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Some Congruence Modulo 2 Statements of Primitive Conway Vassiliev Invariants.

James M. Dawson

University of Tennessee - Knoxville

To the Graduate Council:

I am submitting herewith a thesis written by James M. Dawson entitled "Some Congruence Modulo 2 Statements of Primitive Conway Vassiliev Invariants.." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Jim Conant, Major Professor

We have read this thesis and recommend its acceptance:

Morwen Thistlethwaite, Nikolay Brodskiy

Accepted for the Council:
Dixie L. Thompson

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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Some Congruence Modulo 2 Statements of Primitive Conway Vassiliev Invariants

A Thesis Presented for
the Master of Science
Degree
The University of Tennessee, Knoxville

James M. Dawson
August 2009

Acknowledgements

Thanks to Dr. James Conant this thesis would not be possible without you.
Special thanks to Denise whose love and support I have always treasured.

Abstract

Polynomial knot invariants can often be used to define Vassiliev invariants on singular knots. Here Vassiliev invariants form the Conway, Jones, HOMFLY, and Kauffman polynomials are explored. Also, some explanation is given about how symbols of the Jones and Conway polynomial can evaluated on suitable chord diagrams. These invariants are further used to find expressions that are congruent modulo 2 to some low degree invariants derived from the Primitive Conway polynomial.

Contents

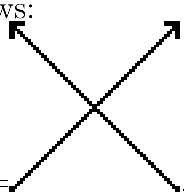
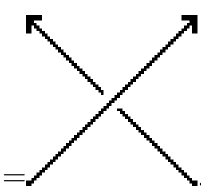
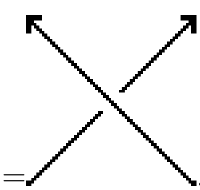
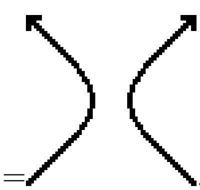
Chapter 1 Introduction

General Information

Definition 1. An *oriented knot* is a smooth embedding of the oriented circle in $\mathbb{R}^3$ (or in the sphere S^3) as well as the image of this embedding.
For a knot, we can switch a crossing in $\mathbb{R}^3$ by continuously moving an over crossing to an under crossing. During this transformation there is some point for which the over crossing and under crossing intersect. At this moment of intersection the classical oriented knot that we started with becomes a *singular knot* of degree one. Singular knots are considered up to flat vertex isotopy. This is similar to isotopy for classical knots except in order for two knots to be isotopic the transversality of the intersection must be preserved by the isotopy.

Definition 2. A *singular knot* of degree n is the immersion of S^1 in $\mathbb{R}^3$ with only n simple transverse intersection points (i.e., points where two branches intersect transversely).

Figure one shows how a classical unknot can be modified in order to represent the simplest singular knot. In this case we have simply required that the crossing be a singular point.
At this point, we should define some shorthand notation for oriented singular knots as follows:

$int =$ , $+ =$ , $- =$ , and $o =$ .

The set of singular knots of degree n considered up to isotopy is χ_n. Our goal now is to define functions on singular knots. These functions on singular knots will be based on knot invariants. Consider a knot invariant f. This invariant is not defined for singular knots, but we can define a derivative f' of the invariant f, which is defined on knots with one singular point.

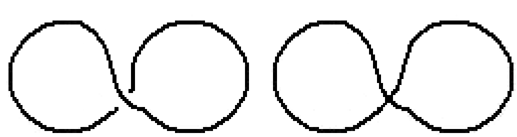


Figure 1: Classical and Singular Unknot

Definition 3. $f'(int) = f(+) - f(-)$

This relation is called *the Vassiliev relation.* It holds for all triples of diagrams that only differ outside of the small circular domain. Here, the first derivative of a knot invariant is only defined for degree 1 singular knots. In addition, we can define higher order derivatives of knot invariants, but as with the first derivative a higher order derivative is only defined for a singular knot of suitable degree. For example suppose that the n-th derivative of a knot invariant is defined for any two singular knots of degree n. We can define the n+1 derivative by using the Vassiliev relation with two singular knots of degree n and one singular knot of degree n+1 (n singular vertices of each of these singular knots lie outside of the "visible" part of the diagram.)

Definition 4. A knot invariant f is said to be a *Vassiliev invariant* of order $\leq n$ if its extension for the set of all (n+1) singular knots equals zero identically.

Definition 5. Denote by $\mathcal{V}_n$ the space of all Vassiliev knot invariants of order less than or equal to n.

Thus for any knot invariant *f*, we can define some invariant on the set of all singular knots. This invariant is called *the extension of f for singular knots.*

Definition 6. A Vassiliev invariant of order (type) $\leq n$ is said to have order n if it is not an invariant of order less than or equal to n-1.

The coefficients of many polynomial knot invariants can be used to define Vassiliev invariants. For example, the coefficients c_n of the Conway Polynomial C are Vassiliev invariants. Under a suitable substitution, the coefficients of the Jones, Homfly, and Kauffman polynomials can also be used to define Vassiliev invariants.

Definition 7. Let v be a Vassiliev invariant of order n, then $v^{(n+1)} = 0$. The function $v^{(n)}$ is called the symbol of v.

Some Examples of Vassiliev Invariants

Example 8. For each natural n, the function c_n, the n^{th} coefficient of the Conway polynomial, is a knot invariant of degree less than or equal to n.

Consider the skein relation of the Conway polynomial:

$$C(+) - C(-) = xC(o).$$

Now we can define the derivative as:

$$C'(int) = C(+) - C(-) = xC(o).$$

This means the the first derivative of the Conway polynomial is divisible by x; so, the constant term of the Conway polynomial disappears after the first differentiation. Similarly, the n-th derivative of the Conway polynomial must be divisible by x^n; so, after $n+1$ differentiations, c_n vanishes. This means that the coefficient c_n is a Vassiliev invariant of order less than or equal to n for each natural number n. In addition we know that

$$c_n'(int) = c_n(+) - c_n(-) = c_{n-1}(o).$$

In other words, evaluating a derivative of this Vassiliev invariant defined in this way only amounts to evaluating the invariant of one degree lower on a suitable knot or link.

Example 9. The coefficients a_n of the Jones Polynomial can also be used to define Vassiliev invariants by making the substitution $t = e^x$. Recall that the skein relation of the Jones polynomial is:

$$t^{-1}V(+) - tV(-) = (t^{1/2} - t^{-1/2})V(o)$$

and the Jones polynomial of the unlink is

$$V(\text{unlink}) = 1.$$

After this substitution is made, we can write the skein relation as:

$$Exp[-x]V(+) - Exp[x]V(-) = (Exp[x/2] - Exp[-x/2])V(o).$$

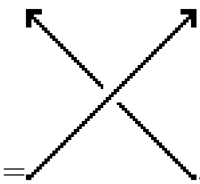
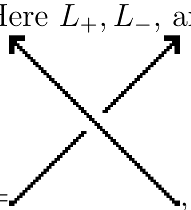
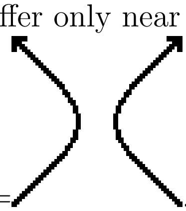
Now we can use the formal Taylor Series in x of the expression above and move all the members of the equation that are divisible by x to the right side of the equation. This will give a sum which is divisible by x on the right side of the equation and the derivative of the Jones polynomial on the left side of the equation; so that:

$$V(+) - V(-) = x\langle \text{power series in x} \rangle$$

We can follow the same argument as above for the second derivative in which case we will find that the second derivative is divisible by x^2. In general after $(n+1)$ differentiations, the n-th member of the series that is obtained after making the substitution above will vanish. Thus, the coefficient of the term x^n after the substitution above has been made can be used to define a Vassiliev invariant of order n. The previous 2 examples are also shown in [4].

Example 10. Vassiliev invariants can be made using the HOMFLY polynomial when a suitable substitution is made. The HOMFLY polynomial is a two variable polynomial in l and m. The HOMFLY polynomial of the unknot is 1, and The skein relation is

$$lP(L_+) + l^{-1}P(L_-) + P(L_0) = 0.$$

Here L_+, L_-, and L_0 differ only near a single crossing where: $L_+ =$ , $L_- =$ , and $l_o =$ .

In order to define a Vassiliev invariant we first make the variable change:

$$l = it^{N/2}$$

$$m = i(t^{-1/2} - t^{1/2}).$$

Using this substitution, the skien relation becomes

$$t^{N/2}P(L_+) - t^{-N/2}P(L_-) = (t^{1/2} - t^{-1/2})P(L_0).$$

Finally in order to derive a Vassiliev invariant, we make the substitution $t = exp(x)$ and write out the fully Taylor expansion of $exp(x)$. Some algebra reveals that

$$P(L_+) - P(L_-) = xS(x).$$

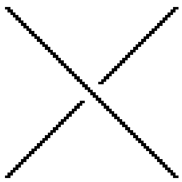
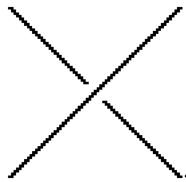
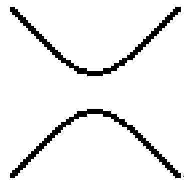
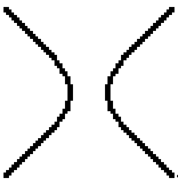
Where $S(x)$ is some polynomial in x. This means that we can define degree n Vassiliev invariants using the coefficients of x^n in the HOMFLY polynomial once the required substitutions are made. This example is shown in [3].

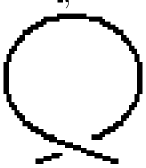
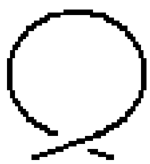
Example 11. It is shown in [4] that we can define a series of Vassiliev invariants using the 2-variable Kauffman polynomial as well. The Kauffman polynomial does not satisfy any skein relation, but it can be expressed in terms of the functions z, a, and $\frac{a-a^{-1}}{z}$ in the following way. First we define a function D on knots that is subject to the following relations:

$$D(L) - D(L') = z(D(L_A) - D(L_B));$$

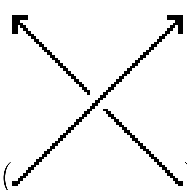
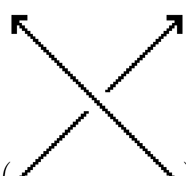
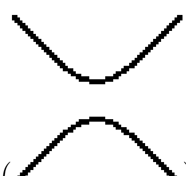
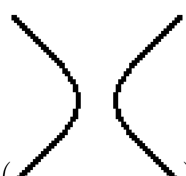
$$D(Unknot) = (1 + \frac{a-a^{-1}}{z});$$

$$D(X\#(R1+)) = aD(X), D(X\#(R1-)) = a^{-1}D(X)$$

Where we define $L =$ , $L' =$ , $L_A =$ , $L_B =$ ,

$R1+ =$ , and $R1- =$ .

Using D as we have defined it above, we can define the Kauffman polynomial to be $Y(L) = a^{-w(L)}D(L)$ where $w(L)$ is the writhe of L. Now we can use the equations above to define a skein relation for the Kauffman polynomial as:

$a^{-1}Y($$) - aY($$) = z(Y($$) - Y($$))$⟨Power of a⟩

In order to define Vassiliev invariants a substitution will be made so that the Kauffman polynomial can be expressed in terms of series of positive powers in the two variables z and p. In order to express the Kauffman polynomial in this way, first we perform the variable change $p = ln(\frac{a-1}{z})$ Then we need to make the substitutions:

$$z = z$$

$$a = zExp[p] + 1$$

$$a^{-1} = 1 - z(1 + p + ...) + z^2(1 + p + ...)^2 + ...$$

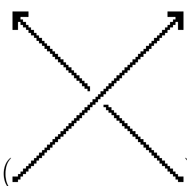
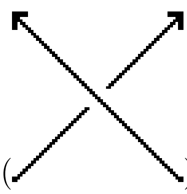
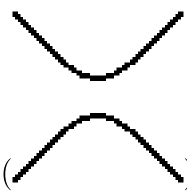
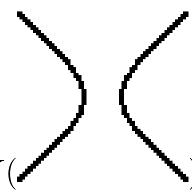
$$\frac{a - a^{-1}}{z} = a^{-1}(a + 1)Exp[p].$$

Each of the sequences on the right can be written as sequences of positive powers of p and z.

This tells us that the Kauffman polynomial in two variables on each knot is represented by positive powers of p and z. Since $a = 1 + z\langle\text{power series in p}\rangle$ and $a^{-1} = 1 + z\langle\text{a different power series in p}\rangle$, it can be deduced using

$$D(X\#(R1+)) = aD(X), D(X\#(R1-)) = a^{-1}D(X)$$

and

$a^{-1}Y($  $) - aY($  $) = z(Y($  $) - Y($  $))\langle\text{Power of a}\rangle$

that $Y' = z\langle\text{a different power series in p}\rangle$.

This tells us that all of the members of our double sequence with a degree that is less than or equal to n in the variable z will vanish after $(n + 1)$ differentiations. This means that we can get Vassiliev invariants from these members.

Chord Diagrams

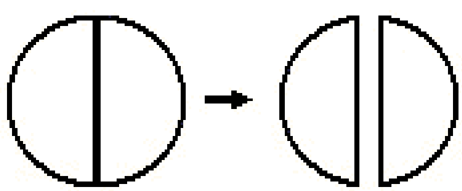
Definition 12. By a *chord diagram* we mean a finite *trivalent* graph consisting of one counterclockwise oriented *cycle* (circle) and unoriented *chords* (edges connecting different points on the cycle). By *trivalent* we mean that there are exactly three rays meeting at a point where a *chord* meets the *cycle*. The *order* of a chord diagram is the number of its chords.

With each singular knot we can associate a chord diagram in the following way. Think of the knot as the image of the standard oriented Euclidian circle S^1 in $\mathbb{R}^3$ then connect the pre-images of singular points using chords. Each invariant of degree n generates a function on the set of chord diagrams with n chords.

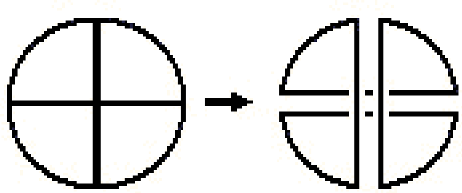
Chord diagrams can be used to evaluate symbols and some Vassiliev invariants of the Jones and Conway polynomial. In order to evaluate these invariants by using chord diagrams we must complete a “doubling” process on the chords to produce a new diagram. The “doubling” process is completed as follows. Consider a chord diagram D of order n. We can “double” each of the chords by inserting two parallel chords and deleting the small arc between them. By doing this, we will create an oriented circle with $2n$ small arcs deleted from it and n pairs of parallel chords. Now we could “walk” along our new diagram by picking an arbitrary point and moving along the oriented circle until we come to one of the $2n$ small deleted arcs. When we come to a deleted arc, we move down the chord to a new point on the oriented circle at the other end of the chord. If we continue this process long enough, we will eventually return to our starting point at which time we can stop our “walk”. There are two possible results from walking along this diagram. Either we have walked over the entire diagram that we created, or some portion of the diagram is left out. In the first case, we have created a new diagram which is connected. If some portion of the diagram is left out then we have created an object that is disconnected.

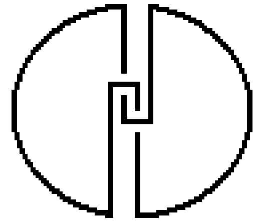
Example 13. We next show how to compute the symbols of the Vassiliev invariants derived from the Conway polynomial. Consider a chord diagram with only one chord. In this case, by doing the “doubling” process described above we must create two disjoint components. On the other hand if we were to start with a chord diagram with two chords that cross in the center of the circle and do the “doubling” process then we will be able to walk along the entire object.

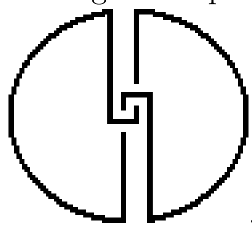
One chord doubled:



Two chords doubled:



More generally, we are not confined to inserting only parallel line for a chord. We could also insert a over crossing or an under crossing. For example, if we started with a chord diagram with only one chord inserting an over crossing would produce 

and inserting an under crossing would produce .

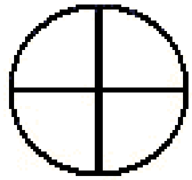
Calculations Using Chord Diagrams

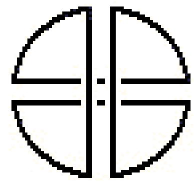
Now we can use this method to evaluate symbols from the Conway and Jones invariants. We will start with the Conway polynomial. The constant in the Conway polynomial, which we call c_0 can be used to define a function on links. In this case, for a link L the value of the constant of the Conway polynomial of that link can be used to define $c_0(L)$, which is a function on links. Specifically,

$$c_0(L) = \begin{cases} 1 & \text{if L has 1 component} \\ 0 & \text{if L has more than 1 component} \end{cases}$$

Since taking the derivative of invariants derived from the Conway polynomial produces the coefficients of lower powers in the Conway polynomial, evaluating the n^{th} derivative of the invariant c_n on a chord diagram is done by evaluating $c_0(L)$ on a the "doubling" of that chord diagram.
Notice that we can evaluate $c_0(L)$ on a chord diagram of any degree. In fact evaluating $c_0(L)$ on a chord diagram of degree n gives the value of the n^{th} derivative of c_n on that chord diagram. This means that for symbols of the Vassiliev invariants from the conway polynomial we know that:

$$c_n^{(n)}(\text{N-chord Diagram}) = \begin{cases} 1 & \text{if the doubled chord diagram has 1 component} \\ 0 & \text{if the doubled chord diagram has more than 1 component} \end{cases}$$

Example 14. Consider the chord diagram . Evaluating the 2^{nd} derivative of c_2 on this chord diagram is the same as evaluating $c_0(00)$ on this same chord diagram.

Which means that we need to compute the value of c_0 on . This diagram has one component; so, $c_0(00) = 1$.

Example 15. We can also compute the symbols of Vassiliev invariants derived form the Jones polynomial quite easily. Suppose that we want to compute the symbol of the invariant a_2. It will be shown later that the derivative of a_2 is given by the expression:

$$a_2' = a_1(0) + a_1(+) + a_1(-),$$

and the derivative of a_1 is given by:

$$a_1' = a_0(0) + a_0(+) + a_0(-).$$

Combining these two calculations shows that

$$a_2'' = a_0(00) + a_0(0+) + a_0(0-) + a_0(+0) + a_0(++) + a_0(+-) + a_0(-0) + a_0(-+) + a_0(--).$$

This means that in order to calculate the symbol of a_2 we need to calculate the value of a_0 on several different realizations of a degree 2 chord diagram. For the Jones polynomial it is true that

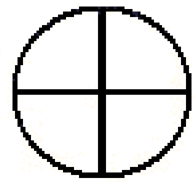
$$V(\text{n Component Unlink}) = (-t^{-1/2} - t^{1/2})^{n-1}.$$

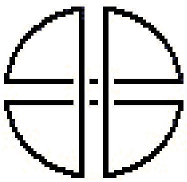
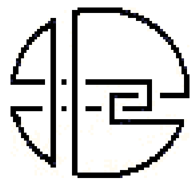
The value of a_0 on an unlink of n components, L, will be the constant term of this polynomial which is given by:

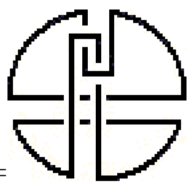
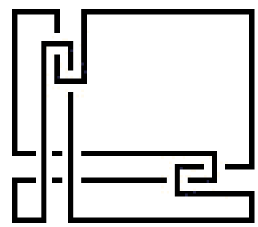
$$a_0(L) = (-2)^{\text{\#components of L}-1}.$$

To compute a_0 on chord diagrams of degree 2, we only need to produce the chord diagrams that correspond to each of the realizations in the second derivative and use the number of components in each to find the value of a_0 on that realization. Since the number of components is not affected if we add an over crossing or an under crossing, we can rewrite a_2'' for chord diagrams as:

$$a_2''(2 \text{ chord diagram}) = a_0(00) + 2a_0(0+) + 2a_0(+0) + 4a_0(++).$$

Now we can compute the value of the a_2'' on the chord diagram . First we notice that the 4 realizations of this chord diagram that we care about are:

(00)=  which has 1 component, (0+)=  which has 2 components,

(+0)=  which has 2 components, and (++) =  which has 1 component.-images to follow-
So, we see that $a_0(00) = (-2)^0 = 1$, $a_0(+0) = (-2)^1 = -2$, $a_0(0+) = (-2)^1 = -2$, and $a_0(++) = (-2)^0 = 1$. Now,

$$a_2''(2 \text{ chord diagram}) = 1 + 2(-2) + 2(-2) + 4(1) = -3$$

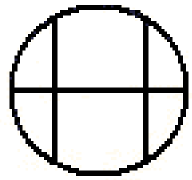
Definition 16. In order to compute the linking number of a two component oriented link we must sum the sign of each crossings between the two components and divide by 2.

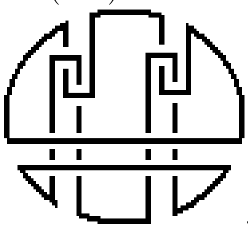
Example 17. We can use a similar technique as we did in the previous examples to determine the value of c_1 on chord diagrams with specified realizations. For an oriented two component link L, $c_1(L)$ is the linking number of the link L. This fact comes from the expression:

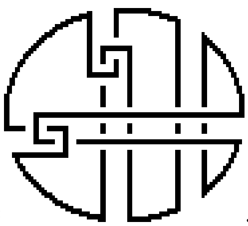
$$c_1(L_+) - c_1(L_-) = c_0(L_0).$$

If we switch a crossing that is contained in one component of the link then $c_1(L_+) - c_1(L_-) = 0$, since we will be computing c_0 on a link that must have more than one component. If the crossing is shared between the two components of a link then

$c_1(L_+) - c_1(L_-) = \pm 1$. The linking number of a two component link reacts the same way when a crossing is switched, and both the linking number and the value of c_1 agree on the two component unlink. This means that c_1 is the linking number for oriented two component links.

Let's compute the value of c_1 for the chord diagram  under the realizations (0++) and (- -0).

(0++)= , which has linking number 2.

(- -0)= , which has linking number -1.
So, $c_1(0++) = 2 \text{and} c_1(--0) = -1$

Relations on Chord Diagrams

Theorem 18. *If a chord diagram C has a isolated chord that does not intersect any other chords, then each symbol of a Vassiliev invariant evaluated at the diagram C equals zero. This fact is called the* 1T-relation *(or one-term relation)*
proof
If v is a Vassiliev invariant of order n then $v^{n+1} = 0$ for any link. This means that if two links are the same except at one crossing where one link has an over-crossing and the other has an under-crossing then the value of v^n is the same on both of these links. This tells us that v^n depends on the order in which singular points are encountered and not the knottedness of the classical knot from which the singular knot was created. Suppose we are evaluating a symbol on a chord diagram, C, with an isolated chord. Then the corresponding chord diagram will appear as shown in figure two where the regions above

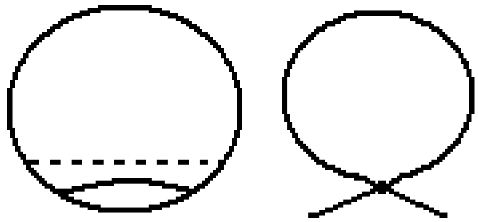
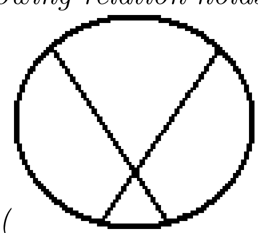
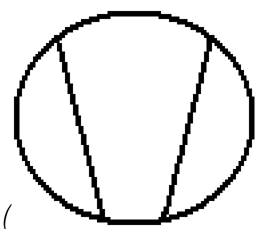
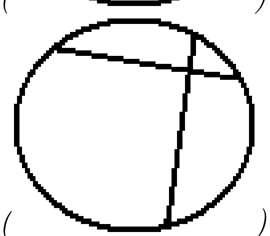
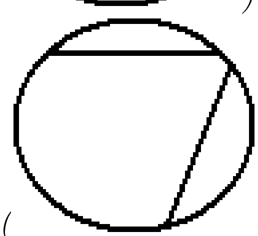


Figure 2: Chord Diagram and Corresponding Knot Segment

the dotted line can contain chords in any configuration, but under the dotted lines we have only one chord. Since we are only concerned with the order that we come to the chords in this diagram, we are essentially dealing with knot with an isolated segment in the knot as seen in figure two. Now at this singular point, the value of v^n on the over-crossing and under-crossing will be the same; so, the value of v^n on this type of chord diagram will be zero.

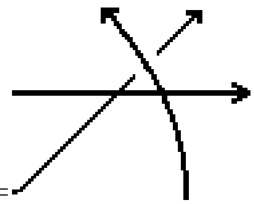
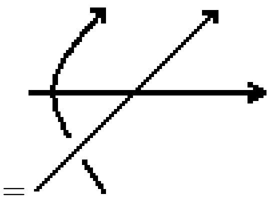
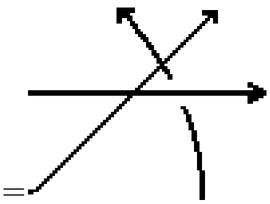
Theorem 19. *(The four-term relation) For each symbol $v^{(n)}$ of the invariant v the following relation holds:*

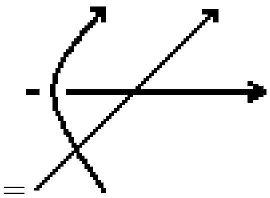
$v^{(n)}$()-$v^{(n)}$()-

$v^{(n)}$()+$v^{(n)}$()=0 *This relation means that for any four diagrams having n chords, where (n-2) chords (not shown in the Figure) are the same for all diagrams and the other two look as shown above, the above equality takes place.*

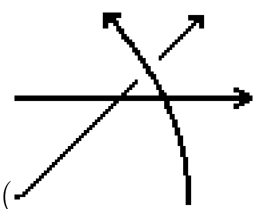
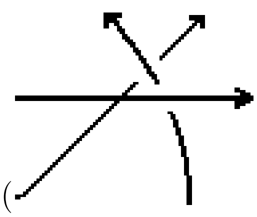
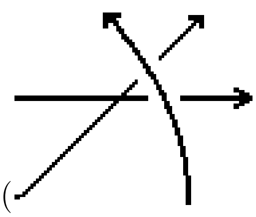
proof

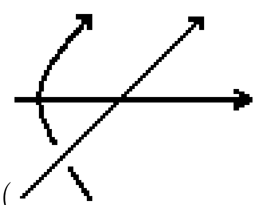
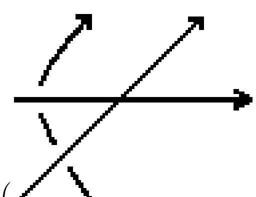
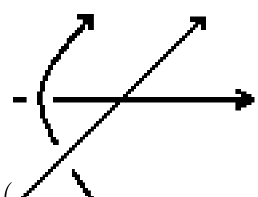
Consider four singular knots $S_1, S_2, S_3,$ andS_4 of the order n, whose diagrams coincide outside of some small circle. Inside the circle these singular knots are the fragments

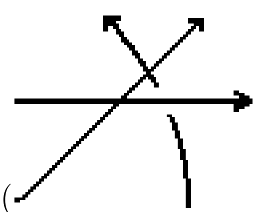
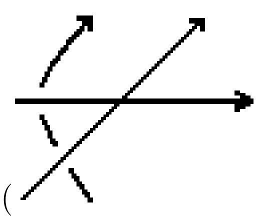
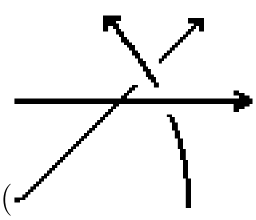
$s_1, s_2, s_3,$ and s_4 where $s_1=$  , $s_2=$  , $s_3=$  ,

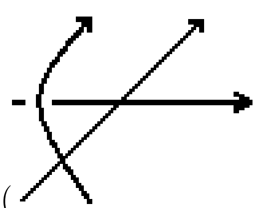
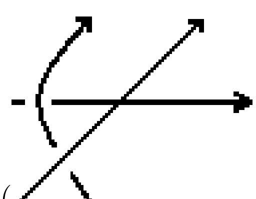
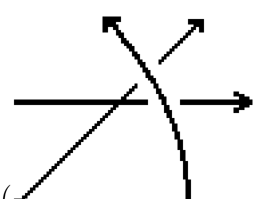
and $s_4=$  .

Consider some invariant v of order n. $v^{(n)}$ will be a symbol on these four singular knots since these knots are also of order n. This means that we can use the definition of the derivative of Vassiliev invariants to define the following set of relations:

$v^{(n)}($  $)=v^{(n-1)}($  $)-v^{(n-1)}($  $)=a-b$

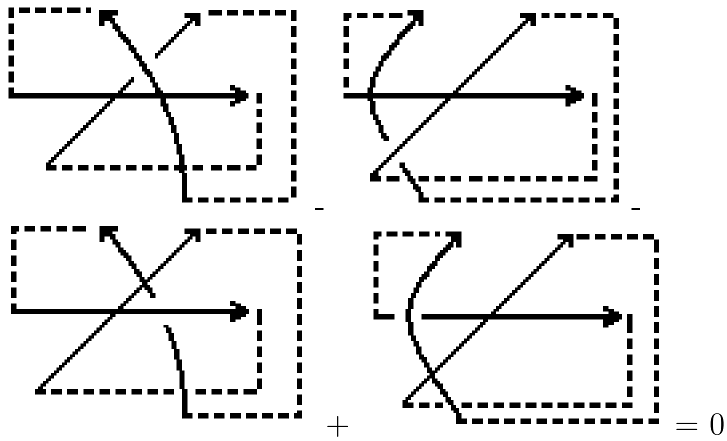
$v^{(n)}($  $)=v^{(n-1)}($  $)-v^{(n-1)}($  $)=c-d$

$v^{(n)}($  $)=v^{(n-1)}($  $)-v^{(n-1)}($  $)=c-a$

$v^{(n)}($  $)=v^{(n-1)}($  $)-v^{(n-1)}($  $)=d-b$

Using these relations we can get the following equality.

$(a-b)-(c-d)+(c-a)-(d-b)=0$. Which tells use that the $4-term$ relation applies for these fragments of diagrams. In order to get the $4-term$ relation, we need to show that these fragments can be thought of as singular knots. In order to turn these fragments into singular knots we need to close off the fragments in a way that gives us a set of singular knots with the chord diagrams that correspond to the relation that we want to prove. One way of closing off these fragments is as follows:



This means that $S_1, S_2, S_3, \text{and} S_4$ satisfy the relation $v^{(n)}(S_1)$-$v^{(n)}(S_2)$-$v^{(n)}(S_3)$+$v^{(n)}(S_4)$=0. Each of the chord diagrams corresponding to the singular knots $S_1, S_2, S_3, \text{and} S_4$ has n chords. $(n-2)$ of these chords are the same for all of these singular knots, and two of the chords are different in the chord diagram for each knot.

In this case, the order of $v^{(n)}$ is n, which means that when we are computing the value of $v^{(n)}$ on these chord diagrams we are computing the value of a symbol. This means that the value of $v^{(n)}$ on the singular knots $S_1, S_2, S_3, \text{and} S_4$ is the same as the value of $v^{(n)}$ on the corresponding chord diagrams. Now, if we let the remaining $(n-2)$ singular points in the diagrams of $S_1, S_2, S_3, \text{and} S_4$ be arbitrary, we get the$4term$ relation.

These two relations are given in [4]. They relations are useful when making calculations on chord diagrams using symbols.

Some Important Results Regarding Vassiliev Invariants and Weight Systems

Definition 20. Each linear function on chord diagrams of order n, satisfying these relations, is said to be a *weight system* (or order n).

Theorem 21. *Each symbol of an invariant of degree n comes from some element of gradation n of the chord diagram algebra (with 1T and 4T-relations).*

Theorem 22. *All the elements of the chord diagram algebra of degree n are symbols of Vassiliev knot invariants of degree n.*

These results are proved in [4]. This tells us that any linear combination of Vassiliev invariants forms a weight system on chord diagrams, which is a symbol of a Vassiliev invariant.

Chapter 2 Important Lemmas and Background Information for a Proof

We now proceed by proving some recursive formulae for low degree Vassiliev invariants. We begin by proving the base case, $i = 1$, for the following conjecture:

Conjecture 23. *There exist integer-valued Vassiliev invariants, v_{4i-1}, of degree $4i - 1$, such that:*

$$pc_{4i} = v_{4i-1} \mod 2$$

Before we prove the base case, we will start by proving three lemmas relating some coefficients in the conway polynomial with coefficients of the Jones polynomial.

Lemma 24. *On two component oriented links, $a_1(0) = -3c_1(0)$*

We begin with the skein relation for the Jones polynomial:

$$e^{-x}v(+) - e^{x}v(-) = (e^{x/2} - e^{-x/2})v(0)$$

Now expanding up to all linear terms in x we get:

$$(1-x)(a_0(+) + a_1(+)x) - (1+x)(a_0(-) + a_1(-)x) = (1 + x/2 - (1 - x/2))(a_0(0) + a_1(0)x)$$

Now we examine all of the coefficients of the x terms to get:

$$-a_0(+)x + a_1(+)x - a_0(-)x - a_1(+)x = a_0(0)x$$

Solving for $a_1(0)$ gives:

$$a_1(0) = a_1(+) - a_1(-) = a_0(0) - 4$$

We proceed by observing what happens when we switch a crossing in this two component link. If we switch a crossing that is contained in one component of the link then we find that $a_0(0) = a_0(3 \text{ comp. link}) = 4$, since resolving the crossing that we switch will result in adding a component. So, $a_1(+) - a_1(-) = 4 - 4 = 0$. If we switch a crossing that is shared between the two components of the link the result will be a knot where $a_0(0) = 1$. So that $a_1(0) = 1 - 4 = -3$

Lemma 25. *On oriented knots, $a_2(+) - a_2(-) = -3c_1(0)$.*

We know from the definition of the derivatives of a_2 and a_1 that:

$$a_2(+) - a_2(-) = a_1(0) + a_1(+) + a_1(-)$$

and

$$a_1(+) - a_1(-) = a_0(0) + a_0(+) + a_0(-).$$

Now, using the previous lemma we can make the substitution $a_1(0) = -3c_1(0)$ and solving for $a_1(+)$ in the first derivative statement allows us to write that:

$$a_2(+) - a_2(-) = -3c_1(0) + a_1(-) + a_0(0) + a_0(+) + a_0(-) + a_1(-)$$

then

$$a_2(+) - a_2(-) = -3c_1(0) + 2a_1(-) + a_0(0) + a_0(+) + a_0(-).$$

For a knot, $a_0(0) = -2$, $a_0(+) = 1$, and $a_0(-) = 1$. This means that for a knot, $a_0(0) + a_0(+) + a_0(-) = 0$, and our second derivative statement becomes:

$$a_2(+) - a_2(-) = -3c_1(0) + 2a_1(-).$$

Now recalling the a_1 is zero on knots gives us that:

$$a_2(+) - a_2(-) = -3c_1(0).$$

Which is what we wanted to show.

Lemma 26. *On oriented knots, $a_2(-) = -3c_2(-)$*

We first consider the Jones and Conway polynomial of the a specific knot. In this case we will use the knot 72 The Conway polynomial of this knot is $3t^2 + 1$ and the Jones polynomial of this knot is $t^{-1} - t^{-2} + 2t^{-3} - 2t^{-4} + 2t^{-5} - t^{-6} + t^{-7} - t^{-8}$. The degree 2 Vassiliev invariant from the Conway polynomial is simply the coefficient of the t^2 term, which is 3. To find the degree 2 Vassiliev invariant from the Jones polynomial we first make the substitution $t = e^x$. We then take the formal logarithm of the polynomial in order to get an additive invariant. The first few terms in the series expansion are $-9x^2 + 36x^3 - 513/4x^4$. The degree 2 Vassiliev invariant is now just the coefficient of the x^2 term in the expansion. So, for the Jones polynomial the value of the invariant is -9. So for this example, $a_2(-) = -3c_2(-)$. Since there is only one Vassiliev invariant of degree 2, we know that these 2 invariants must be multiples of each other, and the only possible choice is the the one that we have calculated.

Now to prove the base case $i = 1$. Can we find some invariant v_3 of degree 3 so that $pc_4 = v_3 \mod 2$? It happens that we can show that the degree 4 primitive conway invariant is congruent mod 2 to a multiple of the degree 3 Jones polynomial invariant. In this case we want to choose $v_3 = 1/6a_3$. We now show by direct calculation that:

$$pc_4 = v_3 \mod 2$$

Chapter 3 A Congruence Modulo 2 Statement for pc_4

Theorem 27. *On Knots*

$$pc_4 = v_3 \mod 2.$$

In order to show this, we must compute several derivatives of both sides of this statement and show that the derivatives are congruent modulo 2 on a suitable set of chord diagrams. More specifically, for the $n-th$ derivative, we must show that equality holds for a generating set of degree n chord diagrams. We can now compute the first derivatives of pc_4 and $1/6a_3$.
Using the skein relation for the Jones polynomial and by making the substitution $t = e^x$, we can define the higher order derivatives of the Jones polynomial. We start by defining the derivative of the Vassiliev invariant given by the third coefficient of the Jones Polynomial when the above substitution is made. To do this, we make the above substitution and find the coefficients of the x^3 term. In this case, we find that

$$a_3(+) - a_2(+) + 1/2a_1(+) - 1/6a_0(+) - (a_3(-) + a_2(-) + 1/2a_1(-) + 1/6a_0(-)) = a_2(0) + 1/24a_0(0).$$

Here "+" and "-" denote positive and negative crossing type and "0" denotes the resolution of a crossing. Now, solving for $a_3(+) - a_3(-)$ will give the desired derivative of the Vassiliev invariant. That calculation tells us that:

$$a_3(+) - a_3(-) = a_2(0) + 2a_2(-) + a_1(0) + 1/24a_0(0) + 1/6a_0(+) + 1/6a_0(-).$$

Now, since the value of $a_0(+), a_0(-), \text{and} a_0(0)$ only depend on the number of components of a link, we know that the value of $1/6(a_0(+) + a_0(-)) + 1/24a_0(0)$ will always be $1/4$ for any knot. So we can write the derivative as:

$$a_3(+) - a_3(-) = a_2(0) + 2a_2(-) + a_1(0) + 1/4.$$

Using a similar calculation for the coefficients of the x^2 and x terms tells us that:

$$a_2(+) - a_2(-) = a_1(0) + a_1(+) + a_1(-)$$

and

$$a_1(+) - a_1(-) = a_0(0) + a_0(+) + a_0(-).$$

Now to the Primitive Conway polynomial, which is defined in terms of the Conway polynomial so that the resulting polynomial is additive on knots. In this case we find that the first few primitive Conway coefficients are defined as:

$$pc_2 = -c_2$$

$$pc_4 = c_4 - 1/2(c_2 + c_2^2)$$

$$pc_6 = -c_6 + c_2c_4 + 1/3(c_2 + c_2^3)$$

For the first derivative of pc_4 we know that we need to compute $pc_4(+) - pc_4(-)$.

$$pc_4(+) - pc_4(-) = c_4(+) - 1/2c_2^2(+) - 1/2c_2(+) - c_4(-) + 1/2c_2^2(-) + 1/2c_2(-)$$

$$= c_3(0) - 1/2c1(0) - 1/2(c_2^2(+) - c_2^2(-))$$

$$= c_3(0) - 1/2c_1(0) - 1/2(c_2(+)c_2(+) - c_2(+)c_2(-) - c_2(-)c_2(-) + c_2(+)c_2(+))$$

$$= c_3(0) - 1/2c_1(0) - 1/2c_2(+)c_1(0) - 1/2c_2(-)c_1(0)$$

Now, we can show that the derivatives

$$pc_4(+) - pc_4(-) = c_3(0) - 1/2c_1(0) - 1/2c_2(+)c_1(0) - 1/2c_2(-)c_1(0)$$

and

$$v_3(+) - v_3(-) = 1/6a_2(0) + 1/6a_1(0) + 1/3a_2(-) + 1/24.$$

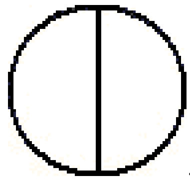
are congruent modulo 2. We begin by, rearranging some terms and making some appropriate substitutions will allow us to right our 1^{st} derivative statement as below. We want to write the statement this way; because, this statement allows us to state a lemma regarding some functions on knots and links, which will be discussed later.
We start this rearrangement with the derivatives as they appear above and substitutions $a_1(0) = -3c_1(0)$ and $a_2(-) = -3c_2(-)$ to get:

$$c_3(0) - 1/2c_1(0) - 1/2c_2(+)c_1(0) - 1/2c_2(-)c_1(0) = 1/6a_2(0) - 1/2c_1(0) - c_2(-) + 1/24$$

$$-1/2c_2(+)c_1(0) - 1/2c_2(-)c_1(0) + 1/2c_2(-) = -c_3(0) + 1/6a_2(+) + 1/6a_2(0) + 1/24$$

We now show congruence for the 1^{st} derivative:

$$c_2(-)(c_1(0)-1) = -1/6a_2(0) + c_3(0) - 1/2(c_1(0))^2 - 1/24 \mod 2$$

We must show this statement is true for . For this chord diagram, the two first derivatives are:

$$c_2(-)(1-c_1(0)) = 0(1-0) = 0$$

$$-1/6a_2(0) - c_1(0) - c_3(0) - 1/2(c_1(0))^2 - 1/24 =$$

$$= -1/6(-1/4) - 0 - (0) - 1/2(0) - 1/24 = 0$$

So the first derivatives are congruent on a suitable set of chord diagrams.
We next compute the second derivative of each side of our first derivative. The second derivative of $c_2(-)(c_1(0)-1)$ is given by:

$$c_2(-+)(c_1(0+)) - c_2(--)(c_1(0-)-1)$$

$$= c_2(-+)(c_1(0+)) - c_2(--)(c_1(0+)-1) + c_2(--)(c_1(0+)-1) - c_2(--)(c_1(0-)-1)$$

$$= (c_1(0+)-1)c_1(-0) + c_2(--)c_0(00).$$

The second derivative of $-1/6a_2(0) + c_3(0) - 1/2(c_1(0))^2 - 1/24$ is given by:

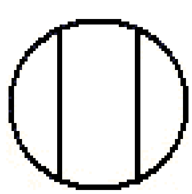
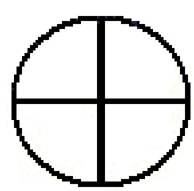
$$c_2(00) - 1/2(c_1(0+)c_1(0+) - c_1(0-)c_1(0-)) - 1/6(a_1(00) + a_1(0+) + a_1(0-)$$

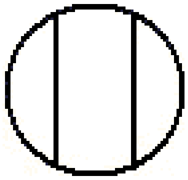
$$= c_2(00) - 1/2(c_1(0+)c_0(00) + c_1(0-)c_0(00) - 1/6(a_1(00) + a_1(0+) + a_1(0-)$$

We now show congruence for the 2^{nd} derivative:

$$c_1(-0)(c_1(0+)-1) + c_2(--)c_0(00) =$$

$$= c_2(00) - 1/2(c_1(0+)c_0(00) + c_1(0-)c_0(00)) - 1/6(a_1(00) + a_1(0+) + a_1(0-)) \mod 2$$

We must show this statement is true for  and .

For  we can compute that:

$c_0(00) = 0$, since this chord diagram under the (00) realization has three components.

$a_1(00) = 0$, since this chord diagram under the (00) realization has linking number zero.

$a_1(0-) = 0$, since this chord diagram under the (0-) realization has linking number zero.

$a_1(0+) = 0$, since this chord diagram under the (0+) realization has linking number zero.

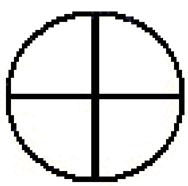
$c_1(-0) = 0$, since this chord diagram under the (-0) realization has linking number 0.

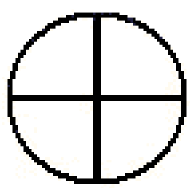
$c_2(00) = 0$, since this chord diagram under the (00) realization is the three component unlink and the value of $c_2(3unlink) = 0$.

So, our statements for the second derivative becomes:

$$c_1(-0)(c_1(0+) - 1) + c_2(--)c_0(00) = 0(0-1) + c_2(--)(0) = 0$$

$$c_2(00) - 1/2(c_1(0+)c_0(00) + c_1(0-)c_0(00)) - 1/6(a_1(00) + a_1(0+) + a_1(0-)) =$$
$$= 0 - 1/2(c_1(0+)(1) + c_1(0-)(1)) - 1/6(0+0-0) = 0$$

Which are congruent modulo 2, which means the second derivative statement works for the chord diagram .

For the chord diagram  we can compute that: $c_0(00) = 1$, since this chord diagram under the (00) realization has one components.

$a_1(00) = 0$, since this chord diagram under the (00) realization has linking number zero.

$a_1(0-) = 3$, since this chord diagram under the (0-) realization has linking number -1.

$a_1(0+) = -3$, since this chord diagram under the (0+) realization has linking number 1.

$c_1(0-) = -1$, since this chord diagram under the (0-) realization has linking number -1.

$c_1(-0) = -1$, since this chord diagram under the (-0) realization has linking number -1.

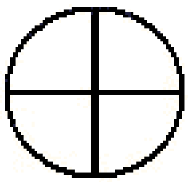
$c_1(0+) = 1$, since this chord diagram under the (-0) realization has linking number 1.

$c_2(--) = 0$, since this chord diagram under the $(--)$ realization is the unknot and $c_2(o) = 0$.

$c_2(00) = 0$, since this chord diagram under the (00) realization is the unknot and $c_2(o) = 0$. So, our statements for the second derivative becomes:

$$c_1(-0)(c_1(0+) - 1) + c_2(--)c_0(00) = 0(1-1) + 0(1) = 0$$

$$c_2(00) - 1/2(c_1(0+)c_0(00) + c_1(0-)c_0(00)) - 1/6(a_1(00) + a_1(0+) + a_1(0-)) =$$
$$= 0 - 1/2(0(1) + 0(1)) - 1/6(0 - 3 + 3) = 0$$

Which are congruent modulo 2, which means the second derivative statement works for the chord diagram . So the second derivatives are congruent on a suitable set of chord diagrams.

Now we must compute the third derivative of each side of the second derivative. The third derivative of $(c_1(0+) - 1)c_1(-0) + c_2(--)c_0(00)$. is given by:

$$(c_1(0++) - 1)c_1(-0+) - (c_1(0+-))c_1(-0-) + c_2(--+)c_0(00+) - c_2(---)c_0(00-)$$

$$= (c_1(0++) - 1)c_0(-00) + c_1(-0-)c_0(0+0) + c_0(00-)c_1(--0)$$
$$= -c_0(-00) + c_1(0++)c_0(-00) + c_1(-0-)c_0(0+0) + c_0(00-)c_1(--)$$

The third derivative of
$c_2(00) - 1/2(c_1(0+)c_0(00) + c_1(0-)c_0(00)) - 1/6(a_1(00) + a_1(0+) + a_1(0-))$ is given by:

$$c_1(000) - 1/2[c_1(0++)c_0(00+) - c_1(0+-)c_0(00-) + c_1(0-+)c_0(00+) - c_1(0--)c_0(00-)$$

$$+1/6[a_0(000)+a_0(00+)+a_0(00-)+a_0(0+0)+a_0(0++)+a_0(0+-)+a_0(0-0)+a_0(0-+)+a_0(0--)]$$

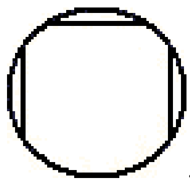
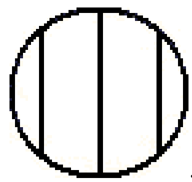
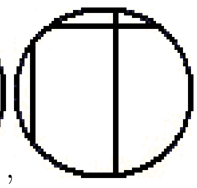
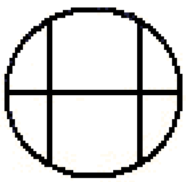
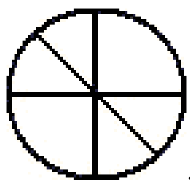
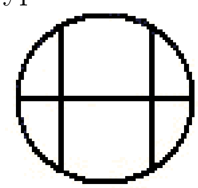
$$= c_1(000)-1/2[c_0(00+)c_0(0+0)+c_0(00-)c_0(00-)]+1/6[a_0(000)+2a_0(00+)+2a_0(0+0)+4a_0(0++)]$$
$$= c_1(000) - c_0(00+)c_0(0+0) + 1/6[a_0(000) + 2a_0(00+) + 2a_0(0+0) + 4a_0(0++)]$$

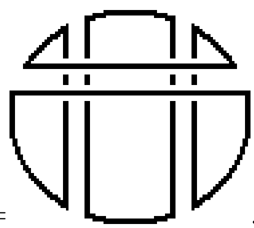
Here we are able to combine terms by change all minus crossings to plus crossings for c_0 and a_0 terms since these terms only count the number of components when we create a realization of chord diagram.

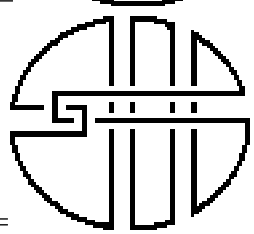
So, the 3^{rd} derivative statement is:

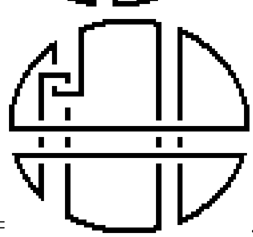
$$-c_0(-00) + c_1(0++)c_0(-00) + c_1(-0-)c_0(0+0) + c_1(--0)c_0(00-) =$$

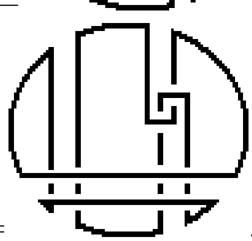
$= c_1(000) - c_0(00+)c_0(0+0) + 1/6[a_0(000) + 2a_0(00+) + 2a_0(0+0) + 4a_0(0++)] \mod 2$

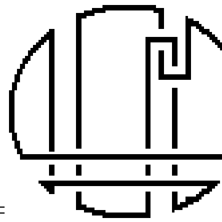
We must show this statement is true for , , , , and . To get an idea of the types of calculations that we need to do, lets show that the 3^{rd} derivatives are congruent for .

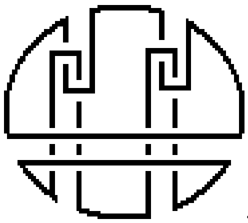
(000)= , which has 2 components and linking number 0.

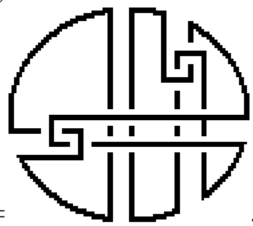
(-00)= , which has 3 components.

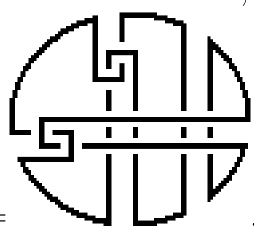
(0+0)= , which has 1 component.

(00-)= , which has 1 component.

(00+)=  , which has 1 component.

(0++)=  , which has 2 components and linking number 2.

(-0-)=  , which has 2 components and linking number -1.

(- -0)=  , which has 2 components and linking number -1.

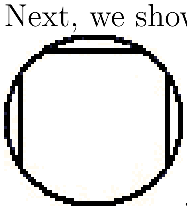
Now we get the 3^{rd} derivative as:

$c_1(000) - c_0(00+)c_0(0+0) + 1/6[a_0(000) + 2a_0(00+) + 2a_0(0+0) + 4a_0(0++)]$=0-(1)(1)+1/6[-2+2(1)+2(1)+4(-2)]=-2

$-c_0(-00) + c_1(0++)c_0(-00) + c_1(-0-)c_0(0+0) + c_1(--0)c_0(00-)$=0+2(0)-1-1=-2

So, the equality statement for the 3^{rd} derivative holds for this chord diagram.

Next, we show that the 3^{rd} derivative statement is true for the chord diagram

 . In this case we will show less detail in the calculation.

Since this diagram has an isolated chord, the values of $-c_0(-00)$, $c_0(-00)$, $c_0(0+0)$, $c_0(00-)$, $c_0(00+)$, and $c_0(0+0)$ are all zero. This follows directly from the fact that all of these realizations have more than one component. We can also compute that: $c_1(000) = 0$, since this chord diagram under the (000) realization has linking number 0.

$a_0(0++) = -2$, since this chord diagram under the (0++) realization has 2 component.

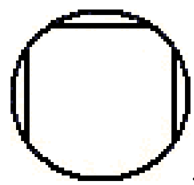
$a_0(000) = -8$, since this chord diagram under the (000) realization has 4 component.
$a_0(00+) = 4$, since this chord diagram under the (00+) realization has 3 component.
$a_0(0+0) = 4$, since this chord diagram under the (0+0) realization has 3 component.
Now, our statements for the third derivatives become:

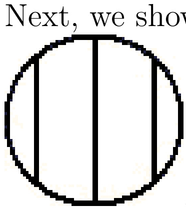
$$-c_0(-00) + c_1(0++)c_0(-00) + c_1(-0-)c_0(0+0) + c_1(--0)c_0(00-)$$

$$= 0 + c_1(0++)(0) + c_1(-0-)(0) + c_1(--0)(0) = 0$$

$$c_1(000) - c_0(00+)c_0(0+0) + 1/6[a_0(000) + 2a_0(00+) + 2a_0(0+0) + 4a_0(0++)]$$

$$= 0 - 0(0) + 1/6[-8 + 2(4) + 2(4) + 4(-2)] = 0$$

So the 3^{rd} derivative statement is true for the chord diagram  .
Next, we show that the 3^{rd} derivative statement is true for the chord diagram

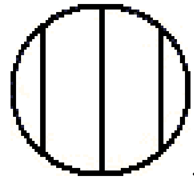
. Since this diagram has an isolated chord, the values of $-c_0(-00)$, $c_0(-00)$, $c_0(0+0)$, $c_0(00-)$, $c_0(00+)$, and $c_0(0+0)$ are all zero. This follows directly from the fact that all of these realizations have more than one component. We can also compute that:
$c_1(000) = 0$, since this chord diagram under the (000) realization has linking number 0.
$a_0(0++) = -2$, since this chord diagram under the (0++) realization has 2 component.
$a_0(000) = -8$, since this chord diagram under the (000) realization has 4 component.
$a_0(00+) = 4$, since this chord diagram under the (00+) realization has 3 component.
$a_0(0+0) = 4$, since this chord diagram under the (0+0) realization has 3 component.
Now, our statements for the third derivatives become:

$$-c_0(-00) + c_1(0++)c_0(-00) + c_1(-0-)c_0(0+0) + c_1(--0)c_0(00-)$$

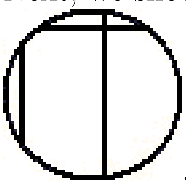
$$= 0 + c_1(0++)(0) + c_1(-0-)(0) + c_1(--0)(0) = 0$$

$$c_1(000) - c_0(00+)c_0(0+0) + 1/6[a_0(000) + 2a_0(00+) + 2a_0(0+0) + 4a_0(0++)]$$

$$= 0 - 0(0) + 1/6[-8 + 2(4) + 2(4) + 4(-2)] = 0$$

So the 3^{rd} derivative statement is true for the chord diagram  .

Next, we show that the 3^{rd} derivative statement is true for the chord diagram

. Since this diagram has an isolated chord, the values of $-c_0(-00)$, $c_0(0+0)$, $c_0(00-)$, $c_0(00+)$, and $c_0(0+0)$ are all zero except $c_0(-00)$. This follows directly from the fact that all of these realizations except $c_0(-00)$ have more than one component. We can also compute that: $c_0(-00) = 1$, since this chord diagram under the (-00) realization has 1 component.

$c_1(0++) = 0$, since this chord diagram under the (0++) realization has linking number 0.

$c_1(000) = 0$, since this chord diagram under the (000) realization has linking number 0.

$a_0(0++) = -2$, since this chord diagram under the (0++) realization has 2 component.

$a_0(000) = -2$, since this chord diagram under the (000) realization has 2 component.

$a_0(00+) = 4$, since this chord diagram under the (00+) realization has 3 component.

$a_0(0+0) = 4$, since this chord diagram under the (0+0) realization has 3 component.

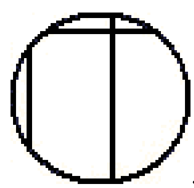
Now, our statements for the third derivatives become:

$$-c_0(-00) + c_1(0++)c_0(-00) + c_1(-0-)c_0(0+0) + c_1(--0)c_0(00-)$$

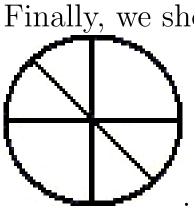
$$= 0 + c_1(0++)(0) + c_1(-0-)(0) + c_1(--0)(0) = 1$$

$$c_1(000) - c_0(00+)c_0(0+0) + 1/6[a_0(000) + 2a_0(00+) + 2a_0(0+0) + 4a_0(0++)]$$

$$= 0 - 0(0) + 1/6[-2 + 2(4) + 2(4) + 4(-2)] = 1$$

So the 3^{rd} derivative statement is true for the chord diagram  .

Finally, we show that the 3^{rd} derivative statement is true for the chord diagram

.

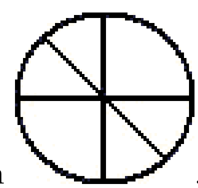
$c_0(00-) = 1$, since this chord diagram under the (00-) realization has 1 component.
$c_0(0+0) = 1$, since this chord diagram under the (0+0) realization has 1 component.
$c_0(-00) = 1$, since this chord diagram under the (-00) realization has 1 component.
$c_1(0++) = 2$, since this chord diagram under the (0++) realization has linking number 2.
$c_1(--0) = -2$, since this chord diagram under the (–0) realization has linking number -2.
$c_1(-0-) = -2$, since this chord diagram under the (-0-) realization has linking number -2.
$c_1(000) = 0$, since this chord diagram under the (000) realization has linking number 0.
$a_0(0++) = -2$, since this chord diagram under the (0++) realization has 2 component.
$a_0(000) = -2$, since this chord diagram under the (000) realization has 2 component.
$a_0(00+) = 1$, since this chord diagram under the (00+) realization has 1 component.
$a_0(0+0) = 1$, since this chord diagram under the (0+0) realization has 1 component.
Now, our statements for the third derivatives become:

$$-c_0(-00) + c_1(0++)c_0(-00) + c_1(-0-)c_0(0+0) + c_1(--0)c_0(00-)$$

$$= -1 + 2(1) - 2(1) - 2(1) = -3$$

$$c_1(000) - c_0(00+)c_0(0+0) + 1/6[a_0(000) + 2a_0(00+) + 2a_0(0+0) + 4a_0(0++)]$$

$$= -1 - 1(1) + 1/6[-2 + 2(1) + 2(1) + 4(-2)] = -1$$

So the 3^{rd} derivative statement is true for the chord diagram  .
Which means that the 3^{rd} derivative statement holds for all of the chord diagrams.
We next show that the 4^{th} derivative equality statement hold for a suitable set of chord diagrams. We start by computing the fourth derivatives. The fourth derivative of $c_1(000) - c_0(00+)c_0(0+0) + 1/6[a_0(000) + 2a_0(00+) + 2a_0(0+0) + 4a_0(0++)]$ is:

$$c_0(0000).$$

The fourth derivative of
$-c_0(-00) + c_1(0++)c_0(-00) + c_1(-0-)c_0(0+0) + c_1(--0)c_0(00-)$ is:

$$c_1(0+++)c_0(-00+) - c_1(0++-)c_0(-00-) + c_1(-0-+)c_0(0+0+) - c_1(-0--)c_{(}0+0-)$$

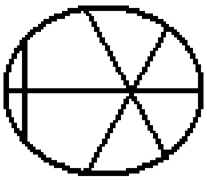
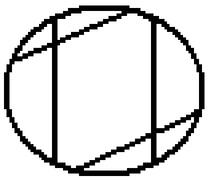
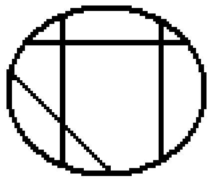
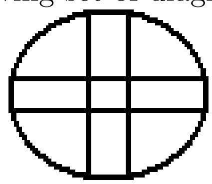
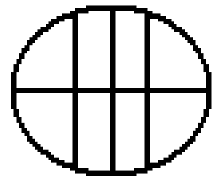
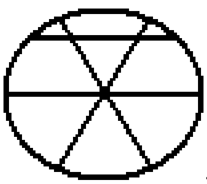
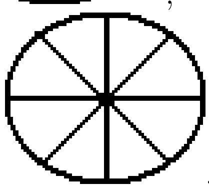
$$+c_0(00-+)c_1(--0+) - c_0(00--)c_1(--0-)$$

$$= c_0(-00+)c_0(0++0) + c_0(0+0-)c_0(-0-0) + c_0(00-+)c_0(--00)$$

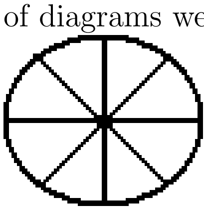
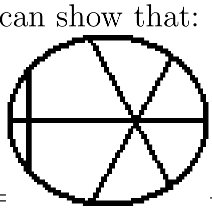
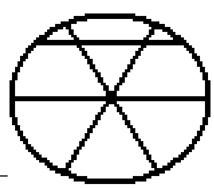
So, the 4^{th} derivative statement is:

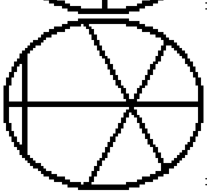
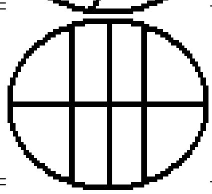
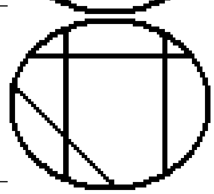
$$c_0(-00+)c_0(0++0) + c_0(0+0-)c_0(-0-0) + c_0(00-+)c_0(--00) = c_0(0000) \mod 2.$$

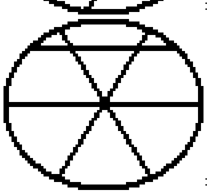
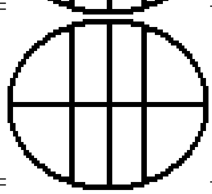
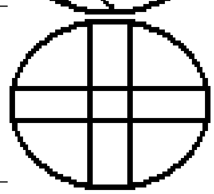
We must show this statement is true for the following set of diagrams with four chords:

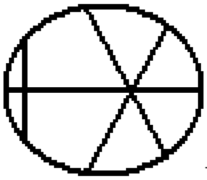
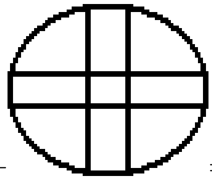
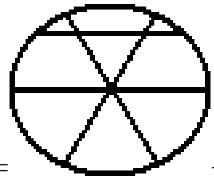
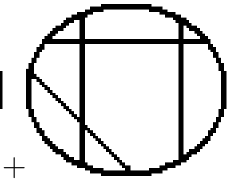
, , , , , , and .

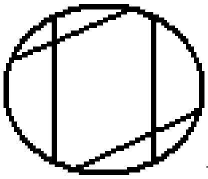
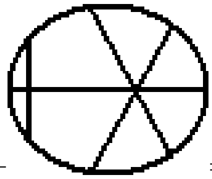
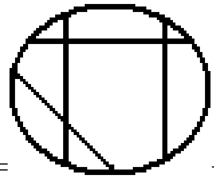
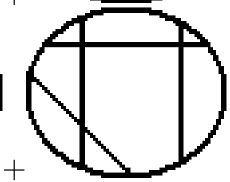
Unlike with the lower order derivatives, we do not have to check for equality on all diagrams with four chords. The 4^{th} derivative is a statement about symbols and according to the $1-term$ relation, the value of any symbol on a chord diagram with an isolated chord will be zero. This means that for the above statement is true for any degree 4 chord diagram with an isolated chord. In addition, we can reduce this list further by using the $4-term$ relation. It turns out that we can generate the entire set of chord diagrams above using only three chord diagrams. Notice that by applying the $4-term$ relation to this set of diagrams we can show that:

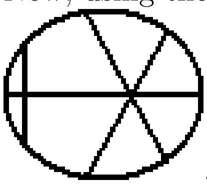
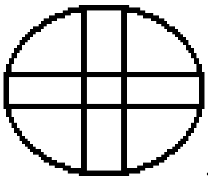
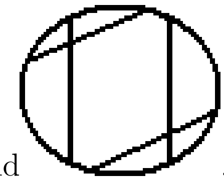
 =  + 

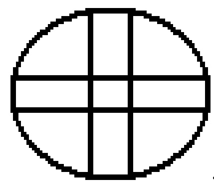
 =  + 

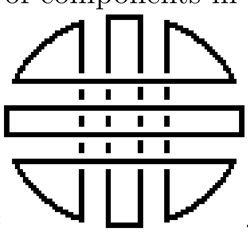
 =  + 

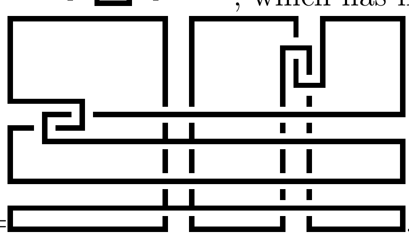
 +  =  + 

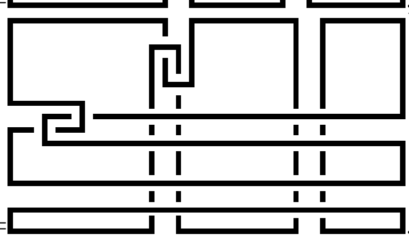
 +  =  + 

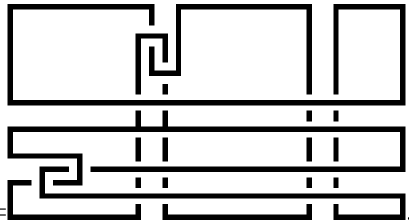
Now, using these relations we can reduce the list above to the generating set:
, , and . Again, once we have shown that the 4^{th} derivative statement is true on this set, we know that it is true for any chord diagram with four chords.

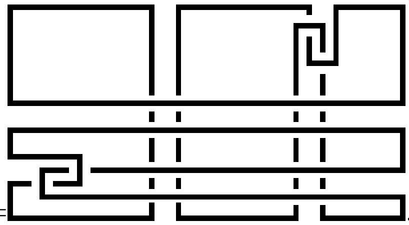
First we show in detail that the 4^{th} derivatives are congruent for . In this case we can resolve the $-$ crossings as $+$ crossing since we are only concerned with the number of components in each diagram. This chord diagram becomes:

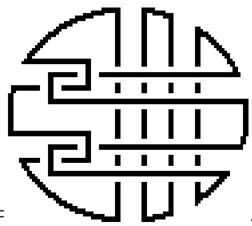
(0000)=, which has more than one component.

(0+0+)=, which has one component.

(0++0)=, which has one component.

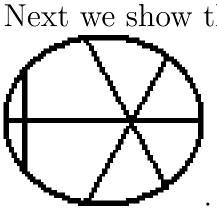
(+0+0)=, which has one component.

(+00+)=, which has one component.

(++00)=, which has more than one component. Using these calculations, we find that the fourth derivatives are:

$c_0(0000) = 0$ and $c_0(+00+)c_0(0++0) + c_0(0+0+)c_0(+0+0) + c_0(00++)c_0(++00) = 1(1) + 1(1) + 0(c_0(00++)) = 2$ which are congruent modulo 2 so the fourth derivatives are congruent for this chord diagram.

Next we show that the 4^{th} derivative statement is true for the chord diagram

. In this case, we will make the calculation showing less detail.

$c_0(0000) = 1$, since the chord diagram under the (0000) realization has one component.

$c_0(+00+) = 0$, since the chord diagram under the (+00+) realization has 3 components.

$c_0(+0+0) = 0$, since the chord diagram under the $(+0+0)$ realization has 3 components.

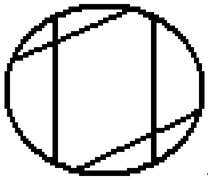
$c_0(00++) = 1$, since the chord diagram under the $(00++)$ realization has one component.

$c_0(++00) = 1$, since the chord diagram under the $(++00)$ realization has one component.

Using these calculations, the 4^{th} derivative statement statement becomes:

$c_0(0000) = 1$ and $c_0(+00+)c_0(0++0) + c_0(0+0+)c_0(+0+0) + c_0(00++)c_0(++00) = 0(c_0(0++0)) + 0(c_0(+0+0)) + 1(1) = 1$ which are congruent modulo 2.

Next we show that the 4^{th} derivative statement is true for the chord diagram

 . $c_0(0000) = 1$, since the chord diagram under the (0000) realization has one component.
$c_0(+00+) = 0$, since the chord diagram under the (+00+) realization has 3 components.
$c_0(+0+0) = 0$, since the chord diagram under the $(+0+0)$ realization has 3 components.
$c_0(00++) = 1$, since the chord diagram under the $(00++)$ realization has one component.
$c_0(++00) = 1$, since the chord diagram under the $(++00)$ realization has one component.
Using these calculations, the 4^{th} derivative statement statement becomes:
$c_0(0000) = 1$ and $c_0(+00+)c_0(0++0) + c_0(0+0+)c_0(+0+0) + c_0(00++)c_0(++00) = 0(c_0(0++0)) + 0(c_0(+0+0)) + 1(1) = 1$ which are congruent modulo 2.
Notice that the 5^{th} derivative of both sides will be 0. This means that our 5^{th} derivative will be congruent modulo 2 for all links. So, we have found a derivative that is true for all links and we have shown that each of our lower order derivatives are true on an appropriate set of chord diagrams.
This completes the proof. We can conclude that:

$$pc_4 = v_3 \mod 2.$$

Corollary 28.

$$c_2(-)(c_1(0) - 1) = -1/6a_2(0) + c_3(0) - 1/2(c_1(0))^2 - 1/24 \mod 2$$

gives a function on knots that is congruent modulo 2 to a function on links.

Chapter 4 A Congruence Modulo 2 Statement for pc_6

Next we will show that the degree 6 primitive Conway Vassiliev invariant is congruent modulo 2 to a linear combination of lower order invariants. In order to show this we start be computing the value of some lower order Vassiliev invariants on 9 knots. Then we will find a linear combination of those invariants that is congruent modulo 2 to the degree 6 primitive Conway invariant, pc_6, for each of the 9 knots.

Computing a Useful Matrix of Vassiliev Invariants

We will start be discussing how we computed the value of the primitive Conway invariants pc_k where k is 2, 4, or 6 for each knot in our list. In this case the degree k invariant is simply the kth coefficient of the primitive conway polynomial.
The Vassiliev invariants from the Jones polynomial, v_3, v_4, and v_5 where computed as follows. For each of our 9 knots, we started with the Jones polynomial of that knot as defined on KnotAtlas. In order to find v_k for a knot we took the logarithm of the knot and make the substitution $t = e^x$ into the knot. The invariant v_k was then a suitable multiple λ of the coefficient of the x^k term when the above substitution is made. For our invariants these multiples were, $\lambda = 1/6$ when $k = 3$, $\lambda = 4$ when $k = 4$, and $\lambda = 2$ when $k = 5$.
In order to compute the invariants form the Kauffman Polynomial, $k_5^{(1)}$ and $k_5^{(2)}$, we started with the polynomial as it is defined on KnotAtlas. We computed the logarithm of that polynomial and made the substitution $a = -ze^p - i$. Once this substitution is made, the coefficients of z^5 are of the form:

$$2ik_5^{(1)}e^p + ik_5^{(2)}e^{2p} + 2ik_5^{(3)}e^{3p} + 2ik_5^{(4)}e^{4p} + ik_5^{(5)}e^{5p}$$

Here, the other three invariants, $k_5^{(3)}, k_5^{(4)}$, and $k_5^{(5)}$, are degree 5 invariants, but they are redundant so they are not needed when we search for our congruent linear combination.
Using these processes we are able to create the following matrix of values:

	3_1	4_1	5_1	5_2	6_1	6_2	6_3	7_1	7_2
pc_2	-1	1	-3	-2	2	1	-1	-6	-3
v_3	1	0	5	3	-1	-1	0	14	6
pc_4	-1	0	-5	-3	-1	-1	0	-16	-6
v_4	-47	-13	-405	-202	-38	23	-35	-1602	-513
v_5	49	0	725	303	59	11	0	4046	966
$k_5^{(1)}$	0	0	1	1	0	-1	0	7	5
$k_5^{(2)}$	-4	-1	-71	-28	0	16	2	-415	-103
pc_6	0	0	-5	-2	2	1	1	-41	-8

Proof That pc_6 is Congruent Modulo 2 to a Degree 5 Invariant

Our goal now is to find a linear combination of these degree 5 and lower invariants that is congruent modulo 2 to the value of pc_6 for all of the knots on our list. In other words, we want to find a list of constants that make the following statement true:

$$a_1pc_2 + a_2v_3 + a_3pc_4 + a_4v_4 + a_5v_5 + a_6k_5^{(1)} + a_7k_5^{(2)} = pc_6 \mod 2.$$

Theorem 29. $-3/4v_3 + v_4 - 1/4v_5 + k_5^{(2)} = pc_6 \mod 2.$ *on all knots of up to 11 crossings*

In order to find a list of coefficients that is a solution, we used a computer program that searched through all combinations of coefficients from the list: (-3/4, -1/2, -1/4, 0, 1/4, 1/2, 3/4, 1). We found that solutions abound. From this procedure alone, we found well over 50 solutions. We then checked these solutions on all knots of up to 11 crossings. In other words we computed the values of $pc_2, v_3, pc_4, v_4, v_5, k_5^{(1)}, k_5^{(2)},$ and pc_6 for each knot using the process described above. We then checked that the statement above was true when we plugged in the appropriate coefficients. All of the solutions that we found using our computer program worked for all knots of up to 11 crossings. Some other coefficient lists that we found were: (-3/4, -3/4, 0, -3/4, 3/4, 0, 1), (1, 1/4, 0, 0, 3/4, 0, 1), and (1/2, -1/4, 1, -1/2, 1/4, 0, 1). Since we have shown that our solution works for such a large set of knots, we hope that the solution must work for all knots.

Chapter 5 Implications and Further Questions

Recall the conjecture, which we proved the base case of earlier:

Conjecture 30. *There exist integer-valued Vassiliev invariants, v_{4i-1}, of degree $4i-1$, such that:*

$$pc_{4i} = v_{4i-1} \mod 2$$

It is possible that the invariants v_{4i-1} in this conjecture can be chosen to be odd. Which means, that these invariants change sign under mirror image. This forces these invariants to vanish on amphicheiral knots. By amphicheiral we mean a knots that is isotopic to its mirror image.

Conjecture 31. *If the invariants v_{4i-1} in the previous conjecture can be chosen to be odd, then $pc_{4i} = 0 \mod 2$ on amphicheiral knots.*

By proving that $pc_4 = v_3 \mod 2$ we have shown that pc_4 is congruent to an odd invariant, which allows us to conclude that:

$$pc_4 = c_4 - 1/2(c_2 + c_2^2) = 0 \mod 2 \text{ on amphicheiral knots.}$$

If we were able to find an odd invariant that is congruent modulo 2 to pc_8 when we could conclude that:

$$c_8 + c_2(c_6 + c_4) + 1/2(c_4^2 - c_4) + 1/4(c_2^4 + c_2^2 + 2c_2) = 0 \mod 2 \text{ on amphicheiral knots.}$$

In order to show this we would start by finding some higher degree Vassiliev invariants than the invariants that we derived from the Kauffman polynomial. These two results come from [2].

List of References

[1] **D Bar-Natan**, *On the Vassiliev Knot Invariants*

[2] **J Conant**, *Chirality and the Conway Polynomial*

[3] **R Lickorish**, *An Introduction to Knot Theory*

[4] **Manturov**, **Vassily**, *Knot Theory*

Vita

James Dawson is a life long Tennessean. He received his High School Diploma from Portland High School in Portland, TN. His studies continued at Western Kentucky University and the University of Tennessee. Backpacking in the Smokey Mountains is one of his favorite pass times in Knoxville. The wonders of the natural world never cease to amaze, and the Smokey Mountains contain some of natures greatest beauty.