(s)ds = 0.$$

Thus, a_n, b_n, and c_n satisfy the equations

$$q_n(1) = 0 = \frac{1}{2}\int_0^1 (1-s)^2[d_n(s) - y'''(s)]ds + a_n + b_n + c_n, \tag{36}$$

$$q_n'(1) = 0 = \int_0^1 (1-s)[d_n(s) - y'''(s)]ds + a_n + 3b_n + 5c_n, \tag{37}$$

$$q_n''(1) = 0 = \int_0^1 [d_n(s) - y'''(s)]ds + 6b_n + 20c_n, \tag{38}$$

Note that the determinant of the coefficients a_n, b_n, and c_n is

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 0 & 6 & 20 \end{vmatrix} = 16 \neq 0.$$

From $\|d_n - y'''\|_2 \longrightarrow 0$ as $n \longrightarrow \infty$, it is clear from (36), (37), and (38) and Kramer's rule that a_n, b_n,c_n $\longrightarrow 0$ as $n \longrightarrow \infty$. The uniform convergence of q_n to y, q_n' to y', and q_n'' to y'' now follows from the equations:

$$q_n(x) - y(x) = \frac{1}{2}\int_0^x (x-s)^2[d_n(s) - y'''(s)]ds + a_n x + b_n x^3 + c_n x^5,$$

$$q_n'(x) = y'(x) = \int_0^x (x-s)[d_n(s) - y'''(s)]ds + a_n + 3b_n x^2 + 5c_n x^4,$$

$$q_n''(x) - y''(x) = \int_0^x [d_n(s) - y'''(s)]ds + 6b_n x + 20c_n x^3.$$

This finishes the proof of Lemma 2.

8 In L^p Space ($[a,b]=[0,1]$)

Let us examine a very general case. Define

$$J_p(y) := \frac{(\int_0^1 |y'''(x)|^p dx)^{2/p}}{\int_0^1 |y(x)y'(x)|dx} \tag{39}$$

and

$$K_{0,p} := \inf_{y \epsilon D_0 \setminus \{0\}} J_p(y) \tag{40}$$

for $1 < p < \infty$. First we need to derive the necessary conditions. Let $y_0, h \epsilon D_0$. We calculate

$$\frac{d}{d\epsilon} J(y_0 + \epsilon h)|_{\epsilon=0} = 0$$

where

$$J(y_0 + \epsilon h) = \frac{(\int_0^1 (y_0''' + \epsilon h''')^p dx)^{2/p}}{\int_0^1 |y_0 y_0' + \epsilon h y_0' + \epsilon h' y_0 + \epsilon^2 h h'| dx}.$$

Using the quotient rule for derivatives and some algebra, we get

$$0 = \frac{N}{D},$$

where

$$N = 2\int_0^1 |y_0 y_0'| dx (\int_0^1 |y_0'''|^p dx)^{2/p-1} \int_0^1 h'''(y_0''')^{p-1}[sgn(y_0''')]dx$$

$$-\left(\int_0^1 |y_0'''|^p dx\right)^{2/p} \int_0^1 [sgn(y_0 y_0')](y_0 h)' dx$$

and

$$D = \int_0^1 |y_0 y_0'| dx.$$

A fraction can only be 0 if the numerator is 0. Thus, our equation becomes

$$0 = 2\int_0^1 |y_0 y_0'|dx \int_0^1 h'''(y_0''')^{p-1}[sgn(y_0''')]dx$$

$$-\Big(\int_0^1 |y_0'''|^p dx\Big)^{2/p} \int_0^1 [sgn(y_0 y_0')](y_0 h)'dx. \tag{41}$$

Definitions (39) and (40) give us the fact that

$$\Big(\int_0^1 |y_0'''|^p dx\Big)^{2/p} = K_{0,p}\int_0^1 |y_0 y_0'|dx. \tag{42}$$

We substitute (42) into (41) to get

$$0 = 2\int_0^1 |y_0 y_0'|dx \Big(\int_0^1 |y_0'''|^p dx)^{2/p-1}\Big) \int_0^1 h'''(y_0''')^{p-1}[sgn)(y_0''')]dx$$

$$-K_{0,p}\int_0^1 |y_0 y_0'|dx \int_0^1 [sgn(y_0 y_0')](y_0 h)'dx.$$

Because division by zero is not allowed, (39) gives us that $\int_0^1 |y_0 y_0'|dx \neq 0$. This tell us

$$0 = 2\int_0^1 h'''(y_0''')^{p-1}[sgn(y_0''')]dx - K_{0,p}\int_0^1 [sgn(y_0 y_0')](y_0 h)'dx. \tag{43}$$

On an interval where $y_0 y_0'$ and y_0''' have constant sign, integration by parts shows that $(|y_0'''|^{p-1}[sgn(y_0''')])''' = 0$, i.e., $|y_0'''|^{p-1}[sgn(y_0''')]$ is a quadratic polynomial.

We now show the existence proof of Chapter 3 carries over to the case $p \neq 2$. Let y_n be a sequence in D_0 such that $J_p(y_n) \longrightarrow K_{0,p}$ as $n \longrightarrow \infty$, where $K_{0,p}$ is the infimum in (40) over $\hat{D}_0 \setminus \{0\}$,

$$\hat{D}_0 = \{y \epsilon B : y(0) = y'(0) = y''(0) = y(1) = y'(1) = y''(1) = 0\} \tag{44}$$

and

$$B = \{\mathrm{y : y\,is\,real\,on\,[a,b], y,\ y',\ y''\,continuous}, y'''\epsilon L^p(\mathit{0}, \mathit{1})\}. \qquad (45)$$

Without loss of generality, we may assume $\int_0^1 |y_n y_n'|dx = 1$. Then we obtain

$$\Big(\int_0^1 |y_n'''|^p dx\Big)^{2/p} \longrightarrow K_{0,p}$$

or

$$\Big(\int_0^1 |y_n'''|^p dx\Big)^{1/p} \longrightarrow (K_{0,p})^{1/2}.$$

Then $\{\|y_n'''\|_p\}$ is a bounded sequence. Therefore [1] $\{y_n'''\}$ has a weakly convergent subsequence in $L^p(0,1)$, say $y_n''' \rightharpoonup m$ as $n \longrightarrow \infty$. From $y_n''(t) = \int_0^t y_n''' dx$, we obtain

$$|y_n''(t) - y_n''(s)|^2 = |\int_s^t y_n''' dx|^2 \le |t-s| \int_0^1 |y_n'''|^p dx. \qquad (46)$$

Equation (43) implies that $\{y_n''\}$ is equicontinuous and uniformly bounded (set $s = 0$). Then the Ascoli-Arzela Theorem tells us that $\{y_n''\}$ has a uniformly convergent subsequence. Without loss of generality, assume $y_n'' \longrightarrow g$ uniformly as $n \longrightarrow \infty$. Letting $n \longrightarrow \infty$ in $y_n''(t) = \int_0^t y_n''' dx$, we have $g(t) = \int_0^t m dx$. Since $y_n'' \longrightarrow g$ uniformly as $n \longrightarrow \infty$, and $y_n'(x) = \int_0^x y_n''(t)dt$, we know that $y_n' \longrightarrow h$ uniformly as $n \longrightarrow \infty$ where $h(x) = \int_0^x g(t)dt$. Set $y(t) = \int_0^t (t-s)g(s)ds$. Then $y(0) = 0$. And $y'(t) = \int_0^t g(s)ds = h(t)$ tells us that $y'(0) = h(0) = 0$. This implies

$$y'(1) = h(1) = \int_0^1 g(t)dt = \lim_{n\longrightarrow\infty} \int_0^1 y_n''(t)dt = \lim_{n\longrightarrow\infty}[y_n'(1) - y_n'(0)] = 0$$

since $y_n \epsilon \hat{D}_0$. Similarly, we have

$$y(1) = \int_0^1 h dx = \lim_{n \longrightarrow \infty} \int_0^1 y_n' dx = \lim_{n \longrightarrow \infty} [y_n(1) - y_n(0)] = 0$$

since $y_n \epsilon \tilde{D}_0$. Also, $y'(t) = \int_0^t g(x)dx$ gives $y''(t) = \frac{d}{dt} \int_0^t g(x)dx = g(t)$. Then $y''(0) = g(0) = \lim_{n \longrightarrow \infty} y_n''(0) = 0$ since $y_n \epsilon \hat{D}_0$. Similarly,

$$y''(1) = g(1) = \lim_{n \longrightarrow \infty} y_n''(1) = 0.$$

Thus, $y \epsilon \hat{D}_0$ with $y''' = h'' = g' = m$. The uniform convergence of y_n' to h implies $\{y_n\}$ converges uniformly to $y(x)$; thus,

$$\int_0^1 |yy'| dx = \lim_{n \longrightarrow \infty} \int_0^1 |y_n y_n'| dx = 1.$$

$y_n''' \rightharpoonup m = y'''$. Therefore, since $y_n''' \rightharpoonup m$ [1],

$$\|m\|_p^2 = \|y'''\|_p^2 = \left(\int_0^1 |y'''|^p dx \right)^{2/p} \leq \liminf_{n \longrightarrow \infty} \left(\int_0^1 |y_n'''|^p dx \right)^{2/p} = K_{0,p}.$$

On the other hand, we have

$$\|y'''\|_p^2 = \left(\int_0^1 |y'''|^p dx \right)^{2/p} \geq K_{0,p}$$

since $y \epsilon \hat{D}_0 \setminus \{0\}$. Therefore, $\left(\int_0^1 |y'''|^p \right)^{2/p} = K_{0,p}$ and y is an extremal for (39) and (40). Finally, since $\int_0^1 |yy'| dx = 1$, y is nonzero on a set of positive measure. The initial conditions are $y(0) = y'(0) = y''(0) = 0$. To show $K_{0,p} > 0$, we suppose that $y''' \equiv 0$ so that y is a nontrivial quadratic function, say $y = mx^2 + bx + c$. Then $y' = 2mx + b$ and $y'' = 2m$. But $y(0) = 0 = c$, $y'(0) = 0 = b$, and $y''(0) = 0 = 2m$ implies that $y \equiv 0$, the trivial quadratic solution. Thus, $y''' \not\equiv 0$. Thus, $\left(\int_0^1 |y'''|^p dx \right)^{2/p} \neq 0$. Thus, $K_{0,p} > 0$ and we are finished.

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Vita

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