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Transient Response of Linear Elastic Structures Determined by the Matrix Exponential Method

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To the Graduate Council:

I am submitting herewith a thesis written by William Christopher Terrill Stoddart entitled "Transient Response of Linear Elastic Structures Determined by the Matrix Exponential Method." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mechanical Engineering.

John E. Akin, Major Professor

We have read this thesis and recommend its acceptance:

L. Raymon Shobe, Paul Nelson, Jr.

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

October 13, 1970

To the Graduate Council:

I am submitting herewith a thesis written by William Christopher Terrill Stoddart entitled "Transient Response of Linear Elastic Structures Determined by the Matrix Exponential Method." I recommend that it be accepted for nine quarter hours of credit in partial fulfillment of the requirements for the degree of Master of Science, with a major in Engineering Mechanics.

John E. Akin
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We have read this thesis
and recommend its acceptance:

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TRANSIENT RESPONSE OF LINEAR ELASTIC STRUCTURES DETERMINED BY THE MATRIX EXPONENTIAL METHOD

A Thesis

Presented to

the Graduate Council of

The University of Tennessee

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

William Christopher Terrill Stoddart

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ABSTRACT

This investigation was undertaken to develop a numerical solution for the transient response of linear, elastic structures based on the matrix exponential solution for first order, linear, constant coefficient differential equations. The investigation was prompted by the need for an economical technique that can be used to analyze multi-degree of freedom systems exemplified by piping and structural components associated with nuclear power plants.

A mathematical model characterizing the behavior of linear, elastic structures was developed by using state variables of displacement and velocity. The structure consists of beam elements of uniformly distributed mass, weightless springs, and rigid masses. The stiffness and mass matrices for the beam elements and techniques for treating boundary conditions were investigated. A digital computer program was written to perform the transient solution. The transient response was determined for three simple structures by using the computer program, and the results obtained agree favorably with previously reported analytical and experimental data.

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1. INTRODUCTION

Several areas in structural design confronting the nuclear industry can generally be classified as transient or time varying. Examples of these are aseismic design, emergency action such as blowdown, or accidents involving the shipment of radioactive material. Designers must consider the circumstances and consequences of the situation and take appropriate steps to insure safe operation of the system involved. In doing so, the designer faces several difficulties: the time available to obtain a solution is limited, the problems can generally be classified as complex, and the assumptions made to obtain a model that can be readily analyzed may greatly affect the answers obtained. Fortunately, large and fast digital computers have become widely available, and this availability results in some reduction of the difficulties caused by limited time.

Several methods are currently used to develop a model of the physical system and to select a solution technique. Quite often, the structure is modeled as a collection of rigid masses and weightless springs. An alternate choice involves finite element methods to minimize error. When selecting a solution technique, the designer must decide what information is to be obtained as a result of the analysis. This may be a complete time history of displacements or simply estimates of the maximum relative displacements. If only estimates of maximum relative displacement are required, the widely known modal superposition methods in combination with a response spectrum may be used. If a complete time history is required, some form of integration of the equations of motion will be needed. If an economical, easy to use, and accurate method for performing the direct integration were available, this technique would appear to be the logical choice under all circumstances in that all the data of interest to the designer would be available in the results of the analysis.

One of the many possible numerical procedures is presented in the following sections of this document. The findings of a literature review relative to methods for determining the transient response of multi-degree of freedom systems are discussed in Section 2. A mathematical model for a complete structure is developed in Section 3, and a derivation of the stiffness and mass matrices which describe a single beam element of the structure is presented in Section 4. The development of a computer program for the matrix exponential solution is described in Section 5, use of this computer program is demonstrated in Section 6, and the conclusions and recommendations resulting from this investigation are presented in Section 7.

2. REVIEW OF LITERATURE

Interest in the transient response of linear, elastic mechanical systems occurs in many fields. However, the literature surveyed in the course of this investigation was limited primarily to research documents sponsored by the United States Atomic Energy Commission and the National Aeronautics and Space Administration and to standard textbooks.

Most current methods for determining the transient response of multi-degree of freedom systems may be separated into two categories. The first is superposition of modal response patterns, and the second is direct integration. The application of both of these methods is illustrated in a recent review of seismic design analysis methods (1)* wherein a linear elastic structural model is formulated by either the lumped parameter or finite element method and the modal analysis technique is recommended for computing both steady state and transient dynamic responses.

The dynamic equations for linear, elastic mechanical structures are characterized by constant coefficients and may be quite readily expressed in matrix form. Since these equations are second order, the solution algorithms generally found in textbooks do not fully exploit the constant coefficient characteristic. The matrix exponential method has been presented (2) as a means of solving a set of first order differential equations that are constant coefficient and linear. This method

*Numbers within parentheses in the text designate numbered references given in the List of References.

has recently received wide attention because of the availability of digital computers. Numerical techniques used in the time domain and in the frequency domain analyses of linear time-invariant systems have been reported by M. L. Liou (3,4). A bound for round-off error involved in digital computation of the transition matrix of a system of linear time-invariant differential equations has been developed and a method of computer selection of the step size and number of series terms in transition matrices has been presented by J. B. Mankin, Jr., and J. C. Hung (5,6).

A technique for determining the transient response of structures that is based on a Taylor series expansion for displacement and velocity has been presented by A. Craggs (7,8). However, the solution presented was developed only for simple mechanical systems, and the definite relation to the matrix exponential method was not presented. The dynamic equations are rewritten as a coupled set of first order equations in Section 3 of this thesis, and it is shown that the solution methods presented by Craggs (7,8) are simply an approximation to the matrix exponential solution.

3. MATHEMATICAL MODEL FOR A COMPLETE STRUCTURE

In order to apply the matrix exponential solution method to the problem of determining transient structural response, the equations of motion for the structure must be written as a coupled set of first order, linear differential equations. Since only linear elastic structures are considered in this investigation, these equations will have constant coefficients. The equations of motion for the structure are presented in a form compatible with the matrix exponential method in this section, and the matrix exponential solution for these equations is derived.

3.1 Dynamic Equations

The equations of motion for a multi-degree of freedom system may be conveniently written in matrix equation form as

$$M\ddot{x} + C\dot{x} + Kx = f(t) \ , \tag{3.1}$$

where

M is the structure mass matrix,

C is the structure damping matrix,

K is the structure stiffness matrix,

x is the structure displacement vector,

$\dot{x}$ is the structure velocity vector,

$\ddot{x}$ is the structure acceleration vector, and

f(t) is the time varying vector of applied loads.

Unless noted otherwise, capital letters are used to denote matrices and lower-case letters are used to denote vectors and scalars. Where necessary to improve clarity of presentation, brackets, $[\;]$, and braces, $\{\;\}$, are also used to denote matrices and vectors.

3.2 Introduction of State Variables

To mathematically simplify the dynamic equations, it is desirable to develop a set of coupled first order differential equations that is equivalent to the set of second order differential equations. This may be accomplished by solving explicitly for the acceleration vector in Equation 3.1 and incorporating an identity relationship involving the velocity vector. Solving Equation 3.1, the acceleration vector

$$\ddot{x} = -M^{-1}C\dot{x} - M^{-1}Kx + M^{-1}f(t) \;, \tag{3.2}$$

where the superscript -1 denotes inversion. The necessary identity is

$$\dot{x} = I\dot{x} \;, \tag{3.3}$$

where I is the identity matrix. By combining Equations 3.2 and 3.3, the following set of first order coupled differential equations is obtained.

$$\begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} = \begin{bmatrix} \Phi & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} + \begin{Bmatrix} \phi \\ M^{-1}f(t) \end{Bmatrix} \;, \tag{3.4}$$

where Φ and ϕ denote the null matrix and null vector, respectively.

3.3 Matrix Exponential Solution

For the free vibration case, $f(t) = \phi$, the solution to Equation 3.4 is as follows. Let

$$A = \begin{bmatrix} \Phi & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} . \tag{3.5}$$

Integrating from time t to t + τ yields

$$\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}_{t + \tau} = \begin{bmatrix} \exp A\tau \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}_{t} , \qquad (3.6)$$

where

t is time,

τ is the time increment, and

[exp Aτ] is the matrix exponential function of A.

The subscripts t and τ are used to denote the point of evaluation in time.

A complete development of this solution has been presented by Zadeh and Desoer (2, Chapter 5). A less rigorous proof is as follows. Let

$$\dot{y} = By \qquad (3.7)$$

represent any linear, constant coefficient set of coupled differential equations. Then

$$\begin{aligned} \ddot{y} &= B\dot{y} \\ &= B^2 y \ . \end{aligned} \qquad (3.8)$$

Similarly,

$$\frac{d^3y}{dt^3} = B^3 y \ , \qquad (3.9)$$

and

$$\frac{d^m y}{dt^m} = B^m y \ . \qquad (3.10)$$

Let y be expanded in a Taylor series at time t + τ.

$$y_{t + \tau} = y_t + \tau\dot{y}_t + \frac{\tau^2}{2!}\ddot{y}_t + \ldots + \frac{\tau^m}{m!}\left.\frac{d^m y}{dt^m}\right|_t + \ldots \qquad (3.11)$$

After substituting the appropriate derivatives, Equation 3.11 becomes

$$y_{t+\tau} = \left[I + \tau B + \frac{\tau^2}{2!} B^2 + \ldots + \frac{\tau^m}{m!} B^m + \ldots\right] y_t , \quad (3.12)$$

which is by definition

$$y_{t+\tau} = [\exp B\tau] y_t . \quad (3.13)$$

The exponential matrix, $[\exp A\tau]$, is also called the transition matrix and is the same as that discussed by Craggs (7, page 2) and labeled as T.

For time increments, τ, such that the forcing function may be considered constant within the time step, the solution to the forced vibration problem is as follows. Consider the set of nonhomogeneous, linear, constant coefficient differential equations

$$\dot{y} = By + g(t) \quad (3.14)$$

where g(t) denotes the vector of time-dependent forcing functions. The solution is developed through a variation of parameters. Assume a solution of the form

$$y = [\exp Bt]u \quad (3.15)$$

where u is a yet undetermined vector. Substituting this into Equation 3.14 yields

$$[\exp Bt]\dot{u} + B[\exp Bt]u = B[\exp Bt]u + g(t) , \quad (3.16)$$

or

$$\dot{u} = [\exp -Bt]\, g(t) . \quad (3.17)$$

The solution to Equation 3.17 is as follows:

$$u_t = u_o + \int_o^t [\exp -Bt']\, g(t')\, dt' . \quad (3.18)$$

Equation 3.18 may be substituted into Equation 3.15 to yield

$$y = [\exp Bt]u_o + [\exp Bt]\int_o^t [\exp -Bt']\, g(t')\, dt' \,. \qquad (3.19)$$

The initial value of u, u_o, may be determined by evaluating the assumed behavior of y at time zero.

$$y_o = u_o \,. \qquad (3.20)$$

Thus,

$$y_t = [\exp Bt]y_o + [\exp Bt]\int_o^t [\exp -Bt']\, g(t')\, dt' \,. \qquad (3.21)$$

If g(t) is a constant vector, g, we may write

$$y_t = [\exp Bt]y_o + [\exp Bt]\int_o^t [\exp -Bt']\, dt'\, g \,. \qquad (3.22)$$

The integral may be evaluated to yield

$$\begin{aligned} y_t &= [\exp Bt]y_o + [\exp Bt]\Big[-[B]^{-1}[\exp -Bt']\Big]\Big|_o^t\, g \\ &= [\exp Bt]y_o + [\exp Bt]B^{-1}g - B^{-1}g \\ &= [\exp Bt]y_o + \Big[[\exp Bt] - I\Big]B^{-1}g \\ &= [\exp Bt]y_o + \Big[I + Bt + B^2\frac{t^2}{2!} + \ldots - I\Big]B^{-1}g \\ &= [\exp Bt]y_o + \Big[Bt + \frac{B^2t^2}{2!} + \ldots\Big]B^{-1}g \\ &= [\exp Bt]y_o + t\left[\sum_{k=1}^{\infty} \frac{B^{k-1}t^{k-1}}{k!}\right]g \,. \qquad (3.23) \end{aligned}$$

Applying the results of the solution given in Equation 3.23 to the coupled equation of motion given in Equation 3.4 yields Equation 3.24.

$$\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}_{t+\tau} = \begin{bmatrix} \exp A\tau \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}_t + \tau \begin{bmatrix} \sum_{k=1}^{\infty} \frac{[A]^{k-1}\tau^{k-1}}{k!} \end{bmatrix} \begin{Bmatrix} \phi \\ M^{-1}f(t) \end{Bmatrix}_t \qquad (3.24)$$

3.4 Boundary Conditions

All that remains to be done to develop a complete set of algorithms is to present a method of treating prescribed zero displacement, velocity, and acceleration boundary conditions as are found at restrained node points in structures. In finite element programs for static analysis, it is common practice to accommodate boundary conditions by modifying the stiffness matrix and applied load vectors to incorporate known nodal displacements. All that is required to accommodate a zero displacement is to delete all of the off-diagonal row and column elements of the stiffness matrix, set the diagonal element equal to unity, and set the applied load associated with that particular node equal to zero.

A parallel procedure may be used to accommodate zero displacement and velocity boundary conditions. For any degree of freedom of the structure for which the prescribed displacement and velocity are zero, the associated off-diagonal row and column elements of the A matrix are deleted, the diagonal element is set equal to unity, and the proper terms in the $M^{-1}f$ vector are deleted.

3.5 Formation of Structure Matrices

The stiffness matrix for the structure may be readily determined by using the principle of superposition commonly relied upon in elementary mechanics. If a point within the structure is designated as a node point and all the structural elements connected to that node point are considered in sequence, the stiffness associated with this node point may be determined by linear superposition (addition) of the appropriate portions of the stiffness matrices of each individual element for all connected elements.

The mass matrix for the structure may be determined by using a procedure identical to that used to determine the stiffness matrix. In the case of the stiffness matrix, the potential energy of the structure is related to the node point displacements. The stiffness matrix and the node point displacement may be used to compute the potential energy of the structure. In a similar manner, the velocity of the structure node points and the mass matrix of the structure determine the kinetic energy of the structure. Thus, linear superposition of the appropriate inertial properties of all elements connected to a given node may be used to determine the mass matrix of the structure.

Because of the general lack of knowledge about the exact velocity dependence of energy dissipative processes in structures, it is common practice to assume that the damping in the structure is a linear function of node point velocities. This may be readily incorporated into the mathematical model of the structure when modal analysis procedures are used. The same procedure used in modal analysis could be used with the matrix exponential solution, but that course was not followed in this investigation. An approximate representation of damping may be incorporated into the structure by considering two sets of dampers: one associated with the node point inertial characteristics and the other associated with the node point stiffness characteristics, as suggested by Biggs (9, pages 140-147). The magnitude of the inertial associated damping coefficient matrix, C_r, is

$$C_r = c_r M \ , \tag{3.25}$$

where c_r is a scalar constant defined explicitly later. The magnitude of the stiffness associated damping coefficient matrix, C_g, is

$$C_g = c_g K \ , \tag{3.26}$$

where c_g is a scalar constant defined explicitly later. Biggs (9, pages 140-147) presents a method for determining these two sets of coefficients by substitution into the following equation.

$$c_g \omega^2 + c_r = \eta 2\omega \ , \tag{3.27}$$

where η is the ratio of actual to critical damping at the circular frequency ω. Thus, the damping ratio, η, may be set at any desired level at two separated frequencies. This determines the damping ratio at all other frequencies. The total structure damping matrix is therefore determined by

$$\begin{aligned} C &= C_r + C_g \\ &= c_r M + c_g K \ . \end{aligned} \tag{3.28}$$

An example of the use of this approximate method of representing structural damping is presented in the third example problem in Section 6 of this document.

4. MATHEMATICAL MODEL OF STRUCTURE ELEMENTS

As discussed in Section 3, the relationship between applied forces, displacements, velocities, and accelerations of node points of a structure may be expressed in matrix form. The matrices used were the structure stiffness matrix and the structure mass matrix. The structure stiffness matrix and the structure mass matrix are completely determined by the properties of the elements which make up the structure and by the boundary conditions of the structure. Boundary conditions were considered in Section 3. A derivation of the stiffness and mass matrices which describe a single beam element of the structure is presented in this section.

The stiffness and mass matrices derived are neither original nor the most general possible for the particular element considered. They were derived and included in this document to insure completeness for the reader unacquainted with finite element techniques. Several authors have derived beam element stiffness and mass matrices under assumptions similar to those made herein, and the reader is directed to the work reported by Archer (10), McCalley (11), Kapur (12), and Gallagher and Lee (13) for comparison. Under similar assumptions, the derived matrices agree with those given in the cited references in all cases.

The beam element matrices may best be developed if the axial and transverse portions of the motion of the beam are considered separately. The incorporation of rigid masses and weightless springs into the mass and stiffness matrices of the structure is not presented in this document because of its simplicity.

4.1 Stiffness and Mass Matrices for Axial Motion

Consider the beam element illustrated in Figure 4.1. Assume that the axial displacement, w(z), of any point on the beam may be represented by

$$w(z) = m + nz \ , \tag{4.1}$$

where m and n are arbitrary constants and z is the position on the beam, as illustrated in Figure 4.1. Substituting for the axial displacement of Ends 1 and 2 of the beam results in the equation

$$w(z) = w_1 + \frac{w_2 - w_1}{L} z \ , \tag{4.2}$$

where w_1 and w_2 are the axial displacements of Ends 1 and 2 of the beam, respectively. Equation 4.2 may be rewritten in matrix form as follows.

$$w(z) = \left\lfloor 1 - \frac{z}{L} \quad \frac{z}{L} \right\rfloor \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} \ . \tag{4.3}$$

From the strain-displacement relations, the axial strain, $\epsilon(z)$, at any point in the beam is obtained by differentiating the displacement with respect to z. The result of this operation is given in Equation 4.4.

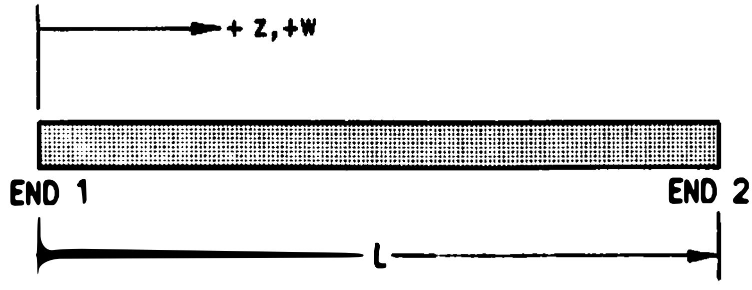


Figure 4.1. Beam Element for Axial Motion.

$$\epsilon(z) = \left\lfloor -\frac{1}{L} \quad \frac{1}{L} \right\rfloor \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} . \tag{4.4}$$

The strain energy, U_A, absorbed within the beam element may be expressed as

$$U_A = \frac{1}{2} \int_{volume} \epsilon(z)\ \sigma(z)\ dV , \tag{4.5}$$

where $\sigma(z)$ is the axial stress at any point on the beam and dV is the increment of volume. Within the linear elastic region, Equation 4.5 may be rewritten as

$$U_A = \frac{1}{2} \int_{volume} \epsilon(z) E\ \epsilon(z)\ dV , \tag{4.6}$$

where E is Young's modulus for the beam material. Substituting for $\epsilon(z)$ from Equation 4.4 into Equation 4.6 yields

$$U_A = \frac{1}{2} \int_0^L E \left\lfloor w_1 \quad w_2 \right\rfloor \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} \left\lfloor -\frac{1}{L} \quad \frac{1}{L} \right\rfloor \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} a\ dz , \tag{4.7}$$

where dV has been replaced by "a dz" and the integration ranges over the beam length L. The cross-sectional area of the beam is represented by "a" and dz is an increment of beam length. Upon integration, Equation 4.7 yields

$$U_A = \frac{1}{2} \left\lfloor w_1 \quad w_2 \right\rfloor \begin{bmatrix} \frac{aE}{L} & -\frac{aE}{L} \\ -\frac{aE}{L} & \frac{aE}{L} \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} , \tag{4.8}$$

where the modulus of elasticity and the cross-sectional area have been assumed to be constant over the length of the beam. By definition, the stiffness matrix for axial displacement of the beam element is

$$K_a = \frac{aE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} . \tag{4.9}$$

The axial velocity, $\dot{w}(z)$, of any point on the beam may be determined by differentiating with respect to time.

$$\dot{w}(z) = \left\lfloor 1 - \frac{z}{L} \quad \frac{z}{L} \right\rfloor \begin{Bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{Bmatrix} , \tag{4.10}$$

where $\dot{w}_1$ and $\dot{w}_2$ are the axial velocities at Ends 1 and 2 of the beam, respectively. The kinetic energy, T_A, brought about by the axial velocity is

$$T_A = \frac{1}{2}\int_{volume} \rho \; \dot{w}(z)^2 \, dV , \tag{4.11}$$

where ρ is the density of the beam material. Substituting for velocity and rewriting Equation 4.11 in matrix form,

$$T_A = \frac{1}{2}\int_o^L \rho a \left\lfloor \dot{w}_1 \quad \dot{w}_2 \right\rfloor \begin{Bmatrix} 1 - \frac{z}{L} \\ \frac{z}{L} \end{Bmatrix} \left\lfloor 1 - \frac{z}{L} \quad \frac{z}{L} \right\rfloor \begin{Bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{Bmatrix} dz . \tag{4.12}$$

After integrating and substituting limits in Equation 4.12, the kinetic energy

$$T_A = \frac{1}{2}\left\lfloor \dot{w}_1 \quad \dot{w}_2 \right\rfloor \begin{bmatrix} \frac{\rho a L}{3} & \frac{\rho a L}{6} \\ \frac{\rho a L}{6} & \frac{\rho a L}{3} \end{bmatrix} \begin{Bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{Bmatrix} . \tag{4.13}$$

The mass matrix, M_a, for axial velocity of the beam element is

$$M_a = \rho a L \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} . \tag{4.14}$$

4.2 Stiffness and Mass Matrices for Transverse Motion

Shear deformation will be neglected but the effect of rotary inertia will be included in the derivation of the element stiffness and mass matrices. Consider the transversely displaced beam element illustrated in Figure 4.2. The slope of the neutral axis of the beam, dy/dx, is represented by θ in Figure 4.2. Assume that the transverse displacement, v(z), may be represented by

$$v(z) = m + nz + oz^2 + pz^3 , \tag{4.15}$$

where m, n, o, and p are arbitrary constants. Substituting the transverse displacements, v_1 and v_2, and rotations, θ_1 and θ_2, at Ends 1 and 2 of the beam, respectively, we may write Equation 4.16.

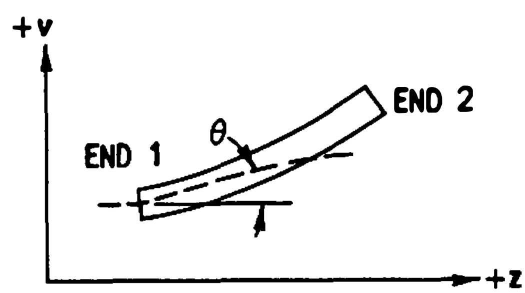


Figure 4.2. Beam Element for Transverse Motion.

$$v(z) = v_1 + \theta_1 z + \left(-3\frac{v_1}{L^2} - 2\frac{\theta_1}{L} + 3\frac{v_2}{L^2} - \frac{\theta_2}{L}\right)z^2$$

$$+ \left(2\frac{v_1}{L^3} + \frac{\theta_1}{L^2} - 2\frac{v_2}{L^3} + \frac{\theta_2}{L^2}\right)z^3 \; . \qquad (4.16)$$

Equation 4.16 may be written in matrix form as

$$v(z) = \left\lfloor 1 - \frac{3z^2}{L^2} + \frac{2z^3}{L^4};\; z - \frac{2z^2}{L} + \frac{z^3}{L^2};\; \frac{3z^2}{L^2} - \frac{2z^3}{L^3};\; -\frac{z^2}{L} + \frac{z^3}{L^2} \right\rfloor \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} . \qquad (4.17)$$

Differentiating Equation 4.17 with respect to z yields

$$\frac{\partial v(z)}{\partial z} = \left\lfloor \frac{6z^2}{L^4} - \frac{6z}{L^2};\; \frac{3z^2}{L^2} - \frac{4z}{L} + 1;\; -\frac{6z^2}{L^3} + \frac{6z}{L};\; \frac{3z^2}{L^2} - \frac{2z}{L} \right\rfloor \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} . \qquad (4.18)$$

Differentiating Equation 4.18 with respect to z yields

$$\frac{\partial^2 v(z)}{\partial z^2} = \left\lfloor \frac{12z}{L^4} - \frac{6}{L^2};\; \frac{6z}{L^2} - \frac{4}{L};\; -\frac{12z}{L^3} + \frac{6}{L^2};\; \frac{6z}{L^2} - \frac{2}{L} \right\rfloor \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} . \qquad (4.19)$$

Equation 4.19 may be rewritten as

$$\frac{\partial^2 v(z)}{\partial z^2} = \left\lfloor f_1(z)\;\; f_2(z)\;\; f_3(z)\;\; f_4(z) \right\rfloor \{\delta\} \; , \qquad (4.20)$$

where

$$f_1(z) = \frac{12z}{L^4} - \frac{6}{L^2} \; , \qquad (4.21)$$

$$f_2(z) = \frac{6z}{L^2} - \frac{4}{L} , \tag{4.22}$$

$$f_3(z) = -\frac{12z}{L^3} + \frac{6}{L^2} , \tag{4.23}$$

$$f_4(z) = \frac{6z}{L^2} - \frac{2}{L} , \text{ and} \tag{4.24}$$

$$\left\{\delta\right\} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} . \tag{4.25}$$

If the shear deformation is neglected, the strain energy, U_B, absorbed in the beam because of bending is

$$U_B = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 v(z)}{\partial z^2}\right)^2 dz , \tag{4.26}$$

where I is the second moment of area of the cross section of the beam. Substituting $\partial^2 v(z)/\partial z^2$ into the bending energy equation (Equation 4.26) yields

$$U_B = \frac{1}{2}\int_0^L EI\lfloor\delta^T\rfloor \begin{Bmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \\ f_4(z) \end{Bmatrix} \lfloor f_1(z) \ f_2(z) \ f_4(z) \ f_5(z)\rfloor \left\{\delta\right\} dz , \tag{4.27}$$

where $\lfloor\delta^T\rfloor$ is the transpose of $\left\{\delta\right\}$. If the moment of inertia, I, and Young's modulus, E, are independent of position, the resulting equation upon integration and substitution of limits is given in Equation 4.28.

$$U_B = \frac{1}{2} EI \lfloor \delta^T \rfloor \begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ & & \frac{12}{L^3} & -\frac{6}{L^2} \\ \text{Symmetric} & & & \frac{4}{L} \end{bmatrix} \{\delta\} \ . \tag{4.28}$$

The beam element stiffness matrix, K_B, for transverse displacements may be written as

$$K_B = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{Symmetric} & & & 4L^2 \end{bmatrix} . \tag{4.29}$$

If the shear deformation is neglected, the kinetic energy, T_B, associated with transverse motion of the beam element is

$$T_B = \frac{1}{2}\int_0^L \rho a \left(\dot{v}(z)\right)^2 dz + \frac{1}{2}\int_0^L \rho I \left(\frac{\partial \dot{v}(z)}{\partial z}\right)^2 dz \ , \tag{4.30}$$

where $\dot{v}(z)$ is the transverse velocity at any point on the beam and may be found by differentiating the transverse displacement with respect to time.

$$\dot{v}(z) = \left\lfloor 1 - \frac{3z^2}{L^2} + \frac{2z^3}{L^3};\ z - \frac{2z^2}{L} + \frac{z^3}{L^2};\ \frac{3z^2}{L^2} - \frac{2z^3}{L^3};\ -\frac{z^2}{L} + \frac{z^3}{L^2} \right\rfloor \begin{Bmatrix} \dot{v}_1 \\ \dot{\theta}_1 \\ \dot{v}_2 \\ \dot{\theta}_2 \end{Bmatrix} \tag{4.31}$$

where $\dot{v}_1$ and $\dot{v}_2$ are the transverse velocities and $\dot{\theta}_1$ and $\dot{\theta}_2$ are the angular velocities at Ends 1 and 2 of the beam.

The first integral in Equation 4.30 is associated with translational inertia and the second integral is associated with rotatory inertia. To evaluate the first integral, let

$$\dot{v}(z) = \left\lfloor f_5(z)\ f_6(z)\ f_7(z)\ f_8(z) \right\rfloor \left\{\dot{\delta}\right\} \tag{4.32}$$

where

$$f_5(z) = 1 - \frac{3z^2}{L^2} + \frac{2z^3}{L^3}\ , \tag{4.33}$$

$$f_6(z) = z - \frac{2z^2}{L} + \frac{z^3}{L^2}\ , \tag{4.34}$$

$$f_7(z) = \frac{3z^2}{L^2} - \frac{2z^3}{L^3}\ , \tag{4.35}$$

$$f_8(z) = -\frac{z^2}{L} + \frac{z^3}{L^2}\ , \text{ and} \tag{4.36}$$

$$\left\{\dot{\delta}\right\} = \begin{Bmatrix} \dot{v}_1 \\ \dot{\theta}_1 \\ \dot{v}_2 \\ \dot{\theta}_2 \end{Bmatrix}\ . \tag{4.37}$$

The expression for the first integral may be written as follows.

$$\frac{1}{2}\int_0^L \rho a \left(\dot{v}(z)\right)^2 dz$$

$$= \frac{1}{2}\int_0^L \rho a \left\lfloor \dot{\delta}^T \right\rfloor \begin{Bmatrix} f_5(z) \\ f_6(z) \\ f_7(z) \\ f_8(z) \end{Bmatrix} \left\lfloor f_5(z)\ f_6(z)\ f_7(z)\ f_8(z) \right\rfloor \left\{\dot{\delta}\right\}\ dz\ . \tag{4.38}$$

By assuming constant cross-sectional area and constant density after integrating Equation 4.38, the first integral becomes

$$\frac{1}{2}\int_{0}^{L} \rho a\left(\dot{v}(z)\right)^{2}\, dz = \frac{1}{2}\left\lfloor \dot{\delta}^{T} \right\rfloor \frac{\rho a}{420}\begin{bmatrix} 156L & 22L^{2} & 54L & -13L^{2} \\ & 4L^{3} & 13L^{2} & -3L^{2} \\ & & 156L & -22L \\ \text{Symmetric} & & & 4L^{3} \end{bmatrix}\left\{\dot{\delta}\right\} . \quad (4.39)$$

Thus, the mass matrix associated with the translational portion of the transverse motion is

$$M_{BT} = \rho a L\begin{bmatrix} \frac{13}{35} & \frac{11}{210}L & \frac{9}{70} & -\frac{13}{420}L \\ & \frac{1}{105}L^{2} & \frac{13}{420}L & -\frac{1}{140}L^{2} \\ & & \frac{13}{35} & -\frac{11}{210}L \\ \text{Symmetric} & & & \frac{1}{105}L^{2} \end{bmatrix} . \quad (4.40)$$

To evaluate the second integral, which is associated with rotatory inertia, in Equation 4.30; it is necessary to differentiate Equation 4.18 with respect to time.

$$\frac{\partial \dot{v}(z)}{\partial z} = \left\lfloor \frac{6z^{2}}{L^{3}} - \frac{6z}{L^{2}};\ \frac{3z^{2}}{L^{2}} - \frac{4z}{L} + 1;\ -\frac{6z^{2}}{L^{3}} + \frac{6z}{L^{2}};\ \frac{3z^{2}}{L^{2}} - \frac{2z}{L} \right\rfloor \begin{Bmatrix} \dot{v}_{1} \\ \dot{\theta}_{1} \\ \dot{v}_{2} \\ \dot{\theta}_{2} \end{Bmatrix} . \quad (4.41)$$

Equation 4.41 may be rewritten as

$$\frac{\partial \dot{v}(z)}{\partial z} = \left\lfloor f_{9}(z)\ f_{10}(z)\ f_{11}(z)\ f_{12}(z) \right\rfloor \left\{\dot{\delta}\right\} , \quad (4.42)$$

where

$$f_9(z) = \frac{6z^2}{L^3} - \frac{6z}{L^2} \,, \tag{4.43}$$

$$f_{10}(z) = \frac{3z^2}{L^2} - \frac{4z}{L} \,, \tag{4.44}$$

$$f_{11}(z) = \frac{6z^2}{L^3} + \frac{6z}{L^2} \,, \text{ and} \tag{4.45}$$

$$f_{12}(z) = \frac{3z^2}{L^2} - \frac{2z}{L} \,. \tag{4.46}$$

The second integral in Equation 4.30 may be written as

$$\frac{1}{2}\int_0^L \rho I \left(\frac{\partial \dot{v}(z)}{\partial z}\right)^2 dz$$

$$= \frac{1}{2}\int_0^L \rho I \lfloor \dot{\delta}^T \rfloor \begin{Bmatrix} f_9(z) \\ f_{10}(z) \\ f_{11}(z) \\ f_{12}(z) \end{Bmatrix} \lfloor f_9(z) \;\; f_{10}(z) \;\; f_{11}(z) \;\; f_{12}(z) \rfloor \{\dot{\delta}\} \, dz \,. \tag{4.47}$$

After integration and substitution of limits in Equation 4.47, the second integral

$$\frac{1}{2}\int_0^L \rho I \left(\frac{\partial \dot{v}(z)}{\partial z}\right)^2 dz = \frac{1}{2}\lfloor \dot{\delta}^T \rfloor \rho I \begin{bmatrix} \frac{6}{5L} & \frac{1}{10} & -\frac{6}{5L} & \frac{1}{10} \\ & \frac{2}{15}L & -\frac{1}{10} & \frac{L}{30} \\ & & \frac{6}{5L} & -\frac{1}{10} \\ \text{Symmetric} & & & \frac{2}{15}L \end{bmatrix} \{\dot{\delta}\} \,. \tag{4.48}$$

Thus, the mass matrix associated with the rotational portion of the transverse motion is

$$M_{BR} = \rho I \begin{bmatrix} \frac{6}{5L} & \frac{1}{10} & -\frac{6}{5L} & \frac{1}{10} \\ & \frac{2L}{15} & -\frac{1}{10} & \frac{L}{30} \\ & & \frac{6}{5L} & -\frac{1}{10} \\ \text{Symmetric} & & & \frac{2L}{15} \end{bmatrix} . \quad (4.49)$$

4.3 Stiffness and Mass Matrices for Single Beam Element

The stiffness matrices derived in Subsections 4.1 and 4.2 (Equations 4.9 and 4.29) may be combined to form a single element stiffness matrix by superposition. The resulting stiffness matrix for the beam element is given in Equation 4.50.

$$K_E = \begin{bmatrix} \frac{aE}{L} & 0 & 0 & -\frac{aE}{L} & 0 & 0 \\ & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ & & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ & & & \frac{aE}{L} & 0 & 0 \\ & & & & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \text{Symmetric} & & & & & \frac{4EI}{L} \end{bmatrix} . \quad (4.50)$$

The appropriate displacement vector is formed from w_1, v_1, θ_1, w_2, v_2, and θ_2 in this order.

The mass matrices derived in Subsections 4.1 and 4.2 (Equations 4.14 and 4.49) may be combined to form a single element mass matrix by superposition. The resulting mass matrix for the beam element is given in Equation 4.51.

$$M_E = \begin{bmatrix} \frac{\rho a L}{3} & 0 & 0 & \frac{\rho a L}{6} & 0 & 0 \\ & \frac{13\rho a L}{35} + \frac{6\rho I}{5L} & \frac{11\rho a L^2}{210} + \frac{\rho I}{10} & 0 & \frac{9\rho a L}{70} - \frac{6\rho I}{5L} & -\frac{13\rho a L^2}{420} + \frac{\rho I}{10} \\ & & \frac{\rho a L^2}{105} + \frac{2\rho I L}{15} & 0 & \frac{13\rho a L^3}{420} - \frac{\rho I}{10} & -\frac{\rho a L^3}{140} + \frac{\rho I L}{30} \\ & & & \frac{\rho a L}{3} & 0 & 0 \\ & & & & \frac{13\rho a L}{35} + \frac{6\rho I}{5L} & -\frac{11\rho a L^2}{210} - \frac{\rho I}{10} \\ \text{Symmetric} & & & & & \frac{\rho a L^3}{105} + \frac{2\rho I L}{15} \end{bmatrix} \tag{4.51}$$

The appropriate velocity vector is formed from $\dot{w}_1$, $\dot{v}_1$, $\dot{\theta}_1$, $\dot{w}_2$, $\dot{v}_2$, and $\dot{\theta}_2$ in this order.

5. DEVELOPMENT OF COMPUTER PROGRAM

The mathematical form of the matrix exponential solution method makes it necessary that all but the simplest of solutions be performed by computer methods. A high-speed digital computer is well suited for this purpose. With this thought in mind, a computer program was developed to implement the solution of the transient dynamics of plane structures composed of beam elements of uniformly distributed mass, weightless springs, and rigid masses.

The logical flow of the computer program is given in flow-chart form in Appendix A, and the major steps in the program are as follows. Data describing the geometric and structural characteristics are input for the program, which in turn formulates the structure stiffness and mass matrices. The structure stiffness and mass matrices are modified for boundary conditions, as discussed in Subsection 3.4. The mass matrix is inverted and post multiplied by the structure stiffness matrix. The coupling matrix, A, is then formed, and the effect of damping is incorporated by using input damping coefficients c_r and c_g. For a given time increment and number of terms in the series approximation, the matrix exponential, [exp Aτ], and the forcing function transition matrix,

$$\tau \left[\sum_{k=1} \frac{A^{k-1}\tau^{k-1}}{k!} \right] ,$$

are next formed. This completes the preliminary steps directed toward problem solution. The solution is then developed incrementally, as indicated by Equation 3.24 which is repeated here for convenience.

$$\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}_{t+\tau} = [\exp A\tau] \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}_t + \tau \left[\sum_{k=1}^{\infty} \frac{A^{k-1}\tau^{k-1}}{k!} \right] \begin{Bmatrix} \phi \\ M^{-1} f(t) \end{Bmatrix}_t , \qquad (3.24)$$

where approximations have been made for the matrix functions indicated.

It should be noted that the number of terms in the matrix exponential approximation must be limited, as is the case with all series approximations. In this case, an upper limit on the number of terms or lower limit on the time increment exists because of the possibility of exceeding the capability of the digital computer to represent very small floating point numbers. An estimate of the maximum number of terms permissible may be obtained from Equation 5.1.

$$\xi \leq N \ln \tau - \ln (N!) , \qquad (5.1)$$

where ξ is the exponent associated with the smallest number that may be represented within the machine and N is the number of terms used in the approximation. In turn, τ should be chosen to insure accuracy; that is, it should be small enough to permit the necessary transient response details to be represented. For most problems for which this computer program was developed, N will be less than 10 and τ will be chosen to be one-twentieth of the smallest significant structure period. A solution so limited will be in error by less than

$$\left[\frac{A^{N+1}\tau^{N+1}}{(N+1)!} \right] \begin{Bmatrix} \dfrac{d^{N+1}x}{dt^{N+1}} \\ \dfrac{d^{N+2}x}{dt^{N+2}} \end{Bmatrix}_t \qquad (5.2)$$

for free vibration analysis. This error may be made as small as may be represented within the machine by the argument previously presented.

After this investigation was completed, the error criteria reported by Liou (3, 4) and by Mankin and Hung (5, 6) were examined but were not incorporated into this study because of time limitations.

The matrix inversion used was a version of the Gauss-Jordan algorithm as presented by Wang (14). The matrix function approximations and step-by-step solution were re-programmed from programs presented by Ball and Adams (15). The limitations of the computer program are presented in Appendix B, the input data format is presented in Appendix C, the output data format is presented in Appendix D, and the computer program listing is presented in Appendix E.

6. TRANSIENT RESPONSE OF SIMPLE STRUCTURES

To demonstrate the use of the computer program developed in this investigation, three example problems are presented and compared with known solutions.

6.1 First Example Problem

The first example problem involves the determination of the time history of displacements for the three-degrees-of-freedom problem illustrated in Figure 6.1. The displacements indicated in Figure 6.1 are measured from the static equilibrium position of the node points indicated as circled numerals. The time relationship and magnitudes of the applied loads $f_2(t)$, $f_3(t)$, and $f_4(t)$ are indicated in Figure 6.2.

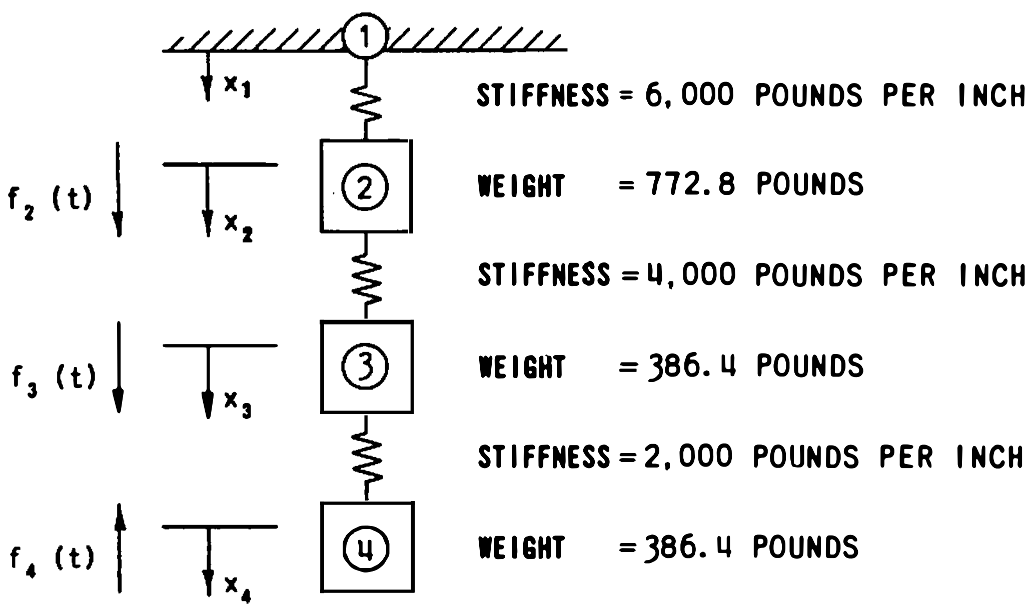


Figure 6.1. Three-Degrees-of-Freedom Model With Weightless Springs and Lumped Masses for First Example Problem.

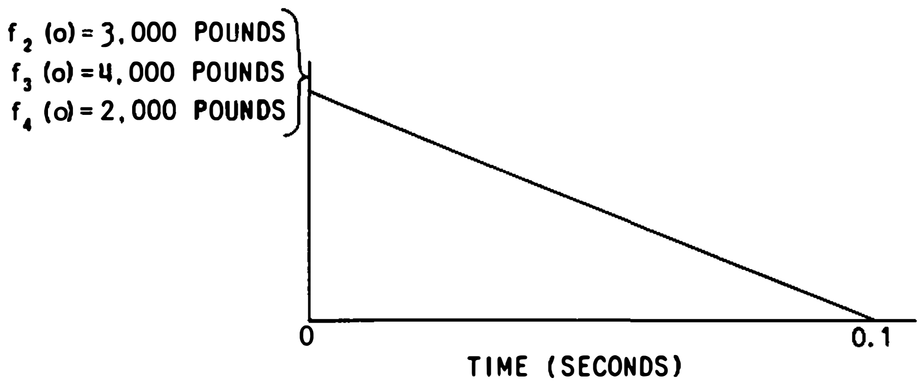


Figure 6.2. Applied Loads for the Three-Degrees-of-Freedom Model of the First Example Program.

The data from this problem were supplied to the computer program and used to write a forcing function subroutine DISTURB, which is the version of DISTURB presented in Appendix E. The time increment used in the solution of the problem was 0.005 second, and the number of terms in the series approximation of the matrix exponential was ten. The displacement of node point 3 as determined with the computer program is plotted in Figure 6.3 and may be compared with the solution developed through the use of modal methods reported by Biggs (9, pages 121-123). The solution for this example problem was plotted by using the computer program XYPLOT presented by Tobias and Jung (16). The smooth line in Figure 6.3 represents the theoretical solution and the symbols "X" represent the approximate solution as output from the computer program.

6.2 Second Example Problem

The distributed mass beam elements developed in Section 4 of this document are used in the second example problem. In this problem, the response of the point of dynamic load application for a simply supported

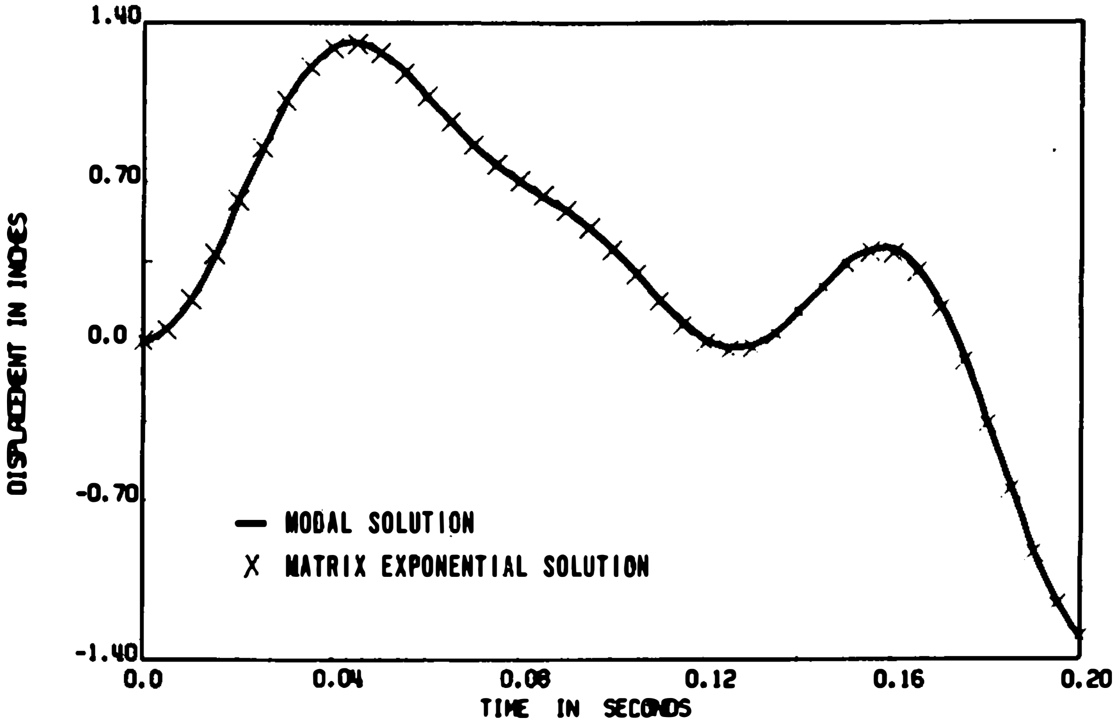


Figure 6.3. Example One Response of Three-Degree System.

beam, as illustrated in Figure 6.4, is to be determined. The beam is a wide-flange steel section 14 inches deep that weighs 142 pounds per lineal foot. The dynamic load, f(t), is initially 50,000 pounds, decreases linearly to zero at 0.01 second, and remains zero for all later time.

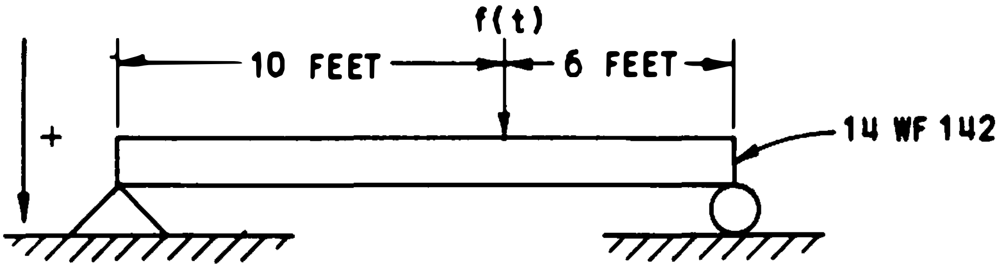


Figure 6.4. Simply Supported Beam of Second Example Problem.

The response of this beam was determined by using two combinations of beam elements connected in series. The time increment used in the solution of the problem was 0.0001 second, and eight terms were used in the series approximation. A comparison of the predicted response and that determined through modal analysis methods is illustrated in Figure 6.5. The smooth line represents the theoretical solution obtained by superposition of the first three modes. The computer solutions for two- and four-beam elements are plotted with the symbols X and Δ, respectively.

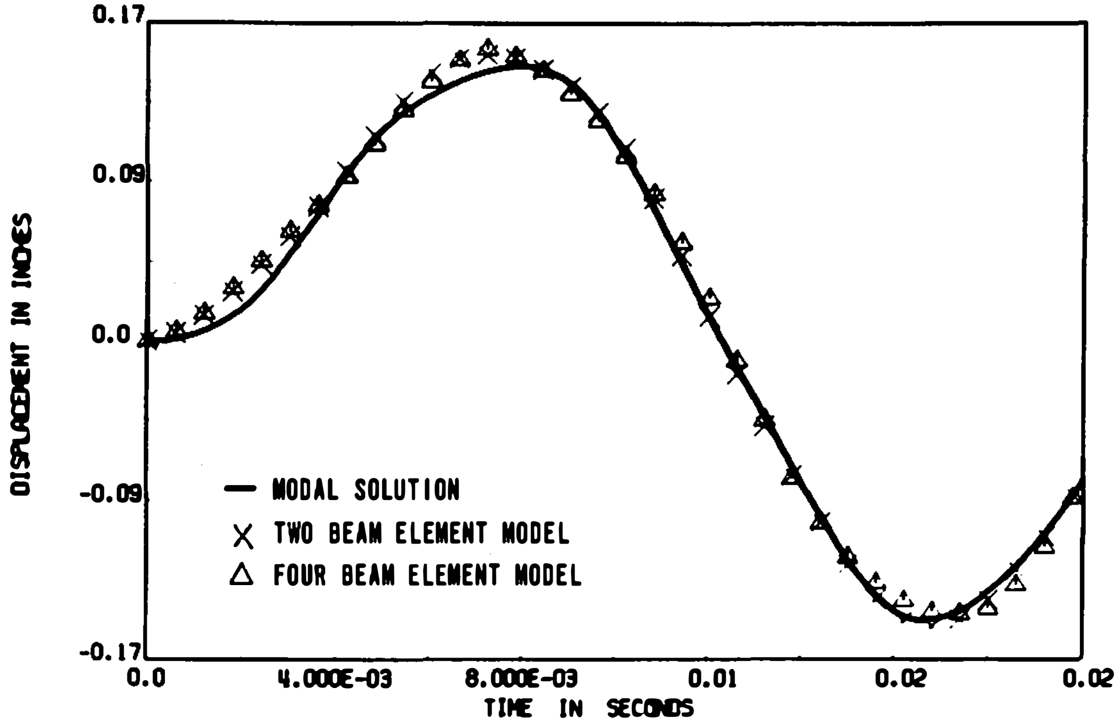


Figure 6.5. Response at Point of Loading for a Simply Supported Beam.

6.3 Third Example Problem

The third example problem is an attempt to predict the transient response of a concrete and steel tower for which experimental data were reported by Takahashi, Gates, and Benuska (17). This tower is diagrammatically illustrated in Figure 6.6, and the model used in the computer analysis is illustrated in Figure 6.7. The data on the structural properties of the tower were taken from that reported by Takahashi, Gates, and Benuska (17). The node points used in modeling the structure are indicated by the circled numerals in Figure 6.7. The small tower was subjected to a base motion acceleration that is a pseudo half sine wave pulse. A multi-linear approximation of this pulse is illustrated in Figure 6.8. The data resulting from tests of this structure indicate a first mode frequency of 125 cycles per second and a fourth mode frequency of 1,300 cycles per second (17).

To analyze the behavior of a system for which a specified base motion is prescribed, a transformation of the basic equations of motion is useful. Let x represent the structure displacement vector relative to its foundation displacement, and let u represent the vector of foundation displacement. The equations of motion may then be written as follows:

$$M\left\{\ddot{x} + \ddot{u}\right\} + C\left\{\dot{x}\right\} + K\left\{x\right\} = 0 \ , \qquad (6.1)$$

where the damping matrix C is assumed to be associated with relative motion only and $\ddot{u}$ is the foundation acceleration vector. Transposition of the base motion terms to the right-hand side of Equation 6.1 yields

$$M\ddot{x} + C\dot{x} + Kx = -M\ddot{u} \ . \qquad (6.2)$$

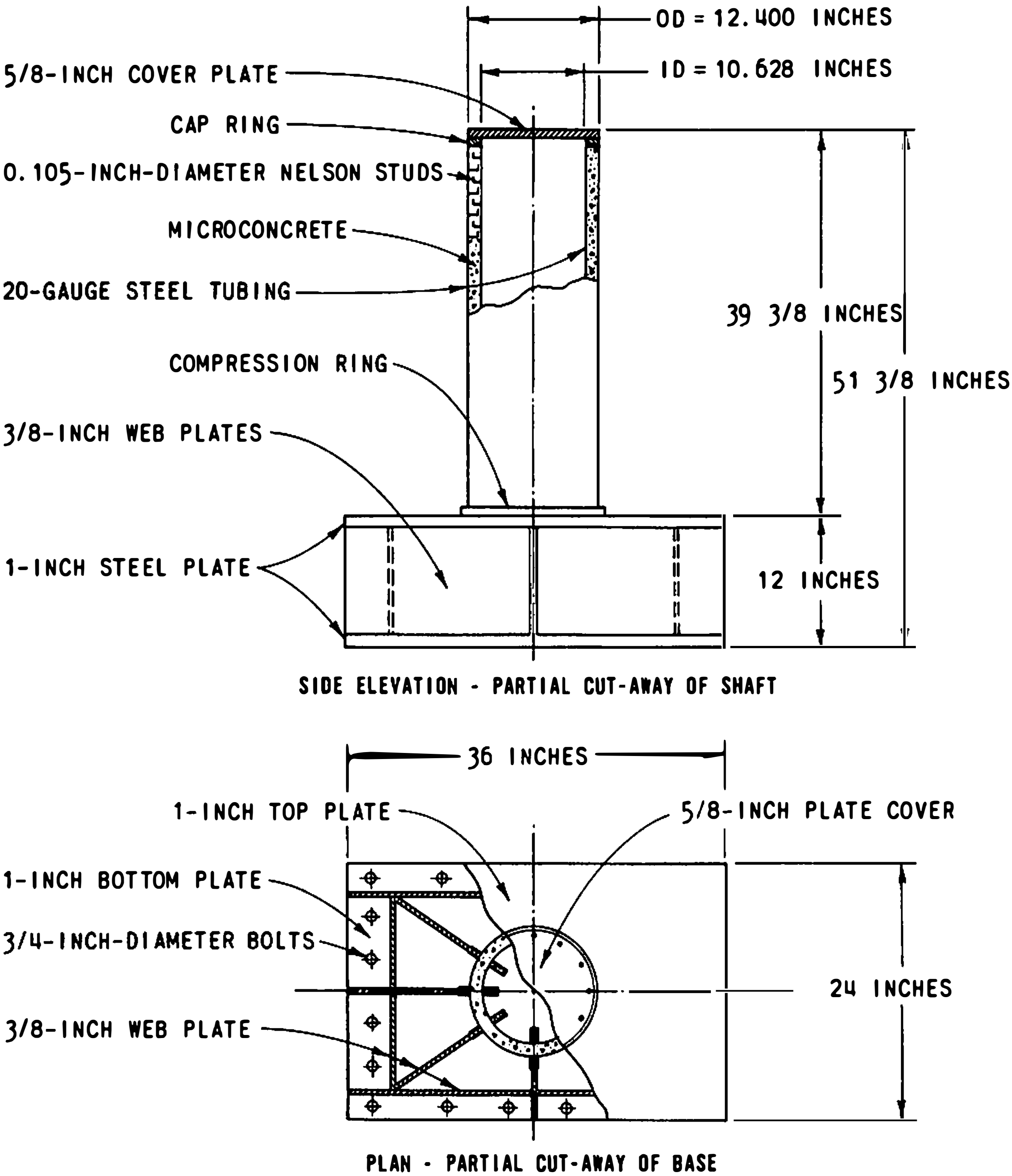
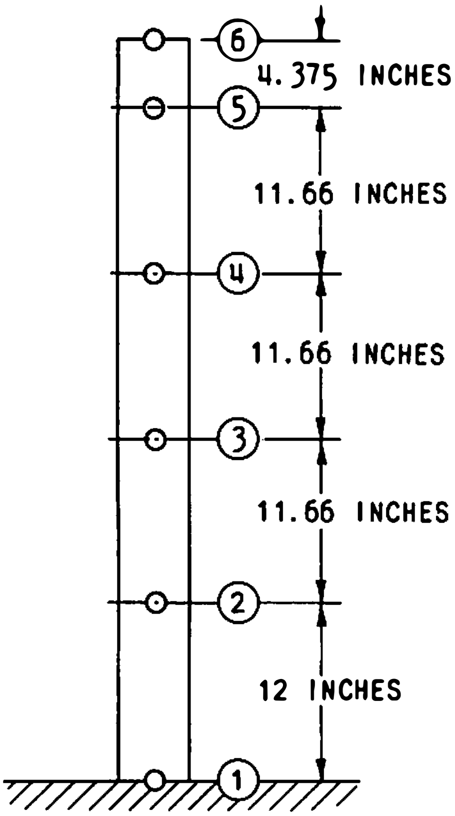


Figure 6.6. Elevation and Plan Views of Small Tower of Third Example Problem.



STRUCTURAL PROPERTIES OF SMALL TOWER

CROSS-SECTIONAL AREA (IN.^{2})	MOMENT OF INERTIA (IN.^{4})	MODULUS OF ELASTICITY (LB./IN.^{2})	UNIT WEIGHT (LB./IN.)
30.3	706	2.5×10^6	2.85
30.3	706	2.5×10^6	2.85
30.3	706	2.5×10^6	2.85
30.3	706	2.5×10^6	2.85
32.0	4000	30×10^6	57.7

Figure 6.7. Model for Small Tower of Third Example Problem.

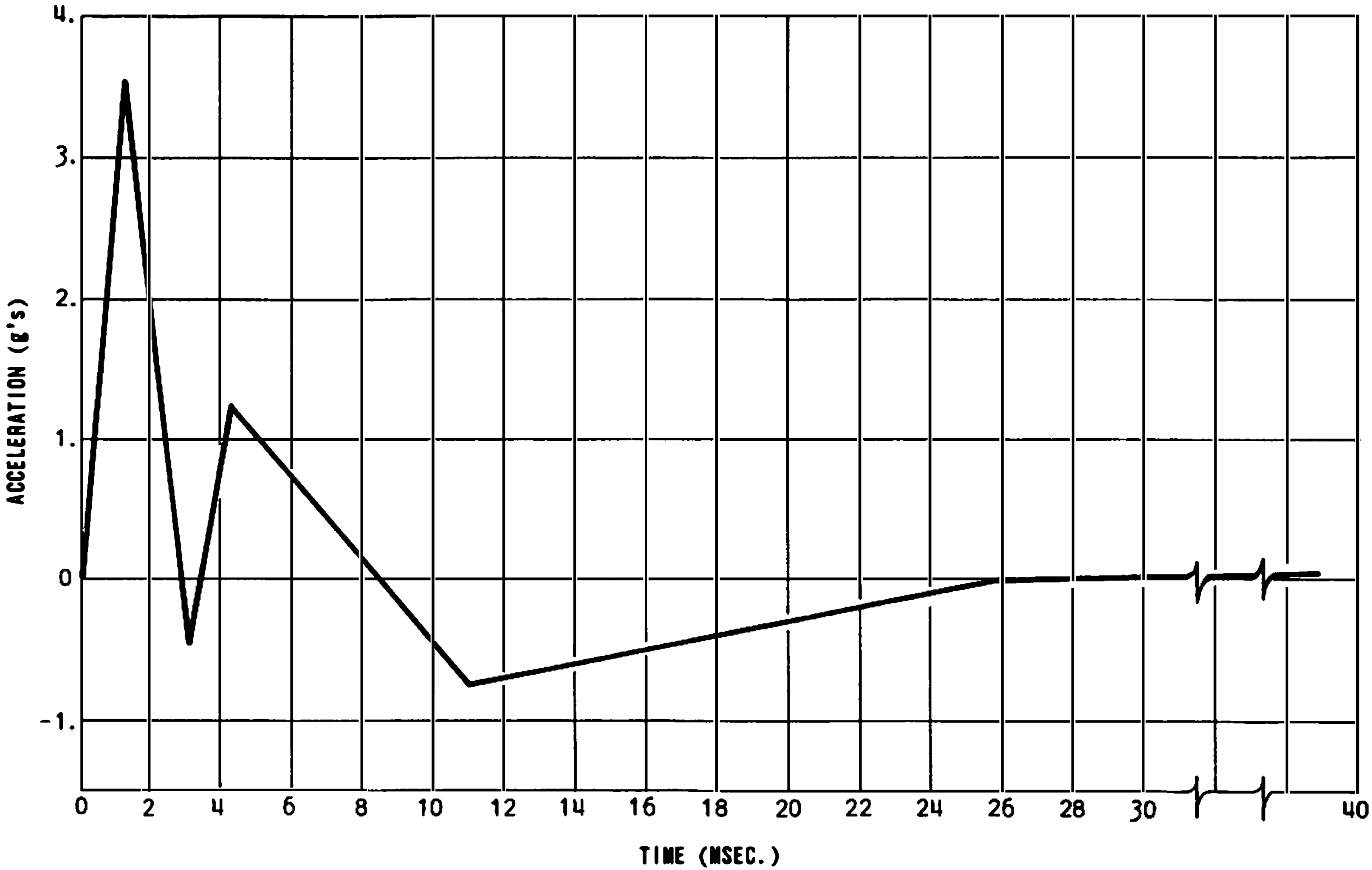


Figure 6.8. Base Motion Accelogram for Small Tower of Third Example Problem.

From a comparison of Equation 6.2 with Equation 3.1, it is apparent that the procedure presented in Section 3 may be used to solve Equation 6.2 if $-M\ddot{u}$ is substituted for f(t).

To model the behavior of the structure, the time increment for solution was chosen as 50 microseconds and six terms were used in the series representation of the matrix functions. The structure damping determined in experiments was approximately 2% of critical in all modes. An approximate representation of this damping is provided by using

$$c_g = 4.75 \times 10^{-6} \text{ seconds.}$$

Using these values as constants in Equation 3.27, the maximum damping is 2% of critical and the minimum damping is 0.2% of critical in the frequency range of interest.

The output data from the computer indicate a dominant frequency of 124 cycles per second, which is a very good agreement with the experimental data. The maximum relative displacement between the base and the top of the tower given by the experimental data (17) is 0.0028 inch, and the maximum relative displacement predicted by the computer program is 0.0024 inch.

7. CONCLUSIONS AND RECOMMENDATIONS

It has been shown in this investigation that the dynamic equations for a linear, elastic structure may be written as a set of coupled first order differential equations with constant coefficients. The matrix exponential solution method was developed to show the close similarity between it and the solution of a single first order constant coefficient differential equation.

The coefficients of the dynamic equations were shown to be related to the stiffness and inertial characteristics of the structure. That these coefficients may be determined by a process of linear superposition was demonstrated. A technique for the incorporation of structural damping was also presented. The stiffness and inertial characteristics of individual beam elements were derived by assuming a compatible deformation pattern for the beam and then determining the strain energy and kinetic energy in the beam. This then defined the stiffness and mass matrices for the beam element.

A computer program based on the equations derived in this document was developed, and the transient response of three simple structures was determined through the use of this program. The transient responses determined in this manner were compared with previously reported analytical and experimental data.

7.1 Conclusions

The objective of this investigation was to develop a numerical solution for the transient response of linear, elastic mechanical systems by using the matrix exponential method. With regard to this objective, the following conclusions may be drawn.

1. The matrix exponential solution method was applied successfully to determine the structural response of linear, elastic mechanical systems.

2. The computer program developed in this investigation provided accurate solutions to the response of simple mechanical systems.

3. This computer program was used and modified with little difficulty, requiring only that one subroutine be rewritten for each system analyzed.

7.2 Recommendations

A comparison was made in this investigation between computer solutions and experimental data to evaluate the ease of program use and modification under realistic circumstances. This effort was severely limited by a lack of sufficient experimental data. Therefore, it is recommended that a minor experimental program be initiated to obtain transient response data for linear, elastic mechanical structures.

It is well known that shear deformation effects can become quite important as the ratio of beam length to depth decreases. It is therefore recommended that the beam element stiffness and mass matrices be modified to include the effect of shear deformation. This could be accomplished by using the modified Timoshenko beam theory presented by Egle (18).

In view of the need to analyze mechanical systems with up to 1,000 degrees of freedom, it is further recommended that the sparse matrix characteristics of the transition matrix be fully utilized by rewriting the computer program in the computer language MATLAN (19). The MATLAN language is a flexible problem-oriented language designed to carry out matrix and scalar operations. Storage management is accomplished automatically in that MATLAN may control both core and direct access devices. Routines for sparse matrix operations are built into MATLAN.

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APPENDIXES

APPENDIX A

FLOW CHART FOR COMPUTER PROGRAM

As discussed in Section 5 of this document, a computer program was developed to implement solution of the transient dynamics of plane structures composed of beam elements of uniformly distributed mass, weightless springs, and rigid masses. The logical flow of this computer program is presented in flow-chart form in this appendix. The symbols used in the flow chart are illustrated and defined in Figure A.1, and the flow chart is presented in Figure A.2.

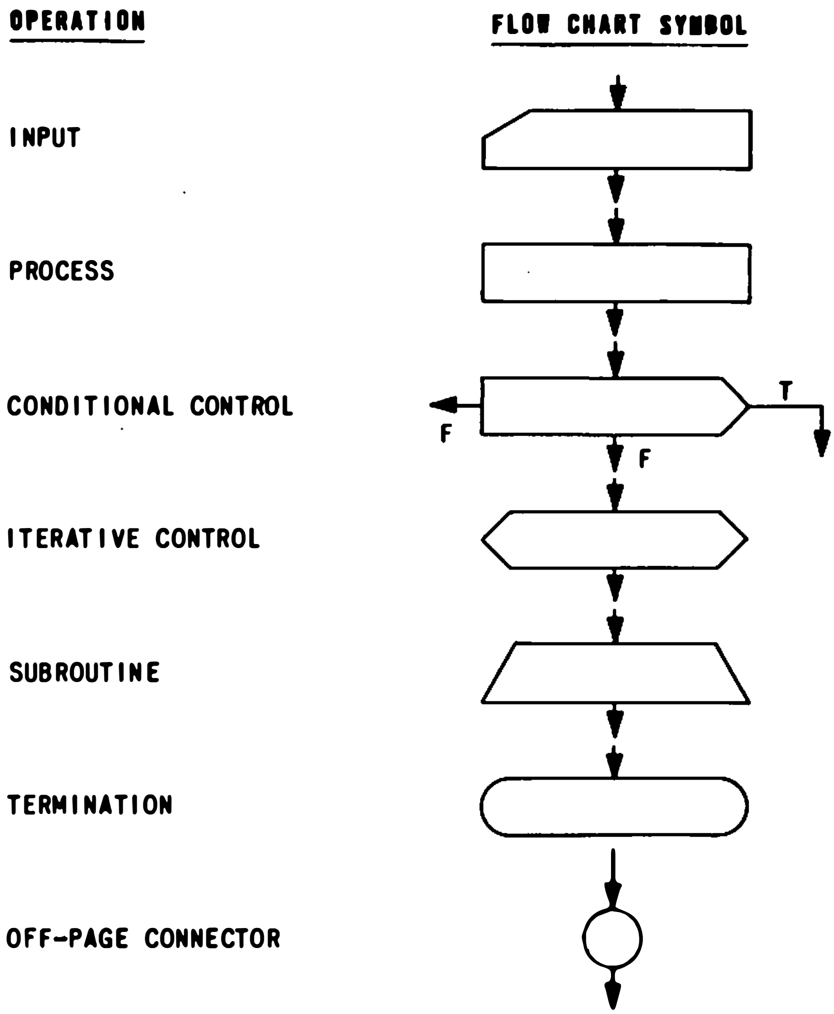

Figure A.1. Symbols Used in Flow Chart for Computer Program.

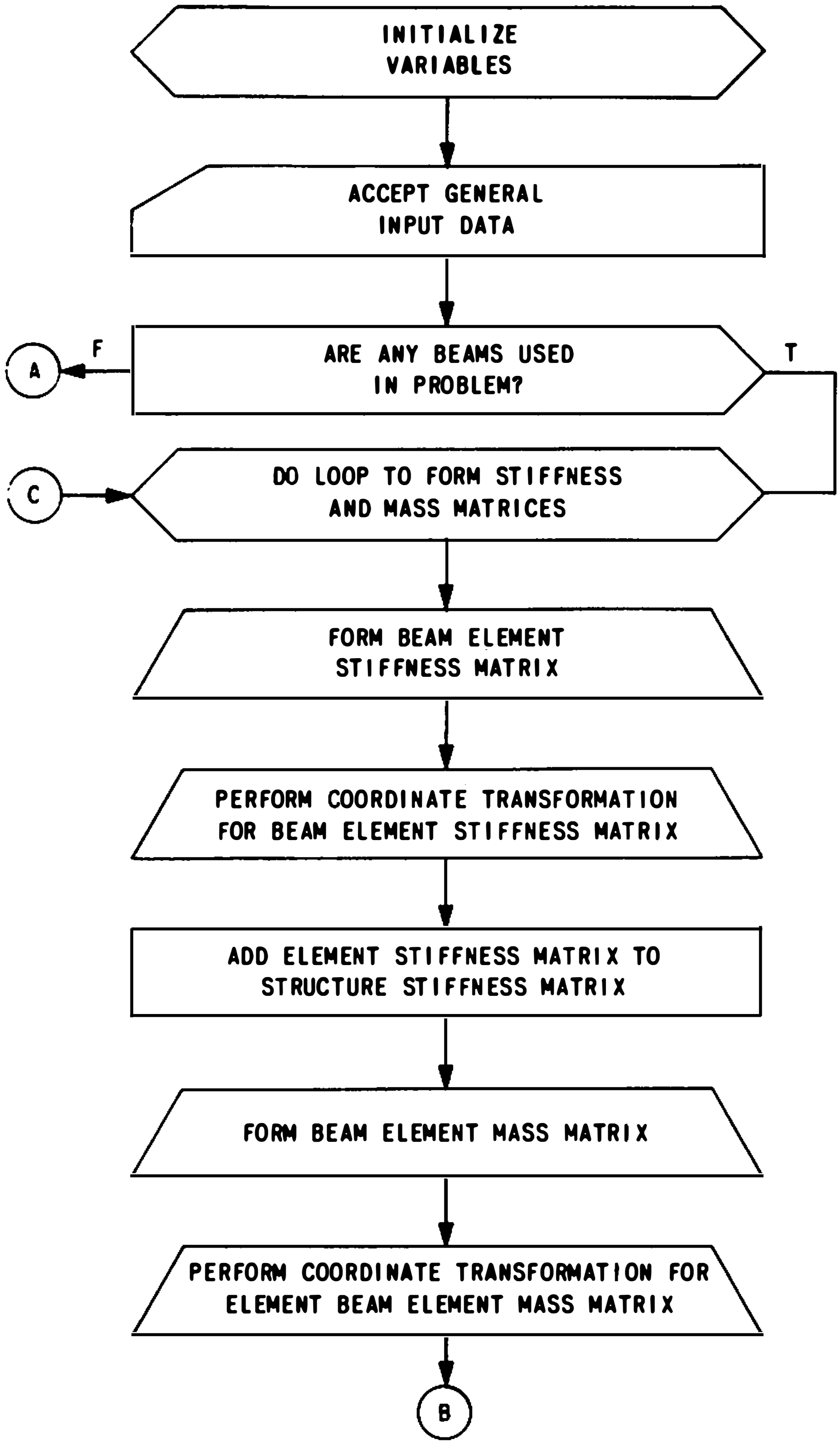

Figure A.2. Flow Chart for Computer Program Developed to Implement Solution of Transient Dynamics of Plane Structures.

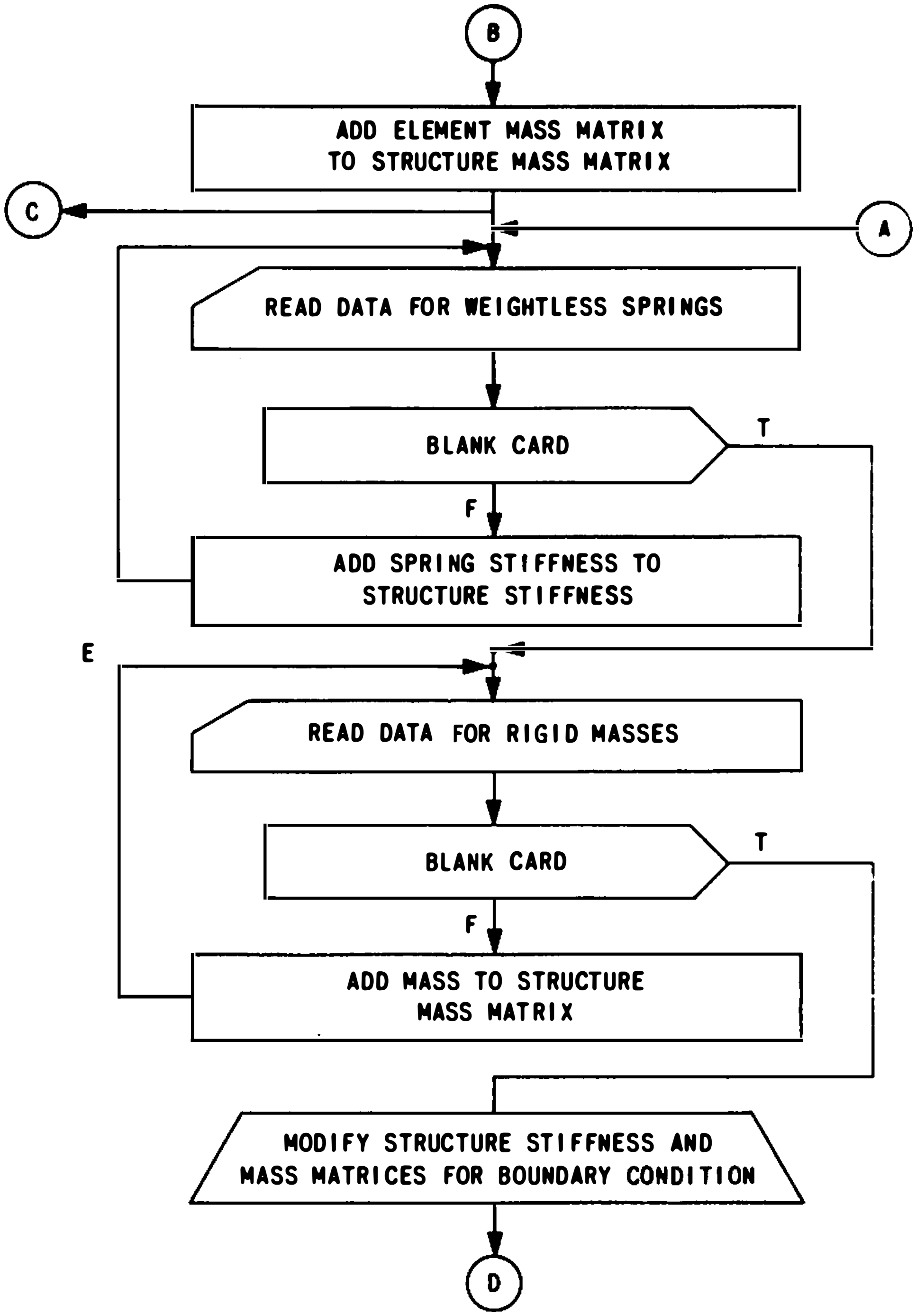

Figure A.2 (continued).

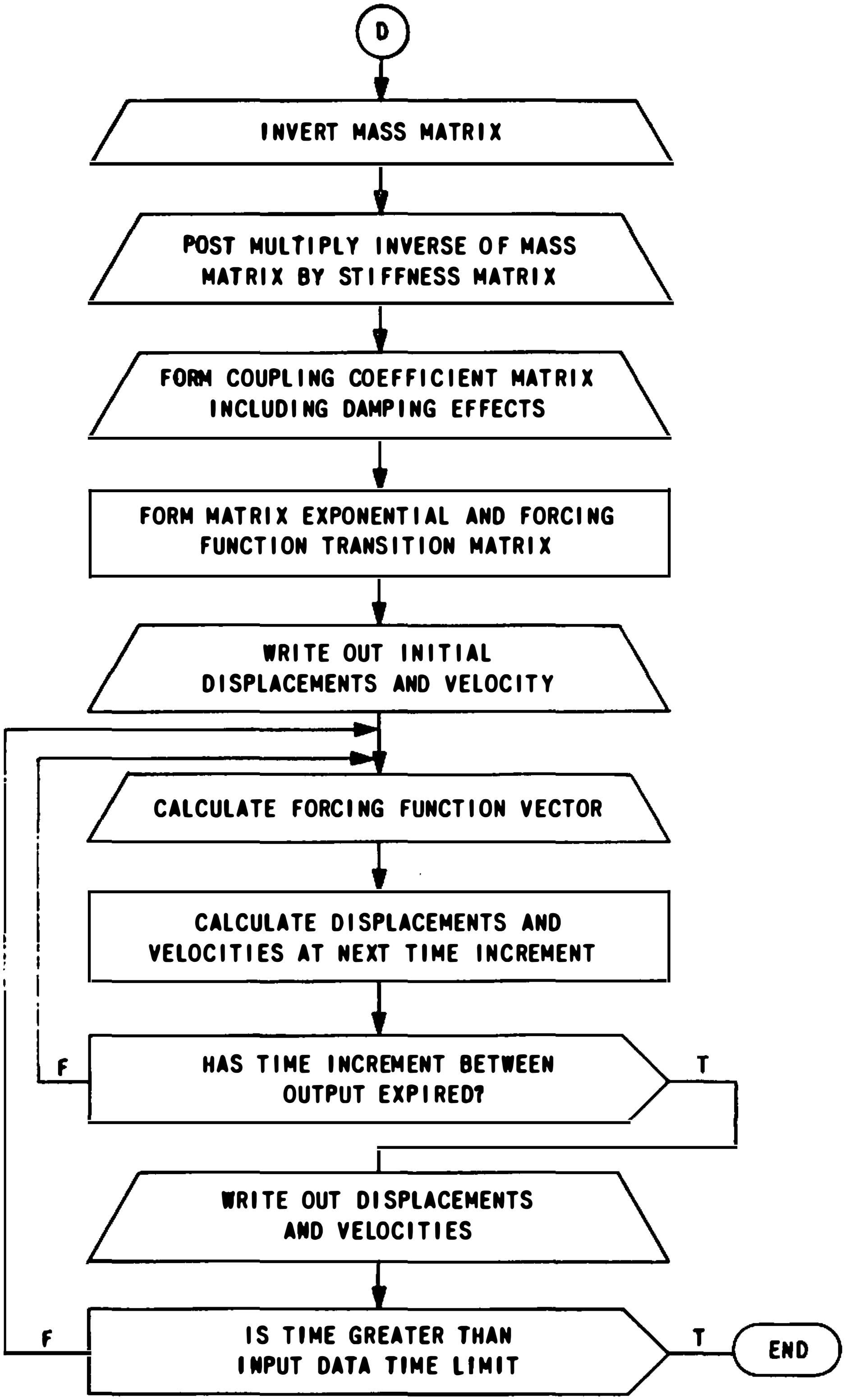

Figure A.2 (continued).

APPENDIX B

PROGRAM LIMITATIONS

The limitations of the computer program developed to implement solution of the transient dynamics of plane structures composed of beam elements of uniformly distributed mass, weightless springs, and rigid masses are as follows.

Maximum number of node points: ten.

Maximum number of beam elements: nine.

Maximum number of data points for plotted output: one.

The number of node points may be increased by changing the dimension statements in blank common and common block /MATEXP/. If the number of node points is N, the common blocks would appear as follows.

```
 COMMON TITLE(18), NUMNP, NUMEL, XNP(N), YNP(N), IRX(N), IRY(N),
1      IRT(N), EE(N-1), EA(N-1), EEI(N-1), ESW(N-1), INP(N-1), JNP(N-1),
2       R(6,6), ESM(6,6), ESG(6,6), SSG(3N,3N), EMM(6,6), EMG(6,6),
3      SMG(3N,3N), SMSG(3N,3N), L, EL, E, ECA, EI, U, RG, CR, CG
 COMMON /MATEXP/ C(6N,6N), HP(6N,6N), A(6N,6N), QPT(6N,6N), X(6N),
1          F(3N), Z(6N), Y(6N), XIC(6N), TQP(6N), ITMAX, KK, LL, MM,
2                JJFLAG, NI, TIME, TMAX, TZERO, NE, T, I1Z, ICONTR,
3                PLTINC, MATYES, ICCS, JFLAG, PLT, IONODE
```

APPENDIX C

INPUT DATA FORMAT

The type designation, contents, and format of the input data cards for the computer program developed to implement solution of transient dynamics of plane structures composed of beam elements of uniformly distributed mass, weightless springs, and rigid masses are given in Table C.1.

Table C.1. Type, Contents, and Format of Input Data Cards for Computer Program

Card Type	Contents	Format
I	Title	18A4
II	Number of node points	I5
	Number of beam elements	I5
	Number of node for which x displacement is to be plotted; zero if no plotted output is desired	I5
III	Coefficient for damping proportional to mass matrices (sec.$^{-1}$)	E10.3
	Coefficient for damping proportional to stiffness matrices (sec.)	E10.3
IV	Initial time for problem (sec.)	F10.0
	Final time for problem (sec.)	F10.0
	Time increment to be used in solution (sec.)	F10.0
	Time increment between printed/plotted output (sec.)	F10.0
	Number of terms to be used in series approximation of matrix exponential	I10
V[a]	Node number	I5
	X coordinate of node (in.)	F10.0
	Y coordinate of node (in.)	F10.0
	X restraint flag	I5
	Y restraint flag	I5
	Theta restraint flag	I5
VI[b]	Beam number	I5
	Young's modulus (p.s.i.)	F10.0
	Beam cross-sectional area (in.^{2})	F10.0
	Beam moment of inertia (in.^{4})	F10.0
	Beam weight per unit of length (lb./in.)	F10.0
	Node point number at first end	I5
	Node point number at opposite end	I5
VII[c]	Node point number at first end of weightless spring	I5
	Node point number at opposite end of weightless spring	I5
	Spring modulus associated with the X direction (lb./in.)	F10.0
	Spring modulus associated with the Y direction (lb./in.)	F10.0
	Spring modulus associated with angular displacement (in.-lb./radian)	F10.0

Table C.1 (continued)

Card Type	Contents	Format
VIII[d]	Node point number for location of rigid mass	I5
	Weight of rigid mass (lb.)	F10.0
	Mass moment of inertia of rigid mass (lb./in.^{2})	F10.0

[a]Node is restrained if restraint flag is not zero. The number of Type V cards is equal to the number of node points given on card Type II.

[b]The number of Type VI cards is equal to the number of beam elements given on card Type II. If no beam elements are used, no Type VI cards appear in the input data.

[c]Terminate entry of Type VII cards with a blank card.

[d]Terminate entry of Type VIII cards with a blank card.

APPENDIX D

COMPUTER PROGRAM OUTPUT

The computer program developed to implement solution of the transient dynamics of plane structures composed of beam elements of uniformly distributed mass, weightless springs, and rigid masses prints out all input data. The element stiffness and mass matrices, the assembled coupling matrix (A), and the series approximations to the matrix functions are printed. The major output of the program is the printout of the node point displacements and velocities at each point in time, as specified on the input cards. The x displacement for the specified node point is punched on cards for computer plotting.

APPENDIX E

COMPUTER PROGRAM LISTING

The listing for the computer program developed to implement solution of the transient dynamics of plane structures composed of beam elements of uniformly distributed mass, weightless springs, and rigid masses is given on the following pages of this appendix.

```
C
C     MAIN PROGRAM
C
      COMMON TITLE(18),NUMNP,NUMEL,XNP(10),YNP(10),IRX(10),IRY(10),
     1      IRT(10),EE(9),EA(9),EEI(9),ESW(9),INP(9),JNP(9),
     2       R(6,6),ESM(6,6),ESG(6,6),SSG(30,30),EMM(6,6),EMG(6,6),
     3      SMG(30,30),SMSG(30,30),L,EL,E,ECA,EI,U,RG,CR,CG
      COMMON /MATEXP/ C(60,60),HP(60,60),A(60,60),OPT(60,60),X(60),
     1          F(30),Z(60),Y(60),XIC(60),TOP(60),ITMAX,KK,LL,MM,
     2                JJFLAG,NI,TIME,TMAX,TZERO,NE,T,IIZ,ICONTR,
     3                PLTINC,MATYES,ICCS,JFLAG,PLT,IONODE
      COMMON /PLOT/ TPLOT(99),XPLOT(99)
      REAL * 4 MXY,MMI
C
C     INITIALIZE ARRAYS
C
      DO 1 I=1,10
      XNP(I)=0.0
      YNP(I)=0.0
      IRX(I)=0
      IRY(I)=0
    1 IRT(I)=0
      DO 2 I=1,9
      EE(I)=0.0
      EA(I)=0.0
      EEI(I)=0.0
      ESW(I)=0.0
      INP(I)=0
    2 JNP(I)=0
      DO 3 I=1,6
      DO 3 J=1,6
      R(I,J)=0.0
      ESM(I,J)=0.0
      ESG(I,J)=0.0
      EMM(I,J)=0.0
    3 EMG(I,J)=0.0
      DO 4 I=1,30
      F(I) = 0.0
      DO 4 J=1,30
      SSG(I,J)=0.0
      SMG(I,J)=0.0
    4 SMSG(I,J)=0.0
      DO 5 I=1,60
      DO 5 J=1,60
      C(I,J)=0.0
      HP(I,J)=0.0
      A(I,J)=0.0
    5 OPT(I,J)=0.0
      DO 6 I=1,60
      X(I)=0.0
      Z(I)=0.0
      Y(I)=0.0
      XIC(I)=0.0
    6 TOP(I)=0.0
C
C     READ AND PRINT INPUT DATA
C
    7 READ (5,1001) (TITLE(I),I=1,18)
 1001 FORMAT (18A4)
      WRITE (6,1002) (TITLE(I),I=1,18)
 1002 FORMAT (1H1,18A4)
      READ(5,1003) NUMNP,NUMEL,IONODE
 1003 FORMAT (3I5)
      WRITE (6,1004) NUMNP
 1004 FORMAT(1H0,22HNUMBER OF NODE POINTS ,I4)
      WRITE (6,1005) NUMEL
 1005 FORMAT(1H0,24HNUMBER OF BEAM ELEMENTS ,I4)
```

```
      WRITE (6,1006) IONODE
 1006 FORMAT (1H0,32HX DISPLACEMENT PLOTTED FOR NODE ,I5)
      READ (5,1007) CR,CG
 1007 FORMAT (2E10.3)
      WRITE (6,1008) CR,CG
 1008 FORMAT(1H0,30HABSOLUTE DAMPING COEFFICENT = ,E11.4,2X,
     130HRELATIVE DAMPING COEFFICENT = ,E11.4)
      READ (5,3001) TZERO,TMAX,T,PLTINC,ITMAX
 3001 FORMAT (4F10.0,5X,I5)
      WRITE(6,2000) TZERO
 2000 FORMAT(1H0,23HINITIAL PROBLEM TIME = ,F10.4)
      WRITE(6,2001) TMAX
 2001 FORMAT(1H0,21HFINAL PROBLEM TIME = ,F10.4)
      WRITE(6,2002) T
 2002 FORMAT(1H0,34HTIME INCREMENT USED FOR EXP(AT) = ,F10.4)
      WRITE(6,2003) PLTINC
 2003 FORMAT(1H0,36HTIME INCREMENT FOR PRINTED OUTPUT = ,F10.4)
      WRITE(6,2004) ITMAX
 2004 FORMAT(1H0,41HNUMBER OF TERMS IN SERIES APPROXIMATION = ,I3)
      WRITE (6,1009)
 1009 FORMAT(54H0NODE NUMBER  X-COORDINATE  Y-COORDINATE  X-RESTRAINT ,
     127HY-RESTRAINT THETA-RESTRAINT)
      DO 8  I = 1,NUMNP
      READ(5,1010) I,XNP(I),YNP(I),IRX(I),IRY(I),IRT(I)
 1010 FORMAT (I5,2F10.0,3I5)
 8    WRITE(6,1011)I,XNP(I),YNP(I),IRX(I),IRY(I),IRT(I)
 1011 FORMAT(1H0,5X,I5,10X,F7.3,7X,F7.3,3X,I1,10X,I1,14X,I1)
C
C     START LOOP TO DETERMINE STRUCTURE PROPERTIES
C
      IF (NUMEL .EQ. 0) GO TO 12
      WRITE (6,1012)
 1012 FORMAT(1H0,44HBEAM NUMBER    ELASTIC MODULUS AREA INERTIA ,
     125HWEIGHT/INCH I-NODE J-NODE)
      DO 9  I = 1,NUMEL
      READ(5,1013) I,EE(I),EA(I),EEI(I),ESW(I),INP(I),JNP(I)
 1013 FORMAT (I5,4F10.0,2I5)
 9    WRITE (6,1014)I,EE(I),EA(I),EEI(I),ESW(I),INP(I),JNP(I)
 1014 FORMAT(4X,I4,8X,E12.4,3X,F5.0,F9.2,F10.2,4X,I3,4X,I3)
      DO 11 L=1,NUMEL
      I  = INP(L)
      J  = JNP(L)
      X1 = XNP(I)
      Y1 = YNP(I)
      X2 = XNP(J)
      Y2 = YNP(J)
      EL=SQRT((X2-X1)**2+(Y2-Y1)**2)
      E = EE(L)
      ECA = EA(L)
      EI = EEI(L)
      U = ESW(L)/386.4
      RG = SQRT(EI/ECA)
C
C     DETERMINE STIFFNESS MATRIX
C
      CALL  ELSTIF
      WRITE (6,1015)
 1015 FORMAT (40H1ELEMENT STIFFNESS MATRIX IN MEMBER AXIS)
      WRITE (6,1016)  ((ESM(II,JJ), JJ=1,6 ), II=1,6)
 1016 FORMAT (1H ,6(5X,F10.4))
C
C     DETERMINE ROTATION TRANSFORMATION MATRIX
C
      CSANG=(X2-X1)/EL
      SNANG=(Y2-Y1)/EL
 10   R(1,1) = CSANG
      R(1,2) = -SNANG
```

```
      R(2,1) =  SNANG
      R(2,2) = CSANG
      R(3,3) = 1.0
      R(4,4) = CSANG
      R(4,5) = -SNANG
      R(5,4) = SNANG
      R(5,5) = CSANG
      R(6,6)= 1.0
C
C     TRANSFORM STIFFNESS MATRIX FROM ELEMENT TO STRUCTURE AXIS
C
      CALL MMULT(R,ESM,ESG,6)
      CALL MTMUL(ESM,R,ESG,6)
C
C      ADD  ELEMENT STIFFNESS MATRIX TO STRUCTURE STIFFNESS MATRIX
CC
      I1 = 3*I -2
      I2 = 3*I -1
      I3 = 3*I
      J1 = 3*J -2
      J2 = 3*J -1
      J3 = 3*J
      SSG(I1,I1) = SSG(I1,I1) + ESG(1,1)
      SSG(I1,I2) = SSG(I1,I2) + ESG(1,2)
      SSG(I1,I3) = SSG(I1,I3) + ESG(1,3)
      SSG(I2,I1) = SSG(I2,I1) + ESG(2,1)
      SSG(I2,I2) = SSG(I2,I2) + ESG(2,2)
      SSG(I2,I3) = SSG(I2,I3) + ESG(2,3)
      SSG(I3,I1) = SSG(I3,I1) + ESG(3,1)
      SSG(I3,I2) = SSG(I3,I2) + ESG(3,2)
      SSG(I3,I3) = SSG(I3,I3) + ESG(3,3)
      SSG(I1,J1) = SSG(I1,J1) + ESG(1,4)
      SSG(I1,J2) = SSG(I1,J2) + ESG(1,5)
      SSG(I1,J3) = SSG(I1,J3) + ESG(1,6)
      SSG(I2,J1) = SSG(I2,J1) + ESG(2,4)
      SSG(I2,J2) = SSG(I2,J2) + ESG(2,5)
      SSG(I2,J3) = SSG(I2,J3) + ESG(2,6)
      SSG(I3,J1) = SSG(I3,J1) + ESG(3,4)
      SSG(I3,J2) = SSG(I3,J2) + ESG(3,5)
      SSG(I3,J3) = SSG(I3,J3) + ESG(3,6)
      SSG(J1,I1) = SSG(J1,I1) + ESG(4,1)
      SSG(J1,I2) = SSG(J1,I2) + ESG(4,2)
      SSG(J1,I3) = SSG(J1,I3) + ESG(4,3)
      SSG(J2,I1) = SSG(J2,I1) + ESG(5,1)
      SSG(J2,I2) = SSG(J2,I2) + ESG(5,2)
      SSG(J2,I3) = SSG(J2,I3) + ESG(5,3)
      SSG(J3,I1) = SSG(J3,I1) + ESG(6,1)
      SSG(J3,I2) = SSG(J3,I2) + ESG(6,2)
      SSG(J3,I3) = SSG(J3,I3) + ESG(6,3)
      SSG(J1,J1) = SSG(J1,J1) + ESG(4,4)
      SSG(J1,J2) = SSG(J1,J2) + ESG(4,5)
      SSG(J1,J3) = SSG(J1,J3) + ESG(4,6)
      SSG(J2,J1) = SSG(J2,J1) + ESG(5,4)
      SSG(J2,J2) = SSG(J2,J2) + ESG(5,5)
      SSG(J2,J3) = SSG(J2,J3) + ESG(5,6)
      SSG(J3,J1) = SSG(J3,J1) + ESG(6,4)
      SSG(J3,J2) = SSG(J3,J2) + ESG(6,5)
      SSG(J3,J3) = SSG(J3,J3) + ESG(6,6)
C
C     DETERMINE MASS MATRIX
      CALL  ELMASS
      WRITE (6,1017)
 1017 FORMAT (35HOELEMENT MASS MATRIX IN MEMBER AXIS)
      WRITE (6,1018)  ((EMM(II,JJ), JJ=1,6 ), II=1,6)
 1018 FORMAT(1H ,6(5X,E10.4))
C
C     TRANSFORM MASS MATRIX FROM ELEMENT TO STRUCTURE AXIS
```

```
      CALL  MMULT(R,EMM,EMG,6)
      CALL MTMUL(EMM,R,EMG,6)
C      ADD  ELEMENT MASS MATRIX TO STRUCTURE MASS MATRIX
      SMG(I1,I1) = SMG(I1,I1) + EMG(1,1)
      SMG(I1,I2) = SMG(I1,I2) + EMG(1,2)
      SMG(I1,I3) = SMG(I1,I3) + EMG(1,3)
      SMG(I2,I1) = SMG(I2,I1) + EMG(2,1)
      SMG(I2,I2) = SMG(I2,I2) + EMG(2,2)
      SMG(I2,I3) = SMG(I2,I3) + EMG(2,3)
      SMG(I3,I1) = SMG(I3,I1) + EMG(3,1)
      SMG(I3,I2) = SMG(I3,I2) + EMG(3,2)
      SMG(I3,I3) = SMG(I3,I3) + EMG(3,3)
      SMG(I1,J1) = SMG(I1,J1) + EMG(1,4)
      SMG(I1,J2) = SMG(I1,J2) + EMG(1,5)
      SMG(I1,J3) = SMG(I1,J3) + EMG(1,6)
      SMG(I2,J1) = SMG(I2,J1) + EMG(2,4)
      SMG(I2,J2) = SMG(I2,J2) + EMG(2,5)
      SMG(I2,J3) = SMG(I2,J3) + EMG(2,6)
      SMG(I3,J1) = SMG(I3,J1) + EMG(3,4)
      SMG(I3,J2) = SMG(I3,J2) + EMG(3,5)
      SMG(I3,J3) = SMG(I3,J3) + EMG(3,6)
      SMG(J1,I1) = SMG(J1,I1) + EMG(4,1)
      SMG(J1,I2) = SMG(J1,I2) + EMG(4,2)
      SMG(J1,I3) = SMG(J1,I3) + EMG(4,3)
      SMG(J2,I1) = SMG(J2,I1) + EMG(5,1)
      SMG(J2,I2) = SMG(J2,I2) + EMG(5,2)
      SMG(J2,I3) = SMG(J2,I3) + EMG(5,3)
      SMG(J3,I1) = SMG(J3,I1) + EMG(6,1)
      SMG(J3,I2) = SMG(J3,I2) + EMG(6,2)
      SMG(J3,I3) = SMG(J3,I3) + EMG(6,3)
      SMG(J1,J1) = SMG(J1,J1) + EMG(4,4)
      SMG(J1,J2) = SMG(J1,J2) + EMG(4,5)
      SMG(J1,J3) = SMG(J1,J3) + EMG(4,6)
      SMG(J2,J1) = SMG(J2,J1) + EMG(5,4)
      SMG(J2,J2) = SMG(J2,J2) + EMG(5,5)
      SMG(J2,J3) = SMG(J2,J3) + EMG(5,6)
      SMG(J3,J1) = SMG(J3,J1) + EMG(6,4)
      SMG(J3,J2) = SMG(J3,J2) + EMG(6,5)
      SMG(J3,J3) = SMG(J3,J3) + EMG(6,6)
   11 CONTINUE
C     READ AND PRINT INPUT DATA FOR LINEAR SPRINGS
   12 READ (5,1019) INODE , JNODE , SX , SY , STHETA
 1019 FORMAT ( 2I5,3F10.0)
      IF(INODE.EQ.0) GO TO 13
      WRITE (6,1020) INODE,JNODE,SX,SY,STHETA
 1020 FORMAT (31HOADDED SPRINGS CONNECTING NODE ,I5, 8HAND NODE,I5,
     1         11HX-DIRECTION,E10.4,11HY-DIRECTION,E10.4,
     2          8HROTATION,E10.4)
      INODE = 3*INODE - 2
      JNODE = 3*JNODE - 2
      SSG(INODE,INODE) = SSG(INODE,INODE) + SX
      SSG(INODE,JNODE) = SSG(INODE,JNODE) - SX
      SSG(JNODE,JNODE) = SSG(JNODE,JNODE) + SX
      SSG(JNODE,INODE) = SSG(JNODE,INODE) - SX
      SSG(INODE+1,INODE+1) = SSG(INODE+1,INODE+1) + SY
      SSG(INODE+1,JNODE+1) = SSG(INODE+1,JNODE+1) - SY
      SSG(JNODE+1,JNODE+1) = SSG(JNODE+1,JNODE+1) + SY
      SSG(JNODE+1,INODE+1) = SSG(JNODE+1,INODE+1) - SY
      SSG(INODE+2,INODE+2) = SSG(INODE+2,INODE+2) + STHETA
      SSG(INODE+2,JNODE+2) = SSG(INODE+2,JNODE+2) - STHETA
      SSG(JNODE+2,JNODE+2) = SSG(JNODE+2,JNODE+2) + STHETA
      SSG(JNODE+2,INODE+2) = SSG(JNODE+2,INODE+2) - STHETA
      GO TO 12
   13 CONTINUE
C
C     READ AND PRINT INPUT DATA FOR LUMPED MASSES
   14 READ (5,1021) INODE, MXY , MMI
```

```
 1021 FORMAT (I5,2F10.0)
      IF(INODE.EQ.0) GO TO 15
      WRITE (6,1022) INODE,MXY,MMI
 1022 FORMAT (28HOADDED LUMPED MASSES TO NODE,I5,12HTRANSLATION ,
     1        E10.4, 8HROTATION,E10.4)
      MXY = MXY/386.4
      MMI = MMI/386.4
      INODE = 3*INODE - 2
      SMG(INODE,INODE) = SMG(INODE,INODE) + MXY
      SMG(INODE+1,INODE+1) = SMG(INODE+1,INODE+1) + MXY
      SMG(INODE+2,INODE+2) = SMG(INODE+2,INODE+2) + MMI
      GO TO 14
 15   CONTINUE
C     MODIFY STRUCTURE STIFFNESS AND MASS MATRICES FOR CONSTRAINTS
      DO  16 I = 1,NUMNP
      M1 = 3*I - 2
      M2 = 3*I - 1
      M3 = 3*I
      IF(IRX(I).NE.0) CALL MODIFY(M1)
      IF(IRY(I).NE.0) CALL MODIFY(M2)
      IF(IRT(I).NE.0) CALL MODIFY(M3)
 16   CONTINUE
      WRITE (6,1023)
 1023 FORMAT (27H1STRUCTURE STIFFNESS MATRIX)
      NC = 10
      DO 17 NCM = 1,21,10
      WRITE (6,1024) ((SSG(I,J),J=NCM,NC),I=1,M3)
 1024 FORMAT(1H ,1P10E11.3)
      IF(M3-NC) 18,18,17
 17   NC = NC + 10
 18   NC = 10
      WRITE (6,1025)
 1025 FORMAT (22H1STRUCTURE MASS MATRIX)
      DO 19 NCM = 1,21,10
      WRITE (6,1024) ((SMG(I,J),J=NCM,NC),I=1,M3)
      IF(M3-NC) 20,20,19
 19   NC = NC + 10
 20   CONTINUE
C
C     INVERT STRUCTURE MASS MATRIX AND POST MULTIPLY BY
C     STRUCTURE STIFFNESS MATRIX
      CALL MIV (SMG,SMSG,M3,30)
C
      CALL MMLLT(SMG,SSG,SMSG,30)
C
      CALL MTXP
      STOP
      END
```

```
      SUBROUTINE MODIFY (M)
      COMMON TITLE(18),NUMNP,NUMEL,XNP(10),YNP(10),IRX(10),IRY(10),
     1       IRT(10),EE(9),EA(9),FEI(9),ESW(9),INP(9),JNP(9),
     2        R(6,6),ESM(6,6),ESG(6,6),SSG(30,30),EMM(6,6),EMG(6,6),
     3       SMG(30,30),SMSG(30,30),L,EL,E,ECA,EI,U,RG
      N = 3*NUMNP
      DO 1 I=1,N
      SSG(I,M) = 0.0
      SSG(M,I) = 0.0
      SMG(I,M) = 0.0
      SMG(M,I) = 0.0
 1    CONTINUE
      SSG(M,M) = 1.0
      SMG(M,M) = 1.0
      RETURN
      END
```

```
      SUBROUTINE ELSTIF
      COMMON TITLE(18),NUMNP,NUMEL,XNP(10),YNP(10),IRX(10),IRY(10),
     1       IRT(10),EE(9),EA(9),EEI(9),ESW(9),INP(9),JNP(9),
     2        R(6,6),ESM(6,6),ESG(6,6),SSG(30,30),EMM(6,6),EMG(6,6),
     3       SMG(30,30),SMSG(30,30),L,EL,E,ECA,EI,U,RG,CR,CG
      DO 1 I = 1,6
      DO 1 J = 1,6
    1 ESM(I,J) = 0.0
      ESM(1,1) = ECA*E/EL
      ESM(1,4) = -ESM(1,1)
      ESM(4,1) = -ESM(1,1)
      ESM(4,4) =  ESM(1,1)
      ESM(2,2) = 12.*E*EI/EL**3
      ESM(5,5) = ESM(2,2)
      ESM(2,5) =-ESM(2,2)
      ESM(5,2) =-ESM(2,2)
      ESM(2,3) = 6.*E*EI/(EL*EL)
      ESM(2,6) = ESM(2,3)
      ESM(3,2) = ESM(2,3)
      ESM(6,2) = ESM(2,3)
      ESM(3,5) =-ESM(2,3)
      ESM(5,3) =-ESM(2,3)
      ESM(5,6) =-ESM(2,3)
      ESM(6,5) =-ESM(2,3)
      ESM(3,3) = ESM(2,2)*EL*EL/3.
      ESM(6,6) = ESM(3,3)
      ESM(3,6) = ESM(3,3)/2.
      ESM(6,3) = ESM(3,6)
      RETURN
      END
```

```
      SUBROUTINE  ELMASS
      COMMON TITLE(18),NUMNP,NUMEL,XNP(10),YNP(10),IRX(10),IRY(10),
     1       IRT(10),EE(9),EA(9),EEI(9),ESW(9),INP(9),JNP(9),
     2        R(6,6),ESM(6,6),ESG(6,6),SSG(30,30),EMM(6,6),EMG(6,6),
     3       SMG(30,30),SMSG(30,30),L,EL,E,ECA,EI,U,RG,CR,CG
      DO 1 I = 1,6
      DO 1 J = 1,6
    1 EMM(I,J) = 0.0
      EMM(1,1) = U*EL/3
      EMM(4,4) = EMM(1,1)
      EMM(1,4) = EMM(1,1)/2.
      EMM(4,1) = EMM(1,4)
      EMM(2,2) = U*EL*(13./35. +((RG/EL)**2.)*6./5.)
      EMM(5,5) = EMM(2,2)
      EMM(2,5) = U*EL*(9./70. - ((RG/EL)**2.)*6./5.)
      EMM(5,2) = EMM(2,5)
      EMM(2,3) = U*EL*(11./210.*EL +((RG/EL)**2.)*EL/10.)
      EMM(3,2) = EMM(2,3)
      EMM(5,6) = -EMM(2,3)
      EMM(6,5) = -EMM(2,3)
      EMM(2,6) = U*EL*(-13.*EL/420.+((RG/EL)**2)*EL/10.)
      EMM(6,2) = EMM(2,6)
      EMM(3,3) = U*EL*(EL*EL/105. +((RG/EL)**2)*EL*EL*2./15.)
      EMM(6,6) = EMM(3,3)
      EMM(3,5) = -EMM(2,6)
      EMM(5,3) = -EMM(2,6)
      EMM(3,6) = U*EL*(-EL*EL/140. -((RG/EL)**2)*EL*EL/30.)
      EMM(6,3) = EMM(3,6)
      RETURN
      END
```

```
      SUBROUTINE MIV (A,U,NM,M)
C     MATRIX INVERSION BY GAUSS-JORDAN METHOD
      DIMENSION A(M,M),U(M,M)
      DO 1 I=1,M
      DO 1 J=1,M
      U(I,J)=0.
      IF(I.EQ.J) U(I,J)=1.0
    1 CONTINUE
      EPS=0.0000001
      DO 11 I=1,NM
      K=I
      IF(I-NM)2,6,2
    2 IF(A(I,I)-EPS)3,4,6
    3 IF(-A(I,I)-EPS)4,4,6
    4 K=K+1
      DO 5 J=1,NM
      U(I,J)=U(I,J)+U(K,J)
    5 A(I,J)=A(I,J)+A(K,J)
      GO TO 2
    6 DIV=A(I,I)
      DO 7 J=1,NM
      U(I,J)=U(I,J)/DIV
    7 A(I,J)=A(I,J)/DIV
      DO 11 MM=1,NM
      DELT=A(MM,I)
      IF(ABS(DELT)-EPS)11,11,8
    8 IF(MM-I)9,11,9
    9 DO 10 J=1,NM
      U(MM,J)=U(MM,J)-U(I,J)*DELT
   10 A(MM,J)=A(MM,J)-A(I,J)*DELT
   11 CONTINUE
      DO 12 I=1,NM
      DO 12 J=1,NM
   12 A(I,J)=U(I,J)
      RETURN
      END


      SUBROUTINE MMULT  (A,B,C,N)
      DIMENSION A(N,N),B(N,N),C(N,N)
C     MATRIX MULTIPLICATION MCTS
      DO 1 I = 1,N
      DO 1 J = 1,N
      C(I,J) = 0.0
      DO 1 K = 1,N
    1 C(I,J) = C(I,J) + A(I,K)*B(K,J)
      DO 2 I=1,N
      DO 2 J=1,N
    2 B(I,J) = C(I,J)
      RETURN
      END


      SUBROUTINE MTMUL  (A,B,C,N)
      DIMENSION A(N,N),B(N,N),C(N,N)
C      TRANSPOSE MULTIPLICATION
      DO 1 I = 1,N
      DO 1 J = 1,N
      C(I,J) = 0.0
      DO 1 K = 1,N
    1 C(I,J) = C(I,J) + A(I,K)*B(J,K)
      RETURN
      END
```

```
      SUBROUTINE MTXP
      COMMON TITLE(18),NUMNP,NUMEL,XNP(10),YNP(10),IRX(10),IRY(10),
     1       IRT(10),EE(9),EA(9),EEI(9),FSM(9),INP(9),JNP(9),
     2        R(6,6),ESM(6,6),ESG(6,6),SSG(30,30),EMM(6,6),EMG(6,6),
     3       SMG(30,30),SMSG(30,30),L,EL,E,ECA,EI,U,RG,CR,CG
      COMMON /MATEXP/ C(60,60),HP(60,60),A(60,60),OPT(60,60),X(60),
     1            F(30),Z(60),Y(60),XIE(60),TOP(60),ITMAX,KK,LL,MM,
     2                 JJFLAG,NI,TIME,TMAX,TZERO,NE,T,IIZ,ICONTR,
     3                 PLTINC,MATYES,ICCS,JFLAG,PLT,IONODE
      COMMON /PLOT/ TPLOT(99),XPLOT(99)
C
C
C     THIS PROGRAM CALCULATES THE SOLUTION OF A MATRIX OF FIRST
C     ORDER, SIMULTANEOUS DIFFERENTIAL EQUATIONS W/ CONSTANT COEFFICIENT
C     OF THE FORM DX/DT = AX + Z.
C
C     THE METHOD IS PAYNTER-S MATRIX EXPONENTIAL METHOD
C
C     THE SOLUTION IS GIVEN FOR INCREMENTS OF THE INDEPENDENT
C     VARIABLE (T) FROM TZERO THROUGH TMAX
C
C     COMPUTES MATRICES C = EXP(A*T) AND
C                       HP = (C-I)*A INVERSE
C     SOLUTION X(N*T) = C*X((N-1)*T)+HP*Z((N-1)*T)
C
C     OUTPUT FROM THE PROGRAM IS PRINTED AT INTERVALS PLTINC.
C     THE PROGRAM USES SUBROUTINES DISTRB AND OUTPUT
C
C
C
C     NI=0 ON 1-ST PASS.  SET TO 1 ON 1-ST CALL OF OUTPUT.
C
      NI=0
      NE = 6*NUMNP
      M3 = 3*NUMNP
      DO 2 I = 1,M3
      J = I + M3
 2    A(I,J) = 1.0
      DO 3 I = 1,M3
      IM3 = I + M3
      A(IM3,IM3) = -CR
      DO 3 J = 1,M3
      JM3 = J + M3
      A(IM3,JM3) = A(IM3,JM3) - CG*SMSG(I,J)
 3    A(IM3,J) = -SMSG(I,J)
 5    JJFLAG=0
C     CALCULATION OF MATRIX EXPONENTIALS C AND HP
      DO 6 I=1,NE
 6    C(I,I)=1.
      DO 7 I=1,NE
 7    HP(I,I)=T
C
C
      DO 9 I=1,NE
      DO 9 J=1,NE
 9    OPT(I,J)=C(I,J)
C
C NOW FORM THE MATRIX EXPONENTIALS C=EXP(A*T) AND HP=((C-I)*A INVERSE)
C
      AL=1.0
C
 10   DO 18 KL=1,ITMAX
C
      KLM=KL
      ALL=T/AL
      AL=AL+1.0
      TALLL=T/AL
```

```
C
      DO 12 I=1,NE
C
C
      DO 11 J=1,NE
      TOP(J)=0.0
      DO 11 KX=1,NE
 11   TOP(J)=TOP(J)+OPT(I,KX)*A(KX,J)
C
      DO 12 J=1,NE
 12   OPT(I,J)=TOP(J)*ALL
C
C     OPT=MATRIX TERM IN SERIES APPROX. =((A*T)**K)/K FACTORIAL
C
      DO 13 I=1,NE
      DO 13 J=1,NE
 13   C(I,J)=C(I,J)+OPT(I,J)
 14   IF(ITMAX-KL)17,17,15
 15   DO 16 I=1,NE
      DO 16 J=1,NE
 16   HP(I,J)=HP(I,J)+OPT(I,J)*TALLL
 17   CONTINUE
 18   CONTINUE
C
C     C(I,J) IS THE MATRIX EXPONENTIAL C=EXP(A*T)
C     AND HP(I,J) IS THE ((C-I)*A INVERSE) MATRIX
C NOW WE READ (OR CALL SUBROUTINE FOR) DISTURBANCE VECTOR
C
 19   TIME=TZERO
      PLT=0.
 20   CALL DISTRB
C     ON 1-ST CALL OF OUTPUT NI SET TO 1
 21   CALL OUTPUT
C
C NOW COMES THE EQUATION SOLUTION BASED ON
C     X(NT)=M*X(NT-1)+((M-I)A INV.)*Z(NT-1)
 22   CONTINUE
C
 23   IF(JJFLAG)24,25,24
 24   CALL DISTRB
 25   CONTINUE
 26   DO 27 I=1,NE
      Y(I)=C(I,1)*X(1)+HP(I,1)*Z(1)
      DO 27 J=2,NE
 27   Y(I)=Y(I)+C(I,J)*X(J)+HP(I,J)*Z(J)
 28   DO 29 I=1,NE
 29   X(I)=Y(I)
C
C ONE TIME INCREMENT OF THE SOLUTION HAS JUST BEEN FOUND
C
C NOW PLOT AND PRINT IF PLTINC INTERVAL HAS ELAPSED
C
      JJFLAG=1
      TIME=TIME+T
      PLT=PLT+T
      IF(PLT-PLTINC)31,30,30
 30   CALL OUTPUT
      PLT=0.
 31   IF(TIME-TMAX)22,32,32
 32   PLT=0.0
      NI = NI - 1
 34   WRITE (7,1002) NI
 1002 FORMAT (I2)
      IF(ICNODE.EQ.0) GO TO 40
      WRITE (7,1003) ((TPLOT(I),XPLOT(I)),I=1,NI)
 1003 FORMAT (8E10.3/(8E10.3))
 40   RETURN
      END
```

```
      SUBROUTINE OUTPUT
      COMMON TITLE(18),NUMNP,NUMEL,XNP(10),YNP(10),IRX(10),IRY(10),
     1       IRT(10),EE(9),EA(9),EEI(9),ESH(9),INP(9),JNP(9),
     2        R(6,6),ESM(6,6),ESG(6,6),SSG(30,30),EMM(6,6),EMG(6,6),
     3       SMG(30,30),SMSG(30,30),L,EL,E,ECA,EI,U,RG,CR,CG
      COMMON /MATEXP/ C(60,60),HP(60,60),A(60,60),OPT(60,60),X(60),
     1            F(30),Z(60),Y(60),XIC(60),TOP(60),ITMAX,KK,LL,MM,
     2                  JJFLAG,NI,TIME,TMAX,TZERO,NE,T,TIZ,ICONTR,
     3                  PLTINC,MATYES,ICCS,JFLAG,PLT,IONODE
      COMMON /PLOT/ TPLOT(99),XPLOT(99)
C
C
C
      IF(NI)7,1,7
 1    NI=1
      NC=10
      DO 2 NCM=1,51,10
      WRITE(6,1001)     ((A(I,J),J=NCM,NC),I=1,NE)
 1001 FORMAT (2H1A    /(1H ,1P10E11.3))
      IF(NE-NC) 3,3,2
 2    NC=NC+10
C
 3    NC=10
      DO 4 NCM=1,51,10
      WRITE(6,1002) ((C(I,J),J=NCM,NC),I=1,NE)
 1002 FORMAT (2H0C/(1H ,1P10E11.3))
      IF(NE-NC) 5,5,4
 4    NC=NC+10
C
 5    NC=10
      DO 6 NCM=1,51,10
      WRITE(6,1003) ((HP(I,J),J=NCM,NC),I=1,NE)
 1003 FORMAT (3H0HP/(1H ,1P10E11.3))
      IF(NE-NC) 7,7,6
 6    NC=NC+10
C
 7    WRITE (6,1004) TIME
 1004 FORMAT(1H1,6HTIME =,1PE10.3,1X,29HX-DISPLACEMENT Y-DISPLACEMENT,
     16X,8HROTATION,5X,10HX-VELOCITY,5X,10HY-VELOCITY,5X,10HT-VELOCITY)
      DO 8 I = 1,NUMNP
      K6 = NE/2 + 3*I
      K5 = K6 -1
      K4 = K5 - 1
      K3 = 3*I
      K2 = K3 - 1
      K1 = K2 - 1
      IF(IONODE.EQ.0) GO TO 8
      TPLOT(NI) = TIME
      XPLOT(NI) = X(3*IONODE - 2)
 8    WRITE (6,1005) I,X(K1),X(K2),X(K3),X(K4),X(K5),X(K6)
 1005 FORMAT (1H ,11HNODE NUMBER,I5,6(5X,1PE10.3))
      NI = NI + 1
 9    RETURN
      END
```

```
      SUBROUTINE DISTRB                                                   DISTR  10
      COMMON TITLE(18),NUMNP,NUMEL,XNP(10),YNP(10),IRX(10),IRY(10),       DISTR  20
     1      IPT(10),EE(9),EA(9),EEI(9),ESW(9),INP(9),JNP(9),              DISTR  21
     2      R(6,6),ESM(6,6),ESG(6,6),SSG(30,30),EMM(6,6),EMG(6,6),        DISTR  22
     3      SMG(30,30),SMSG(30,30),L,EL,E,ECA,EI,U,RG,CR,CG               DISTR  23
      COMMON /MATEXP/ C(60,60),HP(60,60),A(60,60),OPT(60,60),X(60),       DISTR  30
     1          F(30),Z(60),Y(60),XIC(60),TCP(60),ITMAX,KK,LL,MM,         DISTR  31
     2                JJFLAG,NI,TIME,TMAX,TZERO,NE,T,N1Z,ICONTR,          DISTR  32
     3                PLTINC,MATYES,ICCS,JFLAG,PLT,IONODE                 DISTR  33
C                                                                         DISTR  40
      M3 = 3*NUMNP                                                        DISTR  50
      FT = 1. - 10.*(TIME + T/2.)                                         DISTR  60
      IF(TIME.GT.0.1) FT = 0.0                                            DISTR  70
      F1 = 2000.*FT                                                       DISTR  80
      F2 = 4000.*FT                                                       DISTR  90
      F3 = -2000.*FT                                                      DIST  100
      F(4) = F1                                                           DIST  110
      F(7) = F2                                                           DIST  120
      F(10) = F3                                                          DIST  130
      DO 1 I = 1,M3                                                       DIST  140
      J = I + M3                                                          DIST  150
      Z(J) = 0.0                                                          DIST  160
      DO 1 K = 1,M3                                                       DIST  170
    1 Z(J) = Z(J) + SMG(I,K)*F(K)                                         DIST  180
      RETURN                                                              DIST  190
      END                                                                 DIST  200
```

VITA

William Christopher Terrill Stoddart was born in Detroit, Michigan, on August 25, 1941. He attended high school in Reading, Ohio, and received his undergraduate education at the University of Cincinnati and The University of Tennessee. He received a Bachelor of Science degree in Mechanical Engineering from The University of Tennessee in August, 1963. While attending the University of Cincinnati, he was employed at Wright Field in Dayton, Ohio; and after graduating from The University of Tennessee, he was employed by Pratt and Whitney Aircraft Corporation in East Hartford, Connecticut. He has been employed at Oak Ridge National Laboratory, which is operated by Union Carbide Corporation under contract with the United States Atomic Energy Commission, since September, 1965.

He entered the Graduate School at The University of Tennessee in June, 1966, and received the Master of Science degree in Engineering Mechanics in December, 1970.