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On the Existence and Uniqueness of Static, Spherically Symmetric Stellar Models in General Relativity

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I am submitting herewith a thesis written by Josh Michael Lipsmeyer entitled "On the Existence and Uniqueness of Static, Spherically Symmetric Stellar Models in General Relativity." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

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On the Existence and Uniqueness of Static, Spherically Symmetric Stellar Models in General Relativity

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I would like to dedicate this work to my family:

Mom, Dad, Joey, Brandi, Tim, and Shannon

Your love, support, and encouragement has never waned for one second and for that I am truly thankful.

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“*Everybody can be great. Because anybody can serve. You don’t have to have a college degree to serve. You don’t have to make your subject and your verb agree to serve. You don’t have to know about Plato and Aristotle to serve. You don’t have to know Einstein’s theory of relativity to serve. You don’t have to know the second theory of thermodynamics to serve. You only need a heart full of grace. A soul generated by love.*”
- Martin Luther King, Jr.

“*We all die. The goal isn’t to live forever, the goal is to create something that will.*”
- Chuck Palahniuk

Abstract

The "Fluid Ball Conjecture" states that a static stellar model in General Relativity is spherically symmetric. This conjecture has been the motivation of much work since first studied by Avez in 1964. There have been many partial results(ul-Alam, Lindblom, Beig and Simon, etc) which rely heavily on arguments using the Positive Mass Theorem and the equivalence of conformal flatness and spherical symmetry. The purpose of this thesis is to outline the general problem, analyze and compare the key differences in several of the partial results, and give existence and uniqueness proofs for a particular class of equations of state which represents the most recent progress towards a fully generalized solution.

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Chapter 1

Introduction

The “Fluid Ball Conjecture” seems to have first been addressed by Avez in 1964.[23] The conjecture is concerned with equilibrium configurations for static stellar models. As in the Newtonian case [21], the belief is that a static stellar model equilibrium configuration always attains spherical symmetry. Partial results to the conjecture were attained in the 1970's and 1980's. Kunzle [20], Kunzle and Savage [15] made contributions to the conjecture but used very restrictive equations of state. ul-Alam [24],[25] brought in the use of the Posititve Mass Theorem [28]. ul-Alam's work in the 1980's unfortunately relied on unphysical equations of state to complete the proof. Lindblom [22] succeeded in proving spherical symmetry in the case of uniform density stars using ul-Alam's work and the use of Robinson-type identities that had previously been used in proving uniqueness of static black holes.[27],[26]. A drawback to the work of ul-Alam and Lindblom in the 1980's was that their method of proof required the existence of a “reference spherical stellar model” with the same mass and surface potential V_s as their static stellar model. To complete the proof using their method they assumed the existence of the “reference model”, without proof. In 1994 joint work of ul-Alam and Lindblom [29] proved the existence of the ”reference spherical stellar model” under certain restrictions on the equation of state. This represented the most complete work up to that point. Given a static stellar model and assuming an equation of state satisfying certain properties, the "reference stellar model" existed and the procedure developed in the 1980's utilizing the Positive Mass Theorem showed that the static stellar model must in fact be spherical. In 2007 the most recent result pertaining to the "Fluid Ball Conjecture" by ul-Alam used a spinor norm weighted scalar curvature integral. It was this integral that had been used by Witten to prove the Positive Energy Theorem in n-dimensions[30].

The method of proving spherical symmetry using the Positive Mass Theorem has been the standard method of proof since it was first utilized by ul-Alam. The procedure is to start with a static stellar model with a certain given equation of state. The goal is to find a conformal factor so that the mass of the conformal metric is zero and the conformal scalar curvature is non-negative. The Positive Mass Theorem then implies that the conformal metric must in fact be flat. In the conformally flat case , Avez [23] showed that the original geometry had to be spherical. The difficulty in this method is showing the non-negativity of the conformal scalar curvature. The conformal factor is modeled after the conformal factor for the "reference spherical model". This was the source of difficulty in ul-Alam and Lindblom's work in the 1980's. In order to show existence of the "reference stellar model" and the non-negativity of the conformal scalar curvature certain restrictions were placed on the equation of state. All of the modifications to this method revolved around restrictions on the equation of state.

Point-wise non-negativity of the conformal scalar curvature is a strict requirement. In an effort to relax this condition, the use of a spinor norm weighted scalar curvature integral was utilized by ul-Alam in 2007. This allows the point-wise non-negativity to be relaxed as long as the overall negative contribution to the integral of the conformal scalar curvature is small. The scalar curvature integral is precisely the integral used by Witten in his proof of the Positive Energy Theorem. In ul-Alam's work a conformal factor is defined as a limit of conformal factors. In the limit, the scalar curvature integral with the scalar curvature in the original metric is shown to go to zero. The scalar curvature integral equaling zero implies the existence of a global convariantly constant spinor field. It is known that spinors are a type of "square root" of a vector so the global covariantly constant spinor field allows us to define a global covariantly constant frame field. This implies that the space is flat. Classical arguments [23] then imply that since the conformal geometry is flat, then the original geometry must be spherically symmetric.

Let us now give a couple of standard definitions and then rigorously define the static stellar model. We assume a metric signature of $(-,+,+,+)$. Let Greek indices run from 0 to 3 and let Latin indices run from 1 to 3.

Energy-Momentum Tensor for a perfect fluid:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \tag{1.1}$$

where u is a unit time-like vector field representing the 4-velocity of the fluid, ρ is the density and p is the pressure of the fluid.

Einstein Equation:

$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}T^\lambda_\lambda g_{\mu_\nu}) \tag{1.2}$$

where ,$R_{\mu\nu}$ is the Ricci Tensor and $g_{\mu\nu}$ is the metric tensor.

A Static Stellar Model is a static, asymptotically flat space-time that satisfies the Einstein equation coupled with a perfect-fluid matter model. Physically, "static" means that the metric is time independent and the star is non-rotating. This corresponds to the mathematical definition.[20]

A space-time M is called *static* if and only if there exists a 3-dimensional manifold Σ and a diffeomorphism $\Psi : M \to \Sigma \times \mathbb{R}$ such that (i.) $C_x = \Psi^{-1}(\{x\} \times \mathbb{R})$ are time-like curves for all $x \in \Sigma$

(ii.) $\Sigma_t = \Psi^{-1}(\Sigma \times \{t\})$ are globally space-like hyper-surfaces for all $t \in \mathbb{R}$

(iii.) C_x for all $x \in \Sigma$ are tangent to a Killing vector field k on M that is orthogonal to all Σ_t

The main objective of this thesis is expository. This is a compilation of the work of brilliant men over the last 50 years. We hope to give a complete picture of existence and uniqueness of the static stellar model that is current to date. Simply put, the "Fluid Ball Conjecture" pertains to the actual shape of a star. It seems intuitively obvious that a highly idealized star modeled as a perfect fluid in equilibrium which does not rotate or change over time would be spherical in shape. In the context of General Relativity proving this expectation is not straight forward. This paper deals with this problem as a whole. We give a complete proof of the existence of spherically symmetric stellar models with an equation of state of acceptable regularity. It is a rather straightforward proof that utilizes a theorem from the theory of ordinary differential equations (O.D.E) which handles the singularity that arises in the center of the star. We also give a complete proof of the most recent result on the uniqueness of a static stellar model given by Masood ul-Alam. This proof constructs a conformal factor as a limit of constructed conformal factors and shows the spinor norm weighted scalar curvature integral goes to zero in the limit, implying in this case conformal flatness. Finally, we analyze constraints on certain physical quantities that occur in the framework of General Relativity. These constraints lend themselves as support to the "Fluid Ball Conjecture".

Chapter 2

Existence and Uniqueness of Static, Spherically Symmetric Solutions

Existence and uniqueness proofs for spherically symmetric static stellar models with perfect fluid source are scattered throughout literature. Lindblom and ul-Alam [29] proved existence in their joint work for a given equation of state, mass, and surface potential V_s. Pfister [5] proved a general existence theorem for a certain class of equations of state using a Banach fixed point method. Mak and Harko [6] proved an existence theorem using a Riccati type first order O.D.E. with a solution expressed in the form of an infinite power series. The goal of this section is to discuss an existence theorem given by Rendall and Schmidt [7]. Although not the most general existence theorem for the spherically symmetric static stellar model with perfect fluid source, the proof is rather straightforward. We hold fast to the intuitive geometry of standard coordinates and the result follows from a modified existence theorem for singular ordinary differential equations. We start with a given central pressure and prove global existence and uniqueness for the Einstein equations representing the spherically symmetric static stellar model. Since we start from the center of the star, it is possible that the star's radius is infinite. In this case the vacuum region will be empty. If the star has a finite radius the boundary will occur at $r = R$ where $p(R) = 0$ and p denotes the pressure. This particular existence and uniqueness theorem allows for stars of infinite radius, i.e. where the pressure does not have compact support. Certain results on the finiteness of the star can be derived from the given equation of state. We discuss this in a different section. We now state the main theorem.

Theorem 2.1 [Rendall and Schmidt (1991)]
Let an equation of state $\rho(p)$ be given such that ρ is defined for $p \geq 0$, non-negative, and continuous for $p \geq 0$, C^∞ for $p > 0$ and suppose that $\frac{d\rho}{dp} > 0$ for $p > 0$. Then there exists for any value of the central density ρ_0 a unique inextendible static, spherically symmetric solution of Einstein's field equations with a perfect fluid source and equation of state $\rho(p)$. The matter either has finite extent, in which case a unique Schwarzschild solution is joined on as an exterior field, or the matter occupies the whole of space, with ρ tending to zero as r tends to infinity.

We note that the constraint of an equation of state $\rho(p)$ being C^∞ is one of convenience. This proof works for equations of state with lesser regularity. We now want to set the problem up.

2.1 Derivation of the System of Equations

The metric in Schwarzschild coordinates for a static, spherically symmetric space-time is given by

$$ds^2 = -c^2 e^{a(r)} dt^2 + e^{b(r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \tag{2.1}$$

where c represents the speed of light in vacuum and b, a are functions that only depend on r, the "area radius". Notice that we use the metric signature $(-,+,+,+)$. The Einstein field equations are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^2}T_{\mu\nu} \tag{2.2}$$

where $\mu, \nu = 0, 1, 2, 3$, $G_{\mu\nu}$ denotes the Einstein tensor, $R_{\mu\nu}$ is the Ricci curvature, $R = R^{\mu}_{\mu}$ is the scalar curvature, and G is the gravitational constant. We take the cosmological constant Λ to be zero. The perfect fluid stress energy tensor is given by

$$T_{\mu\nu} = \rho u_\mu u_\nu + \frac{p}{c^2}(u_\mu u_\nu + g_{\mu\nu}) \tag{2.3}$$

where $u = (u_0, u_1, u_2, u_3)$ represents the components for the 4-velocity of our static fluid. Aligning the 4-velocity with the static Killing field, we have $u_0 = -\sqrt{e^a}$ and since the fluid is in equilibrium the spatial components are all zero, i.e. $u_i = 0$ for $i = 1, 2, 3$. This gives us

$$g^{\nu\mu}u_\mu u_\nu = -1 \tag{2.4}$$

Other variables include ρ, which is the proper energy density, and p, the proper pressure. Combining equations (2.1)-(2.3) we are able to derive the system of field equations with respect to the above coordinates.[8]

$$\frac{8\pi G}{c^2}\rho r^2 = \frac{rb' - 1}{e^b} + 1 \tag{2.5}$$

$$\frac{8\pi G}{c^4}pr^2 = \frac{ra' + 1}{e^b} - 1 \tag{2.6}$$

$$\frac{8\pi G}{c^4}p = \frac{1}{2e^b}(a'' + \frac{1}{2}(a')^2 + \frac{a' - b'}{r} - \frac{1}{2}a'b') \tag{2.7}$$

where $'$ denotes differentiation with respect to r. Also, we note that the field equations for G_{22} and G_{33} are the same. This provides us with three independent equations and four functions. We follow a strategy outlined in [8]. We want to eliminate p from equations (2.6) and (2.7). Setting these two equations equal and solving gives us the expression

$$\frac{e^b}{r^2} = \frac{a'b'}{4} - \frac{(a')^2}{4} - \frac{1}{2}a'' + \frac{a'+b'}{2r} + \frac{1}{r^2} \tag{2.8}$$

Next, we add equations (2.5) and (2.6) together, which gives us

$$\frac{8\pi G}{c^2}(\rho + \frac{p}{c^2}) = \frac{a'+b'}{re^b} \tag{2.9}$$

We now want to divide both sides of (2.6) by r^2 for $r \neq 0$ and differentiate both sides with respect to r. We have the expression

$$\frac{8\pi G}{c^4}p' = \frac{2}{r^3} - e^{-b}[\frac{a'}{r^2} + \frac{a'b'}{r} + \frac{2}{r^3} + \frac{b'}{r^2} - \frac{a''}{r}] \tag{2.10}$$

We can eliminate a'' from equation (2.10) by using equation (8). Rearranging terms and simplifying gives us the expression

$$\frac{8\pi G}{c^4}p' = -\frac{a'}{2}\frac{(a'+b')}{re^b} \tag{2.11}$$

We combine equation (2.11) with (2.9). This gives us

$$2p' = -c^2 a'(\rho + \frac{p}{c^2}) \tag{2.12}$$

This equation represents the conservation of energy-momentum for a static perfect fluid. Equations (2.5),(2.6),and (2.12) now contain all of the constraint information for our functions. We have only three equations for four functions for our system. We close this system by specifying an equation of state, $\rho(p)$.

Now, equation (2.5) can be rewritten as

$$\frac{8\pi G}{c^2}r^2\rho(r) = \frac{d}{dr}(r - re^{-b} + const.) \tag{2.13}$$

Integrating both sides with respect to r gives us

$$\frac{8\pi G}{c^2}\int r^2\rho(r)dr = r - re^{-b} \implies e^{-b} = 1 - \frac{8\pi G}{c^2}\frac{1}{r}\int r^2\rho(r)dr + const. \tag{2.14}$$

For our solutions we are seeking regular centers of spherical symmetry. In order to avoid a "conical singularity" in the metric at $r = 0$, we set the constant equal to zero.[16] This gives us the following expression

$$e^{-b(r)} = 1 - \frac{8\pi G}{c^2}\frac{1}{r}\int_0^r s^2\rho(s)ds \tag{2.15}$$

We recall the expression for the Newtonian mass given up to radius r by the expression

$$m(r) = 4\pi\int_0^r s^2\rho(s)ds \tag{2.16}$$

Its derivative with respect to r is given by

$$m'(r) = 4\pi r^2\rho(r) \tag{2.17}$$

We combine equations (2.15) and (2.16) to get an expression for e^{-b}

$$e^{-b} = 1 - \frac{2G}{c^2}\frac{m}{r} \tag{2.18}$$

This is the spatial metric potential. Now, if we differentiate equation (2.18) with respect to r we get

$$e^{-b}b' = \frac{2G}{c^2}\frac{rm' - m}{r^2} \tag{2.19}$$

and inserting equation (2.17) into (2.19) gives us

$$e^{-b}b' = \frac{2G}{c^2}\frac{4\pi r^3\rho - m}{r^2} \tag{2.20}$$

If we solve equation (2.9) for $a'e^{-b}$ we get

$$\frac{a'}{e^b} = \frac{8\pi G}{c^2}(\rho + \frac{p}{c^2})r - e^{-b}b' \tag{2.21}$$

We insert (2.20) into (2.21) and we get the expression

$$\begin{aligned} a'e^{-b} &= \frac{8\pi G}{c^2}(\rho + \frac{p}{c^2})r - \frac{2G}{c^2}\frac{(4\pi r^3\rho - m)}{r^2} && (2.22) \\ &= \frac{8\pi G}{c^2}\rho r + \frac{8\pi G}{c^4}pr - \frac{8\pi G}{c^2}\rho r + \frac{2Gm}{c^2 r^2} && (2.23) \\ &= \frac{2}{c^2 r^2}(4\pi G p r^3 + Gm) && (2.24) \end{aligned}$$

Solving for a' we get

$$a' = \frac{2}{c^2 r^2 e^{-b}}(4\pi G p r^3 + Gm) \tag{2.25}$$

If we insert the expression for a' into the equation for energy-momentum conservation, which is equation (2.12) we get

$$\begin{aligned} 2p' &= -c^2(\frac{2}{c^2 r^2 e^{-b}}(4\pi G p r^3 + Gm))(\rho + \frac{p}{c^2}) && (2.26) \\ &= -(\frac{2}{r^2 e^{-b}}(4\pi G p r^3 + Gm))(\rho + \frac{p}{c^2}) && (2.27) \end{aligned}$$

Finally, if we insert the expression in equation (2.18) into (2.27) we have the following expression.

$$p'(r) = -\frac{1}{r^2}(4\pi G p(r) r^3 + Gm(r))(\rho(r) + \frac{p(r)}{c^2})(\frac{1}{1 - \frac{2G}{c^2}\frac{m(r)}{r}}) \tag{2.28}$$

Equation (2.28) is known as the Tolman-Oppenheimer-Volkoff (T.O.V.) equation of hydrostatic equilibrium. We collect some of the equations that are standard in deriving interior solutions for static, spherically symmetric perfect fluid stellar models.

$$p'(r) = -\frac{1}{r^2}(4\pi G p(r) r^3 + G m(r))(\rho(r) + \frac{p(r)}{c^2})(\frac{1}{1 - \frac{2G}{c^2}\frac{m(r)}{r}}) \tag{2.29}$$

$$m'(r) = 4\pi r^2 \rho(r) \tag{2.30}$$

$$e^{-b(r)} = 1 - \frac{2G}{c^2}\frac{m(r)}{r} \tag{2.31}$$

$$2p'(r) = -c^2 a'(r)(\rho(r) + \frac{p(r)}{c^2}) \tag{2.32}$$

We introduce an equation that represents a normalized mean density. It is given by

$$w(r) = \frac{m(r)}{r^3} \tag{2.33}$$

Inserting this into equation (2.18) gives us

$$e^{-b} = 1 - \frac{2G}{c^2} r^2 w(r) \tag{2.34}$$

We can express the T.O.V. equation in terms of w using equations (2.33) and (2.34) which yields

$$p' = -\frac{1}{r^2}(4\pi G p r^3 + G r^3 w)(\rho + \frac{p}{c^2})(\frac{1}{1 - \frac{2G}{c^2} r^2 w}) \tag{2.35}$$

$$= -Gr(\frac{1}{1 - \frac{2G}{c^2} r^2 w})(4\pi p + w)(\rho + \frac{p}{c^2}) \tag{2.36}$$

If we differentiate (2.33) with respect to r and use equation (2.30) we get

$$w'(r) = \frac{r^3 m'(r) - 3m(r)r^2}{r^6} \implies r^3 w'(r) + \frac{3m(r)}{r} = m'(r) \tag{2.37}$$

$$= 4\pi r^2 \rho(r) \tag{2.38}$$

$$\tag{2.39}$$

Simplifying this expression we get

$$r^3 w'(r) + 3r^2 w(r) = 4\pi r^2 \rho(r) \implies w'(r) = \frac{1}{r}(4\pi\rho(r) - 3w(r) \tag{2.40}$$

If we are given an equation of state, $\rho(p)$ then we can use it to integrate equation (2.2). This gives us

$$a(r) = -\int_{p_0}^{p(r)} \frac{2}{\rho(p) + \frac{p}{c^2}} dp \tag{2.41}$$

where p_0 is the pressure at the center of the star.

Finally, equations (2.35) and (2.40) form a system of equations for the functions $p(r)$ and $w(r)$. If we solve this system, then we can determine $b(r)$ from equation (2.34) and $a(r)$ from equation (2.41). We collect the system here

$$p' = -Gr(\frac{1}{1 - \frac{2G}{c^2} r^2 w})(4\pi p + w)(\rho + \frac{p}{c^2}) \tag{2.42}$$

$$w'(r) = \frac{1}{r}(4\pi\rho(r) - 3w(r) \tag{2.43}$$

$$e^{b(r)} = \frac{1}{1 - \frac{2G}{c^2} r^2 w(r)} \tag{2.44}$$

$$a(r) = -\int_{p_0}^{p(r)} \frac{2}{\rho(p) + \frac{p}{c^2}} dp \tag{2.45}$$

The derivation of the T.O.V equation was motivated and executed by eliminating a' from equations (2.6) and (2.12). Instead of eliminating a' from (2.6) and (2.12) we could eliminate p. For this purpose we consider the following auxiliary functions [9].

$$y^2(r) = e^{-b(r)} = 1 - \frac{2G}{c^2} r^2 w(r) \tag{2.46}$$

$$z(r) = e^{\frac{a(r)}{2}} \tag{2.47}$$

$$x(r) = r^2 \tag{2.48}$$

The goal now is to express equations (2.6) and (2.12) in terms of the new variables just given and then combine the equations, eliminating p. For (2.6) we need an expression for $\frac{da}{dr}$. From (2.47) we get

$$2\ln(z(r)) = a(r) \tag{2.49}$$

Differentiating both sides of (2.49) with respect to x gives us

$$2\frac{1}{z}\frac{dz}{dx} = \frac{da}{dr}\frac{dr}{dx} = \frac{da}{dr}\frac{1}{2\sqrt{x}} \tag{2.50}$$

Solving (2.50) for $\frac{da}{dr}$ gives us

$$\frac{da}{dr} = 4\sqrt{x}\frac{1}{z}\frac{dz}{dx} \tag{2.51}$$

We also have

$$\frac{da}{dx} = \frac{2}{z}\frac{dz}{dx} \tag{2.52}$$

We want to express ρ in terms of w and the auxiliary functions (2.46)-(2.48). Recall equation (2.43). For this we have the following derivation

$$\rho = \frac{4\pi}{4\pi}\rho \tag{2.53}$$
$$= \frac{1}{4\pi}[\frac{3m}{r^3} - \frac{3m}{r^3} + 4\pi\rho] \tag{2.54}$$
$$= \frac{1}{4\pi}[\frac{3m}{r^3} + \frac{2r^2}{2r^2}(4\pi\rho - \frac{3m}{r^3})] \tag{2.55}$$
$$= \frac{1}{4\pi}[3w + 2r^2\frac{1}{r}(4\pi\rho - 3w)\frac{1}{2r}] \tag{2.56}$$
$$= \frac{1}{4\pi}[3w + 2r^2\frac{dw}{dr}\frac{1}{2r}] \tag{2.57}$$
$$= \frac{1}{4\pi}[3w + 2x\frac{dw}{dr}\frac{dr}{dx}] \tag{2.58}$$
$$= \frac{1}{4\pi}[3w + 2x\frac{dw}{dx}] \tag{2.59}$$

We want to express (2.6) using functions (2.46)-(2.48) and (2.51). We express (2.12) using functions (2.46)-(2.48), (2.52) and (2.59). This gives us the following two equations.

$$\frac{8\pi G}{c^4}p = 4y^2\frac{1}{z}\frac{dz}{dx} - \frac{2G}{c^2}w \tag{2.60}$$
$$\frac{dp}{dx} = -\frac{1}{z}\frac{dz}{dx}(\rho + \frac{p}{c^2}) = -\frac{1}{z}\frac{dz}{dx}(\frac{1}{4\pi}[3w + 2x\frac{dw}{dx}] + \frac{p}{c^2}) \tag{2.61}$$

We want to now eliminate dependence on p from (2.60) and (2.61) and set the equation to equal zero. This equation can be expressed in the following way by means of calculation and simplification [7], [9].

$$0 = (1 - \frac{2G}{c^2}xw)\frac{d^2z}{dx^2} - \frac{G}{c^2}\frac{dz}{dx}[\frac{d}{dx}(w + x\frac{dw}{dx})] - \frac{G}{2c^2}\frac{dw}{dx}z \tag{2.62}$$
$$= y^2\frac{d^2z}{dx^2} + y\frac{dy}{dx}\frac{dz}{dx} - \frac{G}{2c^2}z\frac{dw}{dx} \tag{2.63}$$
$$= \frac{d}{dx}(y\frac{dz}{dx}) - \frac{G}{2c^2}\frac{z}{y}\frac{dw}{dx} \tag{2.64}$$

If ρ is given then equation (2.62) is linear in the variable z and if z is given, then (2.62) is linear in w.[9] We next want expressions for $\frac{d\rho}{dx}$ and $\frac{dw}{dx}$. Using equation (2.35) and $r^2 = x$ we get

$$\frac{d\rho}{dx} = (\frac{dp}{d\rho})^{-1}\frac{dp}{dx} \tag{2.65}$$

$$= (\frac{dp}{d\rho})^{-1}\frac{dp}{dr}\frac{dr}{dx} \tag{2.66}$$

$$= (\frac{dp}{d\rho})^{-1}[-Gr(\frac{1}{1-\frac{2G}{c^2}r^2w})(4\pi p + w)(\rho + \frac{p}{c^2})]\frac{1}{2\sqrt{x}} \tag{2.67}$$

$$= -\frac{G}{2}(\frac{dp}{d\rho})^{-1}(\frac{1}{1-\frac{2G}{c^2}xw})(4\pi p + w)(\rho + \frac{p}{c^2}) \tag{2.68}$$

and using equation (40) and $r^2 = x$ we get

$$\frac{dw}{dx} = \frac{dw}{dr}\frac{dr}{dx} \tag{2.69}$$

$$= \frac{1}{r}(4\pi\rho - 3w)\frac{1}{2\sqrt{x}} \tag{2.70}$$

$$= \frac{1}{2x}(4\pi\rho - 3w) \tag{2.71}$$

Equations (2.68) and (2.71) provide a system of equations for unknown functions $w(x)$ and $\rho(x)$. Solving for w allows us to recover b from (2.46). Given an equation of state, and having solved the system for ρ we then solve (2.45) for a. We collect this system here.

$$\frac{d\rho}{dx} = -\frac{G}{2}(\frac{dp}{d\rho})^{-1}(\frac{1}{1-\frac{2G}{c^2}xw})(4\pi p + w)(\rho + \frac{p}{c^2}) \tag{2.72}$$

$$\frac{dw}{dx} = \frac{1}{2x}(4\pi\rho - 3w) \tag{2.73}$$

$$a(r) = -\int_{p_0}^{p(r)} \frac{2}{\rho(p)+\frac{p}{c^2}}dp \tag{2.74}$$

$$e^b = \frac{1}{1-\frac{2G}{c^2}xw} \tag{2.75}$$

Finally, equations (2.72)-(2.75) represents the system our proof will deal with. First note that while equation (2.72) is regular at $x = 0$, (2.73) is not. In an effort to have uniform properties for both equations we want to make (2.72) singular at $x = 0$. Define a function ρ_1 so that

$$\rho = \rho_0 + x\rho_1 \tag{2.76}$$

where ρ_0 denotes the density at the center of symmetry. Similar to equation (2.77) we have the corresponding relationship

$$p(\rho) = p_0 + xp_1(\rho_1) \tag{2.77}$$

where p_0 denotes the central pressure. Substituting (2.76) and (2.77) into equation (2.72),(2.73) and algebraically manipulating (2.73) gives us

$$x\frac{d\rho_1}{dx} + \rho_1 = -\frac{G}{2}(\frac{dp}{d\rho}(\rho_0) + xh(\rho_1))^{-1}(\frac{1}{1-\frac{2G}{c^2}xw})(4\pi p_0 + 4\pi x p_1(\rho_1) + w) \tag{2.78}$$

$$\cdot (\rho_0 + x\rho_1 + \frac{p_0}{c^2} + \frac{xp_1(\rho_1)}{c^2})$$

$$x\frac{dw}{dx} + \frac{3}{2}w = 2\pi x\rho_1 + 2\pi\rho_0 \tag{2.79}$$

where $\frac{dp}{d\rho} = \frac{dp}{d\rho}(\rho_0) + xh(\rho_1)$ for some smooth function h.

2.2 The Singular Point

Rendall and Schmidt prove an existence theorem for singular ordinary differential equations. We state it now.

Theorem 2.2 [Rendall and Schmidt (1991)]
Let V be a finite-dimensional real vector space, $N : V \to V$ a linear mapping, $F : V \times I \to V$ a smooth (i.e. C^∞) mapping and $g : I \to V$ a smooth mapping, where I is an open interval in $\mathbb{R}$ containing zero. Consider the equation

$$s\frac{df}{ds} + Nf = sF(s, f(s)) + g(s) \tag{2.80}$$

for a function f defined on a neighborhood of 0 in I and taking values in V. Suppose that each eigenvalue of N has positive real part. Then there exists an open interval J with $0 \in J \subset I$ and a unique bounded C^1 function f on $J \setminus \{0\}$ satisfying (2.80). Moreover f extends to a C^∞ solution of (2.80) on J. If N, G and g depend smoothly on a parameter γ and the eigenvalues of N are distinct then the solutions also depends smoothly on γ.

We want to apply Theorem 2.2 to the set of equations (2.78) and (2.79). We first want to put it into the proper form. We first define $f : \mathbb{R} \to \mathbb{R}^2$ by

$$f(x) = \begin{pmatrix} \rho_1(x) \\ w(x) \end{pmatrix} \tag{2.81}$$

We now want to look at equation (2.78). Note that we have the following equivalence.

$$(1 - \frac{2G}{c^2}xw)^{-1} = 1 + \frac{2G}{c^2}xw(1 - \frac{2G}{c^2}xw)^{-1} \tag{2.82}$$

Inserting (2.82) into (2.78) gives us

$$x\frac{d\rho_1}{dx}+\rho_1=-\frac{G}{2}(\frac{dp}{d\rho}(\rho_0)+xh(\rho_1))^{-1}[1+\frac{2G}{c^2}xw(1-\frac{2G}{c^2}xw)^{-1}](4\pi p_0+4\pi xp_1(\rho_1)+w) \tag{2.83}$$

$$\cdot(\rho_0+x\rho_1+\frac{p_0}{c^2}+\frac{xp_1(\rho_1)}{c^2})$$

We expand the right-hand side of (2.83) and collect terms. We then define functions $F_1 : I \times \mathbb{R}^2 \to \mathbb{R}$ and $g_1 : I \to \mathbb{R}$ where $I \subset \mathbb{R}$ is an open interval containing 0. This way we can write (2.83) as

$$x\frac{d\rho_1}{dx}+\rho_1=-\frac{G}{2}\frac{dp}{d\rho}(\rho_0)(\frac{p_0}{c^2}+\rho_0)w+xF_1(x,f(x))+g_1(x) \tag{2.84}$$

Further still, we write (2.84) as

$$x\frac{d\rho_1}{dx}+\rho_1+\frac{G}{2}\frac{dp}{d\rho}(\rho_0)(\frac{p_0}{c^2}+\rho_0)w=xF_1(x,f(x)))+g_1(x) \tag{2.85}$$

Now, for equation (2.79) we define functions $F_2 : I \times \mathbb{R}^2 \to \mathbb{R}$ and $g_2 : I \to \mathbb{R}$ by

$$F_2(x,f(x))=2\pi\rho_1(x) \tag{2.86}$$

$$g_2(x)=2\pi\rho_0 \tag{2.87}$$

This allows us to express (2.79) as

$$x\frac{dw}{dx}+\frac{3}{2}w=2\pi x\rho_1+2\pi\rho_0=xF_2(x,f(x))+g_2(x) \tag{2.88}$$

We can now define our function $N : \mathbb{R}^2 \to \mathbb{R}^2$ in matrix form. It is given by

$$N=\begin{pmatrix} 1 & \frac{G(\frac{p_0}{c^2}+\rho_0)}{2\frac{dp}{d\rho}(\rho_0)} \\ 0 & \frac{3}{2} \end{pmatrix} \tag{2.89}$$

We can now write out our system of equations (2.78),(2.79) in the form given by Theorem 2.2. We use the expression in (2.85),(2.88), and (2.89). This gives us

$$x\frac{df}{dx} + Nf(x) = xF(x, f(x)) + g(x) \tag{2.90}$$

$$x\frac{d}{dx}\begin{pmatrix} \rho_1(x) \\ w(x) \end{pmatrix} + \begin{pmatrix} 1 & \frac{G(\frac{p_0}{c^2}+\rho_0)}{2\frac{dp}{d\rho}(\rho_0)} \\ 0 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} \rho_1(x) \\ w(x) \end{pmatrix} = x\begin{pmatrix} F_1(x, f(x)) \\ F_2(x, f(x)) \end{pmatrix} + \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix} \tag{2.91}$$

We now just note that F is a combination of smooth functions, and hence smooth. g is a combination of smooth functions, and hence smooth. The matrix N is upper triangular with eigenvalues 1 and $\frac{3}{2}$ hence all its eigenvalues are real and positive. Therefore, by theorem 2 there exists an open interval J with $0 \in J \subset I$ and a unique bounded C^1 function f on $J \setminus \{0\}$ satisfying (2.91). Moreover, f extends to a C^∞ solution of (2.91) on the entire interval J. This means that we have a unique smooth solution of equations (2.72) and (2.73) in a neighborhood of 0 for the given central density ρ_0 and equation of state $\rho(p)$.

2.3 Extending the Solution Uniquely

Our final step in the proof is showing that our unique, smooth solution f can be extended in a unique way. If our star is finite, there will be some x_b such that $p(x_b) = 0$ where x_b corresponds to R which is the radius of the star. In this case, the Schwarzschild vacuum solution will connect to the interior solution at the boundary region.[10] If the star is infinite, then $p(x) > 0$ for all x,$\rho \to 0$ as $x \to \infty$, and the vacuum region is empty. Standard existence and uniqueness theorems for ordinary differential equations imply that f can be extended in a unique way as long as the right-hand sides of (2.72) and (2.73) are well-defined.[7]. We first look at (2.72).

We want to make sure that as $x \to \infty$ $\frac{d\rho}{dx} > 0$ for all x in the interior solution, i.e. the region where $p > 0$. If the star is infinite then the "interior" solution will occupy all of space. Our hypothesis for Theorem 2.1 includes $\frac{d\rho}{dp} > 0$ and $\rho(p) \geq 0$ for $p \geq 0$. We also know that $G, c > 0$ and $w > 0$ for $p > 0$. These imply that the first, second, and fourth factors in equation (2.72) are positive in the "interior" region.

We first want to show that $y^2 = 1 - \frac{2G}{c^2}xw$ cannot vanish while $p > 0$. If this were to occur, then $p > 0$ implies we are still in the "interior" solution. If y^2 vanished in the interior, then $\frac{d\rho}{dx} = 0$ and we would lose uniqueness of our extension. Let $0 \leq x < x_b$ be an interval where $y^2 > 0$ and $p > 0$. We know that since $\frac{d\rho}{dp} > 0$ that the density $\rho(p)$ does not increase outward, i.e. when p decreases. This implies that $\frac{dw}{dx} \leq 0$. Equation (2.64) then gives us

$$\frac{d}{dx}(y\frac{dz}{dx}) = \frac{G}{2c^2}\frac{z}{y}\frac{dw}{dx} \leq 0 \tag{2.92}$$

Since $z, y > 0$ and G, c are positive constants. Equation (2.60) can be rewritten as

$$y\frac{dz}{dx} = \frac{G}{c^2}\frac{z}{y}[\frac{4\pi p}{c^2} + w] \tag{2.93}$$

Equation (2.92) gives us that the derivative of $y\frac{dz}{dx}$ is non-positive, which means that the expression in (2.93) is non-increasing. Therefore we have

$$y\frac{dz}{dx} \leq y\frac{dz}{dx}|_{x=0} \tag{2.94}$$

This gives us the following inequality, recalling (2.46)-(2.48).

$$\frac{G}{c^2}\frac{z}{y}[\frac{4\pi p}{c^2}+w] \leq \frac{G}{c^2}[\frac{4\pi p_c}{c^2}+w_0] \tag{2.95}$$

$$\frac{z}{y}[\frac{4\pi p}{c^2}+w] \leq \frac{4\pi p_c}{c^2}+w_0 \tag{2.96}$$

$$z\frac{[\frac{4\pi p}{c^2}+w]}{\frac{4\pi p_c}{c^2}+w_0} \leq y \tag{2.97}$$

$$\frac{[\frac{4\pi p}{c^2}+w]}{\frac{4\pi p_c}{c^2}+w_0} \leq y \tag{2.98}$$

Therefore, (2.98) shows that it is impossible for y to vanish before p. This of course implies that $y^2 = 1 - \frac{2G}{c^2}xw$ cannot vanish in the region where $p > 0$. Hence $\frac{d\rho}{dx} > 0$ for $p > 0$ and ρ extends uniquely to the boundary of the star. For equation (2.73) we recall the expression in (2.79) and note that $\rho = \rho_0 + x\rho_1$ is unique and smooth up to the boundary and so w must also be smooth and unique up to the boundary.

Consider the case where the star is finite. The radius is given by R which corresponds to some x_b. The exterior solution is the Schwarzschild solution and is uniquely given by the following metric potentials.

$$e^{a(r)} = e^{-b(r)} = 1 - \frac{2G}{c^2}\frac{m(R)}{r} \tag{2.99}$$

The spatial metric potential for the interior solution is given by equation (2.75) by the following.

$$e^{-b(r)} = 1 - \frac{2G}{c^2}r^2 w(r) = 1 - \frac{2G}{c^2}\frac{m(r)}{r} \tag{2.100}$$

So we see by equation (2.99) and (2.100) that the spatial potential for the interior solution matches up to the spatial potential of the exterior vacuum solution at the boundary, $r = R$. For the space-time potential in the interior, we have by equation

(2.74) that

$$e^{a(r)} = \exp[-\int_{p_0}^{p(r)} \frac{2}{\rho(p) + \frac{p}{c^2}} dp] \tag{2.101}$$

at the boundary, $r = R$, the space-time potential can only be joined together continuously, i.e. in C^0 fashion. This is because it may be the case that $\rho(R) > 0$. If ρ does not vanish at the boundary then we can get at most a C^1 space-time potential. This follows from equation (2.2) and the fact that if ρ does not equal zero at the boundary then the Ricci tensor must be discontinuous at the boundary. We also make note that the metric cannot be extended because as r increases so does the area of the $r = const.$ group orbits.

In the case of an infinite star, we have $p(r) > 0$ for all $r > 0$. We know that $\lim_{r\to\infty} p(r)$ exists since $p(r)$ is bounded below by 0 and monotonically decreasing. This implies that $\frac{dp}{dr}$ must vanish at infinity. Recall equation (2.46). Note that $y^2 \leq 1$. (2.42) give us

$$\frac{dp}{dr} = -Gr\frac{1}{y^2}(4\pi p + w)(\rho + \frac{p}{c^2}) \tag{2.102}$$

We know w vanishes at infinity by definition since $m(r) \leq m(R) < \infty$. Also, $\rho \geq 0$ for $p \geq 0$ The fact that $\lim_{r\to\infty} \frac{dp}{dr} = 0$, $-Gr \to -\infty$ as $r \to \infty$ and $\frac{1}{y^2} \geq 1$ gives us

$$0 = \lim_{r\to\infty} \frac{dp}{dr} = -\lim_{r\to\infty} Gr\frac{1}{y^2}(4\pi p + w)(\rho + \frac{p}{c^2}) \tag{2.103}$$

$$\leq -\lim_{r\to\infty} Gr(4\pi p + w)(\rho + \frac{p}{c^2}) \tag{2.104}$$

Clearly we see that $\lim_{r\to\infty} p(r) = 0$. This implies that $\lim_{r\to\infty} \rho(r) = 0$. This completes the proof of Theorem 2.1.

Chapter 3

Uniqueness of the Static Stellar Model

The metric tensor in a static model in General Relativity has the form:

$$ds^2 = -V^2 dt^2 + g_{ab} dx^a dx^b \tag{3.1}$$

where g_{ab}, $a, b = 1, 2, 3$, denotes the spatial metric for the $t = const.$ hyper-surfaces and V and g_{ab} are time-independent. Coupling Einstein's equation with a perfect fluid matter model gives us the following pair of equations [31]

$$D^a D_a V = 4\pi V(\rho + 3p) \tag{3.2}$$

$$R_{ab} = V^{-1} D_a D_b V + 4\pi(\rho - p) g_{ab} \tag{3.3}$$

where the density and pressure are denoted by ρ and p, respectively. D_a and R_{ab} are with respect to the spatial metric, and the density is assumed to be a given function of pressure, $\rho(p)$, which is referred to as the equation of state. Combining the two equations above and the Bianchi identity yields:

$$D_a p = -V^{-1}(\rho + p) D_a V \tag{3.4}$$

If this static stellar model was spherically symmetric, then this equation would be equivalent to the so-called Tolman-Oppenheimer-Volkoff (T.O.V.) equation for hydrostatic fluids for a metric tensor of the above form. These equations have become standard over the evolution of the attempt to solve this problem.

The idea of using the Positive Mass Theorem to prove rotational symmetry of static stellar models which we analyze here was introduced by Masood ul-Alam in the late 1980's. Slight variations have been used in order to obtain uniqueness while lessening the restrictions on the acceptable equations of state. The scheme is to first parametrize the system in terms of the potential V, where the surface potential V_s represents the boundary of the star. The task is then to derive a conformal factor which transforms the spatial metric of the $t = const.$ hyper-surface of the static stellar model into a metric with zero mass and non-negative scalar curvature. The rigidity part of the Positive Mass Theorem then implies that the conformal metric must be flat. An old result due to Avez [23] says that a spatially conformally flat static perfect-fluid solution is necessarily spherically symmetric. The technical difficulties with this method revolve around showing the that the scalar curvature of the conformal metric is non-negative. The scalar curvature associated with conformal metric parameterized by V is given by the equation

$$\widehat{R} = (\widetilde{W} - W)\frac{8}{\psi^5}\frac{d^2\psi}{dV^2} \tag{3.5}$$

where $W = D^a D_a V$ is the square of the field intensity for the static stellar model, ψ is the conformal factor which transforms the spatial metric, and $\widetilde{W}$ is the norm squared of the field intensity for a "reference model". More precisely, $\widetilde{W}$ is related to ψ via the second-order linear differential equation:

$$\widetilde{W}\frac{d^2\psi}{dV^2}(V) = 2\pi[\rho\psi - 2V(\rho + 3p)\frac{d\psi}{dV}(V)] \tag{3.6}$$

In general, the conformally transformed scalar curvature is related to the original scalar curvature by the formula

$$\widehat{R} = \psi^{-4}(R - 8\psi^{-1}D^a D_a \psi) \tag{3.7}$$

where the scalar curvature R and the covariant derivative D_a are with respect to the spatial metric.[32]. The variations on this overall prescription have revolved around choosing ψ and $\widetilde{W}$ in a clever way. Outside the boundary of our finite static star, the metric is necessarily Schwarzschild, which follows from the fact that outside of our star we have a vacuum solution of Einstein's equation. In the case rotational symmetry holds (as we hope to prove), Birkhoff's Theorem then implies that the metric is Schwarzschild. Therefore, outside of the star, we choose the conformal factor

$$\psi = \frac{1}{2}(1 + V) \tag{3.8}$$

for $V \geq V_s$. (That his corresponds to the Schwarzschild metric is easily shown, see [31].) Using this conformal factor the scalar curvature with respect to the conformal metric outside of the star will be zero. The mass of the conformal metric will also be zero. What is left is to determine $\widetilde{W}$ and ψ for the interior solution. Choosing $\widetilde{W}$ and ψ for the interior solution is what directs us to add restrictions to the equation of state. The challenge of guaranteeing the sign of $\widehat{R}$ in the interior of the star be non-negative is the heart of the variations of this method.

Lindblom in his 1988 paper [22] dealt with constant density models (see also [31]). This allowed the equation of hydrostatic equilibrium to be integrated explicitly. He then assumed without proof the existence of a "reference spherical model" which possesses the same equation of state, in this case the same constant density, and the same surface potential V_s. This "reference spherical model" is then used to define ψ and $\widetilde{W}$ inside the star. This leads to $\frac{8}{\psi^5}\frac{d^2\psi}{dV^2}$ being non-negative inside the star and vanishing on the outside. The sign of $(\widetilde{W} - W)$ in this case can then be determined using Robinson type identities and the maximum principle for elliptic operators.

In 1993 [29], Lindblom and ul-Alam demonstrated the existence of a "reference spherical model" for an appropriately chosen mass parameter $\nu = M(V_s)$ which possesses the same equation of state and surface potential value as the static stellar model. This means finding 'area radius' and 'mass' functions $r_\nu(V), m_\nu(V)$, solving a system of ordinary differential equations described below, and from them computing the 'field strength' function $W_\nu(V)$. The conformal factor is then found inside the star to by solving the first-order linear differential equation

$$\frac{d\psi}{dV} = \frac{\psi_\nu}{2r_\nu\sqrt{W_\nu}}[1 - \sqrt{1 - \frac{2m_\nu}{r_\nu}}] \tag{3.9}$$

with the boundary condition $\psi_\nu(V_s) = \frac{1}{2}(1 + V_s)$. The parameters subscripted by ν are the parameters of the "reference spherical model". The significance of this equation is that in the case of a spherically symmetric model with a mass parameter of ν, the conformal factor which transforms the spatial metric to a flat spatial metric satisfies this equation [31]. Unique solutions $r_\nu(V)$ and $m_\nu(V)$ exist (inside the star) if we assume that the equation of state is at least C^1. Specifically, the equation of state being C^1 implies that $\rho(V)$ and $p(V)$ are also C^1 for $V < V_s$, and then standard existence-uniqueness for O.D.E. systems give solutions on a maximal domain $(V_\nu, V_s]$.[29] In this case we set $\widetilde{W} = W_\nu$. One of the pivotal points of this variable mass modification being successful is monotonicity with respect to the mass parameter μ of the function $W_\mu(V)$ for a fixed $V \in (V_\mu, V_s)$. This monotonicity result follows from showing that the sign of the expression

$$\Sigma_\mu = \frac{dW_\mu}{dV} - \frac{8\pi}{3}V(\rho + 3p) + \frac{4W_\mu}{5V}\frac{\rho + p}{\rho + 3p}\frac{d\rho}{dp} \tag{3.10}$$

is non-negative. The sign of Σ_μ is guaranteed to be non-negative if the equation of state satisfies one of two conditions, Condition A or Condition B. Assuming that the equation of state is at least C^1 and satisfies either Condition A or Condition B ensures that Σ_ν is non-negative. This implies that W_μ is monotonic with respect to μ. Monotonicity then implies that for $\widetilde{W} = W_\nu$, we have $\widetilde{W} - W \geq 0$. For the quantity $\frac{8}{\psi^5}\frac{d^2\psi}{dV^2}$ to be guaranteed non-negative the equation of state must satisfy another condition

$$5\rho^2 - 6p(\rho + 3p)\kappa \geq 0 \tag{3.11}$$

where $\kappa = \frac{\rho+p}{\rho+3p}\frac{d\rho}{dp}$. If it happens to be the case that $\lim_{V\to V_\nu^+} r_\nu > 0$ for the "reference spherical model", then a slight perturbation is made in the mass parameter ν. ψ and $\widetilde{W}$ are then chosen based on the perturbed mass $\nu + \delta$. The two factors of the conformal scalar curvature seem to not be directly related. However, the conditions on the equation of state are. In fact, the inequality just mentioned is exactly Condition B. Moreover, it is shown that Condition A implies Condition B. So an equation of state which satisfies either Condition A or Condition B will imply that both factors of the conformal scalar curvature are non-negative. Then the line of argument described above implies the result. Uniqueness, in fact, implies that the mass parameter ν is actually equal to the ADM mass M of the given static stellar model.

The most recent uniqueness proof by ul-Alam [4] takes a more localized approach. The basic outline of his strategy is the same. The biggest modification made by ul-Alam is by utilizing the spinor norm weighted scalar curvature integral that appears in Witten's proof of the Positive mass theorem. This proof moves away from trying to construct a conformal factor that forces the point-wise non-negativity of the conformal scalar curvature and instead constructs a conformal factor such that the negative contribution of the conformal scalar curvature to the spinor norm weighted integrated scalar curvature can be made as small as we like. This relaxes the need for the scalar curvature of the conformal metric to be non-negative everywhere. We allow areas of the conformal geometry to have negative scalar curvature but the contribution to the integral must be small. Instead of defining a global conformal factor and a global $\widetilde{W}$ he constructs a sequence of conformal factors and a corresponding sequence of $\widetilde{W}$ starting at $V = V_s$ and working inward. This approach entails many more technical details but strengthens the overall result in that the only restriction on the equation of state is that it is piecewise C^1. The remainder of this section will be devoted to outlining the proof of ul-Alam's result which constitutes the most recent uniqueness proof for the static stellar model.

We first give the statement of the theorem and list the hypothesis.

3.1 Statement of the theorem.

Theorem 3.1 [M. ul-Alam (2007)]

A static stellar model satisfying the following assumptions is necessarily spherically symmetric.

i.) The space-time 4-manifold is $M^4 = N^3 \times \mathbb{R}$ with line element

$$ds^2 = -V^2 dt^2 + g_{ij} dx^i dx^j$$

g_{ij} is a complete Riemannian metric on N and $V : N \rightarrow [V_{min}, 1)$.

ii.) The spatial metric and gravitational potential satisfy the asymptotic conditions:

$$g_{ij} = (1 + \frac{2m}{r})\delta_{ij} + O(r^{-2})$$

$$V = 1 - \frac{m}{r} + O(r^{-2})$$

iii.) The density $\rho = \rho(p)$ is a piecewise C^1 positive, non-decreasing function of p for $p > 0$. $\rho = 0$ in the exterior region.

vi.) The sets on the spatial hyper-surface N along which ρ has discontinuity are smooth 2-surfaces. There are at most a finite number of these surfaces.

v.) The pressure $p = p(r)$ is globally Lipschitz and $p > 0$ in the fluid region and $p = 0$ in the exterior vacuum. p is a non-negative, bounded, measurable function.

vi.) The boundary $V = V_s < 1$ of the interior fluid region and the vacuum region are both level sets of V. The level set of V_s is a smooth 2-surface.

vii.) The gravitational potential V and the metric g_{ij} are $C^{1,1}$ globally and locally C^3 in the complement of the smooth 2-surfaces where ρ has discontinuity and the level set of V_s.

The Einstein equations for a static Lorentzian metric:

$$ds^2 = -V^2 dt^2 + g_{ij} dx^i dx^j \tag{3.12}$$

with the perfect fluid energy-momentum tensor reduce [31] to the following system of equations:

$$D^i D_i V = 4\pi V(\rho + 3p) \tag{3.13}$$

$$R_{ij} = V^{-1} D_i D_j V + 4\pi(\rho - p) g_{ij} \tag{3.14}$$

where D_i and R_{ij} are with respect to the Riemannian 3-metric, g_{ij}. The differential Bianchi identity for g_{ij} implies the equation for hydrostatic equilibrium

$$D_i p = -V^{-1}(\rho + p) D_i V \tag{3.15}$$

We carry out the integration of the equation for hydrostatic equilibrium on an interval $[V, V_s]$ where ρ is a C^1 function of p which gives us

$$\ln(\frac{V_s}{V}) = \int_0^p \frac{ds}{\rho(s) + s} = h(p) \tag{3.16}$$

The right hand side of this equation is invertible so we can consider p and ρ as functions of the potential

$$p(V) = h^{-1}(\ln(\frac{V_s}{V})) \tag{3.17}$$

$$\rho(V) = \rho(p(V)) \tag{3.18}$$

Since $\rho(p)$ is C^1 in this interval, it follows that $p(V)$ and $\rho(V)$ are also C^1 for $V < V_s$ which follows from the fact that V is assumed to be C^3 away from the boundary and on the complement of the discontinuities of ρ.

3.2 The "reference system" of O.D.E.

We state now a local existence lemma of the spherically symmetric equations from ul-Alam and Lindblom's joint work.[29]

Consider the following system of equations.

$$\frac{dr}{dV} = \frac{r(r-2m)}{V(m+4\pi r^3 p)}$$
$$\frac{dm}{dV} = \frac{4\pi r^3(r-2m)\rho}{V(m+4\pi r^3 p)}$$

Here $\rho(V), p(V)$ are given C^1 functions in $[V_m, b]$, and initial conditions are given at $V = b$, satisfying $r > 0, r > 2m > -8\pi r^3 p$. (Where $b \in (V_m, V_s]$ is arbitrary). For a spherically symmetric solution (with metric coefficients depending only on V) the squared field strength $D_i V D^i V$ admits the expression:

$$\widetilde{W} = (1 - \frac{2m}{r})(\frac{dr}{dV})^{-2} = \frac{V^2(m+4\pi r^3 p)^2}{r^3(r-2m)}.$$

Let $(V_c, b]$ be the maximal interval of existence for a solution with given initial conditions $r(b), m(b)$. Then $r > 0, r > 2m > -8\pi r^3 p$ and $\widetilde{W} > 0$ on this interval, and furthermore we are assuming $p(V) < \infty$ for $V \in [V_m, 1)$ (where V_m is the minimum value of V in the interior).

Lemma 3.1 [ul-Alam (2007)]

Either the maximal interval of existence $(V_c, b]$ contains $[V_m, b]$, or $V_c \in [V_m, b)$. In either case, r and m are monotonic functions of V and $\sup_{[V_c,b]}(2m/r) < 1$ on the corresponding interval ($[V_m, b]$ or $(V_c, b]$). In the second case $\lim_{V\to V_c^+} \widetilde{W} = 0$. Then either $\lim_{r\to V_c^+} r(V) = 0$ ("regular zero" of $\widetilde{W}$) or $\lim_{r\to V_c^+} r(V) > 0$ ("irregular zero"). If the former, we have $mr^{-3} \to (4\pi/3)\rho(V_c)$ as $V \downarrow V_c$; if the latter, $mr^{-3} \to -4\pi p(V_c)$. In particular, $m < 0$ on approach to an irregular zero of $\widetilde{W}$.

3.3 Spinor approach to the Positive Mass Theorem.

In order to understand the strategy of our proof it is necessary to consider Bartnik's version [17] of Witten's Positive Mass Theorem[30]. We state it here.

Theorem 3.2 [Bartnik (1985)]

Suppose that (M^n, g) is a complete spin manifold satisfying the asymptotic flatness conditions:

$$i.)\quad (\Phi_* g - \delta) \in W^{2,q}_{-\tau}(E_{R_0}) \text{ for some asymptotic structure } \Phi,\ R_0 > 1,\ q > n, \tag{3.19}$$
$$\text{and } \tau \geq \frac{1}{2}(n-2)$$

$$ii.)\quad R(g) \in L^1(M^n, g) \tag{3.20}$$

with non-negative scalar curvature: $R(g) \geq 0$. Let ξ_0 be a spinor, constant near infinity and normalized by $|\xi_0|^2 \to 1$ at infinity, and let ξ be the unique solution of Dirac's equation satisfying:

$$\xi - \xi_0 \in W^{2,q}_{-\tau}.$$

Then the mass of M^n is non-negative and is given by

$$c(n)\text{mass}(g) = \int_M (4|\nabla\xi|^2 + R|\xi|^2)dvol_g \tag{3.21}$$

Furthermore, if mass$(g) = 0$, then M is flat. (Here $c(n) = 2(n-1)\omega_{n-1}$.)

A desirable approach along the lines of ul-Alam's method for a uniqueness proof would be to construct a conformal factor $\widetilde{\psi}$ which makes mass$(\widehat{g}) = 0$ (where $\widehat{g} = \widetilde{\psi}^4 g$) and use the theorem above. We do in fact have a spin manifold since any orientable 3-manifold has a spin structure.[13] The problem is that we do not know the sign of $\widehat{R}$ at this point. The sign of $\widehat{R}$ is necessary in proving that vanishing of the mass implies flatness.

It is important now to recall the argument used by Bartnik to define his mass integral and to conclude that the metric was flat in the case that the mass of the metric was zero. The key to Witten's method is the Lichnerowicz-type identity

$$(|\nabla\psi|^2 + \frac{1}{4}R|\psi|^2 - |\mathcal{D}\psi|^2) * 1 = d(< \psi, \sigma_{ij}.\nabla_j\psi > *e_i) \tag{3.22}$$

where $\mathcal{D}$ is the Dirac operator and

$$\sigma_{ij} = \frac{1}{2}[e_i, e_j] = e_i e_j + \delta_{ij} \tag{3.23}$$

(Here (e_i) is a local orthonormal frame, and we use Clifford multiplication.) Its derivation can be found in the appendix of this paper. The goal was to find an asymptotically constant spinor field, ψ, satisfying $\mathcal{D}\psi = 0$, and then identify the right hand side of Witten's identity, (3.22), with the mass. The first step that Bartnik took was to show that the Dirac operator was in fact an isomorphism between certain weighted Sobolev spaces. This was Proposition 6.1 in [17]. In proving that these the two weighted Sobolev spaces were isomorphic non-negative scalar curvature was used in conjunction with the strong maximum principle to show that the kernel of $\mathcal{D}$ and its adjoint was trivial. Secondly, it was shown that for an arbitrary spinor field ψ_0, which is constant at infinity, there exist a spinor field ψ which satisfies $\mathcal{D}\psi = 0$ and that $\psi - \psi_0$ are elements of a weighted Sobolev space where the weight is equal to the rate of the mass decay condition for the manifold. This was given in Corollary 6.2 in [17]. This shows existence of the spinor occurring in Bartnik's integral expression for the ADM mass, after identifying the boundary term in Witten's identity,(3.22), with the ADM mass. The positivity of mass is then an easy consequence assuming the non-negativity of the scalar curvature. Finally, in the case that the mass was zero we have

$$c(n)\text{mass}(g) = \int_M (4|\nabla\psi|^2 + R|\psi|^2) * 1 = 0 \tag{3.24}$$

This implies that $\nabla\psi \equiv 0$. From the spinor ψ we can define a vector v_ψ via the surjective map

$$< v_\psi, X >= Im < \psi, X.\psi > \text{ for } X \in R^n \tag{3.25}$$

Since $\nabla\psi = 0$ then $\nabla v_\psi = 0$. Since ψ_0 was an arbitrary constant spinor at infinity in our construction we can find a basis for TM consisting of covariantly constant vector fields. This means that M must be flat.

3.4 Strategy of proof of the Main Theorem.

The importance of outlining Bartnik's proof is that we will use the same technique in order to prove that with a properly defined conformal factor we too will have a flat metric for our static stellar model. Namely, by choosing an appropriate conformal factor, we can make the spinor norm weighted scalar curvature as small as we like. Taking limits, we find three linearly independent parallel spinors, which allows us to find a basis for TM consisting of covariantly constant vector fields. We use this to conclude that the conformal metric is flat.

We denote the scalar curvature of the conformal metric $\widehat{g} = \widetilde{\psi}^4 g$ by $\widehat{R}$, given by the following equation:

$$\widehat{R} = (\widetilde{W} - W)\frac{8}{\widetilde{\psi}^5}\frac{d^2\widetilde{\psi}}{dV^2} \tag{3.26}$$

(See (3.6) for $\widetilde{W}$.) The sign of $\widehat{R}$ is not yet known at this point so we cannot apply Bartnik's argument for existence of such a spinor with respect to our conformal metric $\widehat{g}$ and hence cannot yet define mass($\widehat{g}$). However, an easy calculation shows that the scalar curvature R with respect to our original metric g is given by

$$R = 16\pi\rho \tag{3.27}$$

Since $\rho \geq 0$ we know that $R \geq 0$. Therefore, we can apply Bartnik's argument for the existence of the needed Dirac spinor with regards to the metric g. Let ξ_0 be an arbitrary spinor field that is constant at infinity. Corollary 6.2 from Bartnik [17] then says that there exists a spinor field $\xi = \xi_0 + \xi_1$ which satisfies $\mathcal{D}\xi = 0$ and decays to ξ_0 at the needed rate. In other words, ξ is constant at infinity and ξ_1 falls off to the order of $O(r^{-\tau})$ for some $\tau > \frac{1}{2}$. Therefore the mass of our metric g admits the expression:

$$c(n)\text{mass}(g) = \int_M (4|\nabla\xi|^2 + R|\xi|^2) dvol_g \tag{3.28}$$

We want to define a conformal factor $\widetilde{\psi}(V)$ for $V \in [V_m, 1)$. Outside of our static star Birkhoff's theorem says that our space-time should be Schwarzschild. In this case, we know the form of the conformal transformation which sets the mass and scalar curvature to zero. The conformal transformation is

$$\psi(V) = \frac{1}{2}(1 + V) \tag{3.29}$$

for $V \in [V_s, 1)$. We start defining $\widetilde{\psi}$ by setting $\widetilde{\psi} = \psi$ for $V \in [V_s, 1)$. Therefore, $\widehat{R} = 0$ on $[V_s, 1)$ and the mass of $\widehat{g}$ is zero. Under a conformal transformation the spinor $\Theta = \widetilde{\psi}^{-2}\xi$ satisfies the Dirac equation $\mathcal{D}\Theta = 0$ relative to the conformal metric $\widehat{g}$.[3] The spinor Θ is given by

$$\Theta = \widetilde{\psi}^{-2}\xi = \widetilde{\psi}^{-2}(\xi_0 + \xi_1) = \xi_0 + \Theta_1 \tag{3.30}$$

$\Theta_1 = \widetilde{\psi}^{-2}\xi_1$ will fall off like $O(r^{-\tau})$ which follows from our assumptions on ξ_1 and the conformal factor approaching a constant at infinity. Using the spinor Θ with its falloff we can identity the right hand side of Witten's identity with the mass of $\widehat{g}$ just as Bartnik did in his proof. Since the mass of $\widehat{g}$ is zero, the integral formula with respect to $\widehat{g}$ becomes

$$\int_M (\widehat{R}\|\Theta\|^2 + 4\|\nabla_{\widehat{g}}\Theta\|^2)dvol_{\widehat{g}} = 0 \tag{3.31}$$

We express the first term with respect to the original metric g. This gives us

$$0 = \int_M (\widehat{R}\|\Theta\|^2 + 4\|\nabla_{\widehat{g}}\Theta\|^2)dvol_{\widehat{g}} = \int_M \widehat{R}\widetilde{\psi}^2\|\xi\|^2 dvol_g + \int_M 4\|\nabla_{\widehat{g}}\Theta\|^2 dvol_{\widehat{g}}. \tag{3.32}$$

We will divide $[V_m, V_s]$ into two sets, A and B. On the set A we define $\widetilde{\psi}$ to satisfy the second order linear O.D.E.:

$$\frac{d^2\widetilde{\psi}}{dV^2} = \frac{2\pi}{\widetilde{W}}[\rho\widetilde{\psi} - 2V(\rho + 3p)\frac{d\widetilde{\psi}}{dV}] \tag{3.33}$$

assuming $\widetilde{W} > 0$ is given. The significance of the second order ODE is as follows. In the case that the stellar model is spherically symmetric Einstein's equation coupled with the perfect fluid matter model will yield the following equations

$$\frac{dr}{dV} = \frac{r(r-2m)}{V(m+4\pi r^3 p)} \tag{3.34}$$

$$\frac{dm}{dV} = \frac{4\pi r^3(r-2m)\rho}{V(m+4\pi r^3 p)} \tag{3.35}$$

and then the squared field intensity is:

$$W(V) = (1 - \frac{2m}{r})(\frac{dr}{dV})^{-2} = \frac{V^2(m + 4\pi r^3 p)^2}{r^3(r - 2m)} \tag{3.36}$$

Solutions for these equations are given for $V \in [a, b]$ with initial values given at $V = b$. Also, for a stellar model which is spherically symmetric the conformal factor $\widetilde{\psi}$ which makes $\widetilde{\psi}^4 g$ flat will satisfy the linear ODE

$$\frac{d\widetilde{\psi}}{dV} = \frac{\widetilde{\psi}}{2r\sqrt{\widetilde{W}}}[1 - \sqrt{\frac{2m}{r}}] \tag{3.37}$$

Differentiating this whenever possible will yield the second order O.D.E., (3.33). On the set B we define $\widetilde{\psi} = u$ where u is a function for which we have control over the scalar curvature of $u^4 g$. If we break the integral up along the sets A and B, where $A \cup B = [V_m, V_s]$, and recall that for $V \in (V_s, 1]$ we have $\widehat{R} = 0$ we get

$$0 = \int_M \widehat{R}\widetilde{\psi}^2 \|\xi\|^2 dvol_g + \int_M 4\|\nabla_{\widehat{g}}\Theta\|^2 dvol_{\widehat{g}} \tag{3.38}$$

$$= \int_{V^{-1}(A)} \widehat{R}\widetilde{\psi}^2 \|\xi\|^2 dvol_g + \int_{V^{-1}(B)} \widehat{R}\widetilde{\psi}^2 \|\xi\|^2 dvol_g + \int_{V^{-1}((V_s,1))} \widehat{R}\widetilde{\psi}^2 \|\xi\|^2 dvol_g \tag{3.39}$$

$$+ \int_M 4\|\nabla_{\widehat{g}}\Theta\|^2 dvol_{\widehat{g}}$$

$$= \int_{V^{-1}(A)} \widehat{R}\widetilde{\psi}^2 \|\xi\|^2 dvol_g + \int_{V^{-1}(B)} \widehat{R}\widetilde{\psi}^2 \|\xi\|^2 dvol_g + \int_M 4\|\nabla_{\widehat{g}}\Theta\|^2 dvol_{\widehat{g}} \tag{3.40}$$

$$= \int_{V^{-1}(A)} (\widetilde{W} - W)\frac{8}{\widetilde{\psi}^5}\frac{d^2\widetilde{\psi}}{dV^2}\widetilde{\psi}^2 \|\xi\|^2 dvol_g + \int_{V^{-1}(B)} R_{u^4 g} u^2 \|\xi\|^2 dvol_g + \tag{3.41}$$

$$\int_M 4\|\nabla_{\widehat{g}}\Theta\|^2 dvol_{\widehat{g}}$$

$$= \int_{V^{-1}(A)} (\widetilde{W} - W)\frac{8}{\widetilde{\psi}^3}\frac{d^2\widetilde{\psi}}{dV^2} \|\xi\|^2 dvol_g + \int_{V^{-1}(B)} R_{u^4 g} u^2 \|\xi\|^2 dvol_g + \int_M 4\|\nabla_{\widehat{g}}\Theta\|^2 dvol_{\widehat{g}} \tag{3.42}$$

From this vantage point our ultimate goal can be seen. We want to be able to make the third integral as small as we like. What is left is to define the sets A and B, the function $\widetilde{\psi}$ and $\widetilde{W}$ on A and u on B.

We want to look at the integral in equation (3.42) and construct the functions $\widetilde{W}$, $\widetilde{\psi}$, and u on intervals contained in $[V_m, V_s]$ depending on what set the interval falls in and the behavior of $\frac{d^2\widetilde{\psi}}{dV^2}$ at the right endpoints. First, we have a lemma gives existence of solutions for the spherically symmetric ODE system near V_s and starting properties for the solutions at $V_1 \in [V_m, V_s)$.

Lemma 3.2 [ul-Alam (2007)]
There exists a noncritical value $V_1 < V_s$, and solutions $(r, m, \widetilde{W}, \widetilde{\psi})$ of equations (3.34)-(3.37) on $[V_1, V_s]$ with $\widetilde{\psi} > 0$, and if for $V \geq V_s$ we define $\widetilde{\psi}$ as in (3.29), then $\widetilde{\psi}$ is $C^{1,1}$ on $[V_1, 1)$. Furthermore on $(V_1, V_s]$ $0 < 2\frac{d\ln\widetilde{\psi}}{dV} < 1$, $\widetilde{W} - W_{ave} > 0$, and $\frac{d^2\widetilde{\psi}}{dV^2} > 0$. In particular, $\int_{V^{-1}([V_1,V_s])}(\widetilde{W} - W)\frac{8}{\widetilde{\psi}^3}\frac{d^2\widetilde{\psi}}{dV^2}\|\xi\|^2 dvol_g > 0$. ($W_{ave}$ is defined later.)

3.5 Auxiliary system of differential equations.

Lemma 3.2 gives us positivity of the integral on the set $[V_1, 1)$ and the starting conditions of $\widetilde{W}$ and $\frac{d^2\widetilde{\psi}}{dV^2}$. Our goal is to continue into the star constructing the conformal factor $\widetilde{\psi}$. Some intervals we will use $\widetilde{W}$ to define $\widetilde{\psi}$ by means of equation (3.33). On certain intervals in $[V_m, V_1]$, however, we will define $\widetilde{\psi}$ according to the solution to one of two differential equations. On these certain intervals, which we will define later, we want $\widetilde{\psi}$ to have certain properties which depend on the properties of the interval it is defined on. So on certain intervals we will set $\widetilde{\psi} = u$ where u is the solution to this DE we not describe.

Let γ denote the adiabatic index

$$\gamma := \frac{\rho + p}{p} \frac{dp}{d\rho} \tag{3.43}$$

We know that for our model star $\frac{d\rho}{dp} \geq 0$ which means that $\gamma > 0$ on the interior of the star, $V < V_s$. A simple computation yields the next lemma.

Lemma 3.3 [ul-Alam (2007)]
At $V < V_s$ where $\frac{d^2\widetilde{\psi}}{dV^2} = 0$ and $\widetilde{\psi}$ is three times differentiable we have

$$\frac{d^3\widetilde{\psi}}{dV^3} = \frac{10\pi\rho}{\gamma\widetilde{W}}[\gamma - \frac{6}{5}(1+\frac{p}{\rho})^2]\frac{d\widetilde{\psi}}{dV} = \frac{5\pi\rho^2\widetilde{\psi}}{\gamma\widetilde{W}V(\rho+3p)}[\gamma - \frac{6}{5}(1+\frac{p}{\rho})^2] \tag{3.44}$$

Let α, β be constants. We define two alternative O.D.E. for the function u

$$\alpha\rho u - 2V(\rho+3p)\frac{du}{dV} = 0 \tag{3.45}$$

$$\beta u + \rho u - 2V(\rho+3p)\frac{du}{dV} = 0 \tag{3.46}$$

If we integrate these two equations on an interval $[a, b]$ using $(\hat{\alpha}, \hat{\beta}) = (\alpha, 0)$ for (3.45) and $(\hat{\alpha}, \hat{\beta}) = (1, \beta)$ for (3.46) we get

$$\frac{1}{u(V)} = \frac{1}{u(b)} \exp(\int_V^b \frac{\hat{\alpha}\rho(\nu) + \hat{\beta}}{2\nu(\rho(\nu) = 3p(\nu))} d\nu) \tag{3.47}$$

α is chosen in equation (3.45) so that u and $\widetilde{\psi}$ will match in a $C^{1,1}$ way at $V = b$. This implies that
$\alpha = 2b(1 + \frac{3p}{\rho})\frac{dln\widetilde{\psi}}{dV}(b)$. In the case that $\lim_{V\to b^+} \frac{d^2\widetilde{\psi}}{dV^2} = 0$, a simple calculation using (3.33) shows that $\alpha = 1$. If $\lim_{V\to b^+} \frac{d^2\widetilde{\psi}}{dV^2} > 0$ a similar calculation shows that $\alpha < 1$. In both cases, $0 < \alpha \leq 1$, we have that $0 < 2V\frac{d\ln\widetilde{\psi}}{dV} < \alpha \leq 1$ at $V = a$. This fact is key in choosing $\widetilde{W}$ on certain intervals, which we will discuss more in detail later. If

$\lim_{V\to b^+} \frac{d^2\widetilde{\psi}}{dV^2} < 0$, then an α that ensures a $C^{1,1}$ match of u and $\widetilde{\psi}$ at b will be greater than 1. Therefore we lose the inequality $0 < 2V\frac{d\ln\widetilde{\psi}}{dV} < 1$ at a. This is precisely where the equation (3.46) comes in. In the case that $\lim_{V\to b^+} \frac{d^2\widetilde{\psi}}{dV^2} < 0$ we use (3.46). If we assume that $\widetilde{\psi}$ was chosen so that $0 < 2b\frac{d\ln\widetilde{\psi}}{dV}(b) < 1$ at $V = b$ and choose β so that $0 < \beta < 3p(b)$, then at $V = a$ (3.46) gives us

$$0 = \beta u + \rho(a)u - 2V(\rho(a) + 3p(a))\frac{du}{dV} \implies 2V(\rho(a) + 3p(a))\frac{du}{dV} = \beta u + \rho(a)u \tag{3.48}$$

$$\tag{3.49}$$

Therefore

$$2V(\rho(a) + 3p(a))\frac{du}{dV} = \beta u + \rho(a)u \leq (3p(b) + \rho(a))u \tag{3.50}$$

$$\implies 2V\frac{d\ln u}{dV} \leq \frac{(3p(b) + \rho(a))}{(\rho(a) + 3p(a))} < 1 \tag{3.51}$$

where the last inequality follows from the fact that p increases into the star.

Now, if u satisfies equation (3.45), then we have an explicit expression involving the scalar curvature R_{u^4g}

$$u^4R_{u^4g} = \frac{2W_\alpha}{\gamma V^2(1+\frac{3p}{\rho})^2}(6(1+\frac{p}{\rho})^2 - 5\gamma + \gamma(1-\alpha)) + 16\pi\rho(1-\alpha) \tag{3.52}$$

In particular, note that if $\alpha = 1$ and $\gamma \leq \frac{6}{5}(1+\frac{p}{\rho})^2$ we have $R_{u^4g} \geq 0$. When u satisfies equation (3.46) we have an explicit expression involving the scalar curvature of the metric u^4g given by

$$u^4R_{u^4g} = \frac{2W}{\gamma V^2(1+\frac{3p}{\rho})^2}(6(1+\frac{p}{\rho})^2(1-\frac{\beta}{3p}) - \gamma(5+\frac{6\beta}{\rho}+\frac{\beta^2}{\rho^2})) - 16\pi\beta \tag{3.53}$$

3.6 The critical set and the oscillation set.

The function u described above will be used to define $\widetilde{\psi}$ on two types of intervals, the intervals containing critical values of V and intervals where $\frac{d^2\widetilde{\psi}}{dV^2}$ oscillates indefinitely. We first describe the set of intervals which contain the critical values of V. We call this set U. The next lemma describes the construction of the sets in U.

Lemma 3.4 [ul-Alam (2007)]
Suppose $V_1 < V_s$ is not a critical value of V. Given any $\epsilon > 0$ we can ensure that critical values of V in $[V_m, V_1]$ are contained in a union of a finite number of disjoint intervals $[V_m, j_0) \cup (\bigcup_{n=1}^{k}(i_n, j_n)) = U$ such that the 1-dimensional Lebesgue measure of U, and 3-dimensional Hausdorff measure of $V^{-1}(U)$ satisfy

$$\max\{L^1(U), H^3(V^{-1}(U))\} < \epsilon \tag{3.54}$$

Proof.
Let $\epsilon > 0$. We first want to show that the set $C = \{x \in V^{-1}([V_m, V_1]) : W(x) = 0\}$ has 3-dimensional Hausdorff measure 0. Suppose for contradiction that C has positive 3-dimensional Hausdorff measure. Then there exists an open 3-dim ball $O(r)$ with radius r which contains C and that $O(r) \setminus C < \frac{\epsilon_2}{2}$ for any given $\epsilon_2 > 0$. Therefore C cannot have positive H^3 measure. Next, for any $[a, b] \subset [V_m, V_s)$ the co-area formula gives us [2]

$$H^3(\{x \in V^{-1}([a,b])|W(x) \neq 0\}) = \int_a^b (\oint_{V=\tau, W\neq 0} W^{\frac{-1}{2}})d\tau = \int_a^b f(\tau)d\tau \tag{3.55}$$

where we let $\oint_{V=\tau,W\neq 0} W^{\frac{-1}{2}} = f(\tau)$. This equality given by the co-area formula shows that the function $f(\tau)$ must be integrable on any interval $[a, b] \subset [V_m, V_s)$ since the $H^3([V_m, V_s)) < \infty$. Using the continuity of integration we know that for our given $\epsilon > 0$ there exists a $\delta_1 > 0$ such that if $S \subset [V_m, V_s)$ and $L^1(S) < \delta_1$ then

$\int_S f(\tau)d\tau < \epsilon$. The equality given by the co-area formula then tells us that

$$H^3(\{x \in V^{-1}(S)|W(x) \neq 0\}) = \int_S (\oint_{V=\tau, W\neq 0} W^{\frac{-1}{2}})d\tau = \int_S f(\tau)d\tau < \epsilon \tag{3.56}$$

Putting these things together we have that for any $S \subset [V_m, V_s)$ such that $L^1(S) < \delta_1$ then

$$\begin{aligned} H^3(V^{-1}(S)) &= H^3((C \cap V^{-1}(S)) \cup (C^c \cap V^{-1}(S))) && (3.57)\\ &= H^3(C \cap V^{-1}(S)) + H^3(C^c \cap V^{-1}(S)) && (3.58)\\ &= 0 + H^3(C^c \cap V^{-1}(S)) && (3.59)\\ &< \epsilon && (3.60) \end{aligned}$$

Our assumption that V is a C^3 function on the complement of the 2-surfaces where ρ has discontinuity gives us by Sard's theorem [1] that the critical values of V in $[V_m, V_1]$ form a set of measure zero. Since this set has measure zero it can be contained in a countable union, denoted U, of disjoint open intervals such $L^1(U) < \delta$ for any given $\delta > 0$. Since the interval (V_m, V_1) is bounded we can arrange the open intervals so that $U = [V_m, j_0) \cup (\bigcup_{n=1}^k (i_n, j_n))$. If we choose $\delta < \min(\epsilon, \delta_1)$ then we have $L^1(U) < \delta < \epsilon$ and $L^1(U) < \delta < \delta_1$ implies $H^3(V^{-1}(U)) < \epsilon$.

Let U be the set given by Lemma 4 with respect to the V_1 given by Lemma 3.2. For the set U we set out to make the spinor norm weighted scalar curvature integral over this set as small as we like. This is accomplished by choosing the function u described in the previous section. On intervals in U, we use equation (3.45) and $0 < \alpha < 1$ if $\frac{d^2\widetilde{\psi}}{dV^2} > 0$ at the right endpoint and we use equation (3.46) and $0 < \beta < 3p(b)$ if $\frac{d^2\widetilde{\psi}}{dV^2} < 0$ at the right endpoint, and α or β is chosen so that u and $\widetilde{\psi}$ agree in $C^{1,1}$ fashion at the right endpoint.

Let $C_{\alpha,\beta}$ be the constant given by

$$C_{\alpha,\beta} = \max\{ \sup |\frac{2W_\alpha}{\gamma V^2(1+\frac{3p}{\rho})^2}(6(1+\frac{p}{\rho})^2 - 5\gamma + \gamma(1-\alpha)) + 16\pi\rho(1-\alpha)|, \quad (3.61)$$

$$\sup|\frac{2W}{\gamma V^2(1+\frac{3p}{\rho})^2}(6(1+\frac{p}{\rho})^2(1-\frac{\beta}{3p}) - \gamma(5+\frac{6\beta}{\rho}+\frac{\beta^2}{\rho^2})) - 16\pi\beta|\}$$

where the sup is taken over $V^{-1}([V_m, V_1])$. This constant appears in the next lemma which allows us to control the size of $\int R_{u^4 g} u^2 \|\xi\|^2$ on intervals in U.

Lemma 3.5 [ul-Alam (2007)]
Suppose that on $[a,b] \subset (V_m, V_1]$ u satisfies equation (3.45) or (3.46), with initial conditions $u(b) = \widetilde{\psi}(b)$, and $\frac{du}{dV}(b) = \frac{d\widetilde{\psi}}{dV}(b)$ where on $[b, V_1]$, $\widetilde{\psi}(V)$ is $C^{1,1}$ and $0 < 2V\frac{d\ln\widetilde{\psi}}{dV} < 1$. Suppose further that for $V \geq V_1$, $\widetilde{\psi}(V)$ is as in Lemma 3.2. Then

$$|\int_{V^{-1}([a,b])} R_{u^4 g} u^2 \|\xi\|^2 dvol_g| \leq 4C_{\alpha,\beta} V_m^{-1} H^3(V^{-1}([a,b])) \quad (3.62)$$

where the constant V_1 is as in Lemma 3.2, and the constant $C_{\alpha,\beta}$ is as in (3.61).

Recall that on the set U there are only a finite number of intervals, i.e. $U = [V_m, j_0) \cup (\bigcup_{n=1}^{k}(i_n, j_n)$. Therefore, using Lemma 3.5 we can control the integral $\int R_{u^4 g} u^2 \|\xi\|^2$ on all of U using the next lemma. On U we set $\widetilde{\psi} = u$ where u was defined by the construction above.

Lemma 3.6 [ul-Alam (2007)]
We have a constant C_5 independent of ϵ and U from Lemma 3.4 such that for a $\widetilde{\psi}$ constructed above the total contribution in the spinor norm weighted scalar curvature integral from the set U is bounded by

$$|\int_{V^{-1}(U)} R_{u^4 g} u^2 \|\xi\|^2 dvol_g| = |\int_{V^{-1}(U)} R_{\widetilde{\psi}^4 g} \widetilde{\psi}^2 \|\xi\|^2 dvol_g| < C_5 \epsilon \quad (3.63)$$

where ϵ is from Lemma 3.4. In case V_s is not a critical level set we can choose C_5 to be independent of the chose of V_1

We want to define another set of intervals, $\overline{U} \subset [V_m, V_1]$, and describe how we choose the conformal factor on $\overline{U}$. The set $\overline{U}$ is defined in order to capture intervals where $\frac{d^2\widetilde{\psi}}{dV^2}$ oscillates indefinitely. As in the set U, for $\overline{U}$ we define the function u which satisfy the DE described in the previous section. Let $\overline{U} \subset [V_m, V_1]$ be a set of intervals where $|\gamma - \frac{6}{5}(1+\frac{p}{\rho})^2| < \epsilon_2$ for some $\epsilon_2 > 0$ which we will shortly define with motivation. Lemma 3.3 and the regularity conditions on the $\rho(p)$ imply that intervals where $\frac{d^2\widetilde{\psi}}{dV^2}$ changes sign rapidly will be included in $\overline{U}$.

On intervals in $\overline{U}$, we will have $\frac{d^2\widetilde{\psi}}{dV^2} = 0$ at the right endpoint. We use equation (3.45) with $\alpha = 1$ to define our function u, and the set $\widetilde{\psi} = u$ on intervals in $\overline{U}$. The purpose of defining $\widetilde{\psi} = u$ on intervals in $\overline{U}$ is that we want to control the size of $\int_{V^{-1}(\overline{U})} R_{u^4 g} u^2 \|\xi\|^2 dvol_g$. To this end, and for motivating our definition of ϵ_2, we state a lemma.

Lemma 3.7 [ul-Alam (2007)]
Suppose on $[a,b] \subset [V_m, V_1)$, $|\gamma - \frac{6}{5}(1+\frac{p}{\rho})^2|\widetilde{\psi}(b) \leq \gamma\epsilon_2$, for some $\epsilon_2 > 0$. Suppose further that $\frac{d^2\widetilde{\psi}}{dV^2}(b) = 0$. Then on $[a,b]$ we can find a positive function u with $u(b) = \widetilde{\psi}(b)$, $\frac{du}{dV}(b) = \frac{d\widetilde{\psi}}{dV}(b)$ such that

$$|\int_{V^{-1}([a,b])} R_{u^4 g} u^2 \|\xi\|^2 dvol_g| < 40\pi M \epsilon_2 (V_m \widetilde{\psi}(b))^{-3} H^3(V^{-1}([a,b])))$$

Using Lemma 3.7 we get an integral bound on each interval in $\overline{U}$. In order to extend this to all of $\overline{U}$ we need a global bound on $(\widetilde{\psi}(b))^{-2}$, i.e. a bound that holds for every right endpoint of intervals in $\overline{U}$. This is given by following.

Lemma 3.8 [ul-Alam (2007)]
Suppose on $[k,l]$, $\widetilde{\psi}$ is $C^{1,1}$. On some subintervals $[k_1,l_1]$ of $[k,l]$, $\widetilde{\psi}$ coincides with u satisfying (3.45) or (3.46) with $0<\alpha\leq 1$ or $3p(l_1)>\beta>0$ and on the rest of $[k,l]$, $0\leq 2V\frac{d\ln\widetilde{\psi}}{dV}\leq 1$. If $\widetilde{\psi}(l)>0$, then $\frac{1}{(\widetilde{\psi}(V))^2}\leq\frac{l}{V(\widetilde{\psi}(l))^2}$ on $[k,l]$.

In light of Lemma 3.2, we view Lemma 3.8 with interval $[a,b]$ from Lemma 3.7 being a subinterval of $[V_m,V_s]$. We can then replace $\frac{l}{V(\widetilde{\psi}(l))^2}$ with $\frac{V_s}{V(\widetilde{\psi}(V_s))^2}$. This gives us the following control on the integral from Lemma 3.7

$$|\int_{V^{-1}([a,b])}R_{u^4g}u^2\|\xi\|^2dvol_g|<40\pi M\epsilon_2(V_m\widetilde{\psi}(b))^{-3}H^3(V^{-1}([a,b]))$$

$$\leq 40\pi M\epsilon_2\frac{V_s}{V_m^3\widetilde{\psi}(V_s)^2\widetilde{\psi}(b)}H^3(V^{-1}([a,b])) \tag{3.64}$$

$$\leq 40\pi M\epsilon_2\frac{V_s^{3/2}}{V_m^{9/2}\widetilde{\psi}(V_s)^3}H^3(V^{-1}([a,b]) \tag{3.65}$$

$$=40\pi M\epsilon_2 C_{V_s,V_m}H^3(V^{-1}([a,b])) \tag{3.66}$$

where $C_{V_s,V_m}=\frac{V_s^{3/2}}{V_m^{9/2}\widetilde{\psi}(V_s)^3}$. Lemma 3.7 classifies what sets belong to $\overline{U}$ by choice of ϵ_2. Setting ϵ_2 properly, we can use this inequality in order to mitigate the negative contribution on intervals in $\overline{U}$. We know that the sum of $H^3(V^{-1}([a,b]))$ over all possible intervals in $\overline{U}$ must be less than the total measure $H^3(V^{-1}([V_m,V_s]))$. Defining $\epsilon_2<MC_{V_s,V_m}H^3(V^{-1}([V_m,V_s])$ and using this to define our set of intervals $\overline{U}$ in Lemma 3.7 shows that we can make $\int_{V^{-1}(\overline{U})}R_{u^4g}u^2\|\xi\|^2dvol_g$ as small as we like when we properly define what constitutes intervals in the set $\overline{U}$ by specifying an ϵ_2 for Lemma 3.7. Therefore, we define $\overline{U}$ to be the set of intervals $\{[a,b]\subset[V_m,V_1)\}$ in which $\frac{d^2\widetilde{\psi}}{dV^2}(b)=0$ and $|\gamma-\frac{6}{5}(1+\frac{p}{\rho})^2|<\epsilon_2$ where $\epsilon_2<MC_{V_s,V_m}H^3(V^{-1}([V_m,V_s]))$

3.7 The conformal factor on the regular set.

We have defined $\widetilde{\psi}$ on the set U and $\overline{U}$. Our goal is to define $\widetilde{\psi}$ on $[V_m, V_s]$. We now describe the process for choosing $\widetilde{W}$ on $(V_m, V_1] \setminus U$. This will aid us in defining $\widetilde{\psi}$ on the remaining parts of $[V_m, V_s]$. On these intervals $\widetilde{W}$ is used to define a $\widetilde{\psi}$ which satisfies equation (3.33). For a non-critical value $\tau \in (V_m, V_s]$ we define the function

$$W_{ave}(\tau) = \frac{\oint_{V^{-1}(\tau)} \sqrt{W}\|\xi\|^2}{\oint_{V^{-1}(\tau)} \frac{\|\xi\|^2}{\sqrt{W}}} \tag{3.67}$$

Using the co-area formula we see that with this definition, and for the set U given in Lemma 3.4 and V_1 given in Lemma 3.2

$$\int_{V^{-1}((V_m,V_1]\setminus U)} \frac{W_{ave} \circ V - W}{\widetilde{\psi}^3} \frac{8d^2\widetilde{\psi}}{dV^2}\|\xi\|^2 dvol_g = 0 \tag{3.68}$$

Using the fact that $W_{ave}(\tau) - W(x) = W_{ave}(\tau) - \widetilde{W}(x) + \widetilde{W}(x) - W(x)$ and the above equation we have

$$\int_{V^{-1}((V_m,V_1]\setminus U)} \frac{\widetilde{W} \circ V - W}{\widetilde{\psi}^3} \frac{8d^2\widetilde{\psi}}{dV^2}\|\xi\|^2 = \int_{V^{-1}((V_m,V_1]\setminus U)} \frac{\widetilde{W} \circ V - W_{ave} \circ V}{\widetilde{\psi}^3} \frac{8d^2\widetilde{\psi}}{dV^2}\|\xi\|^2 \tag{3.69}$$

We want to now describe the method for defining $\widetilde{W}$ on sets in $(V_m, V_1] \setminus U$. Our goal is to keep $\frac{\widetilde{W}-W_{ave}}{\widetilde{\psi}^3}\frac{8d^2\widetilde{\psi}}{dV^2} \geq 0$. We define discontinuities in $\widetilde{W}$ in order to accomplish this, as well as other goals. Specifically, suppose $\widetilde{\psi}$ has been defined on an interval $[b, a]$ and we must give $\widetilde{W}$ a discontinuity at b. For the interval $[c, b]$, $\widetilde{W}$ is defined so that either $\widetilde{W} < W_{ave}$ if $\frac{d^2\widetilde{\psi}}{dV^2}(b) < 0$ or so that $\widetilde{W} \geq W_{ave}$ if $\frac{d^2\widetilde{\psi}}{dV^2}(b) \geq 0$. For this purpose, if $\widetilde{W}$ has been defined on $[b, a]$ and it happens that $\widetilde{W} - W_{ave}$ changed signs at b and $\frac{d^2\widetilde{\psi}}{dV^2}$ did not, then we give $\widetilde{W}$ a discontinuity at b. That is, we pick new initial data $r_-(b), m_-(b)$ for the ODE system, and compute $\widetilde{W}(b_-)$ from those (slightly greater or slightly smaller than $\widetilde{W}(b_+)$, $\widetilde{W} = W_{ave} \pm \delta$). Then we solve the ODE system with the new data at b to find, on some interval $[c, b]$, $r(V), m(V)$.We then use $r(V)$ and

$m(V)$ to compute $\widetilde{W}(V)$ on $[c,b]$. If $\frac{d^2\widetilde{\psi}}{dV^2}$ changes signs continuously at b and $\widetilde{W}-W_{ave}$ does not (so $\frac{d^2\widetilde{\psi}}{dV^2}(b)=0$), then we define $\widetilde{\psi}$ to be u on $[c,b]$ where u satisfies equation (3.45) with $\alpha=1$. Note that $\frac{d^2\widetilde{\psi}}{dV^2}(V)=0$ persists on $[c,b]$, while $R_{u^4g}\geq 0$ on this interval. A big difference in this proof is that we allow intervals where $\frac{d^2\widetilde{\psi}}{dV^2}<0$. In previous work this was circumvented by placing restrictions on the equation of state. In this construction, when $\frac{d^2\widetilde{\psi}}{dV^2}<0$ we ensure that $W_{ave}-\delta\leq\widetilde{W}<W_{ave}$, giving a discontinuity to $\widetilde{W}$ if needed. We also ensure that $\widetilde{\psi}$ remains $C^{1,1}$ at b. In order to describe this process more rigorously, we start with a lemma.

Lemma 3.9 [ul-Alam (2007)]
On any V-interval where m and r are positive solutions of equations (3.34)-(3.35), we have $0<2V\frac{d\ln\widetilde{\psi}}{dV}<1$.

The value of $\frac{d\ln\widetilde{\psi}}{dV}$ can be derived by using equation (3.37). In the case where $m(b), r(b)$ are both positive, Lemma 3.9 is needed to ensure that we are able to define proper initial values at b when a discontinuity in $\widetilde{W}$ is needed. The next lemma guarantees we can change the initial data for (3.34),(3.35) at $V=b$, so as to produce the desired change in $\widetilde{W}$, while preserving the condition given in Lemma 3.9.

Lemma 3.10 [ul-Alam (2007)]
Suppose a set of solutions of equations (3.34)-(3.37) exists on $[b,a]$ for some a and $0<2V\frac{d\ln\widetilde{\psi}}{dV}(b)<1$, $\widetilde{\psi}(b)>0$.

Suppose $\lim_{V\to b^+}\widetilde{W}:=\widetilde{W}_+=\widetilde{W}(b)\geq 0$. Given $\delta>0$ we can find positive constants r_-, and m_-, $\widetilde{W}_-$ such that $\widetilde{W}_-$,r_-, and m_- satisfy equation (3.36) at $V=b$, $\widetilde{W}_++\delta>\widetilde{W}_->\widetilde{W}_+$ and the value of $\frac{d\ln\widetilde{\psi}}{dV}$ computed from these constants using (3.37) remains the same at b.

Similarly, if $\widetilde{W}_+ > 0$, given $\delta > 0$ we can find positive constants $r_- m_-$ and $\widetilde{W}_-$,such that $\widetilde{W}_-, r_-$, and m_- satisfy equation (3.36) at $V = b$, $\widetilde{W}_+ - \delta < \widetilde{W}_- < \widetilde{W}_+$, and the value of $\frac{d \ln \widetilde{\psi}}{dV}$ computed from these constants using (3.37) remains the same at b..

Lemma 3.10 requires $0 < 2V \frac{d \ln \widetilde{\psi}}{dV} < 1$ at b. By Lemma 3.9 this is true if m and r are both positive. In the case that $m(b) = 0$ but $r(b) > 0$ we have the following lemma.

Lemma 3.11 [ul-Alam (2007)]
Suppose for initially positive solutions $r(V), m(V), \widetilde{W}(V), \widetilde{\psi}(V)$ of equations (3.34)-(3.37) on $[b, a]$, $r(b) > 0$, $\widetilde{\psi}$ is positive and $C^{1,1}$ on $[b, a]$. Suppose further that $m(b) = 0$ or $V \frac{d \ln \widetilde{\psi}}{dV}(b) = 0$. Then we can find a positive $C^{1,1}$ conformal function $\widetilde{\psi}$ on the region $V_m \leq V \leq a$ for which the scalar curvature is nonnegative for $V_m \leq V \leq b$.

Note that in the case that $m(b) = 0$ but $r(b) > 0$ we must have $\frac{d^2 \widetilde{\psi}}{dV^2} > 0$ just before b. In this case we require $\widetilde{W} - W_{ave} > 0$. Since W_{ave} is bounded away from zero, we know that $\widetilde{W}$ cannot equal zero at a point where $m(b) = 0$ unless $r(b) = 0$. This rules out the case of "irregular zeros" (defined earlier). Gathering the previous lemmas we can now state the main lemma for giving needed discontinuities to $\widetilde{W}$ on the set $(V_m, V_1] \setminus U$.

Lemma 3.12 [ul-Alam (2007)]
Suppose regular solutions $r(V), m(V), \widetilde{W}(V)$, and $\widetilde{\psi}(V)$ of equations (3.34)-(3.37) exists on $[b, a] \subset [V_m, V_s]$, $r(V), m(V)$, $\widetilde{W}(V), \widetilde{\psi}(V)$ are positive on $[b, a]$ and $\widetilde{\psi}$ is $C^{1,1}$ on $[b, a]$. Suppose further that $0 < 2V \frac{d \ln \widetilde{\psi}}{dV}(b) < 1$. Then we can give a jump discontinuity to $\widetilde{W}$ at b (as small as desired, in either direction) so that $\widetilde{\psi}(V)$ is a $C^{1,1}$ positive function on $[d, b]$ for some $d < b$. When ρ is continuous at b the sign of $\frac{d^2 \widetilde{\psi}}{dV^2}$

does not change at the discontinuity. When ρ is discontinuous at b, the limit from the left of $\lim_{V\to b^-} \frac{d^2\widetilde{\psi}}{dV^2}$ has the same sign as it would be without the given discontinuity of $\widetilde{W}$ at b.

3.7.1 Accumulation of discontinuities.

There is an issue we need to address. There is a possibility that an infinite number of discontinuities are needed in $\widetilde{W}$ and they accumulate at a point. If k is the number of discontinuities and they accumulate at a point b then it is possible that $\lim_{k\to\infty} 2V\frac{d\ln\widetilde{\psi}_k}{dV} = 1$. This would keep us from continuing to define $\widetilde{W}$ to the left of b using Lemma 3.12. It is also possible that in the case of an infinite number of discontinuities we may not be able to control the sign of $\widetilde{W} - W_{ave}$ no matter how small the initial value we give $\widetilde{W}$. This issue is handled in two cases: when $\widetilde{W}$ is increased above W_{ave} to make $\widetilde{W} - W_{ave}$ positive and when $\widetilde{W}$ is decreased below W_{ave} to make $\widetilde{W} - W_{ave}$ negative. For the first case we have the following lemma.

Lemma 3.13 [ul-Alam (2007)]
Suppose on $[a,b] \subset (V_m, V_s] \setminus U$ that $0 < 2m < r$ and $W_{ave} + \delta \geq \widetilde{W} \geq W_{ave}$ for some $\delta > 0$. If $\widetilde{W}(a) = W_{ave}(a)$, $\widetilde{W}(b) = W_{ave}(b) + \delta$, and c_1 a Lipschitz constant of W_{ave} on $[a,b]$ then $b - a > \frac{\delta}{c_2+c_1}$, where $c_2 = 8\pi(\rho(V_m) + p(V_m)) > 0$.

This lemma prevents the accumulation of intervals in $(V_m, V_s] \setminus U$ such that $\widetilde{W}$ needs to be raised by placing a lower bound on the width of the interval on which the need to raise $\widetilde{W}$ can arise. However, note that this lower bound degenerates as $\delta \to 0$. Therefore, we can conclude that there can be at most a finite number of times in $[V_m, V_s]$ we will need to raise $\widetilde{W}$ above W_{ave}. In the second case a similar result is not available. This follows from the fact that for the above lemma an upper bound on $\frac{d\widetilde{W}}{dV}$ was available and utilized. For a similar result in the case that $\widetilde{W}$ is lowered in order to make $\widetilde{W} < W_{ave}$ we need a lower bound on $\frac{d\widetilde{W}}{dV}$ which is not available. We must handle this case in a different way. There is no way to guarantee that an accumulation of intervals in which $\widetilde{W}$ needs to be lowered will not happen. In the case an accumulation point occurs we show that the negative contribution to the spinor norm weighted scalar curvature integral which occurs in $(V_m, V_s] \setminus U$ can be offset locally. This gives us non-negativity of the integral over an interval containing the accumulation point. The lemma also ensures that we can proceed further into the star using Lemma 3.12.

Lemma 3.14 [ul-Alam (2007)]
Let $b \in [i,j] \subset (V_m, V_1] \setminus U$. Suppose we have a convergent sequence $\{b_k\}$, $b_k \in [i,j]$ such that $b_1 = b$ and on $(b_{k+1}, b_k]$ we have $r_k(V), m_k(V), \widetilde{W}_k(V)$, and $\widetilde{\psi}_k(V)$ are all positive solutions to equations (3.34)-(3.37), $\frac{d^2\widetilde{\psi}_k}{dV^2} \leq 0$, $\widetilde{W}_k < W_{ave}$, $0 < 2m_k < r_k$, and at b_{k+1} $(r_k, m_k, \widetilde{W}_k)$ is related to $(r_{k+1}, m_{k+1}, \widetilde{W}_{k+1})$ by the discontinuity constructed in Lemma 10 with $\widetilde{W}_+(b_{k+1}) = W_{ave}(b_{k+1})$ and $\widetilde{W}_-(b_{k+1}) = \lambda_k > 0$. Denote $\lim_{k\to\infty} b_k$ by b_∞. Then we can find an N such that for the solutions $(r_k(V), m_k(V), \widetilde{W}_k(V))$ and $k \leq N$ we have

$$\int_{V^{-1}([a,b_N])} \frac{\widetilde{W}_N - W_{ave}}{\widetilde{\psi}_N^3} \frac{d^2\widetilde{\psi}_N}{dV^2} \|\xi\|^2 dvol_g + \sum_{k=1}^{N-1} \int_{V^{-1}([b_{k+1},b_k])} \frac{\widetilde{W}_N - W_{ave}}{\widetilde{\psi}_N^3} \frac{d^2\widetilde{\psi}_N}{dV^2} \|\xi\|^2 dvol_g \geq 0$$

for some $a < b_\infty$, and at a: $0 < 2V\frac{d\ln\widetilde{\psi}_N}{dV} < 1$.

Remark: In the first integral we assume N is taken large enough, so that the solution $(r_N, m_N, \widetilde{W}_N)$ is defined slightly to the right of b_∞.

This result is precisely what we need in the case where an accumulation point of a sequence of discontinuities occur when trying to keep $W_{ave} - \delta \leq \widetilde{W} < W_{ave}$ in order to guarantee positivity of the integral on the interval and continue the construction of $\widetilde{W}$ inward.

3.8 Convergence of a sequence of conformal factors.

Our goal on $(V_m, V_1]\backslash U$ is to define $\widetilde{W}$ in such a way as to get a conformal factor which satisfies equation (3.33) with $\widetilde{W}$ replaced by W_{ave}. This will give us $\widehat{R} = \frac{W_{ave}-W}{\widetilde{\psi}^3}\frac{d^2\widetilde{\psi}}{dV^2}$. The spinor norm weighted scalar curvature integral over such an interval with this scalar curvature will be zero (see (3.68)). This involves creating a sequence $(\widetilde{W}_n)$ on $(V_m, V_1] \setminus U$ and proving convergence for a subsequence of $(\widetilde{\psi}_n)$.

This process is as follows. Fix an interval $[a, b] \subset (V_m, V_1] \setminus U$. In the above construction we started $\widetilde{W}$ at b $\widetilde{W} < W_{ave}$ or $\widetilde{W} \geq W_{ave}$, depending on the sign of $\frac{d^2\widetilde{\psi}}{dV^2}$. If $\widetilde{W} < W_{ave}$ on $[a, b]$ and $|\widetilde{W} - W_{ave}| \geq \delta$ at some point in $[a, b]$ then we introduce a discontinuity in $\widetilde{W}$. At the left endpoint of intervals in $U \cup \overline{U}$ we start defining $\widetilde{W} = W_{ave} \pm \delta$ depending on the sign of $\frac{d^2\widetilde{\psi}}{dV^2}$ at the left endpoint of the interval in $U \cup \overline{U}$. If $\widetilde{W} - W_{ave}$ vanishes and changes signs while $\frac{d^2\widetilde{\psi}}{dV^2}$ remains the same sign, then we give a discontinuity to $\widetilde{W}$. All of which uses Lemma 3.12. Notice that δ is used to squeeze $\widetilde{W}$ closed to W_{ave} when $\frac{d^2\widetilde{\psi}}{dV^2} < 0$. To define the sequence we replace δ above with $\frac{1}{n}$. In this case $\delta \to 0$ as $n \to \infty$. Now, we denote the solution to equation (3.33) using $\widetilde{W}_n$ on $[a, b]$ as $\widetilde{\psi}_n$. For each n, $[a, b]$ is the union of two sets, $S^n_\pm$ where S^n_+ denotes the set where $\frac{d^2\widetilde{\psi}_n}{dV^2} \geq 0$ and S^n_- denotes the set where $\frac{d^2\widetilde{\psi}_n}{dV^2} < 0$. The construction of $\widetilde{W}_n$ gives us that

$$\int_{V^{-1}([a,b]\cap S^n_\pm)} \frac{\widetilde{W}_n - W_{ave}}{\widetilde{\psi}_n^3} \frac{8d^2\widetilde{\psi}_n}{dV^2} \|\xi\|^2 \geq 0 \tag{3.70}$$

Lemma 3.14 guarantees that the integral over S^n_- is non-negative. The other is non-negative by construction. We now state a needed lemma.

Lemma 3.15 [ul-Alam (2007)]

Suppose equation (3.70) holds and,

$$\int_{V^{-1}([a,b])} \frac{\widetilde{W}_n - W_{ave}}{\widetilde{\psi}_n^3} \frac{8d^2\widetilde{\psi}_n}{dV^2} \|\xi\|^2 \to 0 \text{ as } n \to \infty \tag{3.71}$$

Then

$$\int_a^b \left| \frac{(1 - \frac{W_{ave}}{\widetilde{W}_n})(\rho\widetilde{\psi}_n - 2V(\rho + 3p)\frac{d\widetilde{\psi}_n}{dV})}{W_{ave}} \right| dV \to 0 \text{ as } n \to \infty \tag{3.72}$$

This lemma helps us satisfy a necessary hypothesis for our convergence lemma, which we state now.

Lemma 3.16 [ul-Alam (2007)]

Suppose in an interval $[a, b]$, the functions $e_n, f_1, f_2, \psi_1^{(n)}$ and $\psi_2^{(n)}$ satisfy

$$\frac{d\psi_1^{(n)}}{ds} = \psi_2^{(n)} \tag{3.73}$$

$$\frac{d\psi_2^{(n)}}{ds} = (1 - e_n)(f_1\psi_1^{(n)} + f_2\psi_2^{(n)}) \tag{3.74}$$

where f_1 and f_2 are piecewise continuous, and e_n are uniformly bounded measurable functions such that

$\int_a^b |e_n(f_1\psi_1^{(n)} + f_2\psi_2^{(n)})|ds \to 0$ as $n \to \infty$. Suppose further that $\psi_i^{(n)}(a) = \psi_i^{(0)}(a)$ and $|\psi_i^{(n)}(s)|$ are all bounded by a number independent of n. Then there is a subsequence $\{\psi_1^{(k)}\}$ which converges to a $C^{1,1}$ solutions ψ of the following equation with the initial conditions $\psi(a) = \psi_1^{(0)}(a)$ and $\frac{d\psi}{ds} = \psi_2^{(0)}(a)$,

$$\frac{d^2\psi}{ds^2} = f_1\psi + f_2\frac{d\psi}{ds} \tag{3.75}$$

Lemma 3.16 is used to get the conformal factor $\widetilde{\psi}$ on $[a,b] \subset (V_m, V_1] \setminus (U \cup \overline{U})$ which satisfies equation (3.33) with $\widetilde{W}$ replaced by W_{ave}. In using Lemma 3.16 in conjunction with Lemma 3.15 we have $\psi_1^{(n)}$ a solution of equation (3.33) with respect to $\widetilde{W}_n$, $f_1 = \frac{2\pi\rho}{W_{ave}}$, $f_2 = \frac{4\pi V(\rho+3p)}{W_{ave}}$, and $e_n = 1 - \frac{W_{ave}}{\widetilde{W}_n}$. Also, $\frac{d}{ds} = -\frac{d}{dV}$ which causes the sign in front of f_2 in equation (3.75) to be positive. The conclusion of Lemma 3.16 gives us our desired conformal factor $\widetilde{\psi}$ on $[a,b]$

We perform a similar scheme to intervals in $\overline{U}$. We want $\widetilde{\psi}$ defined on intervals in $\overline{U}$ to satisfy equation (3.33) with $\widetilde{W}$ replaced with W_{ave}. Recall that on intervals in $\overline{U}$ u is chosen to satisfy equation (3.45) with $\alpha = 1$. We want to adapt Lemma 3.16 for this situation. Suppose $[a,b] \subset \overline{U}$. Let $[a, s_n]$ be an interval such that $s_1 = b$ and $s_n \to a$ as $n \to \infty$. Let χ_n be the characteristic function for the interval $[a, s_n]$. For the function u chosen on intervals in $\overline{U}$ we have

$$\frac{d^2u}{dV^2} = \frac{1}{2V(1+\frac{3p}{\rho})^2}[\gamma - \frac{6}{5}(1+\frac{p}{\rho})^2]u \tag{3.76}$$

Define $f_3\psi_1^{(n)} = \frac{d^2u}{dV^2}$ and $f_1^{(n)} = f_1 + f_3\chi_n$ where f_1 is defined as before, $f_1 = \frac{2\pi\rho}{W_{ave}}$. We also define f_2 as before, $f_2 = \frac{4\pi V(\rho+3p)}{W_{ave}}$. e_n we define as before, $e_n = 1 - \frac{W_{ave}}{\widetilde{W}_n}$, except on $[a, s_n]$ where we set $e_n = 0$. We now have the following lemma for intervals in $\overline{U}$.

Lemma 3.17 [ul-Alam (2007)]
The conclusion of Lemma 3.16 remains valid if we replace f_1 by the function $f_1^{(n)}$.

This will give us the desired conformal factor $\widetilde{\psi}$ for intervals in $\overline{U}$.

3.9 Conclusion of the proof.

Now, we put all of the lemmas above into our final argument to prove the main theorem. Let $0 < \delta_n = \frac{1}{n}$. First, we choose $\widetilde{\psi}_n(V)$ for $V \geq V_s$ to be equal to $\frac{1}{2}(1+V)$. We start with equation (3.42) for the spinor norm weighted scalar curvature integral. By Lemma 3.2 we know there exists a noncritical value $V_1 < V_s$, which does not depend on n, and solutions $(r_n, m_n, \widetilde{W}_n, \widetilde{\psi}_n)$ of the spherically symmetric equations (3.34)-(3.37) on $[V_1, V_s]$, with $\widetilde{\psi}_n > 0$ which is $C^{1,1}$ on $[V_1, 1)$ when adjoined to $\widetilde{\psi}_n$ defined above for $[V_s, 1)$. Also, on $[V_1, V_s]$ we have $0 < 2V\frac{d\ln\widetilde{\psi}_n}{dV} < 1$, $\widetilde{W}_n > W_{ave}$, and $\frac{d^2\widetilde{\psi}_n}{dV^2} > 0$. This gives us

$$\int_{V^{-1}([V_1,V_s])} \frac{\widetilde{W}_n - W_{ave}}{\widetilde{\psi}_n^3} \frac{d^2\widetilde{\psi}_n}{dV^2} \|\xi\|^2 > 0 \tag{3.77}$$

Let $0 < \epsilon < \frac{\delta_n}{2C_5}$ be the chosen ϵ for Lemma 3.4 where C_5 is the constant that appears in Lemma 3.6. This gives us the set U which contains a finite number of intervals which contain the critical values of V. We can choose this ϵ to involve C_5 to bound the measure of $L^1(U)$ and $H^3(V^{-1}(U))$ since the constant C_5 is independent of which ϵ and U is chosen in Lemma 3.4. We define $\overline{U}$ to be precisely those intervals $\{[a, b]\}$ where $\frac{d^2\widetilde{\psi}}{dV^2}(b) = 0$ and $|\gamma - \frac{6}{5}(1 + \frac{p}{\rho})^2| < \epsilon_2$ where $\epsilon_2 < MC_{V_s,V_m}H^3(V^{-1}([V_m, V_s]))$. Once we have the interval $[V_1, V_s]$ in place, we progress inward into the star with initial conditions given at V_1. Intervals will either lie in U, $\overline{U}$ or $(V_m, V_1]\backslash(U\cup\overline{U})$. We will only have to deal with intervals by consequence of our regularity properties.

We start at V_1 with $\frac{d^2\widetilde{\psi}_n}{dV^2} > 0$. This means that apart from intervals in U, for as long as $\frac{d^2\widetilde{\psi}_n}{dV^2} \geq 0$ we must have $\widetilde{W}_n - W_{ave} \geq 0$. If we come to a point b where $\widetilde{W}_n - W_{ave} = 0$ and changes signs to $\widetilde{W}_n - W_{ave} < 0$ to the left of b but $\frac{d^2\widetilde{\psi}_n}{dV^2}$ remains non-negative, we give $\widetilde{W}_n$ a small discontinuity upwards (to make it larger than W_{ave}) and start $(r_n, m_n\widetilde{W}_n)$ again at b, using Lemma 3.12.

If $\frac{d^2\widetilde{\psi}_n}{dV^2} = 0$ at a point b and goes negative continuously to the left of b while $\widetilde{W}_n - W_{ave} \geq 0$ then our action to the left of b depends on the sign of $\gamma - \frac{6}{5}(1+\frac{p}{\rho})^2$ at b. If $\gamma - \frac{6}{5}(1+\frac{p}{\rho})^2 > 0$ at b (note that this means the third derivative $\frac{d^3\widetilde{\psi}}{dV^3}(V) > 0$ at $V = b$) then we give a discontinuity to $\widetilde{W}_n$ at b and set $\widetilde{W}_n(b_-) = W_{ave}(b) - \delta_n$. If $\gamma - \frac{6}{5}(1+\frac{p}{\rho})^2 \leq 0$ at b then we use equation (3.45) and with $\alpha = 1$ and define $\widetilde{\psi}_n = u$ as long as $\gamma - \frac{6}{5}(1+\frac{p}{\rho})^2 \leq 0$. The reason for the this is that if $\frac{d^2\widetilde{\psi}_n}{dV^2}$ has a zero and then goes negative, then we want $\frac{d^3\widetilde{\psi}_n}{dV^3} > 0$ for continuity. If this is not the case then we want to redefine $\widetilde{\psi}_n = u$ as an interval in $\overline{U}$ since a point where $\gamma - \frac{6}{5}(1+\frac{p}{\rho})^2 = 0$ will always be in an interval of $\overline{U}$. Now, for as long as $\frac{d^2\widetilde{\psi}_n}{dV^2} < 0$ we want to keep $W_{ave} - \delta_n \leq \widetilde{W}_n < W_{ave}$. If this changes while $\frac{d^2\widetilde{\psi}_n}{dV^2} < 0$ we introduce a discontinuity according to Lemma 3.12. Lemma 3.14 handles this case where the possibility of accumulation of discontinuity occurs by ensuring that the integral is positive on some interval containing the accumulation point. We note that if $\frac{d^2\widetilde{\psi}_n}{dV^2} > 0$, then by Lemma 3.13 there can be at most a finite number of times we will need to raise $\widetilde{W}_n$ so that $\widetilde{W}_n - W_{ave} > 0$. At each of the points we need to raise $\widetilde{W}_n$ we use Lemma 3.12. At the right endpoint of these intervals we ensure that $\widetilde{\psi}_n$ remains $C^{1,1}$. At the left endpoint of these intervals we maintain $0 < 2V\frac{d\ln\widetilde{\psi}_n}{dV} < 1$.

If we arrive at an interval in $\overline{U}$ then we define $\widetilde{\psi}_n = u$ with u satisfying equation (3.45) with $\alpha = 1$. $\alpha = 1$ ensures that u connects to $\widetilde{\psi}_n$ in a $C^{1,1}$ fashion at the right endpoint of the interval in $\overline{U}$. At the left endpoint of this interval we have $0 < 2V\frac{d\ln\widetilde{\psi}_n}{dV} < 1$. If we come to an interval $[i,j]$ in U, then we set $\widetilde{\psi}_n = u$ and define u on $[i,j]$ according to equation (3.45) or (3.46), depending on the sign of $\frac{d^2\widetilde{\psi}_n}{dV}$ at the right endpoint, j. If $\lim_{V\to j^+}\frac{d^2\widetilde{\psi}_n}{dV} > 0$ then we use equation (3.45) with $0 < \alpha < 1$ to define u. If $\lim_{V\to j^+}\frac{d^2\widetilde{\psi}_n}{dV} < 0$ then we use equation (3.46) with $0 < \beta < 3p(j)$ to define u. If $\lim_{V\to j^+}\frac{d^2\widetilde{\psi}_n}{dV} = 0$ then we use equation (3.45) with $\alpha = 1$ to define u. At i we start $\widetilde{W}_n$ by setting, $\widetilde{W}_n = W_{ave} - \delta_n$ if $\frac{d^2\widetilde{\psi}_n}{dV^2} < 0$ at i or we set $\widetilde{W}_n \leq= W_{ave} + \delta_n$

if $\frac{d^2\widetilde{\psi}_n}{dV^2} > 0$ at i. The proper α, β chosen ensures u connects to $\widetilde{\psi}_n$ in a $C^{1,1}$ fashion at j. At the left endpoint of this interval, i, we have $0 < 2V\frac{d\ln\widetilde{\psi}_n}{dV} < 1$.

Suppose we have defined $\widetilde{W}_n$ and $\widetilde{\psi}_n$ up until the interval $[b, 1]$ and that $\widetilde{\psi}_n$ is $C^{1,1}$ on $[b, 1)$, and $0 < 2V\frac{d\ln\widetilde{\psi}}{dV} < 1$ at b. We now want to continue to the next interval which has b as a right end point. The interval will be contained in one of three sets; U,$\overline{U}$, or $(V_m, V_1]\backslash(U\cup\overline{U})$. Suppose the next interval is an open interval in U, say (a, b). The intervals in U are the intervals given by Lemma 3.4 which contain critical values of V.(Note that for the sake of notation we use (a, b) where $(a, b) = (i_n, j_n)$ for some $n = 1..k$.) There are only a finite number of these intervals. If $\lim_{V\to b^+}\frac{d^2\widetilde{\psi}_n}{dV^2} \geq 0$ then we define u to be a solution to equation (3.45) with $0 < \alpha \leq 1$. This ensures that $u(b)$ agrees with $\widetilde{\psi}_n(b)$ in a $C^{1,1}$ fashion and that $0 < 2V\frac{d\ln\widetilde{\psi}_n}{dV} < 1$ at a. If $\lim_{V\to b^+}\frac{d^2\widetilde{\psi}_n}{dV^2} < 0$ then we define u to be a solution to equation (3.46) with $0 < \beta < 3p(b)$. This ensures that $u(b)$ agrees with $\widetilde{\psi}_n(b)$ in a $C^{1,1}$ fashion and that $0 < 2V\frac{d\ln\widetilde{\psi}_n}{dV} < 1$ at a. In both cases, equation (3.47) implies that u is positive on (a, b). On this interval we set $u(V) = \widetilde{\psi}_n(V)$ for $V \in (a, b)$ and continue inward. Sets in U are disjoint so we know the next interval cannot be in U. Depending on the sign of $\frac{d^2\widetilde{\psi}_n}{dV^2}$ at a and which set the next interval lies in we progress inward. If the next interval $[c, a]$ happens to be an interval in $(V_m, V_1] \setminus (U \cup \overline{U})$ we start $\widetilde{W}_n$ at a with $\widetilde{W}_n = W_{ave} \pm \delta_n$ depending on the sign of $\frac{d^2\widetilde{\psi}}{dV^2}$.

Now, suppose the interval $[a, b]$ is an interval in $(V_m, V_1] \setminus (U \cup \overline{U})$. In this case the endpoint a we do not fix for this demonstration. The left endpoint a will be the first point in which we must alter $\widetilde{W}_n$. We start with $\widetilde{W}_n$ at b according to Lemma 3.12, unless the previous interval was in U or $\overline{U}$. If the previous interval was in U or $\overline{U}$ then we define $\widetilde{W}_n = W_{ave} \pm \delta_n$, where the $\pm$ is determined by the sign of $\frac{d^2\widetilde{\psi}_n}{dV^2}$ at b. If $\frac{d^2\widetilde{\psi}_n}{dV^2} > 0$ at b then we choose $\widetilde{W}_n$ so that $\widetilde{W}_n = W_{ave} + \delta_n$ at b. If $\frac{d^2\widetilde{\psi}_n}{dV^2} < 0$ at b then we choose $\widetilde{W}_n$ so that $\widetilde{W}_n = W_{ave} - \delta_n$ at b. If $\frac{d^2\widetilde{\psi}_n}{dV^2} = 0$ and ρ, hence $\frac{d^2\widetilde{\psi}_n}{dV^2}$, is continuous at b, then on the interval $[a, b]$ $\frac{d^2\widetilde{\psi}_n}{dV^2}$ will be positive or negative depending on the equation of state using equation (3.44). If $\gamma > \frac{6}{5}(1+\frac{p}{\rho})^2$ at b then $\frac{d^2\widetilde{\psi}_n}{dV^2} < 0$ on $[a,b]$ and if $\gamma < \frac{6}{5}(1+\frac{p}{\rho})^2$ at b then $\frac{d^2\widetilde{\psi}_n}{dV^2} > 0$ on $[a,b]$ for some $a < b$. We must also address the issue of accumulation points. Suppose we have sequential intervals $I_j = [a_{j+1}, a_j] \subset (V_m, V_1] \setminus (U \cup \overline{U})$ where $a_1 = b$, $\frac{d^2\widetilde{\psi}_n}{dV^2} \geq 0$, and at each a_j a discontinuity is introduced to $\widetilde{W}_n$ in order to raise $\widetilde{W}_n$ above W_{ave} at a_j to maintain non-negativity of the integral over the interval. We know from lemma 3.13 that there can be at most a finite number of $(I_j)_{j=1}^M$. Lemma 3.12 is used on each interval $I_j = [a_{j+1}, a_j]$. Suppose we have sequential intervals $J_i = [b_{i+1}, b_i] \subset (V_m, V_1] \setminus (U \cup \overline{U})$ where $b_1 = b$, $\frac{d^2\widetilde{\psi}_n}{dV^2} \geq 0$, and at each b_i a discontinuity is introduced to $\widetilde{W}_n$ in order to maintain $W_{ave} - \delta_n \leq \widetilde{W}_n < W_{ave}$ at b_i which keeps the integral over the interval nonnegative. We know that the number of these intervals may tend to infinity and an accumulation of discontinuities (b_i) can occur. Lemma 3.12 is used on each interval $J_i = [b_{i+1}, b_i]$. In this case, Lemma 3.14 gives us an N such that for all $i \leq N$ we have

$$\int_{V^{-1}([a,b_N])} \frac{\widetilde{W}_{n_N} - W_{ave}}{\widetilde{\psi}^3_{n_N}} \frac{8d^2\widetilde{\psi}_{n_N}}{dV^2}\|\xi\|^2 + \sum_{i=1}^{N-1} \int_{V^{-1}([b_{i+1},b_i])} \frac{\widetilde{W}_{n_i} - W_{ave}}{\widetilde{\psi}^3_{n_i}} \frac{8d^2\widetilde{\psi}_{n_i}}{dV^2}\|\xi\|^2 \geq 0 \tag{3.78}$$

for some $a < \lim_{i\to\infty} b_i$. This lemma ensures that the contribution of the integral is non-negative and at a $0 < 2V\frac{d\ln\widetilde{\psi}_N}{dV} < 1$ which allows us to continue construction inwards of $\widetilde{W}_n$.

Finally, suppose that our interval $[a,b]$ is contained in $\overline{U}$ with $0 < \epsilon_2 < \frac{\delta_n}{80\pi C_{V_s,V_m} H^3(V^{-1}([V_m,V_s]))}$ in Lemma 3.7 where $C_{V_s,V_m} = \frac{V_s^{3/2}}{V_m^{9/2}\widetilde{\psi}(V_s)^3}$. By definition this means that

$$|\gamma - \frac{6}{5}(1+\frac{p}{\rho})^2|\widetilde{\psi}(b) \leq \gamma\epsilon_2$$

and that $\frac{d^2\widetilde{\psi}}{dV^2}(b) = 0$. In this case we define u to be a solution of equation (3.45) with $\alpha = 1$. $\alpha = 1$ u connects to $\widetilde{\psi}_n$ at b in a $C^{1,1}$ fashion and a we have $0 < 2V\frac{d\ln\widetilde{\psi}_n}{dV} < 1$.

On this interval we set $u(V) = \widetilde{\psi}_n(V)$ for $V \in [a,b]$ and continue inward.

Now, we have defined a conformal factor $\widetilde{\psi}_n$ which is globally $C^{1,1}$ and uniformly bounded away from zero. We refer back to the expression in equation (3.42) and split the integral along the different types of sets in order to address each type individually.

$$\begin{aligned} 0 = &\int_{V^{-1}(([V_m,V_1]\setminus U)\cap \overline{U}^c)} (\widetilde{W}_n - W_{ave}(\tau))\frac{8}{\widetilde{\psi}_n^3}\frac{d^2\widetilde{\psi}_n}{dV^2}\|\xi\|^2 dvol_g + \\ &\sum_\alpha \int_{V^{-1}(([V_m,V_1]\setminus U)\cap \overline{U}\cap[i_\alpha,j_\alpha])} R_{u^4 g} u^2 \|\xi\|^2 dvol_g + \\ &\int_{V^{-1}([V_1,V_s])} (\widetilde{W}_n - W_{ave}(\tau))\frac{8}{\widetilde{\psi}_n^3}\frac{d^2\widetilde{\psi}_n}{dV^2}\|\xi\|^2 dvol_g + \int_{V^{-1}(U)} R_{u^4 g} u^2 \|\xi\|^2 dvol_g \\ &+ \int_M 4\|\nabla_{\widehat{g}}\Theta_n\|^2 dvol_{\widehat{g}} \end{aligned} \tag{3.79}$$

The integral over $V^{-1}(([V_m,V_1]\setminus U)\cap \overline{U}^c)$ when $\widetilde{W}_n - W_{ave} \geq 0$ is nonnegative by construction and when $\widetilde{W}_n - W_{ave} < 0$ it is non-negative by Lemma 3.14. The integral over $V^{-1}(U)$ we can make greater than $\frac{-\delta_n}{2}$ by Lemma 3.6. The integral over $V^{-1}(([V_m,V_1]\setminus U)\cap \overline{U})$ is greater than $\frac{-\delta_n}{2}$ by Lemma 3.7 and the global bound from Lemma 3.8. The integral over $V^{-1}([V_1,V_s])$ is non-negative by Lemma 3.2. This gives us

$$\begin{aligned} 0 = &\int_{V^{-1}(([V_m,V_1]\setminus U)\cap \overline{U}^c)} (\widetilde{W}_n - W_{ave}(\tau))\frac{8}{\widetilde{\psi}_n^3}\frac{d^2\widetilde{\psi}_n}{dV^2}\|\xi\|^2 dvol_g + \\ &\sum_\alpha \int_{V^{-1}(([V_m,V_1]\setminus U)\cap \overline{U}\cap[i_\alpha,j_\alpha])} R_{u^4 g} u^2 \|\xi\|^2 dvol_g + \\ &\int_{V^{-1}([V_1,V_s])} (\widetilde{W}_n - W_{ave}(\tau))\frac{8}{\widetilde{\psi}_n^3}\frac{d^2\widetilde{\psi}_n}{dV^2}\|\xi\|^2 dvol_g + \\ &\int_{V^{-1}(U)} R_{u^4 g} u^2 \|\xi\|^2 dvol_g + \int_M 4\|\nabla_{\widehat{g}}\Theta_n\|^2 dvol_{\widehat{g}} \end{aligned} \tag{3.80}$$

$$> -\delta_n \tag{3.81}$$

Now we want a conformal factor which satisfies equation (3.33) with W_{ave} replacing $\widetilde{W}$ on intervals in $(V_m, V_s] \setminus U$. Suppose $[a,b] \subset (V_m, V_s] \setminus (U \cup \overline{U})$. From equation (3.80) and (3.81) we see that

$$0 \leq \int_{V^{-1}(([a,b])} (\widetilde{W}_n - W_{ave}(\tau)) \frac{8}{\widetilde{\psi}_n^3} \frac{d^2\widetilde{\psi}_n}{dV^2} \|\xi\|^2 dvol_g \tag{3.82}$$
$$\leq \int_{V^{-1}(([V_m,V_1]\setminus U)\cap \overline{U}^c)} (\widetilde{W}_n - W_{ave}(\tau)) \frac{8}{\widetilde{\psi}_n^3} \frac{d^2\widetilde{\psi}_n}{dV^2} \|\xi\|^2 dvol_g \to 0$$

and by Lemma 3.15 this gives us that

$$\int_a^b \left| \frac{(1 - \frac{W_{ave}}{\widetilde{W}_n})(\rho\psi_n - 2V(\rho + 3p)\frac{d\psi_n}{dV})}{W_{ave}} \right| dV \to 0 \tag{3.83}$$

as $n \to \infty$. This allows us to use Lemma 3.16 with $\psi_1^n = \widetilde{\psi}_n$, $f_1 = \frac{2\pi\rho}{W_{ave}}$, $f_2 = \frac{4\pi V(\rho+3p)}{W_{ave}}$, and $e_n = 1 - \frac{W_{ave}}{\widetilde{W}_n}$. The conclusion of Lemma 3.16 give us the desired conformal factor $\widetilde{\psi}$ on $[a,b]$.

Next, suppose $[a,b]$ is an interval in $\overline{U}$ with the above construction. We again want a conformal factor $\widetilde{\psi}$ defined on $[a,b]$ which satisfies equation (3.33) with W_{ave} replacing $\widetilde{W}$ as discussed previously. Following the hypothesis given in Lemma 3.16, on the interval $[a,b]$ with χ_n the characteristic function on $[a, s_n]$ we define

$$e_n = \chi_n(1 - \frac{W_{ave}}{\widetilde{W}_n}) \tag{3.84}$$
$$f_1 = \frac{2\pi\rho}{W_{ave}} \tag{3.85}$$
$$f_2 = \frac{4\pi V(\rho + 3p)}{W_{ave}} \tag{3.86}$$
$$\chi_n f_3 u = \chi_n \frac{d^2 u}{dV^2} \tag{3.87}$$
$$f_1^{(n)} = f_1 + \chi_n f_3 \tag{3.88}$$

Then for each n, on $[a, s_n]$ we have

$$\psi_1^{(n)} = u \tag{3.89}$$

$$\frac{d\psi_1^{(n)}}{ds} = \frac{du}{dV} := \psi_2^{(n)} \tag{3.90}$$

$$\frac{d\psi_2^{(n)}}{ds} = \frac{d^2u}{dV^2}\chi_n \tag{3.91}$$

$$= \frac{d^2u}{dV^2}\chi_n + \frac{2\pi}{W_{ave}}(\rho u - 2V(\rho + 3p)\frac{du}{dV}) \tag{3.92}$$

$$= \frac{2\pi\rho u}{W_{ave}} + \frac{d^2u}{dV^2}\chi_n - \frac{4\pi V(\rho + 3p)}{W_ave}\frac{du}{dV} \tag{3.93}$$

$$= f_1 u + \chi_n f_3 u + f_2\frac{du}{dV} \tag{3.94}$$

$$= f_1^{(n)}\psi_1^{(n)} + f_2\psi_2^{(n)} \tag{3.95}$$

Also,

$$\int_a^b |e_n(f_1^{(n)}\psi_1^{(n)} + f_2\psi_2^{(n)})|ds = \int_{s_n}^b |(1 - \frac{W_{ave}}{\widetilde{W}})(\frac{2\pi\rho}{W_{ave}}u - \frac{4\pi V(\rho + 3p)}{W_{ave}}\frac{du}{dV}|ds \tag{3.96}$$

$$= \int_{s_n}^b |(1 - \frac{W_{ave}}{\widetilde{W}})(\frac{2\pi}{W_{ave}})(\rho u - 2V(\rho + 3p)\frac{du}{dV})|ds \tag{3.97}$$

$$= 0 \tag{3.98}$$

where the last equality follows from the fact that u satisfies equation (3.45) with $\alpha = 1$. The above shows that on the interval $[a, b]$ in $\overline{U}$ with $f_1^{(n)}$ replacing f_1 in Lemma 3.16, all of the hypothesis from Lemma 3.16 are met. Lemma 3.17 then gives us the desired conformal factor $\widetilde{\psi}$ on $[a, b]$ in this case when $[a, b] \subset \overline{U}$.

Finally, $\widetilde{\psi}_n(V)$ for $V \geq V_s$ is fixed for all n by equation (3.29). $\widehat{R}$ on $V \geq V_s$ is zero. This means for the integral over $V^{-1}([V_s, 1))$ we have

$$\int_{V^{-1}([V_s,1)} 4\|\nabla_{\widehat{g}}\Theta_n\| dvol_{\widehat{g}} = \int_{V^{-1}([V_s,1)} 4\|\nabla_{\widehat{g}}\Theta\|^2 dvol_{\widehat{g}} \tag{3.99}$$

for all n. For $V \leq V_s$ on intervals in $(V_m, V_s] \setminus U$ we have $\Theta_n = \widetilde{\psi}_n^{-2}\xi$ and $\Theta = \lim_{n\to\infty} \widetilde{\psi}_n^{-2}\xi = \widetilde{\psi}\xi$ where $\widetilde{\psi}$ is the limiting conformal factor. On compact intervals in $(V_m, V_s] \setminus U$ this convergence is uniform. On compact intervals in $((V_m, V_s] \setminus (U \cup \overline{U}))$ the total integral is given by

$$\int (\widetilde{W}_n - W_{ave}) \frac{8}{\widetilde{\psi}_n^3} \frac{d^2\widetilde{\psi}_n}{dV^2} \|\xi\|^2 dvol_g + \int 4\|\nabla_{\widehat{g}}\Theta_n\|^2 dvol_{\widehat{g}} \tag{3.100}$$

Taking the limit as $n \to \infty$ on these intervals gives us

$$\lim_{n\to\infty} \int (\widetilde{W}_n - W_{ave}) \frac{8}{\widetilde{\psi}_n^3} \frac{d^2\widetilde{\psi}_n}{dV^2} \|\xi\|^2 dvol_g + \lim_{n\to\infty} \int 4\|\nabla_{\widehat{g}}\Theta_n\|^2 dvol_{\widehat{g}} = \int 4\|\nabla_{\widehat{g}}\Theta\|^2 dvol_{\widehat{g}} \tag{3.101}$$

where the first integral goes to zero by Lemma 3.16. For $V \leq V_s$ on intervals in $\overline{U}$ we have

$$\int_{V^{-1}(\overline{U})} R_{n_u^4 g} u^2 \|\xi\|^2 dvol_g + \int_M 4\|\nabla_{\widehat{g}}\Theta_n\|^2 dvol_{\widehat{g}} \tag{3.102}$$

Taking the limit as $n \to \infty$ on each of these intervals gives us

$$\lim_{n\to\infty} \int_{V^{-1}(\overline{U})} R_{n_u^4 g} u^2 \|\xi\|^2 dvol_g + \lim_{n\to\infty} \int_M 4\|\nabla_{\widehat{g}}\Theta_n\|^2 dvol_{\widehat{g}} = \int_M 4\|\nabla_{\widehat{g}}\Theta\|^2 dvol_{\widehat{g}} \tag{3.103}$$

where the first integral goes zero by Lemma 3.17. This means, that on $[V_m, 1) \setminus U$ with $\widetilde{\psi} = \lim_{n\to\infty} \widetilde{\psi}_n$ we have

$$0 = \int_M 4\|\nabla_{\widehat{g}}\Theta\|^2 dvol_{\widehat{g}} \tag{3.104}$$

Now, we know that $0 = \int_M 4\|\nabla_{\widehat{g}}\Theta\|^2 dvol_{\widehat{g}}$ on $[V_m, 1) \setminus U$ implies that $\nabla_{\widehat{g}}\Theta = 0$ on $[V_m, 1) \setminus U$. Therefore $\widetilde{\psi}^2 = \|\xi\|$ by our choice of ξ and continuity. We have a covariantly constant spinor field $\Theta = \frac{\xi}{\|\xi\|}$, with respect to the conformal metric $\widehat{g}$, defined on $[V_m, 1) \setminus U$. We note here that our choice of ξ involved an arbitrary spinor ξ_0 which was constant at infinity. By definition of "constant at infinity" ξ_0 is constant with respect to an orthonormal frame e_i near infinity where $e_i = e_i^j \partial_{x_j}$ and (x_j) are asymptotically flat coordinates satisfying the mass decay conditions. e_i^j being the "vielbein". [17] We have a surjective map from the space of spinors to $\mathbb{R}^3$ given by

$$< v_\Theta, X >= Im < \Theta, X \cdot \Theta > \text{ for } X \in \mathbb{R}^{\not\Vdash} \tag{3.105}$$

where $v_\Theta \in \mathbb{R}^3$. The fact that the map is surjective follows from the fact that $Spin(3)$ is the double cover of $SO(3)$. If Θ is a spinor field generated by our construction then it is covariantly constant on $V^{-1}([V_m, 1) \setminus U)$. This means that v_Θ is also covariantly constant on $V^{-1}([V_m, 1) \setminus U)$. Choose three spinors γ_i, $i = 1..3$ such that $\gamma_i \mapsto e_i$ with the map above. (e_i) is an orthonormal frame of T_pM at some point p near infinity. $\Theta_i = \gamma_i$, $i = 1..3$ are covariantly constant with respect to the conformal metric $\widehat{g}$ on $V^{-1}([V_m, 1) \setminus U)$. Take a diagonal sequence in the conformal factors converging to Θ_i. This diagonal sequence provides a conformal factor which makes all three spinors covariantly constant. This implies that all three vectors e_i are covariantly constant. In particular, this gives us a that the frame (e_i) is covariantly constant on $V^{-1}([V_m, 1) \setminus U)$. We can extend the constant frame field (e_i) to the set $V^{-1}(U)$ by continuity. Therefore, we have a constant frame for TM, globally. Hence the conformal metric is flat.

We recall the tensor used to prove spherical symmetry from conformal flatness [22]

$$V^4 R_{abc} R^{abc} = 8W^2\|\Omega\|^2 + \|\nabla_T W\|^2 \tag{3.106}$$

In the case that the conformal metric is flat, this tensor vanishes and standard arguments imply that the original metric has to be spherical. In the vacuum, V is a connected level set. When the above tensor vanishes in the vacuum, $V \geq V_s$, we see that W must be constant, which follows from the second term vanishing in equation (3.106). We assume without loss of generality that V_s has only one connected component. This implies that V_s is a noncritical level set of W. Lemma 3.6 in this case implies that the constant C_5 is independent of V_1.

We started with a static stellar model with equation of state piecewise C^1. We were able to construct a conformal factor $\widetilde{\psi}$ which forced $\nabla_{\widetilde{\psi}^4 g}\Theta = 0$ everywhere outside of sets which contain critical values of V. This allows us to define an orthonormal frame field that is covariantly constant everywhere outside of U. We extend the frame field to U which gives us a global, constant frame field on TM. We conclude that the conformal metric is flat. Standard arguments using the vanishing of the tensor equation in (3.106) in the case of a flat conformal metric implies that our original metric g must be spherically symmetric. This concludes the proof that given a static stellar model it must be spherically symmetric.

Chapter 4

Physical Constraints on Stellar Models

The mathematical framework of General Relativity provides physical constraints to stellar models. Assuming the validity of the theory of general relativity, we can exclude certain physical phenomena from existing. We may also draw conclusions regarding the physically unlikely. In this section we derive a lower bound on $\Delta = 1 - \frac{2Gm(R)}{c^2 R}$ in the case of a finite, static, spherically symmetric stellar model in which the density is a monotonically decreasing function of r. In the Newtonian limit $c \to \infty$, $\Delta^{1/2}$ corresponds to the Newtonian gravitational potential. The lower bound on Δ gives us bounds on the value of the metric potential at the boundary, as well as an upper bound on the mass in the case of a fixed radius R, known as "Buchdahl's inequality". We also explore a constraint on the adiabatic index of a finite, static stellar model which is not necessarily spherically symmetric. This is directly related to the proof of uniqueness for a static star. We show that the constraint on the adiabatic index offers support to the "Fluid Ball Conjecture". Finally, we look at necessary and sufficient conditions ensuring the finiteness of the radius of a static, spherically symmetric stellar model. This section follows work from Buchdahl [9], Lindblom and ul-Alam [33], and Rendall and Schmidt [7].

4.1 Buchdahl's Inequality

Suppose we have a finite, static, spherically symmetric stellar Model with a perfect fluid source, a radius R and mass $m(R)$. The metric for such a stellar model is given by

$$ds^2 = -c^2 e^{a(r)} dt^2 + e^{b(r)} dr^2 + r^2 (d\theta^2 + sin^2\theta d\phi^2) \tag{4.1}$$

In this section we will use many of the expressions derived in chapter 2. The motivation of these expressions follow from the work of Buchdahl [9]. We now recall some of these expressions. We consider the system derived in equations (2.5)-(2.7). We also consider the auxiliary equations given in (2.16) and (2.33).

$$m(r) = 4\pi \int_0^r s^2 \rho(s) ds \tag{4.2}$$

$$w(r) = \frac{m(r)}{r^3} \tag{4.3}$$

We also recall expression (2.46)-(2.48).

$$y^2(r) = e^{-b(r)} = 1 - \frac{2G}{c^2} r^2 w(r) \tag{4.4}$$

$$z(r) = e^{\frac{a(r)}{2}} \tag{4.5}$$

$$x(r) = r^2 \tag{4.6}$$

Finally, recall the expression from (2.64).

$$\frac{d}{dx}(y\frac{dz}{dx}) - (\frac{G}{2c^2}\frac{1}{y}\frac{dw}{dx})z = 0 \tag{4.7}$$

Theorem 4.1 [Buchdahl (1959)]

Given a finite, static, spherically symmetric perfect fluid stellar model of radius R and mass $m(R)$ such that the density of the perfect fluid does not increase outward, i.e. $\frac{d\rho}{dr} \leq 0$ we have

$$\Delta \geq \frac{1}{9}$$

where $\Delta = 1 - \frac{2G}{c^2}\frac{m(R)}{R}$

Proof.

We first consider the expression in (4.7) for $r \leq R$ and do a change of variable. Let

$$\xi(x) = \int_0^x \frac{1}{y(s)} dx \tag{4.8}$$

This way, we can expression (4.7) as

$$\frac{d}{d\xi}(\frac{dz}{d\xi}) - g(\xi)z \tag{4.9}$$

where $g(\xi)$ is equal to $\frac{G}{2c^2}\frac{1}{y}\frac{dw}{dx}$ in terms of ξ. Since we assume that the density of the star does not increase outward it follows that

$$0 \geq \frac{dw}{dr} = \frac{dw}{d\xi}\frac{d\xi}{dx}\frac{dx}{dr} = \frac{dw}{d\xi}\frac{1}{y(x)}2r \tag{4.10}$$

which implies that $\frac{dw}{d\xi} \leq 0$. Therefore $g(\xi) \leq 0$ since $y \geq 0$. Since $g(\xi) \leq 0$ we have the inequality

$$0 = \frac{d}{d\xi}(\frac{dz}{d\xi}) - g(\xi)z \geq \frac{d}{d\xi}(\frac{dz}{d\xi}) \tag{4.11}$$

This inequality, of course, implies that

$$\frac{dz}{d\xi}|_{r=R} \leq \frac{dz}{d\xi} \leq \frac{dz}{d\xi}|_{r=0} \tag{4.12}$$

We can need an expression for $\frac{dz}{d\xi}$. Recall equation (4.5).

$$\frac{dz}{d\xi} = \frac{dz}{dr}\frac{dr}{dx}\frac{dx}{d\xi} \tag{4.13}$$

$$= \frac{dz}{dr}\frac{y(r)}{2r} \tag{4.14}$$

Next, we want to evaluate $\frac{dz}{d\xi}$ at the boundary of our star, i.e. $r = R$. Recall Birkhoff's theorem [10]. Continuity implies that at the boundary our interior solution must match the exterior solution, Schwarzschild's vacuum solution in this case. So $y(R) = e^{\frac{a(R)}{2}}$ and $e^{a(R)} = 1 - \frac{2G}{c^2}\frac{m(R)}{R}$. Therefore, we have

$$\frac{dz}{d\xi}|_{r=R} = \frac{dz}{dr}\frac{y(R)}{2R} \tag{4.15}$$

$$= e^{a(R)}\frac{a'(R)}{2}\frac{1}{2R} \tag{4.16}$$

$$= [1 - \frac{2G}{c^2}\frac{m(R)}{R}]\frac{1}{4R}a'(R) \tag{4.17}$$

$$= [1 - \frac{2G}{c^2}\frac{m(R)}{R}]\frac{1}{4R}\frac{1}{[1 - \frac{2G}{c^2}\frac{m(R)}{R}]}[\frac{2G}{c^2}\frac{m(R)}{R^2}] \tag{4.18}$$

$$= \frac{G}{c^2}\frac{m(R)}{2R^3} \tag{4.19}$$

$$= \frac{G}{c^2}\frac{1}{2}w(R) \tag{4.20}$$

Going back to the inequality in (4.12) and using (4.20), we get

$$\frac{G}{c^2}\frac{1}{2}w(R) = \frac{dz}{d\xi}|_{r=R} \leq \frac{dz}{d\xi} = \frac{dz}{dr}\frac{dr}{dx}\frac{dx}{d\xi} = \frac{dz}{dr}\frac{y}{2r} \tag{4.21}$$

$$\frac{G}{c^2}w(R)\frac{r}{y} \leq \frac{dz}{dr} \tag{4.22}$$

If we integrate (4.22) with respect to r from $r = 0$ to $r = R$ we get

$$\frac{G}{c^2}w(R)\int_0^R \frac{r}{y}dr \leq z(r)|_{r=0}^{r=R} = \Delta^{\frac{1}{2}} - z(0) \tag{4.23}$$

Recall the definition of y^2 from (4.4) and the fact that $w' \leq 0$. This means that

$$w(R) \leq w(r) \tag{4.24}$$

and

$$y^2(r) = 1 - \frac{2G}{c^2} r^2 w(r) \leq 1 - \frac{2G}{c^2} r^2 w(R) \tag{4.25}$$

which implies that

$$y(r) \leq [1 - \frac{2G}{c^2} r^2 w(R)]^{\frac{1}{2}} \tag{4.26}$$

Let $\widetilde{C} = \frac{2Gw(R)}{c^2}$. Using (4.26) in (4.23) gives us

$$\Delta^{\frac{1}{2}} - z(0) \geq w(R) \int_0^R \frac{r}{y} dr \tag{4.27}$$

$$\geq w(R) \int_0^R \frac{r}{[1 - \frac{2G}{c^2} r^2 w(R)]^{\frac{1}{2}}} dr \tag{4.28}$$

$$= -\frac{w(R)}{2\widetilde{C}} \int_1^{1-\widetilde{C}R^2} \frac{1}{u^{\frac{1}{2}}} du \tag{4.29}$$

$$= \frac{w(R)}{\widetilde{C}} 1 - (1 - \widetilde{C}R^2)^{\frac{1}{2}} \tag{4.30}$$

$$= \frac{c^2}{2G} [1 - y(R)] \tag{4.31}$$

$$= \frac{c^2}{2G} [1 - \Delta^{\frac{1}{2}}] \tag{4.32}$$

Note that $z(0) \geq 0$ and therefore gives us the inequality

$$\Delta^{\frac{1}{2}} \geq \frac{c^2}{2G} [1 - \Delta^{\frac{1}{2}}] \tag{4.33}$$

$$\frac{2G}{c^2} \Delta^{\frac{1}{2}} + \frac{c^2}{c^2} \Delta^{\frac{1}{2}} \geq 1 \tag{4.34}$$

$$\Delta^{\frac{1}{2}} \geq \frac{c^2}{2G + c^2} \geq \frac{1}{3} \tag{4.35}$$

Finally, we see that

$$\Delta \geq \frac{1}{9} \tag{4.36}$$

As a corollary to Theorem 4.1, we can just note the definition of Δ and algebraically manipulate equation (4.36) to find that

$$\frac{m(R)}{R} \leq \frac{c^2}{G}\frac{4}{9} \tag{4.37}$$

$$m(R) \leq \frac{c^2}{G}\frac{4}{9}R \tag{4.38}$$

Therefore, we have an upper bound on the mass of a finite, static, spherically symmetric stellar model given a fixed radius R. As a side note, the derivation of the upper bound on the mass in the case of a fixed radius is completely independent of pressure.

If we assume that we have a finite, static, spherically symmetric stellar model but parametrized by the potential V so that the line element takes the form

$$ds^2 = -V^2 dt^2 + g_{ij} dx^i dx^j \tag{4.39}$$

for $i, j = 1, 2, 3$ and V, g_{ij} only depends on r we can make another observation. Let $V(R) = V_s$ denote the potential at the surface of the star. By continuity of the metric and Birkhoff's theorem we know that

$$V_s^2 = \Delta \geq \frac{1}{9} \tag{4.40}$$

Hence, we have a lower bound on the surface potential

$$V_s \geq \frac{1}{3} \tag{4.41}$$

4.2 Constraints on the Adiabatic Index

Suppose now that we have a finite, static stellar model that is not necessarily spherically symmetric with a perfect fluid matter model. In this case we have the metric given by

$$ds^2 = -V^2 dt^2 + g_{ij} dx^i dx^j \tag{4.42}$$

for $i, j = 1, 2, 3$ and V and g_{ab} are time-independent. We assume that we also have an equation of state, $p(\rho)$. We denote the adiabatic index of our fluid by γ and define it by

$$\gamma(p) := \frac{\rho + p}{p} \frac{dp}{d\rho} \tag{4.43}$$

We now state a theorem.

Theorem 4.2 [Lindblom and Masood-ul-Alam (1993)]

Consider a static stellar model in general relativity theory whose surface occurs at a finite radius.(This assumes asymptotic conditions on V and g_{ij}) Assume that the equation of state $\rho = \rho(p)$ is a positive and non-decreasing C^1 function of the pressure. Assume that $\frac{1}{\gamma(p)}$ is bounded as $p \to 0^+$. Then the adiabatic index must satisfy the inequality

$$\gamma > \frac{6}{5}(1 + \frac{p}{\rho})^2 \geq \frac{6}{5}$$

at some point within the star.

We suppose that our static stellar model has a surface potential of $0 < V_s < 1$. Recall from chapter 3 expressions (3.16), and (3.17)-(3.18). We can express the pressure and density in terms of the potential,V.

$$h(p) := \int_0^p \frac{1}{\rho(s) + s} ds = \ln(\frac{V_s}{V}) \tag{4.44}$$

$$p(V) = h^{-1}[\ln(\frac{V_s}{V})] \tag{4.45}$$

$$\rho(V) = \rho(p(V)) \tag{4.46}$$

The proof for theorem 4.2 follows the technique we used to derive spherical symmetry in chapter 3. We will define a conformal factor which transforms the spatial metric g_{ij} and then make an observation about the adiabatic index in the case that the spatial metric is conformally flat.

First, we define our conformal factor. The conformal factor for $V \geq V_s$ is defined to be

$$\psi(V) = \frac{1}{2}(1 + V) \tag{4.47}$$

This conformal factor showed up in chapter 3 during our proof of uniqueness of the static stellar model. The conformal factor we use for $0 \leq V \leq V_s$ is given by the expression [33]

$$\psi(V) = \frac{1}{2}(1 + V_s) \exp\left[-\frac{V_s}{1 + V_s} \int_V^{V_s} \frac{\rho(s)}{s[\rho(s) + 3p(s)]} ds\right] \tag{4.48}$$

First note that (4.47) and (4.48) agree at the boundary, V_S. To prove continuity of the first derivative of the conformal factor we need a lemma.

Lemma 4.1 [Lindblom and Masood-ul-Alam (1993)]
Consider a static stellar model in general relativity theory whose surface occurs at a finite radius. Assume that the equation of state $\rho = \rho(p)$ is a positive and non-decreasing function in some open neighborhood of the surface of the star: i.e. for pressures $0 < p < \epsilon$. Then $\lim p \to 0^+ \frac{p}{\rho} = 0$.

Proof.
Recall equation (4.44). Since the equation of state $\rho(p)$ is a non-decreasing function we can estimate the integral.

$$h(p) = \int_0^p \frac{1}{\rho(s) + s} ds \tag{4.49}$$

$$\geq \int_0^p \frac{1}{\rho(p) + s} ds \tag{4.50}$$

$$= \ln(1 + \frac{p}{\rho}) \tag{4.51}$$

We know that $\lim_{p \to 0^+} h(p) = 0$, so raising (4.49) and (4.51) using exp and taking the limit we get

$$0 = \lim_{p \to 0^+} e^{h(p)} - 1 \geq \lim_{p \to 0^+} \frac{p}{\rho} \tag{4.52}$$

since $p, \rho \geq 0$ we see that

$$\lim_{p \to 0^+} \frac{p}{\rho} = 0 \tag{4.53}$$

Proving Lemma 4.1.

Now, we want to show that the first derivative of ψ is continuous at V_S. Computing the derivative of ψ in the interior of the star gives us [33]

$$\frac{d\psi}{dV} = \frac{V_s \psi(V)}{V(1 + V_s)(1 + \frac{3p}{\rho})} \tag{4.54}$$

Recall that V_s is defined to be where $p(V) = 0$. So taking the limit of (4.54) as $V \to V_s$ and recalling the lemma gives us

$$\frac{d\psi}{dV}|_{V_s} = \lim_{V \to V_s^-} \frac{V_s \psi(V)}{V(1+V_s)(1+\frac{3p}{\rho})} \tag{4.55}$$

$$= \frac{V_s \psi(V_s)}{V_s(1+V_s)} \tag{4.56}$$

$$= \frac{1}{2} \tag{4.57}$$

Comparing this to the derivative of ψ for $V \geq V_s$ defined in (4.47) reveals that $\frac{d\psi}{dV}$ is also continuous at the boundary. The second derivative of ψ vanished for $V \geq V_S$. For $V \leq V_s$ we have[33]

$$\frac{d^2\psi}{dV^2} = \frac{V_s \psi(V)}{V(1+V_s)(1+\frac{3p}{\rho})} \left[\frac{2+3V_s}{1+V_s} - \frac{3}{\gamma}(1+\frac{p}{\rho})^2 \right] \tag{4.58}$$

We can see from this expression that $\frac{d^2\psi}{dV^2}$ is continuous on the interior of the star and is bounded since we assume that $\frac{1}{\gamma}$ is bounded. Therefore the first derivative is Lipshitz. Hence the conformal factor we have defined ψ is $C^{1,1}$.

Proof of Theorem 4.2.

We define our conformal metric to be $\widetilde{g}_{ij} = \psi^4(V) g_{ij}$. Our choice of conformal metric for $V \geq V_s$ ensures that the mass of $\widetilde{g}_{ij}$ equals zero. The expression for the conformal scalar curvature $\widetilde{R}$ is given by the following expression.[33]

$$\widetilde{R} = \frac{8}{\psi}^5 (2\pi\rho\psi - D^i D_i \psi) \tag{4.59}$$

where D_i is the covariant derivative with respect to the spatial metric g_{ij}. Using the expression in (3.13) and expressions (4.47)-(4.48), (4.54), and (4.58) we can derive an expression for the conformal scalar curvature. The result is given by the following.[33]

$$\psi^4(1+V_s)\widetilde{R} = 16\pi\rho(1-V_s) + \frac{8\rho^2 V_s D^i V D_i V}{\gamma V^2(\rho+3p)^2}\left[3(1+\frac{p}{\rho})^2 - \gamma\frac{2+3V_s}{1+V_s}\right] \tag{4.60}$$

This expression holds on the interior of the star. Our chose of conformal factor for $V \geq V_s$ gives us that $\widetilde{R} = 0$ for $V \geq V_s$. Recall that for any stellar model that has finite radius the potential on the surface, V_S is strictly less that one [33]. The expression in (4.60) reveals that the conformal scalar curvature $\widetilde{R}$ is non-negative as long as

$$\left[3(1+\frac{p}{\rho})^2 - \gamma\frac{2+3V_s}{1+V_s}\right] \geq 0 \tag{4.61}$$

or equivalently,

$$\gamma \leq 3(1+\frac{p}{\rho})^2\frac{1+V_s}{2+3V_s} \tag{4.62}$$

We recall the Positive Mass Theorem discussed in chapter 3. We note that with our defined conformal factor ψ the conformal metric $\widetilde{g}_{ij}$ and asymptotic conditions assumed for g_{ij} allows us to apply the rigidity statement of the Postive Mass Theorem if $\widetilde{R}$ is nonnegative. $\widetilde{R}$ is non-negative precisely when the adiabatic index satisfies the inequality in (4.62). If the inequality in (4.62) is satisfied, then the Positive Mass Theorem implies that $\widetilde{g}_{ij}$ is flat. In this case $\widetilde{R}$ must vanish. If $\widetilde{R} = 0$ then by equation (4.60) we see from the first term on the right that $V_s = 1$ and in the second term on the right that $3(1+\frac{p}{\rho})^2 = \gamma\frac{2+3V_s}{1+V_s}$, which implies equality in (4.62). However, this contradicts the fact that we have a finite stellar model since V_s must be strictly less than 1 if the star is finite. Therefore, we must assume that $\widetilde{R}$ cannot be everywhere non-negative. Hence, there must be at lease one point where $\widetilde{R} < 0$. At this point, (4.62) is violated. Hence, at the point where $\widetilde{R} < 0$ we must have the inequality

$$\gamma > 3(1+\frac{p}{\rho})^2\frac{1+V_s}{2+3V_s} \tag{4.63}$$

Futhermore, noting that $V_s < 1$ in the case of a finite star, we have

$$\gamma > 3(1+\frac{p}{\rho})^2\frac{1+V_s}{2+3V_s} \tag{4.64}$$
$$\geq \frac{6}{5}(1+\frac{p}{\rho})^2 \tag{4.65}$$
$$\geq \frac{6}{5} \tag{4.66}$$

This concludes the proof of Theorem 4.2.

As a corollary to this theorem, we consider the case where our star has a polytropic equation of state, i.e. an equation of state of the form

$$p(\rho) = \kappa\rho^{1+\frac{1}{n}} \tag{4.67}$$

where κ and n are constants. The constant n is known as the "polytropic index". The constraint on the adiabatic index γ in equation (4.65) gives us a constraint on the polytropic index.

$$n < \frac{5}{1+\frac{6p}{\rho}} \leq 5 \tag{4.68}$$

Therefore, this offers a direct analog to the Newtonian limit on the polytropic index, $n < 5$ [34].

We make one further observation in this section. Recall the expression K defined by Beig and Simon in [35] and recall "Condition B" from Lindblom and Masood-ul-Alam [29]. "Condition B" was needed in the proof of spherical symmetry of static stellar models. In fact, "Condition A" in [29] implies the first of the inequalities of "Condition B". "Condition B" is given by the inequality [29]

$$\frac{5\rho^2}{6p(\rho+3p)} \geq \kappa > \frac{10V_s^2}{\exp[2h(p)]-1} \tag{4.69}$$

where $\kappa := \frac{\rho+p}{\rho+3p}\frac{d\rho}{dp}$. "Condition B" implies the upper bound on κ. This upper bound on κ is equivalent to $K \leq 0$ given by Beig and Simon[35]. If we analyze the upper bound on κ we see that

$$\frac{5\rho^2}{6p(\rho+3p)} \geq \kappa = \frac{\rho+p}{\rho+3p}\frac{d\rho}{dp} \tag{4.70}$$

$$\frac{5}{6} \geq \frac{\rho+p}{\rho+3p}\frac{d\rho}{dp}\frac{p(\rho+3p)}{\rho^2} = \frac{\rho+p}{\rho^2}p\frac{d\rho}{dp} \tag{4.71}$$

$$\frac{6}{5} \leq \frac{\rho^2}{\rho+p}\frac{1}{p}\frac{dp}{d\rho} < \frac{\rho^2+2\rho p+p^2}{\rho+p}\frac{1}{p}\frac{dp}{d\rho} = (\rho+p)\frac{1}{p}\frac{dp}{d\rho} = \tag{4.72}$$

$$\frac{6}{5} < \frac{\rho+p}{p}\frac{dp}{d\rho} = \gamma \tag{4.73}$$

So we see here that Buchdahl's inequality manifest itself in the condition necessary for spherical symmetry. Therefore, Buchdahl's inequality rules out the existence of a large class of potential counter examples to the Fluid Ball Conjecture.

4.3 Necessary and Sufficient Conditions for a Finite Radius

In chapter 2 we proved the existence of spherically symmetric, static stellar models with perfect fluid source. These model stars were either finite or infinite. In the case the model star was finite the pressure became zero at an $R < \infty$ and the exterior Schwarzschild solution matched the interior solution at the boundary. In the case the model stars was infinite the density $\rho > 0$ for all $r \geq 0$ and $\lim_{r\to\infty}\rho(r) = 0$. This section looks at criteria based on the equation of state which supplies sufficient and necessary conditions for a star to posses a finite radius.

In the case that $\rho \to 0$ as $r \to \infty$, if $p(R) = 0$ for some $R < \infty$ and $\rho(R) > 0$ then the star is obviously finite. So consider the case when p and ρ vanish simultaneously. Our first goal is to derive a sufficient condition for a spherically symmetric, static stellar model to have a finite radius. We recall several equations from chapter 2: (2.16), (2.33), and the T.O.V. equation in (2.36).

$$m(r) = 4\pi \int_0^r s^2 \rho(s) ds \tag{4.74}$$

$$w(r) = \frac{m(r)}{r^3} \tag{4.75}$$

$$\frac{dp}{dr} = -Gr \left[1 - \frac{2G}{c^2} r^2 w\right]^{-1} (\frac{4\pi p}{c^2} + w)(\frac{p}{c^2} + \rho) \tag{4.76}$$

Theorem 4.3 [Rendall and Schmidt(1991)]
Consider a spherically symmetric, static stellar model such that $\frac{d\rho}{dr} < 0$,the equation of state is C^∞, and has a central pressure given by p_0. Then a sufficient condition for a finite radius is

$$\int_0^{p_0} \frac{1}{(\rho(s))^2} < \infty$$

Proof.
Since we assume that $\frac{d\rho}{dr} < 0$ then equation (4.74) gives us the inequality

$$m(r) = 4\pi \int_0^r s^2 \rho(s) ds \tag{4.77}$$

$$\geq 4\pi \int_0^r s^2 \rho(r) ds \tag{4.78}$$

$$= \frac{4}{3}\pi \rho(r) r^3 \tag{4.79}$$

If we take this inequality and recall (4.75), this gives us another inequality

$$m(r) \geq \frac{4}{3}\pi\rho(r)r^3 \implies w(r) = \frac{m(r)}{r^3} \geq \frac{4}{3}\pi\rho(r) \tag{4.80}$$

Using the T.O.V. equation in (4.76) and a change of variable $x = r^2$, in conjunction with the inequality in (4.80) we get

$$\frac{dp}{dx}\frac{dx}{dr} = -Gr\left[1-\frac{2G}{c^2}r^2w\right]^{-1}\left(\frac{4\pi p}{c^2}+w\right)\left(\frac{p}{c^2}+\rho\right) \tag{4.81}$$

$$< -Grw\rho \tag{4.82}$$

$$\leq -Gr\rho\frac{4}{3}\pi\rho = -\frac{4G\pi}{3}r\rho^2 \tag{4.83}$$

Since $\frac{dx}{dr} = 2r$, the above inequality gives us

$$\frac{dp}{dx} < -\frac{2G\pi}{3}\rho^2 \tag{4.84}$$

We now define $\overline{p}(x)$ to be the solution of

$$\frac{d\overline{p}}{dx} = -\frac{2G\pi}{3}\rho^2 \tag{4.85}$$

given the same central pressure p_0 and the same equation of state, $\rho(p)$. This gives us the inequality

$$p(x) < \overline{p}(x) \tag{4.86}$$

The differential equation for $\overline{p}$ given in (4.85) can easily be solved.

$$\int_{p_0}^{\overline{p}(x)} \frac{1}{(\rho(s))^2}ds = -\frac{2G\pi}{3}x \tag{4.87}$$

Now, if $\overline{p}(x_1) = 0$ for some $x_1 < \infty$, then there must be some $x_2 \leq x_1$ such that $p(x_2) = 0$ by equation (4.86). But, if the integral

$$\int_{0}^{p_0} \frac{1}{(\rho(s))^2}ds < \infty \tag{4.88}$$

then there must be an $x_1 < \infty$ where $\overline{p}(x_1) = 0$. Therefore, for some $x_2 \leq x_1 < \infty$ we have $p(x_2) = 0$. Hence, the star is finite.

We now look at a necessary condition for a spherically symmetric, static star to be finite.

Theorem 4.4 [Rendall and Schmidt(1991)]
Consider a spherically symmetric, static stellar model such that $\frac{d\rho}{dr} < 0$,the equation of state is C^{∞}, and has a central pressure given by p_0. Then a necessary condition for a finite radius is

$$\int_0^{p_0} \frac{1}{c^{-2}s + \rho(s)} ds < \infty$$

Proof.
Suppose we have a solution to the spherically symmetric stellar model guaranteed by Theorem 2.2 from chapter 2 with a central pressure given by p_0. Suppose also that the boundary of this star occurs at a finite radius $x = x_1$ where $x = r^2$. So by definition the pressure $p(x_1) = 0$. Consider the compact interval $[0, x_1]$ and the continuous quantity

$$1 - \frac{2G}{c^2} xw \tag{4.89}$$

This continuous quantity attains a minimum as some x_0. Let this minimum value be denoted A. Looking at the quantity it is not hard to see that $A \geq 0$. Recalling once more the T.O.V. equation given in this chapter by (4.76), but in terms of x, we have the inequality

$$\frac{dp}{dx} = -\frac{G}{2}\left[1 - \frac{2G}{c^2}xw\right]^{-1}\left(\frac{4\pi p}{c^2} + w\right)\left(\frac{p}{c^2} + \rho\right) \tag{4.90}$$

$$> -G\frac{1}{A}\left(\frac{4\pi p}{c^2} + w\right)\left(\frac{p}{c^2} + \rho\right) \tag{4.91}$$

$$= -B\left(\frac{p}{c^2} + \rho\right) \tag{4.92}$$

where $B = -\frac{G}{A}(\frac{4\pi p_0}{c^2} + w(0))$. Similar to the proof of Theorem 4.3, we define $\overline{p}$ to be the solution of

$$\frac{d\overline{p}}{dx} = -B(\frac{p}{c^2} + \rho) \tag{4.93}$$

with a central pressure given by p_0 and the same equation of state. We can easily solve equation (4.93).

$$\int_{p_0}^{\overline{p}} \frac{1}{c^{-2}s + \rho(s)} = -Bx \tag{4.94}$$

In this case, we see that $\overline{p}(x) < p(x)$. Since we assume that $p(x_1) = 0$, then $\overline{p}(x_1) = 0$ since pressure is non-negative. This implies that

$$\int_0^{p_0} \frac{1}{c^{-2}s + \rho(s)} = Bx_1 < \infty \tag{4.95}$$

This completes the proof.

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Appendix

Appendix A

Spin Geometry

We want to give a brief overview of spin geometry and use it to derive Witten's expression for the ADM mass of an asymptotically flat Riemannian manifold. This appendix follows closely to Friedrich's book "Dirac Operators in Riemannian Geometry". We begin with the construction of spinors in the setting of Minkowski space. We then generalize this construction to the setting of bundles where we describe the connection and curvature for the spinor bundle. Lastly, we derive the formula of Lichnerowicz and show how Witten used this in his construction of the formula for mass in his proof of the Positive Mass Theorem.

A.1 Dirac Spinors

Let (M^4, η) denote Minkowski space. Our first goal is the construction of the Dirac Spinors. To this end, we start with the construction of the Clifford algebra. The Clifford algebra for (M^4, η) is an associative algebra over $\mathbb{R}$ with unit and a linear map $c : M^4 \to CL(M^4)$ which satisfies the relationship

$$c(x).c(y) + c(y)c(x) = -2 < x, y >_\eta .Id \tag{A.1}$$

where the "low dot" represents Clifford Multiplication and $x, y \in M^4$.[14]. Let $\{e_i\}_{i=0}^3$ be an orthonormal basis for M^4. The Clifford algebra contains the relation

$$c(e_i).c(e_j) + c(e_j).c(e_i) = 0 \text{ for } i \neq j \tag{A.2}$$

In this case, we also get a basis for the $2^4 = 16$ dimensional vector space $CL(M^4)$. This basis of $CL(M^4)$ is formed by the elements Id and $e_{i_1}.e_{i_2}.\cdots e_{i_s}$ where $1 \leq i_1 < i_2 < \cdots < i_s \leq n$ with $1 \leq s \leq n$.[14]

Our goal is the construction of the Dirac spinors. The Dirac spinors, which we denote S, is a 4 dimensional complex vector space which acts as a representation space for an irreducible representation of a complexifed Clifford algebra. There are two metric signatures that are common when describing Minkowski space. They are $(-, +, +, +)$, which is common in math literature, and $(+, -, -, -)$ which is common in physics literature. In the setting of spinors it is more natural to work with the signature $(+, -, -, -)$. Each one of these metric signatures will generate a Clifford algebra. These Clifford algebras are not the same. In our case it is not necessary to prescribe a metric signature for η since we are constructing a complexified Clifford algebra. This is no loss of generality since

$$\mathbb{C}L_4 \equiv CL(M^4)_{1,3} \otimes_{\mathbb{R}} \mathbb{C} \equiv CL(M^4)_{3,1} \otimes_{\mathbb{R}} \mathbb{C} \tag{A.3}$$

In other words, the complexification of the Clifford algebras generated by both metric signatures will be isomorphic.[13]

The Clifford algebra $\mathbb{C}L(M^4)$ has a representation space of a 4 dimensional complex vector space. We denote κ to be the so-called Spin representation of the Clifford algebra $\mathbb{C}L(M^4)$. In other words, our representation is simply the following isomorphism.[14]

$$\kappa : \mathbb{C}L(M^4) \to End(S) \tag{A.4}$$

We can now define the notion of Clifford multiplication of vectors and spinors. Let $v \in M^4$. Under the representation, $\kappa(c(v))$ is an endomorphism of the space of Dirac spinors, S. We define a linear map

$$\mu : c(M^4) \otimes_{\mathbb{R}} S \to S \tag{A.5}$$

For $v \in M^4$ and $\psi \in S$ we have

$$\mu(c(v) \otimes \psi) = \kappa(c(v))\psi = c(v).\psi \tag{A.6}$$

We now want to describe a certain subgroup of the Clifford algebra, $Spin(4) \subset \mathbb{C}L(M^4)$. Consider the sub-group of $CL(M^4)$, the non-complexified Clifford algebra generated by (M^4, η), which is multiplicatively generated by $\{c(v)|v \in M^4$ and $\|v\| = 1\}$. We call this group $Pin(4) \subset CL(M^4)$. The elements in $Pin(4)$ are the products $c(v_1).c(v_2)\cdots .c(v_m)$ with $v_i \in M^4$ and $\|v\| = 1$. Let $O(4)$ denote the group of orthogonal transformations of M^4. There exists a continuous surjective group homomorphism from $Pin(4)$ onto the group $O(4)$.[14] We denote this map

$$\lambda : Pin(4) \to O(4) \tag{A.7}$$

In order to define this map we must first define another map. Every Clifford algebra carries an anti-involution denoted

$$\gamma : CL(M^4) \to CL(M^4) \tag{A.8}$$

such that

$$c(v) = \gamma(c(v))\text{for } v \in M^4 \tag{A.9}$$

γ possess several properties.[14]

1.) γ is Linear.

2.) $\gamma \circ \gamma = Id$

3.) $\gamma(c(v)) = c(v)$ for all $v \in M^4$.

4.) $\gamma(x.y) = \gamma(y).\gamma(x)$ for $x, y \in CL(M^4)$.

The homomorphism $\lambda : Pin(4) \to O(4)$ is then given by

$$\lambda(x) : M^4 \to M^4, \qquad \lambda(x)v = x.c(v).\gamma(x) \tag{A.10}$$

for $x \in Pin(4) \subset CL(M^4)$ and $v \in M^4$. Now, let $SO(4)$ denote the orthogonal transformations of M^4 which have unit determinant. Then the group $Spin(4) \subset CL(M^4)$ is given by

$$Spin(4) = \lambda^{-1}(SO(4)) \tag{A.11}$$

and $\ker(\lambda) = \{-1, 1\} \equiv \mathbb{Z}_2$. We also state the fact that $Spin(4)$ is simply connected. Therefore $Spin(4)$ is the universal double cover of the group $SO(4)$ and λ is the covering map. Now, for the complexified Clifford algebra $\mathbb{C}L(M^4)$ the group $Spin(4)^{\mathbb{C}}$ is given by the following.

$$Spin(4)^{\mathbb{C}} = Spin(4) \times_{\mathbb{Z}_2} S^1 \tag{A.12}$$

In other words, $Spin(4)^{\mathbb{C}}$ is the collection of equivalence classes $[g, z] \in Spin(4)^{\mathbb{C}}$ with the equivalence relationship $(g, z) = (-g, -z)$. We want to list a couple of homomorphisms.[14]

1.) $\lambda : Spin(4)^{\mathbb{C}} \to SO(4)$ given by $\lambda([g,z]) = \lambda(g)$ (A.13)

2.) $i : Spin(4) \to Spin(4)^{\mathbb{C}}$ inclusion map (A.14)

3.) $j : S^1 \to Spin(4)^{\mathbb{C}}$ inclusion map (A.15)

4.) $l : Spin(4)^{\mathbb{C}} \to S^1$ given by $l([g,z]) = z^2$ (A.16)

5.) $p : Spin(4)^{\mathbb{C}} \to SO(4) \times S^1$ given by $p([g,z]) = (\lambda(g), z^2)$ i.e. $p = \lambda \times l$ (A.17)

Minkowski space is a four dimensional vector space. The dimension being an even number means that the space of Dirac spinors, S decomposes into the direct sum of the so-called positive and negative Weyl spinors.[14] This is denoted

$$S = S^+ \oplus S^- \tag{A.18}$$

In this case, Clifford multiplication by a non-zero vector v is a bijection, $S^{\pm} \mapsto S^{\mp}$.

In the space of Dirac spinors there exists a positive definite Hermitian inner product such that

$$(c(v).\psi, \phi) = -(\psi, c(v).\phi) \tag{A.19}$$

for $v \in M^4$ and $\psi, \phi \in S$. The spin representation $\kappa : \mathbb{C}L(M^4) \to End(S)$ when restricted to the group $Spin(4) \subset Spin(4)^{\mathbb{C}} \subset \mathbb{C}L(M^4)$ is unitary with respect to this inner product. Furthermore,

$$det(\kappa(g)) = 1 \tag{A.20}$$

for every $g \in Spin(4)$. Therefore, the spin representation of the group $Spin(4)$ is the group $SU(S)$.[14] For example, if we have the metric signature $(-,+,+,+)$ then the spin representation of $Spin(4) = Spin_{3,1}$ is given by the group $SL(2,\mathbb{C})$ and has a representation space of Weyl spinors, i.e two component spinors $S^{\pm}$. In the general case of $[g, z] \in Spin(4)^{\mathbb{C}}$ we have the representation

$$\kappa([g, z])\psi = z.\kappa(g)\psi \tag{A.21}$$

with determinant given by

$$det(\kappa([g, z])) = z^{dim(S)} = z^4 \tag{A.22}$$

A.2 Spin Manifold and the Differential Structure of the Dirac Bundle

We want to extend the structure described above to the setting of manifolds and fiber bundles. A spin manifold is an oriented Riemannian manifold with a spin structure on its tangent bundle. An oriented Riemannian manifold M admits a spin structure (and is a spin manifold) if and only if its second Stiefel-Whitney class is zero.[13]

Let (M, g) be a 4 dimensional spin manifold. Let $\mathcal{F}$ denote the bundle of positively oriented orthonormal frames of the tangent bundle TM. $\mathcal{F}$ is a principal bundle with structure group $SO(4)$. Suppose we have a $Spin(4)^{\mathbb{C}}$ structure on TM. Since we have a $Spin(4)^{\mathbb{C}}$ structure, we know that there exists an $S^1 = SO(2)$-principal bundle P_1 such that the fiber product $\mathcal{F}\widetilde{\times}P_1$ has a $Spin(4)^{\mathbb{C}}$ structure.[14] Our $Spin(4)^{\mathbb{C}}$ structure is given by the pair $(\mathcal{W}, \Lambda)$ where $\mathcal{W}$ is a $Spin(4)^{\mathbb{C}}$ principal bundle and Λ is a double covering map given by

$$\Lambda : \mathcal{W} \to \mathcal{F}\widetilde{\times}P_1 \tag{A.23}$$

Recall from equation (A.17) that we also have the covering map $p : Spin(4)^{\mathbb{C}} \to SO(4) \times S^1$. The $SO(4)$ principal bundle allows us to define an associated four dimensional vector bundle.

$$T = \mathcal{F} \times_{SO(4)} M^4 = \mathcal{W} \times \lambda M^4 \tag{A.24}$$

T is isomorphic to the tangent bundle TM. The fibers of T are Minkowski. As described above, we can view T as an associated vector bundle to the $Spin(4)^{\mathbb{C}}$ principal bundle. Recall our spin representation κ. We can define another associated vector bundle to the $Spin(4)^{\mathbb{C}}$ principal bundle by

$$\Delta = \mathcal{W} \times_{Spin^{\mathbb{C}}(4)} S \tag{A.25}$$

The fibers of this associated vector bundle are a four complex dimensional space with a Hermitian metric. These fibers are precisely the space of Dirac spinors. For this reason, we call Δ the Dirac spinor bundle.

Finally, recall that each orthogonal transformation of M^4 induces and orthogonal transformation of $\mathbb{C}L(4)$. Hence we get a representation of $SO(4)$ denoted

$$\rho : SO(4) \to Aut(\mathbb{C}L(4)) \tag{A.26}$$

This allows us to define an associated bundle of Clifford algebras.[13]

$$\mathcal{C}(T) = \mathcal{F} \times_{\rho} \mathbb{C}L(4) \tag{A.27}$$

In this way, at each point $p \in M$ there is a complexified Clifford algebra that is generated by vectors in $T|_p \equiv T_pM$. This allows us the use of Clifford multiplication fiber-wise since we can view the tangent space as a fiber of the vector bundle associated with the $Spin(4)^{\mathbb{C}}$ principal bundle.

The Lie algebra of $SO(4)$ we denote by $\mathbf{so}(4)$. The Lie algebra of S^1 is $i\mathbb{R}$ and so the Lie algebra of the product $SO(4) \times S^1$ is $\mathbf{so}(4) \oplus i\mathbb{R}$. The Lie algebra of $Spin(4)^{\mathbb{C}}$ we denote by $\mathbf{spin}_{\mathbb{C}}(4)$. Let $x \in M$ and $\{e_i\}_{i=0}^3$ denotes the standard basis of $T|_x = M^4$. Then $\mathbf{spin}_{\mathbb{C}}(4) = \mathbf{m}_2 \oplus i\mathbb{R}$ where

$$\mathbf{m}_2 = span\{c(e_i).c(e_j)|1 \leq i < j \leq 4\} \tag{A.28}$$

$\mathbf{m}$ is a Lie algebra equipped with the commutator bracket which coincides with Clifford multiplication

$$[x, y] = x.y - y.x \tag{A.29}$$

We take the connection on the $SO(4)$ principal bundle to be the Levi-Civita connection, which is the unique torsion-free metric connection. This is an $\mathbf{so}(4)$-valued 1-form

$$Z : T\mathcal{F} \to \mathbf{so}(4) \tag{A.30}$$

We also fix a connection on the S^1 principal bundle P_1. It is given by

$$A : TP_1 \to i\mathbb{R} \tag{A.31}$$

We use the product of these connections to define a connection in the fiber product $\mathcal{F}\widetilde{\times}P_1$.

$$Z \times A : T(\mathcal{F}\widetilde{\times}P_1) \to \mathbf{so}(4) \oplus i\mathbb{R} \tag{A.32}$$

The map $d\Lambda : T(\mathcal{W}) \to T(\mathcal{F}\widetilde{\times}P_1)$ and the differential $p_* : \mathbf{m}_2 \oplus i\mathbb{R} \to \mathbf{so}(4) \oplus i\mathbb{R}$ allow us to lift the connection $Z \times A$ to a connection

$$\widetilde{Z \times A} : T(\mathcal{W}) \to \mathbf{m}_2 \oplus i\mathbb{R} \tag{A.33}$$

which completes a commuting diagram. The connection $\widetilde{Z \times A}$ determines a covariant derivative in the Dirac bundle.[14]

Let X, Y be vector fields defined on M and let ψ be a spinor field. Then for the spinor covariant derivative with respect to any connection A in (A.31) we have

$$\nabla_Y^A(c(X).\psi) = c(X).(\nabla_Y^A.\psi) + c(\nabla_Y X).\psi \tag{A.34}$$

where ∇ is the Levi-Civita covariant derivative. [14] Also, the spinor covariant derivative is metric with respect to the Hermitian inner product on the Dirac bundle. In other words, if X is a vector field and ψ, ψ_1 are both spinor fields then the vector field acts as a derivation on the Hermitian inner product, i.e.

$$X(\psi, \psi_1) = (\nabla_X^A \psi, \psi_1) + (\psi, \nabla_X^A \psi_1) \tag{A.35}$$

Finally, we want to give local expressions for the covariant derivative and curvature operators. Let $e : U \subset M \to \mathcal{F}$ be a local section of the frame bundle given on an open set U. This is a positively oriented orthonormal frame field. Let $\{E_{ij} = e_i \wedge e_j\}$ denote the standard basis of $\mathbf{so}(4)$, with matrix representation given by a 1 in the i^{th} column, j^{th} row and a -1 in the j^{th} column, i^{th} row. The local connection form of Z with respect to e we denote Z^e. It is given by

$$Z^e = \sum_{i<j} \omega_{ij} E_{ij} \tag{A.36}$$

where $\omega_{ij} = g(\nabla_{e_i}, e_j)$ are the standard 1-forms for the Levi-Civita connection. Let $s : U \subset M \to P_1$ be a fixed section of the S^1 principal bundle. Then the local connection form of A with respect to s we denote A^s. It is given by a map

$$A^s : TU \to i\mathbb{R} \tag{A.37}$$

Then we have the local connection form of the product $Z \times A$ with respect to the product of sections $e \times s : U \to \mathcal{F}\widetilde{\times}P_1$. It is given by

$$Z \times A^{e \times s} = (\sum_{i<j} \omega_{ij} E_{ij}, A^s) \tag{A.38}$$

Let the lift of the product of sections $e \times s$ be given by $\widetilde{e \times s} : U \to \mathcal{W}$. It is a fact [13] that the Lie algebras $\mathbf{spin}(4) = \mathbf{m}_2$ and $\mathbf{so}(4)$ are isomorphic, with isomorphism given by

$$E_{ij} \mapsto \frac{1}{2} c(e_i).c(e_j) \tag{A.39}$$

With (A.38) and (A.39) we have the local connection form of $\widetilde{Z \times A}$ with respect to $\widetilde{e \times s}$. It is denoted $\widetilde{Z \times A}^{\widetilde{e \times s}}$ and given by

$$\widetilde{Z \times A}^{\widetilde{e \times s}} = (\frac{1}{2}\omega_{ij} c(e_i).c(e_j), \frac{1}{2} A^s) \tag{A.40}$$

(A.40) allows us to give a local expression of the covariant derivative on the Dirac bundle. It is the following.[14]

$$\nabla^A \psi = d\psi + \frac{1}{2} \sum_{i<j} \omega_{ij} c(e_i).c(e_j).\psi + \frac{1}{2} A^s \psi \tag{A.41}$$

We next want to define the curvature 2-form for the Dirac bundle. We start with the curvature form for the Levi-Civita connection. It is

$$\Omega^Z : T\mathcal{F} \times \mathcal{F} \to \mathbf{so}(4) \tag{A.42}$$

given by

$$\Omega^Z = \sum_{i<j} \Omega_{ij} E_{ij} \tag{A.43}$$

where $\Omega_{ij} = d\omega_{ij} + \sum_{k=0}^{3} \omega_{ik} \wedge \omega_{kj}$. The curvature 2-form for the connection A is given by

$$\Omega^A = dA \tag{A.44}$$

We are then able to lift the product to a curvature 2-form for the connection $\widetilde{Z \times A}$. It is given by

$$\Omega^{\widetilde{Z\times A}} = \frac{1}{2}\sum_{i<j} \Lambda^*(\Omega_{ij}) c(e_i).c(e_j) \oplus \frac{1}{2}\Lambda^*(dA) \tag{A.45}$$

$$= \frac{1}{4}\sum_{i<j}(\sum_{k,l} R_{ijkl}\sigma^k \wedge \sigma^l) c(e_i).c(e_j) + \frac{1}{2}dA \tag{A.46}$$

where $\{\sigma^k\}$ is the dual frame. This allows us use of the following expression for the second covariant derivative of a Dirac spinor field.[14]

$$\nabla^A(\nabla^A \psi) = \frac{1}{2}\sum_{i<j} \Omega_{ij} c(e_i).c(e_j).\psi + \frac{1}{2}dA.\psi \tag{A.47}$$

$$= \frac{1}{4}\sum_{i<j}(\sum_{k,l} R_{ijkl}\sigma^k \wedge \sigma^l) c(e_i).c(e_j).\psi + \frac{1}{2}dA\psi \tag{A.48}$$

A.3 Dirac Operator and the Schrödinger-Lichnerowicz formula

Let ∇^A denote the covariant derivative on the Dirac bundle. Recall (A.34) and (A.35). These equations show how the covariant derivative ∇^A interacts with the Hermitian inner product $(,)$ defined on the Dirac bundle. Let $\{e_i\} = e$ be the positively oriented orthonormal frame field given above. We first define the Laplace operator on the Dirac bundle. If ψ is a Dirac spinor field, then the Laplace operator on the Dirac bundle, which we denote $\Delta^A(\psi)$, is given by

$$\Delta^A(\psi) = -\sum_i \nabla^A_{e_i}\nabla^A_{e_i}\psi - \sum_i div(e_i)\nabla^A_{e_i}\psi \tag{A.49}$$

where the divergence is given by

$$-div(e_j) = \sum_k g(\nabla_{e_k} e_j, e_k)e_k \tag{A.50}$$

Next we define the Dirac operator. The Dirac operator, denoted $\not{D}$, is a differential operator on the space of spinors which is defined as the contraction of covariant differentiation and Clifford multiplication.[14] Namely,

$$\not{D}\psi = \sum_i c(e_i).\nabla^A_{e_i}\psi \tag{A.51}$$

We now give the statement of the Schrödinger-Lichnerowicz Formula.

Proposition A.1[Schrödinger-Lichnerowicz Formula]
Denote by R the scalar curvature of the Riemannian manifold and let $dA = \Omega^A$ be the imaginary-valued curvature 2-form of the connection A in the S^1-principal bundle associated with the $Spin(4)^{\mathbb{C}}$ structure. Then one has for a Dirac spinor field ψ,

$$\not{D}^2\psi = \Delta^A\psi + \frac{1}{4}R\psi + \frac{1}{2}dA.\psi$$

The goal now is to establish the Schrödinger-Lichnerowicz formula. The formula is a way to relate the square of the Dirac operator and the Laplace operator on the Dirac bundle. Let $P \in M$. We have an orthonormal frame $\{e_i\}$ on T_PM. Recall the Clifford relationships from (A.1) and (A.2). In particular, at P we have

$$c(e_i).c(e_i) = -1 \tag{A.52}$$

$$c(e_i).c(e_j) + c(e_j).c(e_i) = -2\delta_{ij} \tag{A.53}$$

We now compute the difference between the square of the Dirac operator and the Spin connection Laplacian.

$$\not{D}^2 - \Delta^A = \sum_i c(e_i).\nabla^A_{e_i}(\sum_j c(e_j).\nabla^A_{e_j}) + \sum_i \nabla^A_{e_i}\nabla^A_{e_i} + \sum_i div(e_i)\nabla^A_{e_i} \tag{A.54}$$

$$= \sum_{i,j}\{c(e_i).c(\nabla_{e_i}e_j).\nabla^A_{e_j} + c(e_i).c(e_j).\nabla^A_{e_i}\nabla^A_{e_j}\} + \sum_i \nabla^A_{e_i}\nabla^A_{e_i} + \sum_i div(e_i)\nabla^A_{e_i} \tag{A.55}$$

$$= \sum_{i,j,k} g(\nabla_{e_i}e_j, e_k)c(e_i).c(e_k).\nabla^A_{e_j} + \sum_{i,j} c(e_i).c(e_j).\nabla^A_{e_i}\nabla^A_{e_j} + \sum_i \nabla^A_{e_i}\nabla^A_{e_i} \tag{A.56}$$

$$+ \sum_i div(e_i)\nabla^A_{e_i}$$

$$= \sum_{j,i\neq k} g(\nabla_{e_i}e_j, e_k)c(e_i).c(e_k).\nabla^A_{e_j} + \sum_{j,i=k} g(\nabla_{e_i}e_j, e_k)c(e_i).c(e_k).\nabla^A_{e_j} + \tag{A.57}$$

$$\sum_{i\neq j} c(e_i).c(e_j).\nabla^A_{e_i}\nabla^A_{e_j} + \sum_{i=j} c(e_i).c(e_j).\nabla^A_{e_i}\nabla^A_{e_j} + \sum_i \nabla^A_{e_i}\nabla^A_{e_i} + \sum_i div(e_i)\nabla^A_{e_i}$$

$$= \sum_{j,i\neq k} g(\nabla_{e_i}e_j, e_k)c(e_i).c(e_k).\nabla^A_{e_j} - \sum_i div(e_i)\nabla^A_{e_i} + \sum_{i\neq j} c(e_i).c(e_j).\nabla^A_{e_i}\nabla^A_{e_j} - \tag{A.58}$$

$$\sum_i \nabla^A_{e_i}\nabla^A_{e_i} + \sum_i \nabla^A_{e_i}\nabla^A_{e_i} + \sum_i div(e_i)\nabla^A_{e_i}$$

$$= \sum_{j,i\neq k} g(\nabla_{e_i}e_j, e_k)c(e_i).c(e_k).\nabla^A_{e_j} + \sum_{i\neq j} c(e_i).c(e_j).\nabla^A_{e_i}\nabla^A_{e_j} \tag{A.59}$$

$$= -\sum_{j,i\neq k} g(e_j, \nabla_{e_i}e_k)c(e_i).c(e_k).\nabla^A_{e_j} + \sum_{i\neq j} c(e_i).c(e_j).\nabla^A_{e_i}\nabla^A_{e_j} \tag{A.60}$$

$$= -\sum_{j,i<k} g(e_j, \nabla_{e_i}e_k - \nabla_{e_k}e_i)c(e_i).c(e_k).\nabla^A_{e_j} + \sum_{i<j} c(e_i).c(e_j).[\nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{e_j}\nabla^A_{e_i}] \tag{A.61}$$

$$= \sum_{j,i<k} g(e_j,[e_k,e_i])c(e_i).c(e_k).\nabla^A_{e_j} + \sum_{i<j} c(e_i).c(e_j).[\nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{e_j}\nabla^A_{e_i}] \tag{A.62}$$

$$= \sum_{i<j} c(e_i).c(e_j).[\nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{e_j}\nabla^A_{e_i} + \nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{e_i}\nabla^A_{e_j}] \tag{A.63}$$

$$= \sum_{i<j} c(e_i).c(e_j).[\nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{[e_i,e_j]}] \tag{A.64}$$

$$= \sum_{i<j} c(e_i).c(e_j).[\nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{[e_i,e_j]}]+ \tag{A.65}$$

$$\sum_{i\geq j} c(e_i).c(e_j).[\nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{[e_i,e_j]}] - \sum_{i\geq j} c(e_i).c(e_j).[\nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{[e_i,e_j]}]$$

$$= \frac{1}{2}\sum_{i,j} c(e_i).c(e_j).(\nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{e_j}\nabla^A_{e_i} - \nabla^A_{[e_i,e_j]}) \tag{A.66}$$

$$= \frac{1}{2}\sum_{i,j} c(e_i).c(e_j).R^A(e_i,e_j) \tag{A.67}$$

Recall equation (A.48) and that

$$(\nabla^A\nabla^A\psi)(e_i,e_j) = R^A(e_i,e_j)\psi \tag{A.68}$$

This gives us for (A.67)

$$\frac{1}{2}\sum_i c(e_i).[-\frac{1}{2}Ric(e_i) + \frac{1}{2}(e_i \lrcorner dA)] = -\frac{1}{4}\sum_i c(e_i)Ric(e_i) + \frac{1}{4}\sum_i c(e_i)(e_i \lrcorner dA) \tag{A.69}$$

$$= \frac{1}{4}R + \frac{1}{2}dA \tag{A.70}$$

Therefore,

$$\not{D}^2 = \Delta^A + \frac{1}{4}R + \frac{1}{2}dA \tag{A.71}$$

A.4 Witten's expression for the mass

The above formula in equation (A.71) is key to the derivation of Witten's expression for mass. Suppose that M is an asymptotically flat,complete, n-dimensional, spin manifold with metric

$$g_{ij} = \delta_{ij} + h_{ij} \tag{A.72}$$

with $h_{ij} \in W^{2,\alpha}_{-\tau}(M)$ for $\tau > \frac{n-2}{2}$ and $R \in L^1$. Then Witten's expression for the mass in n-dimensions is given by

$$c(n)mass(g) = \int_M (4\|\widetilde{\nabla}\psi\|^2 + R\|\psi\|^2)dV_g \tag{A.73}$$

We are considering a general spin structure here instead of the more complicated complexification. This means we do not specifically have a "Dirac" spinor, i.e. 4 component spinor. In this case, the Schrödinger-Lichnerowicz Formula takes the following form when applied to a spinor ψ.[19]

$$\not{D}^2\psi = \nabla^*\nabla\psi + \frac{1}{4}R\psi \tag{A.74}$$

In an orthonormal frame $\{e_i\}_{i=0}^3$, we recall the isomorphism between $\mathbf{so}(4)$ and $\mathbf{spin}(4)$ given by

$$e_i \wedge e_j \mapsto \frac{1}{2}c(e_i).c(e_j) \tag{A.75}$$

This allows us to lift the Levi-Civita connection Z^e to a connection on an associated spin bundle. The connection on the spin bundle is

$$\widetilde{Z}^{\widetilde{e}} = \frac{1}{2}\omega_{ij}c(e_i).c(e_j) \tag{A.76}$$

where ω_{ij} are the Levi-Civita connection 1-forms. This gives us the expression for the covariant derivative of spinors in a local frame. We denote the covariant derivative that acts on spinors as $\widetilde{\nabla}$. The formula for the covariant derivative is given by

$$\widetilde{\nabla}\psi = d\psi + \frac{1}{2}\sum_{i<j}\omega_{ij}c(e_i).c(e_j).\psi \tag{A.77}$$

$$= d\psi - \frac{1}{4}\sum_{i,j}\omega_{ij}c(e_i).c(e_j).\psi \tag{A.78}$$

$$\tag{A.79}$$

Now, let ψ be a spinor that is a solution to Dirac's equation, i.e.

$$\not{D}\psi = 0 \tag{A.80}$$

that is asymptotic to a given constant spinor normalized by $|\psi_0|^2 \to 1$. The required spinor $\psi = \psi_0 + \psi_1$, where $\psi_1 \in W^{2,\alpha}_{-\tau}$, exists.[17]

We need to prove an identity. Let $(,)$ denote the positive definite inner product on the bundle of spinors, i.e. from (A.19). Recall that the Laplacian and Dirac operator are both self-adjoint. Let $\sigma_{ij} = \frac{1}{2}[c(e^i), c(e^j)]$ so that

$$\sigma_{ij}.\widetilde{\nabla_j} = \frac{1}{2}[c(e^i), c(e^j)].\widetilde{\nabla_j} \tag{A.81}$$

$$= (\delta^{ij} + c(e^i).c(e^j)).\widetilde{\nabla_j} \tag{A.82}$$

$$= \widetilde{\nabla}_i + e^i.\not{D} \tag{A.83}$$

Then,

$$
\begin{aligned}
d\{(\psi, \sigma_{ij}.\widetilde{\nabla}_j\psi)e_i\lrcorner dVol_g\} &= d((\psi, \widetilde{\nabla}_i\psi)e_i\lrcorner dVol_g) + d((\psi, e^i.\not{D}\psi)e_i\lrcorner dVol_g\} && \text{(A.84)}\\
&= \{(\widetilde{\nabla}_i\psi, \widetilde{\nabla}_i\psi) + (\psi, \widetilde{\nabla}_i\widetilde{\nabla}_i\psi) + (\widetilde{\nabla}_i\psi, e^i.\not{D}\psi) + && \text{(A.85)}\\
&\quad (\psi, c(\nabla_i e^i).\not{D}\psi) + (\psi, \not{D}^2\psi)\}dVol_g\\
&= \{\|\widetilde{\nabla}\psi\|^2 + (\psi, \widetilde{\nabla}_i\widetilde{\nabla}_i\psi) - (e^i.\widetilde{\nabla}_i\psi, \not{D}\psi) + && \text{(A.86)}\\
&\quad (\psi, c(\nabla_i e^i).\not{D}\psi) + (\psi, \not{D}^2\psi)\}dVol_g\\
&= \{\|\widetilde{\nabla}\psi\|^2 + (\psi, \widetilde{\nabla}_i\widetilde{\nabla}_i\psi) - (\not{D}\psi, \not{D}\psi) + && \text{(A.87)}\\
&\quad (\psi, c(\nabla_i e^i).\not{D}\psi) + (\psi, \not{D}^2\psi)\}dVol_g\\
&= \{\|\widetilde{\nabla}\psi\|^2 + (\psi, \widetilde{\nabla}_i\widetilde{\nabla}_i\psi) - \|\not{D}\psi\|^2 + && \text{(A.88)}\\
&\quad (\psi, c(\nabla_i e^i).\not{D}\psi) + (\psi, \not{D}^2\psi)\}dVol_g\\
&= \{\|\widetilde{\nabla}\psi\|^2 - (\psi, -\widetilde{\nabla}_i\widetilde{\nabla}_i\psi) - \|\not{D}\psi\|^2 + (\psi, \not{D}^2\psi) + && \text{(A.89)}\\
&\quad (\psi, \frac{1}{4}R\psi) - (\psi, \frac{1}{4}R\psi) + (\psi, c(\nabla_i e^i).\not{D}\psi)\}dVol_g\\
&= \{\|\widetilde{\nabla}\psi\|^2 - (\psi, \nabla^*\nabla\psi) + (\psi, \frac{1}{4}R\psi) - (\psi, \frac{1}{4}R\psi) - && \text{(A.90)}\\
&\quad \|\not{D}\psi\|^2 + (\psi, \not{D}^2\psi)\}dVol_g\\
&= \{\|\widetilde{\nabla}\psi\|^2 + \frac{R}{4}\|\psi\|^2 - \|\not{D}\psi\|^2\}dVol_g, && \text{(A.91)}
\end{aligned}
$$

using (A.74). This is the identity we wanted to prove. We state it here for clarity.

$$
\{\|\widetilde{\nabla}\psi\|^2 + \frac{R}{4}\|\psi\|^2 - \|\not{D}\psi\|^2\}dVol_g = d\{(\psi, \sigma_{ij}.\widetilde{\nabla}_j\psi)e_i\lrcorner dVol_g\} \tag{A.92}
$$

This is the key to Witten's formula for mass. We use this identity for our spinor described above, i.e. $\psi = \psi_0 + \psi_1$. We integrate this over M and identify the boundary term on the right with the ADM mass. Recall

$$
c(n)mass(g) = \lim_{R\to\infty}\int_{\partial B_R}(\partial_j g_{ij} - \partial_i g_{jj})e_i\lrcorner dVol_g \tag{A.93}
$$

Our next step is to integrate (A.91) with $\psi = \psi_0 + \psi_1$ over the region $M_R = \{r \leq R\}$ and use Stokes Theorem. Note that the left hand side is real, and so we are only concerned with the real part of the right hand side.

$$\int_{M_R} \|\widetilde{\nabla}\psi\|^2 + \frac{R}{4}\|\psi\|^2 - \|\not{D}\psi\|^2\} dVol_g = \int_{M_R} d\{(\psi, \sigma_{ij}.\widetilde{\nabla}_j\psi)e_i \lrcorner dVol_g\} \tag{A.94}$$

$$= \int_{\partial M_R} (\psi, \sigma_{ij}.\widetilde{\nabla}_j\psi)e_i \lrcorner dVol_g \tag{A.95}$$

$$= \int_{\partial M_R} (\psi_0 + \psi_1, \sigma_{ij}.\widetilde{\nabla}_j\psi_0 + \sigma_{ij}.\widetilde{\nabla}_j\psi_1)e_i \lrcorner dVol_g \tag{A.96}$$

We now reference an identity given by Bartnik in [17]. It is the following relationship

$$d\{(\phi, \sigma_{ij}.\chi)(e_i \wedge e_j)\lrcorner dVol_g\} = \{(\phi, \sigma_{ij}.\widetilde{\nabla}_j\chi) - (\sigma_{ij}.\widetilde{\nabla}_j\phi, \chi)\}e_i \lrcorner dVol_g \tag{A.97}$$

The identity in (A.97) give us an equivalent expression for (A.96).

$$\int_{\partial M_R} (\psi_0, \sigma_{ij}.\widetilde{\nabla}_j\psi_0) + d\{(\psi_0, \sigma_{ij}.\psi_1)(e_i \wedge e_j)\lrcorner dVol_g\} + (\psi_1, \sigma_{ij}.\widetilde{\nabla}_j\psi) + (\sigma_{ij}.\widetilde{\nabla}_j\psi_0, \psi_1)e_i \lrcorner dVol_g \tag{A.98}$$

In (A.98) we note that the last two terms will not contribute to the integral at infinity because of the falloff conditions on ψ_1. The second term disappears because $d^2 = 0$. This leaves us, at infinity, with the term $(\psi_0, \sigma_{ij}.\widetilde{\nabla}_i\psi_0)$. Evaluating this term using (A.78) and recalling that ψ_0 is constant at infinity gives us

$$(\psi_0, \sigma_{ij}.\widetilde{\nabla}_j\psi_0) = (\psi_0, \sigma_{ij}d\psi_0 - \frac{1}{4}\sum_{k,l}\omega_{kl}(e_j)\sigma_{ij}c(e_k).c(e_l).\psi_0) \tag{A.99}$$

$$= (\psi_0, -\frac{1}{4}\sum_{k,l}\omega_{k,l}(e_j)\sigma_{ij}.c(e_k).c(e_l).\psi_0) \tag{A.100}$$

$$= (\psi_0, -\frac{1}{4}\sum_{i,j,k,l}\omega_{k,l}(e_j)\sigma_{ij}.\sigma_{kl}.\psi_0) \tag{A.101}$$

$$= -\frac{1}{4}\sum_{i,j,k,l}\omega_{k,l}(e_j)(\psi_0, \sigma_{ij}.\sigma_{kl}.\psi_0) \tag{A.102}$$

Let $\sigma_{ijkl} = c(e_i).c(e_j).c(e_k).c(e_l)$ if $i \neq j \neq k \neq l$ and 0 otherwise. We can easily verify the fact that σ_{ij} is skew hermitian. Also, looking at the left hand side of (A.94) we see that we are only interested in the real part of the right hand side. Therefore, equation (A.102) simplifies to the following.[17]

$$\frac{1}{2}\omega_{ij}(e_j)\|\psi_0\|^2 - \frac{1}{4}\omega_{kl}(e_j)(\psi_0, \sigma_{ijkl}.\psi_0) \tag{A.103}$$

For the second term note that σ_{ijkl} is antisymmetric. This term turns out to be equal to the divergence plus terms on the order or $r^{-2\tau-1}$. Therefore it will not contribute to the boundary integral at infinity.

Now we introduce coordinates that were used by Bartnik.[17]. Let $e_i = e_i^j\partial_j$ be an orthonormal frame near infinity satisfying:

$$e_i^j - \delta_{ij} \in W^{2,\alpha}_{-\tau}(M_R) \tag{A.104}$$

This frame is called asymptotically constant. For the first term in (A.103) we want to write out ω_{ij} in terms of Christoffel symbols and the frame.

$$\omega_{ij}(e_j) = \Gamma_{ijj} + \partial_j(e_j^i) + O(r^{-2\tau-1}) \tag{A.105}$$

Next we want to decompose the frame as $e = (e^i_j) = \delta + s + a$ where s is a symmetric part and a is the antisymmetric part. This gives us the following expression.[17]

$$\omega_{ij}(e_j) = \frac{1}{2}(\partial_j g_{ij} - \partial_i g_{jj}) + \partial_j a_{ji} + O(r^{-2\tau-1}) \tag{A.106}$$

Finally note that

$$\partial_j a_{ji} e_i \lrcorner dVol_g = d(a_{ij}(dx^i \wedge dx^j) \lrcorner dVol_g) + O(R^{-2\tau-1}) \tag{A.107}$$

Therefore, taking $R \to \infty$ in (A.94)-(A.96), recalling (A.103), (A.106)-(A.107) and the fact that our spinor ψ is in the kernel of the Dirac operator gives us

$$\int_M \|\widetilde{\nabla}\psi\|^2 + \frac{R}{4}\|\psi\|^2 dVol_g = \lim_{R\to\infty} \frac{1}{4} \int_{\partial M_R} (\partial_j g_{ij} - \partial_i g_{jj}) e_i \lrcorner dVol_g = \frac{c(n)}{4} mass(g) \tag{A.108}$$

We note that under the decay conditions of asymptotic flatness, this integral converges.

Vita

Joshua Michael Lipsmeyer was born on October 29, 1983 in Bigelow, Arkansas. He was raised in a small town by loving parents Taneau and Gary Lipsmeyer. He was the oldest of four children, with siblings Brandi Edwards, Joey Lipsmeyer, and Shannon Howell. Josh received his high school diploma from Bigelow High School in 2002 with honors. After a successful career in industry, Josh went on to pursue undergraduate studies at the University of Arkansas at Little Rock where he graduated Magna Cum Laude with a Bachelor of Science degree in mathematics with a minor in physics. In 2012, Josh received a graduate teaching assistantship from the University of Tennessee. There, his area of study was geometric analysis and general relativity under the guidance of Dr. Alex Freire. In August 2015 he was awarded his Masters of Science in mathematics.