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The Dimension of the Restricted Hitchin Component for Triangle Groups

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The Dimension of the Restricted Hitchin Component for Triangle Groups

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Notice. *Chapters* 2 *and* 3*, in full, are each currently being prepared for submission for publication of their material. The dissertation author is the sole investigator and author of these papers.*

Abstract

Given positive integers p, q, r satisfying $1/p + 1/q + 1/r < 1$, the hyperbolic **triangle group** $T(p, q, r)$ is the group of orientation-preserving isometries of a tiling of the hyperbolic plane by triangles congruent to a geodesic triangle with angles π/p, π/q, and π/r. We will examine representations of triangle groups in the Hitchin component, a topologically connected component of the representation variety where representations are always discrete and faithful.

We begin by giving a formula for the dimension of a subset of the Hitchin component of an arbitrary hyperbolic triangle $T(p, q, r)$ for general degree $n > 2$. Depending on whether n is even or odd, we will consider only those Hitchin representations whose images lie in $Sp(2m)$ or $SO(m, m + 1)$, respectively. We call the space of representations satisfying this criterion the **restricted Hitchin component**.

We then provide two new families of representations of the specific triangle group $T(3, 3, 4)$ into $SL(5, R)$; the image groups of these families are each shown to be Zariski dense in $SL(5, R)$. Further, we consider a restriction to a surface subgroup of finite index in $T(3, 3, 4)$. For each family, we will demonstrate the existence of a subsequence of representations whose images are pairwise non-conjugate in $SL(5, Z)$ when restricted to a surface subgroup.

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Chapter 1

Background and Introduction

1.1 Triangle Groups as Geometric, Topological and Algebraic Objects

A **triangle group** $T(p,q,r)$ is a group with the presentation

$$T(p,q,r) := \langle a,b,c \,|\, a^p = b^q = c^r = abc = 1 \rangle \,.$$

This algebraic description will be the source of our methods for studying triangle groups, but it is important to note that these groups can also be understood in a geometric and topological context, and such interpretations give strong motivation for studying triangle groups.

Geometrically, triangle groups $T(p,q,r)$ can be interpreted as the rotational symmetries of a tiling of a constant-curvature space by geodesic triangles with angles π/p, π/q, and π/r. Which constant-curvature space this tiling inhabits depends on whether $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}$ is greater than, equal to, or less than 1.

Topologically, $T(p,q,r)$ is the orbifold fundamental group of a sphere with cone points of order p, q, and r. As a result, triangle groups contain fundamental groups of surfaces (called **surface groups**) as subgroups of finite index. Surface groups have been a historical object of interest to mathematicians, so part of the motivation for studying triangle groups is that they automatically provide information about these finite index surface subgroups.

1.1.1 Geometry: Rotations

Our goal for the following section is to connect each triangle group $T(p,q,r)$ to a triangle $\Delta(p,q,r)$ whose tiling has relevant geometric properties. In particular, we need to be able to say something about the ambient space M where we will embed our triangle $\Delta(p,q,r)$. To that end, we cite the following classical theorem from low-dimensional differential geometry:

Theorem 1.1 (Gauss-Bonnet). *Suppose M is a compact two-dimensional Riemannian manifold with boundary ∂M. Let K be the Gaussian curvature of M, and let κ_g be the geodesic curvature of ∂M. Then*

$$\int_M K dA + \int_{\partial M} \kappa_g ds = 2\pi\chi(M).$$

Now suppose we wish to embed $\Delta(p,q,r)$ in a surface M with Gaussian curvature K. An immediate corollary of the Gauss-Bonnet Theorem, as stated in [14], gives us constraints on the curvature arising from the interior angles of the triangle:

Corollary 1.2. *If Δ is a triangle in a geometric surface M, then*

$$\int\int_\Delta K dM + \int_{\partial\Delta} \kappa_g ds = (i_1 + i_2 + i_3) - \pi,$$

where i_1, i_2, and i_3 are the interior angles of the triangle.

In particular, let $\Delta(p,q,r)$ be a triangle with geodesic edges and interior angles π/p, π/q, and π/r, and assume that M has constant Gaussian curvature K. If A is the area of the triangle, Corollary 1.2 above becomes

$$KA = (\pi/p + \pi/q + \pi/r) - \pi.$$

Thus, the curvature of M will be positive, zero, or negative depending on whether the sum of the interior angles is greater than, equal to, or less than π, respectively. For angle sums above π, we will use the sphere $M = \mathbb{S}^2$; for angle sums equal to π, we will use Euclidean space $M = \mathbb{R}^2$; for angle sums below π, we will use hyperbolic space $M = \mathbb{H}^2$.

Now that we have a constant curvature space M for our tiling to inhabit, we can define the tiling itself. To do so, start with the triangle $\Delta(p, q, r)$ in the appropriate M, and reflect the triangle over each of its edges, resulting in a total of 4 triangles. Next, reflect each of these triangles over their edges, and continue this process until the triangles tile all of M. Note that composing any two reflections results in an orientation-preserving rotation about a point. Also, if this point is adjacent to an angle π/p on a triangle, precisely p of these rotations will take a triangle back to itself (since the sum of angles around a point must be 2π).

To identify the triangle group $T(p, q, r)$ with rotations in this tiling, assign the generator a to the $2\pi/p$ clockwise rotation of a triangle around its vertex with angle π/p, and assign the other two generators similarly. See Figure 1.1 for an example of this tiling when $p = 2$, $q = 3$, and $r = 7$.

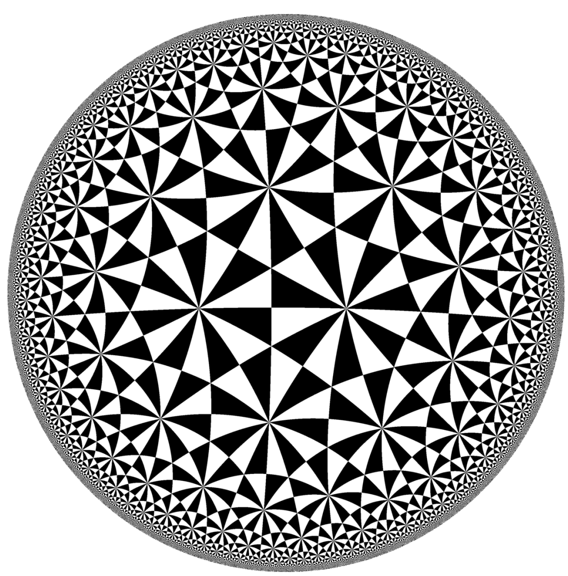


Figure 1.1: Tiling of the Poincaré disk model of $\mathbb{H}^2$ for $T(2, 3, 7)$

In the case where M is a sphere, note that spheres have finite area, and so any such tilings on the sphere will also be finite. As a consequence, the triangle group $T(p, q, r)$ will be finite when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$. Finite groups are relatively well-understood, so we will not include these triangle groups in our consideration from this point forward.

When M is Euclidean, we no longer encounter the finite area constraint, but Corollary 1.2 implies that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1,$$

so there are only 3 possible triangle groups: $T(2, 3, 6)$, $T(2, 4, 4)$, and $T(3, 3, 3)$. Unlike their spherical cousins, these groups are infinite, but the fact that there are only three of them means that we can exclude them as well without losing too much content.

Going forward, we will consider only **hyperbolic** triangle groups: those $T(p, q, r)$ which correspond to a tiling of the hyperbolic plane $M = \mathbb{H}^2$ (or equivalently, which satisfy $1/p + 1/q + 1/r < 1$.) There are infinitely many hyperbolic triangle groups, and each group has countably infinite order.

1.1.2 Topology: Orbifold Groups

A triangle group can also be understood as the fundamental group of an orbifold; to understand this construction, we first define an orbifold following Thurston in Chapter 13 of [17]. This definition has the advantage of being both geometrically intuitive and sufficient for our purposes, but some later authors (e.g., Henriques in [5]) give a broader definition than the one we provide here.

Intuitively speaking, a 2-orbifold $\mathcal{O}$ is a generalization of the idea of a 2-dimensional surface. As a surface looks locally like $\mathbb{R}^2$, an orbifold looks locally like $\mathbb{R}^2$ under the quotient of a finite group action. The case where the group action is everywhere trivial produces a surface. Otherwise, $\mathcal{O}$ has local behavior that falls into one of three categories: mirrors, corner reflectors, and cone points. A mirror is exactly what it sounds like: reflection along an axis in $\mathbb{R}^2$, where mirror images are identified. If two mirrors meet at an angle of π/p, then there is a corner reflector, where reflecting over both axes gives an identification through rotation by $2\pi/p$.

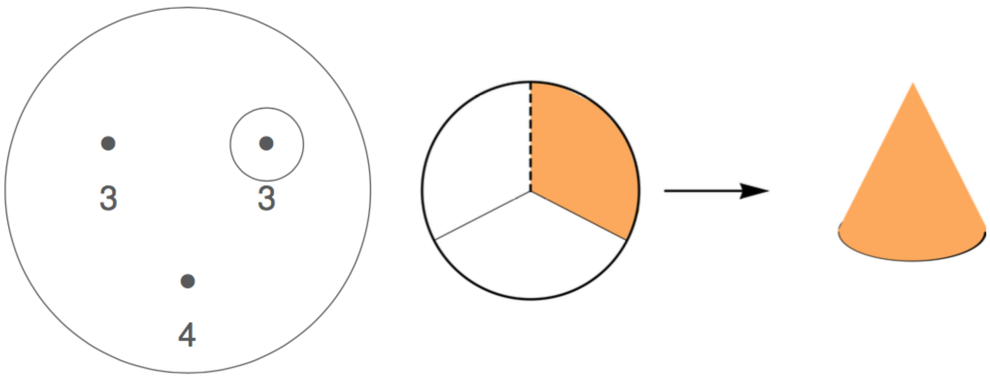


Figure 1.2: A cone point of order $p = 3$ on the orbifold for $T(3, 3, 4)$.

The last possibility for local behavior, and the only one that appears in the context of triangle groups, is a cone point of order p. In this case, $\mathbb{Z}_p$ acts on $\mathbb{R}^2$ by angle $2\pi/p$ rotations around a point. Visually, one can think of taking a disc around the cone point, cutting along a radius, and wrapping the disc onto a cone where only $\frac{1}{p}^{th}$ of the disc is visible from the outside. In the case of the triangle group $T(3, 3, 4)$, the associated orbifold is a sphere with cone points of order 3, 3, and 4; Figure 1.2 shows how a disc near a cone point of order $p = 3$ can be wrapped around itself to quotient out by the group action.

The formal definition of an orbifold given by Thurston in [17] follows this intuitive construction:

Definition 1.3 (Thurston)**.** *An* ***orbifold*** $\mathcal{O}$ *is a Hausdorff space* $X_{\mathcal{O}}$*, along with a covering of* $X_{\mathcal{O}}$ *by open sets* $\{U_i\}$ *which are closed under finite intersections. Each* U_i *is assigned a finite group* Γ_i*, an action of* Γ_i *on an open* $\tilde{U} \subset \mathbb{R}^2$*, and a homeomorphism* $\phi_i : U_i \xrightarrow{\approx} \tilde{U}/\Gamma_i$*.*

Whenever $U_i \subset U_j$*, there is an injective homomorphism* $f_{ij} : \Gamma_i \hookrightarrow \Gamma_j$ *and an embedding* $\tilde{\phi} : \tilde{U}_i \hookrightarrow \tilde{U}_j$ *equivariant with respect to* f_{ij} *such that the following diagram commutes:*

$$\begin{array}{ccc}
\tilde{U}_i & \xrightarrow{\tilde{\varphi}_{ij}} & \tilde{U}_j \\
\downarrow & & \downarrow \\
\tilde{U}_i/\Gamma_i & \xrightarrow{\varphi_{ij} = \tilde{\varphi}_{ij}/\Gamma_i} & \tilde{U}_j/\Gamma_i \\
 & & \downarrow f_{ij} \\
\varphi_i \Big\uparrow & & \tilde{U}_j/\Gamma_j \\
 & & \varphi_j \uparrow \\
U_i & \subset & U_j
\end{array}$$

We regard $\tilde{\varphi}_{ij}$ as being defined only up to composition with elements of Γ_j, and f_{ij} as being defined up to conjugation by elements of Γ_j.

We say that two coverings have the same orbifold structure if they can be combined consistently to give a larger cover that satisfies the criteria above.

If we can find a universal covering space of an orbifold, then we could define its fundamental group to be its group of covering transformations. To do this, we begin by defining a (not necessarily universal) covering orbifold:

Definition 1.4. *A **covering orbifold** for a 2-orbifold $\mathcal{O}$ is an orbifold $\tilde{\mathcal{O}}$ with a continuous surjection $p : X_{\tilde{\mathcal{O}}} \to X_{\mathcal{O}}$ between the underlying spaces of $\tilde{\mathcal{O}}$ and $\mathcal{O}$, respectively. Further, each point x in the image of this map must have a neighborhood $U = \tilde{U}/\Gamma$ such that $\tilde{U} \subset \mathbb{R}^2$ is open and each component V_i of $p^{-1}(U)$ is isomorphic to $\tilde{U}/\Gamma_i$, where Γ_i is a subgroup of Γ. The isomorphism must respect the projection.*

Any orbifold has a universal cover (see Proposition 13.2.4 of [17]), and for an orbifold with underlying space S^2 and at least three singular points, this cover is a simply connected manifold. However, Thurston points out the existence of "bad" orbifolds–i.e., orbifolds which do not have a manifold as a covering orbifold. One simple example of a bad orbifold is the teardrop, which is a sphere with a single cone point of order $p > 1$. Since a sphere with cone points of order (p, q, r) is a good orbifold, this is not a concern for our purposes. In fact, this is not really a concern generally speaking; Scott points out in [15] that there are only 4 types of bad 2-orbifolds without boundary, and many authors avoid these examples entirely by only considering good orbifolds.

So, as desired, we can define the fundamental group of a (good) orbifold to be its group of covering transformations. As is the case with surface groups, though, this is often not the best way to compute π_1 in practice. If we want to find the fundamental group of a sphere with cone points of order (p, q, r), then an approach similar to the one Scott uses in Section 2 of [15] incorporating the Seifert van Kampen theorem is more straightforward than relying on the definition.

To begin, we place the cone points as in Figure 1.3, so that the cone point of order p is in the upper hemisphere U and the other cone points are in the lower hemisphere V; the cone

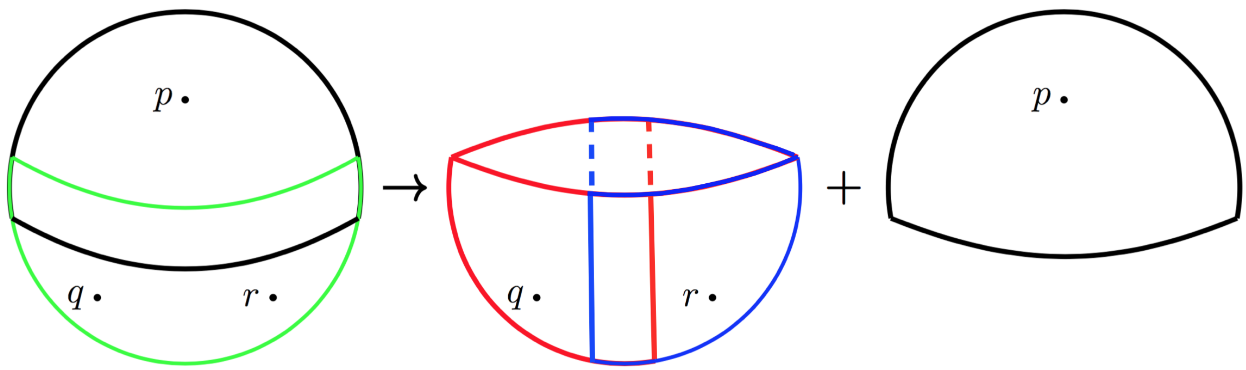


Figure 1.3: Seifert van Kampen decomposition for S^2 with cone points (p, q, r)

points of order q and r should also be in the left V_L and right V_R hemisphere, respectively. Divide the sphere into upper and lower pieces so that there is an annulus of overlap, then divide the lower piece into left and right pieces so that they overlap along a band. This process results in three pieces homeomorphic to a disk, each with a single cone point inside.

Consider a disk with cone point of order p in the center, and denote it B_p. Its universal cover is a disk, and the action that produces B_p is rotation by $2\pi/p$. So the fundamental group of B_p is isomorphic to $\mathbb{Z}_p$, and we can take a representative of the homotopy class to be a small circle around the cone point.

Now, compute the orbifold fundamental group of U and V. $\pi_1(U) = \langle a \,|\, a^p = 1 \rangle$ and since $V_L \cap V_R$ has trivial fundamental group, $\pi_1(V) = \langle b, c \,|\, b^q = c^r = 1 \rangle$. Finally, consider a loop around the equator of $U \cup V$ contained in $U \cap V$. If we include the loop in U, then it will be a or a^{-1} (depending on the orientation chosen for a). Including the loop in V gives, with the proper choice of orientation for b and c, the loop bc. So by the Seifert van Kampen theorem, the fundamental group for our sphere with cone points of order p, q, and r is

$$\pi_1(U \cup V) = \langle a, b, c \,|\, a^p = b^q = c^r = abc = 1 \rangle \,,$$

which is also the triangle group $T(p, q, r)$. So each hyperbolic triangle group is the orbifold fundamental group of a sphere with 3 cone points.

1.1.3 Algebra: Surface Groups

Since the algebraic model of triangle groups will be explored extensively in Chapter 2, we will restrict our study here to one particular property of triangle groups; namely, that they contain surface groups as subgroups of finite index. A surface group is the fundamental group of a surface, and their representations have been historical objects of interest. If we are able to find representations of triangle groups, we can restrict them to representations of surface groups, and this process is considered in more detail in Chapter 3. Given this, it is important to understand which surface groups can be found in a particular triangle group.

We begin by constructing a surface from copies of $T(3,3,4)$ using surgery. This construction only works on triangle groups when at least two of the cone points have the same order, and at least one cone point has even order, so we will have to use other methods to demonstrate the existence of surface subgroups more generally. Still, the procedure gives a geometric intuition that is absent elsewhere, so it is worth considering. A schematic for this process is given in Figure 1.4; note that for the sake of simplicity, only cone points are depicted and the absence of handles in the drawing does not indicate an absence of handles in reality.

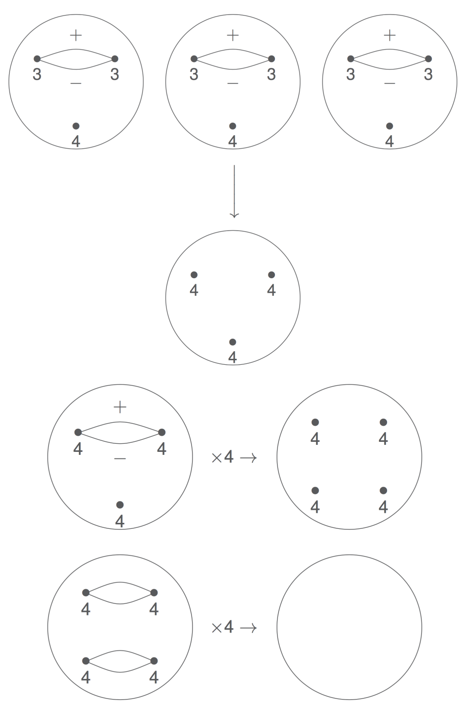


Figure 1.4: Constructing a manifold from the $T(3,3,4)$ orbifold

Begin with 3 copies of a sphere, each with cone points of order 3, 3, and 4. For each copy, cut along a straight line between the cone points of equal order, and label one side of this cut as "+" and the other as "-". Now, glue the positive edge on the first copy to the negative edge on the second, then the positive edge on the second to the negative edge on the third, and finally the positive edge on the third to the negative edge of the first, so that we are left with an orbifold without boundary. The cone points of order 4 (one from each of the 3 original copies) are left untouched, and these are the only points that prevent the new orbifold from being a manifold.

Now, make 4 copies of this new orbifold, and cut along a straight line between two of the cone points on each one. Gluing as in the previous step will again give us an orbifold with 4 cone points of order 4. Make 4 copies of this newest orbifold, and cut along a straight line between each of the two pairs of cone points, so that the slits do not intersect. On each copy, label one slit as upper and one as lower, and then glue as before along all the upper slits, then along all the lower slits. At this point, all cone points have been unwound and we are left with some genus g surface; its fundamental group will be a subgroup of finite index in $T(3,3,4)$.

As mentioned before, however, this geometric construction is only effective under very specific conditions. To find surface groups in other cases, we will rely on the work of Edmonds, Ewing, and Kulkarni in [4]. Their results apply more generally to any finitely generated Fuchsian group, which can be presented in the standard form

$$\langle a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_r, y_1, \ldots, y_s \,|[a_1, b_1] \ldots [a_g, b_g]\, x_1 \ldots x_r y_1 \ldots y_s = 1 = x_1^{m_1} \ldots x_r^{m_r} \rangle \, .$$

In the case where $g = 0$, $s = 0$, and $r = 3$, we obtain a triangle group with cone points of order m_1, m_2, and m_3. The theorem both shows that triangle groups have finite-index surface subgroups, and also enumerates the possibilities for the index.

Theorem 1.5 (Edmonds, Ewing, Kulkarni). *Let G be an infinite, finitely generated Fuchsian group in standard form. Then G has a torsion-free subgroup of finite index $k \geq 1$ if and only if k is divisible by $2^\epsilon \lambda$, where $\lambda = LCM(m_1, \ldots, m_r)$. If $s = 0$, λ is even, and λ/m_i is odd for exactly an odd number of m_i, then $\epsilon = 1$; otherwise, $\epsilon = 0$.*

Given a triangle group $T(p, q, r)$, we can use Theorem 1.5 to figure out which surfaces have fundamental groups appearing as index k subgroups. To do so, we will use the Euler characteristic of the orbifold with fundamental group $T(p, q, r)$, along with the fact that a k-fold cover will multiply the Euler characteristic of the orbifold by k. Let $\mathcal{O}$ be a compact 2-dimensional orbifold with a universal cover, let $X_{\mathcal{O}}$ be its underlying surface, and suppose that $\mathcal{O}$ has cone points of order q_i and corner reflectors with angle π/r_j. Then the most general version of the Riemann-Hurwitz formula (found, e.g., in [15]) is

$$\chi(\mathcal{O}) := \chi(X_{\mathcal{O}}) - \sum_{i=1}^{m}\left(1 - \frac{1}{q_i}\right) - \frac{1}{2}\sum_{i=1}^{n}\left(1 - \frac{1}{r_j}\right).$$

The orbifold associated with $T(p, q, r)$, which we will call $\mathcal{O}_T$, is a sphere with cone points of order p, q, and r, so $X_{\mathcal{O}_T} = S^2$, $m = 3$, $n = 0$, and $(q_1, q_2, q_3) = (p, q, r)$. So the formula above becomes

$$\chi(\mathcal{O}_T) = -1 + \frac{1}{p} + \frac{1}{q} + \frac{1}{r}.$$

Fix a triangle group $T(p, q, r)$; we will consider the specific case $T(3, 3, 4)$. Find the least common multiple of p, q, r, which is 12. Then $\lambda/m_1 = \lambda/m_2 = 4$ are even, and $\lambda/m_3 = 3$ is odd, so $\epsilon = 1$ and k must be divisible by $2^\epsilon\lambda = 24$. So, for a surface group to have finite index in $T(3, 3, 4)$, the surface must have an Euler characteristic that is a multiple of

$$2^\epsilon\lambda\chi(\mathcal{O}_T) = -2,$$

which includes all g-handled tori for $g \geq 2$.

1.2 The Hitchin Component

To better understand these hyperbolic triangle groups, we will be considering their representations. A **representation** of a group G is a group homomorphism $\rho : G \to GL(V)$ that takes each group element to an invertible linear transformation. For our purposes, V will always be $\mathbb{R}^n$ or $\mathbb{C}^n$, and since these vector spaces are finite-dimensional, we can primarily deal with matrix representatives instead of abstract linear transformations.

To ensure that no information is lost by replacing a group with its image under a representation, representations should be **faithful**, meaning that the map ρ should be injective. Representations of triangle groups should also be **discrete**, meaning that the subspace topology for $\rho(G) \subseteq GL(V)$ should be the discrete topology. The latter requirement reflects the fact that the action of G on the hyperbolic plane is properly discontinuous.

How can we guarantee that a representation of a triangle group is both discrete and faithful? For a specific representation, it would not be too difficult to verify, but we will be examining infinite families of representations in the following chapters. Fortunately, the following result of Labourie in [9] makes this task tractable:

Theorem 1.6 (Labourie, 2006)**.** *Every representation in the Hitchin component is discrete, faithful, and purely loxodromic.*

The idea of the Hitchin component is introduced by Hitchin in [6], though he calls it the Teichmüller component by analog with Teichmüller space.

Definition 1.7. *Let S be a compact oriented surface of genus $g > 1$, and $Hom^+(\pi_1(\Sigma), PSL(n, \mathbb{R}))$ denote the space of homomorphisms which act completely reducibly on the Lie algebra of $PSL(n, \mathbb{R})$, and let $Rep(\pi_1(S), PSL(n, \mathbb{R}))$ denote its quotient by the conjugation action of $PSL(n, \mathbb{R})$.*

*Define an **n-Fuchsian** representation to be one that can be written as a composition of a cocompact representation $\pi_1(S) \to PSL(2, \mathbb{R})$ with the unique irreducible representation $PSL(2, \mathbb{R}) \to PSL(n, \mathbb{R})$. Then the **Hitchin component** is the connected component of $Rep(\pi_1(S), PSL(n, \mathbb{R}))$ containing n-Fuchsian representations.*

Essentially, the Hitchin component is a special connected component of the space of representations which contains a base representation $\rho_0 : G \to PSL(n, \mathbb{R})$, which we define for triangle groups in Section 1.2.2. Since our discussions will often involve representation varieties, it is worth emphasizing at this point that word "component" will always refer to a connected component in the topological sense rather than an irreducible component as used in algebraic geometry; this usage is consistent with that of Hitchin [6] and Labourie [9].

1.2.1 History of the Hitchin Component

Our goal for this section will be to provide a brief history of the study of the Hitchin component. The Hitchin component is a generalization of Teichmüller space, which in turn was motivated by Riemann surfaces. A **Riemann surface** is an orientable surface equipped with a conformal structure. A **Riemann moduli space** for a Riemann surface of genus g is the set of isomorphism classes of complex structures for that surface. One can further select a finite set of points on a Riemann surface, called marked points, and require that these isomorphisms preserve them; **Teichmüller space** is the deformation space of these marked surfaces.

Theorem 1.8 (Teichmüller, 1939)**.** *Let S be an genus $g > 1$ closed oriented surface, without boundary or marked points, along with a conformal structure. Then the Teichmüller space of S is a topological manifold homeomorphic to $\mathbb{R}^\sigma$, with*

$$\sigma = 6g - 6.$$

It should be noted that this is a simplified version of Teichmüller's statement (which does not require orientability and allows boundary components and marked points), and that the process of proving the statement rigorously was the work of quite a few mathematicians. See [8] for a survey of these and other contributions to Teichmüller theory.

Hitchin mentions in the introduction to [6] that a given conformal structure on a compact surface S can be uniformized as the quotient of $\mathbb{H}^2$ by a subgroup of $PSL(2, \mathbb{R})$ isomorphic to $\pi_1(S)$. So there is a homomorphism $\pi_1(S) \to PSL(2, \mathbb{R})$, well-defined up to conjugation, which gives a direct correspondence between Teichmüller space and the Hitchin component in the case where $n = 2$. This means that the Hitchin component is a degree n representation generalization of Teichmüller space, and it is for this reason that Hitchin refers to this connected component as the "Teichmüller component" in [6].

Theorem 1.9 (Hitchin, 1992)**.** *The Hitchin component for a compact oriented surface of genus $g > 1$ is homeomorphic to a Euclidean space of dimension $(2g - 2) \dim PSL(n, \mathbb{R})$.*

Indeed, Hitchin points out that we can replace $PSL(n,\mathbb{R})$ in Definition 1.7 by the adjoint group of a split real form G^r of a complex simple Lie group G^c (including $SO(m+1,m)$ and $Sp(2n,\mathbb{R})$, which will feature prominently in Chapter 2), and Theorem 1.9 holds so long as we replace $\dim PSL(n,\mathbb{R})$ with $\dim G^r$.

Increasing the degree of the representation from $PSL(2,\mathbb{R})$ to $PSL(n,\mathbb{R})$ is one way to generalize the idea of Teichmüller space, but given our discussion in 1.1.2, it would also make sense to expand our understanding from surfaces to orbifolds. Thurston provides this next step in Corollary 13.3.7 of [17].

Theorem 1.10 (Thurston, 1979)**.** *The Teichmüller space of an orbifold Σ with $\chi(\Sigma) < 0$ is homeomorphic to Euclidean space of dimension $-3\chi(X_\Sigma) + 2c + r$, where c is the number of cone points and r is the number of corner reflectors.*

But this result applies only to Teichmüller space; what can be said about the dimension of the Hitchin component for orbifolds? Choi and Goldman answered that question for the $n = 3$ case (i.e., $PSL(3,\mathbb{R})$) in [2] using geometric cutting and sewing methods borrowed in part from Thurston.

Theorem 1.11 (Choi and Goldman, 2005)**.** *Let Σ be a compact 2-orbifold with negative Euler characteristic and without boundary. Then the deformation space of convex $\mathbb{RP}^2$ structures on Σ is homeomorphic to a cell of dimension*

$$-8\chi(X_\Sigma) + (6k_c - 2b_c) + (3k_r - b_r)$$

where X_Σ is the underlying space of Σ, k_c is the number of cone points, k_r the number of corner reflectors, b_c the number of cone points of order two, and b_r the number of corner-reflectors of order two.

While the result is posed in the language of convex $\mathbb{RP}^2$ structures, this is equivalent to finding the dimension of the Hitchin component of an orbifold in $PSL(3,\mathbb{R})$. Note that in the case of a genus g surface S, this is $-8\chi(S) = 8(2g-2) = (2g-2)\dim PSL(3,\mathbb{R})$, which matches Hitchin's formula. Unfortunately, their technique is not easily generalized for $n > 3$.

The dimension of the Hitchin component for $n > 3$ and a general orbifold Σ remains an open question. However, recently Long and Thistlethwaite gave a partial answer in the case where $\pi_1(\Sigma)$ is a hyperbolic triangle group in their paper [11]. While techniques for handling the Hitchin component have historically been geometric in nature (e.g., Hitchin's use of Higgs bundles, and Choi and Goldman's techniques described above), Long and Thistlethwaite exploit the algebraic structure of triangle groups in their approach. We will discuss their result further in Chapter 2, and our main theorem will provide an analogous result for triangle groups when the representations into $PSL(n, \mathbb{R})$ found in Definition 1.7 of the Hitchin component are replaced with special orthogonal or symplectic representations of the same degree.

1.2.2 The Base Representation for the Hitchin Component

In each of the following chapters, our work will begin with a base representation

$$\rho_0 : T(p, q, r) \xrightarrow{\rho} PSL(2, \mathbb{R}) \xrightarrow{\rho_n} PSL(n, \mathbb{R})$$

inside the Hitchin component. To that end, we describe each of the maps in the composition and reference programs in the Appendix for finding their matrices.

The representation $\rho : T(p, q, r) \to PSL(2, \mathbb{R})$ comes directly from the interpretation of a triangle group as the group of rotational symmetries of a tiling of constant-curvature space by geodesic triangles with angles π/p, π/q, and π/r. Each rotation is an orientation-preserving isometry of the hyperbolic plane $\mathbb{H}^2$, and so can be written as a Möbius transformation. The map ρ takes this transformation to its coefficient matrix, which will have determinant 1 as required.

$$\rho\left(z \mapsto \frac{fz + g}{hz + k}\right) = \begin{pmatrix} f & g \\ h & k \end{pmatrix}.$$

To determine an appropriate Möbius transformation on the upper half plane model of $\mathbb{H}^2$, we pick vertices for an initial geodesic triangle in standard coordinates (where the first coordinate is the real part of a complex number, and the second is the imaginary part.) The vertex with angle π/p is $(0, \sin(\pi/p))$, and the vertex with angle π/q is found by moving up

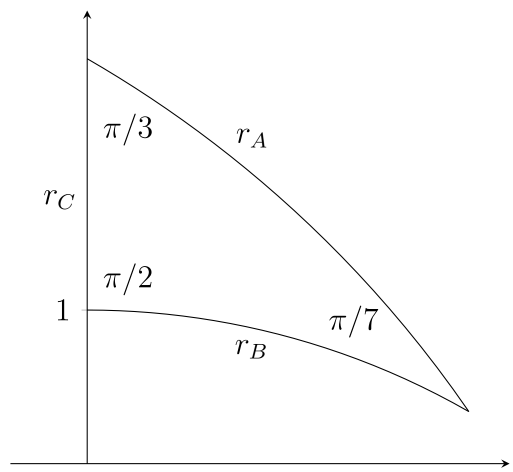


Figure 1.5: $(2, 3, 7)$ triangle in upper half plane model for $\mathbb{H}^2$

the imaginary axis by 1 unit, using the Poincaré metric on the upper half plane

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

and the second hyperbolic law of cosines; see [1] for a reference on hyperbolic geometry.

From there, we find rotations by first finding matrices for reflections over each of the sides; our choice of vertices ensures that one reflection r_C (refer to Figure 1.5 for a diagram of these labels on a triangle for $T(2, 3, 7)$) will be over the imaginary axis, with reflection matrix

$$r_C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Another reflection, r_B, will be inversion over a circle of radius $\sin(\pi/p)$. Finally, we find a circle with center along the real axis that crosses the imaginary axis at an angle of π/q, and reflect over it to get the last reflection r_A. To find the matrices for the rotations, we compose these reflections as seen in the Mathematica notebook in Appendix A.1.

The other representation, $\rho_n : PSL(2, \mathbb{R}) \to PSL(n, \mathbb{R})$, is a projectivized version of the unique irreducible representation $\sigma_n : SL(2, \mathbb{R}) \to SL(n, \mathbb{R})$. One way to construct this map is using degree $n - 1$ homogeneous polynomials in two variables with basis

$$\{x^{n-1}, x^{n-2}y, \ldots, xy^{n-2}, y^{n-1}\}.$$

We start with the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then perform the following substitution on each basis element, and write down the corresponding coefficient matrix:

$$x \mapsto ax + cy$$
$$y \mapsto bx + dy.$$

For example, consider the $n = 3$ case with $\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$. The substitution $x \mapsto x + 3y$, $y \mapsto 2x + 7y$ takes

$$\begin{array}{lll} x^2 & \mapsto (x+3y)^2 & = x^2 + 6xy + 9y^2 \\ xy & \mapsto (x+3y)(2x+7y) & = 2x^2 + 13xy + 21y^2 \\ y^2 & \mapsto (2x+7y)^2 & = 4x^2 + 28xy + 49y^2 \end{array}.$$

So

$$\rho_n \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 6 & 13 & 28 \\ 9 & 21 & 49 \end{pmatrix}.$$

See Appendix A.2 for a Mathematica notebook implementation of this algorithm.

Chapter 2

The Restricted Hitchin Component

Notice. *Chapter* 2*, in full, is currently being prepared for submission for publication of the material. The dissertation author was the sole investigator and author of this paper.*

2.1 Background and Definitions

In their recent paper [11], Long and Thistlethwaite gave a formula for the dimension of the Hitchin component for hyperbolic triangle groups $T(p, q, r)$. Another way to phrase this is that they found the dimension for all representations $T(p, q, r) \to PSL(n, \mathbb{R})$ in the deformation space that are in the same connected component as the base representation $\rho_0 : T(p, q, r) \to PSL(n, \mathbb{R})$.

Theorem 2.1 (Long and Thistlethwaite)**.** *Let Q and R be the quotient and remainder of dividing n by k, and define $\sigma(n, k) := (n + R)Q + R - 1$.*

Then the dimension of the Hitchin component $\mathcal{H}^*$ of the triangle group $T(p, q, r)$ is

$$\dim \mathcal{H}^* = \dim SL(n, \mathbb{R}) - (\sigma(n, p) + \sigma(n, q) + \sigma(n, r)) .$$

Note that $\mathcal{H}^*$ is referred to as the moduli space of essential deformations in [11]; we refer to it here as the Hitchin component to be consistent with Definition 1.7 and definitions used in [2], [6], and [9]. Our definition of $\sigma(n, k)$ above is also off by 1 compared with theirs; this is to emphasize the similarities to our result in Theorem 2.4, and the statement above is equivalent to the one presented in [11].

It is straightforward to show that, given some $g \in T(p,q,r)$, its image $g_0 := \rho_0(g)$ satisfies the relation $g_0^T F g_0 = F$, where F is the antidiagonal matrix

$$F = \begin{pmatrix} 0 & & & & & \binom{n-1}{0} \\ & & & & -\binom{n-1}{1} & \\ & & & \binom{n-1}{2} & & \\ & & -\binom{n-1}{3} & & & \\ & \iddots & & & & \\ \pm\binom{n-1}{n-1} & & & & & 0 \end{pmatrix},$$

which is either symmetric or skew-symmetric, depending on whether n is odd or even, respectively. This means that, depending on whether the degree n of the representation is even or odd, $\rho_0(g)$ is in either a symplectic or special orthogonal group:

$$\rho_0(g) \in \begin{cases} Sp(2m) & \text{for } n = 2m \text{ even} \\ SO(m, m+1) & \text{for } n = 2m+1 \text{ odd} \end{cases}.$$

From this, it is natural to wonder which of the representations in the deformation space also have images in either $Sp(2m)$ or $SO(m, m+1)$. In particular, recall that the definition of the Hitchin component introduced in Hitchin's paper [6] applies not only to $PSL(n, \mathbb{R})$, but also to the adjoint group of a split real form G^r of a complex simple Lie group G^c, which includes both $SO(m+1, m)$ and $Sp(2m)$. However, to avoid confusion, we will continue using Definition 1.7 of the Hitchin component, and define the following additional term to refer to groups other than $PSL(n, \mathbb{R})$.

Definition 2.2. *Let Σ be a 2-orbifold, let G be the adjoint group of a split real form G^r of a complex simple Lie group G^c, and let $\mathcal{H}$ be the Hitchin component of $\pi_1(\Sigma)$ in $PSL(n, \mathbb{R})$. The* ***restricted Hitchin component*** *$\mathcal{H}_G$ is the set of representations in the Hitchin component $\mathcal{H}$ whose images are contained within the group G, up to conjugation by G.*

When we refer to the restricted Hitchin component, G will be understood to be $Sp(2m)$ if we are considering even degree representations, or $SO(m, m+1)$ for odd degree representations. The goal of this chapter (and the main result of this dissertation) will then be to determine the dimension of the restricted Hitchin component for general degree $n > 2$ and for any hyperbolic triangle group $T(p, q, r)$.

In addition to this definition, the statement of the main theorem relies on an arithmetic function $\sigma_G(n, k)$, which arises from the process of counting the multiplicities of real and complex eigenvalues for the image of a generator of $T(p, q, r)$ in G:

Definition 2.3. *Let $n, k \geq 2$ be integers, and let Q be the quotient and R the remainder of integer division of n by k:*

$$n = Qk + R \qquad\qquad 0 \leq R \leq k - 1.$$

Further, denote the parity of each of n, k, and Q by $n_{\mathcal{E}} := n \pmod 2$, $k_{\mathcal{E}} := k \pmod 2$, and $Q_{\mathcal{E}} := Q \pmod 2$. Then

$$\sigma_G(n, k) := \frac{1}{2}\left((n+R)Q + R + k_{\mathcal{E}}\left(Q + Q_{\mathcal{E}}\right) - n_{\mathcal{E}}\left(2Q+1\right)\right).$$

In the sections that follow, we will establish the following result giving the dimension of the restricted Hitchin component:

Theorem 2.4. *Let $n \geq 3$ be an integer, m be integer division of n by 2, and G be one of the following subgroups of $SL(n, \mathbb{R})$:*

$$G = \begin{cases} SO(m, m+1) & \textit{if } n = 2m+1 \\ Sp(2m) & \textit{if } n = 2m \end{cases}.$$

Let $\mathcal{H}_G$ be the restricted Hitchin component for the base representation $\rho_0 : T(p, q, r) \to G$. Then

$$\dim \mathcal{H}_G = \dim G - \left(\sigma_G(n, p) + \sigma_G(n, q) + \sigma_G(n, r)\right).$$

Remark. For $k = n$, $Q = 1$ and $R = 0$, so $n_{\mathcal{E}} = k_{\mathcal{E}}$ and $Q_{\mathcal{E}} = 1$.

For $k > n$, $Q = 0$ and $R = n$, so $Q_{\mathcal{E}} = 0$.

In either case, if $k \geq n$,

$$\sigma_G(n,k) = \begin{cases} \frac{1}{2}(n-1) & \text{if } n \text{ odd} \\ \frac{1}{2}n & \text{if } n \text{ even.} \end{cases} \tag{2.1}$$

So if all of p, q, and r are no less than n,

$$\dim \mathcal{H}_G = \begin{cases} \frac{1}{2}(n-1)(n-3) & \text{if } n \text{ odd} \\ \frac{1}{2}n(n-2) & \text{if } n \text{ even.} \end{cases}$$

In Section 2.2, we will give a sense of Long and Thistlethwaite's work in [11] and how it relates to our results, and in Section 2.3, we deal with a certain technical obstruction to using matrix representatives. From there, the proof of Theorem 2.4 will proceed as follows: in Section 2.4, we determine the contribution of cyclic generators to the dimension of the restricted Hitchin component, and in Section 2.5, we handle the final, non-cyclic, group relation.

2.2 The (Unrestricted) Hitchin Component

The argument presented in this chapter is inspired by those in [11], with several key differences. To make these contributions clear, we will begin by giving a brief outline of the ideas in Long and Thistlethwaite's paper. As mentioned in the introduction, their approach differs from earlier work on the Hitchin component in that it is fundamentally algebraic, in contrast to earlier geometric approaches. Their argument is based on the group structure of hyperbolic triangle groups $T(p,q,r)$, in particular the presentation

$$T(p,q,r) = \langle a,b,c \,|\, a^p = b^q = c^r = abc = 1 \rangle \,.$$

They begin by examining the contribution of each cyclic generator to the dimension of the Hitchin component separately. Let g be a cyclic generator of order k in $T(p,q,r)$, and consider the restriction of the base representation to the cyclic subgroup generated by g (i.e., consider $\rho_0|_{<g>}$.) Denote the connected component of the representation variety of $< g >$ in $SL(n, \mathbb{R})$ that contains $\rho_0|_{<g>}$ by $\mathcal{D}(g)$; we will also refer to this as the deformation space.

As we allow representations ρ' to vary continuously away from ρ_0 in the deformation space, the order of $\rho'(g)$ must still be k. Because of this, $\rho_0(g)$ and $\rho'(g)$ must have the same Jordan Canonical Form, and thus will always be in the same conjugacy class. Further, any element of this conjugacy class can be reached by such a variation, since $SL(n, \mathbb{R})$ is connected. So we can define a diffeomorphism between the deformation space $\mathcal{D}(g)$ and the conjugacy class of $\rho_0(g)$ by the map $\rho' \mapsto \rho'(g)$. Thus, the dimension of the deformation space is precisely that of the conjugacy class of $\rho_0(g)$.

Since the conjugacy class of $\rho_0(g)$ is in 1-to-1 correspondence with right cosets of the centralizer $C(\rho_0(g))$, we can determine the dimension of the deformation space using the dimension of the centralizer of $\rho_0(g)$:

$$\dim \mathcal{D}(g) = \dim SL(n, \mathbb{R}) - \dim C(\rho_0(g)) \,.$$

Notice that the only argument above that depends specifically on $SL(n, \mathbb{R})$ is the fact that it is connected, which is also true of $Sp(2m)$ and $SO(m, m+1)$. Thus, we have the following lemma:

Lemma 2.5. *Let G be $SL(n, \mathbb{R})$, $Sp(2m)$, or $SO(m, m+1)$, and let g be a cyclic generator in $T(p, q, r)$. Then the dimension of the deformation space $\mathcal{D}(g)$ in G is given by*

$$\dim \mathcal{D}(g) = \dim G - \dim C(\rho_0(g)) \,,$$

where $C(\rho_0(g))$ is the centralizer of $\rho_0(g)$.

Long and Thistlethwaite work out the dimension of the centralizer of $\rho_0(g)$ by examining its eigenvalues. The image of the cyclic generator g in 2×2 matrices can be diagonalized to $\operatorname{diag}(\zeta, \zeta^{-1})$, where ζ is a root of unity. Thus, the diagonalized image of g in $n \times n$ matrices will cycle through powers of ζ, allowing us to count the number of repetitions for each eigenvalue. If we rearrange the eigenvalues $\lambda_1, \ldots, \lambda_s$ so that repetitions are placed next to one another, we obtain a matrix that looks like the following:

$$\begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{matrix}} & & \\ & \ddots & \\ & & \boxed{\begin{matrix} \lambda_s & & \\ & \ddots & \\ & & \lambda_s \end{matrix}} \end{pmatrix}$$

This arrangement is advantageous because scalar matrices are in the center of $SL(n, \mathbb{R})$, so each block of identical eigenvalues contributes its size to the dimension of the centralizer. That is, if we let $n_1 \times n_1, \ldots, n_s \times n_s$ be the size of the block for $\lambda_1, \ldots, \lambda_s$ respectively, we obtain

$$\dim C(\rho_0(g)) = n_1^2 + \ldots + n_s^2 - 1,$$

where the -1 ensures that matrices are in $SL(n, \mathbb{R})$ rather than $GL(n, \mathbb{R})$ by forcing the determinant to be 1.

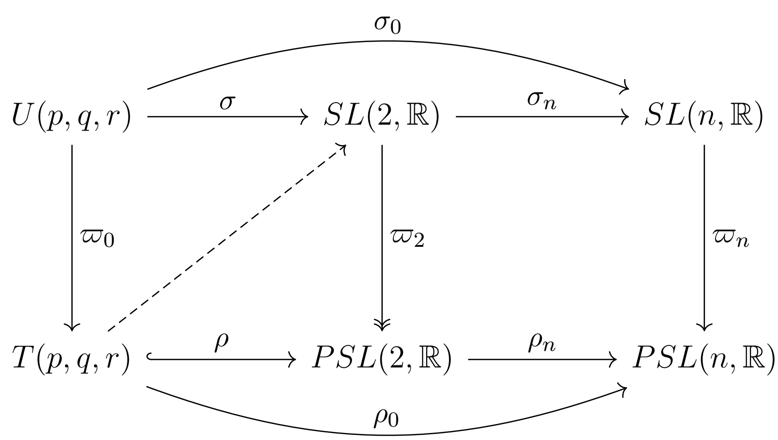
However, this kind of rearrangement does not work more generally for $Sp(2m)$ or $SO(m, m+1)$ because, in general, it is not compatible with the group relation involving the (skew-)symmetric matrix F. We will see that treating real and complex eigenvalues differently will be needed to adjust for this fact. We will also need to make adjustments to the argument made in the last section of Long and Thistlethwaite's paper [11] regarding the contribution of the non-cyclic group relation $abc = 1$; that argument is left to a later section both for brevity and because there are a significant number of adjustments that need to be made.

2.3 Going from $PSL(n, \mathbb{R})$ to $SL(n, \mathbb{R})$

The content in this section does not differ significantly from Section 2 of Long and Thistlethwaite's paper [11]. The following summary is included both for the sake of completeness of our proof, and to establish notation which is slightly different from theirs and which will be used for the rest of the argument.

Theorem 2.4 is concerned with deformations of the base representation ρ_0 in $PSL(n,\mathbb{R})$, but for computational reasons we prefer to deal with matrix representatives of $PSL(n,\mathbb{R})$, keeping in mind that each matrix is identified with its negative. This is not a problem in the case where n is odd, since in that case $PSL(n,\mathbb{R}) = SL(n,\mathbb{R})$. However, for n even and at least one of p, q, or r even, we encounter a certain technical issue that must be resolved.

Consider the pullback of the injective map $\rho : T(p,q,r) \to PSL(2,\mathbb{R})$ (defined in Section 1.2.2) and the surjective map $\varpi_2 : SL(n,\mathbb{R}) \to PSL(n,\mathbb{R})$ (which identifies a matrix with its negative.) The pullback, which we call $U(p,q,r)$, comes equipped with an injective map $\sigma : U(p,q,r) \to SL(2,\mathbb{R})$ and a surjective 2-to-1 map $\varpi_0 : U(p,q,r) \to T(p,q,r)$, such that Diagram 2.2 commutes. σ_n is the unique irreducible representation discussed in 1.2.2, and ρ_n is its projectivization.



(2.2)

ϖ_n is the map identifying a matrix with its negative; this will be the identity map when n is odd, since the determinant of a matrix in $SL(n,\mathbb{R})$ must be 1. However, if n is even, this is not the case, and if at least one of the cyclic generators has even order, we encounter a problem in defining the lift indicated by the dotted line in Diagram 2.2. The matrix for a rotation by angle $2\pi/k$ in $\mathbb{H}^2$ will have eigenvalues $e^{\pm i\pi/k}$, and so its order will be $2k$, rather than k. When k is an odd number, we can correct for this by multiplying the matrix by -1, so that we get $\left(-e^{\pm i\pi/k}\right)^k = (-1)^k \left(e^{\pm i\pi/k}\right)^k = (-1)(-1) = 1$. But if k is even, $(-1)^k = 1$, and this correction will not work.

One solution to this problem is to concern ourselves with representations $\sigma_0 : U(p,q,r) \to SL(n,\mathbb{R})$ instead, in the hope that there is some relationship between its deformation space and the restricted Hitchin component of $T(p,q,r)$. In doing so, we use the presentation given in Proposition 2.1 of [11]:

Proposition 2.6 (Long and Thistlethwaite)**.** *For appropriately chosen generators α, β, γ, $U(p,q,r)$ admits the presentation*

$$U(p,q,r) = \left\langle \alpha, \beta, \gamma, z \,\middle|\, \alpha^p = \beta^q = \gamma^r = \alpha\beta\gamma = z, z^2 = 1 \right\rangle,$$

where the images of α, β, γ under $\varpi_2 \circ \sigma$ are $\rho(a), \rho(b), \rho(c)$ respectively and $\sigma(z) = -I$.

Thus, $T(p,q,r)$ can be identified with the quotient $U(p,q,r)/ < z >$. The relationship between the deformation spaces of $\rho_0 : T(p,q,r) \to PSL(n,\mathbb{R})$ and $\sigma_0 : U(p,q,r) \to SL(n,\mathbb{R})$ is given in Proposition 2.2 of [11]; we reproduce the proof here with minor alterations to demonstrate that considering $Sp(2m)$ or $SO(m,m+1)$ instead of $SL(n,\mathbb{R})$ does not cause problems.

Lemma 2.7. *Let $\mathcal{H}$, $\mathcal{K}$ be the deformation spaces of ρ_0, σ_0 respectively. Then $\mathcal{H}$, $\mathcal{K}$ are diffeomorphic.*

Proof. Let G be $Sp(2m)$ or $SO(m,m+1)$ as appropriate, and note that both are path connected. First, we define a smooth map $F : \mathcal{H} \to \mathcal{K}$.

Let $\rho' \in \mathcal{H}$, and let P_t be a smooth path between ρ_0 and ρ' in $\mathcal{H}$ for $t \in [0,1]$. For each $g \in T(p,q,r)$, $P_t(g)$ can be associated with a pair of paths $\{m_t(g), -m_t(g)\}$ in $G \subseteq SL(n,\mathbb{R})$ which are well-defined by continuity. Define a path $S_t(g)$ in $\mathcal{K}$ by

$$S_t(g) = \begin{cases} m_t(g) & \text{if } \sigma_0(g) = m_0(g) \\ -m_t(g) & \text{if } \sigma_0(g) = -m_0(g) \end{cases}.$$

Then we can define $F(\rho')$ by $F(\rho'(g)) = S_1(g)$.

To define a smooth inverse $G : \mathcal{K} \to \mathcal{H}$ for F, let $\sigma' \in \mathcal{K}$ and let S_t be a smooth path between σ_0 and σ' in $\mathcal{K}$ for $t \in [0,1]$. Since $\sigma_0(z) = \pm I$ is in the center of G, $S_t(z) \equiv \pm I$ for all $t \in [0,1]$. Thus, S_t induces a path P_t such that $\varpi_n \circ S_t = P_t \circ \varpi_0$ for all $t \in [0,1]$. So we can define $G(\sigma') = P_1$. □

2.4 Contribution of Cyclic Subgroups

We wish to determine the dimension of the deformation space of each cyclic generator in

$$U(p,q,r) = \left\langle \alpha, \beta, \gamma, z \,\middle|\, \alpha^p = \beta^q = \gamma^r = \alpha\beta\gamma = z, z^2 = 1 \right\rangle .$$

One might hope to apply Long and Thistlethwaite's method of counting eigenvalues for each cyclic generator, but as mentioned earlier, following their approach directly would fail to preserve the group relation for $Sp(2m)$ and $SO(m, m+1)$ involving the antidiagonal matrix F. Instead, we exploit a certain symmetry in the eigenvalues of cyclic generators that mirrors the (skew-)symmetry of F. This will allow us to break generators into block matrices that remain compatible with F and have centralizers with easily-computed dimensions. Finally, we will arrive at the task of actually counting the repeated eigenvalues for a cyclic generator, and we will see shortly that it is necessary to do so in a way that distinguishes between real and complex (i.e., $\mathbb{C} \setminus \mathbb{R}$) eigenvalues.

Let G be $Sp(2m)$ or $SO(m, m+1)$ as appropriate, let g be one of α, β, γ in $U(p,q,r)$ with order $2k$, and let $\mathcal{D}(g)$ be the component of the representation variety of $< g >$ in G that contains $\rho_0|_{<g>}$, i.e., the deformation space of g. Consider the image of g in $SL(2,\mathbb{R})$; for $\zeta := -e^{\pi i/k}$, we can conjugate this 2×2 matrix into the diagonal form $\mathrm{diag}\left(\zeta, \overline{\zeta}\right)$. Denote by D the image of $\mathrm{diag}\left(\zeta, \overline{\zeta}\right)$ under σ_n in $SL(n,\mathbb{R})$. Then

$$D = (-1)^{n-1}\mathrm{diag}\left(\zeta^{n-1}, \zeta^{n-3}, \ldots, \zeta^{n-2i+1}, \ldots, \zeta^{-(n-3)}, \zeta^{-(n-1)}\right). \tag{2.3}$$

Note that each diagonal entry of D is a (not necessarily primitive) $2k^{th}$ root of unity, so the only real eigenvalues are ± 1, and complex conjugates are the same as their inverses. Further, we know that entries on opposite sides of D will be complex conjugates (here, D_i denotes the ith entry along the diagonal):

$$
\begin{aligned}
D_i &= (-1)^{n-1}\zeta^{n-2i+1} \\
&= (-1)^{n-1}\zeta^{-1(-n+2i-1)} \\
&= (-1)^{n-1}\zeta^{-1(n-2(n-i+1)+1)} \\
&= (D_{n-i+1})^{-1} \\
&= \overline{D_{n-i+1}}.
\end{aligned} \tag{2.4}
$$

In the case where $n = 2m + 1$, this means the middle entry must be a root of unity and its own complex conjugate, meaning that it will be ± 1; this fact will play an important role in Proposition 2.11.

Using this symmetry, we can reorder (by conjugation) the eigenvalues of D in a way that respects the relation $D^T FD = F$. First, collect the real eigenvalues ± 1 as done in [11]. The rest of the blocks will be collections of pairs of complex conjugates; since they appear in pairs on opposite ends of D, this ensures that conjugating F in the same way will give it skew-symmetric (for n even) or symmetric (for n odd) blocks that match up with D. This breaks the problem of finding the dimension of the centralizer into smaller centralizer computations, which we will see are much easier to handle.

Lemma 2.8. *Suppose D is written in the form given in Equation 2.3. Then the dimension of the centralizer of D in G can be written as the sum*

$$
\dim C_G(D) = \dim C_{G_-}\left(-I_{n_-}\right) + \dim C_{G_+}\left(I_{n_+}\right) + \sum_{i=1}^{j} \dim C_{G_i}\left(D_i\right),
$$

where each G_i is a subgroup of $SL(n, \mathbb{R})$ respecting a (skew-)symmetric non-degenerate bilinear form F_i.

Proof. Since F is skew-symmetric when n is even and symmetric when n is odd, we use the term (skew-)symmetric to reflect precisely this situation below for the sake of brevity.

Suppose -1 has multiplicity $n_- \geq 0$ in D, and 1 has multiplicity $n_+ \geq 0$. Then define the (possibly empty) matrices $D_{n_-} := I_{n_-}$ and $D_{n_+} := I_{n_+}$. Construct an $n_- \times n_-$ antidiagonal matrix F_- by making the only entry on its i^{th} row equal to the nonzero entry of F on the same row as the i^{th} instance of -1 in D. Construct F_1 similarly. For example, if $n = 5$ and $k = 2$, then for $\zeta := -e^{\pi i/k}$,

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } F = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so

$$F_- = \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix} \text{ and } F_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 6 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since real numbers are their own complex conjugates and F is (skew-)symmetric, F_- and F_+ will also be (skew-)symmetric. Of course, while the matrix above only contains real eigenvalues, in general D will also have non-real eigenvalues.

Now, suppose some $\omega \in \mathbb{C} \setminus \mathbb{R}$ has multiplicity d_i in D. Because of Equation (2.4), $\overline{\omega}$ must have the same multiplicity. Note that the entries of D in Equation (2.3) cycle through its distinct k eigenvalues in a fixed order; in particular, ω and $\overline{\omega}$ will alternate as we proceed down the diagonal of D. To define a diagonal D_i, we will pull out each of the ω and $\overline{\omega}$ eigenvalues in order from D; do the same with the corresponding entries from F. This will give us a $2d_i \times 2d_i$ matrix

$$\operatorname{diag}(\omega, \overline{\omega}, \omega, \overline{\omega}, \dots, \omega, \overline{\omega}),$$

but we would prefer for computational reasons that ω and $\overline{\omega}$ be grouped with their duplicates. Fortunately, it is possible to swap eigenvalues on opposite ends as necessary to make this happen, doing the same with the corresponding entries of F. This will define D_i and F_i, with

$$D_i = \left(\begin{array}{ccc|ccc} \omega & & & & & \\ & \ddots & & & 0 & \\ & & \omega & & & \\ \hline & & & \overline{\omega} & & \\ & 0 & & & \ddots & \\ & & & & & \overline{\omega} \end{array}\right),$$

and F_i (skew-)symmetric.

Continue this process until you exhaust all eigenvalues of D and arrive at a decomposition

$$D_- \oplus D_+ \oplus D_1 \oplus \ldots \oplus D_j,$$

for D with corresponding antidiagonal (skew-)symmetric $F_-, F_+, F_1, \ldots, F_j$. Denote the groups containing all invertible matrices h of appropriate size satisfying $h^T F_i h = F_i$ by G_i (or G_-, G_+ as appropriate.)

For example, if $n = 7$ and $k = 3$, we get that

$$D = \begin{pmatrix} 1 & & & & & & \\ & \zeta & & & & 0 & \\ & & \overline{\zeta} & & & & \\ & & & 1 & & & \\ & & & & \zeta & & \\ & 0 & & & & \overline{\zeta} & \\ & & & & & & 1 \end{pmatrix} \text{ and } F = \begin{pmatrix} & & & & & & 1 \\ & 0 & & & & -6 & \\ & & & & 15 & & \\ & & & -20 & & & \\ & & 15 & & & & \\ & -6 & & & & 0 & \\ 1 & & & & & & \end{pmatrix}$$

have decompositions (respectively)

$$\left(\begin{array}{ccc|cccc} 1 & & & & & & \\ & 1 & & & 0 & & \\ & & 1 & & & & \\ \hline & & & \zeta & & & \\ & & & & \zeta & & \\ & 0 & & & & \overline{\zeta} & \\ & & & & & & \overline{\zeta} \end{array}\right) \quad \text{and} \quad \left(\begin{array}{ccc|cccc} & & 1 & & & & \\ & -20 & & & 0 & & \\ 1 & & & & & & \\ \hline & & & & & & -6 \\ & & & & & 15 & \\ & 0 & & & 15 & & \\ & & & -6 & & & \end{array}\right).$$

Now that we have the decomposition, we need to show that finding the dimension of each centralizer individually and adding them up is equivalent to finding the dimension of the centralizer for our original diagonal matrix D. Suppose that h is in the centralizer of D in the group $SL(n, \mathbb{R})$. Then h can be written as $h = h_- \oplus h_+ \oplus h_1 \oplus \ldots \oplus h_j$, with the sizes of h_i corresponding to those of D_i and F_i, and each h_i invertible. Any such h satisfying $h^T F h = F$ will also be in the centralizer of G; this is equivalent to

$$h_i^T F_i h_i = F_i \text{ for } i = 0, 1, \ldots, k,$$

which gives us the desired sum for $\dim C_G(D)$. □

Now that we have the ability to break images of generators in $SL(n, \mathbb{R})$ into blocks that are compatible with the group relation for $SO(m, m+1)$ or $Sp(2m)$, we need to know how to find the dimension of the centralizer for each block. In the case of real eigenvalues ± 1, this is straightforward; since $\pm I$ is in the center, the dimension of its centralizer will be the same as the group that contains it. A quick block matrix computation will handle the case where blocks consist of conjugate pairs of non-real eigenvalues.

Lemma 2.9. *Let $d \in \mathbb{Z}_{>0}$, let $\omega \in \mathbb{C}$, and define*

$$D_\omega := \begin{cases} \omega I_d & \text{if } \omega \in \mathbb{R} \\ \omega I_d \oplus \overline{\omega} I_d & \text{if } \omega \in \mathbb{C} \setminus \mathbb{R} \end{cases}.$$

Let F' be an anti-diagonal symmetric or skew-symmetric non-degenerate bilinear form of the same size as D_ω, and let G' be the subgroup of the special linear group consisting of all g such that $g^T F' g = F'$.

Then the dimension of the centralizer of D_ω in G' is

$$\dim C_{G'}(D_\omega) := \begin{cases} \frac{1}{2}d(d-1) & \text{if } \omega \in \mathbb{R},\ F' \text{ symmetric} \\ \frac{1}{2}d(d+1) & \text{if } \omega \in \mathbb{R},\ F' \text{ skew-symmetric} \\ d^2 & \text{if } \omega \in \mathbb{C} \setminus \mathbb{R} \end{cases} \cdot$$

Proof. Suppose first that $\omega \in \mathbb{R}$, so $D_\omega = \omega I_d$. Scalar matrices commute with all other matrices of the same size, so its centralizer in G' will be all of G'. If F' is symmetric, then G' is an indefinite special orthogonal group, which has the same dimension as the special orthogonal group where $\dim SO(d, \mathbb{R}) = \frac{1}{2}d(d-1)$. If F' is skew-symmetric, G' is an indefinite symplectic group with dimension $\dim Sp(d, \mathbb{R}) = \frac{1}{2}d(d+1)$. This completes the case for ω real.

Now, let $\omega \in \mathbb{C} \setminus \mathbb{R}$ and F' be an antidiagonal symmetric or skew-symmetric $2d \times 2d$ matrix. Then we can write

$$F' = \begin{pmatrix} 0 & F_0 \\ \pm F_0^T & 0 \end{pmatrix}$$

for some $d \times d$ antidiagonal matrix F_0.

Let h be in the centralizer of D_ω in $GL(2d, \mathbb{R})$. Then $h = h_1 \oplus h_2$, where h_1 and h_2 can be any elements of $GL(d, \mathbb{R})$. h will be in the centralizer of D_ω in G' precisely when $h^T F' h = F'$. Breaking this equation into blocks,

$$\begin{pmatrix} h_1^T & 0 \\ 0 & h_2^T \end{pmatrix} \begin{pmatrix} 0 & F_0 \\ \pm F_0^T & 0 \end{pmatrix} \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} = \begin{pmatrix} 0 & h_1^T F_0 h_2 \\ \pm h_2^T F_0^T h_1 & 0 \end{pmatrix} = F',$$

which is true if and only if $h_2^{-1} = F_0^{-1} h_1^T F_0$. If $h_1 \in GL(d, \mathbb{R})$ is arbitrary, h_2 is fully determined, and so

$$\dim C_{G'}(D_\omega) = \dim GL(d, \mathbb{R}) = d^2.$$

□

Our final task for this section will be to answer the following question:

Given a generator g in $U(p,q,r)$ (i.e., α, β, or γ) and a degree n, how many eigenvalues in its diagonalized image under $\sigma_0 : U(p,q,r) \to SL(n,\mathbb{R})$ are 1s, how many are -1s, and how many distinct complex pairs of eigenvalues are there?

If we can answer that question, then Lemma 2.9 along with Lemma 2.8 tell us the dimension of its centralizer. In Long and Thistlethwaite's paper [11], they determine the number of distinct eigenvalues for $\sigma_0(g) \in SL(n,\mathbb{R})$ along with their multiplicity, without regard to whether these eigenvalues are real or not (since such a distinction is not necessary in their case.) We provide a somewhat expanded version of their argument below.

Recall that the diagonalized image of $\sigma_0(g)$, which we call D, is as follows:

$$D = (-1)^{n-1}\mathrm{diag}\left(\zeta^{n-1}, \zeta^{n-3}, \ldots, \zeta^{n-2i+1}, \ldots, \zeta^{-(n-3)}, \zeta^{-(n-1)}\right),$$

where $\zeta = -e^{\pi i/k}$ for a generator of order $2k$. The function $f(z) = -e^{\pi i z/k}$ is periodic with period $2k$, and injective when restricted to a single period. Since the powers in the eigenvalues above are in increments of two, the eigenvalues will repeat after k entries, so the maximum possible number of distinct eigenvalues is k. When $k \geq n$, there are no duplicates in the list of eigenvalues, so there are n eigenvalues, each with multiplicity 1.

When $k < n$, there will be repeated eigenvalues, and we can write $n = Qk + R$ with $0 \leq R < p$, so that Q is integer division $n \backslash k$ and R is the remainder. There will be Q full cycles through all k eigenvalues, and then another partial cycle through the first R eigenvalues. As a result, R eigenvalues occur with multiplicity $Q+1$ and $k-R$ occur with multiplicity Q.

At this point, our approach diverges from that of [11]. In Proposition 2.11, we will need to determine which powers of ζ produce real entries. Our approach requires the following lemma about powers of roots of unity:

Lemma 2.10. *Let $k \geq 2$ and s be integers, and define $\zeta := -e^{\pi i/k}$.*

Then for k odd, $\zeta^s = 1$ if and only if s is an integer multiple of k, and $\zeta^s \in \mathbb{C} \setminus \mathbb{R}$ otherwise.

For k even,

$$\zeta^s = \begin{cases} -1 & \textit{if } s \textit{ is an odd multiple of } k \\ +1 & \textit{if } s \textit{ is an even multiple of } k \end{cases},$$

and for all other k, ζ^s is not real.

Proof. Note that $\zeta^s = (-1)^s e^{s\pi i/k}$, and that $e^{s\pi i/k}$ is real precisely when k divides s, i.e.,

$$e^{s\pi i/k} = \begin{cases} -1 & \text{if } s \text{ is an odd multiple of } k \\ +1 & \text{if } s \text{ is an even multiple of } k \end{cases}. \tag{2.5}$$

Suppose first that k is odd. Then

$$(-1)^s = \begin{cases} -1 & \text{when } s \text{ is an odd multiple of } k \\ +1 & \text{when } s \text{ is an even multiple of } k \end{cases},$$

so combining this with Equation (2.5), we have that $\zeta^s = 1$ if and only if s is a multiple of k.

Now suppose that k is even. Equation (2.5) still holds, but now for any s a multiple of k, $(-1)^s = 1$. Thus, for k even,

$$\zeta^s = \begin{cases} -1 & \text{when } s \text{ is an odd multiple of } k \\ +1 & \text{when } s \text{ is an even multiple of } k \end{cases}.$$

□

Proposition 2.11. *Let g be a generator of order $2k$ for some $k \geq 2$ in $U(p,q,r)$, and D its diagonalized image in $SL(n,\mathbb{R})$. Let Q be the quotient and R be the remainder of dividing n by k:*

$$n = Qk + R \qquad\qquad 0 \leq R \leq k-1,$$

and define $n_{\mathcal{E}} := n \pmod 2$*,* $k_{\mathcal{E}} := k \pmod 2$*, and* $Q_{\mathcal{E}} := Q \pmod 2$*. Then the dimension of the centralizer of* D *in* G *is*

$$\sigma_G(n,k) := \dim C_G(D) = \frac{1}{2}\left((n+R)Q + R + k_{\mathcal{E}}\left(Q + Q_{\mathcal{E}}\right) - n_{\mathcal{E}}\left(2Q+1\right)\right).$$

Proof. Lemmas 2.8 and 2.9 reduce the problem to counting the multiplicities of 1, -1, and pairs of complex conjugates. The main difficulty in doing so is determining the number of real entries in D; non-real entries are considerably easier to handle.

Since the diagonal entries of D are

$$D = (-1)^{n-1}\mathrm{diag}\left(\zeta^{n-1}, \zeta^{n-3}, \dots, \zeta^{n-2i+1}, \dots, \zeta^{-(n-3)}, \zeta^{-(n-1)}\right),$$

any real entries will be 1 or -1. In particular, ± 1 can only be diagonal entries of D if n is odd (forcing the middle entry to be 1) or if k divides one of

$$n-1, n-3, \dots, n-2m+1.$$

We proceed by cases, considering in turn whether n, k, and Q are odd or even. The parity of n will tell us whether the middle entry must be real (since, when n is odd, the middle entry must be self-conjugate). The parity of k as compared with n will tell us whether at least one of the powers $n-1, n-3, \dots, n-2i+1, \dots, -(n-3), -(n-1)$ are divisible by k, and thus produce a real eigenvalue. The parity of the quotient Q will provide us with the multiplicity of each eigenvalue.

Case 1. $n = 2m$ *even* $(n_{\mathcal{E}} = 0)$*:*

(a) $k < n$ even $(k_{\mathcal{E}} = 0)$:

If k is even, there will be no real entries, since $n-1, n-3, \dots, n-2m+1$ are odd, and k is even, so k cannot divide any of these powers.

From [11], we know that there will be R eigenvalues in D which occur with multiplicity $Q+1$, and $k-R$ eigenvalues which occur with multiplicity Q. From Equation (2.4), and the fact that n is even, we know that each eigenvalue must be paired up with its complex conjugate, so in particular, an eigenvalue ω and its complex conjugate $\overline{\omega}$ must occur with the same multiplicity.

From Lemma 2.9, we know that the dimension of the centralizer of a $2d \times 2d$ matrix is d^2. If we group all of one eigenvalue together with all instances of its complex conjugate, we get that the dimension of the centralizer of D in G is

$$\dim C_G(D) = \frac{R}{2}\left(Q+1\right)^2 + \frac{k-R}{2}Q^2. \tag{2.6}$$

(b) $k < n$ odd ($k_{\mathcal{E}} = 1$):

Since n is even,

$$D = (-1)\mathrm{diag}\left(\zeta^{n-1}, \zeta^{n-3}, \ldots, \zeta^{n-2i+1}, \ldots, \zeta^{-(n-3)}, \zeta^{-(n-1)}\right),$$

and we know from Lemma 2.10 that $\zeta^s = 1$ when s is a multiple of k, and ζ^s is complex otherwise. Thus, we can infer that there will be no instances of 1 along the diagonal.

As long as $k < n$, k will equal one of $n-1, n-3, \ldots, n-2m+1$, and so we know there is at least one -1 along the diagonal. Also, since -1 is its own complex conjugate, its multiplicity must be even.

(i) Q even ($Q_{\mathcal{E}} = 0$):

-1 must have multiplicity Q, and we know from Lemma 2.9 that the centralizer of $-I_Q$ has dimension $\frac{1}{2}Q(Q+1)$. Grouping eigenvalues as in Case 1(a):

$$\dim C_G(D) = \frac{1}{2}Q(Q+1) + \frac{R}{2}(Q+1)^2 + \frac{k-R-1}{2}Q^2. \tag{2.7}$$

(ii) Q odd ($Q_{\mathcal{E}} = 1$):

-1 must have multiplicity $Q+1$. Group as before:

$$\dim C_G(D) = \frac{1}{2}(Q+1)(Q+2) + \frac{R-1}{2}(Q+1)^2 + \frac{k-R}{2}Q^2. \tag{2.8}$$

(c) $k \geq n$:

We will see below that Equations (2.6) and (2.7) are already sufficient.

If $k \geq n$, then k cannot divide any of $n-1, n-3, \ldots, n-2m+1$, so there will be no real entries along the diagonal.

(i) $k = n$:

The quotient of n divided by k is $Q = 1$, with remainder $R = 0$. So there are $n = k - R$ complex eigenvalues occurring with multiplicity $1 = Q$, and

$$\dim C_G(D) = \frac{k-R}{2} Q^2,$$

which, for $n = k$ even and $R = 0$, is the same as Equation (2.6).

(ii) $k > n$:

The quotient of n divided by k will be $Q = 0$, and thus $R = n$. So there are $R = n$ complex eigenvalues occurring with multiplicity $1 = Q + 1$, and

$$\dim C_G(D) = \frac{R}{2} (Q+1)^2,$$

which, for $Q = 0$, is the same as Equations (2.6) and (2.7).

Case 2. $n = 2m+1$ *odd* $(n_{\mathcal{E}} = 1)$

When n is odd, every possible D must have a 1 in the center of its diagonal to satisfy Equation (2.4). Any additional 1s must be paired together due to the same equation, so 1 will always have odd multiplicity. Also, since n is odd,

$$D = \operatorname{diag}\left(\zeta^{n-1}, \zeta^{n-3}, \ldots, \zeta^{n-2i+1}, \ldots, \zeta^{-(n-3)}, \zeta^{-(n-1)}\right).$$

(a) $k < n$ even $(k_{\mathcal{E}} = 0)$:

Lemma 2.10 tells us that $\zeta^s = -1$ when s is an odd multiple of k. If $k < n$, then one of $n-1, n-3, \ldots, n-2m+1$ will equal k, so we are guaranteed a -1; further, since -1 is its own conjugate, its multiplicity is even. Then ± 1 has multiplicity Q and ∓ 1 has multiplicity $Q+1$, so

$$\dim C_G(D) = \frac{1}{2}Q(Q-1) + \frac{1}{2}Q(Q+1) + \frac{R-1}{2}(Q+1)^2 + \frac{k-R-1}{2}Q^2. \quad (2.9)$$

(b) $k < n$ odd ($k_{\mathcal{E}} = 1$):

Lemma 2.10 tells us that ζ^s is never -1 when k is odd, so there are no -1 entries on the diagonal.

(i) Q even ($Q_{\mathcal{E}} = 0$):

1 has multiplicity $Q+1$, so

$$\dim C_G(D) = \frac{1}{2}Q(Q+1) + \frac{R-1}{2}(Q+1)^2 + \frac{k-R}{2}Q^2. \quad (2.10)$$

(ii) Q odd ($Q_{\mathcal{E}} = 1$):

1 has multiplicity Q, so

$$\dim C_G(D) = \frac{1}{2}Q(Q-1) + \frac{R}{2}(Q+1)^2 + \frac{k-R-1}{2}Q^2. \quad (2.11)$$

(c) $k \geq n$:

We will see below that Equations (2.9), (2.10), and (2.11) are already sufficient.

If $k \geq n$, then k cannot divide any of $n-1, n-3, \ldots, n-2m+1$, so there are no -1 entries. We have already established that 1 must be along the diagonal with odd multiplicity since n is odd.

(i) $k = n$:

$Q = 1$, $R = 0$, and we know that 1 must be one of the eigenvalues. So there are $k - R = k$ eigenvalues with multiplicity Q, and

$$\dim C_G(D) = \frac{1}{2}Q(Q-1) + \frac{k-R-1}{2}Q^2,$$

which matches up with Equation (2.11) for $R = 0$.

(ii) $k > n$:

$Q = 0$, and there are $R = n$ distinct eigenvalues, one of which must be a 1. Thus,

$$\dim C_G(D) = \frac{1}{2}Q(Q+1) + \frac{R-1}{2}(Q+1)^2 ,$$

which matches up with Equations (2.9) and (2.10) for $Q = 0$.

It is straightforward, if a bit tedious, to check that

$$\begin{aligned} \dim C_G(g) &= \frac{1}{2}\left(kQ^2 + 2QR + R + k_{\mathcal{E}}\left(Q + Q_{\mathcal{E}}\right) - n_{\mathcal{E}}\left(2Q+1\right)\right) \\ &= \frac{1}{2}\left((n+R)Q + R + k_{\mathcal{E}}\left(Q + Q_{\mathcal{E}}\right) - n_{\mathcal{E}}\left(2Q+1\right)\right) \end{aligned}$$

satisfies each of Equations (2.6), (2.7), (2.8), (2.9), (2.10), and (2.11). One can think of this equation as a parameterization of the previous equations by parameters $n_{\mathcal{E}}$, $k_{\mathcal{E}}$, and $Q_{\mathcal{E}}$; it is included for the sake of brevity in the statement of the main theorem, and does not to the author's knowledge have any additional significance beyond that of the individual cases. □

Now that we are able to accurately count eigenvalues (and as a result, the dimension of the centralizer), we can summarize the contribution of the cyclic generators to the dimension of the restricted Hitchin component.

Proposition 2.12. *Let g be a generator of order $2k$ in $U(p,q,r)$. Then*

$$\dim \mathcal{D}_n\left(\langle g \rangle\right) = \begin{cases} \frac{1}{2}n(n-1) - \sigma_G(n,k) & \text{if } n \text{ odd} \\ \frac{1}{2}n(n+1) - \sigma_G(n,k) & \text{if } n \text{ even.} \end{cases}$$

Proof. Define $g_n := \sigma_0(g) \in G$, where G is $Sp(2m)$ for $n = 2m$ even, and $SO(m, m+1)$ for $n = 2m+1$ odd. There exists some $c \in GL(n, \mathbb{C})$ such that $D = c^{-1}g_n c$ is diagonal.

D is not, in general, in G, but it is in the complexified version of G, which we call $G_{\mathbb{C}}$. Define $\phi_c : G_{\mathbb{C}} \to G_{\mathbb{C}}$ by the automorphism $\phi_c(g) = c^{-1}gc$. So $\phi_c(G) =: G'$ is isomorphic as a group to G. If h' is in the centralizer of D for the group D', then $h = ch'c^{-1}$ is in the centralizer of g_n for G. So finding the dimension of the centralizer of D in G' is equivalent to finding the dimension of the centralizer of g_n in G.

Because of this, the statement follows directly from Proposition 2.11 and the fact that

$$\dim G = \begin{cases} \dim\left(SO(m,m+1)\right) & \text{if } n = 2m+1 \\ \dim\left(Sp(2m)\right) & \text{if } n = 2m. \end{cases}$$

□

2.5 Proof of Theorem 2.4

Recall that $U(p,q,r)$ has presentation

$$U(p,q,r) = \left\langle \alpha, \beta, \gamma, z \,\middle|\, \alpha^p = \beta^q = \gamma^r = \alpha\beta\gamma = z, z^2 = 1 \right\rangle.$$

In the previous section, we determined the contribution of cyclic generators α, β, and γ to the dimension of the restricted Hitchin component for the base representation $\sigma_0 : U(p,q,r) \to G$. In this section, we will deal with the relation $\alpha\beta\gamma = z$ to complete the proof of our main theorem, Theorem 2.4.

Let $n > 2$ and let G be either $Sp(2m)$ or $SO(m, m+1)$ as appropriate. For $g \in G$, denote the conjugacy class of g by $[g] = \{xgx^{-1} \,|x \in G\}$, and the images of $\alpha, \beta, \gamma, z \in U(p,q,r)$ under the base representation $\sigma_0 : U(p,q,r) \to G$ by $\alpha_n, \beta_n, \gamma_n, z_n$, respectively.

Define the natural product map

$$\Pi : [\alpha_n] \times [\beta_n] \times [\gamma_n] \to G$$
$$\Pi(\alpha'_n, \beta'_n, \gamma'_n) := \alpha'_n\beta'_n\gamma'_n.$$

We will show in Proposition 2.13 that Π is a submersion near the image of our base representation, σ_0. Since our deformation space $\mathcal{K}$ for $U(p,q,r)$ is diffeomorphic to $\Pi^{-1}(z_n)$,

$$\dim \mathcal{K} = \dim\,[\alpha_n] + \dim\,[\beta_n] + \dim\,[\gamma_n] - \dim G.$$

Since $\mathcal{K}$ is diffeomorphic to the deformation space $\mathcal{H}$ of $T(p,q,r)$ in G, and the restricted Hitchin component $\mathcal{H}_G$ is $\mathcal{H}$ modded out by G-conjugation, the proof of Theorem 2.4 will be complete.

Our last task, then, is to show that for $P = \alpha_n\beta_n\gamma_n$, the differential map $D\Pi_P$ is surjective. Indeed, we shall see that fixing γ_n and allowing each of α_n and β_n to vary individually within their respective conjugacy classes will generate the full tangent space of G at $\alpha_n\beta_n\gamma_n$; this approach was established in Long and Thistlethwaite's paper [11]. Since we will deal entirely with degree n representations for the duration of this chapter, we will omit the subscript n when there is no risk of confusion.

Proposition 2.13. *Define*

$$S = \{\alpha'\beta' : \alpha' \in [\alpha], \beta' \in [\beta]\}.$$

Then the tangent spaces of S and G are equal at $\alpha\beta$.

It will be sufficient to show that the tangent spaces of each of the following sets generate the tangent space of G at $\alpha\beta$:

$$S_1 = \{\alpha'\beta : \alpha' \in [\alpha]\}, S_2 = \{\alpha\beta' : \beta' \in [\beta]\}.$$

To do so, we will need to write arbitrary elements of each set in terms of the exponential map. Then, we will need to show that the images of these adjoint representations generate the Lie algebra of G, which we denote $\mathfrak{g}$. This last step will require a few small technical lemmas, which we provide next.

Lemma 2.14. *The Lie algebra of G is*

$$\mathfrak{g} := \left\{X \in M_n(\mathbb{R}) : X^T F = -FX\right\}.$$

Proof. Let $X \in M_n(\mathbb{R})$ and recall that, for F the (skew-)symmetric bilinear form discussed earlier, $X \in G$ if and only if $X^T F X = F$. Define a smooth path in $\mathfrak{g}$ starting at the identity by $A(t) := I + tX + \mathcal{O}\left(t^2\right)$. $A(t)$ must also satisfy the group relation, so

$$\begin{aligned} A^T F A &= \left(I + tX^T\right) F \left(I + tX\right) \\ &= F + t\left(X^T F + FX\right) \end{aligned}$$

must equal F, and thus $X^T F = -FX$. □

Finding the dimension of S will also entail the use of an indefinite bilinear form on $\mathfrak{g}$:

$$\langle x, y \rangle := tr(xy).$$

We will need to establish that this form is non-degenerate and allows us to break $\mathfrak{g}$ up using orthogonal complements.

Lemma 2.15. *The indefinite bilinear form $\langle x, y \rangle := tr(xy)$ on $\mathfrak{g}$ is non-degenerate, and respects Ad_g for all $g \in G$.*

Proof. Cartan's criterion states that a Lie algebra is semisimple if and only if the Killing form is non-degenerate. The Lie algebras for the symplectic group and special orthogonal group are both classical simple Lie algebras, so the Killing form is non-degenerate. Further, simple Lie algebras contained in the general linear Lie algebra (which includes both possibilities for $\mathfrak{g}$) have a Killing form that is proportional to the trace form given above.

Let $g \in G$ and $x, y \in \mathfrak{g}$. Then, by an argument given in Section 4 of Long and Thistlethwaite's paper [11], we have that

$$\langle Ad_g(x), Ad_g(y) \rangle = tr\left(gxg^{-1}gyg^{-1}\right) = tr\left(gxyg^{-1}\right) = tr(xy) = \langle x, y \rangle,$$

so $\langle\ ,\ \rangle$ respects Ad_g. □

Lemma 2.16. *Let W be a linear subspace of $\mathfrak{g}$, and $W^\perp$ its orthogonal complement under $\langle\ ,\ \rangle$. Then* $\dim(W) + \dim\left(W^\perp\right) = \dim(\mathfrak{g})$.

Proof. Define

$$\phi : \mathfrak{g} \to \mathfrak{g}^*$$
$$x \mapsto f_x$$
$$f_x(y) := \langle x, y \rangle .$$

Since $\langle \ , \ \rangle$ is non-degenerate by Lemma 2.15, $\ker(\phi) = 0$. The annihilator of W in $\mathfrak{g}$, which we denote $Ann(W)$, is the set of all linear functionals $f \in \mathfrak{g}^*$ which are zero on W. This is precisely the image of $W^\perp$ under ϕ. So $W^\perp \cong Ann(W)$, and

$$\dim W + \dim W^\perp = \dim W + \dim Ann(W) = \mathfrak{g}.$$

□

Then, in particular, we can use the bilinear form $\langle \ , \ \rangle$ to break up $\mathfrak{g}$ into a sum of the kernel and image of the map $Ad_g - I$:

Lemma 2.17. *Let g be in the image of $U(p, q, r)$ under σ_0. Then $\mathfrak{g}$ can be written as the direct sum*

$$\mathfrak{g} = \ker\left(Ad_g - I\right) \oplus Im\left(Ad_g - I\right),$$

orthogonal with respect to $\langle \ , \ \rangle$.

Proof. Let $g_0 \in U(p, q, r)$. $g_2 := \sigma\left(g_0\right)$ is diagonalizable in $SL(2, \mathbb{C})$ since triangle groups have no parabolic elements, and $g := \sigma_0(g_0)$ retains this property. We need to demonstrate that Ad_g is also diagonalizable; to do so, we will construct bases for $\mathfrak{g}$ when n is odd, and for when n is even. Recall from Lemma 2.14 that $\mathfrak{g}$ can be characterized as all $n \times n$ matrices satisfying the relation $X^T F = -FX$.

First, suppose n is odd. For simplicity, first consider what happens when F is just the identity matrix. In this case, the relation becomes $X^T + X = 0$, so we would get all skew-symmetric matrices, meaning also that the diagonal must consist of zeroes. Now, consider what happens when F has 1 on the antidiagonal and 0 elsewhere; we would get matrices that have zeroes on the antidiagonal instead, and the rest of the entries would be skew-symmetric about the antidiagonal. Using

$$F = \text{antidiag}\left(\binom{n-1}{0}, -\binom{n-1}{1}, \binom{n-1}{2}, \dots, \binom{n-1}{n-1}\right)$$

will introduce constants, but entries will still be symmetric about the antidiagonal, up to a fixed constant for each antidiagonal pair. So one basis for $\mathfrak{g}$ when n is odd would be

$$\{B_{ij} := E_{ij} + c_{ij}E_{n-j+1,n-i+1}\}$$

for all i, j above the antidiagonal, where $c_{ij} \in \mathbb{Q}$ is a carefully-chosen constant and E_{ij} is the matrix with a 1 in the (i, j)th place and zeroes elsewhere.

Now, suppose n is even. Using an argument similar to the above, $\mathfrak{g}$ will still be matrices symmetric about the antidiagonal up to a constant, but now the diagonal is permitted to be nonzero. So a basis for $\mathfrak{g}$ with n even would include the basis elements listed before, along with basis elements along the antidiagonal,

$$\{A_i := E_{i,n-i+1}\}.$$

Up to conjugacy, we can write g as $\text{diag}(g_1, \dots, g_n)$. Then

$$\begin{aligned} Ad_g(A_i) &= \frac{g_i}{g_{n-i+1}}A_i \\ Ad_g(E_{ij} + c_{ij}E_{n-j+1,n-i+1}) &= \frac{g_i}{g_j}E_{ij} + \frac{g_{n-j+1}}{g_{n-i+1}}c_{ij}E_{n-j+1,n-i+1}. \end{aligned}$$

The second equation would pose a problem for a general diagonal g, since $\frac{g_i}{g_j}$ and $\frac{g_{n-j+1}}{g_{n-i+1}}$ would not necessarily be equal. However, the diagonalized form D given in Equation 2.3 can also be written as $g = (-1)^{n-1}\text{diag}\left(\zeta^{n-1}, \zeta^{n-3}, \zeta^{n-5}, \dots, \zeta^{n-(2n-1)}\right)$, and so $i - j = (n - j + 1) - (n - i + 1)$. Thus,

$$\frac{g_i}{g_j} = \zeta^{2(i-j)} = \frac{g_{n-j+1}}{g_{n-i+1}},$$

and $Ad_g(B_{ij}) = \zeta^{2(i-j)} B_{ij}$. So Ad_g is diagonal under our basis. In particular, $Ad_g - I$ will have a full eigenspace for the eigenvalue zero, which is the kernel of $Ad_g - I$, and so we have the desired direct sum decomposition as long as the summands are indeed orthogonal under $\langle\ ,\ \rangle$.

To see this, we reproduce an argument found in the proof for Lemma 4.1.1 of Long and Thistlethwaite's paper [11]. Let $\xi \in \ker(Ad_g - I)$ and $\eta \in Im(Ad_g - I)$. Then $\xi = Ad_g \xi$ and $\eta = (Ad_g - I)\zeta$ for some $\zeta \in \mathfrak{g}$.

$$\begin{aligned}
\langle \xi, \eta \rangle &= \langle \xi, (Ad_g - I)\zeta \rangle \\
&= \langle \xi, Ad_g \zeta \rangle - \langle \xi, \zeta \rangle \\
&= \langle Ad_g \xi, Ad_g \zeta \rangle - \langle \xi, \zeta \rangle \\
&= \langle \xi, \zeta \rangle - \langle \xi, \zeta \rangle \\
&= 0.
\end{aligned}$$

□

Proof of Theorem 2.4

We will now show that the tangent spaces of $S_1 = \{\alpha'\beta : \alpha' \in [\alpha]\}$ and $S_2 = \{\alpha\beta' : \beta' \in [\beta]\}$ generate the tangent space of G at $\alpha\beta$. This will show that the tangent spaces of

$$S' = \{\alpha'\beta' : \alpha' \in [\alpha], \beta' \in [\beta]\}$$

and G are equal at the same point, and thus that Π is a submersion nearby. The ideas used in this argument do not differ significantly from those found in the proof of Proposition 4.1 in Long and Thistlethwaite's paper [11]; it is included here so that certain elementary details omitted from their proof can be included (namely, Equations 2.12 and 2.13).

Any element of S_1, S_2 can be written as $g\alpha g^{-1}\beta$, $\alpha h\beta h^{-1}$ respectively for some $g, h \in G$ be close to the identity I. We can also write $g = \exp(\xi)$ and $h = \exp(\eta)$ for some $\xi, \eta \in \mathfrak{g}$. Then we have the following equations (up to first order terms):

$$
\begin{aligned}
g\alpha g^{-1}\beta &= (I+\xi)\,\alpha\,(I-\xi)\,\beta \\
&= \alpha\beta + \xi\alpha\beta - \alpha\xi\beta \\
&= \left(I + \xi - \alpha\xi\alpha^{-1}\right)\alpha\beta \\
&= (I + \xi - Ad_\alpha\xi)\,\alpha\beta \\
&= \exp\left((1 - Ad_\alpha)\,\xi\right)\alpha\beta, \qquad (2.12)
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha h\beta h^{-1} &= \alpha\,(I+\eta)\,\beta\,(I-\eta) \\
&= \alpha\beta + \alpha\eta\beta - \alpha\beta\eta \\
&= \left(I + \alpha\eta\alpha^{-1} - \alpha\beta\eta\beta^{-1}\alpha^{-1}\right)\alpha\beta \\
&= (I + Ad_\alpha\eta - Ad_{\alpha\beta}\eta)\,\alpha\beta \\
&= \exp\left(Ad_\alpha\,(I - Ad_\beta)\,\eta\right)\alpha\beta. \qquad (2.13)
\end{aligned}
$$

So every element of S_1 can be written as $\exp(v)\alpha\beta$ for some $v \in \mathfrak{g}$, and every element of S_2 an be written as $\exp(w)\alpha\beta$ for some $w \in \mathfrak{g}$. It remains to show that the images of $(1 - Ad_\alpha)$ and $Ad_\alpha\,(I - Ad_\beta)$ generate all of $\mathfrak{g}$.

Suppose that $Im\,(I - Ad_\alpha) + Ad_\alpha Im\,(I - Ad_\beta) \subsetneq \mathfrak{g}$, and pick some $\xi \neq 0$ in the orthogonal complement of the left hand side. Then $\xi \in \ker(I - Ad_\alpha)$, so $\xi = Ad_\alpha\xi$.

ξ is also in the orthogonal complement of $Ad_\alpha Im\,(I - Ad_\beta)$. Ad_α preserves $\langle\ ,\ \rangle$, so $\xi \in Ad_\alpha \ker(I - Ad_\beta)$. Therefore, $Ad_\alpha\xi \in Ad_\alpha \ker(I - Ad_\beta)$, and thus $\xi \in \ker(I - Ad_\beta)$.

Since σ_0 is irreducible, Schur's Lemma gives us that

$$
\ker(I - Ad_\alpha) \cap \ker(I - Ad_\beta) = \{0\}\,,
$$

so $\xi = 0$, which is a contradiction. □

Chapter 3

Representations of $T(3,3,4)$ in $SL(5,\mathbb{Z})$

Notice. *Chapter* 3*, in full, is currently being prepared for submission for publication of the material. The dissertation author was the sole investigator and author of this paper.*

3.1 Background and Main Theorem

In Chapter 2, we were able to find the dimension of the restricted Hitchin component, but the arguments used were not constructive. That is, the proof tells us nothing of how to actually find the representations that constitute the Hitchin component (restricted or otherwise). For some purposes, dimension is enough; in [2], Choi and Goldman point out a correspondence between representations of an orbifold Σ in the Hitchin component and convex $\mathbb{RP}^2$ structures on Σ, so the dimension of the Hitchin component gives us meaningful information about the geometry of the orbifold. One can also interpret the dimension of the Hitchin component in terms of projective structures in the degree $n = 4$ case; see the introduction of [3] for a concise treatment of the subject.

Even setting aside these connections, the Hitchin component remains useful as a tool to study, e.g., $SL(n,\mathbb{R})$ and $SL(n,\mathbb{Z})$; the existence and construction of Zariski dense surface subgroups is a topic of particular difficulty and interest. With that in mind, we continue our exploration of the Hitchin component by seeking out Zariski dense representations of the triangle group $T(3,3,4)$ in $SL(5,\mathbb{Z})$; by Theorem 1.5, we can then restrict those representations to surface subgroups of $SL(5,\mathbb{Z})$. Since our approach relies heavily on the machinery developed in [3] and applied specifically to triangle groups in [10] and in Long and Thistlethwaite's paper [12], we begin by stating one of these results.

In [12] Theorem 1.1, Long and Thistlethwaite give a 1-parameter family of discrete, faithful representations of the triangle group $T(3,3,4)$ into $PSL(4,\mathbb{R})$, indexed over $\mathbb{R}$. They also demonstrate that, for parameter values which are non-negative integers, the images of these representations are Zariski dense in $SL(4,\mathbb{R})$. Using similar methods, they are also able to produce such a 1-parameter family for $T(3,3,4)$ in $PSL(5,\mathbb{R})$; in both cases, all of their representations lie in the Hitchin component of the corresponding representation variety. Their family for the degree 5 case is Theorem 3.1 of [12]:

Theorem 3.1 (Long and Thistlethwaite)**.** *The family of representations of the triangle group*

$$\rho_k : T(3,3,4) = \langle a, b | a^3 = b^3 = (ab)^4 = 1 \rangle \to PSL(5,\mathbb{R})$$

given by

$$\rho_k(a) = \begin{bmatrix} 1 & 0 & -3-2k-8k^2 & -1+10k+32k^3 & -5-16k^2 \\ 0 & 4(-1+k) & -13-4k & 3+16(1+k)^2 & -4+16k \\ 0 & 1-k+4k^2 & 3-2k+8k^2 & -2(1+3k+16k^3) & 3+16k^2 \\ 0 & k & 2k & 1-2k-8k^2 & 1+4k \\ 0 & 0 & 3k & 3(-1+k-4k^2) & -2 \end{bmatrix},$$

$$\rho_k(b) = \begin{bmatrix} 0 & 0 & -3-2k-8k^2 & -1+10k+32k^3 & -5-16k^2 \\ 0 & 1 & 3+4k & -13-8k-16k^2 & 4-16k \\ 0 & 0 & -2(1+k+4k^2) & 6k+32k^3 & -3-16k^2 \\ 1 & 0 & -2(1+k) & -1+2k+8k^2 & -1-4k \\ 2k & 0 & 1-2k & -4k & 1 \end{bmatrix}$$

are discrete and faithful for every $k \in \mathbb{R}$.

Using Theorem 1.1 of [11], we observe that the Hitchin component in either case is 2-dimensional. As a result, these 1-parameter families cannot possibly exhaust all representations in the Hitchin component, and one might hope that there are other infinite families of Zariski dense representations in $SL(4,\mathbb{Z})$ or $SL(5,\mathbb{Z})$. Indeed, Long and Thistlethwaite hint at the possibility of such a family of symplectic representations in the degree 4 case.

Our task for this chapter will be to continue the work started in [12] by producing two additional 1-parameter families of discrete, faithful representations $T(3,3,4) \to PSL(5,\mathbb{R})$, whose images are Zariski dense for non-negative integer parameter values:

Theorem 3.2. *Define* $\omega := 4k^2+2k+2$. *Then both families of representations of the triangle group* $T(3,3,4)$ *given below are discrete and faithful for every* $k \in \mathbb{R}$.

$$\tau_k, \upsilon_k : T(3,3,4) = \langle a, b | a^3 = b^3 = (ab)^4 = 1 \rangle \to PSL(5,\mathbb{R})$$

$$\tau_k(a) = \begin{bmatrix} -1 & -1 & 0 & 0 & 3 \\ 4 & -3 & 3 & 0 & 3 \\ 3 & 0 & 1 & 0 & -3 \\ 2-4\omega & 0 & 0 & 1 & -2 \\ 1 & -1 & 1 & 0 & 1 \end{bmatrix}, \tau_k(b) = \begin{bmatrix} 0 & -1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & -3+4k \\ -1 & 0 & 0 & -2k & -4-8k^2 \\ 0 & 0 & 0 & 2k(2\omega-1) & 16\omega k^2+4k+1 \\ 0 & 0 & 0 & 1-\omega & -4\omega k+2k-1 \end{bmatrix}$$

$$\upsilon_k(a) = \begin{bmatrix} -1 & 1 & 0 & 0 & 3 \\ -1 & -3 & -3 & 0 & -3 \\ 1 & 1 & 1 & 0 & 0 \\ -2(7+12k+8k^2) & 5 & 5 & 1 & -2 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\upsilon_k(b) = \begin{bmatrix} 0 & 1 & 0 & 0 & 3 \\ 1 & 0 & -1 & -1 & 6-4k \\ -1 & 1 & 0 & -1-2k & 2+2k-8k^2 \\ 0 & 0 & 0 & 4(-2-k+3k^2+4k^3) & 19-18k-44k^2+64k^4 \\ 0 & 0 & 0 & -3-6k-4k^2 & 7+4k-12k^2-16k^3 \end{bmatrix}$$

We would like the images of these representations to be Zariski dense in $SL(5,\mathbb{R})$; to prove this, we will need Proposition 1 of Lubotzky's paper [13]:

Proposition 3.3 (Lubotzky)**.** *Let H be a subset of $SL(n,\mathbb{Z})$. Assume that for some prime p, H generates $SL(n,\mathbb{Z}_p)$ under reduction modulo p. If $n=2$, assume $p\neq 2,3$. If $n=3,4$, assume $p\neq 2$. Then for all but finitely many primes q, H generates $SL(n,\mathbb{Z}_q)$.*

Further, H is dense in $SL(n,\mathbb{Z}_p)$, thus H is Zariski dense in $SL(n,\mathbb{R})$.

Using this, we obtain the following Corollary of Theorem 3.2:

Corollary 3.4. *For each $k\in\mathbb{Z}_{\geq 0}$, the images of $T(3,3,4)$ under τ_k and υ_k are Zariski dense in $SL(5,\mathbb{R})$.*

Proof. In particular, the prime $p=3$ works for both families of representations. One can verify by computer program (e.g. Sage [16], see the code in Appendix B.1) that for each of the residue classes of k modulo 3, $\tau_k\,(T(3,3,4))$ and $\upsilon_k\,(T(3,3,4))$ surject $SL(5,\mathbb{Z}_3)$, so each of these image groups are Zariski dense in $SL(5,\mathbb{R})$.

Since we will concern ourselves with surface subgroups in Section 3.5, we would like a similar result for images of finite-index surface subgroups. In the discussion that follows, we will use ρ_k to refer to τ_k and υ_k interchangeably, since the argument is the same in either case. Note that the argument in the following paragraph is very similar to one that appears in the introduction of [12].

Let Γ be a subgroup of finite index in $T(3,3,4)$, and let N be a normal subgroup of finite index in $T(3,3,4)$ such that $N\subseteq\Gamma$. By Proposition 3.3, we can choose a prime q sufficiently large so that $\rho_k(T(3,3,4))$ generates $PSL(5,\mathbb{Z}_q)$, and such that $|T(3,3,4):N|<|PSL(5,\mathbb{Z}_q)|$. Suppose that $\rho_k(N)$ is trivial. Then $\rho_k(T/N)=\rho_k(T)=PSL(5,q)$, but ρ_k is faithful and $|T(3,3,4):N|<|PSL(5,\mathbb{Z}_q)|$, so this is a contradiction. Thus, $\rho_k(N)\trianglelefteq PSL(5,\mathbb{Z}_q)$ is nontrivial, and since $PSL(5,\mathbb{Z}_q)$ is simple, $\rho_k(N)$ must be all of $PSL(5,\mathbb{Z}_q)$. Since Γ contains N, it follows that $\rho_k(\Gamma)$ is also $PSL(5,\mathbb{Z}_q)$, and so $\rho_k(\Gamma)$ is Zariski dense in $PSL(5,\mathbb{R})$ as desired. ☐

The representations given in Theorem 3.2 are discrete and faithful from their place in the Hitchin component, and for $k\in\mathbb{Z}_{\geq 0}$, they are pairwise non-conjugate. We can see the latter by looking at traces

$$\begin{aligned}\operatorname{tr}\left(\tau_k\left(a^{-1}b\right)\right) &= 34+102k+232k^2+240k^3+192k^4\\ \operatorname{tr}\left(\upsilon_k\left(a^{-1}b\right)\right) &= 185+610k+880k^2+624k^3+192k^4.\end{aligned}$$

Since these are both strictly increasing functions for $k \in \mathbb{Z}_{\geq 0}$, the characters of each τ_k and υ_k are distinct; further, these polynomials are not equal for any two integer parameters, so there are no characters shared between the two families. Thus, for $k \in \mathbb{Z}_{\geq 0}$, none of the representations from either family are conjugate. However, this is not sufficient to guarantee that images of surface subgroups are also pairwise non-conjugate, so we will need the following Corollary:

Corollary 3.5. *There exists a surface subgroup Γ of $T(3,3,4)$ and subsequences $\{\tau_{k_i}\}_{i\in\mathbb{N}}$ and $\{\upsilon_{k_i}\}_{i\in\mathbb{N}}$ such that the images $\tau_{k_i}(\Gamma)$ and $\upsilon_{k_i}(\Gamma)$ are pairwise non-conjugate surface subgroups in $SL(5,\mathbb{Z})$.*

We will begin in Section 3.2 by giving a summary of the methods used in [12] to produce representations $T(3,3,4) \to PSL(5,\mathbb{C})$ with entries in a degree 4 extension over $\mathbb{Q}(u,v)$ for parameters u, v, and which are not necessarily in the Hitchin component. In Section 3.3, we discuss how to restrict the parameters u and v to only those representations contained in the Hitchin component. Next, we describe and apply an algorithm developed by Long and Thistlethwaite to conjugate to representations over the rational numbers, and use conjugation by elementary row operations and polynomial interpolation to produce matrices with integral entries for parameters $k \in \mathbb{Z}$ in Section 3.4. Finally, in Section 3.5, we prove Corollary 3.5 and show that the representations in Theorem 3.2 produce a subsequence of pairwise non-conjugate surface subgroups in $SL(5,\mathbb{Z})$.

3.2 Complex Representatons

The task of finding a 2-parameter family of representations $T(3,3,4) \to PSL(5,\mathbb{C})$ was accomplished in Long and Thistlethwaite's paper [12] in the course of proving their Theorem 3.1. In this section, we provide a brief summary of their methods (with some further details gathered from Cooper, Long, and Thistlethwaite's paper [3]), since we will be using this 2-parameter family as our starting point for the proof of our Theorem 3.2. We begin by noting that since we are considering representations of degree $n = 5$, the diagram in Equation 2.2 admits the lift $T(3,3,4) \to SL(2,\mathbb{R})$ indicated by the dotted line in the figure, so we may consider the image of the base representation $\rho_0 : T(3,3,4) \to PSL(5,\mathbb{R})$ to lie naturally in $SL(5,\mathbb{R})$ and avoid the discussion required in Section 2.3.

They begin with the base representation $\rho_0 : T(3,3,4) \to SL(5,\mathbb{R})$, and apply small random perturbations to the images of generators a and b. The matrices produced will no longer form a representation of $T(3,3,4)$, so Newton's method is used to converge to numerical representations $\rho_1 : T(3,3,4) \to SL(5,\mathbb{R})$ near ρ_0 but not conjugate to it. In doing so, one notices that characteristic polynomials for certain elements of $T(3,3,4)$ are of the form

$$\begin{aligned}\text{charpoly}\left(\rho_1\left(ba^{-1}\right)\right) &= 1 - vx + (u+3v-3)x^2 - (3u+v-3)x^3 + ux^4 - x^5\\ \text{charpoly}\left(\rho_1\left(ab^{-1}\right)\right) &= 1 - ux + (3u+v-3)x^2 - (u+3v-3)x^3 + vx^4 - x^5,\end{aligned}$$

where the parameters u and v seem to be able to vary independently of one another.

For the base representation, $u = v = 9 + 6\sqrt{2}$, so further constrain Newton's method so that u and v as defined above are rational numbers close to $9 + 6\sqrt{2}$ and adjust step lengths for each iteration to ensure convergence. The next goal is to find representations for $T(3,3,4)$ given by matrices for a and b in terms of the parameters $u = \mathrm{Tr}\left(ba^{-1}\right)$ and $v = \mathrm{Tr}\left(ab^{-1}\right)$. This is achieved by finding a large "grid" of numerical representations over the (u,v) plane, applying a suitable change of basis, and using polynomial interpolation to find expressions for each entry.

Doing so produces a 2-parameter family of representations $\rho_{(u,v)} : T(3,3,4) \to SL(5,\mathbb{C})$. However, the entries for $\rho_{(u,v)}(a)$ and $\rho_{(u,v)}(b)$ are quite lengthy. To address this, we use the reparameterization $s = u - 1$ and $t = v - 1$, and we define

$$\alpha = \sqrt{-4s^3 - 4st^2 - 4t^2(1+t) + s^2(-4-4t+t^2)}$$
$$\beta = -s^2 + 2t + si(s+t+2)$$
$$\gamma = -4it(t+1) + 4s(t-it+1) + s^2(it+2).$$

Then $\rho_{(s,t)}(a)$ is

$$\begin{bmatrix} 1 & 0 & \frac{2(\alpha+\beta)}{(s^2-2t)(s-it)} & 1 & \frac{4(\overline{\beta}-\alpha)}{(s^2-2t)(s+t+2)(s-it)} \\ 0 & -1 & -\frac{s+t+2}{2} & \frac{i(\alpha+\overline{\beta})}{8} & 1 \\ 0 & 1 & \frac{s}{2} & \frac{(i-1)(s^2-2t)}{8} & \frac{(i+1)(\beta-\alpha)}{2(s+t+2)(s-it)} \\ 0 & \frac{(i-1)(\overline{\beta}-\alpha)}{(s^2-2t)(s+t+2)} & 1 & \frac{(i-1)(s+2i)}{4} & \frac{-2s^3+(i+1)(-\alpha(s+2i)+\gamma)}{(s^2-2t)(s+t+2)(s-it)} \\ 0 & \frac{(i+1)(s-it)}{4} & -\frac{i(\alpha+\beta)}{8} & \frac{-2s^3+(i+1)(\alpha(s+2i)+\gamma)}{32} & \frac{(i+1)(2i-s)}{4} \end{bmatrix},$$

and $\rho_{(s,t)}(b)$ is

$$\begin{bmatrix} 0 & 0 & \frac{2(\alpha+\beta)}{(s^2-2t)(s-it)} & 1 & \frac{4(\alpha+\overline{\beta})}{(s^2-2t)(s+t+2)(s-it)} \\ -\frac{\alpha(2i+s+is)-\gamma+s(2s+st+2it)}{16} & 1 & \frac{s+t+2}{2} & \frac{\alpha+\overline{\beta}}{8} & i \\ \frac{i(\alpha-\beta)}{8} & 0 & -\frac{s+2}{2} & \frac{(i+1)(s^2-2t)}{8} & \frac{(i-1)(\beta-\alpha)}{2(s+t+2)(s-it)} \\ \frac{(i+1)(s-it)}{4} & 0 & -1 & \frac{(i+1)(s-2)}{4} & \frac{-2is^3+(i-1)(-\alpha(2i+s)+\gamma)}{(s^2-2t)(s+t+2)(s-it)} \\ -\frac{(i+1)(s-it)(\alpha-\overline{\beta})}{32} & 0 & \frac{i(\alpha+\beta)}{8} & \frac{2is^3+(i-1)(\alpha(2i+s)+\gamma)}{32} & -\frac{(i-1)(s-2)}{4} \end{bmatrix}.$$

Observe that β and γ are both in $\mathbb{Q}(s,t)(i)$, and so matrix entries lie in

$$\mathbb{Q}(s,t)(i,\alpha) = \mathbb{Q}(u,v)(i,\alpha),$$

where α under the original parameterization is

$$\alpha = \sqrt{9 - 18u + 13u^2 - 4u^3 - 18v + 20uv - 6u^2v + 13v^2 - 6uv^2 + u^2v^2 - 4v^3};$$

so the traces for these representations must lie in $\mathbb{Q}(u,v)(i,\alpha)$. However, the trace field could be contained in a proper subfield. We can easily demonstrate that α is actually necessary by taking

$$\operatorname{tr}\left(\rho_{(u,v)}\left(aba^{-1}b^{-1}\right)\right) = \frac{1}{2}(uv - 3u - 3v - \alpha + 3),$$

but does the trace field contain complex numbers?

To see that the trace field is actually $\mathbb{Q}(u,v)(\alpha)$, consider the representation $\rho_{(u,v)}$ where $u = v = 9 + 6\sqrt{2}$. Find a conjugation matrix that takes the images of a and b under this representation to the images under the base representation ρ_0. Since images of ρ_0 are always in $PSL(5,\mathbb{R})$, and traces are invariant under conjugation, the trace field for $\rho_{(u,v)}$ when $u = v = 9 + 6\sqrt{2}$ must also be real. Further, deforming ρ_0 continuously in the Hitchin component keeps traces in $\mathbb{R}$; similarly, continuously varying (u,v) cannot introduce non-real traces. Thus, the trace field of $\rho_{(u,v)}$ is real, so it must be $\mathbb{Q}(u,v)(\alpha)$.

3.3 Hitchin Representations

While the procedure described in Section 3.2 produces representations in $SL(5,\mathbb{C})$, some parameter values (u,v) will give representations outside the Hitchin component. To ensure we can use the results of Labourie in [9], we will have to select parameter values that land us in the Hitchin component. Recall that we currently have representations $\rho_{(u,v)}$ with entries in $\mathbb{Q}(u,v)(i,\alpha)$, where α is the square root of a polynomial in u and v.

Representations in the Hitchin component must be conjugate to representations in $SL(5,\mathbb{R})$, so their trace fields (which are invariant under conjugation) must be real. Thus, a necessary condition for a particular $\rho_{(u,v)}$ to be in the Hitchin component is that α must be real. Figure 3.1 shows where $\alpha = 0$ occurs in the (u,v) plane.

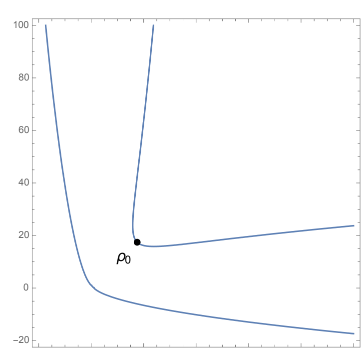


Figure 3.1: Contour plot for $\alpha = 0$ in the (u,v) plane

The base representation ρ_0 occurs when $u = v = 9 + 6\sqrt{2}$, and must by definition be contained in the Hitchin component. We can consider two half-planes glued along the component of $\alpha = 0$ containing ρ_0 to be the Hitchin component, which project to the northeast portion of the Figure 3.1. While representations southwest of the other component of $\alpha = 0$ are real-valued, they are not in the Hitchin component, and so are not guaranteed to be discrete or faithful.

To guarantee that a representation has entries in the integers, we must require at minimum that α be an integer. Figure 3.2 shows a scatter plot of integer coordinates (u, v) such that $\alpha \in \mathbb{Z}$. Notice that some of the points in the Hitchin component fall along a parabola, indicated by the red points along the grey parabola. If we parameterize the parabola by

$$u(k) = 20 + 19k + 124k^2 + 48k^3 + 192k^4$$
$$v(k) = 20 - 19k + 124k^2 - 48k^3 + 192k^4,$$

integer values of the parameter k will correspond to some, but not all, of the points along the parabola.

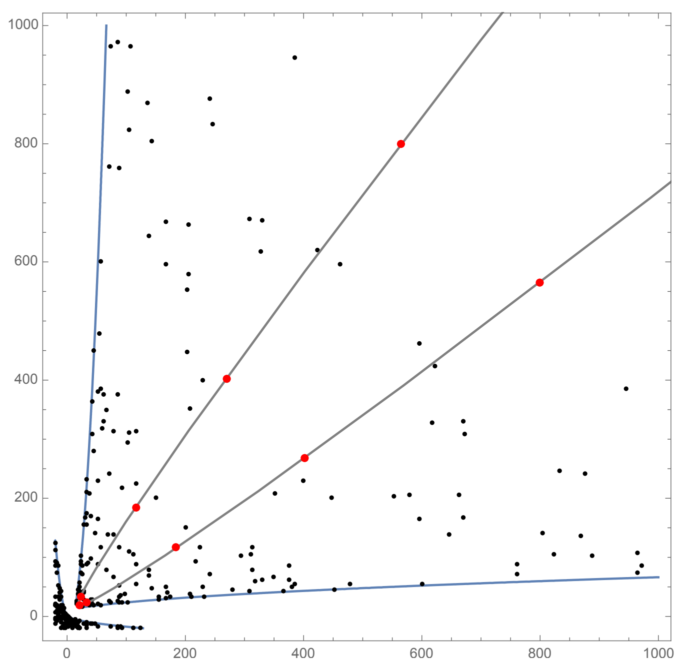


Figure 3.2: Points in the (u, v) integral lattice where $\alpha \in \mathbb{Z}$

These parameterizations result in the family of representations $T(3,3,4) \to SL(5,\mathbb{R})$ presented in Theorem 3.1 of [12]. After removing these points, there are two other families of regularly-spaced points falling along the same parabola, so it would make sense that a reparameterization might hit these points.

In particular, we find that (u,v) which make α an integer occur for parameter values

$$\{k \in \mathbb{Z} : k = \frac{\ell}{4}, \ell \in \mathbb{Z}, 2 \nmid \ell\},$$

i.e., quarters, but not halves. Replacing the parameter k with $k + \frac{1}{4}$ gives us

$$u_1(k) := 34 + 102k + 232k^2 + 240k^3 + 192k^4$$
$$v_1(k) := 23 + 46k + 160k^2 + 144k^3 + 192k^4,$$

and replacing k with $k + \frac{3}{4}$ yields

$$u_2(k) := 185 + 610k + 880k^2 + 624k^3 + 192k^4$$
$$v_2(k) := 116 + 410k + 664k^2 + 528k^3 + 192k^4.$$

Both $(u_1(k), v_1(k))$ and $(u_2(k), v_2(k))$ produce an integral α when $k \in \mathbb{Z}$, and substituting either pair in for (u,v) in the matrices for a and b will give us two families of representations, which have entries in $\mathbb{Q}[i]$ for each $k \in \mathbb{Z}$.

3.4 Rational Representations

Ultimately, our goal will be to find two infinite one-parameter families of representations, which have integer entries when the parameter is also an integer. To show that this is possible, it will be sufficient to find an appropriate change of basis such that each $\rho_{(u_i,v_i)}$ for $i = 1,2$ has rational entries. As long as traces of group elements are integral, an argument similar to Proposition 2.1 of [10] will guarantee that these rational matrices are conjugate to matrices over $\mathbb{Z}$.

To find a good change of basis, we will rely on a tool established in Proposition 2.1 of [12], which we summarize here. Let ρ_k be one of the $\rho_{(u_i,v_i)}$ for $i = 1, 2$ and $k \in \mathbb{Z}$. Suppose that there is some $\mathbb{Q}$-linear combination S of matrices in $\rho_k(T(3, 3, 4))$ such that the dimension of the nullspace of S is 1. In practice, a linear combination of $\{I, a, b, a^{-1}, b^{-1}, ab\}$ seems to work frequently for the degree 5 case, and the coefficients are found by brute force; we assume from now on that such a combination has been found. Let v be a vector in the kernel of S; then the orbit of v under the action $\rho_k(T(3, 3, 4))$ is all of $\mathbb{Q}^5$. Then, if we multiply each of $\{a, b, a^{-1}, b^{-1}, ab\}$ by v, we obtain a basis for $\mathbb{Q}^5$ where the matrices for ρ_k have entries in $\mathbb{Q}$.

To obtain τ_k in Theorem 3.2, begin by selecting a handful of parameter values $k \in \mathbb{Z}$. For each one, perform the algorithm above (see Appendix B.2 for an implementation in Mathematica), and pick the solution that has the smallest denominators. Check that the coefficients for the linear combination S are the same for each chosen parameter (meaning that the corresponding matrices were obtained by the same change of basis).

Next, we need to conjugate these matrices so that their entries are over $\mathbb{Z}$ rather than $\mathbb{Q}$. Long and Thistlethwaite point out in [12] that it is often advantageous to use eigenvectors in this process; however, in our case, the denominators obtained in the previous step are 2 at worst, so trial and error quickly produces a short sequence of elementary matrix operations that will work. For τ_k, we conjugate by

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 1 \end{bmatrix},$$

and for υ_k, we use

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & 0 \\ 1 & 0 & \frac{1}{2} & 0 & 2 \end{bmatrix}.$$

Now, we have a handful of matrices over the integers. To get the $(i,j)^{th}$ entry of $\tau_k(a)$, use polynomial interpolation on the $(i,j)^{th}$ entries in the matrices for a; do the same for $\tau_k(b)$. This will give you matrices with entries in $\mathbb{Z}[k]$, and one can check that they still satisfy the group relations. Follow the same procedure for υ_k, then simplify both using conjugation by elementary row operations to get the families in the statement of Theorem 3.2.

3.5 Pairwise Non-Conjugate Surface Subgroups

In this section, we will prove Corollary 3.5: given a surface subgroup Γ of $T(3,3,4)$, there are subsequences for each of our representations such that the images of Γ are pairwise non-conjugate surface subgroups in $SL(5,\mathbb{Z})$. Further, by our discussion at the end of Section 3.1, these images are Zariski dense in $SL(5,\mathbb{R})$. To begin, we note that this Corollary is not vacuous, i.e., there exists at least one surface subgroup Γ in $T(3,3,4)$. We saw this directly in the beginning of Section 1.1.3 by constructing a manifold cover whose fundamental group is a surface subgroup of $T(3,3,4)$. More generally, Theorem 1.5 provides us with a method for determining which indices $|T(p,q,r):\Gamma|$ are possible for an arbitrary hyperbolic triangle group. In the case of $T(3,3,4)$, we saw at the end of 1.1.3 that for any $g \geq 2$, we could obtain the surface group for a g-handled torus.

Given this, all that remains for Corollary 3.5 is to prove the following Proposition:

Proposition 3.6. *Let Γ be a subgroup of finite index in $T(3,3,4)$. There exist subsequences $\{\tau_{k_i}\}_{i\in\mathbb{N}}$ and $\{\upsilon_{k_i}\}_{i\in\mathbb{N}}$ such that the images $\tau_{k_i}(\Gamma)$ and $\upsilon_{k_i}(\Gamma)$ are pairwise non-conjugate subgroups in $SL(5,\mathbb{Z})$.*

To find these subsequences, we will follow the approach of Long and Thistlethwaite in [11], though we make some substantive adjustments to Lemma 3.7 below.

Lemma 3.7. *There exists a strictly increasing sequence of primes $\{p_j\}$ and an associated strictly increasing sequence of positive integers $\{k_{p_j}\}$ such that*

$$T(3,3,4) \xrightarrow{\tau_{k_{p_j}}} SL(5,\mathbb{Z}) \to SL\left(5,\mathbb{Z}_{p_j}\right)$$

is not surjective, and similarly for $\upsilon_{k_{p_j}}$.

Proof. Let $\pi_p : SL(5,\mathbb{Z}) \to SL(5,\mathbb{Z}_p)$ denote reduction modulo p on individual entries. We claim that if $p \equiv 1 \pmod{228}$ is prime and if $k \in \mathbb{N}$ satisfies $24k^2 + 12k + 11 \equiv 0 \pmod{p}$, then the image of $T(3,3,4)$ under $\pi_p \circ \tau_k$ has an invariant two-dimensional subspace, and thus cannot be all of $SL(5,p)$. The corresponding result holds for υ_k if $p \equiv 1 \pmod{228}$ is prime and $k \in \mathbb{N}$ satisfies $24k^2 + 36k + 23 \equiv 0 \pmod{p}$.

Let $p \equiv 1 \pmod{228}$ prime. We need to demonstrate that $24k^2 + 12k + 11 \equiv 0 \pmod{p}$ and $24k^2 + 36k + 23 \equiv 0 \pmod{p}$ have solutions for any given p. Over the complex numbers, the solutions for these equations are $k = \frac{1}{12}\left(-3 \pm i\sqrt{57}\right)$ and $k = \frac{1}{12}\left(-9 \pm i\sqrt{57}\right)$, respectively. Thus, the equations will both have solutions if we can demonstrate that the square roots of -1 and 57, as well as the multiplicative inverse of 12, are in $\mathbb{Z}_p$. Indeed, $1/12 \in \mathbb{Z}_p$ if and only if $(p,12) = 1$. Since $12 \mid 228$, 12 has a multiplicative inverse in $\mathbb{Z}_p$.

The remaining criteria can be rephrased as follows: are -1 and 57 quadratic residues modulo p? In the argument that follows, we use facts from basic number theory, all of which are covered in Chapter 5 of Ireland and Rosen's textbook [7]. First, -1 is a quadratic residue modulo p if and only if p is of the form $p = 4m + 1$. Since $4 \mid 228$, this condition is satisfied. Next, note that 57 is a quadratic residue modulo p if and only if $57^{(p-1)/2} \equiv 1 \pmod{p}$, and

$$\begin{aligned} 57^{(p-1)/2} &\equiv \left(\frac{57}{p}\right) && \pmod{p} \\ &\equiv \left(\frac{3}{p}\right)\left(\frac{19}{p}\right) && \pmod{p} \end{aligned}$$

where $\left(-\right)$ is the Legendre symbol. Since $p \equiv 1 \pmod{4}$, we can simplify each of the Legendre symbols in the last line above as follows:

$$\begin{aligned}\left(\frac{3}{p}\right) &= \left(\frac{p}{3}\right) \\ &= \left(\frac{1}{3}\right) \\ &\equiv 1^{(p-1)/2} \pmod{p},\end{aligned}$$

and

$$\begin{aligned}\left(\frac{19}{p}\right) &= \left(\frac{1}{19}\right) \\ &\equiv 1^{(p-1)/2} \pmod{p}.\end{aligned}$$

Thus, 57 is a quadratic residue modulo p and both congruences have solutions in $\mathbb{Z}_p$.

Now we must show that plugging in one of these solutions k into τ_k, υ_k produces an invariant two-dimensional subspace. To see this, we will consider the images of generators a and $c = ab$ in $SL(5, \mathbb{Z})$. A Mathematica [18] notebook with the matrix computations required for the rest of this proof is available in Appendix B.3.

For both families of representations, c is diagonalizable over $\mathbb{C}$; perform this change of basis simultaneously on a and c. At this point, it is possible to reorder the basis elements so that the last two entries on the top row of $\tau_k(a)$ vanish if k is a solution of $24k^2+12k+11 = 0$. Indeed, if we substitute one of these solutions into a, we see that it fixes the subspace spanned by the last two basis vectors. c does as well, since it is diagonal, and taking complex entries to their $\mathbb{Z}_p$ counterparts does not change this since the solutions in question are also in $\mathbb{Z}_p$. The same argument works for $\upsilon_k(a)$ when k is a solution of $24k^2 + 36k + 23 = 0$.

By Dirichlet's theorem on arithmetic progressions, there are infinitely many primes p satisfying $p \equiv 1 \pmod{228}$. To complete the proof of the lemma, start with $p_1 = 229$ and choose k_{p_1} to be a solution to $24k^2+12k+11 \equiv 0 \pmod{p}$ (respectively, $24k^2+36k+23 \equiv 0 \pmod{p}$) in $\mathbb{Z}_p$. Since $\pi_{p_1} \circ \tau_{k_{p_1}}$ (resp. $\pi_{p_1} \circ \upsilon_{k_{p_1}}$) fixes a two-dimensional subspace, it cannot be surjective. For each subsequent term p_{i+1}, continue searching in order through larger primes congruent to 1 modulo 228 until one of the corresponding solutions is larger than k_{p_i}, and choose that solution to be $k_{p_{i+1}}$.

Note that this inductive process is possible as a result of Proposition 3.3. For each fixed k, τ_k and υ_k fail to surject onto $\mathbb{Z}_p$ for only finitely many primes, the set of solutions k_{p_i} corresponding to primes $p_i \equiv 1 \pmod{228}$ cannot be bounded, and so we can always choose a prime p_i large enough so that $\left\{k_{p_j}\right\}$ is strictly increasing. □

Proof of Proposition 3.6

Let Γ be a subgroup of finite index in $T(3,3,4)$. To complete the proof of Proposition 3.6, suppose there exist $\tau_{k_1}, \ldots, \tau_{k_n}$ such that $\{\tau_{k_i}(\Gamma)\}_{i=1}^{n}$ are pairwise non-conjugate subgroups of $SL(5,\mathbb{Z})$ for $i \in \{1,\ldots,n\}$. We proceed by induction on n.

By Proposition 3.3, for each $1 \leq i \leq n$

$$\Gamma \xrightarrow{\tau_{k_i}} SL(5,\mathbb{Z}) \to SL(5,\mathbb{Z}_p)$$

fails to be surjective for only finitely many primes p. Denote by p_0 the maximum of all such "bad" primes occurring for at least one of $\tau_{k_1}, \ldots, \tau_{k_n}$, and use Lemma 3.7 to choose a prime $p_j > p_0$ such that $k_{p_j} > k_n$. Then

$$\Gamma \xrightarrow{\tau_{k_i}} SL(5,\mathbb{Z}) \to SL(5,\mathbb{Z}_{p_j})$$

is surjective for all $1 \leq i \leq n$, but

$$T(3,3,4) \xrightarrow{\tau_{k_{p_j}}} SL(5,\mathbb{Z}) \to SL(5,\mathbb{Z}_{p_j})$$

and its restriction to Γ are not. So $\tau_{k_{p_j}}(\Gamma)$ cannot be conjugate to any of the previous $\tau_{k_i}(\Gamma)$, and so we can set $k_{n+1} = k_{p_j}$ as the next representation in our subsequence. □

Chapter 4

Conclusion

4.1 Summary

In Chapter 2, we gave a formula for the dimension of the restricted Hitchin component for degree $n \geq 3$ representations of hyperbolic triangle groups $T(p, q, r)$. In Chapter 3, we found two infinite families of Hitchin representations $T(3, 3, 4) \to PSL(5, \mathbb{R})$, each of which has an infinite subsequence where images of $T(3, 3, 4)$ are Zariski dense in $SL(5, \mathbb{R})$, and for each surface subgroup $\Gamma \leq T(3, 3, 4)$, a further subsequence where images of Γ are pairwise non-conjugate in $SL(5, \mathbb{Z})$. While it is clear that both results concern Hitchin representations of hyperbolic triangle groups, other relationships between the two topics may not be immediately obvious. With this in mind, we will use this section to draw further connections and elaborate on the result in Theorem 2.4.

First, note that in Table 4.1 that while the dimension of the Hitchin component for the triangle group $T(3, 3, 4)$ used in Chapter 3 and degree $n = 5$ has dimension 2, the dimension of the restricted Hitchin component is 0. This tells us that the only Hitchin representation with image in $SO(3, 2)$ is the base representation ρ_0; there is evidence to suggest this happens in only a few isolated instances. Of considerably more interest is the case where the dimension of the Hitchin component is equal to that of the restricted Hitchin component. In that case, there are no representations with Zariski dense images in the Hitchin component, since the closure for each image will be contained in the proper Lie subgroup $G \lneq SL(n, \mathbb{R})$.

Table 4.1: Comparison of Hitchin and restricted Hitchin components for small degree n. Dots indicate any value such that $p \leq q \leq r$ and $T(p, q, r)$ is hyperbolic.

n	p	q	r	$\dim \mathcal{H}$	$\dim \mathcal{H}_G$
3	2	$\cdot$	$\cdot$	0	0
	≥ 3	$\cdot$	$\cdot$	2	0
4	2	3	$\cdot$	0	0
	2	≥ 4	$\cdot$	2	2
	3	3	≥ 4	2	0
	3	≥ 4	$\cdot$	4	2
	4	$\cdot$	$\cdot$	6	4
5	2	3	$\cdot$	0	0
	2	4	$\cdot$	2	2
	2	≥ 5	$\cdot$	4	2
	3	3	4	2	0
	3	3	≥ 5	4	0
	3	4	4	4	2
	3	4	≥ 5	6	2
	3	≥ 5	$\cdot$	8	2
	4	4	4	6	4
	4	4	≥ 5	8	4
	4	≥ 5	$\cdot$	10	4
	≥ 5	$\cdot$	$\cdot$	12	4
6	2	3	$\cdot$	2	2
	2	4	5	2	2
	2	4	≥ 6	4	4
	2	5	5	4	2
	2	5	≥ 6	6	4
	2	≥ 6	$\cdot$	8	6

In the introduction to Long and Thistlethwaite's paper [11], they note in passing that the dimension of the Hitchin component is always even, and the dimensions listed in Table 4.2 would suggest this might also be the case for the restricted Hitchin component. Indeed, this can be proved using the following lemma:

Lemma 4.1. *Suppose m is integer division of n by 2 (i.e., $n = 2m$ if n even, and $n = 2m+1$ if n odd.) Then $\sigma_G(n,k) \equiv m \pmod 2$ for all $n \geq 3$ and all $k \geq 2$.*

The proof is a tedious but not terribly difficult or enlightening divisibility argument, so it has been relegated to Appendix C. Using Lemma 4.1, we have the following:

Proposition 4.2. *The dimension of the restricted Hitchin component $\mathcal{H}_G$ for a hyperbolic triangle group is always an even number if $G = SO(m, m+1)$ or $G = Sp(2m)$.*

Proof. If $G = SO(m, m+1)$, then $\dim G = \frac{1}{2}n(n-1) = \frac{1}{2}(2m+1)(2m) = m(2m+1)$. If $G = Sp(2m)$, then $\dim G = \frac{1}{2}n(n+1) = \frac{1}{2}2m(2m+1) = m(2m+1)$. In either case, if m is odd (respectively, even), so is the dimension of G.

By Theorem 2.4,

$$\dim \mathcal{H}_G = \dim G - (\sigma_G(n,p) + \sigma_G(n,q) + \sigma_G(n,r)).$$

If m is odd, using Lemma 4.1, the above equation modulo 2 becomes

$$\dim \mathcal{H}_G \equiv 1 - (1+1+1) \pmod 2,$$

and similarly, if m is even, we have that

$$\dim \mathcal{H}_G \equiv 0 - (0+0+0) \pmod 2.$$

Thus, the dimension of the Hitchin component for G is even. □

Table 4.2: Dimensions of the restricted Hitchin component for degrees $n \leq 6$.

n	p	q	r	$\dim \mathcal{H}_G$
4	2	3	$\cdot$	0
	2	≥ 4	$\cdot$	2
	3	3	$\cdot$	0
	3	≥ 4	$\cdot$	2
	≥ 4	$\cdot$	$\cdot$	4
5	2	3	$\cdot$	0
	2	≥ 3	$\cdot$	2
	3	3	$\cdot$	0
	3	≥ 4	$\cdot$	2
	≥ 4	$\cdot$	$\cdot$	4
6	2	3	$\cdot$	2
	2	4	5	2
	2	4	≥ 6	4
	2	5	5	2
	2	5	≥ 6	4
	2	≥ 6	$\cdot$	6
	3	3	$4-5$	2
	3	3	≥ 6	4
	3	4	$4-5$	4
	3	4	≥ 6	6
	3	5	5	4
	3	5	≥ 6	6
	3	≥ 6	$\cdot$	8
	4	4	$4-5$	6
	4	4	≥ 6	8
	4	5	5	6
	4	5	≥ 6	8
	4	≥ 6	$\cdot$	10
	5	5	5	6
	5	5	≥ 6	8
	5	≥ 6	$\cdot$	10
	≥ 6	$\cdot$	$\cdot$	12

4.2 Future Work

The methods used by Long and Thistlethwaite to determine the dimension of the Hitchin component in [11] (and by extension, those used in Chapter 2 for the restricted Hitchin component) rely heavily on the algebraic structure of triangle groups. Consequently, these results do not have an obvious extension to the fundamental groups of two-dimensional orbifolds more generally. However, it is likely possible to use a similar approach for other classes of orbifold groups, particularly considering the characterization of local behavior given in this proposition from Chapter 13 of Thurston's notes [17]:

Proposition 4.3 (Thurston)**.** *The singular locus of a two-dimensional orbifold has these types of local models:*

(i) The mirror: $\mathbb{R}^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by reflection in the y-axis.

(ii) Cone points of order n: $\mathbb{R}^2/\mathbb{Z}_n$, with $\mathbb{Z}_n$ acting by rotations.

(iii) Corner reflectors of order n: $\mathbb{R}^2/D_n$, with D_n is the dihedral group of order $2n$ and with presentation

$$\langle a, b : a^2 = b^2 = (ab)^n = 1 \rangle ,$$

where the generators correspond to reflections in lines meeting at angle π/n.

Broadly construed, this leaves us with the following question:

Question 1. *What is the dimension of the Hitchin component for a family of orbifolds with shared algebraic structure?*

Cone points can be handled using the approach of Long and Thistlethwaite described in [11], and note the similarity of the algebraic structure for corner reflectors to the following presentation for triangle groups:

$$T(p, q, r) = \langle a, b \,|a^p = b^q = (ab)^r = 1 \rangle .$$

The main difficulty, then, would be to devise an approach to handle the algebraic contribution of including the local patches described in Proposition 4.3 into the orbifold as a whole. One might hope for an argument analogous to the one used for the triangle group relation $abc = 1$ in [11], but more complicated relations may produce additional difficulties.

If, similar to triangle groups, this family of orbifold groups has a base representation with image contained in the adjoint group G of a split real form for some complex simple Lie group, it makes sense to ask about the restricted Hitchin component:

Question 2. *What is the dimension of the restricted Hitchin component for a family of orbifolds?*

Our next avenue of inquiry focuses back in on triangle groups, but in a slightly different setting. The triangle groups $T(p,q,r)$ we use are the orientation-preserving isometries of a tiling of the hyperbolic plane $\mathbb{H}^2$ by geodesic triangles, but triangle groups are also commonly defined in a way that includes orientation-reversing isometries. This results in a larger triangle group, which has $T(p,q,r)$ as a subgroup of index 2, and with presentation

$$\tilde{T}(p,q,r) = \left\langle a, b, c : (ab)^p = (bc)^q = (ca)^r = a^2 = b^2 = c^2 = 1 \right\rangle,$$

where each of the generators corresponds to reflection over a side of a geodesic triangle with angles π/p, π/q, and π/r. Our orientation-preserving triangle groups $T(p,q,r)$ are sometimes called von Dyck groups, and $\tilde{T}(p,q,r)$ are examples of Coxeter groups. Thus, we arrive at a natural extension of the work done in Chapter 2 on the dimension of the restricted Hitchin component:

Question 3. *What is the relationship between the dimension of the (restricted) Hitchin component of the orientation-preserving triangle group $T(p,q,r)$ and its reflection-inclusive counterpart $\tilde{T}(p,q,r)$?*

Since the group $\tilde{T}(p,q,r)$ is larger, the dimension would be no greater than that for $T(p,q,r)$. We suspect that the dimension might be half that of $T(p,q,r)$; Proposition 4.2 would certainly allow for that possibility.

Finally, while our discussions have focused on the Hitchin component, there is another side to the story told so far. If we complexify $SL(n,\mathbb{R})$ by allowing entries to have complex values, we get $SL(n,\mathbb{C})$. The anti-holomorphic involution that takes entries in $\mathbb{C}$ to their complex conjugates does not change real entries, so the fixed points under this involution give us $SL(n,\mathbb{R})$. However, there is another anti-holomorphic involution we can apply to arrive at a special unitary group $SU(\ell,m)$; our base representation $\sigma_0 : T(p,q,r) \to SL(n,\mathbb{R})$ has images that lie in $SL(n,\mathbb{R}) \cap SU(\ell,m)$. Further, these two Lie groups have the same dimension, leading naturally to our final question:

Question 4. *What can be said about the deformation space for a triangle group (or other orbifold group) into a special unitary group?*

These deformations would no longer be in the Hitchin component (since entries would no longer be strictly real), but may have interesting geometric properties of their own. A computational foray into this deformation space, similar to the one described in Chapter 3 for $SL(5,\mathbb{R})$, may also produce interesting results.

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Appendices

A Mathematica Notebooks for Base Representations

These are Mathematica notebooks created and used by the author to compute the base representation for a given hyperbolic triangle group $T(p, q, r)$ and a given representation degree n. The parameters p, q, r, n must be defined before running the first program, and the second program should be run after the first so that the required inputs are already defined. While the programs below are definitely not the first implementations of these algorithms, they were created by the author.

A.1 Triangle Groups in $SL(2, \mathbb{R})$

```
(*Required inputs: p,q,r; dimension n>2*)
alpha = Pi/p; beta = Pi/q; gamma = Pi/r;
a2 = Sin[alpha];

(*Reflection over B and C:*)
rB = {{1, Cos[alpha]}, {0,1}}.({{0, 1}, {1, 0}}.{{1, -Cos[alpha]}, {0, 1}});
rC = {{-1, 0}, {0, 1}};

(*Find the length of ab and use it to find the coordinates of b.*)
Cc = (Cos[alpha] Cos[beta] + Cos[gamma])/(Sin[alpha] Sin[beta]);

(*Specifically, move up from i by length Cc; we want the second solution
   below because the first one is moving down by that amount.*)
b1 = 0;
b2 = (y /.Last[Solve[Cc == 1 + ((0 - 0)^2 + (y - a2)^2)/(2 (a2) (y)), y]]);
b0 = b2*I;

(*Find the radius and horizontal displacement from the origin of the circle
   making angle beta with the imaginary axis.*)
rad = b2/Sin[beta]; d = rad* Cos[beta];
```

```
tA = {{1, d}, {0, 1}};
scaleA = {{1/Sqrt[rad], 0}, {0, Sqrt[rad]}};
rA = Inverse[tA].Inverse[scaleA].{{0, 1}, {1, 0}}.scaleA.tA;

(*Compose reflections to get your representations:*)
repa = rB.rC;
repb = rC.rA;
repc = rB.rA;
```

A.2 Irreducible Representation from $SL(2,\mathbb{R})$ to $SL(n,\mathbb{R})$

```
(*Required inputs: matrices repa, repb, repc in SL(2,R) from previous
   notebook*)

(*Create a homogeneous degree n-1 polynomial.*)
f=ConstantArray[0,n];
Do[f[[i+1]]=(ToExpression["f"<>ToString[i]]),{i,0,n-1}];
poly=0;
Do[poly=poly+(f[[i+1]])*x0^(n-1-i)*y0^i,{i,0,n-1}];

(*Representation for a in SL(n,R): use rep in SL2 to create variable
   substitution*)
xsub=(repa.{1,0}).{x,y};
ysub=(repa.{0,1}).{x,y};
apoly=poly/.x0->xsub/.y0->ysub//Expand;
a=ConstantArray[0,{n,n}];
Do[a[[i,j+1]]=(*first isolate the variable term you want*)D[
(D[D[apoly,{x,n-i}],{y,i-1}])/((n-i)!*(i-1)!),
(*then isolate the coefficient you want*)
ToExpression[f[[j+1]]]]
,{i,1,n},{j,0,n-1}];
```

```
xsub=(repb.{1,0}).{x,y};
ysub=(repb.{0,1}).{x,y};
bpoly=poly/.x0->xsub/.y0->ysub//Expand;
b=ConstantArray[0,{n,n}];
Do[b[[i,j+1]]=(*first isolate the variable term you want*)D[
(D[D[bpoly,{x,n-i}],{y,i-1}])/((n-i)!*(i-1)!),
(*then isolate the coefficient you want*)
f[[j+1]]]
,{i,1,n},{j,0,n-1}];

xsub=(repc.{1,0}).{x,y};
ysub=(repc.{0,1}).{x,y};
cpoly=poly/.x0->xsub/.y0->ysub//Expand;
c=ConstantArray[0,{n,n}];
Do[c[[i,j+1]]=(*first isolate the variable term you want*)D[
(D[D[cpoly,{x,n-i}],{y,i-1}])/((n-i)!*(i-1)!),
(*then isolate the coefficient you want*)
f[[j+1]]]
,{i,1,n},{j,0,n-1}];

ea=Sign[Tr[MatrixPower[a,p]]//N];
eb=Sign[Tr[MatrixPower[b,q]]//N];
ec=Sign[Tr[MatrixPower[c,r]]//N];
output={Transpose[a],Transpose[b],Transpose[c],ea,eb,ec}
```

B Code for Degree 5 Representations

These are the programs used to find the families of degree 5 representations in Chapter 3, as well as any other programs used in proofs for that Chapter. No parameters need to be pre-defined, but Mathematica notebooks should be run first in their entirety so that relevant output is displayed.

B.1 Sage Code for Corollary 3.4

```
# This program compares the size of image groups of representations mod 3
# to the size of SL(5,Z_3). If they are equal, it displays both sizes;
# if not, it prints the error message "p = 3 does not work."

# generator a for first new representation, in terms of k
def gena(k):
       a = [[-1, -1, 0, 0, 3], [4, -3, 3, 0, 3], [3, 0, 1, 0, -3], [-6 - 8*
          k - 16*k^2, 0, 0, 1, -2], [1, -1, 1, 0, 1]]
       return a

# generator b for first new representation, in terms of k
def genb(k):
       b = [[0, -1, 0, 0, 3], [0, 0, 1, 1, -3 + 4*k], [-1, 0, 0, -2*k, -4 -
           8*k^2], [0, 0, 0, 6*k + 8*k^2 + 16*k^3, 1 + 4*k + 32*k^2 + 32*k
          ^3 + 64*k^4], [0, 0, 0, -1 - 2*k - 4*k^2, -1 - 6*k - 8*k^2 - 16*k
          ^3]]
       return b

# check that for each residue class of k in Z/3Z,
# the group <a,b> is the same size as SL(5,3).

# working over a finite field of order 3 prime
```

```
f = GF(3)
k = 0
kcount = 0

# compute the size of SL(5,3)
sl5p = SL(5,f)
sl5pSIZE = sl5p.order()

# run through all possible k-values for the generators in Z/3Z
for k in range(0,3):
        a = gena(k)
        b = genb(k)

        # create a matrix group for the k-value in Z/3Z
        a = matrix(f,5,a)
        b = matrix(f,5,b)
        newrep = MatrixGroup([a,b])

        # compare the size of the matrix group for fixed k to SL(5,3)
        newrepSIZE = newrep.order()
        if newrepSIZE == sl5pSIZE:
                kcount = kcount + 1
        else:
                break

# if all representations modulo 3 have equal size to SL(5,3), print the
   prime.
if kcount == 3:
        print 'The first rep mod p = 3 has size ' + str(newrepSIZE) + '.
           Size of SL(5,3): ' + str(sl5pSIZE) + '.'
```

```
else:
        print 'p = 3 does not work.'

# generator a for second new representation, in terms of k
def gena(k):
        a = [[-1, 1, 0, 0, 3], [-1, -3, -3, 0, -3], [1, 1, 1, 0, 0], [-14 -
           24*k - 16*k^2, 5, 5, 1, -2], [0, 1, 1, 0, 1]]
        return a

# generator b for second new representation, in terms of k
def genb(k):
        b = [[0, 1, 0, 0, 3], [1, 0, -1, -1, 6 - 4*k], [-1, 1, 0, -1 - 2*k,
           2 + 2*k - 8*k^2], [0, 0, 0, -8 - 4*k + 12*k^2 + 16*k^3, 19 - 18*k
            - 44*k^2 + 64*k^4], [0, 0, 0, -3 - 6*k - 4*k^2, 7 + 4*k - 12*k^2
            - 16*k^3]]
        return b

# check that for each residue class of k in Z/3Z,
# the group <a,b> is the same size as SL(5,3).

# working over a finite field of order 3 prime
f = GF(3)
k = 0
kcount = 0

# compute the size of SL(5,3)
sl5p = SL(5,f)
sl5pSIZE = sl5p.order()

# run through all possible k-values for the generators in Z/3Z
```

```
for k in range(0,3):
        a = gena(k)
        b = genb(k)

        # create a matrix group for the k-value in Z/3Z
        a = matrix(f,5,a)
        b = matrix(f,5,b)
        newrep = MatrixGroup([a,b])

        # compare the size of the matrix group for fixed k to SL(5,3)
        newrepSIZE = newrep.order()
        if newrepSIZE == sl5pSIZE:
                kcount = kcount + 1
        else:
                break

# if all representations modulo 3 have equal size to SL(5,3), print the
   prime.
if kcount == 3:
        print 'The second rep mod p = 3 has size ' + str(newrepSIZE) + '.
           Size of SL(5,3): ' + str(sl5pSIZE) + '.'
else:
        print 'p = 3 does not work.'
```

B.2 Mathematica Function for Matrix Entries over $\mathbb{Q}$

The following function is a cleaned-up version of code written by Long and Thistlethwaite for use in their paper [12], and is included with their permission.

`nullconjugate[a,b,max,minint]` takes degree 5 representations a and b with complex entries, and tries to conjugate them to $\mathbb{Q}$. $\{-max, max\}$ is the range of coefficients used in this computation; minint is the minimum number of non-integer entries that a given representation must have to be displayed. The author used `max = 1` and `minint = 10` for the computations in Chapter 3.

For each rational representation satisfying the minimum integer entry requirement, the function will display the matrices for a and b as well as the change of basis used to obtain them.

```
nullconjugate := Function[{a, b, max, minint},
   Module[{A, B, z1, z2, z3, z4, z5, z6, coeff, elts, elts1, S, ns, v,
      basis, aq, bq, sola, solb, q1, q2, q3, q4, q5},
    A = Inverse[a];
    B = Inverse[b];

    (* Each loop tries a Z-linear combo of group elements, S *)

    Do[coeff = {z1, z2, z3, z4, z5, z6};
     elts = {IdentityMatrix[5], a, b, A, B, a.b};
     S = coeff.elts;

     (* If the combo has a 1D nullspace, use it to form a basis *)

       ns = NullSpace[S];
     If[Length[ns] != 1, Continue[], v = ns[[1]]];
     basis = {elts[[2]].v, elts[[3]].v, elts[[4]].v, elts[[5]].v,
       elts[[6]].v};
     If[Det[basis] == 0, Continue[]];

     (* Attempt to write a and b in this basis *)

     aq = ConstantArray[0, {5, 5}];
     bq = ConstantArray[0, {5, 5}];
     Do[
```

```
 sola =
  Solve[a.basis[[i]] == {q1, q2, q3, q4, q5}.basis, {q1, q2, q3,
    q4, q5}];
 solb =
  Solve[b.basis[[i]] == {q1, q2, q3, q4, q5}.basis, {q1, q2, q3,
    q4, q5}];

 (* Next loop if basis fails or too many non-integer entries *)

      If[Length[sola] || Length[solb] == 0,
  Throw[{"Not a valid basis."}]];
 aq[[i]] = Simplify[{q1, q2, q3, q4, q5} /. sola[[1]]];
 bq[[i]] = Simplify[{q1, q2, q3, q4, q5} /. solb[[1]]];
 , {i, 1, 5}];
If[Catch[testint[aq]] < minint || Catch[testint[bq]] < minint,
 Continue[]];
Print[{aq // MatrixForm, bq // MatrixForm, basis}]
, {z1, -max, max}, {z2, -max, max}, {z3, -max, max}, {z4, -max,
 max}, {z5, -max, max}, {z6, -max, max}];
]];
```

B.3 Mathematica Notebook for Proof of Lemma 3.7

```
(* Define the first representation. We will use a and c=a.b as our
   generators. *)

a = {{-1, -1, 0, 0, 3}, {4, -3, 3, 0, 3}, {3, 0, 1, 0, -3}, {-6 - 8 k - 16 k
   ^2, 0, 0, 1, -2}, {1, -1, 1, 0, 1}};
```

```
b = {{0, -1, 0, 0, 3}, {0, 0, 1, 1, -3 + 4 k}, {-1, 0, 0, -2 k, -4 - 8 k^2},
    {0, 0, 0, 6 k + 8 k^2 + 16 k^3, 1 + 4 k + 32 k^2 + 32 k^3 + 64 k^4},
   {0, 0, 0, -1 - 2 k - 4 k^2, -1 - 6 k - 8 k^2 - 16 k^3}};
c = a.b // Expand;

eigen = Eigensystem[c]

(* Put c into a form that is almost Rational Canonical Form; this will
   simplify computations. Conjugate a into the same basis.*)

e1 = {1, 0, 0, 0, 0};
e2 = c.e1;
e3 = c.e2;
e4 = c.e3;
e5 = 2*eigen[[2, 1]] // Expand;
cj1 = Transpose[{e1, e2, e3, e4, e5}] // Expand;
c1 = Simplify[Inverse[cj1].c.cj1];
a1 = Simplify[Inverse[cj1].a.cj1];

MatrixForm[c1]

(* Diagonalize c, and conjugate a into the same basis. *)

e1 = NullSpace[ c1 + id][[1]];
e2 = NullSpace[ c1 + id][[2]];
e3 = NullSpace[ c1 - id][[1]];
e4 = NullSpace[ c1 + I*id][[1]];
e5 = NullSpace[ c1 - I*id][[1]];
cj2 = Transpose[{e1, e3, e2, e4, e5}];
c2 = Simplify[Inverse[cj2].c1.cj2];
```

```
a2 = Simplify[Inverse[cj2].a1.cj2];
MatrixForm[c2]

(* Look at a in this basis, and notice that entries on the top row and
   columns 4-5 will vanish if 11+ 12k + 24k^2 does. *)

MatrixForm[Take[a2, {1, 1}, {4, 5}] // Expand // Simplify]

Solve[11 + 12 k + 24 k^2 == 0, k]

(* Plug in one of the roots of 11+12k+24k^2, and notice that, for k a root
   of that polynomial, a fixes the last two basis vectors. *)

a2a = Simplify[a2 /. k -> 1/12 (-3 + I Sqrt[57])];
MatrixForm[a2a]

(* Define the second representation. We will use a and c=a.b as our
   generators. *)

Clear[k];
a = {{-1, 1, 0, 0, 3}, {-1, -3, -3, 0, -3}, {1, 1, 1, 0, 0}, {-14 - 24 k -
   16 k^2, 5, 5, 1, -2}, {0, 1, 1, 0, 1}};
b = {{0, 1, 0, 0, 3}, {1, 0, -1, -1, 6 - 4 k}, {-1, 1, 0, -1 - 2 k, 2 + 2 k
   - 8 k^2}, {0, 0, 0, -8 - 4 k + 12 k^2 + 16 k^3, 19 - 18 k - 44 k^2 + 64
   k^4}, {0, 0, 0, -3 - 6 k - 4 k^2, 7 + 4 k - 12 k^2 - 16 k^3}};
c = a.b // Expand;

eigen = Eigensystem[c]
```

```
(* Put c into a form that is almost Rational Canonical Form; this will
   simplify computations. Conjugate a into the same basis.*)

e1 = {0, 0, 0, 1, 0};
e2 = c.e1;
e3 = c.e2;
e4 = c.e3;
e5 = 2*eigen[[2, 1]] // Expand;
cj1 = Transpose[{e1, e2, e3, e4, e5}] // Expand;
c1 = Simplify[Inverse[cj1].c.cj1];
a1 = Simplify[Inverse[cj1].a.cj1];
MatrixForm[c1]

(* Diagonalize c, and conjugate a into the same basis. *)

e1 = NullSpace[ c1 + id][[1]];
e2 = NullSpace[ c1 + id][[2]];
e3 = NullSpace[ c1 - id][[1]];
e4 = NullSpace[ c1 + I*id][[1]];
e5 = NullSpace[ c1 - I*id][[1]];
cj2 = Transpose[{e1, e3, e2, e4, e5}];
c2 = Simplify[Inverse[cj2].c1.cj2];
a2 = Simplify[Inverse[cj2].a1.cj2];
MatrixForm[c2]

(* Look at a in this basis, and notice that entries on the top row and
   columns 4-5 will vanish if 23+36k+24k^2 does. *)

MatrixForm[Take[a2, {1, 1}, {4, 5}] // Expand // Simplify]
```

```
Solve[23 + 36 k + 24 k^2 == 0, k]

(* Plug in one of the roots of 11+12k+24k^2, and notice that, for k a root
   of that polynomial, a fixes the last two basis vectors. *)

a2a = Simplify[a2 /. k -> 1/12 (-9 + I Sqrt[57])];
MatrixForm[a2a]
```

C Proof of Lemma 4.1

Lemma. *Suppose m is integer division of n by 2 (i.e., $n = 2m$ if n even, and $n = 2m+1$ if n odd.) Then $\sigma_G(n,k) \equiv m \pmod 2$ for all $n \geq 3$ and all $k \geq 2$.*

Proof. Let $n \geq 3$ and $k \geq 2$ be integers, and Q, R integers such that $n = Qk + R$. We begin with the definition used for Theorem 2.4

$$\sigma_G(n,k) := \frac{1}{2}\left((n+R)Q + R + k_{\mathcal{E}}\left(Q + Q_{\mathcal{E}}\right) - n_{\mathcal{E}}\left(2Q+1\right)\right)$$

and proceed by cases, considering in turn whether n, k, and Q are even or odd.

1. **n even**: $\sigma_G(n,k) = \frac{1}{2}\left[(n+R)Q + R + k_{\mathcal{E}}(Q + Q_{\mathcal{E}})\right]$.

   (a) **k even**: Since $n = Qk + R$, R is even, and $2\sigma_G(n,k) = nQ + RQ + R$.

      i. **Q even**: Then $n = Qk + R \equiv (2)(2) + R \equiv R \pmod 4$, and so

      $$\begin{aligned} 2\sigma_G(n,k) &= nQ + RQ + R \\ &\equiv nQ + nQ + n \\ &\equiv n(2Q+1) \\ &\equiv 2m(0+1) \pmod 4, \end{aligned}$$

      so $\sigma_G(n,k) \equiv m \pmod 2$.

      ii. **Q odd**:

      $$\begin{aligned} 2\sigma_G(n,k) &= nQ + RQ + R \\ &= 2mQ + R(Q+1) \\ &\equiv 2mQ + 0 \pmod 4, \end{aligned}$$

      so $\sigma_G(n,k) \equiv m \pmod 2$.

   (b) **k odd**: $2\sigma_G(n,k) = (n+R)Q + R + Q + Q_{\mathcal{E}}$.

i. **Q even**: Since $n = Qk + R$, R is even, and so is $n + R$. Thus,

$$\begin{aligned} 2\sigma_G(n,k) &= (n+R)Q + R + Q \\ &\equiv 0 + R + Q \pmod 4, \end{aligned}$$

and $R + Q \equiv 0 \pmod 4$ if and only if $n \equiv Qk - Q \equiv Q(k-1) \equiv 0 \pmod 4$ (i.e., m is even.) Otherwise, m is odd and $R + Q \equiv 2 \pmod 4$, and so $\sigma_G(n,k) \equiv m \pmod 2$.

ii. **Q odd**: Since $n = Qk + R$, R is odd, and since $Q + 1$ and $R + 1$ are even,

$$\begin{aligned} 2\sigma_G(n,k) &= (n+R)Q + R + Q + 1 \\ &= nQ + (R+1)(Q+1) \\ &\equiv 2mQ + 0 \pmod 4. \end{aligned}$$

2. **n odd**: $\sigma_G(n,k) = \frac{1}{2}\left[(n+R)Q + R + k_{\mathcal{E}}(Q + Q_{\mathcal{E}}) - 2Q - 1\right]$.

(a) **k even**: Since $n = Qk + R$, R is odd, and $2\sigma_G(n,k) = (n+R)Q + R - 2Q - 1$.

i. **Q even**: Then since $n + R$ and $R + 1$ are also even,

$$\begin{aligned} 2\sigma_G(n,k) &= nQ + RQ + R - 2Q - 1 \\ &= (n+R)Q - 2Q + R - 1 \\ &\equiv 0 - 0 + R - 1 \pmod 4, \end{aligned}$$

and since $2m + 1 = Qk + R \equiv 0 + R \pmod 4$, $R - 1 \equiv 2m \pmod 4$ and so $\sigma_G(n,k) \equiv m \pmod 2$.

ii. **Q odd**: Since $R - 1$ and $Q + 1$ are even,

$$
\begin{aligned}
2\sigma_G(n,k) &= nQ + RQ + R - 2Q - 1 \\
&= nQ - Q + RQ + R - Q - 1 \\
&= (n-1)Q + (R-1)(Q+1) \\
&= 2mQ + (R-1)(Q+1) \\
&\equiv 2mQ + 0 \pmod 4,
\end{aligned}
$$

so $\sigma_G(n,k) \equiv m \pmod 2$.

(b) **k odd**: $2\sigma_G(n,k) = (n+R)Q+R+Q+Q_{\mathcal{E}}-2Q-1 = nQ+QR+R+Q_{\mathcal{E}}-Q-1$.

i. **Q even**: Since $n = Qk + R$, R is odd. $n + R$ and $R - 1$ are even, and so

$$
\begin{aligned}
2\sigma_G(n,k) &= nQ + QR + R - Q - 1 \\
&= (n+R)Q + R - Q - 1 \\
&\equiv 0 + R - Q - 1 \pmod 4.
\end{aligned}
$$

$R - Q - 1$ is even, and $R - Q - 1 \equiv 0 \pmod 4$ if and only if

$$
2m + 1 = n = Qk + R \equiv Qk + Q + 1 \equiv Q(k+1) + 1 \equiv 1 \pmod 4,
$$

meaning that m is even. Otherwise, m is odd, and $2\sigma_G(n,k) \equiv 2 \pmod 4$, so $\sigma_G(n,k) \equiv m \pmod 2$.

ii. **Q odd**: Since $n = Qk + R$ is odd, R is even. Since $Q + 1$ is also even,

$$
\begin{aligned}
2\sigma_G(n,k) &= nQ + QR + R + 1 - Q - 1 \\
&= (n-1)Q + R(Q+1) \\
&\equiv 2mQ + 0 \pmod 4,
\end{aligned}
$$

and the result follows.

□

Vita

Elise Weir was born in the San Jose area of northern California to Anne and Gary Weir. She has one younger sister, Heather, whose career has included work with nonprofit organizations serving immigrant populations, ESL instruction, and infant photography. The Weir family moved to the Memphis area of Tennessee while she was in late elementary school, and after graduating from Collierville High School, she enrolled as a Chemical Engineering major at the University of Tennessee, Knoxville. During her second year, she took Dr. Grozdena Todorova's Introduction to Abstract Mathematics course, where she fell in love with mathematical reasoning and changed her major to Mathematics. She graduated from UTK summa cum laude with a B.S. in Honors Mathematics, and began her graduate career at the University of North Carolina, Chapel Hill. After taking some time away from academics to work in healthcare software, she returned to UTK to complete her Ph.D. in Mathematics under the direction of Dr. Morwen Thistlethwaite. After graduation, she will spend another year at the University of Tennessee as a Postdoctoral Teaching Associate and continue her work toward a career in academia.