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# Contractible Theta Complexes of Graphs

Chelsea Marian McAmis

*University of Tennessee - Knoxville*, cplaut1@utk.edu

To the Graduate Council:

I am submitting herewith a thesis written by Chelsea Marian McAmis entitled "Contractible Theta Complexes of Graphs." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

James Conant, Major Professor

We have read this thesis and recommend its acceptance:

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Accepted for the Council:
Dixie L. Thompson

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# Contractible Theta Complexes of Graphs

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# Abstract

We examine properties of graphs that result in the graph having a contractible theta complex. We classify such properties for tree graphs and graphs with one loop and we introduce examples of graphs with such properties for tree graphs and graphs with one or two loops. For more general graphs, we show that having a contractible theta complex is not an elusive property, and that any skeleton of a graph with at least three loops can be made to have a contractible theta complex by strategically adding vertices to its skeleton.

# Contents

# List of Figures

# Chapter 1

# Introduction

In this paper we expand on recent research in combinatorial topology. We focus on homotopy types of theta complexes of graphs, which is related to the independence complex that is discussed in [4] and [5]. This paper specifically deals with properties of the graph that may determine when the theta complex is contractible. Before these results are discussed, however, a rigorous definition of the theta complex and the primary method used to calculate its homotopy type is necessary.

Given a graph $G$, define a simplicial complex $\Theta(G)$ whose vertex set is the same as that for $G$ and which has a simplex $\sigma \subset V$ if and only if the vertices that span $\sigma$ are in the complement of at least one edge of $G$. The main tool for calculating the homotopy type of $\Theta(G)$ will be discrete Morse theory, which was introduced in [3]; the next few definitions are related to this theory. Given a simplicial complex $K$ with simplices $S$, a *discrete vector field* on $K$ is a collection of pairs of simplices $(\sigma, \tau)$, called vectors, such that $\sigma \subset \tau$, $\sigma$ has one fewer vertex than $\tau$, and such that no simplex appears in more than one ordered pair. A simplex that does not appear in any vector is called a *critical* simplex, and a vector field is called a *gradient* vector field if there are no closed loops of the form: $\sigma_0, \tau_0, \sigma_1, \tau_1, ..., \sigma_n, \tau_n, \sigma_0, ...$ where $(\sigma_i, \tau_i)$ is in the vector field and $\sigma_{i+1} \neq \sigma_i$ is any subset of $\tau_i$ with one fewer vertex than $\tau_i$.

According to [2] and [3] , if $K$ is a simplicial complex with a gradient field, then it is homotopy equivalent to a cell complex with one $k$-cell for every critical $k$-simplex. In particular, if the only critical simplices are $l$ $k$-simplices and a single 0-simplex, then $\Theta(G)$ is homotopy equivalent to $\vee_l S^k$. The goal in calculating the homotopy type of $\Theta(G)$ then becomes to first define a gradient vector field on the simplicial complex, and then to examine the critical simplices. An outline of this process is given in Example 1.

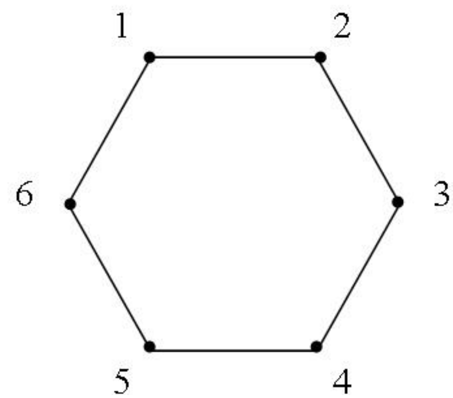


Figure 1.1: The graph of $P_6$ with vertices labeled.

**Example 1** *Consider the graph of $P_6$, or the polygon with six sides pictured in Figure 1.1. In this example we calculate the homotopy type of its theta complex $\Theta(P_6)$.*

*Label the vertices $1, 2, 3, 4, 5, 6$ in the clockwise direction. Now define a vector field by forming all valid vectors of the form $(\sigma, \sigma \cup \{1\})$ and note that the $0$-simplex $\{1\}$ is clearly critical. Also observe that the only other simplices that can be critical will be among those for which $\sigma \cup \{1\}$ hits every edge of the graph, which will only occur if $\sigma$ only avoids edges incident to $1$. The goal therefore becomes to examine all simplices that miss only edges incident to $1$ and determine which of these are critical.*

*We have the following three ways that $\sigma$ can avoid an edge incident to $1$: (1) $\sigma$ misses only the edge between $1$ and $2$; (2) $\sigma$ misses only the edge between $1$ and $6$; (3) $\sigma$ misses both of the aforementioned edges. These three cases are illustrated in Figure 1.2. Note that for each case the vertices that border the missing edge cannot be included in any simplex that is derived from this process. This observation necessitates the following terminology: Any vertex that cannot be included in any simplex derived from a case in this process is called an open vertex. Similarly, any vertex that must be included in any simplex derived from a case in this process is a closed vertex, and any vertex that may or may not be included in a critical simplex is referred to as a vertex that has not yet been determined.*

*Since $\sigma$ can only miss edges incident to $1$, any vertex bordering one of the open vertices in Figure 1.2 must be closed; Figure 1.3 illustrates this step. Now examine vertex $4$ in case (3). If vertex $4$ is open, the resulting simplex $\sigma_1$ will still satisfy the requirement that only edges incident to $1$ may be missed; similarly, the simplex $\sigma_2$ resulting from vertex $4$ being closed will also be legal. Since $\sigma_1$ has only one fewer vertex than $\sigma_2$ and since neither vertex was paired during the first step of this*

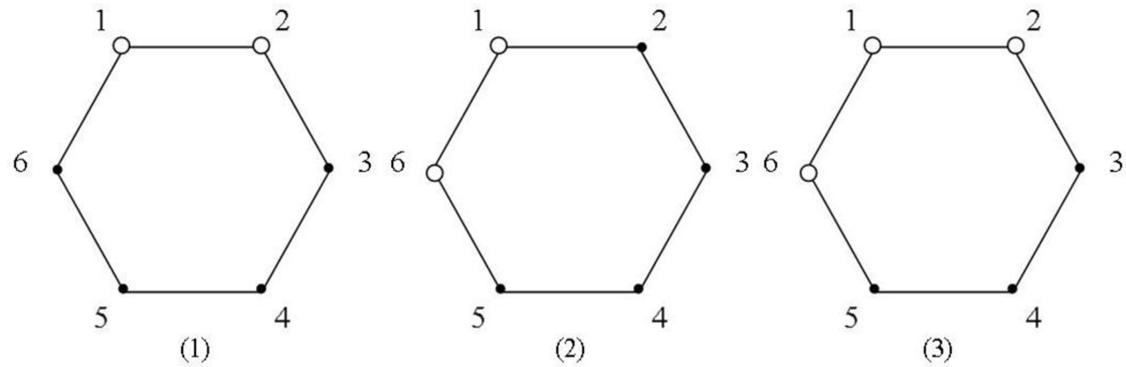


Figure 1.2: Cases (1), (2), and (3) described above. The white vertices represent open vertices and the small black vertices represent vertices that have yet to be determined.

*example, we may pair* $(\sigma_1, \sigma_2)$ *together as a vector. Therefore case (3) does not lead to any critical simplices. When this happens, we call the case contractible.*

*Next examine vertex* $4$ *in cases (1) and (2). In these cases we will also be adding more vectors to the vector field by pairing together some of the simplices that can be derived from those pictured in Figure 1.3. In both cases (1) and (2), form all valid vectors of the form* $(\sigma, \sigma \cup \{4\})$*; note again that none of the simplices derived from these cases were paired in the first step of this example, so these vectors are valid. Observe that the only way* $(\sigma, \sigma \cup \{4\})$ *can be illegal is if* $\sigma$ *misses the edge* $\{4, 5\}$ *in case (1) and* $\{3, 4\}$ *in case (2), so we are left with one unpaired vector in each case. Thus each case produces one critical simplex, both of which are pictured in Figure 1.4. Since we are left with two critical* $2$*-simplices and since this process does in fact produce a gradient vector field, we have that* $\Theta(S_6) = S^2 \vee S^2$*.*

The method used in Example 1 can be generalized to any graph $G$. Depending on the complexity of the graph the process may become more involved; this is particularly true when the graph contains vertices that are neither univalent nor bivalent, as this will result in multiple sub-cases. See [1] for other detailed examples.

Using gradient vector fields, we begin by examining the homotopy types of theta complexes of tree graphs. We then generalize our results from tree graphs to study the homotopy types of more general graphs, beginning with graphs with only one loop.

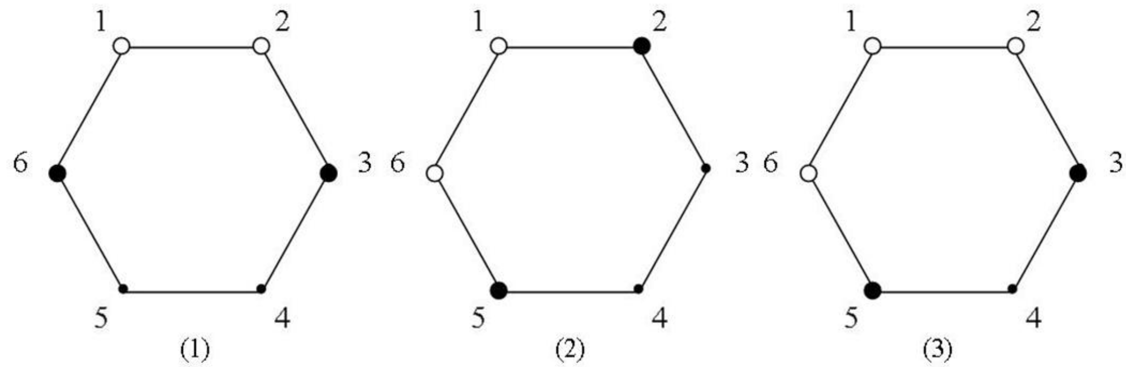


Figure 1.3: The large black vertices represent closed vertices.

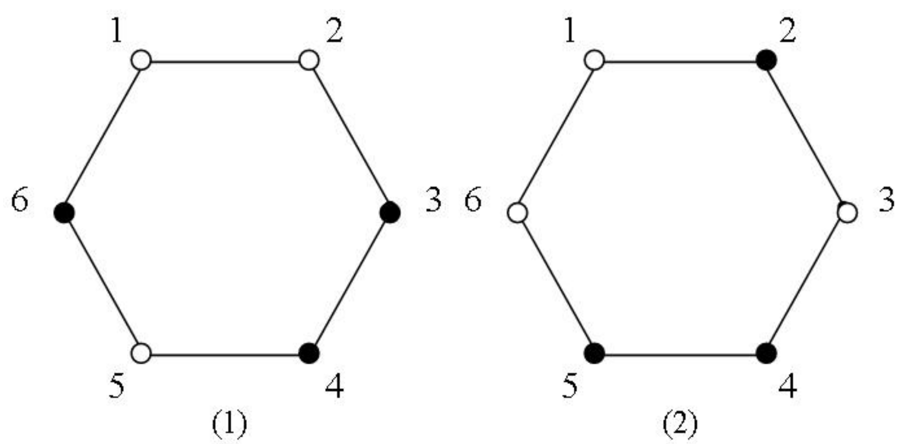


Figure 1.4: The two critical simplices derived from cases (1) and (2).

# Chapter 2

# Contractible Theta Complexes of Tree Graphs

Recall that a tree graph is a connected graph $G$ with no loops. We begin with tree graphs because their relatively simple structure leads to a correspondingly simple classification of the contractibility of their theta complexes. We first give several definitions that will prove useful in the following results.

A chain of edges in a graph $G$ has a *simple poison configuration* if it connects two univalent vertices and contains three edges. Given a simple poison configuration, one can construct the general *poison configuration* by allowing any configuration of vertices and edges to be connected to vertices 2 and 3 of the simple poison configuration, but by only allowing the univalent vertices to be connected to simple poison configurations by univalent vertices. We will also consider the graph consisting of a single vertex to be a poison configuration of length 0. An example of a simple poison configuration and a general poison configuration can be seen in Figures 2.1 and 2.2.

Let $v_1, v_2$ be vertices in a tree. Then the *distance* between $v_1$ and $v_2$ is defined to be the minimal number of edges between $v_1$ and $v_2$. Let $T$ be a tree and let $v_0$ be

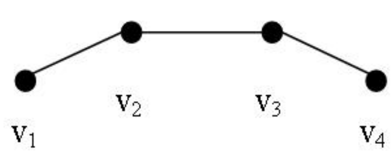


Figure 2.1: A simple poison configuration. Vertices $v_1$ and $v_4$ are univalent.

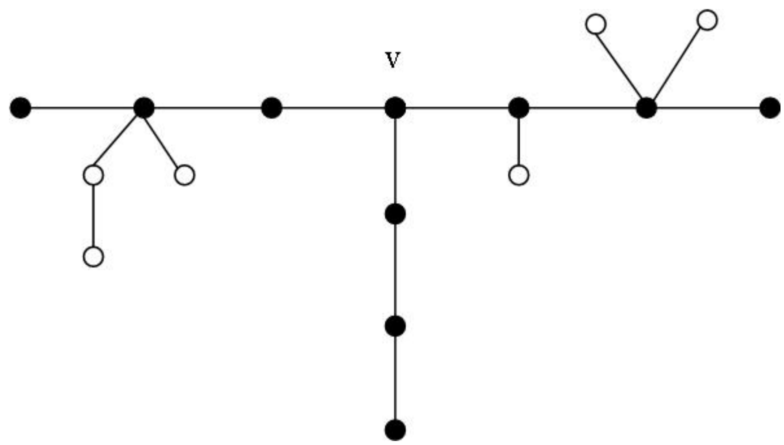


Figure 2.2: A general poison configuration. The black vertices constitute three simple poison configurations connected at the vertex $v$, which, for each simple poison configuration, is univalent. The white vertices illustrate how any configuration of vertices may be connected to vertices 2 and 3 of the simple poison configurations.

some univalent vertex of $T$; call $v_0$ the *root* of $T$. Then if all vertices of $T$ that are of distance less than or equal to two from $v_0$ are removed from $T$, we are left with a disjoint union of tree graphs $T_1, T_2, ..., T_n$, each of which is called a $v_0$-*subtree* of $T$. Lemma 2 relates the $v_0$-subtree to the contractibility of the theta complex of a tree graph.

**Lemma 2** *Let $T$ be a tree with root $v_0$. If $\Theta(T)$ is contractible then $T$ contains a $v_0$-subtree $T_i$ such that $\Theta(T_i)$ is contractible.*
***Proof.*** *Let $\Theta(T)$ be contractible and assume that for all $i$, $\Theta(T_i)$ is not contractible. First note that every $v_0$-subtree $T_i$ of $T$ is connected by its root to a vertex $v_i$ in $T$ that is of distance 2 from $v_0$. If the configuration of every $v_0$-subtree $T_i$ forces $v_i$ to be closed for all $i$, then $\Theta(T)$ is not contractible, since $v_0$ is univalent and since $\Theta(T_i)$ is not contractible for any $i$. So $v_i$ must be open for some $i$. But the only way for $v_i$ to be open is if for all $v_0$-subtrees $T_i$ that are connected to $v_i$, all vertices that are directly connected to the root of $T_i$ are closed; but this implies that the root of any such $v_0$-subtree is isolated by closed vertices. In particular, $\Theta(T_i)$ is contractible, which contradicts our assumption.* ■

Figure 2.3 illustrates Lemma 2. The next theorem fully classifies trees that have a contractible theta complex.

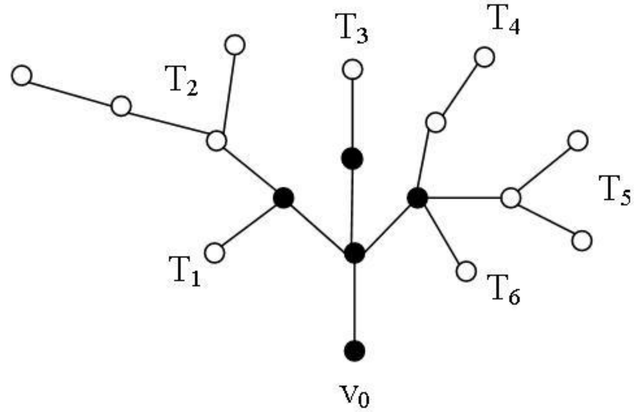


Figure 2.3: A tree graph with root $v_0$ and six $v_0$-subtrees, $T_1, T_2, ..., T_6$, which have been drawn with white vertices. The theta complex of this tree is contractible; as predicted by Lemma 2, $\Theta(T_2)$ is also contractible.

**Theorem 3** *Let $T$ be a tree. Then $\Theta(T)$ is contractible if and only if $T$ contains a poison configuration.*

***Proof.*** ($\Longrightarrow$) *Let $\Theta(T)$ be contractible and proceed by induction on the number of vertices of $T$. If $T$ has only one vertex, then $T$ has a poison configuration of length 0. Next suppose that $T$ has root $v_0$ and can be decomposed into $T_1, T_2, ..., T_m$ $v_0$-subtrees. By Lemma 2 since $\Theta(T)$ is contractible, $\Theta(T_i)$ is contractible for some $i$. Let $v_i$ be the vertex of $T$ that is connected to the root of $T_i$ and is distance 2 from $v_0$. By the inductive hypothesis, $T_i$ has a poison configuration. If none of the univalent vertices of any of the simple poison configurations of the poison configuration are connected to $v_i$, then the poison configuration of $T_i$ is also a poison configuration of $T$, and we are done. If one of the univalent vertices of one of the simple poison configurations is the root of $T_i$ and hence connected to $v_i$, then the path between the root of $T_i$ and the root of $T$ is a simple poison configuration that is connected by a univalent vertex to the univalent vertex of another poison configuration, and $T$ has a poison configuration.*

($\Longleftarrow$) *Let $T$ have a poison configuration. Begin the process of creating a discrete vector field with one of the univalent vertices of the configuration. Then the simple poison configuration that contains this univalent vertex will have vertex 2 open and vertex 3 forced to be closed. If 4 is also univalent, then we are done. Otherwise, this process can be repeated on all simple poison configurations until eventually a vertex*

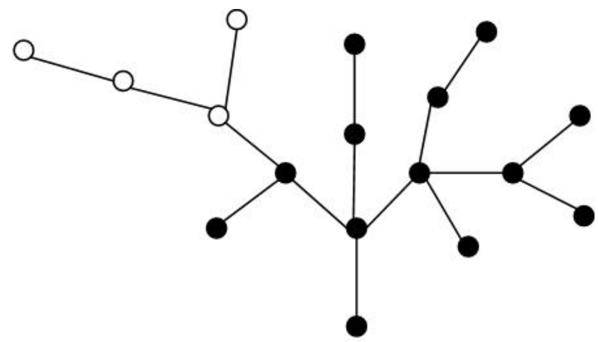


Figure 2.4: The same tree from Figure 2.3 is shown here. Again, this tree has a contractible theta complex; a poison configuration in the tree is highlighted with white vertices.

*is isolated and $T$ is contractible.* ■

An example of Theorem 3 can be viewed in Figure 2.4. With the classification of tree graphs with contractible theta complexes complete, we turn to more interesting cases. Our first goal was to experiment with poison configurations in graphs with one loop; although we could not apply Theorem 3 to graphs with one loop, a nice generalization of the theorem is discussed in Chapter 3.

# Chapter 3

# Contractible Theta Complexes of Graphs with One Loop

By Proposition 5 in [1] we know that the homotopy types of the simplest graphs with only one loop, the polygons, have been classified. Specifically, if $P_n$ is the n-gon, we have the following:

$$\Theta(P_n) \simeq \begin{cases} S^{2k-2} \vee S^{2k-2}, & n = 3k \\ S^{2k-1}, & n = 3k+1 \\ S^{2k-1}, & n = 3k+2 \end{cases}$$

Of particular interest is the fact that no theta complex of a polygon is contractible. We therefore turn our attention to more complicated graphs with one loop, which take the form of polygons whose vertices may be connected to trees.

Let $G$ be a graph with one loop. Then if we remove any vertex $v$ from the loop of $G$, we are left with a disjoint union of subgraphs of $G$. We call a subgraph $T$ of $G$ a *subtree of a looped graph* (or simply a *subtree* if the context is clear) if the only vertex $T$ shares with the loop of $G$ is $v$, which we will call the *root* of the subtree. Call a subtree $T$ of a looped graph $G$ with root $v$ a *closed subtree* if its configuration forces $v$ to be closed; that is, if the process of defining a gradient vector field is performed on $T$, then $v$ is a closed vertex regardless of the choice of initial vertex, provided that the initial vertex is a univalent vertex other than $v$. Similarly, call a subtree $T$ of a looped graph $G$ with root $v$ an *open subtree* if its configuration forces $v$ to be open in the same manner that $v$ is forced to be closed in the closed subtree definition.

Although the classification of contractible theta complexes of trees was relatively simple, the classification of graphs with one loop is relatively harder and requires several more definitions. The following definitions will introduce several important

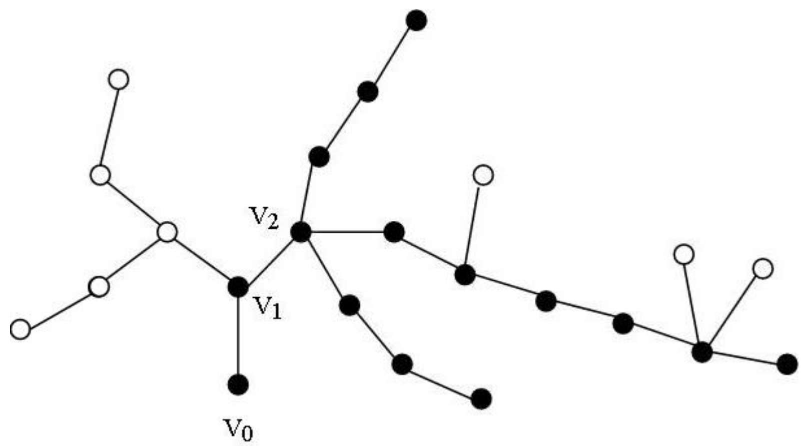


Figure 3.1: $v_0$ is the root of the tree, and $v_0, v_1, v_2$ mark the original simple closed configuration. The remaining black vertices illustrate a chain of simple closed configurations emanating from $v_2$, while the white vertices represent legal configurations that may be attatched to $v_1$.

configurations of graphs. Define a *simple closed configuration* of a tree $T$ to be a chain of three vertices $v_0, v_1, v_2$ such that $v_0$ is the root of $T$, $v_2$ is univalent, and $v_1$ is bivalent. A general *closed configuration* may be constructed in the following way: Given our simple closed configuration, allow any number of simple closed configurations to be attached by the root to $v_2$ by an extra edge; then for each of these simple closed configurations allow any number of simple closed configurations to be attached by the root to their third vertices by an extra edge, and so on. Additionally, for any of the simple closed configurations allow any configuration to be attached to the vertex corresponding to $v_1$ in the original simple closed configuration, except any configuration that would cause $v_1$ to be the root of some simple closed configuration. Any configuration of this type is a closed configuration, and an example is given in Figure 3.1.

An *open configuration* can be constructed in the same way as a closed configuration; in this case, we start with a simple open configuration, or a chain of two vertices $v_0, v_1$ such that $v_0$ is the root of $T$ and $v_1$ is univalent. Then a general open configuration is constructed by only allowing any number of closed configurations to be attached to $v_1$ by an extra edge.

Given a poison configuration of a graph $G$, one can construct the *pseudo poison configuration* in the following way: For any consecutive vertices $v_1, v_2$ in the poison

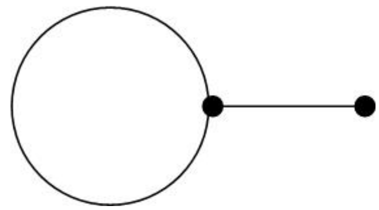

Figure 3.2: This configuration is also considered a pseudo poison configuration.

configuration, allow any number of extra edges to connect the two vertices. Then one may legally add $3k$ vertices to any of the edges connecting $v_1$ and $v_2$ and the resulting configuration will be a pseudo poison configuration; this is due to Proposition 5 in [1]. Additionally, Figure 3.2 illustrates a special configuration which will also be considered to be a pseudo poison configuration. If $v_1$ was univalent before this process, then we call $v_1$ *pseudo univalent* in the new configuration.

The next few results illustrate how the previously defined configurations are related.

**Lemma 4** *A subtree $T_i$ of a looped graph $G$ is a closed subtree if and only if it has a closed configuration.*

***Proof.*** *($\Longrightarrow$) Let $T_i$ be a closed subtree of a looped graph $G$. The root $v_1$ of our closed subtree is forced to be closed, which means that the vertex $v_2$ connected to $v_1$ is open. $v_2$ is forced to be open, which means that some vertex $v_3$ connected to $v_2$ was isolated by the process of creating a discrete vector field. $v_3$ can be isolated either if it is a univalent vertex or if, besides $v_2$, it is connected only to vertices that are forced to be closed. If it is univalent, then we are done; otherwise, every vertex connected to $v_3$ besides $v_2$ is forced to be closed. By the same logic above that we used on the chain of vertices $v_1, v_2, v_3$, we see that $T$ must have a closed configuration.*

*($\Longleftarrow$) Let $T_i$ have a closed configuration. Then it is clear to see that this configuration will force the root of $T_i$ to be closed.* ∎

**Lemma 5** *A subtree $T_i$ of a looped graph $G$ is an open subtree if and only if it has an open configuration.*

***Proof.*** *($\Longrightarrow$) Let $T_i$ be an open subtree of a looped graph $G$. The root $v_1$ of our open subtree is forced to be open, which means that the vertex $v_2$ connected to $v_1$ was isolated by the process of creating a discrete vector field. $v_2$ can be isolated either if it is a univalent vertex or if, besides $v_1$, it is connected only to vertices that are forced to be closed. If it is univalent, then we are done; otherwise, every vertex connected* *to $v_2$ besides $v_1$ is forced to be closed. By the same logic as in Lemma 4 we see that $T$ must have an open configuration.*

($\Longleftarrow$) *Let $T_i$ have an open configuration. Then it is clear to see that this configuration will force the root of $T_i$ to be open.* ■

We now begin the classification of contractible theta complexes of graphs with one loop.

**Lemma 6** *The root of a subtree $T_i$ of a looped graph $G$ is forced to be neither open nor closed if and only if $\Theta(T_i)$ is contractible.*

***Proof.*** ($\Longrightarrow$) *Let the root $v_1$ of $T_i$ be forced to be neither open nor closed. This implies that the vertex $v_2$ of $T_i$ connected to $v_1$ is closed, since if $v_2$ were open $v_1$ would be forced to be closed. But then $v_1$ is an isolated vertex, and $\Theta(T_i)$ is contractible.*

($\Longleftarrow$) *Let $\Theta(T_i)$ be contractible. Assume that the root $v_1$ is forced to be closed, i.e., that $T_i$ has a closed configuration. But $\Theta(T_i)$ is contractible, so $T_i$ must also have a poison configuration. Since simple closed configurations all have two edges, the poison configuration must involve vertices and edges that are not contained in the chain of simple closed configurations. The only vertices in the chain of simple closed configurations that can be connected to configurations besides simple closed ones are those vertices that are forced to be open by the configuration. Let $v$ be one such vertex. Then we have three cases:*

1. *Assume $v$ is part of the configuration, and that it connects a univalent vertex outside the closed configuration to a univalent vertex inside the closed configuration. Then since the distance between any univalent vertex of the closed configuration and $v$ is $3k+1$, the poison configuration must end in a chain $v - v_2 - v_3$ with $v_3$ univalent and $v_2, v_3$ outside the closed configuration. This, however, violates a requirement for a closed configuration. Figure 3.3 illustrates this case.*

2. *Assume $v$ is part of the configuration, but connects two univalent vertices $v_2, v_3$ outside of the closed configuration. This implies that the other univalent vertex is connected to some open vertex $v'$ (which may also be $v$) in the closed configuration. Since the distance between any two open vertices in the closed configuration is $3k$, and since the distance between any univalent vertex in the closed configuration and an open one is $3k+1$, this violates a requirement for a poison configuration. Figure 3.4 illustrates this case.*

3. *Lastly assume that the poison configuration is connected to $v$, but $v$ is not part of the poison configuration. Then $v$ must be connected to vertex 2 or 3 of a*

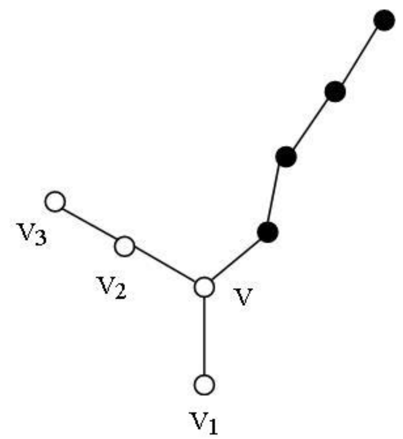


Figure 3.3: An example of case (1). The poison configuration is highlighted in white; notice that the configuration connected to $v$ forces $v$ to be closed, which violates the requirement for a closed configuration.

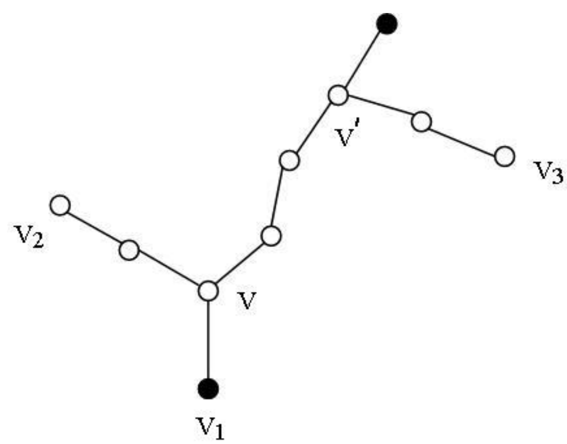


Figure 3.4: An example of case (2). The poison configuration is highlighted in white; notice again that the requirement for $v$ is violated and this is not a closed configuration.

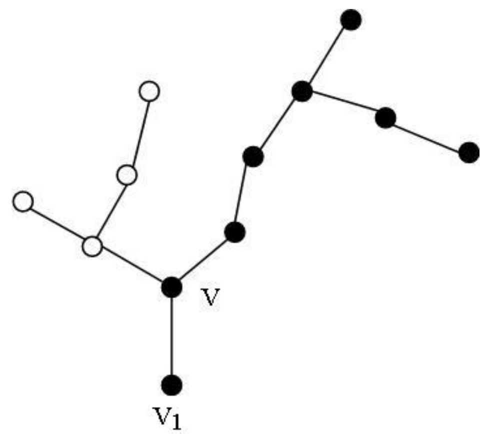


Figure 3.5: An example of case (3). The poison configuration is highlighted in white; note that once again the configuration connected to $v$ violates the requirement for a closed configuration.

*simple poison configuration, which would similarly create a configuration that contradicts a requirement for a closed configuration. Therefore $v_1$ cannot be forced closed. Figure 3.5 illustrates this case.*

*Next assume that $v_1$ is forced to be open, i.e., $T_i$ has an open configuration. Since $\Theta(T_i)$ is contractible, $T_i$ also has a poison configuration. By the same reasons as above, the poison configuration must involve vertices connected to some $v$ distinct from $v_1$ that is forced open by the open configuration. Then we have three cases that are almost identical to those above, all of which give rise to a contradiction. Therefore $v_1$ cannot be forced open, and must be forced to be neither open nor closed.*
■

**Lemma 7** *A vertex $v$ contained in the loop of a looped graph $G$ is forced to be closed if and only if at least one subtree of the looped graph that has $v$ for a root is closed, and all other subtrees that share $v$ as a root are either closed or contractible subtrees.*
***Proof.*** ($\Longrightarrow$) *Let $v$ be forced closed. Then at least one subtree with $v$ for a root must be a closed subtree. If there is an open subtree $T_i$ that has $v$ for a root, then depending on the choice of starting vertex, $v$ may be forced open or closed. Therefore all subtrees with $v$ as a root must be closed or contractible.*

($\Longleftarrow$) *Let one subtree of the looped graph that has $v$ for a root be closed, with all other subtrees that share $v$ as a root closed or contractible. Note that the contractible* *trees do not force $v$ to be open or closed. Since at least one subtree is closed and all others are either closed or contractible, $v$ is forced closed.* ■

**Lemma 8** *A vertex $v$ contained in the loop of a looped graph $G$ is forced to be open if and only if at least one subtree of the looped graph that has $v$ for a root is open, and all other subtrees that share $v$ as a root are either open or contractible subtrees.*
***Proof.*** ($\Longrightarrow$) *Let $v$ be forced open. Then at least one subtree with $v$ for a root must be an open subtree. If there is a closed subtree $T_i$ that has $v$ for a root, then depending on the choice of starting vertex, $v$ may be forced open or closed. Therefore all subtrees with $v$ as a root must be open or contractible.*

($\Longleftarrow$) *Let one subtree of the looped graph that has $v$ for a root be open, with all other subtrees that share $v$ as a root open or contractible. Note that the contractible trees do not force $v$ to be open or closed. Since at least one subtree is open and all others are either open or contractible, $v$ is forced open.* ■

**Lemma 9** *A vertex $v$ contained in the loop of a looped graph $G$ is forced to be neither open nor closed if and only if any subtrees of the looped graph that have $v$ for a root are contractible or if both a closed subtree and an open subtree have $v$ as a root.*
***Proof.*** ($\Longrightarrow$) *Let $v$ be forced to be neither open nor closed. Then if there are only either closed or open subtrees with $v$ as a root (in addition to possible contractible subtrees) but not both, then $v$ would be forced closed or open, so all subtrees with $v$ as a root must be contractible or both an open and a closed subtree must be present.*

($\Longleftarrow$) *Let all subtrees of the looped graph with $v$ for a root be contractible. Then none of the subtrees force $v$ to be open or closed, and so $v$ is forced to be neither open nor closed. Let both an open and a closed subtree have $v$ for a root. Then $v$ is forced neither to be open nor closed.* ■

The following lemma connects the poison configuration from Chapter 2 to the one-loop case, in that it shows that graphs with one loop that have contractible theta complexes appear to "contain" a poison configuration.

**Lemma 10** *Let $G$ be a graph with one loop such that it does not have the pseudo poison configuration featured in Figure 3.2 and such that if a subtree $T_i$ of $G$ has poison configuration, then the poison configuration contains the root of $T_i$. Furthermore, assume that no vertex of the loop of $G$ is the root of both a closed and an open subtree. Then $\Theta(G)$ is contractible if and only if after filling in all of the subtrees of $G$ according to the process of creating a gradient vector field and closing all vertices surrounding the roots of open subtrees, the remaining empty configuration is a poison configuration or $G$ has two open subtrees with consecutive roots.*

***Proof.*** ($\Longrightarrow$) *Let $\Theta(G)$ be contractible. Since $G$ does not have the pseudo poison configuration shown in Figure 3.2, any poison configuration of a subtree $T_i$ of $G$ contains the root of $T_i$, and no vertex of the loop is the root of both an open and a closed subtree, it is clear that the contractibility of $\Theta(G)$ must depend on the configuration of its loop. Therefore we must consider the remaining empty configuration after filling in all subtrees and closing all vertices surrounding the roots of open subtrees. Assume that this remaining empty configuration is not a poison configuration and that $G$ does not have two open subtrees with consecutive roots. Then since this remaining configuration is necessarily a tree, by filling in the remaining empty configuration we see that we are left with a nontrivial critical simplex, a contradiction. Therefore this remaining configuration must have a poison configuration or $G$ has two open subtrees with consecutive roots.*

($\Longleftarrow$) *Let the remaining empty configuration be a poison configuration. Then after filling in this configuration we are left with a vertex that is isolated by closed vertices and $\Theta(G)$ must be contractible. Let $G$ have two open subtrees with consecutive roots. Then $\Theta(G)$ is clearly contractible.* ■

Finally, we introduce the theorem that classifies the contractibility of theta complexes for graphs with one loop.

**Theorem 11** *Let $G$ be a graph with one loop. Then $\Theta(G)$ is contractible if and only if $G$ has a pseudo poison configuration.*

***Proof.*** ($\Longrightarrow$) *Let $\Theta(G)$ be contractible. If $G$ has a subtree with a poison configuration that does not involve its root, then we are done. Similarly, if $G$ has the pseudo poison configuration from Figure 3.2, then we are done. Lastly, if one of the vertices of the loop of $G$ is the root of both an open and a closed subtree, then the structure of both of these trees creates a poison configuration. So assume $G$ has none of the above configurations. Then by Lemma 10 either $G$ has two consecutive open subtrees or the remaining empty configuration after filling in all subtrees of $G$ and closing all vertices surrounding the roots of open subtrees has a poison configuration. If $G$ has two consecutive open subtrees then the structure of the open subtrees plus the edge between their roots creates a poison configuration. Therefore let us assume that the remaining configuration of $G$ has a poison configuration. We have the following two cases:*

1. *Let the poison configuration be bordered by two closed vertices. If these vertices are the roots of closed subtrees, then the structure of the closed subtrees plus the $3k + 2$ edges between them creates a poison configuration. If these vertices are closed because they border the roots of open subtrees, then the structure of the*

*open subtrees plus the* $3k+1$ *edges between them creates a poison configuration. If these vertices are closed because one of them borders an open subtree and the other is the root of a closed subtree, then the structure of the open and closed subtrees and the* $3k$ *edges between them creates a poison configuration.*

2. *Let the poison configuration be formed by only one vertex of the loop forced to be closed and all other vertices forced to be neither open nor closed. Then since the remaining configuration is a poison configuration, this implies that the loop is a polygon with* $3k+2$ *edges. Then since the closed vertex must be the root of a closed subtree, the structure of the closed subtree plus the configuration of the loop imply that* $G$ *has a pseudo poison configuration.*

($\Longleftarrow$) *Let* $G$ *have a pseudo poison configuration. Then, according to Proposition* 5 *in [1], since the extra edges that make the pseudo poison configuration more than a poison configuration do not affect the process described in Theorem 3, we see that* $\Theta(G)$ *is clearly contractible.* ■

The results in this chapter seem to indicate that as the complexity of the graph grows, so does the list of configurations that force the theta complex to be contractible. Indeed, we will see that this tendency continues as we add more loops to the graphs. One trend that will change, however, is the configurations themselves. Up until now the poison configurations that have been added to the list are more or less generalizations of those that were introduced in Chapter 2. We will soon see that entirely new configurations will need to be added to the list in Chapter 4.

# Chapter 4

# Contractible Theta Complexes of General Graphs

Once the classification of the zero- and one-loop cases were complete, the natural next step was to examine the two-loop case. The initial goal was to see if Theorem 11 could be extended to include graphs with two loops, and ultimately to graphs with an arbitrary number of loops. After a systematic examination of graphs with two loops, however, the graph in Figure 4.1 was shown to have a contractible theta complex. This graph does not have a pseudo poison configuration, nor does its configuration follow from any pattern that had been previously discussed.

As with the zero- and one-loop cases, one could continue to classify the contractible theta complexes of graphs for the $n$-loop case by systematically outlining certain configurations that force the theta complex of a graph to be contractible. As one can especially see from the two-loop case, however, the configurations tend to become significantly more complicated with every case. This increase in complexity is not without explanation, however; the final result of this paper reveals that contractibility of the theta complex of a graph is not necessarily an elusive property.

Before we discuss this idea further, consider the graphs in Figure 4.2. These represent the three possible skeletons of a graph with two loops. Any graph with two loops will have one of these three basic shapes, plus extra vertices on the edges shown and possibly trees attached to those vertices. Graphs with two loops that have trees will not, however, be discussed for the following reason: If a graph with two loops has a tree connected to one of its vertices, then we have two cases. First, the tree may be a contractible subtree; in this case it will not affect the contractibility of the theta complex of the whole graph. On the other hand, if the tree is a closed or an open subtree, then after starting the gradient vector field on the tree the remaining

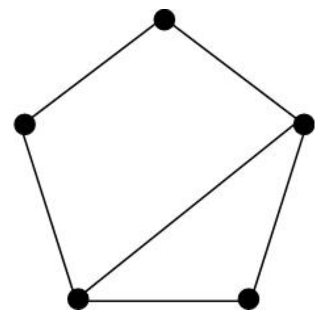

Figure 4.1: A graph with two loops and a contractible theta complex that does not contain a pseudo poison configuration.

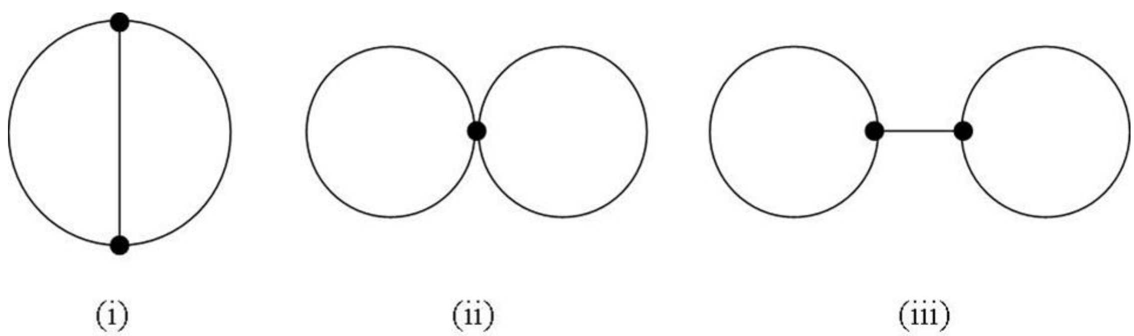


Figure 4.2: The three possible skeletons of a graph with two loops.

empty configuration will be a graph with one loop; thus the contractibility of its theta complex falls into a category in the previous chapter.

Therefore consider the three skeletons in Figure 4.2. For each of these cases we can find a graph with a contractible theta complex; these examples are shown in Figure 4.3. This fact led to the discovery that, given any skeleton for a graph with at least three loops, one can systematically place vertices on the edges of the graph to force it to have a contractible theta complex. This proved that the contractibility of a theta complex is not an elusive property requiring only a small number of very specific configurations. Theorem 13 proves this in general, but we first present a specific example of this process.

**Example 12** *Consider the graph in Figure 4.4; it represents the skeleton of a graph with four loops. The goal of this example is to systematically add vertices to the*

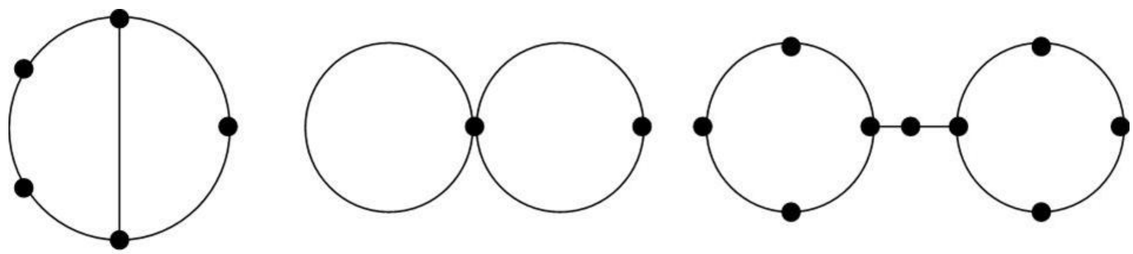


Figure 4.3: Enough vertices can be added to each skeleton to make a graph with a contractible theta complex.

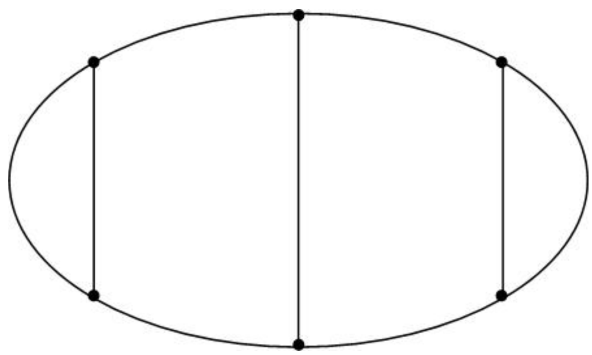

Figure 4.4: The skeleton of a graph with four loops.

*skeleton until we have devised a graph with a contractible theta complex.*

*First add two vertices as shown in Figure 4.5. Next label one of the new vertices as $v$ and begin creating the gradient vector field by using $v$ as the starting vertex. We will first deal with the case where the edge between the two newly added vertices is avoided; Figure 4.6 shows how these vertices must be filled in at this stage.*

*We add one more vertex to the graph on the edge between the two closed vertices, as shown in Figure 4.7. This forces this case to be contractible, as the newly added vertex will be isolated between two closed vertices. This concludes the first case.*

*Note that at this point no vertex that was added in case (1) may be changed and no more vertices may be added to the edge between two vertices that were fixed in case (1). For the second case, the other edge adjacent to $v$ is avoided, while the edge in case (1) is not. We fill in the fixed vertices according to the gradient vector field, as*

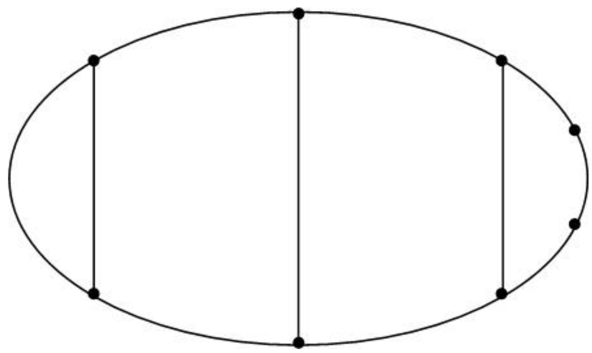

Figure 4.5: We have added two vertices to the graph.

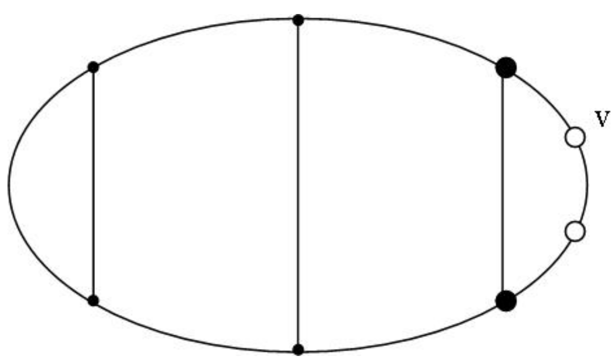


Figure 4.6: Vertex $v$ is the starting vertex as we create the gradient vector field. The large black and white vertices are now fixed and have been filled in as dictated by the gradient vector field.

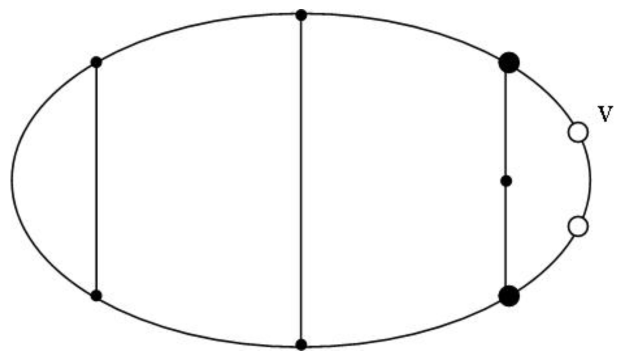


Figure 4.7: The final vertex that will be added in case (1). Note that since this vertex is isolated between two closed vertices, this case will be contractible.

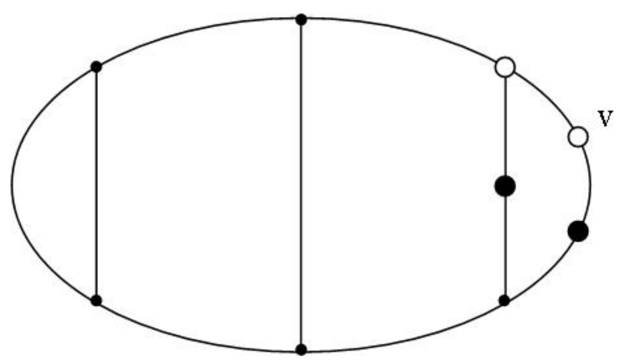


Figure 4.8: We fill in the vertices fixed in case (1) as dictated by the second case in the gradient vector field.

*illustrated in Figure 4.8. To conclude case (2), we add the vertices shown in Figure 4.9. Note that once again a vertex is isolated between two closed vertices and this case is also contractible.*

*We repeat the procedure for the third case, in which both edges incident to $v$ are avoided. Figure 4.10 shows the vertices that are added in this case and how they are filled in according to the gradient vector field. As before, we have isolated a vertex between two closed vertices, and this case is also contractible. Since all three cases are contractible, we have indeed acheived our goal of constructing a graph with the skeleton introduced in Figure 4.4 that has a contractible theta complex.*

We now present the theorem illustrated by Example 12.

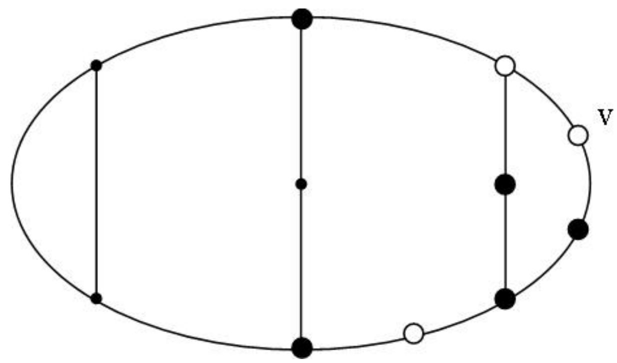


Figure 4.9: All vertices to be added in case (2); note that once again we have a vertex isolated between two closed vertices.

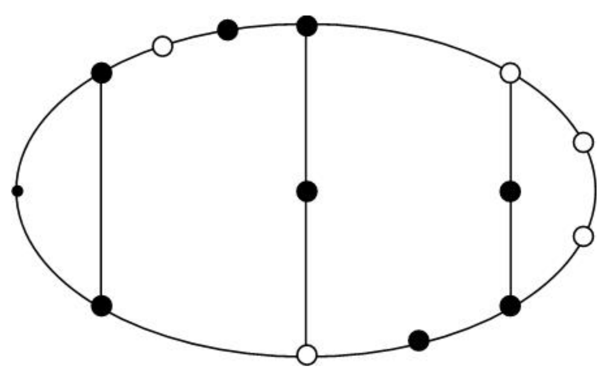

Figure 4.10: The final stage of the third case. As before, we again have an isolated vertex. The vertices shown here are exactly those that will be included in the graph that was constructed in Example 12.

**Theorem 13** *Let $G$ be a graph with at least three loops. Then there exists a graph $G'$ with a contractible theta complex that can be obtained from $G$ by adding a finite number of vertices to the edges of $G$.*

***Proof.*** *Let $G$ be a graph with at least three loops and choose a loop. Choose a bivalent vertex $u$ on the loop; if such a vertex does not exist on the chosen loop, add one. Proceed to create the gradient vector field as previously described in this paper, starting with $u$. Since $u$ is bivalent we start the process of creating the gradient vector field with three cases; pick one case to start with.*

*Choose another bivalent vertex $v$ on the loop; if no other bivalent vertex exists, add one. Starting on one side of $u$, label all vertices that are not bivalent and that lie on the loop between $u$ and $v$ as $v_1, v_2, ..., v_n$. If no such vertices exist, then add enough bivalent vertices between $u$ and $v$ so that the vertex that is consecutive to $v$ on our chosen side is forced closed. Otherwise, add enough vertices to the chosen side so that between each consecutive pair $v_i, v_{i+1}$ there are $3k+2$ vertices. Add enough vertices between $u$ and $v_1$ so that $v_1$ is closed. Then all $v_i$ will be forced closed. Finally add enough vertices so that there are $3k$ vertices between $v_n$ and $v$. Repeat this exact procedure for the other side of the loop. Now $v$ is isolated between two closed vertices, and this first case is contractible.*

*Proceed to the second case and fill in all of the vertices in the first loop, which is now determined and cannot be changed. Choose a neighboring loop and choose a bivalent vertex $v$ on the loop that has not yet been determined. If none exist, add one. Label the two vertices that are closest to $v$ that have already been determined as $u_1$ and $u_2$. Beginning with $u_1$, label all vertices between $u_1$ and $v$ that are not bivalent as $v_1, v_2, ..., v_n$. If no such vertices exist, then add enough bivalent vertices between $u_1$ and $v$ so that the vertex that is consecutive to $v$ on this side is forced closed. Otherwise, add enough vertices so that between each consecutive pair $v_i, v_{i+1}$ there are $3k+2$ vertices. Add enough vertices between $u_1$ and $v_1$ so that $v_1$ is closed. Then all $v_i$ will be forced closed. Finally add enough vertices so that there are $3k$ vertices between $v_n$ and $v$. Repeat this exact procedure beginning with $u_2$. Then $v$ is isolated between two closed vertices, and this second case is contractible.*

*For the third case, fill in all of the vertices in the first and second loops, which are now determined and cannot be changed. Then repeat the exact same procedure outlined for the second loop for the third loop. This will force the third case to also be contractible, and our desired graph $G'$ is defined.* ■

Theorem 13 concludes our findings. Future research on this topic may include an exploration of graphs whose theta complexes are not necessarily contractible, such as the classification of the homotopy types of theta complexes of certain families of graphs. For instance, line graphs and polygons have already been classified in [1]; a future project may involve the classification of more complicated families.

# Bibliography Section

# Bibliography

[1] Conant, James and Thistlethwaite, Oliver, Boolean formulae, hypergraphs and combinatorial topology, Topology Appl. 157 (2010), no. 16, 2449–2461.

[2] Forman, Robin, A user's guide to discrete Morse theory, Sem. Lothar. Combin. 48 (2002), Art. B48c.

[3] Forman, Robin, Morse theory for cell complexes, Adv. Math. 134 (1) (1998) 90-145.

[4] Jonsson, Jakob, Simplicial Complexes of Graphs, Springer-Verlag, 2008.

[5] Kozlov, D.M., Complexes of directed trees, J. Combin. Theory Ser. A 88(1) (1991) 112-122.

[6] Thistlethwaite, Oliver, Masters thesis, University of Tennessee, 2007.

# Vita

Chelsea McAmis was born in Falls Church, Virginia to parents Conrad and Barbara Plaut. She is the youngest of two sisters, following Lena Plaut. She attended Maryville High School in Maryville, Tennessee and graduated in fall of 2007. After graduation she was accepted to the University of Tennessee, where she pursued a math major and a music minor. While at UT Chelsea was very involved in the math department. She was a member of the Math Honors Program, a math tutor, and she gave regular talks at events such as the UT Honors Symposium and the UT Undergraduate Math Conference.

In the summer of 2009 she was accepted into the Wabash Summer Institute of Mathematics and participated in the REU program there. During her eight week stay at Wabash College, Chelsea conducted research in the combined areas of graph theory and commutative algebra; her research resulted in her co-authoring two papers, both of which are now published. One paper appeared in the Rose-Hulman Undergraduate Math Journal in 2010 and the other appeared in Communications in Algebra in 2011.

When Chelsea graduated from the UT in August 2011, she was named the Outstanding Graduate in Mathematics of 2011. After receiving a teaching assistantship at UT, she continued her studies there in the form of the fifth year Master of Science degree. She graduated with her master's in August 2012 and will continue her education at the University of Virginia with a teaching assistantship in fall 2012.