8-1967

Conductive Sphere on a Charged Conductive Plane

Birney Robert Fish
University of Tennessee, Knoxville

To the Graduate Council:

I am submitting herewith a thesis written by Birney Robert Fish entitled "Conductive Sphere on a Charged Conductive Plane." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Physics.

Robert J. Lovell, Major Professor

We have read this thesis and recommend its acceptance:

R. H. Ritchie, Isabel H. Tipton

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

August 4, 1967

To the Graduate Council:

I am submitting herewith a thesis written by Birney Robert Fish entitled "Conductive Sphere on a Charged Conductive Plane." I recommend that it be accepted for nine quarter hours of credit in partial fulfillment of the requirements for the degree of Master of Science, with a major in Physics.

Robert J. Lovell
Major Professor

We have read this thesis and recommend its acceptance:

R.H. Ritchie

Isabel H Tipton

Accepted for the Council:

Hilton A. Smith
Vice President for
Graduate Studies and Research

CONDUCTIVE SPHERE ON A CHARGED CONDUCTIVE PLANE

A Thesis

Presented to

the Graduate Council of

The University of Tennessee

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

---

by

Birney Robert Fish

August 1967

ACKNOWLEDGMENTS

The experimental work described in this study was based on ideas developed by Mr. F. G. Karioris, Associate Professor of Physics, Marquette University. Substantially all of the laboratory work was done by Mr. Karioris with the assistance of Mr. J. L. Thompson, Health Physics Technology Section, Health Physics Division, Oak Ridge National Laboratory*; and with only minor assistance by the author. Their aid has been indispensible in this effort and is gratefully acknowledged. The author is indebted to Miss Blanche Reed, Health Physics Technology Section, for her excellent work in preparing this manuscript under a most demanding schedule. Thanks also are due to the author's graduate committee, Dr. R. J. Lovell, Dr. Isabel H. Tipton, and Dr. R. H. Ritchie, for their thoughtful review of the material presented in this report and for their many valuable suggestions.

*Operated by Union Carbide Corporation for the United States Atomic Energy Commission.

TABLE OF CONTENTS

LIST OF FIGURES

ABSTRACT

The electrostatic problem of a conductive sphere on a charged conductive plane has been solved using theoretical and experimental methods. From elementary electrostatics it is seen that there will be a uniform electric field E_o volts/meter normal to a semi-infinite plane bearing σ_o coulombs/meter^2. Based on an image charge approach, it was determined that a sphere of radius R meters will acquire a total equilibrium charge $q = \frac{2}{3}\pi^3 R^2 \sigma_o$ coulombs at contact with the plane, and the uniform field will be perturbed, resulting in a field strength $E = 4.5 E_o$ volts/meter at the top of the sphere. An approximation, valid to within less than about 1 per cent, for the repulsive force acting on the sphere was obtained by extrapolating to contact, series solutions of the Laplace equation for finite separation between sphere and plane: the force at contact is estimated to be $F_z = 1.537 \times 10^{-10} k R^2 E_o^2$ newtons; where k is the dielectric constant of the surrounding medium. The charge distribution on the sphere was determined experimentally by plating copper on a sphere-plane cathode and, within about 10 per cent, is given by the empirical expression $\sigma(\theta) = (0.8696 \cos^2\frac{\theta}{2} + 3.6304 \cos^4\frac{\theta}{2})\,\sigma_o$; where θ is the angle of colatitude in spherical polar coordinates referred to the axis of contact and concentric with the sphere.

CHAPTER I

INTRODUCTION

I. STATEMENT OF THE PROBLEM

If a conductive sphere is placed in contact with a semi-infinite, conductive plane, the configuration constitutes a three-dimensional equipotential surface. Assuming an isolated plane to be electrically charged to a known surface charge density (σ_o) originally, it is desired to determine the effect produced by placing a sphere on the plane. Specifically, the purpose of this investigation is to determine the force exerted on the sphere normal to the plane, the total charge acquired by the sphere and the perturbation in the electrical field due to its presence on the plane.

II. SCIENTIFIC JUSTIFICATION

Whereas there are classical solutions which are applicable to the problems of a thin disk or af a hemisphere on a plane [e.g., Jeans (1941)], no such solution exists for the sphere on a plane. Historically, the solutions to electrostatic problems involving three-dimensional bodies, such as two finite spheres or a sphere and a plane, have been approached by the method of images described by Maxwell(1892). Weber(1965), says, for example, "The electrostatic field of two finite conducting spheres can be described only by an infinite sequence of images." This restriction is especially damaging in that the infinite series that result from the method of

images are very unwieldy and have been made to converge in only a few comparatively simple cases.

In a recent paper, Taylor (1966), pointed out that, despite the fact the first observations that light objects lying on a surface will jump to a piece of rubbed amber were recorded about 2500 years ago, there remain areas of work on electrostatics not yet completely solved. He went on to indicate that the distribution of charge on axisymmetric bodies is known for only a few cases: Taylor (1964), had earlier published a derivation for the surface charge density over a spheroid of finite eccentricity.

III. TECHNOLOGICAL JUSTIFICATION

Beyond the esthetic challenge presented by the electrostatic problem of a perfect conducting sphere on a charged semi-infinite plane, there are many areas of technology in which a solution to this problem would be, at least, useful and, in some cases, vital. The various mechanisms of adhesion of particles to surfaces are important in understanding the behavior of airborne particles after they come in contact with a solid surface. Electrical and aerodynamic forces are the only ones that can influence particle deposition, contribute to preferred orientations of the deposit and also exert a net repulsion tending to lessen adhesion after deposition. These phenomena are important in connection with the retention and removal of particles in air sampling and air cleaning devices. Obviously, any factor tending to produce preferred locations and to affect orientation of particles collected in a

sampling device could produce significant artifacts in the deposit, thus obscuring important detail, as for example, the agglomeration of particles during deposition yields a deposit that usually cannot be differentiated from a collection of previously agglomerated particles.

Adventitious resuspension of deposited particles is of interest in such widely divergent fields as the dispersion of radioactive surface contamination in laboratory facilities or in open fields, the movement of microbial contaminants in hospital or space environments, the movement of sands and dunes, and soil erosion [e.g., Fish (1967)]. Deliberate decontamination requires knowledge of the adhesion as well as the removal forces. In this connection, a number of persons have attempted to utilize electrostatic forces to remove particles from solid surfaces, unfortunately the required fields are so high that the method has shown very little promise as a primary means of decontamination.

Electrical repulsive forces on spherical particles are important in numerous areas of the space programs carried out by the National Aeronautical and Space Administration. Recent discussions with personnel of the General Electric, Space and Missile Systems Division, have emphasized the possible importance of adhesion and repulsion of bacterial spores and other viable organisms on the surface of the Voyager and other planetary probes. There is even an electrical propulsion device which utilizes small particles repelled from a plane electrode as the working medium [e.g., Schultz (1961)].

A considerable technology, recently discussed by Ralston (1961), has arisen around the use of electrical means to separate minerals during benification of ores. While most of the processes depend upon dielectric phenomena, still the repulsion of conductive particles after contact with a charged electrode also plays a significant role.

In the increasingly important field of air pollution control, it has been recognized, for many years, that highly conductive particles can lead to serious reduction in collection (i.e., retention) efficiency of electrostatic precipitators [e.g., Lowe and Lucas (1953)]. Finally, the intrusion of dust into high voltage systems, e.g., ionization chambers, is obviously undesirable because of the spurious pulses produced when a particle suddenly is repelled from the surface of a charged part of the assembly.

These fields, in addition to others such as aerodynamics, all have significant problems for which an understanding of the sphere-on-a-plane problem would be beneficial.

IV. PLAN OF ATTACK

First, estimates of the total charge acquired by the sphere, of the repulsive force and of the magnitude of the electrical field will be obtained by comparison with solutions of other similar problems. Then, an image charge approach will be attempted. A solution for an equipotential sphere very near, but not touching, a charged plane will be derived based on a solution to the Laplace equation in a

bispherical coordinate system. And, finally, an essentially experimental approach utilizing a current-density-field analog will be used to verify results obtained by other methods.

V. UNITS

When following the derivation of a given author, insofar as possible, the same system of units will be used as employed in the referenced work. However, for work originating in the present study and in computations based on the work of others the rationalized MKS system will be used.

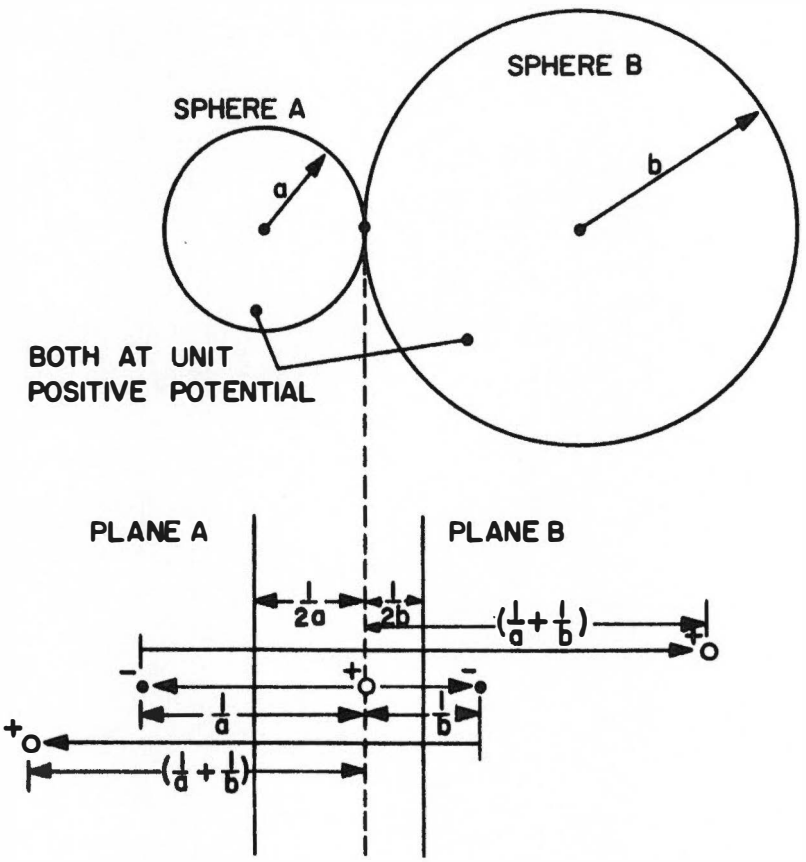
CHAPTER II

ESTIMATION OF CHARGE, FORCE AND FIELD

I. ESTIMATE OF CHARGE

Maxwell (1892) applied the methods of electrical images and electric inversion to the problem of the distribution of charge on two spheres in contact. Consider two contacting spheres of radii a and b, both at unit potential. If the system is inverted with respect to the point of contact, the result is two parallel planes located at $\frac{1}{2a}$ and $\frac{1}{2b}$ from the point of inversion and these planes are influenced by a unit positive point charge situated at the inversion point, Figure I. Similarly, negative images occur at $\frac{1}{a}, \frac{1}{a}+(\frac{1}{a}+\frac{1}{b}), \frac{1}{a}+2(\frac{1}{a}+\frac{1}{b}), \cdots$ $\frac{1}{a}+n(\frac{1}{a}+\frac{1}{b})$ for integral $n \geq 0$. Images of the point charge are located in plane A at $(\frac{1}{a}+\frac{1}{b}),\ 2(\frac{1}{a}+\frac{1}{b}),\ 3(\frac{1}{a}+\frac{1}{b}), \cdots n(\frac{1}{a}+\frac{1}{b})$ for integral values of $n \geq 1$. When the system is inverted back to the original geometry the first set of positive images becomes negative and the amount of charge (assuming the electrostatic system of units) is numerically equal to its distance from the point of contact. This distance, in sphere A, becomes $\frac{1}{n(\frac{1}{a}+\frac{1}{b})}$, $n = 1, 2, \cdots \infty$. The negative images become positive and are located at

$\frac{1}{\frac{1}{a}+n(\frac{1}{a}+\frac{1}{b})}$, $n = 0, 1, \cdots \infty$. The total charge on sphere A is



TWO PLANES RESULT FROM THE INVERSION OF TWO SPHERES

Figure 1. Two contacting spheres showing the electrical images in the system of planes formed as the electrical inversion of the spheres about their point of contact.

$$Q_A = \sum_{n=0}^{\infty} \frac{1}{\frac{1}{a} + n\left(\frac{1}{a}+\frac{1}{b}\right)} - \sum_{n=1}^{\infty} \frac{1}{n\left(\frac{1}{a}+\frac{1}{b}\right)}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n\left(\frac{1}{a}+\frac{1}{b}\right) - \frac{1}{b}} - \sum_{n=1}^{\infty} \frac{ab}{n(a+b)}.$$

Each of these series is divergent, but the combination

$$Q_A = \sum_{n=1}^{\infty} \frac{a^2 b}{n(a+b)\,[n(a+b)-a]}, \qquad (1)$$

is convergent. Similarly, the total charge on sphere B is given by

$$Q_B = \sum_{n=1}^{\infty} \frac{ab^2}{n(a+b)\,[n(a+b)-b]}. \qquad (2)$$

For the special case when sphere A is very small with respect to sphere B

$$Q_A \approx \sum_{n=1}^{\infty} \frac{a^2 b}{nb[nb]} = \frac{a^2}{b} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \frac{a^2}{b},$$

and the charge on sphere B remains practically undisturbed, being approximately the same as it would be at unit potential, uninfluenced by any other charged body. Thus, (in electrostatic units),

$$Q_B \approx b \times 1 = b.$$

Maxwell, pointed out that if one defines the mean surface density as the total charge divided by the total surface area, then

$$\bar{\sigma}_A = \frac{Q_A}{4\pi a^2} \approx \frac{\pi^2 a^2}{6 \times 4\pi a^2 b} = \frac{\pi}{24 b},$$

and

$$\bar{\sigma}_B \approx \frac{b}{4\pi b^2} = \frac{1}{4\pi b},$$

or

$$\bar{\sigma}_A \approx \frac{\pi^2}{6} \bar{\sigma}_B . \qquad (3)$$

If a plane, which incidentally is a very large sphere, bears a surface charge density σ_o, then a contacting sphere of radius r will accumulate a total charge

$$q = 4\pi r^2 \left(\frac{\pi^2}{6} \sigma_o\right) = \frac{2}{3} \pi^3 r^2 \sigma_o . \qquad (4)$$

This result holds for any self consistent system of units and will be referred to as Maxwell's solution.

II. ESTIMATE OF NORMAL FORCE

By comparison with known solutions for other shaped conductors on a charged plane, a plausible argument can be made to produce an estimate of the force on a sphere. Basically the approach is to determine the amount of charge acquired by the movable conductor and to estimate the effective electrical

field of the fixed plane in the neighborhood of this charge. The force then may be approximated as the product of the charge and the effective field strength.

In the case of a disk of radius r on a plane the normal pressure is given by

$$P = \frac{1}{2} \vec{D} \cdot \vec{E}_o ,$$

where $|\vec{D}|$ = the magnitude of the displacement vector = σ_o and $\vec{E}_o$ = undisturbed field above the plane.

At the surface of a conductor, both $\vec{D}$ and $\vec{E}_o$ are normal to the surface; thus the normal force on the disk is given by

$$F_n = \text{area} \times P = \frac{1}{2}(\pi r^2)|D| \times |E_o| \cos 0^\circ = (\pi r^2 \sigma_o)(\frac{1}{2} E_o),$$

but $\pi r^2 \sigma_o$ = total charge on the disk = q;

therefore $F_n = q E_{\text{effective}} = q(\frac{1}{2} E_o)$.

The effective field operating on the charge acquired by the disk is $\frac{1}{2} E_o$.

Similarly for the hemisphere, the total charge and the effective field can be calculated from

$$E(\theta) = 3 E_o \cos \theta,$$

$$D(\theta) = \epsilon E(\theta), \text{ parallel to } E(\theta),$$

and

$$P(\theta) = \frac{1}{2}[\epsilon E(\theta)] E(\theta) \cos 0^\circ = \frac{1}{2} \epsilon E^2(\theta);$$

where ϵ is the permittivity of the surrounding medium. The total charge on the hemisphere is obtained from

$$q = \int_0^{\pi/2} \epsilon E(\theta) \, d(\text{area}) = \int_0^{\pi/2} \epsilon [3E_o \cos \theta][2\pi r^2 \sin \theta \, d\theta] = 3\pi r^2 \epsilon E_o. \quad (5)$$

The vertical component of force on the hemisphere is calculated by summing the vertical component of the normal pressure on each differential area and is given by

$$F_n = \int_0^{\pi/2} \cos\theta \left[\frac{1}{2}\epsilon E^2(\theta)\right] d(\text{area}) = 9\pi\epsilon r^2 E_o^2 \int_0^{\pi/2} \cos^3\theta \sin\theta \, d\theta = \frac{9}{4}\pi r^2 \epsilon E_o^2, \quad (6)$$

or

$$F_n = qE_{\text{effective}} = (3\pi r^2 \epsilon E_o)(\tfrac{3}{4}E_o).$$

Thus the effective field acting on the total charge acquired by the hemisphere is $\frac{3}{4}E_o$. It is seen that, although the total charge on the hemisphere is three times greater than that on a disk of the same radius, the effective field is only 50 per cent greater for the hemisphere. This effect results from the greater average distance of the charge on the hemisphere from that on the plane as compared with that of the charge on the disk.

The effective field arises as the result of all of the charge on the plane except that which resides on the surface of the, presumed small, conductor in question, and is an indication of the perturbation in the charge distribution on the plane brought about by the charge on the small contacting body. If the body is a non-interacting point charge, or if it is far enough away from the plane so as to be treated as a point charge there will be no perturbation of the charge density on the plane and the effective field will be simply E_o.

Although somewhat more than twice as much total charge will be acquired by a sphere than by a hemisphere of equal radius, still the charge is located, on the average, further from the plane in the case of the sphere. Furthermore, in the

case of the hemisphere a disk-shaped area of πr^2 lying under the hemisphere bears no charge, while for the spherical conductor the surface charge density may be expected to remain finite under the sphere, going to zero only at the point of contact. Thus, it seems reasonable to assume that the charge on the sphere will exert more influence on the plane than would a non-interacting point charge hence the effective field is likely to be less than E_o; but, on the other hand, the interaction would be expected to be less than that for a hemisphere consequently the effective field should be greater than $\frac{3}{4}E_o$.

Without more detailed investigation, it would be unwise to guess exactly where in the range $\frac{3}{4}E_o$ to E_o the effective field might be for the sphere. However, it seems unlikely that the factor would lie exactly on the extremes of the range; therefore, it is assumed that a first approximation of the effective field is $\frac{7}{8}E_o$, it is not likely that the error would exceed ±12.5 per cent at most and probably would be less than ±10 per cent. Then, on this basis, within about 10 per cent, the vertical component of force is estimated to be

$$F_n = q E_{\text{effective}},$$

where

$$q = \frac{2}{3}\pi^3 r^2 \epsilon E_o \text{ (Maxwell's solution);}$$

finally

$$F_n \approx \left(\frac{2}{3}\pi^3 r^2 \epsilon E_o\right)\left(\frac{7}{8}E_o\right) = \frac{7\pi^3}{12}\epsilon r^2 E_o^2 . \qquad (7)$$

III. ESTIMATE OF FIELD STRENGTH

As a first approximation the field at the top of a conductive sphere in contact with a charged plane can be expected to be of the order of that at the top of a hemisphere of equal radius. However, the sphere, having twice the surface area of the hemisphere, will acquire somewhat more than twice the amount of charge on the hemisphere. Thus,

$$q_s = 4\pi r^2\left(\frac{\pi^2}{6}\sigma_o\right) = \frac{2}{3}\pi^3 r^2 \sigma_o \text{ (sphere)},$$

and

$$q_h = 3\pi r^2 \sigma_o \text{ (hemisphere)}.$$

The ratio is

$$\frac{q_s}{q_h} = \frac{2}{9}\pi^2 = 2.193.$$

Furthermore, most of the charge would be expected to reside on the upper half sphere as a result of repulsion from the charges on the plane. On the basis of the above it seems plausible to expect the charge density, hence the field, at the top of the sphere to be greater than that at the top of a hemisphere; i.e.

$$E_s > 3E_o.$$

Although more than half of the charge may lie above the midplane of the sphere, It appears reasonable to assume that the charge distribution is continuous

in such a way as to maintain the sphere as an equipotential surface and that not all of the charge would be found on the upper half sphere. In addition, even if all of the charge on the sphere were found on the upper half, the peak in the surface charge distribution function would not be likely to be quite as pronounced as that observed on the hemisphere because of the lessened interaction with the charges on the plane. Following this line of argument, an upper limit to the field strength may be assumed to be that which would be seen if all of the charge were distributed on the upper half-sphere in the same way as on a hemisphere; i.e.

$$E < 2.193 \times 3E_o = 6.57E_o.$$

If the midpoint of the range is chosen as a first approximation, the error should not exceed about ±37.5 per cent. Clearly, this is a poorer quality estimate than that obtained for the repulsion force or for the amount of charge accumulated on the sphere. Nevertheless, it is probably not far wrong and, perhaps, useful for some purposes to assume

$$E \approx 4.8E_o \qquad (8)$$

at the top of a conductive sphere in contact with a charged plane where E_o is the undisturbed field above the plane.

IV. SUMMARY OF ESTIMATES

At this point, estimates have been obtained of the total charge acquired, the repulsion force and the field strength at the top of a spherical conductor

residing on a charged conductive plane. Obviously, these estimates are not of uniform quality, although, even in the worse case, the expected precision of the estimate is probably not so bad as to preclude its use for some purposes. These approximations are summarized in Table I along with estimates of their precision.

TABLE I

SUMMARY OF ESTIMATES FOR CONDUCTIVE SPHERE ON A CHARGED CONDUCTIVE PLANE

Quantity	Estimate	Approximate Error (per cent)
Total Charge (coulombs)	$\frac{2}{3}\pi^3 r^2 \sigma_o$	0
Normal Force (newtons)	$\frac{7\pi^3}{12\epsilon} r^2 \sigma_o^2$	$\pm < 12.5$
Field at Top (volts/meter)	$4.8\, E_o$	$\pm < 37.5$

CHAPTER III

IMAGE CHARGE APPROACH

I. BACKGROUND

Credit for developing the method of electrical images usually is given to Sir William Thomson [e.g. Maxwell (1873)]. Basically, the method provides a means of solving quite complex boundary value problems in electrostatics through a series of approximations, each step of which requires only relatively simple mathematical manipulations.

Justification for use of the method is rather simple and is well based theoretically. Fundamentally, the classical electrostatic problem is to find a charge distribution that will conform to some specified condition on the boundary between a conductor and a charge-free space, and that will satisfy the Laplace equation everywhere outside the charged boundary. The image charge distribution usually consists of a series of point charges selected in such a way that the boundary conditions are met. Since the potential of a point charge satisfies the Laplace equation, and a linear combination of solutions also is a solution, then any such combination of point charges that can be found to conform to the fixed boundary conditions will provide a solution to the potential field in the space outside the boundary. Thus, whereas the electrical charge on an isolated sphere resides on its surface, nevertheless an equivalent, imaginary, point charge located at the center

of the sphere produces a potential on and outside of the spherical boundary which is in every way the same as that produced by a real charge distribution on the surface.

II. BASIC PRINCIPLES

An electric image is defined by Maxwell (1892), "as an electrified point or system of points on one side of a surface which would produce on the other side of that surface the same electrical action which the actual electrification of that surface really does produce."

Consider a semi-infinite plane at zero potential under the influence of a positive point charge q at a perpendicular distance z above the plane as shown in Figure 2. There will be a local accumulation of charge on the plane but the means of computing the charge distribution are quite difficult to apply. It is clear, however, that if the plane is to be maintained as an equipotential surface then the potential at every point on the plane must be nullified by an equal and opposite potential. It is easy to see that this can be accomplished by placing an equal but opposite sign point charge $-q$ at an equal distance z on the other side of the plane surface. This "mirror image" charge is the electrical image of the original point charge in a conductive plane.

Selection of the image charge necessary to maintain a spherical surface as an equipotential under the influence of an external point charge is more difficult than for the plane; however, the steps involved are nonetheless straightforward.

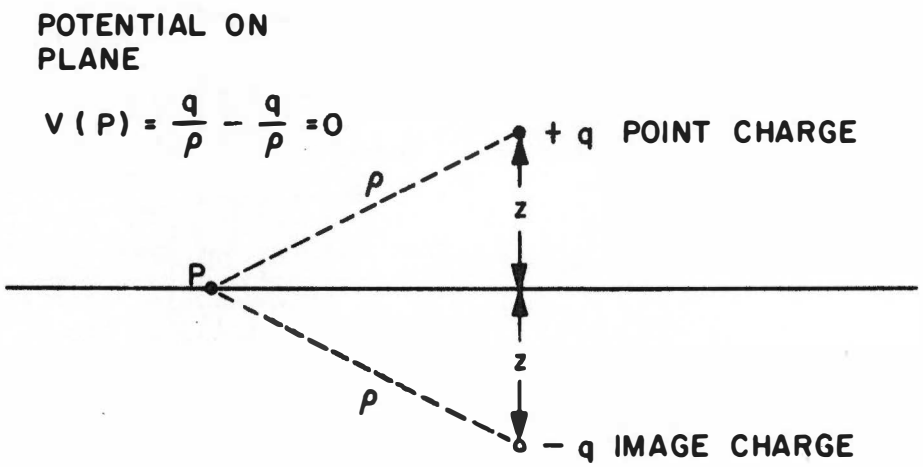


Figure 2. Image charge in a plane under the influence of a point charge outside the plane.

Referring to Figure 3, a positive point charge is located at A, which is an axial distance f from a sphere for which the potential of the surface is to be uniformly zero. A point C can be selected such that an image charge q_i may be found to yield an equal but opposite potential at any point P on the surface as that produced by the given point charge at A. It is seen that triangle ADP and CDP share the same interior angle θ, and one side, r. Thus, from the geometry

$$\rho_1^2 = r^2 + \overline{AD}^2 - 2r\,(\overline{AD})\cos\theta, \tag{9}$$

and

$$\rho_2^2 = r^2 + \overline{CD}^2 - 2r\,(\overline{CD})\cos\theta. \tag{10}$$

Furthermore, to make the potential at P equal zero

$$V(P) = 0 = \frac{q}{\rho_1} + \frac{q_i}{\rho_2}. \tag{11}$$

If we choose $\overline{CD}$ in such a way that triangles APD and PDC are similar, then

$$\frac{\overline{CD}}{r} = \frac{r}{\overline{AD}}, \text{ or } \overline{CD} = \frac{r^2}{\overline{AD}}. \tag{12}$$

Substituting (12) in (10) yields

$$\rho_2^2 = \frac{r^2}{\overline{AD}^2}\left[\overline{AD}^2 + r^2 - 2r\overline{AD}\cos\theta\right] = \frac{r^2}{\overline{AD}^2}\rho_1^2,$$

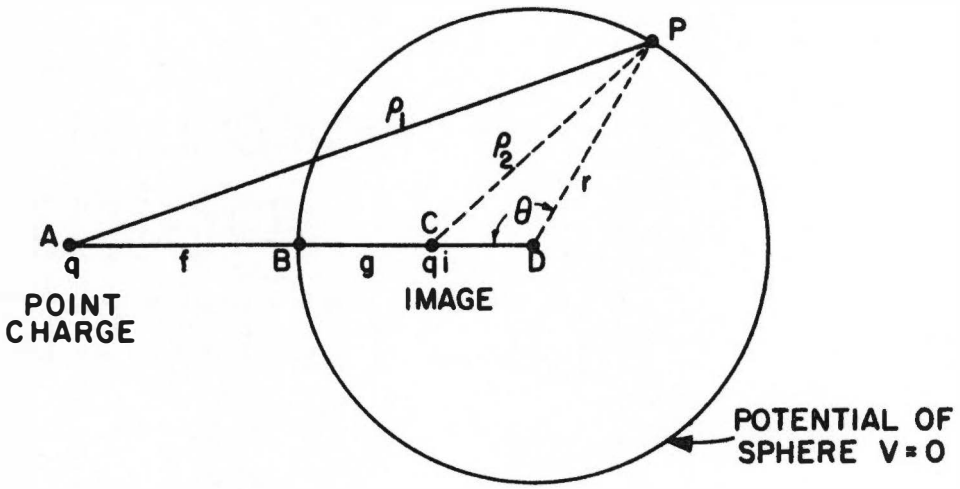


Figure 3. Image charge in a spherical conductor under the influence of a point charge outside the sphere.

which, along with (11), results in

$$0 = \frac{q}{\rho_1} + \left(\frac{\overline{AD}}{r}\right) \frac{q_i}{\rho_1};$$

finally,

$$q_i = -\frac{r}{\overline{AD}}\, q = -\left(\frac{r}{f+r}\right) q = -kq, \tag{13}$$

and

$$g = r - \overline{CD} = \left(\frac{r}{f+r}\right) f = kf; \tag{14}$$

where $$k = \frac{r}{f+r}.$$

The image of a charge q, located at a radial distance f from a sphere of radius r, is a charge -kq located on the axis at a distance kf inside the sphere. Whereas k=1 for a plane, it is less than one for a sphere and depends on the ratio of the radius of the sphere to the distance of the exterior point charge from the center of the sphere.

III. APPLICATION TO SPHERE ON A PLANE

In the case of a sphere near, but not touching, a grounded plane the problem is approached by assuming a point charge to be located at the center of the sphere. The point charge is taken to be the amount q necessary to produce a uniform potential $V = \frac{q}{r}$ on the surface of an isolated sphere, where V is the given potential of the sphere with respect to the plane. Then, beginning with q, an

infinite series of image charges is constructed in the plane and in the sphere as described in section II of this chapter.

For the conductive sphere in contact with a conductive plane, the potential of the spherical surface is zero with respect to the plane. External to the sphere there is a uniform field E_o which produces a variation in potential over a hypothetical sphere. The distribution of charge necessary to maintain an isolated sphere at zero potential in a uniform field is known from classical electrostatics [e.g., Jeans (1941)], to be

$$\sigma(\theta) = 3 \epsilon E_o \cos \theta; \qquad (15)$$

where θ is the angle from the direction of the field at the center of the sphere. For the purpose of the problem at hand it would be conceivable that one could start with either the full surface charge distribution given by (15) and its images, or with an equivalent dipole of moment

$$M = \epsilon r^3 E_o \qquad (16)$$

located at the center of the sphere. In view of the obvious complexity of treating images of three-dimensional charge distributions, it is tempting to begin with the equivalent dipole. Unfortunately, although the first image required to return the plane to zero potential is a mirror image dipole in the plane, the first image in the sphere is no longer a simple dipole, nor are succesive images in either the plane or the sphere. Consequently, it is necessary to examine the images of the three-dimensional charge distribution.

The origin of the uniform field E_o above the undisturbed plane is the uniform surface charge distribution

$$\sigma_o = \epsilon E_o ;$$

thus, from (15)

$$\sigma(\theta) = 3\,\sigma_o \cos\theta .$$

Referring to Figure 4 one can see in principle how the first and successive images in the sphere can be derived from the geometrical distribution of each "generator" image in the plane. Thus, the distribution on the first image in the sphere is given by

$$\sigma(\beta) = -\frac{r}{\rho}(-3\,\sigma_o \cos\theta'), \qquad (17)$$

where

$$\rho = \sqrt{(2r + r\cos\theta')^2 + r^2\sin^2\theta'} = r\sqrt{5 + 4\cos\theta'},$$

and finally

$$\sigma(\beta) = 3\,\sigma_o \frac{\cos\theta'}{\sqrt{5 + 4\cos\theta'}}. \qquad (18)$$

However, a difficulty arises when one tries to express the angle θ' in terms of the angle β in the new image. There is, of course, a relationship between the two angles, and this can be derived from Figure 4; thus

$$\frac{\sin\theta'}{2 + \cos\theta'} = \frac{\sin\beta}{2 - \cos\beta} \qquad (19)$$

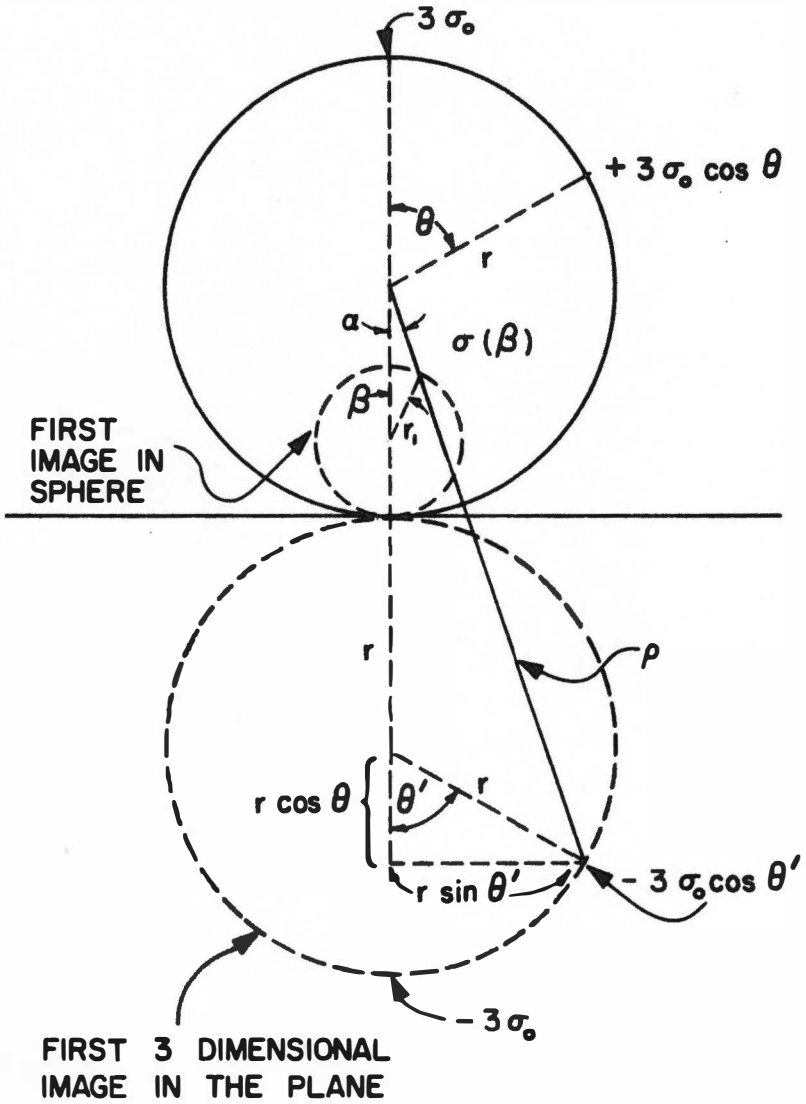


Figure 4. Three dimensional image charge distributions in the plane and in the sphere.

Although (19) may be solved as an explicit function and a graphical relationship between β and θ' may be obtained, there is no simple functional form relating the two which could be used in a recursion equation for successive images in the sphere. It is possible to perform a graphical derivation of each new image, treating each point of the generator surface as a point charge and using the relations given in (13) and (14). This has been done for the first and second images in the sphere, and the new distributions $\sigma_1(\beta_1)$ and $\sigma_2(\beta_2)$ are compared with the starting distribution $\sigma(\theta)$ in Figure 5. This procedure could be carried on through, perhaps, the 5th image, the height of which is only about 0.4 per cent of the height of the original sphere; however, in addition to the tedium and the inaccuracies of the graphical method, there would still remain the problem of determining graphically the potential on and near the spherical surface and, from this, calculating the field at many points of the surface. There seems to be little to recommend this method over a frankly experimental approach such as that of measuring the potential distribution in an electrolytic tank.

Despite the difficulty in applying the method of images to solve the complete problem, it is possible to obtain an exact solution for the electrical field at the top of the sphere. By successive applications of the recursion relations (13) and (14) the total field at the top of the sphere may be readily obtained as a linear superposition of the fields at the top of all of the image charge distributions, and can be expressed as

$$E(0) = 3E_o + 3E_o \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n. \qquad (20)$$

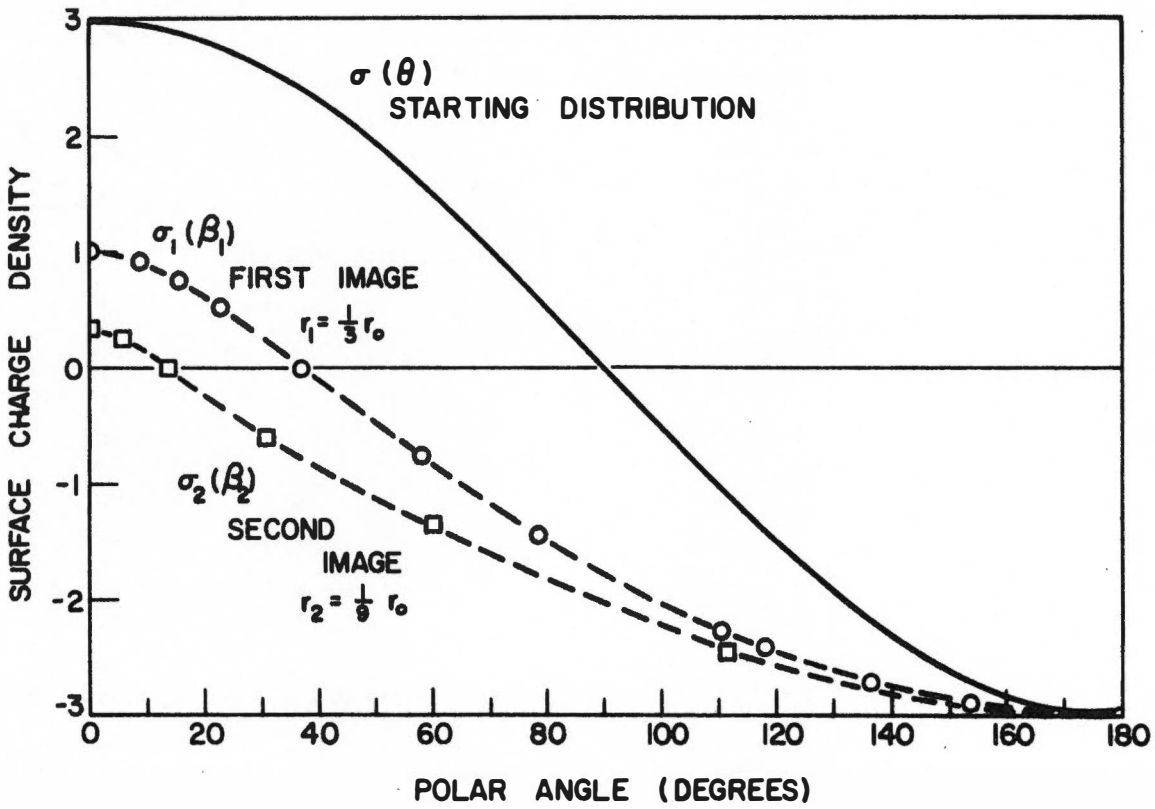


Figure 5. Results of graphic solution for the surface charge distribution on the first and second three-dimensional images in the sphere, as compared with the starting distribution of charge.

This can be written as

$$E(0) = 3E_o \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n,$$

which is in the form

$$1 + x + x^2 + x^3 + x^4 + \ldots = \frac{1}{1-x},$$

and, for $x < 1$, converges to

$$E(0) = 3E_o\left(\frac{3}{2}\right) = 4.5E_o. \qquad (21)$$

The simple linear superposition of fields used in (20) is valid only for points of azimuthal symmetry on the sphere. This is limited to the top and bottom where θ equals 0 and π radians. In the case of the bottom of the sphere, it is seen that there are two equal and opposite series of images in contact and superposition leads to

$$E(\pi) = 0. \qquad (22)$$

The rough estimate, $4.8E_o$, given in section III of Chapter II, was made before the result of (21) was known, and is less than 7 per cent in error despite the broad assumptions used to justify it.

CHAPTER IV

LAPLACE EQUATION IN BISPHERICAL COORDINATES

I. THE LAPLACE EQUATION

In connection with his study of Saturn's rings, Laplace (1785), gave a complete solution of the general problem of the attraction of a spheroid on an external particle and introduced the concept of the potential function and what is termed the Laplacian. It is shown, in the calculus of variations, that the Laplace equation

$$\nabla^2 V = 0 \qquad (23)$$

expresses the condition that the function V have the minimum mean gradient in space. This symbolizes the universal tendency to reduce any departure from uniformity to a minimum. Also equation (23) represents the difference between the local and average values of V in an infinitesimal neighborhood of the point.

The Laplace equation arises in electrostatics as a special case of the Poisson equation

$$\nabla^2 V = -\frac{\rho}{\epsilon};$$

where ρ is the volume density of charge in a homogeneous medium. This equation is a result of applying Gauss' flux law to the displacement vector $\vec{D}$, such that

$$\int_v \nabla \cdot \vec{D}\, dv = Q,$$

but

$$Q = \int_v \rho \, dv;$$

thus

$$\nabla \cdot \vec{D} = \rho.$$

The displacement relationship for the potential V is

$$\vec{D} = \epsilon \vec{E} = -\epsilon \text{ grad } V = -\epsilon \nabla V,$$

from which

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

for a medium in which ϵ is uniform. For charge free regions, such as the space outside of a charged conductor, the volume density of charge ρ is zero and the Laplace equation is seen to apply.

The Laplace equation is an elliptic, linear, homogeneous, partial differential equation and requires either Dirichlet (known potential), or Neumann (known potential gradient), conditions on a closed boundary. For the problem at hand, the potential is known on the semi-infinite conductive boundary ($V = 0$), and the potential gradient is specified on the boundary at infinity ($-\text{grad } V \rightarrow E_o$); thus the boundary conditions are of a mixed type.

Unfortunately the symmetry of the problem also is of a mixed type. The natural coordinate system to use for planar boundaries is the rectangular Cartesian system; on the other hand, when dealing with spherically symmetric boundaries,

spherical polar coordinates would be the system of choice. The closest approximation to a system which includes both a sphere and a plane as natural boundaries is the bispherical coordinate system described by Morse and Feshbach (1953). Although the sphere degenerates into a plane at contact with the base plane in the bispherical coordinate system, still asymptotic solutions may be possible for the sphere almost in contact with the plane.

II. BISPHERICAL COORDINATES

The geometry of the bispherical coordinate system is shown in Figure 6. Following the nomenclature of Morse and Feshbach the coordinates are Φ, η, and μ. It is difficult intuitively ta relate these coordinates to the more familiar rectangular or simple polar systems. In this system, Φ is the azimuthal angle as in spherical polar coordinates and ranges from 0 to 2π. However, η and μ have no direct counterpart in the simple systems, combining the properties of angle and shape in the case of η for the range 0 to π, and the properties of radius and position in the case of μ which ranges from $-\infty$ to $+\infty$.

In Figure 6 the sphere is represented by a surface of constant $\mu = \mu_o$. The sphere has a radius of

$$r = a\,|\mathrm{csh}\,\mu_o|\,, \tag{24}$$

and the center is located on the z-axis at

$$b = a\coth\mu_o. \tag{25}$$

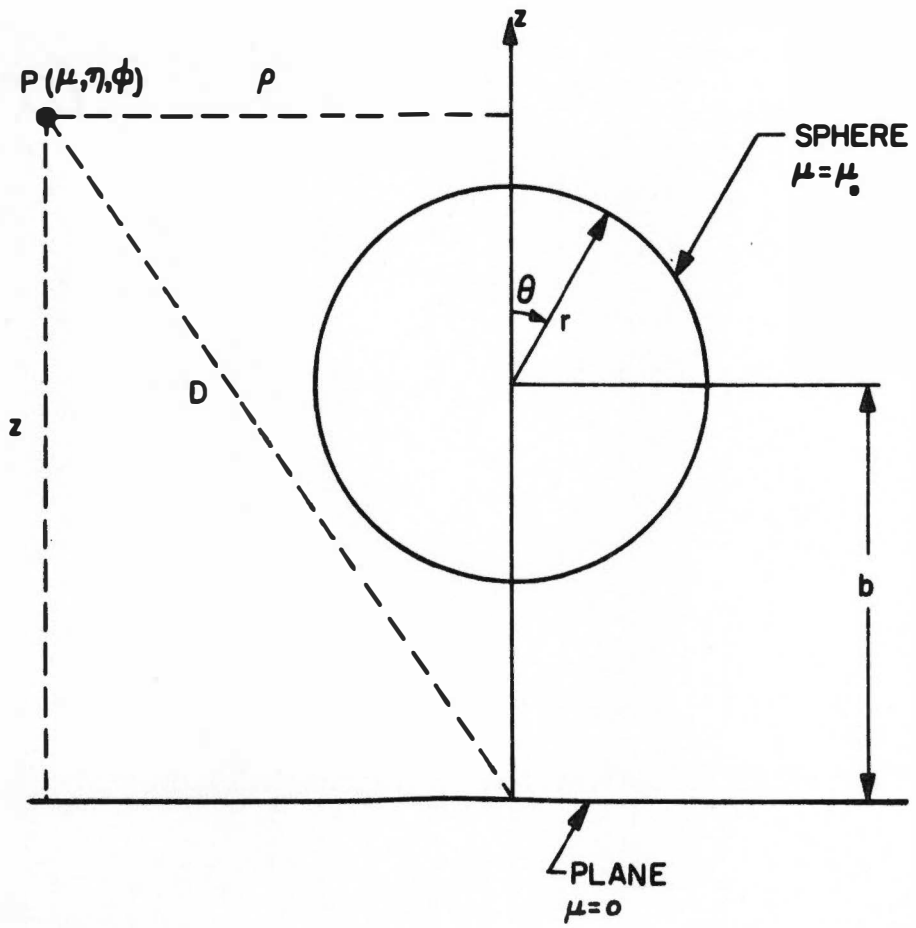


Figure 6. Sphere and plane in the bispherical coordinate system.

Solving (24) and (25) for the coefficient a results in

$$a = \sqrt{b^2 - r^2}\,. \tag{26}$$

The plane is the xy-plane and is given by $\mu = 0$. An arbitrary point P, outside the sphere and above the plane, in cylindrical coordinates, is at a distance

$$z = \sqrt{b^2 - r^2}\,\frac{\sinh\mu}{\cosh\mu - \cos\eta} \tag{27}$$

above the plane, and at a radial distance

$$\rho = \sqrt{b^2 - r^2}\,\frac{\sin\eta}{\cosh\mu - \cos\eta} \tag{28}$$

from the axis of symmetry.

The differential area on the surface of the sphere is given by

$$dA = 2\pi r\,dz,$$

but

$$dz = d\left(\frac{a\sinh\mu_o}{\cosh\mu_o - \cos\eta}\right) = \frac{a\sinh\mu}{(\cosh\mu_o - \cos\eta)^2}\,d\cos\eta\,;$$

thus

$$dA = 2r\,\frac{a(r\sinh\mu_o)d\cos\eta}{(\cosh\mu_o - \cos\eta)^2} = \frac{2\pi a^2}{(\cosh\mu_o - \cos\eta)^2}\,d\cos\eta. \tag{29}$$

In any coordinate system a component of the gradient such as E_μ is given by

$$E_\mu = -\frac{1}{h_\mu}\frac{\partial V}{\partial \mu},$$

where h_μ is the metric scale factor related to the coordinate μ. In bispherical coordinates

$$h_\mu = \frac{a}{\cosh\mu - \cos\eta},$$

hence the field normal to a surface of constant μ is given by

$$E_\mu = -\frac{\cosh\mu - \cos\eta}{a}\left(\frac{\partial V}{\partial \mu}\right).$$

From this it is seen that the field at the spherical surface is

$$E_{\mu_o} = -\frac{\cosh\mu_o - \cos\eta}{\sqrt{b^2 - r^2}}\left(\frac{\partial V}{\partial \mu}\right)_{\mu=\mu_o}, \tag{30}$$

and the field over the plane is

$$E_o = -\frac{1 - \cos\eta}{\sqrt{b^2 - r^2}}\left(\frac{\partial V}{\partial \mu}\right)_{\mu=0}. \tag{31}$$

The Laplace equation separates in bispherical coordinates if a function F is chosen such that

$$V = \sqrt{\cosh\mu - \cos\eta}\ F.$$

Equation (23) transforms to one which separates into three equations, each involving only one of the coordinates; thus, setting

$$F = M(\mu)\, H(\eta)\, \Phi(\theta),$$

leads to

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi,$$

$$\frac{d^2M}{d\mu^2} = (n+\frac{1}{2})^2 M,$$

and

$$\frac{1}{\sin\eta}\frac{d}{d\eta}(\sin\eta\frac{dH}{d\eta}) - \frac{m^2H}{\sin^2\eta} = -n(n+1)H.$$

Because of azimuthal symmetry, the potential is not a function of Φ and typical solutions of the original Laplace equation for V are products such as

$$\sqrt{\cosh\mu - \cos\eta}\; e^{\pm(n+\frac{1}{2})\mu} P_n(\cos\eta), \qquad (32)$$

where $P_n(\cos\eta)$ are the Legendre polynomials.

Application of (32) to the problem of the conductive sphere near a charged conductive plane, limited by the boundary condition that V = 0 on both surfaces, yields

$$V(\mu,\eta) = -E_o z + E_o\sqrt{\cosh\mu - \cos\eta}\sum_{n=0}^{\infty}(A_n e^{(n+\frac{1}{2})\mu} + B_n e^{-(n+\frac{1}{2})\mu})P_n(\cos\eta); \quad (33)$$

where

$$A_n = \frac{2\sqrt{2}\,a(n+\frac{1}{2})}{e^{(n+\frac{1}{2})\mu_o} - 1},$$

and

$$B_n = -\sqrt{2}\,a(n+\tfrac{1}{2})\,\frac{e^{(n+\frac{1}{2})\mu_o}+1}{e^{(n+\frac{1}{2})\mu_o}-1}.$$

If one sets $M = (n+\frac{1}{2})$ for economy of space, (33) can be written

$$\frac{1}{E_o}V = -\frac{a\sinh\mu}{\cosh\mu-\cos\eta} + 2\sqrt{2}\,a\sqrt{\cosh\mu-\cos\eta}\sum_{n=0}^{\infty} M\frac{2e^{M\mu}-e^{-M\mu}(e^{2M\mu_o}+1)}{(e^{2M\mu_o}-1)}\cdot P_n(\cos\eta). \quad (34)$$

This potential function may be tested by determining its value on the plane, where $\mu=0$, $\sinh\mu = 0$, and $\cosh\mu = 1$.

Thus,

$$\frac{1}{E_o}(V)_{\mu=0} = 0 + 2\sqrt{2}\,a\sqrt{1-\cos\eta}\sum_{n=0}^{\infty} MP_n(\cos\eta)\left[\frac{1-e^{2M\mu_o}}{e^{2M\mu_o}-1}\right]$$

$$= -2\sqrt{2}\,a\sqrt{1-\cos\eta}\sum_{n=0}^{\infty} MP_n(\cos\eta). \quad (35)$$

Referring to the generating function of the Legendre polynomials,

$$(1-2\mu x+x^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} x^n P_n(\mu),$$

and setting $x = e^{-\mu}$,

and $\mu = \cos\eta$,

we can arrive at the relation

$$(\cos\mu - \cos\eta)^{-\frac{1}{2}} = \sqrt{2}\sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\mu} P_n(\cos\eta). \tag{36}$$

Differentiating both sides of (36) with respect to μ gives

$$-\frac{1}{2}\sinh\mu\,(\cosh\mu - \cos\eta)^{-\frac{3}{2}} = -\sqrt{2}\sum_{n=0}^{\infty} M e^{-M\mu} P_n(\cos\eta). \tag{37}$$

On the plane, $\mu = 0$ and $\sinh\mu = 0$; thus the summation given in (35) becomes

$$\sum_{n=0}^{\infty} M P_n(\cos\eta) = 0.$$

Therefore, on the plane

$$(V)_{\mu=0} = 0$$

as required.

On the sphere, $\mu = \mu_o$ and the potential function is

$$\frac{1}{E_o}(V)_{\mu=\mu_o} = -\frac{a\sinh\mu_o}{\cosh\mu - \cos\eta} + 2\sqrt{2}\,a\sqrt{\cos\mu_o - \cos\eta}\sum_{n=0}^{\infty} M e^{-M\mu_o} P_n(\cos\eta). \tag{38}$$

From (37) the second term on the right of (38) can be written in the form

$$a\sqrt{\cosh\mu_o - \cos\eta}\left[2\sqrt{2}\sum_{n=0}^{\infty} M e^{-M\mu_o} P_n(\cos\eta)\right] = \frac{a\sinh\mu_o}{\cosh\mu_o - \cos\eta}\,.$$

When this is substituted back into (38) it is seen that, on the sphere,

$$(V)_{\mu=\mu_o} = 0.$$

Next, it is necessary to examine the behavior of the given potential function at distances far from the sphere. From (27) and (28) it is seen that the distance D from the origin is given by

$$D = a\sqrt{\frac{\sinh^2\mu + \sin^2\eta}{(\cosh\mu - \cos\eta)^2}}. \qquad (39)$$

It is noted that

$$\sinh^2\mu + \sin^2\eta = (\cosh^2\mu - 1) + (1 - \cos^2\eta) = \cosh^2\mu - \cos^2\eta$$

$$= (\cosh\mu - \cos\eta)(\cosh\mu + \cos\eta),$$

and (39) becomes

$$D = a\sqrt{\frac{\cosh\mu + \cos\eta}{\cosh\mu - \cos\eta}}.$$

Since the maximum value of $\cos\eta$ is 1.0 it can be seen that

$$D \to a \text{ as } \cosh\mu \to \infty,$$

but a is a constant and not necessarily very large. However,

$$D \to \infty \text{ as } (\cosh\mu - \cos\eta) \to 0.$$

Therefore, at large distances from the origin, the term $(\cosh\mu - \cos\eta)$ becomes very small and the first term of the potential function, (34), predominates, thus

at large distances from the sphere

$$V \to -E_o z,$$

as was required.

To summarize, at this stage the following points may be noted. First, it is well known that a uniform field is a solution of the Laplace equation. Similarly, the indicated linear combination of Legendre polynomials is a solution. Consequently, since a linear sum of solutions is also a solution, the potential function V, given in (34), is a solution of the Laplace equation. Furthermore, V is zero on all portions of the conductive sphere and plane boundary and the gradient of V approaches the constant $-E_o$ at large distances from the sphere. It is clear that V is the appropriate potential function for the electrostatic problem as stated.

III. SOLUTION FOR THE FIELD

Straightforward differentiation of (34) with respect to μ, taking into account the metric scale factor, yields the result

$$E_\mu = E_o \left[\frac{1 - \cosh\mu\cos\eta}{\cosh\mu - \cos\eta} \right.$$

$$-\sqrt{2}\sinh\mu\sqrt{\cosh\mu - \cos\eta}\sum_{n=0}^{\infty} M\left[\frac{2e^{M\mu} - e^{-M\mu}(e^{2M\mu_o}+1}{e^{2M\mu_o}-1}\right]P_n(\cos\eta)$$

$$\left. -2\sqrt{2}(\cosh\mu - \cos\eta)^{\frac{3}{2}}\sum_{n=0}^{\infty} M^2\left[\frac{2e^{M\mu} + e^{-M\mu}(e^{2M\mu_o}+1)}{e^{2M\mu_o}-1}\right]P_n(\cos\eta)\right]. \quad (40)$$

The normal component of the field on the surface of the sphere is

$$E_{\mu_o} = E_o\Big[\frac{1-\cosh\mu_o\cos\eta}{\cosh\mu-\cos\eta}$$

$$-\sqrt{2}\sinh\mu_o\sqrt{\cosh\mu_o-\cos\eta}\sum_{n=0}^{\infty} M e^{-M\mu_o}P_n(\cos\eta)$$

$$-2\sqrt{2}(\cosh\mu-\cos\eta)^{\frac{3}{2}}\sum_{n=0}^{\infty} M^2\Big(\frac{3e^{M\mu_o}+e^{-M\mu_o}}{e^{2M\mu_o}-1}\Big)P_n(\cos\eta)\Big],$$

which, using (37), reduces to

$$E_{\mu_o} = E_o\Big[\frac{2-2\cosh\mu_o\cos\eta-1\sinh^2\mu_o}{2(\cosh\mu_o-\cos\eta)}\Big]$$

$$-2\sqrt{2}E_o(\cosh\mu_o-\cos\eta)^{\frac{3}{2}}\sum_{n=0}^{\infty} M^2\Big(\frac{3e^{M\mu_o}+e^{-M\mu_o}}{e^{2M\mu_o}-1}\Big)P_n(\cos\eta). \quad (41)$$

Similarly, the field normal to the plane is

$$E_{\mu=0} = E_o\Big[1-2\sqrt{2}(1-\cos\eta)^{\frac{3}{2}}\sum_{n=0}^{\infty} M^2\frac{e^{2M\mu_o}+3}{e^{2M\mu_o}-1}P_n(\cos\eta)\Big]. \quad (42)$$

IV. ASYMPTOTIC SOLUTION FOR THE FORCE

The upward component of force acting on the sphere can be found by integration of the z-component of the electrical pressure at the boundary, thus

$$F_z = \int_{\text{sphere}} \frac{1}{2} \epsilon E_{\mu_o}^2 \, d(\text{area}) \cos\theta$$

$$= \pi a^2 \epsilon \int_0^{\pi} \frac{(1 - \cosh\mu_o \cos\eta)}{(\cosh\mu_o - \cos\eta)^3} E_{\mu_o}^2 \, d(\cos\eta). \qquad (43)$$

The solution of (43) leads to a formidable array of infinite series that must be solved by computer; however, the cost of such a solution would be prohibitive.

Fortunately, a similar problem has been solved by Davis (1962), for two equal spheres and computer solutions were tabulated for relative separations from 1 to 0.001 radii. In that work, the force acting on one of a pair of charged spheres in a uniform field is given by

$$F_z = -R^2 E_o^2 F_1 - E_o(Cq_2 - q_1 - Bq_1) - \frac{1}{R^2}(Dq_1^2 - Fq_1q_2 + Gq_1^2), \qquad (44)$$

where F_1, C, B, D, F and G are the results of computer solutions of the relevant series. The geometry of the two-sphere problem and its relationship to the sphere-plane problem are illustrated in Figure 7.

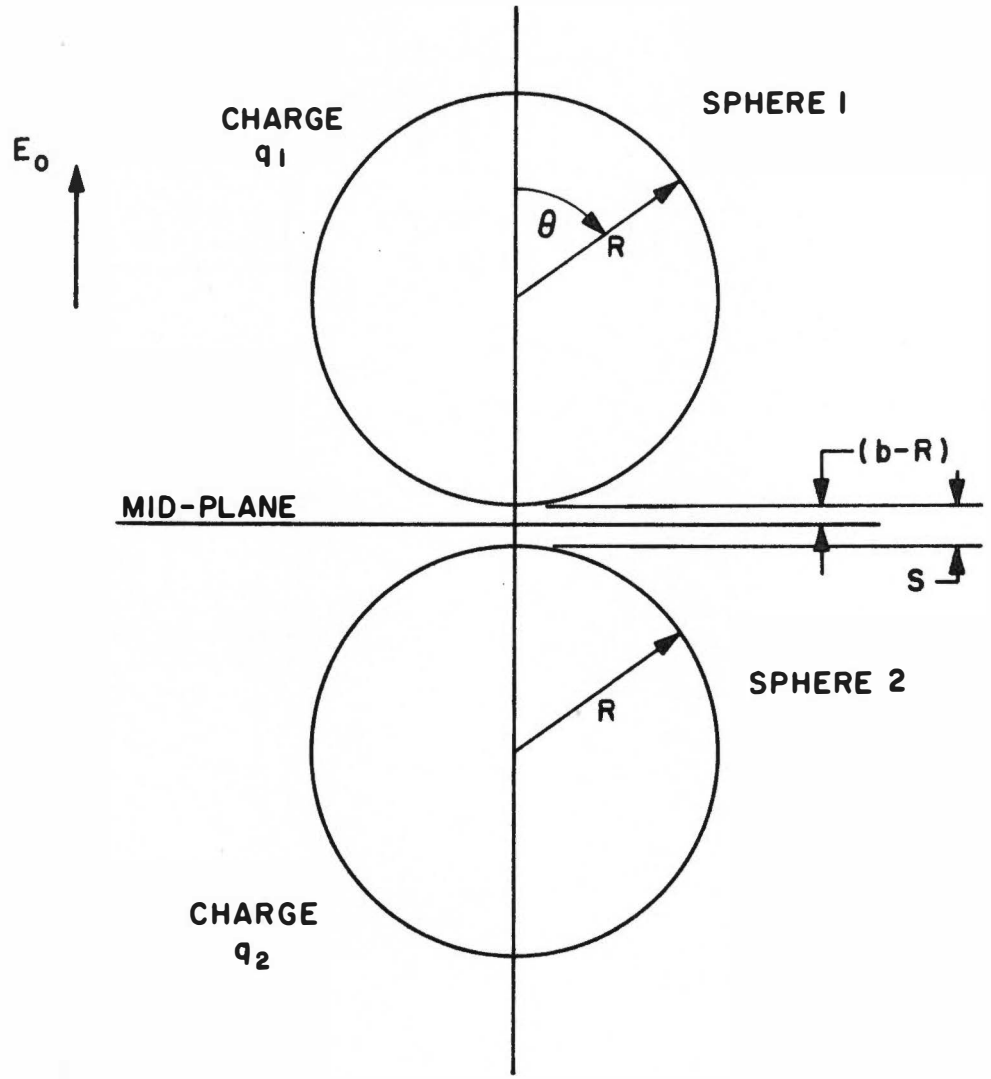


Figure 7. The problem of two charged spheres of equal radii can be related to the problem of a conductive sphere on a charged plane.

Assuming the validity of Maxwell's solution for the charge acquired by a sphere contacting a charged plane, the charge on sphere I is taken to be

$$q_1 = \frac{2}{3}\pi^3 \epsilon R^2 E_o.$$

In analogy with the image charge approach, sphere 2 is assumed to have a charge equal to that of sphere I but of opposite sign; thus

$$q_2 = -q_1.$$

After making the substitutions for q_1 and q_2, and noting that, for equal radii spheres,

$$C = B,$$

and

$$D = G,$$

equation (44) for the force becomes

$$F_z = -R^2 E_o^2 \left[F_1 + (\frac{2}{3}\pi^3\epsilon)^2(2D+F) - (\frac{2}{3}\pi^3\epsilon)(2B+1)\right] = -R^2 E_o^2 [H]. \quad (45)$$

Using the solutions tabulated by Davis (1962), the bracketed quantity H in (45) was calculated and the results are listed in Table II.

Despite some uncertainty in the computed results for the closest spacing, as evidenced by the values of H for S = 0.01 and 0.001, still the new estimate of force is almost certainly within less than 1 per cent, and possibly within 0.5 per cent of the true value. The rough estimate, given in section II of Chapter II is high by only 4 per cent.

TABLE II

ESTIMATE OF FORCE ON A SPHERE IN CONTACT WITH A CHARGED PLANE BASED ON AN ANALOGY WITH THE PROBLEM OF TWO SPHERES

$\frac{(b-R)}{R}$	$\frac{S}{R}$	$(2B+1)$	$(2D+F)$ (meters/farad)	F_1 (farads/meter)	$-H$ (farads/meter)
5	10	1.002	7.12×10^{7}	$\sim 2 \times 10^{-14}$	1.808×10^{-10}
0.5	1	1.190	1.213×10^{9}	1.031×10^{-11}	1.667×10^{-10}
0.05	0.1	2.948	6.831×10^{9}	1.557×10^{-10}	1.549×10^{-10}
0.005	0.01	12.99	3.469×10^{10}	1.062×10^{-9}	1.531×10^{-10}
0.0005	0.001	73.78	2.01×10^{11}	6.616×10^{-9}	1.544×10^{-10}
Estimate for $(b-R) = 0$					1.537×10^{-10}
Rough Estimate given in Table I, page 16					1.599×10^{-10}

CHAPTER V

EXPERIMENTAL STUDIES

I. INTRODUCTION

So far, exact solutions have been obtained for the charge acquired by a sphere in contact with a charged plane and for the electric field at the top of the sphere. Furthermore, the extrapolated value obtained in Chapter IV for the force exerted on the sphere is probably within less than I per cent of the true value. Thus, it is unlikely that an experimental approach will improve on the precision of the estimates. Nevertheless, there is much advantage in confirming, even approximately, the theoretical results by experimentation.

II. CHARGING OF A SPHERE ON A PLANE

A convenient way to study the charge on a small sphere is to observe its movements in a cell such as the one described by Millikan (1924), in connection with his study of the charge on oil droplets. A small cell was constructed of plexiglas having two gold plated electrodes spaced three millimeters apart. There is an advantage in using a fairly large sphere because it is more readily observed microscopically and can more readily be confirmed to be a smooth sphere than can small particles on the order of one micron. Also, to reduce possible artifacts related to contact potential charging, it is necessary to select spheres having the same composition as the surface of the electrodes, in this case gold. The

combination of large size and high specific gravity precludes the use of air as a medium for several reasons, chief among which is the fact that the air resistance to motion deviates significantly from Stokes' law at the Reynolds numbers involved and there are no adequate theoretical treatments of the drag force in this range. Consequently, the cell was filled with a viscous silicone oil and a 50 micron diameter smooth gold sphere was selected as the test object.

The experimental setup is illustrated in Figure 8. A toothed wheel was used to provide a pulsed source of light and the repetition rate was determined by observation of the rotating wheel using a calibrated strobe light. The light source was a high intensity zircon arc lamp which was directed into the chamber at 90° to the line of sight so that only scattered light is observed through the microscope. During an experimental run the microscope substage is moved slowly to insure separation of the images of one traverse of the sphere between the plates from those of successive trips across the field of view. In a typical run, the potential difference between the electrodes was set using a precision DC voltage source. After the light and the microscope focuses were adjusted, and the strobe frequency was set to give well separated images of the sphere, the camera lens was opened long enough to record photographically several trips between electrodes. The photograph in Figure 9 illustrates the appearance of the photographic record. Measurements of the average vertical distance between images on the photograph were made and, with these results and a knowledge of the strobe rate, the vertical velocity was calculated.

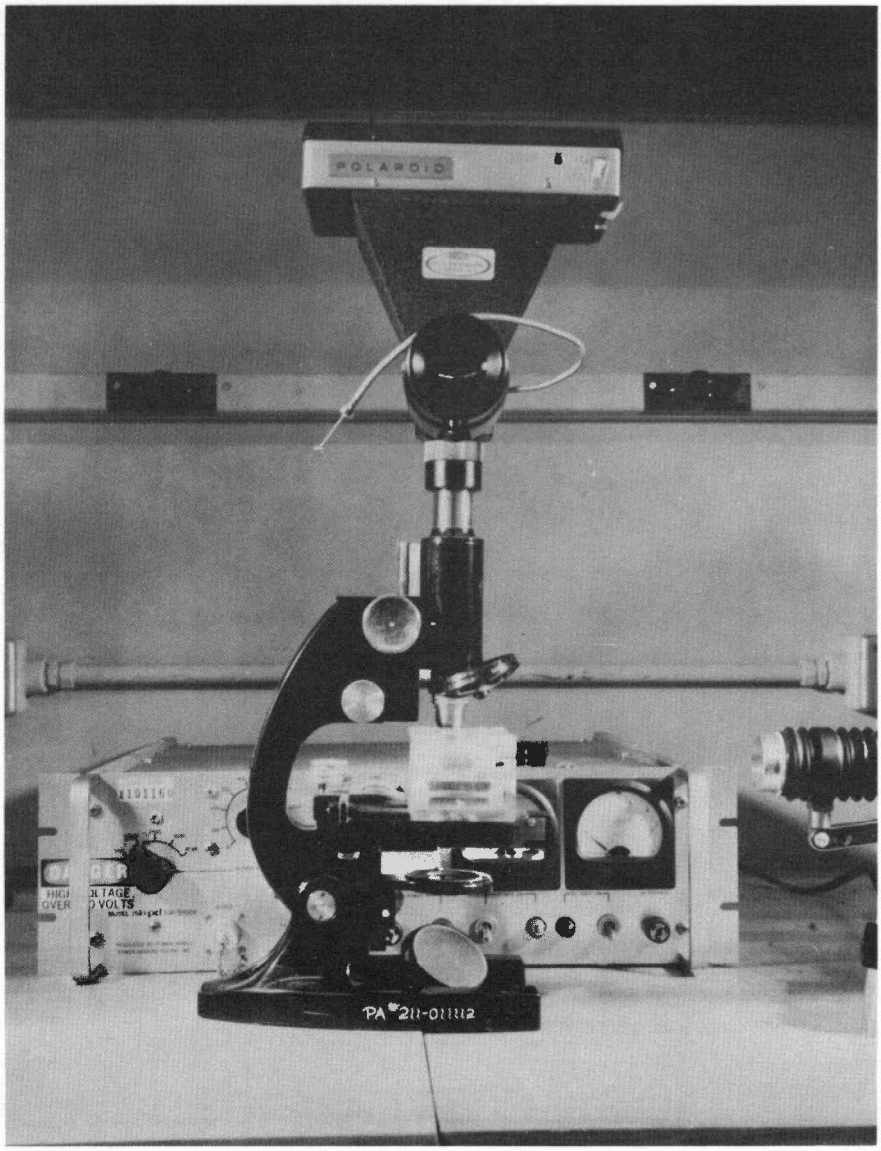


Figure 8. Experimental setup for measurement of charge acquired by sphere contacting a charged plane.

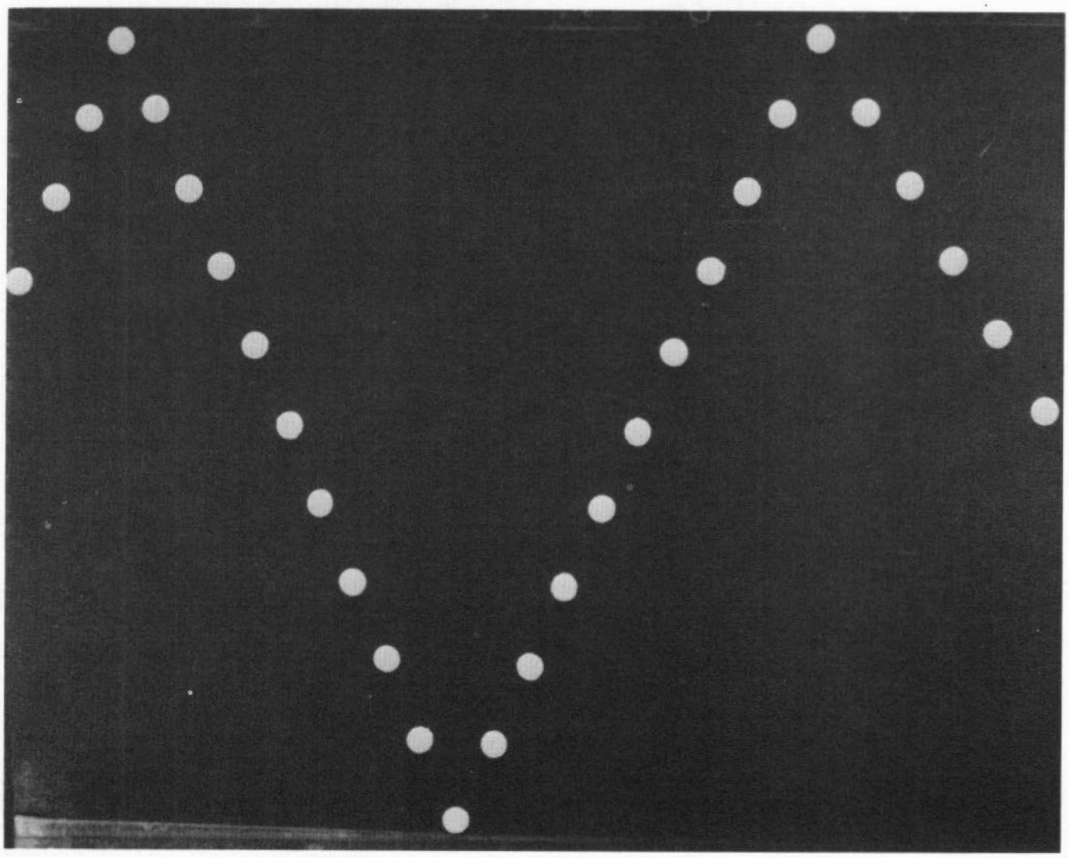


Figure 9. Strobe photograph* of 50 micron diameter gold sphere oscillating (vertically in silicone oil) between gold electrodes held at constant potential.

*This is a simulation of the appearance of the photographs used in the study. Original photographs are no longer available.

The calculation of the charge is illustrated in Figure 10. It is noted that for a sphere having a charge of $1.828 \times 10^{-10} R^2 E_o$, the force, unaffected by image charge interactions with the plane, produced by a uniform field E_o is $1.828 \times 10^{-10} R^2 E^2_o$; whereas, from Table II, page 44, it is seen that the force is $1.808 \times 10^{-10} R^2 E^2_o$, or within one per cent of qE_o, at a separation of only 5 radii from the plane. As a general rule, no measurements were made within 10 diameters of the electrodes; thus the assumption that the electrical force is equal to qE_o is well justified. Comparison of distances between successive images shows that the sphere moves at a constant velocity in the interval used for measurement. In no case did the Reynolds number exceed 0.0001; thus the assumption of Stokes' law for the drag force is well founded. Consequently,

$$\text{Drag Force} = 6\pi\eta Rv;$$

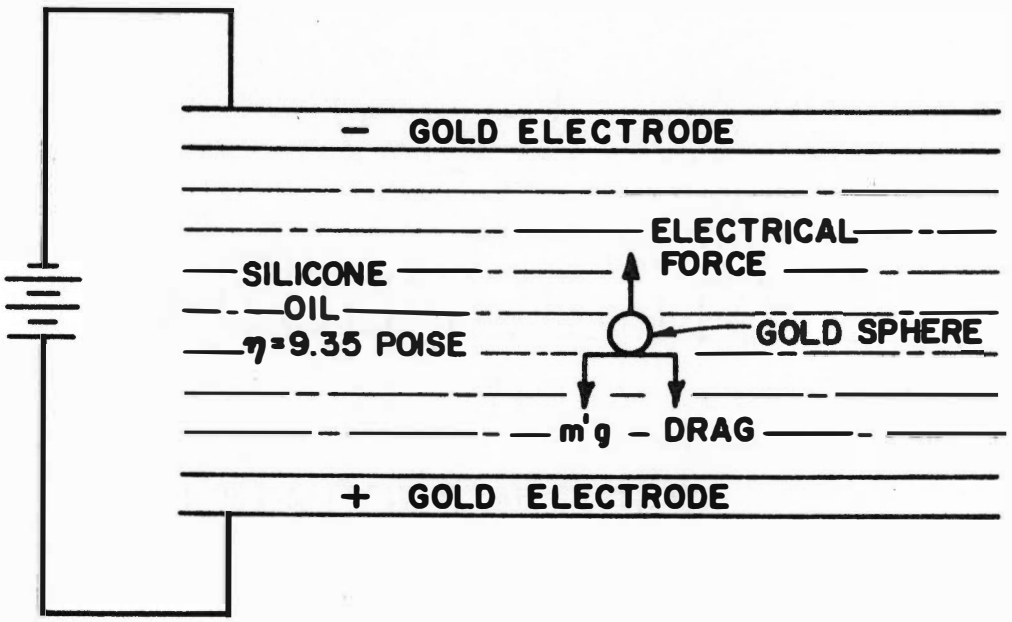
where $\eta = 9.35$ poise (cgs units),

and v = the observed velocity.

Finally, the force due to gravity was calculated, including a correction for buoyancy, and was verified by observing the terminal velocity of the sphere settling under zero electrical field.

The electrical force on the upward moving sphere is equated to the drag plus the gravity forces and an expression for the charge is found to be

$$q = \frac{0.44}{E_o}(v + 2.68 \times 10^{-5}) \text{ coulombs.} \qquad (46)$$



$$F_e = qE$$

Figure 10. Diagram of charged capacitor used to determine velocity of a gold sphere as a function of applied field. (plate spacing = 3 millimeters)

Equation (46) was used to compute the values of q related to the observed velocities listed in Table III. These results are plotted in Figure II and are compared with the amount of charge predicted by Maxwell's solution,

$$q = \frac{2}{3}\pi^3 \epsilon R^2 E_o,$$

which, for $R = 25 \times 10^{-6}$ meters, and $\epsilon = 2.42 \times 10^{-11}$ farads/meter, becomes

$$q = 3.130 \times 10^{-19} E_o.$$

Comparison of the calculated charge based on the observed velocity with that calculated on the assumption of Maxwell's solution suggests that the data are none too good; however the disparities seem to be distributed with little tendency to follow a definite trend. The observed velocities are quite slow, ranging from 1 to, at most, 18 particle diameters per second. No special precautions were taken to prevent thermal convection in the cell and, because of the intense and non-uniform illumination which was used, convective currents are likely to have occurred. The algebraic sum of the deviations from the indicated line is only 0.96. Although the mean deviation from the line, 1.38, and the root-mean-square deviation, 1.53, represent about 8 per cent of the mean ordinate, it may be noted that the ordinate of the line, 18.689, and of the values from Table III, page 52, 18.594, differ by only 0.5 per cent. The conclusion being that, despite obvious error arising from the experimental technique, the data support the estimate of charge accumulation based on Maxwell's solution.

TABLE III

CHARGE ACQUIRED BY A 50 MICRON DIAMETER GOLD SPHERE, CALCULATED FROM THE OBSERVED UPWARD SPEED IN AN APPLIED FIELD

Applied Field (volts/meter)	Observed Speed (meters/second)	Calculated Charge (coulombs)
0.3×10^{6}	5.3×10^{-5}	11.7×10^{-14}
0.333	6.3	11.9
0.4	9.6	13.5
0.5	14.4	15.1
0.577	18.5	16.2
0.667	29.3	21.1
0.833	44.7	25.1
1.167	88.0	34.2

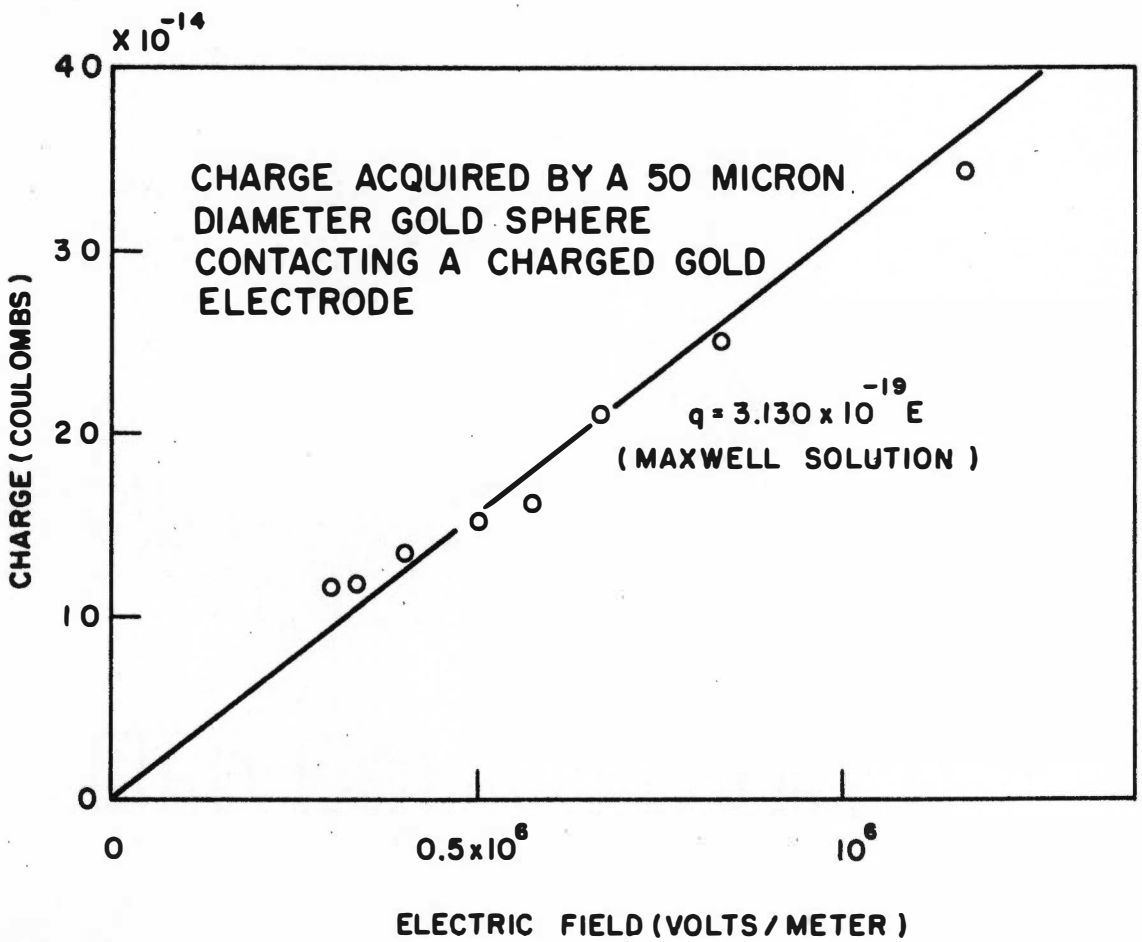


Figure II. Charge acquired by a 50 micron diameter gold sphere contacting a charged gold electrode.

III. COULOMETRIC ANALOG STUDY OF SURFACE CHARGE DENSITY

Of the various methods discussed so far, the only one showing much promise for producing information about the distribution of charge on the sphere and the plane is the technique employing bispherical coordinates. A solution was derived in Chapter IV for the field strength at the surface of the sphere and the plane, and the surface charge density is directly proportional to the field strength. However, numerical solution of the given result would be expensive and there is not much advantage in having a precise determination of the surface distribution of charge, assuming the total charge, the field at the top and the repulsion force are well enough determined. For whatever academic interest there may be in the surface charge distribution, it will probably suffice to obtain a first order approximation using experimental methods.

The most common experimental approach used to solve three-dimensional electrostatic problems is the, so-called, electrolytic trough. The basic theory for the electrolytic trough analog is well known [e.g., Kennedy and Kent (1956), and Hartill, McQueen and Bobson (1957)]. For an isotropic, homogeneous, conducting medium, Ohm's law may be stated as

$$\vec{j} = g\vec{E} = -g\nabla V; \qquad (47)$$

where $\vec{j}$ is the current density in amperes/meter^2, $\vec{E}$ is electric field in volts/meter, g is conductivity in (ohm meters)$^{-1}$, and V is the potential in volts.

Since, in a conductive medium,

$$\nabla \cdot \vec{j} = -g \nabla \cdot \nabla V = 0, \qquad (48)$$

equation (48) reduces to the Laplace equation

$$\nabla^2 V = 0.$$

The electrolytic trough analog has been used extensively to simulate the potential distributions for many physical situations which are governed by the Laplace equation [Dadda (1951), and Hackenschmidt (1963)]. This includes electric and magnetic fields, hydrodynamic and aerodynamic fields, temperature fields, elastic stress fields, diffusion and others. The analog method is used to determine fields which cannot be solved analytically or for which numerical relaxation methods [Southwell (1946)], would be extremely tedious. In practice, precision measurements are made of the spatial distribution of potential in the model, and by numerical or graphical analysis of the observed equipotentials, the field is determined in the space between the given electrodes. When the field intensity at a boundary is required, for example at the surface of an electrode as described by Loeb et al. (1950), the usual procedure is to plot the equipotentials in the intervening space, calculate field intensities by numerical methods and extrapolate to the boundary.

A more direct measurement of the electric field at a complicated boundary is possible with the electrolytic tank if one carries out the procedure so as to electrodeposit a metal on the surface of interest. When equation (47) is

multiplied by the electrochemical equivalent z (kilograms/coulomb) of the metal ion in the electrolyte, the result is

$$z\vec{i} = \frac{d}{dt}\left(\frac{\vec{dm}}{dA}\right) = (g z)\vec{E}, \qquad (49)$$

and the area density of metal electrodeposited during time t seconds on a surface at which the field is E_s, ideally, is given by

$$\left(\frac{dm}{dA}\right) = (g z t)E_s = k\sigma_s . \qquad (50)$$

The area density of metal can be determined by stripping known areas from definite positions and weighing the metal or determining the mass chemically. The constants of proportionality in equation (50) can be determined by measuring the conductivity and time or by determining the area density of metal at places within the model where the field is well known from theoretical considerations.

As in the case of many other simplistic solutions to complex problems, there are difficulties with the electrolytic plating method. Equation (49) only tells part of the story, leaving out the important consideration of ionic diffusion effects. A more complete description of the current density field in an electroplating bath is given by

$$z\vec{i} = -D \text{ grad } C + (g z)\vec{E}; \qquad (51)$$

where D is the diffusion constant of the ion (meters^2/second), and C is its concentration. Although various steps can be taken to minimize the effect of concentration gradient, it cannot be eliminated and, consequently, analog solutions based on this approach must be considered to be only first order approximations.

Where applicable, the method is especially suited to problems for which the field at certain boundaries must be determined and for which the field in the space between electrodes is of no interest. Extensive measurements of the three dimensional potential distribution within the model are not required. Such a coulometric analog tank has been constructed for the determination of surface charge distributions on various shaped conducting particles in contact with a conducting plane. It consists of a cylindrical polyethylene tank, open at the top, 66 centimeters high, 45 centimeters in diameter and filled to a height of about 60 centimeters with a copper plating solution. Circular copper electrodes, 0.64 centimeters thick and closely fitting the inside diameter of the tank, are spaced about 60 centimeters apart in the solution with the lower electrode resting on the bottom. The bottom plate is covered with a polyethylene sheet, painted with silver paint and then very lightly "flashed" with copper in the plating bath to insure a uniform, low resistance equipotential surface. Models are made of the particle shapes to be tested, they are coated with silver paint, "flashed" with copper and attached to the bottom plate. Tests are made to insure zero resistance from any part of the particle surface to representative points on the plane. The electrode assembly is then placed in the tank and copper is electrodeposited on the test configuration at room temperature, with 1 to 3 volts DC, and 2 to 4 amperes, corresponding to about 2 milliamperes/centimeter^2. The plating liquid is periodically circulated through a charcoal and fiber-glass filter. Copper is plated to sufficient thickness that it may be cut from specific areas and weighed directly on an analytical balance.

To explore the validity of the method, preliminary tests were made with a hemisphere at the center of the bottom electrode. The charge distribution for this configuration is well documented in texts on electrostatics [Jeans (1941)]. Using the coordinate system shown in Figure 12, where the radius of the hemisphere is a, and σ_o is the charge density on the plane at a very large distance from the origin, the charge density on the hemisphere as a function of the angle θ from the axis is

$$\sigma(\theta) = 3\sigma_o \cos\theta, \tag{52}$$

while on the plane, at $\theta = 90°$ and a distance $r > a$ from the origin, the charge distribution is

$$\sigma(r) = \sigma_o\left(1 - \frac{a^3}{r^3}\right). \tag{53}$$

Equation (52) can be integrated to yield an expression for the cumulative fraction of the charge residing on the sector from 0 to θ, and by applying the coulometric analog equation, a similar expression can be obtained for the fraction of the total copper mass deposited on the hemisphere which should be found on the sector 0 to θ. Thus,

$$M(\theta) = \frac{\sum_{o}^{\theta} \Delta m}{\text{total } m} = \frac{1}{q}\int_{o}^{\theta} \sigma(\theta)\, d(\text{area}) = \sin^2\theta. \tag{54}$$

By a similar treatment, it can be shown that, on the plane,

$$M(r) = \int_{r_1}^{r_2} dm = k\sigma_o\left\{\pi\left[r_2^2 - r_1^2 - \frac{2a^3(r_2 - r_1)}{r_1 r_2}\right]\right\} = k\sigma_o f(r). \tag{55}$$

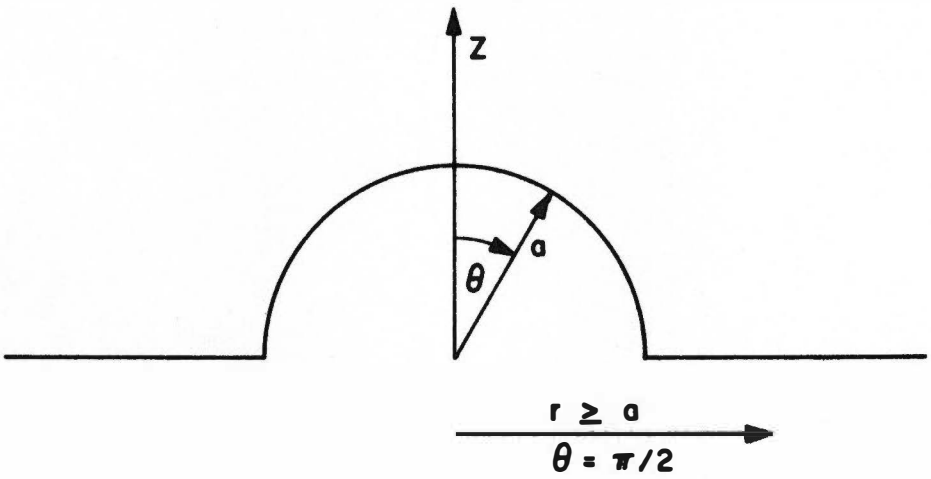


Figure 12. Coordinates for a hemisphere on a plane.

The electrodeposited copper is stripped from the hemisphere and the plane in known sectors as shown in Figure 13, and each segment is weighed. The fractional distribution of the copper on the hemisphere was calculated and is compared with the theoretical distribution in Figure 14. Similarly, the total quantity deposited on the plane from $r_1 = 4$ centimeters to r is compared with the theoretical f(r) in Figure 15.

It is obvious from the results of this study of the hemisphere that the electrodeposition analog is far from perfect; nevertheless, it also seems clear that the degree of error resulting from diffusion effects is not so great as to preclude the use of this method for many purposes.

Application of the electroplating method to the sphere-plane geometry has been made and the results of two separate runs are plotted in Figure 16. With the exception of the four pairs of points indicated in the figure, the results of the two runs are indistinguishable on the scale used for plotting.

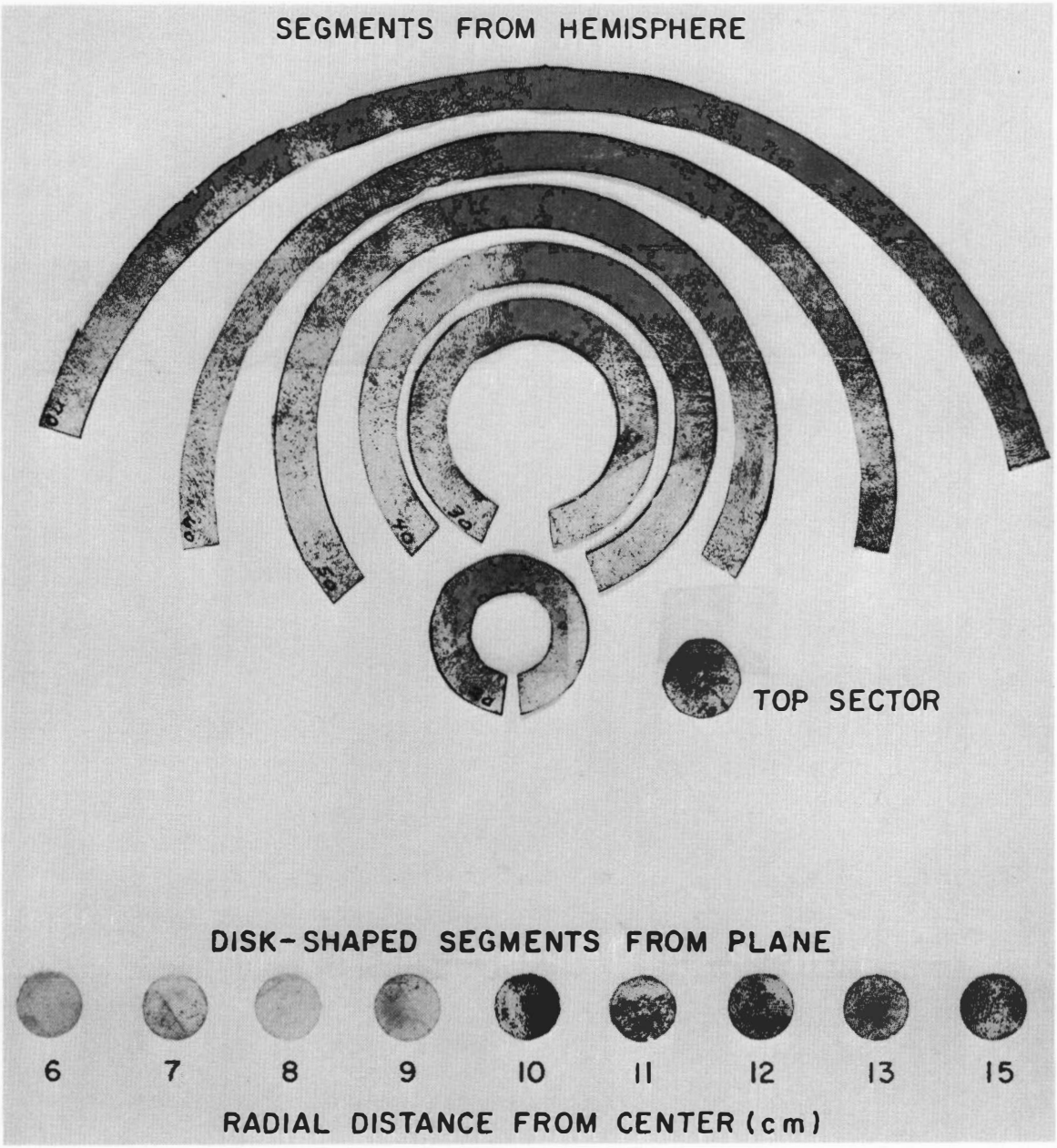


Figure 13. Segments of copper plated on a hemisphere in contact with a plane electrode.

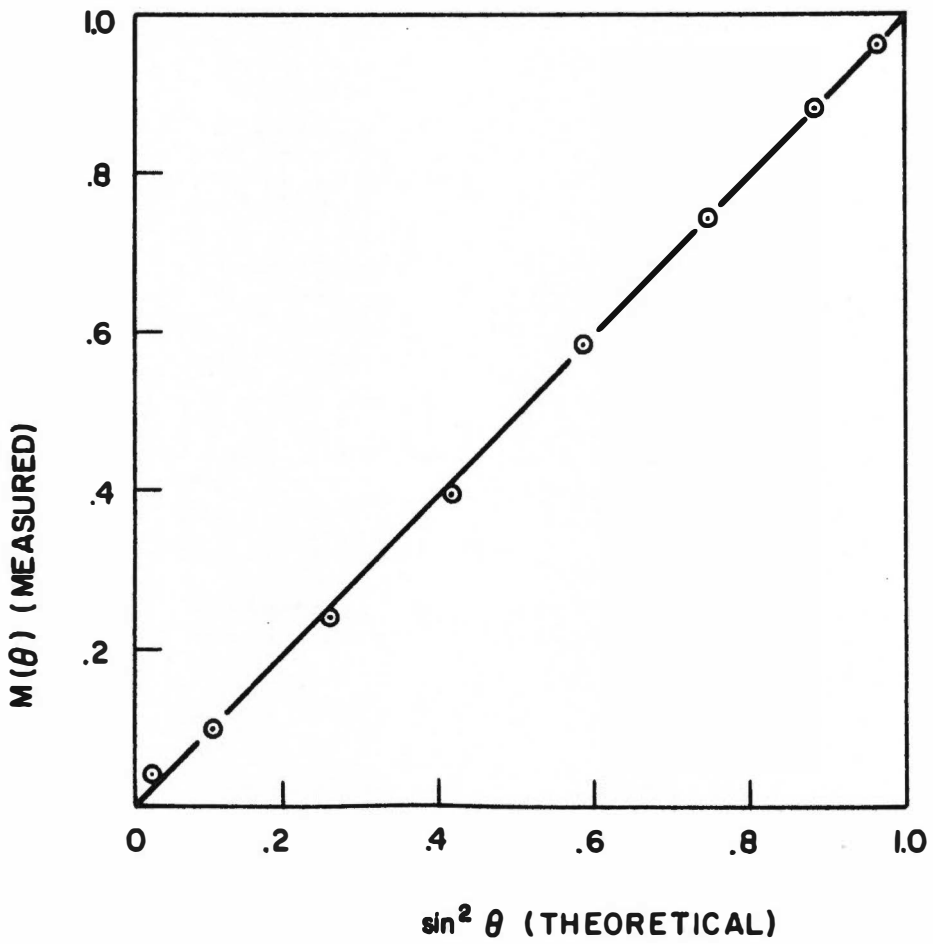


Figure 14. Distribution of electrodeposited copper on a hemisphere in contact with a plane cathode in a plating tank. (hemisphere radius = 3.825 cm.)

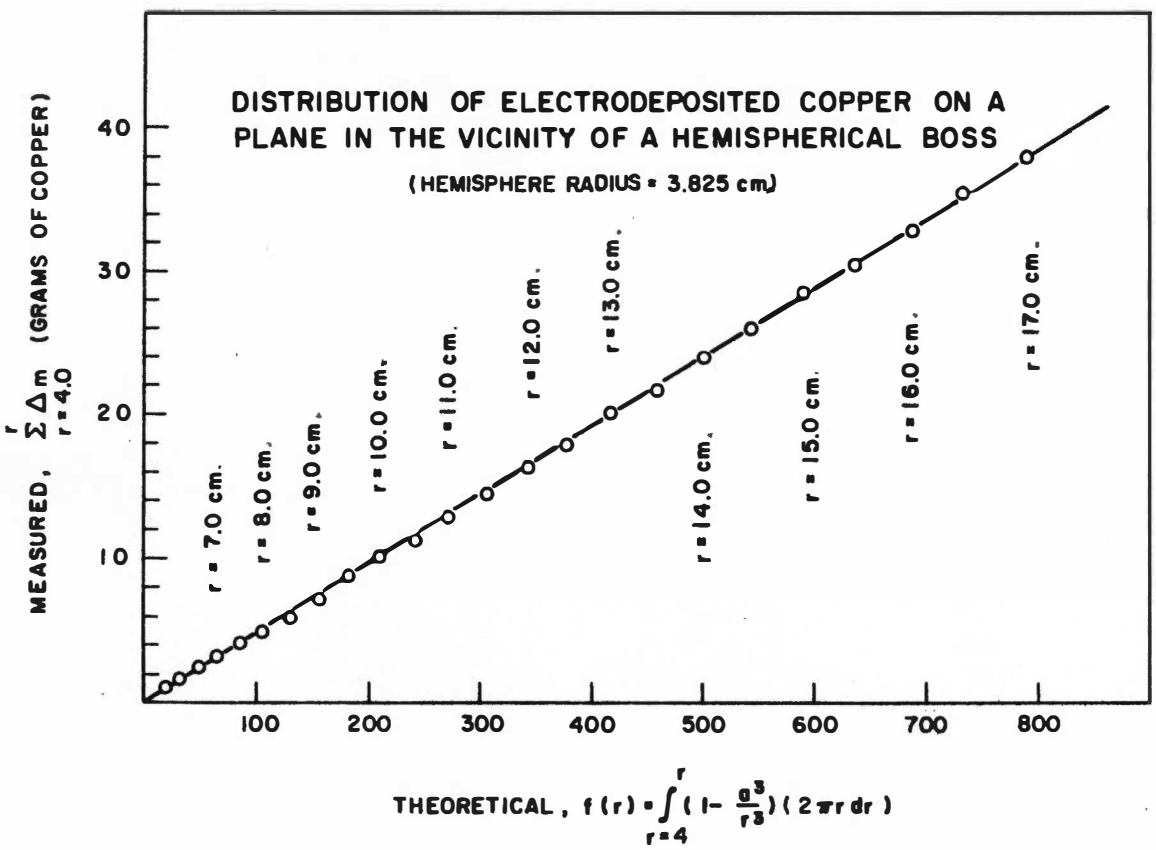


Figure 15. Distribution of electrodeposited copper on a plane in the vicinity of a hemisphere.

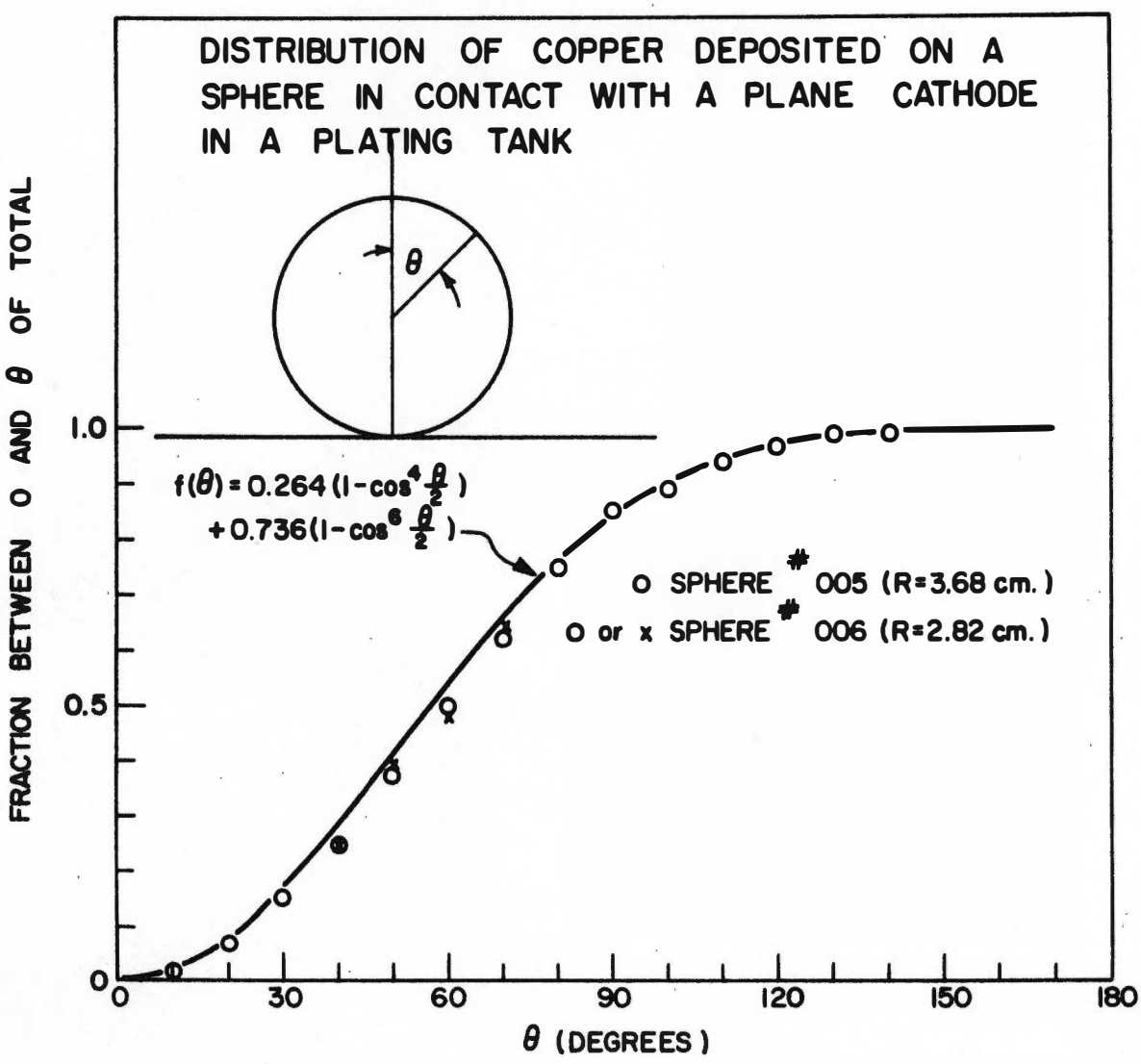


Figure 16. Distribution of copper deposited on a sphere in contact with a plane cathode in a plating tank.

CHAPTER VI

SUMMARY AND CONCLUSION

I. SUMMARY

The three major goals of this study have been met within a useful degree of precision. Exact solutions were obtained for the charge acquired by a conductive sphere in contact with a charged plane, and for the field at the top of the sphere. An approximate solution, with less than about 1 per cent error, has been derived for the repulsive force on the sphere. In addition a first order estimate was developed for the relative charge density as a function of location on the sphere.

II. DISCUSSION

In the application of these results it must be borne in mind that everything that has been said presupposes that the sphere remains in contact with the plane long enough for the equilibrium charge distribution to be attained. This will be realized in the case of good conductors, but for some materials having a finite relaxation time for surface charge, it may be possible for the sphere to be repelled from the surface before the equilibrium charge is transferred.

Further manipulation of the data obtained by plating copper on a sphere-plane cathode suggests that the surface charge density may be approximated by a function of the form

$$\sigma(\theta) = (C_1 \cos^2 \frac{\theta}{2} + C_2 \cos^4 \frac{\theta}{2})\, \sigma_o. \qquad (56)$$

There are two adjustable constants and if the function is forced to be $4.5\,\sigma_o$ at the top of the sphere, and if the total integrated charge is set equal to the charge predicted by Maxwell's solution, then the result is

$$\sigma(\theta) = (0.8696 \cos^2 \frac{\theta}{2} + 3.6304 \cos^4 \frac{\theta}{2})\,\sigma_o. \qquad (57)$$

Using (57), the repulsion force is calculated to be

$$F_z = 1.572 \times 10^{-10} R^2 E_o^2, \qquad (58)$$

which is 2 per cent higher than the best estimate available. Equation (57) also was integrated from 0 to θ and the result was divided by the total charge on the sphere to yield

$$f(\theta) = 0.2643\,(1 - \cos^4 \frac{\theta}{2}) + 0.7357\,(1 - \cos^6 \frac{\theta}{2}). \qquad (59)$$

Equation (59) is plotted in Figure 16 and is seen to be a fair approximation to the experimental data obtained from two separate runs using spheres of different radii.

III. CONCLUSION

In conclusion, for a conductive sphere in contact with a charged conductive plane, the charge acquired by the sphere will be

$$q = \frac{2}{3}\pi^3 \epsilon R^2 E_o = \frac{2}{3}\pi^3 R^2 \sigma_o \text{ (coulombs)},$$

the field at the top of the sphere will be

$$E(0) = 4.5\,E_o \text{(volts/meter)},$$

the repulsive force will be, within about 1 per cent,

$$F_z = 1.537 \times 10^{-10} R^2 E_o^2 \text{ (newtons)},$$

and the charge distribution may be approximated, to within about 10 per cent, by

$$\sigma(\theta) = (0.8696 \cos^2\frac{\theta}{2} + 3.6304 \cos^4\frac{\theta}{2})\,\sigma_o \text{(coulombs/meter}^2\text{)}.$$

BIBLIOGRAPHY

BIBLIOGRAPHY

Dadda, L., L'Energia Elettrica 1, 23 (1951).

Davis, M. H., "The Forces Between Conducting Spheres in a Uniform Electric Field," The RAND Corporation, Santa Monica, California, Memorandum RM-2607-1-PR. (September 1962).

Fish, B. R., ed., Surface Contamination (Pergamon Press, Oxford, 1967).

Hackenschmidt, M., Technik 18, 478 (1963).

Hartill, E. R., J. G. McQueen and P. N. Bobson, Proc. IEE (London) 104A, 401 (1957).

Jeans, Sir James, The Mathematical Theory of Electricity and Magnetism (Cambridge University Press, London, 1941), 5th Edition, pp. 80 and 194.

Kennedy, P. A., and G. Kent, Rev. Sci. Instr. 27, 916 (1956).

Laplace, P. S. de, "Théorie des attractions des sphéroids et de la figure des planetes," Memoirs (Paris) (1785).

Loeb, L. B., J. H. Parker, E. E. Dodd and W. N. English, Rev. Sci. Instr. 21, 42 (1950).

Lowe, H. J., and D. H. Lucas, "The Physics of Electrostatic Precipitation," Brit. J. Appl. Phys., Suppl. 2, S40 (1953).

Maxwell, J. C., Nature 7, 218 (1873), [review of book by Thomson (1872)].

Maxwell, J. C., Electricity and Magnetism (Oxford University Press, London, 1892), 3rd Edition, Vol. 1, pp. 244-276.

Millikan, R. A., The Electron (University of Chicago Press, Chicago, 1924), 2nd Edition, pp. 67ff.

Morse, P. M., and H. Feshbach, Methods of Theoretical Physics (McGraw-Hill Book Co., New York, 1953), p. 1298.

Ralston, O. C., Electrostatic Separation of Mixed Granular Solids (Elsevier Publishing Co., Amsterdam, 1961).

Schultz, R. D., and R. W. Wiech, Jr., "Electrical Propulsion with Colloidal Materials," in Advanced Propulsion Techniques ed. S. S. Penner (Pergamon Press, New York, 1961), p. 98.

Southwell, R. V., Relaxation Methods in Theoretical Physics (Clarendon Press, Oxford, 1946).

Taylor, Sir G., Proc. Roy. Soc. 280A, 383 (1964).

Taylor, Sir G., Proc. Roy. Soc. 291A, 145 (1966).

Thomson, Sir Wm., Reprint of Papers on Electrostatics and Magnetism (MacMillan and Co., London, 1872).

Weber, E., Electromagnetic Theory (Dover Publications, New York, 1965), p. 231.