12-2013

The Green's Function Method for Solutions of Fourth Order Nonlinear Boundary Value Problem.

Olga A. Teterina
University of Tennessee - Knoxville, oteterin@utk.edu

To the Graduate Council:

I am submitting herewith a thesis written by Olga A. Teterina entitled "The Green's Function Method for Solutions of Fourth Order Nonlinear Boundary Value Problem.." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Don B. Hinton, Major Professor

We have read this thesis and recommend its acceptance:

Judy D. Day, Henry C. Simpson

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

The Green's Function Method for Solutions of Fourth Order Nonlinear Boundary Value Problem

A Thesis Presented for the
Master of Science
Degree
The University of Tennessee, Knoxville

Olga A. Teterina
December 2013

Abstract

Title: The Green's Function Method for Solutions of Fourth Order Nonlinear Boundary Value Problem.

This thesis has demonstrated that Green's functions have a wide range of applications with regard to boundary value problems. In particular, existence and uniqueness of solutions of a large class of fourth order boundary value problems has been established. In fact, given any fourth order ODE with homogeneous boundary conditions, as long as the corresponding Green's function exists and f satisfies an appropriate Lipschitz condition, Theorem 2.1 guarantees such a solution under equally mild conditions. Similarly, Theorem 2.2 also guarantees such a solution under equally mild conditions. These theorems are contrasted with classical ODE existence theorems in that they get around the use of classical convergence analysis by assuming the existence of the Green's function. Banach techniques are still used, but the existence of the Green's function is the primary tool in showing existence and uniqueness. This requires, of course, that the Green's function exists for particular problem, but the examples in Section 4 show that this s usually not a severe restriction.

However, as mild as the restrictions seem to be, one should pay particular detail to the range of values on the Lipschitz constant(s). The Lipschitz constants corresponding to f must satisfy an inequality involving bounds on integrals of G and its derivatives, which, if G is badly behaved, may be a severe restriction. The examples of Section 4 illustrate these ideas. For example, Theorems 4.1-4.2 are specific cases in which Theorem 2.2 is applicable.

Table of Contents

Section 1: Introduction and Preliminaries

Introduction

Fourth order differential equations are models for bending or deformation of elastic beams in equilibrium state, whose two ends are simply supported and therefore have important applications in engineering and physical science. The solutions of fourth order boundary value problems are the subject of this work. In particular, the focus will be on applications of the corresponding Green's functions and the resulting qualitative properties of the solutions. After establishing the basic definitions, the first step is to guarantee that solutions to the equations in questions do, in fact, exist and are unique. Thus, a sequence of general uniqueness and existence theorems for fourth order differential equations with homogeneous boundary conditions is established. Next, an extensive collection of Green's functions is derived, many of which are used later on to illustrate various applications. Finally, specific existence and uniqueness theorems are proved by applying many of the previous results. We follow the method of [1] for 2^{th} order equations and that of [4] for 3^{th} order equations.

Preliminary Definition and Theorem

This section will introduce the basic definitions, theorems, and conclusions that will be used throughout.

Definition 1.1 (Norms and Normed Space)**.** Let X be a vector space(=linear space) over the field $\mathbb{C}$ of complex scalars. Then X is a *normed linear space* if for every $f \in X$ there is a real number $\|f\|$, called the *norm* of f, such that:
(a) $\|f\| \geqslant 0$,
(b) $\|f\| = 0$ if and only if $f = 0$
(c) $\|cf\| = |c|\|f\|$ for every scalar c, and
(d) $\|f + g\| \leqslant \|f\| + \|g\|$.

Definition 1.2 (Convergent and Cauchy Sequence)**.** Let X be a normed space, and let$\{f_n\}_{n\in\mathbb{N}}$ be a sequence of elements of X.

(a) $\{f_n\}_{n\in\mathbb{N}}$ converges to $f\in X$ if $\lim_{n\to\infty}\|f - f_n\| = 0$, i.e., if

$$\forall\,\varepsilon > 0\,, \quad \exists\,N > 0, \quad \forall\,n \geqslant N, \quad \|f - f_n\| < \varepsilon.$$

In this case, we write $\lim_{n\to\infty} f_n = f$.

(b) $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy if

$$\forall\,\varepsilon > 0\,, \quad \exists\,N > 0, \quad \forall\,n, m \geqslant N, \quad \|f_m - f_n\| < \varepsilon.$$

Definition 1.3 (Banach Space)**.** A Banach Space, $(X, \|\cdot\|)$ is a complete normed linear space. Let $B = C[a, b]$ with the supreme norm, denoted by $\|\cdot\|_\infty$. Then B is a Banach space. It will be beneficial, however, to use a variation of this norm on some subspace of B. An example of such a space that will be used frequently is given by the following.

Theorem 1.1. *Let w in $C^{(1)}[a, b]$ be a fixed function such that $w(a) = w'(a) = w(b) = w'(b) = 0$ and $w(x) > 0$ for $a < x < b$. Let*

$$B^* = \{u \in B : |u(x)| \leqslant Cw(x) \text{ for some } C = C(u) > 0\}.$$

For $u \in B^*$, define

$$\|u\|^* = \sup_{a<x<b} \frac{|u(x)|}{w(x)}.$$

Then $\|\cdot\|^*$ is a norm on B^* and $(B^*, \|\cdot\|^*)$ is a Banach space.

Proof. It is easy to see that B^* is a subspace of B. By the definition of B^*, u(a)=u(b)=0 for any $u \in B^*$. Moreover, if $\|u(x)\|^* = 0$,then

$$\frac{|u(x)|}{w(x)} \leqslant \sup_{a<x<b} \frac{|u(x)|}{w(x)} = 0 \implies u(x) = 0.$$

The triangle inequality and the fact that scalars can be factored out of $\|\cdot\|^*$ follow easily from the definition. Thus, $\|\cdot\|^*$ is a norm on B^*. Now, it must be shown that B^* is complete. Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(B^*, \|\cdot\|^*)$. Let $M = \sup_{a\leqslant x\leqslant b} w(x)$. Let $\epsilon > 0$ be given. There exists an $N \in \mathbb{N}$ such that for any $n, m \geqslant N$, $\|u_n - u_m\|^* < \frac{\epsilon}{M}$. If n, $m \geqslant N$, then for $a < x < b$ we have $1 \leqslant \frac{M}{w(x)}$, so that

$$|u_n(x) - u_m(x)| \leqslant \sup_{a<x<b} \frac{M|u_n(x) - u_m(x)|}{w(x)} = M\|u_n - u_m\|^* < \epsilon.$$

Thus, $|u_n(x) - u_m(x)| \leqslant \epsilon \,\forall\, x \in [a, b] \implies \|u_n - u_m\|_\infty < \epsilon \quad \forall\, n, m \geqslant N$. Since $\epsilon > 0$ is arbitrary, $\{u_n\}$ is a Cauchy sequence in B, showing that a Cauchy sequence in B^* is also Cauchy in B. B is complete, implying that there exists a $u \in C[a, b]$ such that $u_n \to u$ in $(B, \|\cdot\|)$.

Choose an $x \in (a, b)$. Since $u_n \to u$ in $\|\cdot\|_\infty$, there exists an $N \in \mathbb{N}$ such that $n \geqslant N \implies \|u_n - u\|_\infty < w(x)$. For any $n \geqslant N$,

$$|u_n(x) - u(x)| \leqslant \sup_{a\leqslant x\leqslant b} |u_n(x) - u(x)| = \|u_n - u\|_\infty < w(x).$$

which, along with the reverse triangle inequality, implies

$$\frac{|u(x)|}{w(x)} < \frac{|u_n(x)|}{w(x)} + 1 \leqslant \sup_{a<x<b} \frac{|u_n(x)|}{w(x)} + 1 = \|u_n\|^* + 1.$$

So, $\frac{|u(x)|}{w(x)} < \|u_n\|^* + 1$. Since Cauchy sequences are bounded there exists a constant $K > 0$ such that $\|u_n\|^* \leqslant K$. Hence,

$$\sup_{a<x<b} \frac{|u_n(x)|}{w(x)} \leqslant K + 1 \implies \|u\|^* \leqslant K + 1 < \infty \implies u \in B^*.$$

Finally, for $\epsilon > 0$, there exists an $N \in \mathbb{C}$ such that for $n, m \geqslant N$ and for all $x \in (a, b)$,

$$\frac{|u_n(x) - u_m(x)|}{w(x)} \leqslant \|u_n - u_m\|^* < \epsilon.$$

Let $m \to \infty$ in the previous inequality. Then for $n \geqslant N$ and $\forall\, x$, it follows that

$$\frac{|u_n(x) - u(x)|}{w(x)} \leqslant \epsilon.$$

Thus, $\|u_n - u\|^* \leqslant \epsilon$ for $n \geqslant N$, showing that $u_n \to u$ in $(B^*, \|\cdot\|^*)$. $\square$

Theorem 1.1 can be generalized for a non-identically zero function $w \in C[a, b]$ such that $w(x) \geqslant 0$ on $[a, b]$. Let $B_w = B^*$ in Theorem 1.1. A norm on B_w can be defined by $\|u\|^* = \sup_{x \in S_w} \frac{|u(x)|}{w(x)}$ where $S_w = \{x : w(x) \neq 0\}$. The preceding proof applies without change to show that B_w is complete under $\| \cdot \|^*$. Typically in the following applications, however, $w(x) > 0$ on $a < x < b$.

The *Contraction Mapping Principle* which we study in the present section applies in many function spaces. In particular,it implies the initial value problem for differential equations, under mild hypotheses, has a unique solution.

Contraction Mapping Theorem. Let $T : B \to B$ be a continuous map from the Banach space, B , into itself such that for all $u, v \in B$,

$$\|T(u) - T(v)\| \leqslant \theta \|u - v\|$$

for some fixed $\theta \in (0, 1)$. Then T has a unique fixed point u_0, i.e., $T(u_0) = u_0$ and $T(u) = u$ if and only if $u = u_0$.

The Contraction Mapping Theorem is an important tool in proving existence and uniqueness of solutions to ordinary differential equations, as will be seen later.

The following is some basic material from the theory of ordinary differential equations and boundary value problems. The definitions and the proofs of Theorem 1.2 and 1.3 are given in Walter([Wa],Ch.6). For notational purposes, arbitrary differential operators will be denoted by D.

Definition 1.4. The linear fourth order separated boundary value problem is defined as

$$(Du)(x) := (p(x)u''(x))'' + q(x)u(x) = g(x), \; x \in [a, b], \tag{1.1}$$

with linearly independent separated boundary conditions

$$\begin{aligned} R_1 u &:= \alpha_1 u(a) + \alpha_2 u'(a) + \alpha_3 p(a)u''(a) + \alpha_4 (pu'')'(a) = \eta_1, \\ R_2 u &:= \beta_1 u(a) + \beta_2 u'(a) + \beta_3 p(a)u''(a) + \beta_4 (pu'')'(a) = \eta_2, \\ R_3 u &:= \gamma_1 u(b) + \gamma_2 u'(b) + \gamma_3 p(b)u''(b) + \gamma_4 (pu'')'(b) = \eta_3, \\ R_4 u &:= \xi_1 u(b) + \xi_2 u'(b) + \xi_3 p(b)u''(b) + \xi_4 (pu'')'(b) = \eta_4. \end{aligned} \tag{1.2}$$

assuming that $p \in C^{(2)}[a,b]$ and $q, g \in C^{(0)}[a,b]$ are real-valued functions, that $p(x) > 0$ in [a,b], and that α_i, β_i, γ_i, ξ_i, η_i, $i = 1, 2, 3, 4$ are real constants. The corresponding homogeneous boundary value problem is given by

$$Du = 0 \text{ on } [a,b], \tag{1.3}$$

$$R_1 u = R_2 u = R_3 u = R_4 u = 0. \tag{1.4}$$

Theorem 1.2. *Let $u_1(x)$, $u_2(x)$, $u_3(x), u_4(x)$ be a fundamental system of solutions to the homogeneous differential equation $Du = 0$. The inhomogeneous boundary value problem, (1.1), with boundary conditions, (1.2), is uniquely solvable if and only if the homogeneous problem, (1.3), (1.4), has only the zero solution $u \equiv 0$. The latter is true if and only if the determinant of*

$$[R_i u_j]_{i,j=1}^{4} \text{ is nonzero.}$$

Moreover, the determinant condition does not depend on the choice of fundamental system.

Consequently, it is sufficient to solve (1.1) with the homogeneous boundary conditions, (1.4), instead of (1.1), (1.2). To illustrate this, suppose a function, $w(x)$ in $C^{(2)}[a,b]$ can be found that satisfies (1.2). If v satisfies $Dv = g(x) - Dw$ and (1.4), then $u = v + w$ satisfies $Du = Dv + Dw = g(x)$ and (1.2). Finding such a function, w, is not typically a problem and, in fact, w can be chosen to be a polynomial as in the examples given later.

Theorem 1.3. *Assume $p \in C^{(2)}[a,b]$, $q, g \in C^{(0)}[a,b]$ are real valued functions, and $p(x) > 0$ in $[a,b]$. If the homogeneous boundary value problem*

$$Du = 0 \;\; on\; [a,b], \qquad R_1 u = R_2 u = R_3 u = R_4 u = 0, \tag{1.5}$$

has only the trivial solution(i.e., if the determinant given in Theorem 1.2 is nonzero) then the Green's function for this boundary value problem exists and is unique.
The solution of the "semihomogeneous" boundary value problem

$$Du = g(x) \; on\; [a,b], \quad R_1 u = R_2 u = R_3 u = R_4 u = 0 \tag{1.6}$$

which is unique by Theorem 1.2, is given by

$$u(x) = \int_a^b G(x,s) g(s) ds. \tag{1.7}$$

The focus of this work will be on fourth order ordinary differential equations,

$$y^{(4)} = f(x, y(x), y'(x), y''(x), y'''(x)),$$

satisfying a Lipschitz conditions of the form

$$|f(x, u(x), u'(x), u''(x), u'''(x)) - f(x, v(x), v'(x), v''(x), v'''(x))| \leqslant$$
$$L|u(x) - v(x)| + K|u'(x) - v'(x)| + M|u''(x) - v''(x)| + N|u'''(x) - v'''(x)|, \quad (1.8)$$

where K, L, M and N are fixed positive constants. At a later point, these constants will be replaced by functions h(x), k(x), r(x) and s(x), giving a more general Lipschitz condition.

Theorems 1.2 and 1.3 have extensions for differential system of arbitrary order. In particular, if D is a linear operator of order n, then the nonhomogeneous problem $Du = g(x)$ with n linearly independent boundary conditions has a unique solution if and only if the corresponding homogeneous problem has only the zero solution. In this case, the solution of the nonhomogeneous problem has the representation (1.7), although the Green's function is in general not symmetric. By symmetry we mean $G(x, s) = G(s, x)$ for all $x,\ s \in [a, b]$.

Section 2: Existence and Uniqueness

The following two theorems will form the foundation for what is to come. The remainder of this work will consist of applications of the following existence and uniqueness theorems. The proofs are generalizations of the corresponding second order theorems, which are proved in Bailey,([1], Ch.3).

Consider the fourth order differential equation

$$u^{(4)}(x) \;=\; f(x, u(x), u'(x), u''(x), u'''(x)) \tag{1.9}$$

with linearly independent boundary conditions

$$\begin{aligned} R_1 u \;&:=\; \alpha_1 u(a) \;+\; \alpha_2 u'(a) \;+\; \alpha_3 u''(a) \;+\; \alpha_4 u'''(a) \;=\; 0 \\ R_2 u \;&:=\; \beta_1 u(a) \;+\; \beta_2 u'(a) \;+\; \beta_3 u''(a) \;+\; \beta_4 u'''(a) = \; 0 \\ R_3 u \;&:=\; \gamma_1 u(b) \;+\; \gamma_2 u'(b) \;+\; \gamma_3 u''(b) \;+\; \gamma_4 u'''(b) \;=\; 0 \\ R_4 u \;&:=\; \xi_1 u(b) \;+\; \xi_2 u'(b) \;+\; \xi_3 u''(b) \;+\; \xi_4 u'''(b) \;=\; 0. \end{aligned} \tag{1.10}$$

A test for linear independence is given by Coddington ([3]) to be

$$rank \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & 0 & 0 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ 0 & 0 & 0 & 0 & \xi_1 & \xi_2 & \xi_3 & \xi_4 \end{pmatrix} = 4. \tag{1.11}$$

The boundary conditions space, denoted by S, consists of all u in $C^{(4)}[a, b]$ that satisfy the boundary conditions. Various norms will be assigned throughout to make S a subspace of a Banach space. The following theorem is proved for fourth order equations, though the conclusion holds for other order equations as well.

Theorem 1.4. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfy*

$$|f(x, u_2) \;-\; f(x, u_1)| \;\leqslant\; h(x)|u_2 \;-\; u_1|$$

for some nonnegative continuous function h. Suppose the Green's function G for the boundary value problem $u^{(4)}(x) = g(x)$ and (2.2) exists. Define the operator,
$T : C[a,b] \to S \subset C[a,b]$,by

$$(Tu)(x) = \int_a^b G(x,s) f(s,u(s)) ds.$$

Suppose w is a fixed nontrivial element of $C[a,b]$ with $w(x) \geqslant 0$. Suppose also $T : B_w \to B_w$ where the Banach space B_w is described in the remarks following Theorem 1.1.

a) If the Green's function, G, is of constant sign, and

$$\max_{x \in S_w} \left[\frac{z(x)}{w(x)} \right] < 1,$$

where z is defined by $z(x) = \int_a^b |G(x,s)| h(s) w(s) ds$ and $S_w = \{x \in [a,b] : w(x) \neq 0\}$, then (2.1), (2.2) has a unique solution. Further z satisfies $z^{(4)}(x) = (sign\ G) h(x) w(x)$ with boundary conditions (2.2).

b) If G is possibly not of constant sign and

$$\max_{x \in S_w} \left[\frac{1}{w(x)} \int_a^b |G(x,s)| h(s) w(s) ds \right] < 1,$$

then (2.1), (2.2) has a unique solution.

Proof. **(a)** Consider the case where G is negative (the proof for positive G is similar). Let $\|u\|^* = \sup_{a<x<b} \frac{|u(x)|}{w(x)}$ denote the norm that was defined in Theorem 1.1, but with the maximum taken over S_w. Then

$$\begin{aligned}
|(Tu)(x) - (Tv)(x)| &= |\int_a^b G(x,s)[f(s,u(s)) - f(s,v(s))]ds| \\
&\leqslant \int_a^b |G(x,s)||u(s) - v(s)|h(s)ds \\
&\leqslant \int_a^b \|u - v\|^* |G(x,s)|h(s)w(s)ds \\
&= \|u - v\|^* z(x).
\end{aligned}$$

From the definition of $z(x)$ and the fact that G is a Green's function, it follows that $z^{(4)}(x) = -h(x)w(x)$ with boundary conditions (2.2). Now for $x \in S_w$

$$\frac{|(Tu)(x) - (Tv)(x)|}{w(x)} \leqslant \frac{z(x)\|u - v\|^*}{w(x)}.$$

This implies

$$\|Tu - Tv\|^* \leqslant \|u - v\|^* \max_{x\in S_w} \frac{z(x)}{w(x)},$$

where $\max_{x\in s_w} \frac{z(x)}{w(x)} < 1$ by hypothesis, proving that T is a contraction on B_w which yields a unique fixed point that is the solution of (2.1), (2.2). This proves part (a).

(b) If G is possibly not of one sign, then for $x \in S_w$,

$$\frac{|(Tu)(x) - (Tv)(x)|}{w(x)} \leqslant \|u - v\|^* \frac{1}{w(x)} \int_a^b |G(x,s)|h(s)w(s)ds.$$

Thus,

$$\|Tu - Tv\|^* \leqslant \|u - v\|^* \max_{x\in S_w} \frac{1}{w(x)} \int_a^b |G(x,s)|h(s)w(s)ds.$$

The maximum is less than 1 by hypothesis, so T is a contraction, which yields a unique fixed point that is a solution of (2.1), (2.2). □

Two cases of Theorem 2.1 are needed because if G is of constant sign, the function $z(x)$ is much easier to compute by solving the differential equation $z^{(4)}(x) = \text{sign}(G)h(x)w(x)$. One of the obstacles that can arise in applying Theorem 2.1 is confirming the hypothesis that T maps B_w into B_w. In many cases, $w(x) = 1$, in which case $B_w = C[a,b]$ and $T : B_w \rightarrow B_w$ clearly holds. In general, though, for w having zeros in $[a,b]$, the properties of the corresponding Green's function must be used to establish that this hypothesis holds.

Example of calculating a function h(x).

Consider a function of the two variables $f(x,y) = x^2 y \sin(e^x)$ on $[0,1]$.

By the Mean Value Theorem, for all $x, y \in \mathbb{R}$ (say $y_1 < y_2$)), we have a number $\xi \in (y_1, y_2)$ such that

$$|f(x,y_2) - f(x,y_1)| = \left.\frac{\partial f}{\partial y}\right|_{y=\xi} |y_2 - y_1|.$$

So, $h(x) = \max|\frac{\partial f}{\partial y}|$ on $1 \leqslant x \leqslant 1 \times (-\infty, \infty) \Rightarrow h(x) = f_y = x^2 \sin(e^x) \leqslant x^2$

It will be beneficial, particularly for the examples and applications that will be presented later, to state the analog of Theorem 2.1 for more general fourth order equations.

In Chapter 4, examples will be given showing how the function w may be chosen and how the existence of solutions depends on this choice.

Theorem 1.5. *Let $f : [a,b] \times \mathbb{R}^5 \rightarrow \mathbb{R}$ satisfy General Lipschitz Condition* (1.8)*. Suppose the Green's function $G(x,s), a \leqslant x, s \leqslant b$ exists for boundary value problem $u^{(4)}(x) = g(x)$ and* (2.2)*. Suppose further that there exist constants M_1, M_2, M_3, M_4, such that for all $x \in [a,b]$*

$$\int_a^b |G(x,s)|ds \leqslant M_1, \quad \int_a^b |G_x(x,s)|ds \leqslant M_2, \quad \int_a^b |G_{xx}(x,s)|ds \leqslant M_3, \quad \int_a^b |G_{xxx}(x,s)|ds \leqslant M_4.$$

Assume also that $LM_1 + KM_2 + MM_3 + NM_4 < 1$. Then there exists a unique solution to the boundary value problem

$$y^{(4)}(x) = f(x, y(x), y'(x), y''(x), y'''(x)), \qquad x \in [a,b],$$

with boundary conditions (2.2)*.*

Proof. Let $\|u\|_3 := \max_{a\leqslant x\leqslant b}[L|u(x)| + K|u'(x)| + M|u''(x)| + N|u'''(x)|]$ be the norm on $C^{(3)}[a,b]$ so that $C^{(3)}[a,b]$ is a Banach space. Define the operator $T : C^{(3)}[a,b] \to C^{(4)}[a,b]$ by

$$y = Ty = \int_a^b G(x,s)f(s,y(s),y'(s),y''(s),y'''(s))ds.$$

To see that T does, indeed, map into $C^{(4)}[a,b]$, note first that the differentiability of G allows differentiation under integral sign. Hence,

$$\begin{aligned}
(Tu)'(x) &= \int_a^b G_x(x,s)f(s,u(s),u'(s),u''(s),u'''(s))ds, \\
(Tu)''(x) &= \int_a^b G_{xx}(x,s)f(s,u(s),u'(s),u''(s),u'''(s))ds, \\
(Tu)'''(x) &= \int_a^b G_{xxx}(x,s)f(s,u(s),u'(s),u''(s),u'''(s))ds,
\end{aligned}$$

and the fact that the Green's function exists gives

$$(Tu)^{(4)}(x) = f(s,u(s),u'(s),u''(s),u'''(s)).$$

Now it must be shown that T is a contraction map.

$$\begin{aligned}
|Tu(x) - Tv(x)| &\leqslant \int_a^b |G(x,s)||f(s,u(s),u'(s),u''(s),u'''(s)) - f(s,v(s),v'(s),v''(s),v'''(s))|ds \\
&\leqslant \int_a^b |G(x,s)|[L|u(s) - v(s)| + K|u'(s) - v'(s)| \\
&\quad + M|u''(s) - v''(s)| + N|u'''(s) - v'''(s)|]ds \\
&\leqslant \|u - v\|_3 \int_a^b |G(x,s)|ds \\
&\leqslant \|u - v\|_3 M_1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|(Tu)'(x) - (Tv)'(x)| &\leqslant \|u - v\|_3 \int_a^b |G_x(x,s)|ds \leqslant \|u - v\|_3 M_2, \\
|(Tu)''(x) - (Tv)''(x)| &\leqslant \|u - v\|_3 \int_a^b |G_{xx}(x,s)|ds \leqslant \|u - v\|_3 M_3,
\end{aligned}$$

and

$$|(Tu)'''(x) - (Tv)'''(x)| \leqslant \|u - v\|_3 \int_a^b |G_{xxx}(x,s)|ds \leqslant \|u - v\|_3 M_4.$$

Since x is arbitrary in the previous inequalities, it follows by multiplying the four inequalities by L, K, M, N respectively that

$$\|Tu - Tv\|_3 \leqslant \|u - v\|_3(LM_1 + KM_2 + MM_3 + NM_4).$$

By hypothesis, $LM_1 + KM_2 + MM_3 + NM_4$ is less than 1. Therefore, T is a contraction from the complete space, $C^{(3)}[a,b]$ into itself. Consequently, it has unique fixed point, u, which is the desired solution. $\square$

Example of calculating constants L,K,M,N.

Consider a function $f(x,u,u',u'',u''') = e^x u + \cos(u') - 10u'' + xu'''$ on $[0,1] \times \mathbb{R}^4$. By the Mean Value Theorem

$$\begin{aligned}
\frac{\partial f}{\partial u} &= e^x \leqslant e = L. \\
|\frac{\partial f}{\partial u'}| &= |-\sin(u')| \leqslant 1 = K. \\
|\frac{\partial f}{\partial u''}| &= |-10| = 10 = M. \\
\frac{\partial f}{\partial u'''} &= x \leqslant 1 = N.
\end{aligned}$$

Now that existence and uniqueness of solutions have been established, the next step is to derive the corresponding Green's functions.

Section 3: Green's Functions and First Eigenvalues

This section presents the derivation of Green's function for fourth order boundary value problems. This facilitates the transition into the last part of the chapter, where the relation between these Green's functions and the eigenvalues of the corresponding differential equations is illustrated through some examples. Also in this section, the general interval, [a,b], will be replaced by $[0, a]$ for $a > 0$.

Example 3.1. Consider the following boundary value problem

$$u^{(4)}(x) = g(x) \tag{1.12}$$

with boundary conditions

$$u(0) = u'(0) = u(a) = u'(a) = 0. \tag{1.13}$$

We can rewrite the inhomogeneous problem (3.1) as:

$$D[u] \equiv u^{(4)}(x) = g(x), \qquad \text{for } 0 \leqslant x \leqslant a,$$

subject to the homogeneous boundary conditions

$$B_1[u] \equiv u(0) = 0, \quad B_2[u] \equiv u'(0) = 0, \quad B_3[u] \equiv u(a) = 0 \quad \text{and} \quad B_4[u] \equiv u'(a) = 0.$$

The Green's function $G(x, s)$ is defined as the solution to

$$G_{xxxx}(x, s) = \delta(x - s) \quad \text{subject to} \quad G(0, s) = G_x(0, s) = 0 \quad \textit{and} \quad G(a, s) = G_x(a, s) = 0,$$

where δ is the Dirac delta function.

We can represent the solution to the above inhomogeneous problem as an integral involving the Green's function. To show that

$$u(x) = \int_0^a G(x, s)g(s)ds$$

is the solution, we apply the differential operator to the integral. (Assuming that the integral is uniformly convergent.)

$$\begin{aligned} D\Big[\int_0^a G(x,s)g(s)ds\Big] &= \int_0^a D[G(x,s)]g(s)ds \\ &= \int_0^a \delta(x - s)g(s)ds \\ &= g(s) \end{aligned}$$

The integral also satisfies the boundary conditions for $i = 1, 2, 3, 4$.

$$\begin{aligned} B_i\Big[\int_0^a G(x,s)g(s)ds\Big] &= \int_0^a B_i[G(x,s)]g(s)ds \\ &= \int_0^a (0)g(s)ds \\ &= 0 \end{aligned}$$

One of the advantage of using Green's function is that once you find the Green's function for a differential operator and certain homogeneous boundary conditions,

$$D[G] = \delta(x - s), \qquad B_1[G] = B_2[G] = B_3[G] = B_4[G] = 0.$$

you can write the solution for any inhomogeneity g(x), i.e.,

$$D[f] = g(x), \quad B_1[u] = B_2[u] = B_3[u] = B_4[u] = 0.$$

Before calculating $G(x,s)$, let's see what general conditions it must satisfy for any arbitrary boundary conditions.

(1). For each x and s , $G(x,s)$ satisfies $G_{xxxx} = 0$ except when $x = s$.
(2). G satisfies the boundary conditions $G(0,s) = G_x(0,s) =$
$= G(a,s) = G_x(a,s) = 0$.
(3). G(x,s) must be continuous at all x, as well as its derivatives up to 2^{th} order.
$G(s+,s) - G(s-,s) = 0$, $G_x(s+,s) - G_x(s-,s) = 0$,
and $G_{xx}(s+,s) - G_{xx}(s-,s) = 0$.
(4). At $x = s$, the 3^{th} derivative of G must have a jump discontinuity of magnitude -1 in order that the 3^{th} term match the delta function, i.e.,
$G_{xxx}(s+,s) - G_{xxx}(s-,s) = 1$.

Now, knowing the properties of the Green's function we can construct such a function. It is not hard to see that linear independent solutions for the DE (3.1) are $1,\ x,\ x^2,\ x^3$ and $1,\ a - x$,
$(a - x)^2,\ (a - x)^3$. Thus,the Green's function, $G(x, s)$, satisfies the condition (1) above if

$$G(x,s) = \begin{cases} A_1 + A_2x + A_3x^2 + A_4x^3, & \text{if } 0 \leqslant x \leqslant s \leqslant a \\ B_1 + B_2(a-x) + B_3(a-x)^2 + B_4(a-x)^3 & \text{if } 0 \leqslant s < x \leqslant a \end{cases} \tag{1.14}$$

where $A_1,\ A_2,\ A_3,\ A_4$ and $B_1,\ B_2,\ B_3,\ B_4$ are functions of s. Knowing that the Green's function $G(x, s)$ satisfies the BVP with homogeneous BCs by the condition (2) we have

$$A_1 = A_2 = B_1 = B_2 = 0.$$

We deduce that the Green's function for the problem is

$$G(x,s) = \begin{cases} A_3x^2 + A_4x^3, & \text{if } 0 \leqslant x \leqslant s \leqslant a \\ B_3(a-x)^2 + B_4(a-x)^3 & \text{if } 0 \leqslant s < x \leqslant a \end{cases} \tag{1.15}$$

The continuity conditions (3) yield the following equations

$$\begin{cases} B_3(a-s)^2 + B_4(a-s)^3 = A_3s^2 + A_4s^3 \\ -2B_3(a-s) - 3B_4(a-s)^2 = 2A_3 + 3A_4s^2 \\ 2B_3 + 6B_4(a-s) = 2A_3 + 6A_4s \end{cases} \tag{1.16}$$

And the jump condition (4) gives us the equation

$$-6B_4 - 6A_4 = 1. \tag{1.17}$$

We can find the coefficients $A_3,\ A_4,\ B_3,\ B_4$ solving equations (3.5) and equation (3.6) simultaneously using Maple.

$$A_3 = \frac{s(a^2 - 2as + s^2)}{2a^2}, \quad A_4 = -\frac{2as^3 - 3as^2 + a^3}{6a^3},$$
$$B_3 = -\frac{s^2(s-a)}{2a^2}, \quad \text{and} \quad B_4 = \frac{s^2(2s-3a)}{6a^3}.$$

And finally, substituting the found coefficients into equation (3.4) we arrive to the expression of a Green's function

$$G(x,s) = \begin{cases} x^2(s-a)^2(3as-2sx-xa)/6a^3, & \text{if } 0 \leqslant x \leqslant s \leqslant a \\ -s^2(a-x)^2(as+2sx-3xa)/6a^3, & \text{if } 0 \leqslant s < x \leqslant a \end{cases} \tag{1.18}$$

The Green's function for the Example 3.1 turned out to be symmetric, i.e., $G(x,s) = G(s,x)$. It is also nonnegative as may be verified by calculus to show that

$$f_s(x) := 3as - 2sx - ax \geqslant 0 \quad \text{on} \quad 0 \leqslant x \leqslant s.$$

Example 3.2. Consider the following boundary value problem

$$u^{(4)}(x) = g(x)$$

with boundary conditions

$$u(0) = u'(0) = u'(a) = u'''(a) = 0.$$

Starting again with the general expression of a Green's function for the homogeneous 4$^{\text{th}}$ order differential equation.

$$G(x,s) = \begin{cases} A_1 + A_2 x + A_3 x^2 + A_4 x^3, & \text{if } 0 \leqslant x \leqslant s \leqslant a \\ B_1 + B_2(a-x) + B_3(a-x)^2 + B_4(a-x)^3 & \text{if } 0 \leqslant s < x \leqslant a \end{cases} \tag{1.19}$$

Applying boundary conditions we find that

$$A_1 = A_2 = B_2 = B_4 = 0.$$

This reduces the expression for a Green's function to the following

$$G(x,s) = \begin{cases} A_3 x^2 + A_4 x^3, & \text{if } 0 \leqslant x \leqslant s \leqslant a \\ B_1 + B_3(a-x)^2, & \text{if } 0 \leqslant s < x \leqslant a \end{cases} \tag{1.20}$$

The continuity conditions yield the following equations

$$\begin{cases} A_3 s^2 + A_4 s^3 = B_1 + B_3(a-s)^2 \\ 2A_3 s + 3A_4 s^2 = -2B_3(a-s) \\ 2A_3 + 6A_4 s = 2B_3. \end{cases} \tag{1.21}$$

And the jump condition gives us the equation

$$-6A_4 = 1. \tag{1.22}$$

Again, we can find the coefficients A_3, A_4, B_1, B_3 solving equations (3.10) and equation (3.11) simultaneously using Maple.

$$A_3 = -\frac{s(s-2a)}{4a}, \quad A_4 = -\frac{1}{6}, \quad B_1 = -\frac{1}{6}s^3 + \frac{1}{4}s^2 \quad \text{and} \quad B_3 = -\frac{s^2}{4a}.$$

And substituting the found coefficients into equations (3.9) we arrive at the expression of the Green's function

$$G(x,s) = \begin{cases} -s^2(2as - 6ax + 3x^2)/12a, & \text{if } 0 \leqslant s < x \leqslant a \\ -x^2(3s^2 - 6as + 2ax)/12a, & \text{if } 0 \leqslant x \leqslant s \leqslant a \end{cases} \tag{1.23}$$

The Green's function for the Example 3.2 turned out to be symmetric and nonnegative as well, i.e., $G(x,s) = G(s,x)$.

Example 3.3. Consider the following boundary value problem

$$u^{(4)}(x) = g(x)$$

with boundary conditions

$$u(0) = u'(0) = u''(a) = u'''(a) = 0.$$

Following the same procedure as in examples (3.1) and (3.2) we find the Green's function to be

$$G(x,s) = \begin{cases} x^2(3s - x)/6, & \text{if } 0 \leqslant x \leqslant s \leqslant a \\ -s^2(s - 3x)/6, & \text{if } 0 \leqslant s < x \leqslant a \end{cases} \tag{1.24}$$

The Green's function for the Example 3.3 turned out to be symmetric and nonnegative as well, i.e., $G(x,s) = G(s,x)$.

Example 3.4. Consider the following boundary value problem

$$u^{(4)}(x) = g(x)$$

with boundary conditions

$$u(0) = u''(0) = u(a) = u''(a) = 0.$$

In this case of homogeneous boundary conditions the Green's function is

$$G(x,s) = \begin{cases} x(s-a)(s^2-2as+x^2)/6a, & \text{if } 0 \leqslant x \leqslant s \leqslant a \\ s(a-x)(s^2-2ax+x^2)/6a, & \text{if } 0 \leqslant s < x \leqslant a \end{cases} \tag{1.25}$$

And the Green's function possesses the symmetry property as well as in all previous examples, i.e.,

$G(x,s) = G(s,x)$.

The Green's functions in given examples could have been derived for $a = 1$. We can show that it gives us a general case.

The general form of the Green's function for 4^{th} order nonlinear differential equations $y^{(4)} = g(x)$ with some homogeneous boundary conditions is $y(x) = \int_0^a G_a(x,s)g(s)ds$. We rescale the variable $y(x)$ so that

$$\tilde{y}(\xi) = y(a\xi), \quad 0 \leqslant \xi \leqslant 1, \quad x = a\xi, \quad \tau = \frac{s}{a},$$

then

$$\tilde{y}^{(4)})\xi) = a^4 y^{(4)}(a\xi) = a^4 g(x) = a^4 g(a\xi).$$

Therefore

$$\tilde{y}(\xi) = \int_0^1 G_1(\xi,\, \tau)a^4 g(a\tau)d\tau.$$

Consequently

$$y(x) = \int_0^a G_1(\xi,\, \frac{s}{a})a^4 g(s)\frac{ds}{a} = \int_0^a G_1(\frac{x}{a},\, \frac{s}{a})a^3 g(s)ds.$$

We can see the relationship between the Green's function $G_a(x,s)$ and rescaled one $G_1(\frac{x}{a},\, \frac{s}{a})$ is:

$$G_a(x,\, s) = G_1(\frac{x}{a},\, \frac{s}{a})a^3. \tag{1.26}$$

The last part of this section will focus on the eigenvalues associated with the Green's functions that we just found. Let us find the eigenvalues associated with

$$Du = u^{(4)}(x) = f(x, u(x), u'(x), u''(x), u'''(x))$$

with homogeneous boundary conditions, where f satisfies the conditions of the Theorem (2.1). An eigenvalue of the operator, D, is a constant λ such that $Du = \lambda h(x)u(x)$ for some some nontrivial u satisfying the given boundary conditions. Later we apply our knowledge to the special case where $f(x, u(x), u'(x), u''(x), u'''(x)) = \lambda u(x)$. The corresponding eigenvalue problem is

$$u^{(4)}(x) = \lambda u(x). \tag{1.27}$$

with appropriate boundary conditions. Substituting e^{rx} for u, we find that the characteristic equation is

$$r^4 - \lambda = (r^2 - \sqrt{\lambda})(r^2 + \sqrt{\lambda}) = 0. \tag{1.28}$$

Therefore the roots are $r = \pm\lambda^{1/4}, \pm i\lambda^{1/4}$. This means that the linear space of solutions of Eq. (3.16) is spanned by

$$e^{rx}, \quad e^{-rx}, \quad e^{irx}, \quad \text{and} \quad e^{-irx}.$$

Since

$$\cos(rx) = \frac{e^{irx} + e^{-irx}}{2} \quad \text{and} \quad \sin(rx) = \frac{e^{irx} - e^{-irx}}{2i}$$

we may also change basis and use instead the real-valued functions

$$e^{rx}, \quad e^{-rx}, \quad \cos(rx), \quad \text{and} \quad \sin(rx).$$

And the general solution of Eq. (3.16) is

$$u(x) = c_1 e^{rx} + c_2 e^{-rx} + c_3 \cos(rx) + c_4 \sin(rx). \tag{1.29}$$

Since

$$\cosh(rx) = \frac{e^{rx} + e^{-rx}}{2} \quad \text{and} \quad \sinh(rx) = \frac{e^{rx} - e^{-rx}}{2}$$

we may convert the above general solution of Eq. (3.16) to the following

$$u(x) = c_1 \cosh(rx) + c_2 \sinh(rx) + c_3 \cos(rx) + c_4 \sin(rx). \tag{1.30}$$

Finding the 1$^{\text{st}}$ derivative of $u(x)$ we obtain

$$u'(x) = r[c_1 \sinh(rx) + c_2 \cosh(rx) - c_3 \sin(rx) + c_4 \cos(rx)]. \tag{1.31}$$

It is convenient to rescale the variables in the following examples so that

$$\tilde{u}(s) = u(as) = u(x), \quad x = as, \quad \text{and} \quad 0 \leqslant s \leqslant 1.$$

Therefore, letting $\lambda_0(a)$ be the first positive eigenvalue,

$$\tilde{u}^{(4)}(s) = a^4 u^{(4)}(as) = a^4 u^{(4)}(x) = a^4 \lambda_0(a) u(x) = a^4 \lambda_0(a) \tilde{u}(s) = \lambda_0(1)\tilde{u}(s).$$

We can see the relationship between the first positive eigenvalue $\lambda_0(a)$ and rescaled one $\lambda_0(1)$ is:

$$\lambda_0(a) = \frac{\lambda_0(1)}{a^4}. \tag{1.32}$$

Example 3.5. This example corresponds to Example 3.1.

$$\begin{aligned} u^{(4)}(x) &= \lambda u(x) \\ u(0) = u'(0) &= u(1) = u'(1) = 0. \end{aligned} \tag{1.33}$$

We now find the first eigenvalue for this example using (3.19). If we impose the boundary conditions
$u(0) = u'(0) = 0$ we find that

$$c_3 = -c_1, \quad \text{and} \quad c_4 = -c_2.$$

which reduces the equations of a general solution $u(x)$ and its 1st derivative to the following two equations

$$\begin{aligned} u(x) &= c_1[\cosh(rx) - \cos(rx)] \quad + c_2[\sinh(rx) - \sin(rx)] \\ u'(x) &= rc_1[\sinh(rx) + \sin(rx)] \quad + rc_2[\cosh(rx) - \cos(rx)] \end{aligned} \tag{1.34}$$

And if we impose the boundary conditions $u(1) = u'(1) = 0$ we come down with the system of equations written in the matrix form

$$\begin{bmatrix} \cosh(r) - \cos(r) & \sinh(r) - \sin(r) \\ \sinh(r) + \sin(r) & \cosh(r) - \cos(r) \end{bmatrix} \times \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1.35}$$

From the basic theory of system of linear algebraic equations, compatibility requires the vanishing of the determinant of the matrix, i.e.,

$$(\cosh(r) - \cos(r))^2 - (\sinh^2(r) - \sin^2(r)) = 0.$$

which is equivalent to $\cosh(r)\cos(r) = 1$. Using numerical methods Maple finds the smallest possible r to be 4.7300. And since $r^4 = \lambda$, the smallest possible positive eigenvalue for the Eq.(3.22) is equal to 500.56390.

Example 3.6. This example corresponds to Example 3.2.

$$\begin{aligned} u^{(4)}(x) &= \lambda u(x) \\ u(0) = u'(0) &= u'(1) = u'''(1) = 0. \end{aligned} \tag{1.36}$$

Again we start with the general solution of Eq. (3.16)and its 1$^{\text{th}}$ derivative expressed in terms of hyperbolic and trigonometric functions.

$$u(x) = c_1 \sinh(rx) + c_2 \cosh(rx) + c_3 \cos(rx) + c_4 \sin(rx) \tag{1.37}$$

$$u'(x) = r[c_1 \sinh(rx) + c_2 \cosh(rx) - c_3 \sin(rx) + c_4 \cos(rx)]. \tag{1.38}$$

If we impose the boundary conditions $u(0) = u'(0) = 0$ we find that

$$c_1 = -c_4, \quad \text{and} \quad c_2 = -c_3.$$

Applying the boundary condition $u'(1) = 0$ to the Eq.(3.26) we obtain

$$r[c_1 \cosh(r) + c_2 \sinh(r) - c_3 \sin(r) + c_4 \cos(r)] = 0.$$

The 3$^{\text{th}}$ derivative of $u(x)$ is

$$u'''(x) = r^3[c_1 \cosh(rx) + c_2 \sinh(rx) + c_3 \sin(rx) - c_4 \cos(rx)]. \tag{1.39}$$

Applying the boundary condition $u'''(1) = 0$ to the Eq.(3.28) we obtain

$$r^3[c_1 \cosh(r) + c_2 \sinh(r) + c_3 \sin(r) - c_4 \cos(r)] = 0. \tag{1.40}$$

Knowing that $c_1 = -c_4$, and $c_2 = -c_3$ yields the system of equations of the 1$^{\text{th}}$ and 3$^{\text{th}}$ derivative of the general solution in the matrix form

$$\begin{bmatrix} \sinh(r) + \sin(r) & \cosh(r) - \cos(r) \\ \sinh(r) - \sin(r) & \cosh(r) + \cos(r) \end{bmatrix} \times \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1.41}$$

The determinant of the above matrix must to be equal to zero in order for the system of algebraic equations to have a nontrivial solution, i.e.,

$$\sinh(r)\cos(r) + \sin(r)\cosh(r) = 0.$$

Using numerical methods Maple finds the smallest possible r to be 2.3650. And since $r^4 = \lambda$, the smallest possible positive eigenvalue for the Eq.(3.25) is equal to 31.2852.

Example 3.7. This example corresponds to Example 3.3.

$$\begin{aligned} u^{(4)}(x) &= \lambda u(x) \\ u(0) = u'(0) &= u''(1) = u'''(1) = 0. \end{aligned} \tag{1.42}$$

As previously we start with the general solution of Eq. (3.16)and its 1^{th} derivative expressed in terms of hyperbolic and trigonometric functions.

$$u(x) = c_1 \sinh(rx) + c_2 \cosh(rx) + c_3 \cos(rx) + c_4 \sin(rx) \tag{1.43}$$

$$u'(x) = r[c_1 \sinh(rx) + c_2 \cosh(rx) - c_3 \sin(rx) + c_4 \cos(rx)]. \tag{1.44}$$

If we impose the boundary conditions $u(0) = u'(0) = 0$ we find that

$$c_1 = -c_4, \quad \text{and} \quad c_2 = -c_3.$$

The 2^{th} derivative of $u(x)$ is

$$u''(x) = r^2[c_1 \sinh(rx) + c_2 \cosh(rx) - c_3 \cos(rx) - c_4 \sin(rx)]. \tag{1.45}$$

And the 3^{th} derivative of $u(x)$ is

$$u'''(x) = r^3[c_1 \cosh(rx) + c_2 \sinh(rx) + c_3 \sin(rx) - c_4 \cos(rx)]. \tag{1.46}$$

Applying the boundary condition $u''(1) = 0$ to the Eq.(3.34) we obtain

$$r^2[c_1 \sinh(r) + c_2 \cosh(r) - c_3 \cos(r) - c_4 \sin(r)] = 0. \tag{1.47}$$

Applying the boundary condition $u'''(1) = 0$ to the Eq.(3.33) we obtain

$$r^3[c_1 \cosh(r) + c_2 \sinh(r) + c_3 \sin(r) - c_4 \cos(r)] = 0. \tag{1.48}$$

Knowing that $c_1 = -c_4$, and $c_2 = -c_3$ yields the system of equations of the 2^{th} and 3^{th} derivative of the general solution in the matrix form

$$\begin{bmatrix} \cosh(r) + \cos(r) & \sinh(r) + \sin(r) \\ \sinh(r) - \sin(r) & \cosh(r) + \cos(r) \end{bmatrix} \times \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1.49}$$

The determinant of the above matrix should be zero, i.e.,

$$\cosh(r)\cos(r) + 1 = 0.$$

Using numerical methods Maple finds the smallest possible r to be 1.8751. And since $r^4 = \lambda$, the smallest possible positive eigenvalue for the Eq.(3.31) is equal to 12.3623.

Example 3.8. This example corresponds to Example 3.4.

$$\begin{aligned} u^{(4)}(x) &= \lambda u(x) \\ u(0) = u''(0) &= u(1) = u''(1) = 0. \end{aligned} \tag{1.50}$$

We start with the general solution of Eq. (3.15)and its 2^{th} derivative expressed in terms of hyperbolic and trigonometric functions.

$$u(x) = c_1\sinh(rx) + c_2\cosh(rx) + c_3\cos(rx) + c_4\sin(rx) \tag{1.51}$$

$$u''(x) = -r^2[-c_1\sinh(rx) - c_2\cosh(rx) + c_3\cos(rx) + c_4\sin(rx)]. \tag{1.52}$$

Applying the boundary condition $u(0) = 0$ to the Eq.(3.40) we obtain

$$c_2 + c_3 = 0. \tag{1.53}$$

Applying the boundary condition $u''(0) = 0$ to the Eq.(3.39) we obtain

$$-c_2 + c_3 = 0. \tag{1.54}$$

It is obvious that $c_2 = c_3 = 0$. Substituting the found coefficients into equations (3.40) and applying the boundary condition $u(1) = 0$ renders the following

$$c_1\sinh(r) - c_4\sin(r) = 0.$$

Substituting the found coefficients into equations (3.41) and applying the boundary condition $u''(1) = 0$ gives us the following

$$c_1\sinh(r) - c_4\sin(r) = 0.$$

The last two equations can be written in the matrix form as following

$$\begin{bmatrix} \sinh(r) & \sin(r) \\ \sinh(r) & -\sin(r) \end{bmatrix} \times \begin{bmatrix} c_1 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1.55}$$

Again, the determinant of the above matrix should be zero,i.e.,

$$2\sinh(r)\sin(r) \;=\; 0.$$

It is evident that the smallest possible r to be π. And since $r^4 \;=\; \lambda$, the smallest possible positive eigenvalue for the Eq.(3.37) is equal to $\pi^4 \;=\; 97.4091$.

Section 4: Applications to Boundary Value Problems

In this section, various aspects of the preceding material will be applied to boundary value problems, with the specific goal of illustrating the theorems in Section 2. In particular, with respect to finding the first positive eigenvalue, better bounds on the Lipschitz constant L will be obtained.

Application of Theorem 2.1.

We rescale the variables z, w for the equation $z^{(4)}(x) = Lw(x)$ in the following examples so that

$$\tilde{z}(s) = z(as), \quad \tilde{w}(s) = w(as), \quad \text{and} \quad \tilde{L} := a^4 L.$$

Therefore

$$\tilde{z}^{4)}(s) = a^4 z^{(4)}(as) = a^4 Lw(x) = a^4 L\tilde{w}(s) = \tilde{L}\tilde{w}(s),$$

and writing

$$\max_{0<\leqslant x\leqslant a} \frac{z(x)}{w(x)} = M(a)L, \quad \text{and} \quad \max_{0\leqslant s\leqslant 1} \frac{\tilde{z}(s)}{\tilde{w}(s)} = M(1)a^4 L,$$

we can establish the relationship between bound constant $M(a)$ and rescaled one $M(1)$.

Since $\dfrac{z(x)}{w(x)} = \dfrac{\tilde{z}(s)}{\tilde{w}(s)}$ we can conclude that $M(a)L = M(1)a^4 L$ and $M(a) = M(1)a^4$.

Example 4.1.1. This example corresponds to Example 3.1.
$Du = u^{(4)}(x) = f(x, u(x))$, where f satisfies $|f(x, u_1(x) - f(x, u_2(x))| \leqslant L|u_1(x) - u_2(x)|$ on $[0, 1]$ (where $h(x) = L$ in Theorem 2.1) with boundary conditions $u(0) = u'(0) = u(1) = u'(1) = 0$, choose a polynomial, w, of degree 4 satisfying the boundary conditions. Denote w by

$$w(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4.$$

Then

$$w'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3.$$

Imposing the boundary conditions $w(0) = w'(0) = 0$ yields

$$a_0 = a_1 = 0.$$

Knowing that $a_0 = a_1 = 0$ and taking $a_4 = 1$ simplifies the equation for $w(x)$ and $w'(x)$.

$$w(x) = a_2x^2 + a_3x^3 + x^4 \tag{1.56}$$

$$w'(x) = 2a_2x + 3a_3x^2 + 4x^3 \tag{1.57}$$

And consequently the boundary conditions

$$w(1) = a_2 + a_3 + 1 = 0 \tag{1.58}$$

$$w'(1) = 2a_2 + 3a_3 + 4 = 0. \tag{1.59}$$

Solving simultaneously equations (4.3) and (4.4) for a_2 and a_3 yields

$$a_2 = 1 \quad \text{and} \quad a_3 = -2.$$

Therefore, $w(x) = x^2 - 2x^3 + x^4 = x^2(x-1)^2$, and $z^{(4)}(x) = L(x^2 - 2x^3 + x^4)$. Integrating yields

$$\begin{aligned} z'(x) &= \frac{1}{60}Lx^5 - \frac{1}{60}Lx^6 + \frac{1}{210}Lx^7 + \frac{1}{2}C_3x^2 + C_2x + C_1, \\ z(x) &= \frac{1}{360}Lx^6 - \frac{1}{420}Lx^7 + \frac{1}{1680}Lx^8 + \frac{1}{6}C_3x^3 + \frac{1}{2}C_2x^2 + C_1x + C_0. \end{aligned}$$

Applying the boundary conditions $z(0) = z'(0) = 0$ reveals $C_0 = C_1 = 0$. Therefore,

$$z'(x) = \frac{1}{60}Lx^5 - \frac{1}{60}Lx^6 + \frac{1}{210}Lx^7 + \frac{1}{2}C_3x^2 + C_2x, \tag{1.60}$$

$$z(x) = \frac{1}{360}Lx^6 - \frac{1}{420}Lx^7 + \frac{1}{1680}Lx^8 + \frac{1}{6}C_3x^3 + \frac{1}{2}C_2x^2. \tag{1.61}$$

Applying the boundary conditions $z(1) = z'(1) = 0$ gives us the following couple of equations:

$$z'(1) = \frac{1}{210}L + \frac{1}{2}C_3 + C_2 = 0, \tag{1.62}$$

$$z(1) = \frac{1}{1008}L + \frac{1}{6}C_3 + \frac{1}{2}C_2 = 0. \tag{1.63}$$

Solving simultaneously equations (4.7) and (4.8) for C_2 and C_3 yields

$$C_2 = \frac{1}{280}L, \quad \text{and} \quad C_3 = -\frac{1}{60}L.$$

Knowing all the coefficients we arrive at the expression for $z(x)$.

$$z(x) = \frac{1}{5040}Lx^2(3x^4 - 6x^3 - x^2 + 4x + 9)(x - 1)^2. \tag{1.64}$$

We finally can find the quotient $\frac{z(x)}{w(x)}$:

$$\frac{z(x)}{w(x)} = \frac{1}{5040}(3x^4 - 6x^3 - x^2 + 4x + 9)L.$$

Finding $\max_{0\leqslant x\leqslant 1} \frac{z(x)}{w(x)}$ by Maple shows that the maximum is $\frac{163L}{80640}$ and less than one for $L < \frac{80640}{163} \approx 494.7239$.

It is necessary to show that T maps B_w into B_w in order to apply Theorem 2.1. The Green's function calculated in Example 3.1. is positive and simplifies for $a = 1$ to

$$G(x,s) = \begin{cases} x^2(s-1)^2(3s-2sx-x)/6, & \text{if } 0 \leqslant x \leqslant s \leqslant 1 \\ -s^2(1-x)^2(s+2sx-3x)/6, & \text{if } 0 \leqslant s < x \leqslant 1 \end{cases}$$

Thus,

$$\begin{aligned} Tu(x) &= \int_0^1 G(x,s)f(s,u(s))ds \\ &= \int_0^x \left[\frac{-s^2(1-x)^2)(s+2sx-3x)}{6}\right] f(s,u(s))ds + \\ &+ \int_x^1 \left[\frac{x^2(s-1)^2)(3s-2sx-x)}{6}\right] f(s,u(s))ds. \end{aligned}$$

Since G(x,s) is positive and $|s + 2sx - 3x| \leqslant 6$, then

$$\begin{aligned} |Tu(x)| &\leqslant \int_0^x \left|\frac{-s^2(1-x)^2)(s+2sx-3x)}{6}\right| |f(s,u(s))|ds + \\ &+ \int_x^1 \left|\frac{x^2(s-1)^2)(3s-2sx-x)}{6}\right| |f(s,u(s))|ds \leqslant \\ &\leqslant x^2(1-x)^2 \int_0^x |f(s,u(s))|ds + x^2(x-1)^2 \int_x^1 |f(s,u(s))|ds. \end{aligned}$$

Thus, since $w(x) = x^2(1-x)^2$,

$$\left|\frac{Tu(x)}{w(x)}\right| \leqslant \int_0^1 |f(s,u(s))ds| = C(u).$$

Now the hypotheses of Theorem 2.1. are satisfied. Comparing the bound $L < 494.7239$ to the optimal bound, $L < 500.56390$, obtained in Example 3.5 shows that the function w as a polynomial is a good estimate of the eigenfunction. If w is chosen to be more similar to the eigenfunction, a better bound on L could be achieved.

Example 4.1.2. This example corresponds to Example 3.2.

$Du = u^{(4)}(x) = f(x, u(x))$, where f satisfies $|f(x, u_1(x) - f(x, u_2(x))| \leqslant L|u_1(x) - u_2(x)|$ on $[0, 1]$ with boundary conditions $u(0) = u'(0) = u'(1) = u'''(1) = 0$, choose a polynomial, w, of degree 4 satisfying the boundary conditions.

$$w(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4. \tag{1.65}$$

Then

$$w'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3, \tag{1.66}$$

$$w''(x) = 2a_2 + 6a_3 x + 12a_4 x^2, \tag{1.67}$$

$$w'''(x) = 6a_3 + 24a_4 x \tag{1.68}$$

Imposing the boundary conditions $w(0) = w'(0) = 0$ yields

$$a_0 = a_1 = 0.$$

Knowing that $a_0 = a_1 = 0$ and taking $a_4 = 1$ simplifies the equation for $w(x)$ and $w'(x)$.

$$w(x) = a_2 x^2 + a_3 x^3 + x^4, \tag{1.69}$$

$$w'(x) = 2a_2 x + 3a_3 x^2 + 4x^3, \tag{1.70}$$

$$w'''(x) = 6a_3 + 24x. \tag{1.71}$$

And consequently

$$w'(1) = 2a_2 + 3a_3 + 4, \tag{1.72}$$

$$w'''(1) = 6a_3 + 24. \tag{1.73}$$

Solving simultaneously Eq.(4.17) and Eq.(4.18) for a_2 and a_3 reveals that

$$a_2 = 4 \quad \text{and} \quad a_3 = -4.$$

Therefore, $w(x) = 4x^2 - 4x^3 + x^4 = x^2(x-2)^2$, and $z^{(4)}(x) = L(4x^2 - 4x^3 + x^4)$. Integrating yields

$$z'''(x) = \frac{1}{5}Lx^5 - Lx^4 + \frac{4}{3}Lx^3 + C_3, \tag{1.74}$$

$$z'(x) = \frac{1}{210}Lx^7 - \frac{1}{30}Lx^6 + \frac{1}{15}Lx^5 + \frac{1}{2}C_3x^2 + C_2x + C_1, \tag{1.75}$$

$$z(x) = \frac{1}{1680}Lx^8 - \frac{1}{210}Lx^7 + \frac{1}{90}Lx^6 + \frac{1}{6}C_3x^3 + \frac{1}{2}C_2x^2 + C_0. \tag{1.76}$$

Applying the boundary conditions $z(0) = z'(0) = 0$ reveals $C_0 = C_1 = 0$. Therefore,

$$z'(x) = \frac{1}{210}Lx^7 - \frac{1}{30}Lx^6 + \frac{1}{15}Lx^5 + \frac{1}{2}C_3x^2 + C_2x, \tag{1.77}$$

$$z(x) = \frac{1}{1680}Lx^8 - \frac{1}{210}Lx^7 + \frac{1}{90}Lx^6 + \frac{1}{6}C_3x^3 + \frac{1}{2}C_2x^2. \tag{1.78}$$

Applying the boundary conditions $z'(1) = z'''(1) = 0$ gives us a couple equations:

$$z'(1) = \frac{1}{105}L + \frac{1}{2}C_3 + C_2 = 0, \tag{1.79}$$

$$z'''(1) = \frac{8}{15}L + C_3 = 0. \tag{1.80}$$

Solving simultaneously equations (4.24) and (4.25) for C_2 and C_3 yields

$$C_2 = \frac{2}{35}L, \quad \text{and} \quad C_3 = -\frac{8}{15}L.$$

Knowing all the coefficients we arrive at the expression for $z(x)$.

$$z(x) = \frac{1}{5040}Lx^2(3x^4 - 12x^3 - 4x^2 + 32x + 144)(x-2)^2. \tag{1.81}$$

We finally can find the quotient $\frac{z(x)}{w(x)}$:

$$\frac{z(x)}{w(x)} = \frac{1}{5040}(3x^4 - 12x^3 - 4x^2 + 32x + 144)L.$$

Finding $\max_{0\leqslant x\leqslant 1}\frac{z(x)}{w(x)}$ by Maple shows that the maximum is $\frac{163L}{5040}$ and less than one for $L < \frac{5040}{163} \approx 30.9202$.

Next we show that T maps B_w into B_w in order to apply Theorem 2.1. The Green's function calculated in Example 3.1. is positive and simplifies for $a = 1$ to

$$G(x,s) = \begin{cases} -s^2(2s - 6x + 3x^2)/12, & \text{if} \quad 0 \leqslant x \leqslant s \leqslant 1 \\ -x^2(3s^2 - 6s + 2x)/12, & \text{if} \quad 0 \leqslant s < x \leqslant 1 \end{cases} \tag{1.82}$$

Thus,

$$\begin{aligned}
Tu(x) &= \int_0^1 G(x,s)f(s,u(s))ds \\
&= \int_0^x \left[\frac{-x^2(3s^2 - 6s + 2x)}{12}\right] f(s,u(s))ds + \\
&+ \int_x^1 \left[\frac{-s^2(2s - 6x + 3x^2)}{12}\right] f(s,u(s))ds.
\end{aligned}$$

Since G(x,s) is positive and $|3s^2 - 6s + 2x| \leqslant 11$, then

$$\begin{aligned}
|Tu(x)| &\leqslant \int_0^x \left|\frac{-x^2(3s^2 - 6s + 2x)}{12}\right| |f(s,u(s))|ds + \\
&+ \int_x^1 \left|\frac{-s^2(2s - 6x + 3x^2)}{12}\right| |f(s,u(s))|ds \leqslant \\
&\leqslant \frac{11x^2}{12}\int_0^x |f(s,u(s))|ds + \frac{11x^2}{12}\int_x^1 |f(s,u(s))|ds.
\end{aligned}$$

Thus, since $w(x) = x^2(x-2)^2$ and $(x-2)^2 \geqslant 1$,

$$\left|\frac{Tu(x)}{w(x)}\right| \leqslant \frac{11}{12}\int_0^1 |f(s,u(s))ds| = C(u).$$

Now the hypotheses of Theorem 2.1. are satisfied. Comparing the bound $L < 30.9202$ to the optimal bound, $L < 31.2852$, obtained in Example 3.6 shows that the function w as a polynomial is a very good estimate of the eigenfunction.

Example 4.1.3. This example corresponds to Example 3.3.

$Du = u^{(4)}(x) = f(x, u(x))$, where f satisfies $|f(x,u_1(x) - f(x,u_2(x))| \leqslant L|u_1(x) - u_2(x)|$ on $[0, 1]$ with boundary conditions $u(0) = u'(0) = u''(1) = u'''(1) = 0$, choose a polynomial, w, of degree 4 satisfying the boundary conditions.

$$w(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4. \tag{1.83}$$

Then

$$w'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3, \tag{1.84}$$

$$w''(x) = 2a_2 + 6a_3 x + 12a_4 x^2, \tag{1.85}$$

$$w'''(x) = 6a_3 + 24a_4 x \tag{1.86}$$

Imposing the boundary conditions $w(0) = w'(0) = 0$ yields

$$a_0 = a_1 = 0.$$

Knowing that $a_0 = a_1 = 0$ and taking $a_4 = 1$ simplifies the equation for $w'(x)$, $w''(x)$ and $w'''(x)$.

$$w'(x) = 2a_2x + 3a_3x^2 + 4x^3, \quad (1.87)$$

$$w''(x) = 2a_2 + 6a_3x + 12x^2, \quad (1.88)$$

$$w'''(x) = 6a_3 + 24x. \quad (1.89)$$

And consequently

$$w''(1) = 2a_2 + 6a_3 + 12, \quad (1.90)$$

$$w'''(1) = 6a_3 + 24. \quad (1.91)$$

Solving simultaneously Eq.(4.35) and Eq.(4.36) for a_2 and a_3 reveals that

$$a_2 = 6 \quad \text{and} \quad a_3 = -4.$$

Therefore, $w(x) = x^4 - 4x^3 + 6x^2 = x^2(x^2 - 4x + 6)$, and $z^{(4)}(x) = L(x^4 - 4x^3 + 6x^2)$. Integrating yields

$$z'''(x) = \frac{1}{5}Lx^5 - Lx^4 + 2Lx^3 + C_3, \quad (1.92)$$

$$z''(x) = \frac{1}{30}Lx^6 - \frac{1}{5}Lx^4 + \frac{1}{2}Lx^4 + C_3x + C_2, \quad (1.93)$$

$$z'(x) = \frac{1}{210}Lx^7 - \frac{1}{30}Lx^6 + \frac{1}{10}Lx^5 + \frac{1}{2}C_3x^2 + C_2x + C_1, \quad (1.94)$$

$$z(x) = \frac{1}{1680}Lx^8 - \frac{1}{210}Lx^7 + \frac{1}{60}Lx^6 + \frac{1}{6}C_3x^3 + \frac{1}{2}C_2x^2 + C_1x + C_0. \quad (1.95)$$

Applying the boundary conditions $z(0) = z'(0) = 0$ reveals $C_0 = C_1 = 0$. Therefore,

$$z'(x) = \frac{1}{210}Lx^7 - \frac{1}{30}Lx^6 + \frac{1}{10}Lx^5 + \frac{1}{2}C_3x^2 + C_2x, \quad (1.96)$$

$$z(x) = \frac{1}{1680}Lx^8 - \frac{1}{210}Lx^7 + \frac{1}{60}Lx^6 + \frac{1}{6}C_3x^3 + \frac{1}{2}C_2x^2. \quad (1.97)$$

Applying the boundary conditions $z(1) = z'''(1) = 0$ gives us a couple equations:

$$z''(1) = \frac{1}{3}L + C_3 + C_2 = 0, \quad (1.98)$$

$$z'''(1) = \frac{6}{5}L + C_3 = 0. \quad (1.99)$$

Solving simultaneously equations (4.43) and (4.44) for C_2 and C_3 yields

$$C_2 = \frac{13}{15}L, \quad \text{and} \quad C_3 = -\frac{6}{5}L.$$

Knowing all the coefficients we arrive at the expression for $z(x)$.

$$z(x) = \frac{1}{1680} L x^2 (x^6 - 8x^5 + 28x^4 - 336x + 728). \tag{1.100}$$

We finally can find the quotient $\frac{z(x)}{w(x)}$:

$$\frac{z(x)}{w(x)} = \frac{L(x^6 - 8x^5 + 28x^4 - 336x + 728)}{1680(6 - 4x + x^2)}.$$

Finding $\max_{0 \leqslant x \leqslant 1} \frac{z(x)}{w(x)}$ by Maple shows that the maximum is $\frac{59L}{720}$ and less than one for $L < \frac{720}{59} \approx 12.20338$.

Next we show that T maps B_w into B_w in order to apply Theorem 2.1. The Green's function calculated in Example 3.3. is positive and equal to

$$G(x,s) = \begin{cases} x^2(3s - x)/6, & \text{if } 0 \leqslant x \leqslant s \leqslant a \\ -s^2(s - 3x)/6, & \text{if } 0 \leqslant s < x \leqslant a \end{cases} \tag{1.101}$$

Thus,

$$\begin{aligned} Tu(x) &= \int_0^1 G(x,s) f(s,u(s)) ds \\ &= \int_0^x \left[\frac{-s^2(s - 3x)}{6} \right] f(s,u(s)) ds + \\ &+ \int_x^1 \left[\frac{x^2(-x + 3s)}{6} \right] f(s,u(s)) ds. \end{aligned}$$

Since G(x,s) is positive and $|-x + 3s| \leqslant 4$, then

$$\begin{aligned} |Tu(x)| &\leqslant \int_0^x \left| \frac{s^2(3x - s)}{6} \right| |f(s,u(s))| ds + \\ &+ \int_x^1 \left| \frac{x^2(3s - x)}{6} \right| |f(s,u(s))| ds \leqslant \\ &\leqslant \frac{4x^2}{6} \int_0^x |f(s,u(s))| ds + \frac{4x^2}{6} \int_x^1 |f(s,u(s))| ds. \end{aligned}$$

Thus, since $w(x) = x^2(6 - 4x + x^2)$ and $|6 - 4x + x^2| \geqslant 3$,

$$\left| \frac{Tu(x)}{w(x)} \right| \leqslant \frac{2}{9} \int_0^1 |f(s,u(s)) ds| = C(u).$$

Now the hypotheses of Theorem 2.1. are satisfied. Comparing the bound $L < 12.20338$ to the optimal bound, $L < 12.3623$, obtained in Example 3.7 shows that the function w as a polynomial is a good estimate of the eigenfunction.

Example 4.1.4. This example corresponds to Example 3.4.

$Du = u^{(4)}(x) = f(x, u(x))$, where f satisfies $|f(x, u_1(x) - f(x, u_2(x))| \leqslant L|u_1(x) - u_2(x)|$ on $[0, 1]$ with boundary conditions $u(0) = u''(0) = u(1) = u''(1) = 0$, choose a polynomial, w, of degree 4 satisfying the boundary conditions.

$$w(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4. \tag{1.102}$$

Then

$$w''(x) = 2a_2 + 6a_3x + 12a_4x^2. \tag{1.103}$$

Imposing the boundary conditions $w(0) = w''(0) = 0$ yields

$$a_0 = a_2 = 0.$$

Knowing that $a_0 = a_2 = 0$ and taking $a_4 = 1$ simplifies the equation for $w(x)$ and $w''(x)$.

$$w(x) = a_1x + a_3x^3 + x^4, \tag{1.104}$$

$$w''(x) = 6a_3x + 12x^2. \tag{1.105}$$

And consequently

$$w(1) = a_1 + a_3 + 1, \tag{1.106}$$

$$w''(1) = 6a_3 + 12. \tag{1.107}$$

Solving simultaneously Eq.(4.51) and Eq.(4.52) for a_1 and a_3 reveals that

$$a_1 = 1 \quad \text{and} \quad a_3 = -2.$$

Therefore, $w(x) = x^4 - 2x^3 + x = x(x - 1)(x^2 - x - 1)$, and $z^{(4)}(x) = L(x^4 - 2x^3 + x)$. Integrating yields

$$z''(x) = \frac{1}{30}Lx^6 - \frac{1}{10}Lx^5 + \frac{1}{6}Lx^3 + C_3x + C_2, \tag{1.108}$$

$$z(x) = \frac{1}{1680}Lx^8 - \frac{1}{420}Lx^7 + \frac{1}{120}Lx^5 + \frac{1}{6}C_3x^3 + \frac{1}{2}C_2x^2 + C_1x + C_0. \tag{1.109}$$

Applying the boundary conditions $z(0) = z''(0) = 0$ reveals $C_0 = C_2 = 0$. Therefore,

$$z''(x) = \frac{1}{30}Lx^6 - \frac{1}{10}Lx^5 + \frac{1}{6}Lx^3 + C_3x, \tag{1.110}$$

$$z(x) = \frac{1}{1680}Lx^8 - \frac{1}{420}Lx^7 + \frac{1}{120}Lx^5 + \frac{1}{6}C_3x^3 + C_1x. \tag{1.111}$$

Applying the boundary conditions $z(1) = z''(1) = 0$ gives us a couple equations:

$$z(1) = \frac{L}{1680} - \frac{L}{420} + \frac{L}{120} + \frac{1}{6}C_3 + C_1 = \frac{11}{1680}L + \frac{1}{6}C_3 + C_1 = 0, \tag{1.112}$$

$$z''(1) = \frac{1}{10}L + C_3 = 0. \tag{1.113}$$

Solving simultaneously equations (4.57) and (4.58) for C_1 and C_3 yields

$$C_1 = \frac{17}{1680}L, \quad \text{and} \quad C_3 = -\frac{1}{10}L.$$

Knowing all the coefficients we arrive at the expression for $z(x)$.

$$z(x) = \frac{L}{1680}x(x-1)(x^6 - 3x^5 - 3x^4 + 11x^3 + 11x^2 - 17x - 17). \tag{1.114}$$

We finally can find the quotient $\frac{z(x)}{w(x)}$:

$$\frac{z(x)}{w(x)} = -\frac{L(x^6 - 3x^5 - 3x^4 + 11x^3 + 11x^2 - 17x - 17)}{1680(x^2 - x - 1)}.$$

Finding $\max_{0\leqslant x\leqslant 1} \frac{z(x)}{w(x)}$ by Maple shows that the maximum is $\frac{277L}{26880}$ and less than one for $L < \frac{26880}{277} \approx 97.03971$.

Next we show that T maps B_w into B_w in order to apply Theorem 2.1. The Green's function calculated in Example 3.3. is positive and simplifies for $a = 1$ to

$$G(x,s) = \begin{cases} x(s-1)(s^2 - 2s + x^2)/6, & \text{if } 0 \leqslant x \leqslant s \leqslant 1 \\ s(1-x)(s^2 - 2s + x^2)/6, & \text{if } 0 \leqslant s < x \leqslant 1 \end{cases} \tag{1.115}$$

Thus,

$$\begin{aligned} Tu(x) &= \int_0^1 G(x,s)f(s,u(s))ds \\ &= \int_0^x \left[\frac{s(1-x)(s^2 - 2s + x^2)}{6}\right] f(s,u(s))ds + \\ &+ \int_x^1 \left[\frac{x(s-1)(s^2 - 2s + x^2)}{6}\right] f(s,u(s))ds. \end{aligned}$$

Since G(x,s) is positive and $|s^2 - 2s + x^2| \leqslant 2$, then

$$
\begin{aligned}
|Tu(x)| \leqslant & \int_0^x \left|\frac{s(1-x)(s^2-2s+x^2)}{6}\right| |f(s,u(s))|ds + \\
& + \int_x^1 \left|\frac{x(s-1)(s^2-2s+x^2)}{6}\right| |f(s,u(s))|ds \leqslant \\
& \leqslant \frac{x(x-1)}{3}\int_0^x |f(s,u(s))|ds + \frac{x(x-1)}{3}\int_x^1 |f(s,u(s))|ds.
\end{aligned}
$$

Thus, since $w(x) = x(x-1)(x^2 - x - 1)$ and $|x^2 - x - 1| \geqslant 1$,

$$
\left|\frac{Tu(x)}{w(x)}\right| \leqslant \frac{1}{3}\int_0^1 |f(s,u(s))ds| = C(u).
$$

Now the hypotheses of Theorem 2.1. are satisfied. Comparing the bound $L < 97.03971$ to the optimal bound, $L < 97.4091$, obtained in Example 3.8 shows that the function w as a polynomial is a very good estimate of the eigenfunction.

Example 4.1.4.a This example corresponds to Example 3.4. with the polynomial $w(x)$ chosen to be 1 for simplicity.

$Du = u^{(4)}(x) = f(x, u(x))$, where f satisfies $|f(x,u_1(x) - f(x,u_2(x))| \leqslant L|u_1(x) - u_2(x)|$ on $[0, 1]$ with boundary conditions $u(0) = u''(0) = u(1) = u''(1) = 0$.

We start with the general form of the 4^{th} order nonlinear differential equations

$$
z^{(4)}(x) = L.
$$

Integrating yields

$$
z''(x) = \frac{1}{2}Lx^2 + C_3x + C_2, \tag{1.116}
$$

$$
z(x) = \frac{1}{24}Lx^4 + \frac{1}{6}C_3x^3 + \frac{1}{2}C_2x^2 + C_1x + C_0. \tag{1.117}
$$

Applying the boundary conditions $z(0) = z''(0) = 0$ reveals $C_0 = C_2 = 0$. Therefore,

$$
z''(x) = \frac{1}{2}Lx^2 + C_3x, \tag{1.118}
$$

$$
z(x) = \frac{1}{24}Lx^4 + \frac{1}{6}C_3x^3 + C_1x. \tag{1.119}
$$

Applying the boundary conditions $z(1) = z''(1) = 0$ gives us a couple equations:

$$\frac{1}{2}L + C_3 = 0, \tag{1.120}$$

$$\frac{1}{24}Lx^4 + \frac{1}{6}C_3 + C_1 = 0. \tag{1.121}$$

Solving simultaneously equations (4.63) and (4.64) for C_1 and C_3 yields

$$C_1 = \frac{1}{24}L, \quad \text{and} \quad C_3 = -\frac{1}{2}L.$$

Knowing all the coefficients we arrive at the expression for $z(x)$.

$$z(x) = \frac{L}{24}x(x - 1)(x^2 - x - 1). \tag{1.122}$$

Finding $\max_{0\leqslant x\leqslant 1} z(x)$ by Maple shows that the maximum is $\frac{5}{384}L$ and less than one for $L < \frac{384}{5} \approx 76.80$.

Comparing the new found bound $L < 76.80$ to the optimal bound, $L < 97.4091$, obtained in Example 3.8 shows that the function $w(x) = 1$ as a polynomial is a poor estimate of the eigenfunction.

Application of Theorem 2.2.

The Green's function for $Lu = u^{(4)}(x)$ with the boundary conditions $u(0) = u'(0) = u'(1) = u'''(1) = 0$ was found in Section 3, Example 3.2 is

$$G(x,s) = \begin{cases} -s^2(2s - 6x + 3x^2)/12, & \text{if } 0 \leqslant s < x \leqslant 1 \\ -x^2(3s^2 - 6s + 2x)/12, & \text{if } 0 \leqslant x \leqslant s \leqslant 1 \end{cases} \tag{1.123}$$

Notice that the Green's function is positive. To calculate M_1, consider

$$\int_0^1 |G(x,\, s)|ds = \int_0^1 G(x,\, s)ds = \int_0^x \frac{-s^2(2s - 6x + 3x^2)}{12}ds + \int_x^1 \frac{-x^2(3s^2 - 6s + 2x)}{12}ds =$$

$$= -\frac{1}{24}x^4(2x - 3) + \frac{1}{12}x^2(x - 1)(x^2 - 2) = \frac{1}{24}x^2(x - 2)^2.$$

A maximum occurs at $x = 1$. Thus $\int_0^1 |G(x,s)|ds \leqslant \frac{1}{24}$ implying $M_1 = \frac{1}{24}$. The 1^{th} derivative of $G(x,\, s)$ is

$$G_x(x,s) = \begin{cases} \frac{1}{2}s^2(1 - x), & \text{if } 0 \leqslant s < x \leqslant 1 \\ -\frac{1}{2}x(s^2 - 2s + x), & \text{if } 0 \leqslant x \leqslant s \leqslant 1 \end{cases}$$

To calculate M_2,we make an estimates as $G_x(x,\, s)$ is not of constant sign.

$$\begin{aligned}\int_0^1 |G_x(x,\, s)|ds &= \int_0^x |\frac{1}{2}s^2(1 - x)|ds + \int_x^1 |-\frac{1}{2}x(s^2 - 2s + x)|ds \leqslant \\ &\leqslant \int_0^x \frac{1}{2}s^2(1 - x)ds + \int_x^1 \frac{x}{2}(s^2 + 2s + x)ds \leqslant \\ &\leqslant -\frac{1}{6}(x - 1)x^3 - \frac{1}{6}x(x - 1)(x^2 + 7x + 4) \leqslant -\frac{1}{6}x(x - 1)(2x^2 + 7x + 4).\end{aligned}$$

A maximum occurs at the point $x = 0.6222$. Thus $\int_0^1 |G_x(x,s)|ds \leqslant 0.3577$ implying $M_2 = 0.3577$.

The 2^{th} derivative of $G(x,s)$ is

$$G_{xx}(x,s) = \begin{cases} -\frac{1}{2}s^2, & \text{if } 0 \leqslant s < x \leqslant 1 \\ -\frac{1}{2}s^2 + s - x & \text{if } 0 \leqslant x \leqslant s \leqslant 1 \end{cases}$$

Next, to calculate M_3,

$$\begin{aligned}\int_0^1 |G_{xx}(x,\, s)|ds &= \int_0^x |-\frac{1}{2}s^2|ds + \int_x^1 |-\frac{1}{2}s^2 + s - x|ds = \\ &\leqslant \int_0^x \frac{1}{2}s^2 ds + \int_x^1 (\frac{1}{2}s^2 + s - x)ds \leqslant \\ &\leqslant \frac{1}{6}x^3 - \frac{1}{6}x^3 + \frac{1}{2}x^2 - x + \frac{2}{3} = \frac{1}{2}x^2 - x + \frac{2}{3}.\end{aligned}$$

A maximum occurs at $x = 0$. Thus,

$$\int_0^1 |G_{xx}(x,\, s)|ds \leqslant \frac{2}{3} \quad \text{implying} \quad M_3 = \frac{2}{3}.$$

Next, to calculate M_4, note that G_{xxx} vanishes for $0 \leqslant s < x \leqslant 1$ is equal to -1 otherwise. Thus,

$$\int_0^1 |G_{xxx}(x,s)|ds = \int_x^1 |-1|ds.$$

$$\int_0^1 |G_{xxx}(x,s)|ds = 1 - x \implies M_4 = 1.$$

We have proved the following Theorem 4.1 using Theorem 2.2.

Theorem 4.1. Let $\mathrm{f}\colon [0,1] \times \mathbb{R}^5 \to \mathbb{R}$ satisfy (1.8).

Assume

$$\frac{L}{24} + (0.3577)K + \frac{2}{3}M + N < 1.$$

Then the boundary value problem

$$y^{(4)}(x) = f(x, y(x), y'(x), y''(x), y'''(x))$$

with boundary conditions

$$y(0) = y'(0) = y'(1) = y'''(1) = 0$$

has a unique solution.

The Green's function for $Lu = u^{(4)}(x)$ with the boundary conditions $u(0) = u'(0) = u''(1) = u'''(1) = 0$ was found in Section 3, Example 3.3 is

$$G(x,s) = \begin{cases} x^2(3s - x)/6, & \text{if } 0 \leqslant x \leqslant s \leqslant 1 \\ -s^2(s - 3x)/6, & \text{if } 0 \leqslant s < x \leqslant 1 \end{cases}$$

Notice that the Green's function is positive. To calculate M_1, consider

$$\int_0^1 |G(x,s)|ds = \int_0^1 G(x, s)ds = \int_0^x \frac{-s^2(s - 3x)}{6}ds + \int_x^1 \frac{x^2(3s - x)}{6}ds =$$
$$= \frac{1}{8}x^4 - \frac{1}{12}x^2(x + 3)(x - 1) = \frac{1}{24}x^2(x^2 - 4x + 6).$$

A maximum occurs at $x = 1$. Thus $\int_0^1 |G(x,s)|ds \leqslant \frac{1}{8}$ implying $M_1 = \frac{1}{8}$.
The 1^{th} derivative of $G(x, s)$ is positive as well.

$$G_x(x,s) = \begin{cases} sx - \frac{1}{2}x^2, & \text{if } 0 \leqslant x \leqslant s \leqslant 1 \\ \frac{1}{2}s^2, & \text{if } 0 \leqslant s < x \leqslant 1 \end{cases}$$

To calculate M_2,consider

$$\int_0^1 |G_x(x,s)|ds = \int_0^1 G_x(x, s)ds = \int_0^x \frac{1}{2}s^2 ds + \int_x^1 (xs - \frac{1}{2}x^2)ds =$$
$$= \frac{1}{6}x^3 - \frac{1}{2}x(x - 1) = \frac{1}{6}x(x^2 - 3x + 3).$$

A maximum occurs at $x = 1$. Thus $\int_0^1 |G(x,s)|ds \leqslant \frac{1}{6}$ implying $M_2 = \frac{1}{6}$.
The 2^{th} derivative is also positive

$$G_{xx}(x,s) = \begin{cases} s - x & \text{if } 0 \leqslant x \leqslant s \leqslant 1 \\ 0, & \text{if } 0 \leqslant s < x \leqslant 1 \end{cases}$$

To calculate M_3, note that G_{xx} vanishes for $0 \leqslant s \leqslant x \leqslant 1$. Thus,

$$\int_0^1 |G_{xx}(x,\, s)| ds \;=\; \int_0^1 G_{xx}(x,\, s) ds \;=\; \int_x^1 (s \,-\, x) ds \;=\; \frac{1}{2}(x \,-\, 1)^2.$$

A maximum occurs at $x \,=\, 0$. Thus $M_3 \,=\, \frac{1}{2}$. The 3^{th} derivative is

$$G_{xxx}(x, s) \;=\; \begin{cases} -1, & \text{if } 0 \leqslant x \leqslant s \leqslant 1 \\ 0, & \text{if } 0 \leqslant s < x \leqslant 1 \end{cases}$$

To calculate M_4, consider

$$\int_0^1 |G_{xxx}(x, s)| ds \;=\; \int_0^1 -G_{xxx}(x, s) ds \;=\; \int_x^1 1 ds \;=\; 1 \,-\, x.$$

Thus $M_4 \,=\, 1$.

Using Theorem 2.2., we have proved the following

Theorem 4.2. Let $\mathrm{f}\colon [0,1] \times \mathbb{R}^5 \;\to\; \mathbb{R}$ satisfies (1.8).

Assume

$$\frac{L}{8} \,+\, \frac{K}{6} \,+\, \frac{M}{2} \,+\, N \,<\, 1.$$

Then the boundary value problem

$$y^{(4)}(x) \;=\; f(x, y(x), y'(x), y''(x), y'''(x))$$

with boundary conditions

$$y(0) \;=\; y'(0) \;=\; y''(1) \;=\; y'''(1) \;=\; 0$$

has a unique solution.

Numerical examples

Consider the problem $y^{(4)}(x) = 10x + 300y(x)$ with non-homogeneous boundary conditions $y(0) = y'(0) = y'(1) = 0,\ y(1) = 5$.
The bound on the Lipschitz constant, L , for the homogeneous boundary value problem was found in Example 4.1.1 and is equal to 494.7239. Now let us compute the Lipschitz constant for our specific example. The right hand side is $f(x,y) = 10x + 300y(x)$ on $[0,1] \times \mathbb{R}$. The Lipschitz constant by the Mean Value Theorem is

$$\frac{\partial f}{\partial y} = 300 = L.$$

So, $L = 300 < 494.7239$. Since Theorem 2.1 requires homogeneous boundary conditions, it cannot be applied directly to this example. Define w to be a polynomial satisfying the same boundary conditions as above. We find $w(x) = 2x^3 + x^4 + 5$. Set $z(x) = y(x) - w(x)$. Moveover,

$$z^{(4)}(x) = y^{(4)}(x) - w^{(4)}(x) = 10x + 300y(x) - 24 = (10x - 24) + 300(z(x) + w(x)) = \tilde{f}(x, z(x)).$$

so z satisfies an ODE with the same Lipschitz constant. Also note that

$$z(0) = z'(0) = z(1) = z'(1) = 0.$$

Thus, Theorem 2.1 can be applied directly to the ODE $z^{(4)}(x) = \tilde{f}(x, z(x))$ with the homogeneous boundary conditions. As long as $L < 494.7239$, Theorem 2.1 and Example 4.1.1 give a unique solution z, satisfying the non-homogeneous boundary conditions, which in turn, gives a unique solution to the non-homogeneous problem, y. An illustration of this unique solution is given in Figure 1. The graph was obtained using numerical methods for boundary value problem by Maple.

Next, consider the non-homogeneous boundary value problem

$$y^{(4)}(x) = 3x^2 - 10\cos(y(x))$$

with boundary conditions $y(0) = y'(0) = y''(1) = y'''(1) = 0$. The bound on the Lipschitz constant, L , for the homogeneous boundary value problem was found in Example 4.1.3 and is equal to 12.2. Now let us compute the Lipschitz constant for our specific example. The non-homogeneous term $f(x,y) = 3x^2 - 10\cos(y(x))$ on $[0,1] \times \mathbb{R}$. The Lipschitz constant by the Mean Value Theorem is

$$\max_{0<x<1} \left|\frac{\partial f}{\partial y}\right| = \max_{0<x<1} |10\sin(y(x))| \leqslant 10 = L.$$

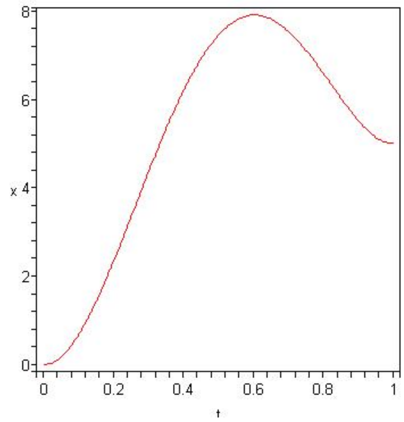

Figure 1: Solution to
$y^{(4)}(x) = 10x + 300y(x)$ with $y(0) = y'(0) = y'(1) = 0$ and $y(1) = 5$

So, $L = 10 < 12.2$. The solution curve was obtained using numerical methods for boundary value problem by Maple and given in Figure 2.

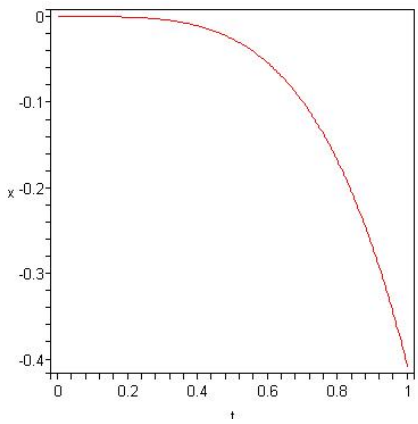

Figure 2: Solution to
$y^{(4)}(x) = 3x^2 - 10\cos y(x)$ with $y(0) = y'(0) = y''(1) = y'''(1) = 0$

Next non-homogeneous boundary value problem is

$$y^{(4)}(x) = 50(\sin(y(x)) - 1) + 40y(x)$$

with boundary conditions $y(0) = y''(0) = y(1) = y''(1) = 0$. The bound on the Lipschitz constant, L , for the non-homogeneous boundary value problem was found in Example 4.1.4 and is equal to 97.03. Now let us compute the Lipschitz constant for our specific example. The nonhomogeneous term $f(x, y) = 50\sin(y(x)) - 50 + 40x(t)$ on $[0, 1] \times \mathbb{R}$. The Lipschitz constant by the Mean Value Theorem is

$$\max_{0<x<1}|\frac{\partial f}{\partial y}| = \max_{0<x<1}|50\cos(y(x)) + 40| \leqslant 90 = L.$$

The solution curve was obtained using numerical methods for boundary value problem by Maple given in Figure 3. The last non-homogeneous boundary value problem is

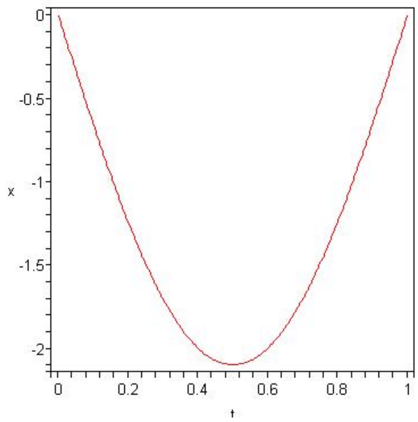

Figure 3: Solution to
$y^{(4)}(x) = 50(\sin(y(x)) - 1) + 40y(x)$ with $y(0) = y''(0) = y(1) = y''(1) = 0$

$$y^{(4)}(x) = -2|y(x)| + 3\cos y'(x) + 0.2y''(x)$$

with boundary conditions $y(0) = y'(0) = y''(1) = y'''(1) = 0$. This is an illustration of Theorem 4.2 with $a = 1$ and

$$f(x, y(x), y'(x), y''(x), y'''(x)) = -2|y(x)| + 3\cos y'(x) + 0.2y''(x).$$

It is necessary to choose K, L, M and N so that f satisfies the Lipschitz condition given in the theorem

$$\frac{L}{8} + \frac{K}{6} + \frac{M}{2} + N < 1.$$

Since f does not depend on y''', N can be chosen to be 0. By the Mean Value Theorem

$$|\frac{\partial f}{\partial y}| = |\frac{\partial}{\partial y} - 2(y^2)^{\frac{1}{2}}| = |2\frac{1}{2}(y^2)^{-\frac{1}{2}}(2y)| = |\frac{2y}{\sqrt{y^2}}| = 2 = L.$$

$$|\frac{\partial f}{\partial y'}| = |\sin(y')| \leqslant 3 = K.$$

$$|\frac{\partial f}{\partial y''}| = 0.2 = M.$$

So, $\frac{L}{8} + \frac{K}{6} + \frac{M}{2} + N = \frac{2}{8} + \frac{3}{6} + 0.2 = 0.95 < 1$. Therefore, the boundary value problem has a unique solution.

The solution curve was obtained using numerical methods for boundary value problem by Maple given in Figure 4.

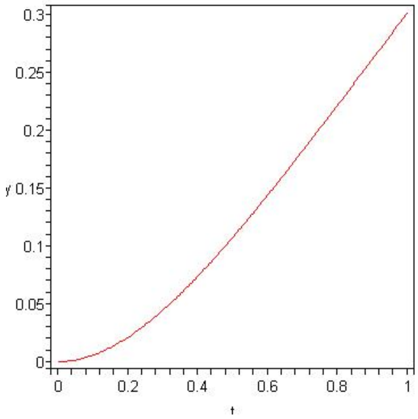

Figure 4: Solution to
$y^{(4)}(x) = -2|y(x)| + 3\cos y'(x) + 0.2y''(x) \quad \text{with} \quad y(0) = y'(0) = y''(1) = y'''(1) = 0$

References

[1] Bailey,P., Shampine,L., and Waltman,P. *Nonlinear Two-Point Boundary-Value Problema*, Ch.3, Academic Press. New York.1997.

[2] Boyce, William E., DiPrima, Richard C. *Elementary Differential Equations and Boundary Value Problems*, Third Edition, Ch. 1. John Wiley and Sons. New York. 1997.

[3] Coddington, E. and Levinson, N. *Theory of Ordinary Differential Equations*, McGraw-Hill.WaWaWa New York. 1995.

[4] Morrison, Shannon M., *Applibation of the Green's Functions for Solutions of Third Order Nonlinear Boundary Value Problems.* Masrer's Thesis, University of Tennessee, 2007.

[5] Walter, Wolfgang. *Ordinary Differential Equations*, Ch. 6, Springer. New York. 1998.

Vita

Olga Aleksandrovna Teterina was born in Retchitsa, Belorussia. After graduating High School in 1970, she entered Mechanical Institute in Leningrad, USSA and graduated with the Bachelor Degree in Radio Engineering in 1980. During graduated with the Bachelor Degree in Radio Engineering in 1980. During the following years, she was employed as an Engineer and Designer at she following years, she was employed as an Engineer and Designer at Izhevsk Productive Company, Motozavod. Upon moving to USA she entered the Graduate School at the University of Tennessee to pursue a degree in Mathematics. Olga graduated with a Master of Science degree in Applied Mathematics in August 2013.