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Nikolski's Approach to the theorems of Beurling and Nyman regarding zeros of the Riemann ζ-function

Jared R. Bunn

University of Tennessee - Knoxville

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To the Graduate Council:

I am submitting herewith a thesis written by Jared R. Bunn entitled "Nikolski's Approach to the theorems of Beurling and Nyman regarding zeros of the Riemann ζ-function." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Stefan Richter, Major Professor

We have read this thesis and recommend its acceptance:

Ken Stephsnson, James Conant

Accepted for the Council:
Dixie L. Thompson

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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Accepted for the Council:

Anne Mayhew

Vice Chancellor and Dean of
Graduate Studies

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Nikolski's Approach to the theorems of Beurling and Nyman regarding zeros of the Riemann ζ-function

A Thesis
Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville

Jared R. Bunn
August 2006

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Abstract

In this thesis we present the proof of a theorem by Nikolai Nikolski. This theorem leads to a more general theorem by Nikolski regarding zero free regions of the Riemann ζ-function. This theorem is an improvement on the theorems that Nyman and Beurling proved in the nineteen fifties. Nikolski's approach uses, in addition to step function approximations introduced by Nyman, distance functions to give more flexibility, including possible numerical experiments. The introduction discusses the Riemann Hypothesis, which always surrounds any study of the Riemann ζ-function.

The background material discussed in this thesis gives all the necessary prerequisites for an understanding of the proof of the main theorem. Topics include infinite products, the Gamma function, the Riemann ζ-function, Fourier series and transforms, the Hardy spaces, reproducing kernels, and Blaschke factors. The focus will be on the Hardy spaces of the upper and right half-planes, whose properties are deduced using the Hardy space of the unit disk via the unitary mapping of Chapter 4. The Mellin transform is also introduced and plays a vital role in the main theorem proven in chapter 6.

Contents

Notation

$\mathbb{N}$	positive integers
$\mathbb{R}$	real numbers
$\mathbb{R}_+$	positive real numbers
$\mathbb{Z}$	integers
$\mathbb{C}$	complex numbers
$\mathbb{T}$	complex numbers with modulus 1
$\mathbb{D}$	complex numbers with modulus less than 1
$\mathcal{R}$	complex numbers with positive real part
$\mathcal{U}$	complex numbers with positive imaginary part
P	the set of prime numbers
$\sim$	asymptotically similar
$\mathcal{F}$	Fourier transform
$\mathcal{F}^{-1}$	inverse Fourier transform
$\mathcal{F}_*$	Mellin transform
J	represents the change of variable $x = e^{-t}$
χ_A	characteristic function on the set A
$\setminus$	set minus
$\subset$	set containment (possibly equality)
log	the logarithm function with base e
$[x]$	the greatest integer less than or equal to x
$\bar{z}$	complex conjugate of z
z_*	defined by $z_* = -\bar{z}$

Chapter 1

Introduction

1.1 Background

The Riemann Hypothesis, which states that the nontrivial zeros of the Riemann Zeta-function all have real part 1/2, has been studied for about a century and a half, still without a fully supported proof being completed. The Clay Mathematics Institute has a 1 million dollar reward available for anyone who can provide rigorous proof of the Riemann Hypothesis. Moreover, the result has to be published in a refereed journal that is respected world-wide, and two years after publication, the Scientific Advisory Board of the Institute must decide whether the proof deserves the prize [Sab03, p. 30]. Clearly, the desire to discover a proof exists within the mathematical community; however, the question is whether any of us will see one presented in our lifetime.

David Hilbert (1862-1943) has been quoted as saying "If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?" The Riemann Hypothesis was one of the 23 problems posed by Hilbert in 1900. The Riemann hypothesis is the only problem from Hilbert's speech to become one of the Clay Institute "millennium" problems. Many of these problems have been solved; however, the Riemann Hypothesis stands as one of a few remaining that lacks unquestionable proof.

The significance of the truth of the Riemann hypothesis stems from its connection to number theory, namely, the prime numbers. The prime number theorem is a notable example of this connection. The prime number theorem gives an approximation of the number of primes less than x (called the pie function π). It states that

$$\pi(x) \sim \frac{x}{\log x},$$

and it was proved independently by Hadamard and de la Vallée Poussin in 1896. In fact, the proof of this theorem uses the fact that the line Re $z = 1$ contains no zeros of the Riemann Zeta-function. The Riemann hypothesis, if true, would give an exact formulation of the distribution of primes. That is, we could write

$$\pi(x) = \mathrm{li}(x) + O(x^{1/2}\log(x)),$$

where the function li is defined by $\mathrm{li}(x) = \int_0^x (\log t)^{-1}\,dt$. The most recent "proposed" proof of the Riemann hypothesis is due to Louis de Branges. An electronic version can be found on his Purdue website. However, many mathematicians dismiss any chance of de Branges having a correct proof, despite de Branges' proof of the Bieberbach conjecture. Some claim that his proofs are always riddled with errors, and others claim that de Branges' approach is incorrect [Sab03, p. 118].

1.2 Purpose

The goal of this thesis is to present a theorem by Nikolski that first appeared in [Nik95], which leads to a generalization of the theorems given by Beurling and Nyman that relate to zero free regions of the Riemann ζ-function. This theorem is proven in Chapter 6. For $x \in \mathbb{R}$, we use $[x]$ to denote the greatest integer less than or equal to x. Also, if $s \in \mathbb{C}$, then $s_* = -\bar{s}$ is the reflection of s with respect to the imaginary axis. Throughout the rest of this thesis, the right half-plane of $\mathbb{C}$ will be denoted by $\mathcal{R}$, that is, $\mathcal{R} = \{z \in \mathbb{C} : \mathrm{Re}\ z > 0\}$.

Theorem 1.1. *(Nikolski) Let $s \in \mathcal{R}$, and let $\gamma > 0$. Also, let*

$$E_{\alpha,\gamma}(x) = x^{\gamma}\left(\left[\frac{\alpha}{x}\right] - \alpha\left[\frac{1}{x}\right]\right), \qquad 0 < x < 1,$$

where $0 \leqslant \alpha \leqslant 1$, and

$$d_{\gamma}^2(s) = \inf \int_0^1 \left| x^s - \sum_{\alpha} a_{\alpha} E_{\alpha,\gamma}(x)\right|^2 \frac{dx}{x},$$

the inf *being taken over all finite linear combinations of $E_{\alpha,\gamma}$, $0 \leqslant \alpha \leqslant 1$. Then the disk*

$$D_{s,\gamma} = \gamma + D_s = \gamma + \left\{ z : \left|\frac{z-s}{z-s_*}\right|^2 < 1 - 2d_{\gamma}^2(s)\,\mathrm{Re}\ s \right\}$$

is free of zeros of the Riemann ζ-function.

A more all-encompassing theorem is the following:

Theorem 1.2. *(Nikolski) Given $\gamma > 0$, the following are equivalent.*

(1) The Riemann ζ-function has no zeros in the half-plane $\{z \in \mathbb{C} : Re\ z > \gamma\}$.

(2) There exists a point s with Re $s > 0$ such that $d_\gamma(s) = 0$.

(3) $d_\gamma(s) = 0$ for every s with Re $s > 0$.

The proof of this theorem is beyond the scope of this paper, but can be found in [Nik02, p. 169]. Beurling and Nyman were the first to suggest using greatest integer function approximations to study the Riemann ζ-function. Nyman's thesis at Uppsala [Nym50] proved the following theorem:

Theorem 1.3. *(Nyman, 1950) The Riemann hypothesis is equivalent to $d_{1/2}(1/2) = 0$, that is*

$$d_{1/2}^2\left(\frac{1}{2}\right) = \inf \int_0^1 \left|1 - \sum_\alpha a_\alpha \left(\left[\frac{\alpha}{x}\right] - \alpha\left[\frac{1}{x}\right]\right)\right|^2 dx = 0, \tag{1.1}$$

the inf *being taken over all finite linear combinations for $0 < \alpha < 1$.*

In other words, the Riemann ζ-function is free of zeros in the half-plane Re $z > 1/2$ if the functions $\left[\frac{\alpha}{x}\right] - \alpha\left[\frac{1}{x}\right]$, $0 < \alpha < 1$, span the space $L^2((0,1), dx)$. Beurling's paper [Beu55] extends this result to the L^p spaces; namely, the Riemann ζ-function is free of zeros in the half-plane Re $z > 1/p$ if the functions $\left[\frac{\alpha}{x}\right] - \alpha\left[\frac{1}{x}\right]$, $0 < \alpha < 1$, span the space $L^p((0,1), dx)$.

Further endeavors have been pursued using similar techniques. In [Vas95], V. Vasyunin performs some numerical experiments using Nikolski's results. From his work, Balazard and Saias deduce further questions in [BS00, pp. 135–137]. This motivates related numerical experiments based on their work. They are shown in [LR02].

Chapter 2

The Riemann ζ-function

2.1 Infinite Products

We open this section with the basic definition.

Definition 2.1. Let $\{z_k\}$ be an infinite sequence of complex numbers. If

$$z = \lim_{n\to\infty} \prod_{k=1}^{n} z_k$$

exists, we define z to be the **infinite product** of the sequence of numbers. We denote this product by

$$z = \prod_{k=1}^{\infty} z_k. \tag{2.1}$$

We would like for

$$\prod_{k=1}^{\infty} a_k = 0 \Longleftrightarrow a_k = 0 \text{ for some } k.$$

So we require $\prod_{k=1}^{\infty} a_k \to s \neq 0$ as $n \to \infty$, if $a_k \neq 0$ for all $k \in \mathbb{N}$. Then, under this requirement,

$$a_n = \frac{\prod_{k=1}^{n} a_k}{\prod_{k=1}^{n-1} a_k} \to \frac{s}{s} = 1.$$

This gives us an analogous theorem to a common one with infinite series, namely,

Theorem 2.2. *If $\prod_{k=1}^{\infty} a_k$ converges to a nonzero number, and if $a_k \neq 0$ for all $k \in \mathbb{N}$, then $a_k \to 1$ as $k \to \infty$.*

Recall the definition for the principal branch of the complex logarithm

$$\log(z) = \ln|z| + i\arg(z) \qquad (-\pi < \arg(z) < \pi),$$

where ln is the real-valued logarithm function defined on $\mathbb{R}_+$. We will need to use this function to be able to get a sufficient condition for which $\prod_{k=1}^{\infty} a_k$ will converge. Note that if $\prod_{k=1}^{n} a_k \to s \neq 0$, then the partial factors, denoted a_n for $n \geqslant 1$, will lie in the right half-plane of $\mathbb{C}$ for sufficiently large n. So the following results with assume that $\mathrm{Re}\, z_k > 0$ for all $k \in \mathbb{N}$.

Definition 2.3. If $\mathrm{Re}\, z_n > 0$ for all n, then the infinite product $\prod z_n$ is said to **converge absolutely** if the series $\sum \log z_n$ converges absolutely.

Proofs of these theorems appear in [Con78, pp. 165–166].

Theorem 2.4. *If* $\mathrm{Re}\, z_n > 0$*, then the product* $\prod z_n$ *converges absolutely if and only if the series* $\sum (z_n - 1)$ *converges absolutely.*

For a sequence of holomorphic functions in a region, we have the following important theorem about products.

Theorem 2.5. *Let* Ω *be a region in* $\mathbb{C}$ *and let* $\{f_n\}$ *be a sequence in* $\mathrm{Hol}(\Omega)$ *such that no* f_n *is identically zero. If* $\sum |f_n(z) - 1|$ *converges uniformly on compact subsets of* Ω*, then* $\prod_{n=1}^{\infty} f_n(z)$ *converges in* $\mathrm{Hol}(\Omega)$ *to an analytic function* $f(z)$*. If* a *is a zero of* f*, then* a *is a zero of only a finite number of the functions* f_n*, and the multiplicity of the zero of* f *at* a *is the sum of the multiplicities of the zeros of the functions* f_n *at* a*.*

2.2 The Gamma Function

Definition 2.6. The **Gamma function** on $\{z \in \mathbb{C} : z \neq 0, -1, -2, \dots\}$ is defined by

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \cdot \frac{1}{\prod_{n=1}^{\infty} (1 + \frac{z}{n}) e^{-z/n}}, \tag{2.2}$$

where γ is called *Euler's constant* and is chosen so that $\Gamma(1) = 1$. The Gamma function has simple poles at all of the points where it is not defined.

It is clear by definition that for all $z \in \mathbb{C}$, $\Gamma(z) \neq 0$. The following formulation of the Gamma function will be useful. It is proved in [Con78, pp. 177–178].

Lemma 2.7. *For* $z \neq 0, -1, -2, \dots$*, we have*

$$\Gamma(z) = \lim_{n \to \infty} \frac{n!\, n^z}{z(z+1) \cdots (z+n)}. \tag{2.3}$$

Using this Lemma, we can easily prove an important functional equation of the Gamma function.

Theorem 2.8. *For $z \neq 0$, we have*

$$\Gamma(z+1) = z\Gamma(z) \tag{2.4}$$

Proof. We will prove (2.4) by evaluating $\Gamma(z+1)/z$ with Lemma 2.7. We write

$$\begin{aligned}
\frac{\Gamma(z+1)}{z} &= \lim_{n\to\infty} \frac{n!\, n^{z+1}}{z(z+1)(z+2)\cdots(z+1+n)} \\
&= \lim_{n\to\infty} \frac{n}{z+1+n} \cdot \lim_{n\to\infty} \frac{n!\, n^{z}}{z(z+1)(z+2)\cdots(z+n)} \\
&= 1 \cdot \Gamma(z) \\
&= \Gamma(z).
\end{aligned}$$

□

Theorem 2.9 provides an integral representation for the Gamma function in the right half-plane. It will be used to obtain an integral representation for the Riemann Zeta-function.

Theorem 2.9. *If* Re $z > 0$, *then*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt.$$

2.3 Riemann's Functional Equation

Definition 2.10. The **Riemann Zeta-function** is the function defined by the infinite series

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}, \qquad \text{Re } z > 1. \tag{2.5}$$

The above defined Riemann ζ-function is defined only when Re $z > 1$. We will use analytic continuation to extend the function to the domain $\mathbb{C} - \{1\}$. Our first Lemma uses Theorem 2.9 to derive an integral representation for the Riemann ζ-function.

Lemma 2.11.

$$\zeta(z)\Gamma(z) = \int_0^\infty \frac{t^{z-1}}{e^t - 1}\, dt, \qquad \text{Re } z > 1. \tag{2.6}$$

It can then be shown that since

$$f_n(z) = \int_{1/n}^{1} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\, dt$$

converges uniformly on compact subsets of the right half-plane, setting $f = \lim_{n\to\infty} f_n$ gives a holomorphic function in the right half-plane. We can then write for Re $z > 1$

$$\zeta(z)\Gamma(z) = \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) t^{z-1}\, dt + \frac{1}{z-1} + \int_1^\infty \frac{t^{z-1}}{e^t - 1}\, dt. \tag{2.7}$$

Note that since $\dfrac{1}{\Gamma(z)(z-1)} = \dfrac{1}{\Gamma(z+1) - \Gamma(z)}$ has a pole at 1, (2.7) defines $\zeta(z)$ as a meromorphic function in $\mathcal{R}$ with a simple pole at 1.

Now restrict z to the domain $0 <$ Re $z < 1$. We can modify (2.7) using the fact

$$\frac{1}{z-1} = -\int_1^\infty t^{z-2}\, dt$$

to get

$$\zeta(z)\Gamma(z) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) t^{z-1}\, dt, \qquad 0 < \text{ Re } z < 1. \tag{2.8}$$

The two integrals

$$\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) t^{z-1}\, dt \qquad \text{and} \qquad \int_1^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) t^{z-1}\, dt$$

converge uniformly on compact subsets of $\{z : \text{Re } z > -1\}$ and $\{z : \text{Re } z < 1\}$, respectively. Hence, using (2.8) and these 2 integrals we can write

$$\zeta(z)\Gamma(z) = \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) t^{z-1}\, dt - \frac{1}{2z} + \int_1^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) t^{z-1}\, dt, \tag{2.9}$$

which gives $\zeta(z)$ analytic for $-1 <$ Re $z < 1$. However, for this to be true, it must be analytic at $z = 0$. Indeed,

$$\frac{1}{2z\Gamma(z)} = \frac{1}{2\Gamma(z+1)}$$

is analytic at $z = 0$. So we have effectively defined $\zeta(z)$ for Re $z > -1$ with a simple pole at $z = 1$ using the combination of equations (2.6) and (2.9).

If we restrict z to the domain $-1 <$ Re $z < 0$, then we can collapse (2.9) to

$$\zeta(z)\Gamma(z) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) t^{z-1}\, dt, \tag{2.10}$$

since

$$\int_1^\infty t^{z-1}\, dt = -\frac{1}{z}.$$

It can be shown that

$$\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)\frac{1}{t}=2\sum_{n=1}^{\infty}\frac{1}{t^2+4n^2\pi^2};$$

hence, we can compute the following:

$$\begin{aligned}
\zeta(z)\Gamma(z) &= 2\int_0^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{t^2+4n^2\pi^2}\right)t^z\,dt\\
&= 2\int_0^{\infty}\left(\sum_{n=1}^{\infty}\frac{t^z}{t^2+4n^2\pi^2}\right)dt\\
&= 2\sum_{n=1}^{\infty}(2\pi n)^{z-1}\int_0^{\infty}\frac{t^z}{t^2+1}\,dt\\
&= 2(2\pi)^{z-1}\zeta(1-z)\int_0^{\infty}\frac{t^z}{t^2+1}\,dt, \qquad (2.11)
\end{aligned}$$

for $-1<\operatorname{Re} z<0$. Using [Con78, pp. 114, 192], for $z=x+iy$, $-1<x<0$, and $c=\frac{1}{2}(1-x)<1$, we have

$$\begin{aligned}
\int_0^{\infty}\frac{t^x}{t^2+1}\,dt &= \frac{1}{2}\int_0^{\infty}\frac{s^{\frac{1}{2}(x-1)}}{s+1}\,ds\\
&= \frac{1}{2}\pi\csc(\pi(1-x)/2)\\
&= \frac{1}{2}\pi\sec(\pi x/2). \qquad (2.12)
\end{aligned}$$

Using Theorem 2.8, we see that

$$\begin{aligned}
\Gamma(x)\Gamma(1-x) &= -x\Gamma(x)\Gamma(-x)\\
&= -x\cdot\frac{e^{-\gamma x}}{x}\cdot\frac{1}{\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-x/n}}\cdot\frac{e^{\gamma x}}{-x}\cdot\frac{1}{\prod_{n=1}^{\infty}\left(1-\frac{x}{n}\right)e^{x/n}}\\
&= \frac{1}{x}\cdot\frac{1}{\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right)}\\
&= \frac{\pi}{\sin(\pi x)}. \qquad (2.13)
\end{aligned}$$

The last equation comes from the factorization of the sine function [Con78, p. 175]. Using this result and the addition formula for sine, we obtain

$$\frac{1}{\Gamma(x)} = \frac{\Gamma(1-x)}{\pi}\sin(\pi x) = \frac{\Gamma(1-x)}{\pi}[2\sin(\pi x/2)\cos(\pi x/2)]. \tag{2.14}$$

Multiplying the right-hand side of (2.11) by the right-hand side of (2.14), we get via (2.12)

$$\zeta(x) = 2(2\pi)^{x-1}\zeta(1-x)\Gamma(1-x)\sin(\pi x/2),$$

for $x \in (-1, 0)$, which has a limit point in $\{z = x+iy : x \in (-1, 1)\}$. So the equation holds on the set $\{z = x + iy : x \in (-1, 1)\}$ by the Identity Theorem [Rud87, p. 209], since both sides of the equation are holomorphic in $\{z = x + iy : x \in (-1, 1)\}$. Furthermore, the right-hand side is holomorphic in $\{z = x+iy : x \in (-\infty, 0)\}$, so we use the same theorem to give $\zeta(z)$ an analytic continuation to this set. Finally, since $\zeta(z)$ is holomorphic in $\{z = x + iy : x \in (-1, \infty)\} \setminus \{1\}$, and the right-hand side is holomorphic there except for possible poles at the positive integers, we get that the equation must hold in $\mathbb{C} \setminus \{1\}$ since $\zeta(z)$ is continuous at $z = 2, 3, 4 \ldots$. Now we can state Riemann's functional equation as a theorem.

Theorem 2.12. *If $z \in \mathbb{C}$ and $z \neq 1$, then*

$$\zeta(z) = 2(2\pi)^{z-1}\zeta(1-z)\Gamma(1-z)\sin(\pi z/2). \tag{2.15}$$

2.4 Trivial Zeros and the Riemann Hypothesis

We can determine certain zeros easily using Riemann's functional equation for the ζ-function

$$\zeta(z) = 2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)\sin(\pi z/2).$$

We do this by noticing that since $\zeta(z)$ is analytic at $z = 2, 3, 4 \ldots$ and $\Gamma(1-z)$ has poles at $z = 1, 2, 3, \ldots$, we must have that $\zeta(1-z)\sin(\pi z/2) = 0$ at $z = 2, 3, \ldots$. Moreover, each zero must be simple since all the poles of Γ are simple. We know exactly when $\sin(\pi z/2)$ is zero, that is, when $z = 2, 4, 6, \ldots$. Hence, $\zeta(1-z)$ has zeros when $z = 3, 5, 7, \ldots$. We then conclude that $\zeta(z)$ has zeros for $z = -2n$, $n \in \mathbb{N}$. These zeros are termed the *trivial zeros* of the Riemann ζ-function.

Furthermore, we can deduce that excluding the trivial zeros, $\zeta(z)$ has no other zeros outside of $\{z : 0 \leqslant \text{Re } z \leqslant 1\}$. For if there exists such a nontrivial zero z_0, where Re $z_0 < 0$, then $\zeta(1-z_0) = 0$ by the functional equation with Re $(1-z_0) > 1$. However,

$$\zeta(z) = \sum_{n\geqslant 1}\frac{1}{n^z} = \prod_{p\in P}\frac{1}{1-p^{-z}} \neq 0, \qquad \text{Re } z > 1,$$

by (2.16) below. This contradicts that $\zeta(1-z_0) = 0$.

At this point, we know that all of the nontrivial zeros of the Riemann ζ-function must lie in the strip $0 \leqslant \operatorname{Re} z \leqslant 1$. It is known that no zeros lie on the line $\operatorname{Re} z = 1$, hence no zeros occur on the line $\operatorname{Re} z = 0$ by the functional equation [Con78, p. 193]. Riemann first stated the Riemann Hypothesis in his now famous paper "*Über die Anzahl der Primzahlen unter einer gegebenen Grösse*," which says that all of the nontrivial zeros of the ζ-function occur when $z = 1/2 + it$, $t \in \mathbb{R}$. To this day, no one has found a counterexample to this hypothesis.

2.5 Further Representations of the ζ-function

The Riemann ζ-function is intimately tied to number theory. This can readily be seen by the following theorem:

Theorem 2.13. *If* $\operatorname{Re}(s) > 1$, *then*

$$\zeta(s) = \prod_{p \in P} \frac{1}{1 - p^{-s}}, \tag{2.16}$$

where P is the set of all prime numbers.

Proof. Since $p^{-s} < 1$ for each $p \in P$, we can write each factor $(1 - p^{-s})^{-1}$ as a convergent geometric series:

$$\frac{1}{1 - p^{-s}} = \sum_{k=0}^{\infty} p^{-ks}. \tag{2.17}$$

We then compute $\prod_{p \in P} \frac{1}{1-p^{-s}}$ by multiplying each sum in (2.17). Using the distributive property of multiplication, the resulting sum is of the form

$$\sum (2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} \cdots p_j^{\alpha_j})^{-s}.$$

The fundamental theorem of arithmetic tells us that for any integer $n > 1$, there exist unique $\alpha_1, \alpha_2, \ldots, \alpha_j$ such that n can be written as one of the terms above. This type of sum is precisely what occurs in $\sum_{n \geqslant 1} n^{-s}$. □

Lemma 2.14. *If* $\operatorname{Re} s > 1$, *we have*

$$\frac{\zeta(s)}{s} = \int_1^{\infty} [t] t^{-s-1}\, dt = \int_1^{\infty} ([t] - t) t^{-s-1}\, dt + \frac{1}{s-1}.$$

The second integral is analytic in $\{s : \operatorname{Re} s > 0\}$.

Proof. For $n \geqslant 1$, we can split up the interval $[1, n+1]$ into subintervals $[k, k+1]$, $k = 1, 2, \ldots, n$, on which $[t]$ is constant. From this we have

$$\begin{aligned} s\int_1^{n+1}[t]t^{-s-1}\,dt &= s\sum_{k=1}^{n} k\int_k^{k+1} t^{-s-1}\,dt \\ &= s\sum_{k=1}^{n} k\left(\frac{t^{-s}}{-s}\right)\Bigg|_k^{k+1} \\ &= \sum_{k=1}^{n} k(k^{-s}-(k+1)^{-s}) \\ &= \sum_{k=1}^{n} kk^{-s} - \sum_{k=1}^{n} k(k+1)^{-s} \\ &= \sum_{k=1}^{n} kk^{-s} - \sum_{k=2}^{n+1}(k-1)k^{-s} \\ &= \sum_{k=1}^{n} k^{-s} - n(n+1)^{-s} \\ &\to \zeta(s), \end{aligned}$$

as $n \to \infty$ for $\operatorname{Re} s > 1$.

To obtain the second integral, we do a simple modification of the first integral:

$$\begin{aligned} \int_1^{\infty}[t]t^{-s-1}\,dt &= \int_1^{\infty}([t]+t-t)t^{-s-1}\,dt \\ &= \int_1^{\infty}([t]-t)t^{-s-1}\,dt + \int_1^{\infty} t^{-s}\,dt \\ &= \int_1^{\infty}([t]-t)t^{-s-1}\,dt + \left(\frac{t^{-s+1}}{-s+1}\right)\Bigg|_1^{\infty} \\ &= \int_1^{\infty}([t]-t)t^{-s-1}\,dt + \frac{1}{s-1}. \end{aligned}$$

The last equality holds since $\operatorname{Re} s > 1$. $\square$

The last Lemma extends the Riemann ζ-function to an analytic function in the right half-plane; this is since $[t]-t$ is bounded for $t > 1$. The first integral representation did not have this property.

Chapter 3

Hardy Spaces and The Fourier Transform

3.1 Fourier Series

We start this chapter with the definition of the L^p-space of the unit circle: $L^p(\mathbb{T})$. This definition, as well as the ones following it, will describe the general case $0 < p < \infty$, while the results focus mainly on the case $p = 2$.

Definition 3.1. If $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle, then for $p > 0$ define

$$\|f\|_p = \left(\int_0^{2\pi} |f(e^{it})|^p \, \frac{dt}{2\pi} \right)^{1/p}.$$

We define $L^p(\mathbb{T})$ to be the space of all measurable functions f such that

$$\|f\|_p < \infty.$$

For a function $f \in L^1(\mathbb{T})$, we define the function $\hat{f}$ on $\mathbb{Z}$ by

$$\hat{f}(n) = \int_0^{2\pi} f(e^{it}) e^{-int} \, \frac{dt}{2\pi}. \tag{3.1}$$

For each n, the complex numbers $\hat{f}(n)$ are called the *Fourier coefficients* of f. They are obtained as the inner product of a function with an element of the orthonormal set $\{e^{int}\}_n$. Since $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$, we can define the Fourier coefficients of an L^2 function as in (3.1). For a function $f \in L^2(\mathbb{T})$, we can write it as a series indexed by $\mathbb{Z}$ as

$$f(e^{it}) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{int}, \tag{3.2}$$

where the convergence of the sum in (3.2) occurs in $L^2(\mathbb{T})$. This is called the *Fourier Series* of f. The Fourier coefficients $\{\hat{f}(n)\}$ satisfy

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \|f\|_2^2 = \int_0^{2\pi} |f(e^{it})|^2 \, \frac{dt}{2\pi}. \tag{3.3}$$

3.2 The Hardy Spaces $H^2(\mathbb{T})$ and $H^2(\mathbb{D})$

We now define the space $H^2(\mathbb{T})$.

Definition 3.2. For $p \geqslant 1$, define the **Hardy space of the unit circle** by

$$H^p(\mathbb{T}) = \{f \in L^p(\mathbb{T}) : \hat{f}(n) = 0 \text{ for all } n < 0\}. \tag{3.4}$$

According to the previous definition, any function in $H^2(\mathbb{T})$ has a Fourier series of the form $\sum_{n=0}^{\infty} \hat{f}(n)e^{int}$. Thus, for each $f \in H^2(\mathbb{T})$, we can define a function $\tilde{f} \in \mathrm{Hol}(\mathbb{D})$ by $\tilde{f}(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$, where $|z| < 1$.

Definition 3.3. If $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk, then for $p \geqslant 1$ define the **Hardy space of the unit disk** by

$$H^p(\mathbb{D}) = \left\{ \tilde{f} \in \mathrm{Hol}(\mathbb{D}) : \sup_{0\leqslant r<1} \int_0^{2\pi} |\tilde{f}(re^{it})|^p \, \frac{dt}{2\pi} < \infty \right\}. \tag{3.5}$$

The space $H^p(\mathbb{D})$ is a normed linear space with its norm defined on any $f \in H^p(\mathbb{D})$ by $\|\tilde{f}\|_{H^p(\mathbb{D})}^p = \sup_{0\leqslant r<1} \int_0^{2\pi} |\tilde{f}(re^{it})|^p \frac{dt}{2\pi}$. If $\tilde{f}(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| < 1$, then we can write

$$\begin{aligned} \|\tilde{f}\|_{H^2(\mathbb{D})}^2 = \sup_{0\leqslant r<1} \int_0^{2\pi} |\tilde{f}(re^{it})|^2 \, \frac{dt}{2\pi} \quad &= \quad \sup_{0\leqslant r<1} \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \\ &= \quad \sum_{n=0}^{\infty} |a_n|^2, \end{aligned} \tag{3.6}$$

where the second equality holds by Theorem 10.22 in [Rud87, p. 211]. So if we take a function $f \in H^2(\mathbb{T})$, then we can define $\tilde{f}(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$, $|z| < 1$, and using (3.3) and (3.6), we obtain

$$\|f\|_2^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 = \|\tilde{f}\|_{H^2(\mathbb{D})}^2.$$

Hence, the map $f \mapsto \tilde{f}$ is an isometric isomorphism between $H^2(\mathbb{T})$ and $H^2(\mathbb{D})$.

3.3 The Fourier Transform $L^2(\mathbb{R}) \to L^2(\mathbb{R})$

Definition 3.4. The **Fourier Transform** of a function $f \in L^1(\mathbb{R})$ is defined for $t \in \mathbb{R}$ by

$$\mathcal{F}f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixt}\, dx. \tag{3.7}$$

Indeed, if $f \in L^1(\mathbb{R})$, we have

$$\begin{aligned} \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixt}\, dx \right| &\leqslant \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|\, |e^{-ixt}|\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|\; dx < \infty. \end{aligned}$$

The formula given in (3.7) defines the Fourier transform of a function $f \in L^1(\mathbb{R})$. However, we want to find a function $\mathcal{F}f \in L^2(\mathbb{R})$ such that the map $f \mapsto \mathcal{F}f$ is an isometry, where $f \in L^2(\mathbb{R})$. This result comes from the Plancherel Theorem:

Theorem 3.5. *For a function $f \in L^2(\mathbb{R})$, there is a unique $\mathcal{F}f \in L^2(\mathbb{R})$ such that*

(a) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\mathcal{F}f$ is defined as in (3.7).

(b) For every $f \in L^2(\mathbb{R})$, $\|\mathcal{F}f\|_2 = \|f\|_2$.

(c) The mapping $f \mapsto \mathcal{F}f$ is a Hilbert space isomorphism of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.

(d) We have the following relationships between f and $\mathcal{F}f$: If

$$\varphi_A(t) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} f(x) e^{-ixt}\, dx \qquad \text{and} \qquad \psi_A(x) = \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \mathcal{F}f(t) e^{ixt}\, dt,$$

then $\|\varphi_A - \mathcal{F}f\|_2 \to 0$ and $\|\psi_A - f\|_2 \to 0$ as $A \to \infty$.

(For a proof, see [Rud87, pp. 186–187].) The function ψ_A can be used to calculate the inverse Fourier transform, which we'll denote by $\mathcal{F}^{-1}$.

Chapter 4

The Unitary Mapping $\widetilde{U_2}$

4.1 The Hardy space $H^2(\mathcal{U})$

Definition 4.1. If $\mathcal{U} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ is the upper half-plane, then for $p > 0$ define the **Hardy space of the upper half-plane** by

$$H^p(\mathcal{U}) = \left\{ f \in \mathrm{Hol}(\mathcal{U}) : \sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^p \, dx < \infty \right\}. \tag{4.1}$$

The norm of this space is defined by $\|f\|_{H^p(\mathcal{U})}^p = \sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^p \, dx$.

This section will discuss the unitary mapping $\widetilde{U_2}$ and how it is used to describe functions in $H^2(\mathcal{U})$ using what we know about functions in $H^2(\mathbb{D})$. First we let ω be the usual conformal mapping from $\mathbb{D}$ to $\mathcal{U}$:

$$\omega(z) = i\frac{1+z}{1-z}, \qquad z \in \mathbb{D}. \tag{4.2}$$

If we think of ω as being defined on the boundary of $\mathbb{D}$, that is $\mathbb{T} \setminus \{1\}$, then the range of the transformation is $\mathbb{R}$, and the simple change of variable

$$\omega^{-1}(x) = \frac{x-i}{x+i}$$

defines the operator $U_p : L^p(\mathbb{T}) \to L^p(\mathbb{R})$ as

$$U_p f(x) = \left(\frac{1}{\pi(x+i)^2}\right)^{1/p} \cdot f\left(\frac{x-i}{x+i}\right), \qquad x \in \mathbb{R}. \tag{4.3}$$

Hence, U_p is an isometric isomorphism from $L^p(\mathbb{T})$ onto $L^p(\mathbb{R})$. In the case $p = 2$, the mapping U_2 is unitary, meaning it preserves inner products [Nik02, p. 143]. Notice that we can also define an operator

$$\widetilde{U_p}f(z) = \left(\frac{1}{\pi(z+i)^2}\right)^{1/p} \cdot f\left(\frac{z-i}{z+i}\right), \qquad z \in \mathcal{U}, \tag{4.4}$$

which defines an analytic function $\widetilde{U_p}f$ in $\mathcal{U}$ given a function $f \in H^p(\mathbb{D})$. We would like to know that $\widetilde{U_p}f \in H^p(\mathcal{U})$. This comes as a result of the following theorem:

Theorem 4.2. *Let $1 \leqslant p < \infty$. Then $U_p H^p(\mathbb{D}) = H^p(\mathcal{U})$.*

A proof is located in [Nik02].

Before moving on to the next section, we need to define the Hardy space $H^\infty(\mathbb{D})$ and explicitly state the operator $\widetilde{U_\infty}$. The last section in this chapter will make use of these definitions.

Definition 4.3. The set of bounded holomorphic functions in the unit disk $\mathbb{D}$ will be denoted by $H^\infty(\mathbb{D})$. The norm

$$\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\} \tag{4.5}$$

makes $H^\infty(\mathbb{D})$ a Banach space.

Definition 4.4. For $p = \infty$ we define the operator $\widetilde{U_\infty}$ by

$$\widetilde{U_\infty}f(z) = f\left(\frac{z-i}{z+i}\right), \qquad z \in \mathcal{U}. \tag{4.6}$$

It is clear that $\widetilde{U_\infty}$ is an isometric isomorphism of $H^\infty(\mathbb{D})$ onto $H^\infty(\mathcal{U})$.

4.2 The Reproducing Kernel of $H^2(\mathcal{U})$

Definition 4.5. Let $\mathcal{H}$ be a Hilbert space of complex-valued analytic functions on a subset $\Omega \subset \mathbb{C}$ with inner product $(\cdot\,,\cdot)$.

(a) Let $\varphi_\lambda : \mathcal{H} \to \mathbb{C}$ be the **point evaluation functional**, that is, the continuous linear map defined by $\varphi_\lambda(f) = f(\lambda)$.

(b) The function k_λ, which we will call the **reproducing kernel** of $\mathcal{H}$, is the unique function in $\mathcal{H}$ such that

$$\varphi_\lambda(f) = (f, k_\lambda).$$

The existence of this function comes from applying the Riesz Representation Theorem to φ_λ.

(c) When the point evaluation functional φ_λ is continuous, the Hilbert space $\mathcal{H}$ is called a **reproducing kernel Hilbert space**.

See [Aro50] for a thorough treatment of reproducing kernels.

The reproducing kernel for $H^2(\mathbb{D})$ is easily calculated (via the correspondence between $H^2(\mathbb{T})$ and $H^2(\mathbb{D})$) using Definition 4.5. It is called the Szegö kernel and is defined by $k_\lambda(z) = (1-\bar{\lambda}z)^{-1}$. To find the reproducing kernel for $H^2(\mathcal{U})$, which we'll call k_ν, we apply the operator $\widetilde{U_2}$ to k_λ and the function ω to the point λ. Set $g = \widetilde{U_2}f$. We want

$$g(\nu) = (g, k_\nu)_{L^2(\mathbb{R})}$$

for each $g \in H^2(\mathcal{U})$, where $\omega(\lambda) = \nu$. Since $\widetilde{U_2}$ is a Hilbert space isomorphism, we can write

$$g(\nu) = (\widetilde{U_2}^{-1} g, \widetilde{U_2}^{-1} k_\nu)_{L^2(\mathbb{T})}.$$

This gives that

$$\widetilde{U_2} f(\nu) = (f, \widetilde{U_2}^{-1} k_\nu)_{L^2(\mathbb{T})},$$

or

$$f\left(\frac{\nu - i}{\nu + i}\right) \frac{1}{\sqrt{\pi}} \frac{1}{\nu + i} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \overline{U_2^{-1} k_\nu(e^{it})}\, dt.$$

This implies that

$$f\left(\frac{\nu - i}{\nu + i}\right) = \frac{(\nu + i)\sqrt{\pi}}{2\pi} \int_0^{2\pi} f(e^{it}) \overline{U_2^{-1} k_\nu(e^{it})}\, dt.$$

But since $\omega(\lambda) = \nu$, we have $\frac{\nu - i}{\nu + i} = \lambda$, so since $f(\lambda) = (f, k_\lambda)$, we get the equality

$$\overline{U_2^{-1} k_\nu(e^{it})}(\nu + i)\sqrt{\pi} = \overline{\left(\frac{1}{1 - \bar{\lambda} e^{it}}\right)},$$

or

$$U_2^{-1} k_\nu(e^{it}) = \frac{1}{\overline{(\nu + i)\sqrt{\pi}}} \cdot \frac{1}{1 - \bar{\lambda} e^{it}}.$$

We can now calculate the reproducing kernel for $H^2(\mathcal{U})$ by operating on each side with U_2 and letting $e^{it} = (x-i)(x+i)^{-1}$:

$$\begin{aligned}
k_\nu(x) &= \frac{1}{(\overline{\nu}-i)\sqrt{\pi}} U_2 k_\lambda(x) \\
&= \frac{1}{(\overline{\nu}-i)\sqrt{\pi}} \cdot \frac{1}{\sqrt{\pi}(x+i)} \cdot \frac{1}{1-\bar{\lambda}\left(\frac{x-i}{x+i}\right)} \\
&= \frac{1}{(\overline{\nu}-i)(x+i)\pi} \cdot \frac{1}{1-\left(\frac{\bar{\nu}+i}{\bar{\nu}-i}\right)\left(\frac{x-i}{x+i}\right)} \\
&= \frac{(\bar{\nu}-i)(x+i)}{\pi(\overline{\nu}-i)(x+i)[(\overline{\nu}-i)(x+i)-(\overline{\nu}+i)(x-i)]} \\
&= \frac{1}{\pi(-2ix+2i\overline{\nu})} \\
&= \frac{1}{2\pi i} \cdot \frac{1}{\overline{\nu}-x}.
\end{aligned}$$

4.3 Blaschke Products in $H^2(\mathcal{U})$

Definition 4.6. The function $B : \mathbb{D} \to \mathbb{C}$ defined by

$$B(z) = \prod_{n=1}^{\infty} \frac{\alpha_n - z}{1-\overline{\alpha}_n z} \frac{|\alpha_n|}{\alpha_n} \tag{4.7}$$

is called a **Blaschke product**. It defines an analytic function in $\mathbb{D}$ if the sequence $\{\alpha_n\} \subset \mathbb{D}$ satisfies

$$\sum_{n=1}^{\infty} (1-|\alpha_n|) < \infty. \tag{4.8}$$

This condition is called the **Blaschke condition**. A standard proof of this fact is located in [Rud87, p. 310].

Using our definition of Blaschke Products for functions in $H^2(\mathbb{D})$, we can derive a Blaschke condition and a Blaschke product for functions in $H^2(\mathcal{U})$.

Theorem 4.7. *If $f \in H^2(\mathcal{U})$, and f is not identically equal to zero, then*

$$\sum_n \frac{\text{Im } \lambda_n}{1+|\lambda_n|^2} < \infty,$$

where λ_n are the zeros of f in $\mathcal{U}$ (counting multiplicities). The corresponding Blaschke product is

$$B(z) = \prod_n \varepsilon_n \frac{z-\lambda_n}{z-\overline{\lambda}_n}, \qquad z \in \mathcal{U},$$

where $\varepsilon_n = |\lambda_n^2+1|(\lambda_n^2+1)^{-1}$ *if* $\lambda_n \neq i$*, and* $\varepsilon_n = 1$ *otherwise.*

Proof. To derive the Blaschke condition, let λ_n be the ω-image of z_n, where z_n is one of the zeros of $\widetilde{U_2}^{-1} f \in H^2(\mathbb{D})$. Then we see that for all n

$$\begin{aligned} \frac{\operatorname{Im}\lambda_n}{1+|\lambda_n|^2} &= \frac{\operatorname{Im}\, i\dfrac{1+z_n}{1-z_n}}{1+\left|i\dfrac{1+z_n}{1-z_n}\right|^2} \\ &= \frac{\left(i\dfrac{1+z_n}{1-z_n}+i\dfrac{1+\overline{z}_n}{1-\overline{z}_n}\right)|1-z_n|^2}{2i(|1-z_n|^2+|1+z_n|^2)} \\ &= \frac{((1+z_n)(1-\overline{z}_n)+(1+\overline{z}_n)(1-z_n))|1-z_n|^2}{4(1+|z_n|^2)(1-z_n)(1-\overline{z}_n)} \\ &= \frac{2(1-|z_n|^2)}{4(1+|z_n|^2)} \\ &< 1-|z_n|^2. \end{aligned}$$

Hence, if $\sum_n 1-|z_n|^2 < \infty$, the comparison test yields that

$$\sum_n \frac{\operatorname{Im}\lambda_n}{1+|\lambda_n|^2} < \infty.$$

Let $f \in H^2(\mathbb{D})$. We will apply the operator $\widetilde{U_\infty}$ to the Blaschke product corresponding to f to obtain a Blaschke product corresponding to the function $\widetilde{U_\infty} f$. This application is possible since $|B(z)| < 1$ for all $z \in \mathbb{D}$ [Rud87, p. 310]. We see that

$$\begin{aligned} \widetilde{U_\infty} B(z) &= B\left(\frac{z-i}{z+i}\right) \\ &= \prod_n \frac{|z_n|}{z_n}\cdot\frac{z_n-\frac{z-i}{z+i}}{1-\overline{z}_n(\frac{z-i}{z+i})} \\ &= \prod_n \frac{|z_n|}{z_n}\cdot\frac{(z+i)z_n-(z-i)}{(z+i)-\overline{z}_n(z-i)}. \end{aligned}$$

Since $z_n \in \mathbb{D}$, we get the corresponding zero of $\widetilde{U_\infty} f$, which we'll call λ_n, by applying ω to z_n for all n. So we have for each n

$$z_n = \frac{\lambda_n - i}{\lambda_n + i} \qquad \text{and} \qquad \overline{z}_n = \frac{\overline{\lambda}_n + i}{\overline{\lambda}_n - i}.$$

Under these substitutions we obtain

$$
\begin{aligned}
\widetilde{U_\infty} B(z) &= \prod_n \frac{\left|\frac{\lambda_n - i}{\lambda_n + i}\right|}{\frac{\lambda_n - i}{\lambda_n + i}} \cdot \frac{(z+i)\frac{\lambda_n - i}{\lambda_n + i} - (z-i)}{z + i - \frac{\overline{\lambda}_n + i}{\overline{\lambda}_n - i}(z-i)} \\
&= \prod_n \frac{|\lambda_n - i|(\lambda_n + i)(\overline{\lambda}_n - i)}{|\lambda_n + i|(\lambda_n - i)(\lambda_n + i)} \cdot \frac{(z+i)(\lambda_n - i) - (z-i)(\lambda_n + i)}{(z+i)(\overline{\lambda}_n - i) - (\overline{\lambda}_n + i)(z-i)} \\
&= \prod_n \frac{|\lambda_n - i|(\overline{\lambda}_n - i)}{|\lambda_n + i|(\lambda_n - i)} \cdot \frac{z - \lambda_n}{z - \overline{\lambda}_n} \\
&= \prod_n \frac{((\lambda_n - i)(\overline{\lambda}_n + i)(\overline{\lambda}_n - i)^2)^{1/2}}{((\lambda_n + i)(\overline{\lambda}_n - i)(\lambda_n - i)^2)^{1/2}} \cdot \frac{z - \lambda_n}{z - \overline{\lambda}_n} \\
&= \prod_n \sqrt{\frac{(\overline{\lambda}_n^2 + 1)}{(\lambda_n^2 + 1)}} \cdot \frac{z - \lambda_n}{z - \overline{\lambda}_n} \\
&= \prod_n \frac{|\lambda_n^2 + 1|}{\lambda_n^2 + 1} \cdot \frac{z - \lambda_n}{z - \overline{\lambda}_n}.
\end{aligned}
$$

□

Chapter 5

The function $g = f/b_\nu$

Let F be a subspace of $H^2(\mathcal{U})$. The goal of this chapter is to find a function $g = f/b_\nu$ such that $\|g\|_{H^2(\mathcal{U})} = \|f\|_{H^2(\mathcal{U})}$, whenever $f \in F$, ν is zero of the subspace F, and b_ν is the corresponding nonnormalized Blaschke factor:

$$b_\nu(z) = (z-\nu)(z-\bar{\nu})^{-1}.$$

The Blaschke factor b_ν comes from Theorem 4.7, and we note that $|b_\nu(x)| = 1$ for all $x \in \mathbb{R}$ by symmetry. We start by introducing the Poisson formula for $H^2(\mathcal{U})$.

Lemma 5.1. *If $F \in H^2(\mathcal{U})$, then for $y > 0$ we have*

$$F(x+iy) = \int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} F^*(t)\,\frac{dt}{\pi}, \tag{5.1}$$

where $F^*(t) = \lim_{\varepsilon\to 0^+} F(t+i\varepsilon)$, which exists for a.e. $x \in \mathbb{R}$.

Proof. Let $F \in H^2(\mathcal{U})$, and let $G(z) = F(z)/(z+i)$. Then $G \in \mathrm{Hol}(\mathcal{U})$, and

$$\begin{aligned}
\sup_{\varepsilon>0}\int_{-\infty}^{\infty} |G(x+i\varepsilon)|^2\,dx &= \sup_{\varepsilon>0}\int_{-\infty}^{\infty}\left|\frac{F(x+i\varepsilon)}{x+i(\varepsilon+1)}\right|^2 dx \\
&= \sup_{\varepsilon>0}\int_{-\infty}^{\infty}\frac{|F(x+i\varepsilon)|^2}{x^2+(\varepsilon+1)^2}\,dx \\
&\leqslant \sup_{\varepsilon>0}\int_{-\infty}^{\infty}\frac{|F(x+i\varepsilon)|^2}{x^2+1}\,dx \\
&\leqslant \sup_{\varepsilon>0}\int_{-\infty}^{\infty}|F(x+i\varepsilon)|^2\,dx < \infty.
\end{aligned}$$

Hence, $G \in H^2(\mathcal{U})$. So there exists a function $g \in H^2(\mathbb{D})$ such that $\widetilde{U_2} g = G$. Thus,

$$G(z) = \frac{1}{\sqrt{\pi}(z+i)}\cdot g\left(\frac{z-i}{z+i}\right). \tag{5.2}$$

Since $g \in H^2(\mathbb{D})$, it has a Poisson integral representation there [Rud87, pp. 244, 247]:

$$g\left(\frac{z-i}{z+i}\right) = \int_0^{2\pi} \frac{1-\left|\frac{z-i}{z+i}\right|^2}{\left|e^{i\theta}-\frac{z-i}{z+i}\right|^2} g^*(e^{i\theta})\,\frac{d\theta}{2\pi}, \tag{5.3}$$

where $g^*(e^{i\theta}) = \lim_{r\to 1} g(re^{i\theta})$ for a.e. $\theta \in [0, 2\pi]$. Let $t = i\dfrac{1+e^{i\theta}}{1-e^{i\theta}}$. Then $e^{i\theta} = \dfrac{t-i}{t+i}$, which allows one to easily see that $dt = \dfrac{2}{t^2+1}\,d\theta$. Also,

$$\begin{aligned}
\sqrt{\pi}\,(t+i)\,G^*(t) &= \lim_{\varepsilon\to 0^+} G(t+i\varepsilon)\sqrt{\pi}(t+i(\varepsilon+1)) \\
&= \lim_{\varepsilon\to 0^+} \frac{g\left(\frac{t+i(\varepsilon+1)}{t+i(\varepsilon-1)}\right)}{\sqrt{\pi}(t+i(\varepsilon+1))}\sqrt{\pi}(t+i(\varepsilon+1)) \\
&= \lim_{\varepsilon\to 0^+} g\left(\frac{t+i(\varepsilon+1)}{t+i(\varepsilon-1)}\right) \\
&= g^*\left(\frac{t+i}{t-i}\right) \\
&= g^*(e^{i\theta}).
\end{aligned}$$

Under these substitutions, (5.2) becomes

$$\begin{aligned}
G(z) &= \frac{\sqrt{\pi}}{\sqrt{\pi}(z+i)} \int_{-\infty}^{\infty} \frac{1-\left|\dfrac{z-i}{z+i}\right|^2}{\left|\dfrac{t-i}{t+i}-\dfrac{z-i}{z+i}\right|^2} G^*(t)\cdot\frac{2(t+i)}{(t^2+1)}\,\frac{dt}{2\pi} \\
&= \frac{1}{z+i}\int_{-\infty}^{\infty} \frac{|t+i|^2(|z+i|^2-|z-i|^2)}{|(t-i)(z+i)-(z-i)(t+i)|^2} G^*(t)\cdot\frac{(t+i)}{t^2+1}\,\frac{dt}{\pi} \\
&= \frac{1}{z+i}\int_{-\infty}^{\infty} \frac{|z+i|^2-|z-i|^2}{|(t-i)(z+i)-(z-i)(t+i)|^2} G^*(t)(t+i)\,\frac{dt}{\pi} \\
&= \frac{1}{z+i}\int_{-\infty}^{\infty} \frac{|z+i|^2-|z-i|^2}{|-2iz+2it|^2} G^*(t)(t+i)\,\frac{dt}{\pi} \\
&= \frac{1}{z+i}\int_{-\infty}^{\infty} \frac{|z+i|^2-|z-i|^2}{4\,|t-z|^2} G^*(t)(t+i)\,\frac{dt}{\pi}.
\end{aligned}$$

Let $z = x + iy$ to get

$$\begin{aligned} G(z) &= \frac{1}{z+i}\int_{-\infty}^{\infty}\frac{|x+i(y+1)|^2-|x-i(1-y)|^2}{4\,|t-x-iy|^2}\,G^*(t)(t+i)\,\frac{dt}{\pi} \\ &= \frac{1}{z+i}\int_{-\infty}^{\infty}\frac{x^2+(y+1)^2-x^2-(1-y)^2}{4((t-x)^2+y^2)}\,G^*(t)(t+i)\,\frac{dt}{\pi} \\ &= \frac{1}{z+i}\int_{-\infty}^{\infty}\frac{y}{(t-x)^2+y^2}\,G^*(t)(t+i)\,\frac{dt}{\pi}. \end{aligned}$$

To complete the proof, we see that

$$\begin{aligned} (t+i)G^*(t) &= (t+i)\lim_{\varepsilon\to 0^+} G(t+i\varepsilon) \\ &= (t+i)\lim_{\varepsilon\to 0^+}\frac{F(t+i\varepsilon)}{t+i(\varepsilon+1)} \\ &= (t+i)\frac{F^*(t)}{t+i} \\ &= F^*(t). \end{aligned}$$

Therefore, (5.1) follows. □

We now use Lemma 5.1 to prove the next Lemma.

Lemma 5.2. *If $f \in H^2(\mathcal{U})$, then*

$$\sup_{y>0}\int_{-\infty}^{\infty}|f(x+iy)|^2\,dx = \lim_{y\to 0^+}\int_{-\infty}^{\infty}|f(x+iy)|^2\,dx. \tag{5.4}$$

Proof. For all $x \in \mathbb{R}$ and $y > 0$, we have that

$$1 = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-t)^2+y^2}\,dt. \tag{5.5}$$

This equation is proven using a simple change of variable: $u = (x-t)/y$, which yields an integral involving the derivative of $\tan^{-1}$. Define a measure $d\mu_{x,y} = \frac{1}{\pi}\frac{y}{(x-t)^2+y^2}\,dt$.

Then using (5.1), we have

$$\begin{aligned}
|f(x+iy)| &= \left|\int_{-\infty}^{\infty} 1 \cdot f^*(t)\, d\mu_{x,y}\right| \\
&\leqslant \int_{-\infty}^{\infty} 1 \cdot |f^*(t)|\, d\mu_{x,y} \\
&\leqslant \sqrt{\int_{-\infty}^{\infty} 1^2\, d\mu_{x,y}}\sqrt{\int_{-\infty}^{\infty} |f^*(t)|^2\, d\mu_{x,y}} \\
&= \sqrt{\int_{-\infty}^{\infty} |f^*(t)|^2\, d\mu_{x,y}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
|f(x+iy)|^2 &\leqslant \int_{-\infty}^{\infty} |f^*(t)|^2\, d\mu_{x,y} \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}|f^*(t)|^2\, dt.
\end{aligned}$$

Then

$$\int_{-\infty}^{\infty} |f(x+iy)|^2\, dx \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}|f^*(t)|^2\, dt\, dx. \tag{5.6}$$

Using Fubini's Theorem and (5.5), we obtain

$$\int_{-\infty}^{\infty} |f(x+iy)|^2\, dx \leqslant \int_{-\infty}^{\infty} |f^*(t)|^2\, dt. \tag{5.7}$$

Now suppose $y_2 > y_1 > 0$. Set $\varepsilon = y_2 - y_1$, and set $g(x+i\varepsilon) = f(x+i(y_1+\varepsilon))$. Then $g \in H^2(\mathcal{U})$, and

$$\begin{aligned}
g^*(x) &= \lim_{\varepsilon\to 0^+} g(x+i\varepsilon) \\
&= \lim_{\varepsilon\to 0^+} f(x+i(y_1+\varepsilon)) \\
&= f(x+iy_1),
\end{aligned}$$

since f is continuous on $\mathcal{U}$. Applying (5.7) to g we have

$$\begin{aligned}\int_{-\infty}^{\infty} |f(x+iy_2)|^2\,dx &\leqslant \int_{-\infty}^{\infty} |f(x+i(y_1+\varepsilon))|^2\,dx \\ &= \int_{-\infty}^{\infty} |g(x+i\varepsilon)|^2\,dx \\ &\leqslant \int_{-\infty}^{\infty} |g^*(x)|^2\,dx \\ &= \int_{-\infty}^{\infty} |f(x+iy_1)|^2\,dx.\end{aligned}$$

Hence, the H^2-norm of $\mathcal{U}$ decreases monotonically as y increases; therefore, the supremum occurs when $y \to 0^+$, and (5.4) follows. □

We would like to apply (5.4) to the function $g = f/b_\nu$. However, we need to know that $g \in H^2(\mathcal{U})$. This is done in the following Lemma:

Lemma 5.3. *If $f \in H^2(\mathcal{U})$, $\nu \in \mathcal{U}$, $f(\nu) = 0$, then $g = f/b_\nu \in H^2(\mathcal{U})$ and $\|g\|_{H^2(\mathcal{U})} = \|f\|_{H^2(\mathcal{U})}$.*

Proof. It is clear that $g \in \mathrm{Hol}(\mathcal{U})$, since $b_\nu = 0$ precisely when $z = \nu$, which is a zero of $f \in \mathrm{Hol}(\mathcal{U})$ by hypothesis. To show that $g \in H^2(\mathcal{U})$, we need to show that

$$\sup_{y>0} \int_{-\infty}^{\infty} |g(x+iy)|^2\,dx < \infty. \tag{5.8}$$

First we see that $1/b_\nu$ is bounded outside a neighborhood of ν. It is clear geometrically that $b_\nu = 0$ if and only if $z = \nu$, and that outside of a neighborhood W of ν, $|b_\nu|$ is bounded below by some positive constant for all $z \in \mathcal{U} \setminus W$. This is because as the denominator of $|b_\nu|$ grows large, so does the numerator at the same rate. Hence, there exists an $M \in \mathbb{R}$ such that $|1/b_\nu(z)| \leqslant M$ for all $z \in \mathcal{U} \setminus \bar{W}$.

To describe an open square in the complex plane with center ξ and side length $2l$, we'll use the notation $C(\xi, l)$, where

$$C(\xi, l) = \{z \in \mathbb{C} : \mathrm{Re}\ z \in (\mathrm{Re}\ \xi - l, \mathrm{Re}\ \xi + l) \text{ and } \mathrm{Im}\ z \in (\mathrm{Im}\ \xi - l, \mathrm{Im}\ \xi + l)\}.$$

It is sufficient to let $W = C(\nu, \frac{1}{2}\mathrm{Im}\ \nu)$. Let $W_1 = \{y > 0 : y < \frac{1}{2}\mathrm{Im}\ \nu \text{ or } y > \frac{3}{2}\mathrm{Im}\ \nu\}$. Then since $\{x+iW_1 : x \in \mathbb{R}\} \cap \bar{W} = \varnothing$, we have that $|1/b_\nu(z)| \leqslant M$ when $\mathrm{Im}\ z \in W_1$. Thus,

$$\sup_{y\in W_1} \int_{-\infty}^{\infty} |g(x+iy)|^2\,dx \leqslant M^2 \sup_{y\in W_1} \int_{-\infty}^{\infty} |f(x+iy)|^2\,dx < \infty. \tag{5.9}$$

Let $W_2 = \{y > 0 \,|\, \frac{1}{2}\text{Im }\nu \leqslant y \leqslant \frac{3}{2}\text{Im }\nu\}$. We still need that

$$\sup_{y \in W_2} \int_{-\infty}^{\infty} |g(x+iy)|^2 \, dx < \infty.$$

This follows by writing $R = \text{Re }\nu + \frac{1}{2}\text{Im }\nu$ and

$$\int_{-\infty}^{\infty} |g(x+iy)|^2 \, dx = \int_{-\infty}^{-R} |g(x+iy)|^2 \, dx + \int_{-R}^{R} |g(x+iy)|^2 \, dx + \int_{R}^{\infty} |g(x+iy)|^2 \, dx$$

and observing that the supremum of the middle integral is finite, say it equals C, since the continuous function g is bounded on the compact set $K = \{x + iy : x \in [-R, R], y \in W_2\}$. Also, the first and last integrals are the same when $|g|$ is replaced with $|f/b_\nu| \leqslant M|f|$, since $z \notin W$. Thus,

$$\begin{aligned}
\sup_{y \in W_2} \int_{-\infty}^{\infty} |g(x+iy)|^2 \, dx &\leqslant \sup_{y \in W_2} \int_{-\infty}^{-R} |g(x+iy)|^2 \, dx + \sup_{y \in W_2} \int_{-R}^{R} |g(x+iy)|^2 \, dx \\
&\quad + \sup_{y \in W_2} \int_{R}^{\infty} |g(x+iy)|^2 \, dx \\
&\leqslant M^2 \sup_{y \in W_2} \int_{-\infty}^{-R} |f(x+iy)|^2 \, dx + C \\
&\quad + M^2 \sup_{y \in W_2} \int_{R}^{\infty} |f(x+iy)|^2 \, dx \\
&< \infty.
\end{aligned}$$

Therefore, (5.8) follows since $\{x + iW_1 : x \in \mathbb{R}\} \cup \{x + iW_2 : x \in \mathbb{R}\} = \mathcal{U}$, and hence $g \in H^2(\mathcal{U})$.

The equation (5.4) is now applicable for g, so we can write

$$\begin{aligned}
\sup_{y>0} \int_{-\infty}^{\infty} |g(x+iy)|^2 \, dx &= \lim_{y \to 0^+} \int_{-\infty}^{\infty} |g(x+iy)|^2 \, dx \\
&= \lim_{y \to 0^+} \int_{-\infty}^{\infty} \left| \frac{f(x+iy)}{b_\nu} \right|^2 \, dx \\
&= \lim_{y \to 0^+} \int_{-\infty}^{\infty} |f(x+iy)|^2 \, dx \\
&= \sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^2 \, dx.
\end{aligned}$$

□

Chapter 6

Nikolski's Theorem

6.1 The Mellin Transform

Definition 6.1. For $p \geqslant 1$ define the **Hardy space of the right half-plane** by

$$H^p(\mathcal{R}) = \left\{ f \in \mathrm{Hol}(\mathcal{R}) : \sup_{x>0} \int_{-\infty}^{\infty} |f(x+iy)|^p \, dy < \infty \right\}. \tag{6.1}$$

This space is obtained directly from $H^p(\mathcal{U})$ by the change of variable $w = iz$. The norm of this space is defined by $\|f\|^p_{H^p(\mathcal{R})} = \sup_{x>0} \int_{-\infty}^{\infty} |f(x+iy)|^p \, dy$.

The change of variable $x = e^{-t}$ gives that $dx = -e^{-t}dt$. So if $g \in L^2(\mathbb{R}_+, dx/x)$, we have

$$\int_0^{\infty} |g(x)|^2 \, \frac{dx}{x} = \int_{-\infty}^{\infty} |g(e^{-t})|^2 \, dt.$$

Then by applying the inverse Fourier transform to $g(e^{-t})$ and rotating by a factor of i, we obtain a function that is in $L^2(i\mathbb{R})$. The map that achieves all three of these transformations is called the *Mellin Transform*.

Definition 6.2. The **Mellin Transform** is a unitary mapping from $L^2(\mathbb{R}_+, dx/x)$ onto $L^2(i\mathbb{R})$ defined by

$$\mathcal{F}_* g(z) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} g(x) x^z \, \frac{dx}{x}, \qquad z \in i\mathbb{R}. \tag{6.2}$$

Consider $\mathcal{F}_* L^2((0,1), dx/x)$. The change of variable $x = e^{-t}$ above gives the space $L^2(\mathbb{R}_+, x)$. Applying the inverse Fourier transform gives the space $H^2(\mathcal{U})$, which results by combining Theorem 4.2 with Lemma 6.2.2 in [Nik02, p. 144]. Finally, the change of variable $w = iz$ rotates this space to give that

$$\mathcal{F}_* L^2((0,1), dx/x) = H^2(\mathcal{R}). \tag{6.3}$$

6.2 Nikolski's Theorem

In Nikolski's paper [Nik95], he introduces the distance function d_E, which is defined below. This idea of using a distance function is what differentiates his approach to the zeros of the Riemann ζ-function from Nyman and Beurling's approach. As mentioned in the introduction, Nyman gives a equivalent statement to the Riemann hypothesis in [Nym50], and in fact, this turns out to be a special case of Theorem 8.4.1 in [Nik02]. The flexibility that arises from using a distance function allows Nikolski to prove more general results. The theorem proved in this section (Theorem 6.6) leads to the most general result, that is, the aforementioned Theorem 8.4.1 in [Nik02]. Next we define the distance function d_E.

Definition 6.3. Let $\mathcal{H}$ be a Hilbert space and suppose $E \subset \mathcal{H}$ is a closed subspace. Define the **distance function** of E by $d_E(\lambda) = \operatorname{dist}(k_\lambda, E) = \inf_{e\in E} \|k_\lambda - e\| = \|P_E^\perp k_\lambda\|$, where $P_E^\perp k_\lambda$ is the projection of k_λ onto the orthogonal complement of E, denoted $E^\perp$.

The fact that $d_E(\lambda) = \|P_E^\perp k_\lambda\|$ comes from Hilbert space theory (see [Rud87, pp. 80–81]). In addition, we know that

$$d_E^2(\lambda) + \|P_E k_\lambda\|^2 = \|k_\lambda\|^2,$$

or equivalently,

$$\|P_E k_\lambda\|^2 \|k_\lambda\|^{-2} = 1 - d_E^2(\lambda)\|k_\lambda\|^{-2}. \tag{6.4}$$

Before we can prove the main theorem of this thesis, we need to prove a Lemma and a preliminary Theorem regarding the space $H^2(\mathcal{R})$.

Lemma 6.4. *Let F be a subspace of $H^2(\mathcal{R})$, $s \in \mathcal{R}$, and*

$$\epsilon_F^2(s) = 1 - d_F^2(s)\|k_s\|^{-2}_{H^2(\mathcal{R})},$$

where k_s is the reproducing kernel of $H^2(\mathcal{R})$. Let further $\nu \in \mathcal{R}$ be a zero of F (i.e. $f(\nu) = 0$ whenever $f \in F$). Then

$$\epsilon_F(s) \leqslant |b_\nu(s)|,$$

where $b_\nu = (z-\nu)(z-\nu_*)^{-1}$ stands for a Blaschke factor. The complex number ν_* is the reflection of ν in the imaginary axis.

Proof. If $f \in F$, the main result of that last chapter applied to $H^2(\mathcal{R})$ allows us to write $f = b_\nu g$, where $\|g\|_{H^2(\mathcal{R})} = \|f\|_{H^2(\mathcal{R})}$, and hence

$$|f(s)| = |b_\nu(s)||g(s)| = |b_\nu(s)|(g, k_s) \leqslant |b_\nu(s)|\|g\|_{H^2(\mathcal{R})}\|k_s\|_{H^2(\mathcal{R})}$$

for all $s \in \mathcal{R}$. We then deduce that

$$|b_\nu(s)| \geqslant \frac{|f(s)|}{\|f\|_{H^2(\mathcal{R})}\|k_s\|_{H^2(\mathcal{R})}},$$

or equivalently,

$$|b_\nu(s)| \geqslant \sup_{f\in F} \frac{|f(s)|}{\|f\|_{H^2(\mathcal{R})}\|k_s\|_{H^2(\mathcal{R})}}.$$

We can now write

$$|b_\nu(s)| \geqslant \sup_{\substack{g\in F\\ \|g\|=1}} |g(s)|\|k_s\|^{-1}_{H^2(\mathcal{R})} = \|\varphi_s|F\|\|k_s\|^{-1}_{H^2(\mathcal{R})} = \|P_E k_s\|\|k_s\|^{-1}_{H^2(\mathcal{R})} = \epsilon_F(s),$$

where the first equality comes from a corollary to the Hahn-Banach Theorem (see [Rud87, p. 108]), the second equality holds by Lemma 8.1.2 in [Nik02], and the third inequality holds by (6.4). □

Corollary 6.5. *Let F be a subspace of $H^2(\mathcal{R})$ and let $s \in \mathcal{R}$. Then the disk*

$$\{z \in \mathcal{R} : |b_s(z)| < \epsilon_F(s)\},$$

where $\epsilon_F^2(s) = 1 - d_F^2(s)\|k_s\|^{-2}_{H^2(\mathcal{R})}$, is free of zeros of the subspace F.

Theorem 6.6. *(Nikolski)*

Let $s \in \mathcal{R}$ and let $\gamma > 0$. Also, let

$$E_{\alpha,\gamma}(x) = x^\gamma \left(\left[\frac{\alpha}{x}\right] - \alpha\left[\frac{1}{x}\right]\right), \qquad 0 < x < 1, \tag{6.5}$$

where $0 \leqslant \alpha \leqslant 1$, and

$$d_\gamma^2(s) = \inf \int_0^1 \left| x^s - \sum_\alpha a_\alpha E_{\alpha,\gamma}(x)\right|^2 \frac{dx}{x}, \tag{6.6}$$

the inf *being taken over all finite linear combinations of $E_{\alpha,\gamma}$, $0 \leqslant \alpha \leqslant 1$. Then the disk*

$$D_{s,\gamma} = \gamma + D_s = \gamma + \left\{ z : \left|\frac{z-s}{z-s_*}\right|^2 < 1 - 2d_\gamma^2(s)\operatorname{Re} s\right\} \tag{6.7}$$

is free of zeros of the Riemann ζ-function.

Proof. We want to apply Corollary 6.5. Using the notation from the corollary, set $F = \mathcal{F}_* K_\gamma$, where

$$K_\gamma = \operatorname{span}_{L^2((0,1),dx/x)}(E_{\alpha,\gamma} : 0 < \alpha < 1).$$

Note that $d_\gamma(s) = \operatorname{dist}(x^s, K_\gamma)$. Since $\mathcal{F}_* L^2((0,1), dx/x) = H^2(\mathcal{R})$, Corollary 6.5 implies that the disk $\{z : |b_s(z)| < \epsilon_F(s)\}$ is free of zeros of the subspace F. If we rotate a Blaschke factor in the previous chapter by $-i$, we get that $|b_s(z)| = |z - s||z - s_*|^{-1}$.

We want to compute the Mellin transform $\mathcal{F}_* K_\gamma$. To do this we need to compute the following:

$$\begin{aligned}
\mathcal{F}_* E_{\alpha,\gamma}(z) &= \frac{1}{\sqrt{2\pi}} \int_0^1 x^{z+\gamma-1}([\alpha/x] - \alpha[1/x])\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_1^\infty u^{-z-\gamma+1}([\alpha u] - \alpha[u])\, \frac{du}{u^2} \\
&= \frac{1}{\sqrt{2\pi}} \int_1^\infty [\alpha u] u^{-z-\gamma-1}\, du - \frac{1}{\sqrt{2\pi}} \int_1^\infty \alpha[u] u^{-z-\gamma-1}\, du \\
&= \frac{1}{\sqrt{2\pi}} \int_\alpha^\infty [t] \left(\frac{t}{\alpha}\right)^{-z-\gamma-1} \frac{dt}{\alpha} - \frac{1}{\sqrt{2\pi}} \int_1^\infty \alpha[t] t^{-z-\gamma-1}\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_1^\infty \alpha^{z+\gamma}[t] t^{-z-\gamma-1}\, dt - \frac{1}{\sqrt{2\pi}} \int_1^\infty \alpha[t] t^{-z-\gamma-1}\, dt \\
&= \frac{1}{\sqrt{2\pi}} (\alpha^{z+\gamma} - \alpha) \int_1^\infty [t] t^{-z-\gamma-1}\, dt \\
&= \frac{1}{\sqrt{2\pi}} (\alpha^{z+\gamma} - \alpha) \frac{\zeta(z+\gamma)}{z+\gamma}. \qquad (6.8)
\end{aligned}$$

The last equality holds using Lemma 2.14.

We want to know the zeros of the subspace $F = \mathcal{F}_* K_\gamma$ of $H^2(\mathcal{R})$, that is, the common zeros of the family of functions $\mathcal{F}_* E_{\alpha,\gamma}$, $0 < \alpha < 1$, where α is our index. Using (6.8), this occurs if $\zeta(z+\gamma) = 0$ or $\alpha^{z+\gamma} - \alpha = 0$. But

$$\begin{aligned}
\alpha^{z+\gamma} - \alpha = 0 &\iff (z+\gamma)\log(\alpha) = \log(\alpha) + 2\pi i k \\
&\iff (z+\gamma-1)\log(\alpha) = 2\pi i k.
\end{aligned}$$

If for $\alpha \in (0,1)$, $k = 0$, then $z + \gamma = 1$. So the zero is cancelled out by the pole of $\zeta(z+\gamma)$. So we need only to solve

$$(z+\gamma-1)\log(\alpha) = 2\pi i k \qquad (6.9)$$

for $k \in \mathbb{Z} \setminus \{0\}$. But, for example, $\alpha_1 = 1/2^{\sqrt{2}}$ and $\alpha_2 = 1/2$ gives that $\frac{\log(\alpha_1)}{\log(\alpha_2)} = \sqrt{2}$. This implies that the solution set of (6.9) is empty. For if z_0 was a zero of F, then (6.9) must hold for all α, in particular for α_1 and α_2. This gives

$$1 - \gamma + \frac{2\pi i k_1}{\log(\alpha_1)} = 1 - \gamma + \frac{2\pi i k_2}{\log(\alpha_2)},$$

or

$$\frac{k_1}{k_2} = \frac{\log(\alpha_1)}{\log(\alpha_2)},$$

which cannot hold since $\log(\alpha_1)/\log(\alpha_2) = \sqrt{2}$. Hence the common zeros of the family $\mathcal{F}_* E_{\alpha,\gamma}$, $0 < \alpha < 1$, are $\{z : \mathrm{Re}\, z > 0, \zeta(z+\gamma) = 0\}$.

Let χ stand for the characteristic function, and let J represent the change of variable $x = e^{-t}$. We then derive

$$k_\lambda = \mathcal{F}^{-1} J \left((2\pi)^{-1/2} x^s \chi_{(0,1)}(x) \right), \tag{6.10}$$

since for $x = e^{-t}$ and $s = i\overline{\lambda}$

$$\begin{aligned}
\mathcal{F}^{-1}\left((2\pi)^{-1/2} e^{-t\bar{\lambda} i} \chi_{(0,\infty)}(x) \right) &= \left(\frac{1}{\sqrt{2\pi}} \right)^2 \int_0^\infty e^{-\bar{\lambda} ix} e^{itx}\, dx \\
&= \frac{1}{2\pi} \int_0^\infty e^{ix(t-\bar{\lambda})}\, dx \\
&= \frac{1}{2\pi i(t-\bar{\lambda})} \left(e^{ix(t-\bar{\lambda})} \right) \Big|_{x=0}^{x=\infty} \\
&= \frac{1}{2\pi i(t-\bar{\lambda})} (0-1) \\
&= \frac{1}{2\pi i(\bar{\lambda}-t)}.
\end{aligned}$$

Also,

$$\begin{aligned}
\|k_\lambda\| = \sqrt{k_\lambda(\lambda)} &= (2\pi i)^{-1/2} (\bar{\lambda} - \lambda)^{-1/2} \\
&= (2\pi i)^{-1/2} (-2i\, \mathrm{Im}\, \lambda)^{-1/2} \\
&= (4\pi\, \mathrm{Im}\, \lambda)^{-1/2} \\
&= (4\pi\, \mathrm{Re}\, s)^{-1/2}.
\end{aligned}$$

Therefore, $\|k_\lambda\|^{-2} = 4\pi\, \mathrm{Re}\, s$. Since $\mathcal{F}_*$ is an isometry, (6.10) yields

$$\begin{aligned}
\epsilon_F^2(s) = 1 - d_F^2(s) 4\pi\, \mathrm{Re}\, s &= 1 - \frac{\|P_{F^\perp} \sqrt{2\pi} k_\lambda\|^2}{2\pi} 4\pi\, \mathrm{Re}\, s \\
&= 1 - \frac{(\mathrm{dist}(K_\gamma, x^s))^2}{2\pi} 4\pi\, \mathrm{Re}\, s \\
&= 1 - \frac{d_\gamma^2(s)}{2\pi} 4\pi\, \mathrm{Re}\, s \\
&= 1 - 2 d_\gamma^2(s)\, \mathrm{Re}\, s,
\end{aligned}$$

and we have

$$D_{s,\gamma} = \gamma + D_s = \gamma + \left\{ z : \left| \frac{z-s}{z-s_*} \right|^2 < 1 - 2d_\gamma^2(s) \operatorname{Re} s \right\}$$

is free of zeros of $\zeta(s)$ by Corollary 6.5. □

The following corollary allows one to more easily conduct numerical experiments by selecting a subspace F of K_γ that is not necessarily invariant, which means that multiplication of a function in F by z need not produced a function in F. The subspace could even be one-dimensional. For an example, see [Nik95, p. 156].

Corollary 6.7. *Let F be any subspace of K_γ. Then the disk*

$$\gamma + \left\{ z : \left| \frac{z-s}{z-s_*} \right|^2 < 1 - 2d_F^2(s) \operatorname{Re} s \right\}$$

is free of zeros of the ζ-function.

Along this same vein, V. I. Vasyunin performs some sophisticated numerical experiments in [Vas95] using the function e_n, which relates to the above $E_{\alpha,\gamma}$. It is defined on $(0, \infty)$ by $e_n(x) = [1/(nx)] - (1/n)[1/x]$. Using these functions, he manages to prove that $\sum_{n=1}^{\infty} \mu(n) e_n(x) = 1$, where the convergence is pointwise and μ is the Möbius function. If this convergence can be shown to occur in $L^2(0, \infty)$, then the Riemann Hypothesis would be true. However, Vasyunin's results indicate that this may not be true. Also in [BS00, p. 135], the authors Balazard and Saias describe the "feeling" that this would be true as a "mirage." However, they proceed to deduce many questions related to this necessary and sufficient condition for the Riemann Hypothesis. Some numerical experiments related to these questions can be found in [LR02].

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Vita

Jared Ross Bunn was born in Knoxville, Tennessee on July 24, 1982. He is the son of Daryl and Debbie Bunn. Jared graduated from Cedartown High School in Cedartown, GA in 2000, and he obtained his Bachelor's degree, *magna cum laude*, in Mathematics from The University of Tennessee at Martin in May 2004. In the fall of 2004, Jared began his studies at The University of Tennessee, where he earned his Master's Degree in the summer of 2006. He plans to continue at The University of Tennessee as a doctoral student in Mathematics. Outside of mathematics, Jared is an avid guitar player and enjoys listening to music.