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On a Multiple Stochastic Integral with Respect to a Strictly Semistable Random Measure

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To the Graduate Council:

I am submitting herewith a dissertation written by P. Xavier Raja Retnam entitled "On a Multiple Stochastic Integral with Respect to a Strictly Semistable Random Measure." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Balram S. Rajput, Major Professor

We have read this dissertation and recommend its acceptance:

Kenneth R. Stephenson, William R. Wade, Robert A. McLean

Accepted for the Council:
Dixie L. Thompson

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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ON A MULTIPLE STOCHASTIC INTEGRAL WITH RESPECT TO A STRICTLY SEMISTABLE RANDOM MEASURE

A Dissertation
Presented for the
Doctor of Philosophy
Degree
The University of Tennessee, Knoxville

P. Xavier Raja Retnam
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DEDICATION

This dissertation is dedicated to my uncle, Moni, whose timely help enabled me to acquire a higher education.

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ABSTRACT

The concept of multiple stochastic integration with respect to Brownian motion was introduced by Wiener (1938). Ito (1951) gave a more general construction of multiple stochastic integrals with regard to Brownian motion. Later the study of multiple stochastic integrals with respect to non-Gaussian processes were considered by some authors (e.g., Lin (1981), Surgailis (1981), Engel (1982)) . Multiple stochastic integrals have found their applications in areas such as statistics and quantum mechanics. Recently, several authors (e.g., Szulga and Woyczynski (1983), Krakowiak and Szulga (1985), Rosinski and Woyczynski (1986), and Surgailis (1985)), using different approaches, have constructed multiple stochastic integrals with respect to symmetric stable random measures. This dissertation is concerned with the development of the multiple stochastic integrals with respect to semistable random measures.

One of the above mentioned approaches used to construct the multiple stochastic integrals with respect to stable random measures is the Lebesgue-Dunford type construction. This approach reduces the problem of stochastic integration to the problem of integration with respect to a vector measure. Using this approach Krakowiak and Szulga (1985) developed multiple stochastic integrals of Banach valued functions with respect to symmetric and also nonsymmetric stable random measures. In this dissertation, using an approach similar to that of Krakowiak and Szulga (1985), we develop multiple stochastic integrals with respect to <u>all</u> symmetric as well as with respect to (nonsymmetric) strictly semistable random measures with index of stability

$\alpha \in (1, 2)$. Our methods, in the nonsymmetric case, yield results on multiple stochastic integrals relative to strictly stable random measure with index $\alpha \in (1, 2)$ considered in [10, 13] .

The most crucial role in the development of the integrals here is played by the inequalities (2.29). In these inequalities we establish a comparison theorem between the moments of the integrals of certain simple functions relative to the strictly semistable random measure and the corresponding moments of integrals of these functions relative to symmetric stable random measure. Once these inequalities are established, the methods of construction of the integrals here are similar to those used by Krakowiak and Szulga in [10, 13] to develop the integrals relative to symmetric stable random measure.

In Chapter I, we collect the notation, definitions, and known results that are basic to this dissertation. In Chapter II, we develop necessary tools and prove the crucial inequalities mentioned above. In the first part of Chapter II, we prove a comparison theorem for tail probabilities of nonsymmetric semistable random measures. This uses a distributional property of a strictly semistable random variable. In Chapter III, we define the multiple stochastic integrals of certain Banach valued Borel measurable functions with respect to a strictly semistable random measure of index α . Then, we show that the class of Banach valued integrable functions relative to a semistable random measure of index α coincides with the class of Banach valued integrable functions relative to a symmetric stable random measure of index α .

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CHAPTER I

PRELIMINARIES

1.1. Introduction

In this chapter, we state some definitions, notations, and known results that are basic to this dissertation. Throughout, $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$ will, respectively, represent the sets of all reals, rationals, and natural numbers. For any topological space X, $\mathcal{B}(X)$ will represent the σ-algebra of Borel subsets of X.

1.2. Random Measures

In this section, we state the definitions of a random measure and certain infinitely divisible random measures. We also state a result from Rosinski [23] which will be needed in the sequel; the material of this section is taken from [13, 19, 23].

1.2.1. Definitions. (i) Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $L_0(\mathbb{R})$ be the class of all real random variables defined on $(\Omega, \mathcal{F}, P)$. Let μ be a measure defined on $\mathcal{B}([0, 1])$, and let $\mathcal{R} = \{A \in \mathcal{B}([0, 1]) : \mu(A) < \infty\}$. A map $M: \mathcal{R} \to L_0(\mathbb{R})$ is called a random measure if, for every sequence $\{A_n\}_{n=1}^{\infty}$ of disjoint sets in $\mathcal{R}$, the random variables $M(A_n)$, $n = 1, 2, 3, \ldots$ are independent and

$$M\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} M(A_n) , \tag{1.1}$$

whenever $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$. The series in (1.1) is assumed to converge in probability (hence, also, because the summands are independent, almost surely).

(i) A random measure M is said to be *symmetric* if, for every $A \in \mathcal{R}$, the distribution of $M(A)$ is symmetric.

(ii) Let $\alpha \in (0, 1) \cup (1, 2)$. A random measure M is called a *strictly stable random measure* of index α (in short, a *strictly* $S(\alpha)$ *random measure*), if for every $A \in \mathcal{R}$, the characteristic (ch.) function $\hat{L}_{M(A)}(\cdot)$ of $M(A)$ is given by

$$\hat{L}_{M(A)}(t) = \exp\{-\mu(A)\, |t|^{\alpha} \,(1 - i\, \beta(A) \tan \tfrac{\pi\alpha}{2}\, \operatorname{sgn}(t))\}\ , \quad t \in \mathbb{R}\, , \qquad (1.2)$$

where $\beta\colon \mathcal{B}([0, 1]) \to [-1, 1]$ is a signed measure. $\beta(A)$ describes the asymmetry of the distribution of $M(A)$.

Throughout, $M_{\alpha,\beta}$ will denote such a random measure. The random measure $M_{\alpha,0}$ is symmetric, and it will be called a *standard* $S(\alpha)$ *random measure*.

1.2.2. Definitions. Let $0 < r < 1$. For $t \neq 0$, define

$$k_{\alpha}(t) = \begin{cases} |t|^{-\alpha} \displaystyle\sum_n r^{-n}\{1 - \cos(r^{\frac{n}{\alpha}} t) - i \sin(r^{\frac{n}{\alpha}} t)\} & \text{if } 0 < \alpha < 1\, , \\[2em] |t|^{-\alpha} \displaystyle\sum_n r^{-n}\{1 - \cos(r^{\frac{n}{\alpha}} t) + i(r^{\frac{n}{\alpha}} t - \sin(r^{\frac{n}{\alpha}} t))\} & \text{if } 1 < \alpha < 2\, , \end{cases}$$

and

$$\overline{k}_\alpha(t) = |t|^{-\alpha} \sum_n r^{-n}\{1 - \cos(r^{\frac{n}{\alpha}} t)\} \quad \text{if} \quad 0 < \alpha < 2 ,$$

where $\sum_n$ stands for $\sum_{n=-\infty}^{\infty}$.

For $r \in (0, 1)$ and $\alpha \in (0, 2)$, let J_n denote the set $\{t: r^{\frac{n+1}{\alpha}} < |t| \leq r^{\frac{n}{\alpha}}\}$, $n = 0, \pm 1, \pm 2, \ldots$.

Let $r \in (0, 1)$ and $\alpha \in (0, 1) \cup (1, 2)$. A random measure M is called a strictly r-semistable random measure of index α (in short, a strictly r-SS(α) random measure), if, for every $A \in \mathcal{R}$, the ch. function $\hat{L}_{M(A)}(\cdot)$ of the random variable $M(A)$ is given by

$$\hat{L}_{M(A)}(t) = \exp\{-\mu(A) \int_{J_0} |ts|^\alpha k_\alpha(ts)\, \Gamma(ds)\} , \quad t \in \mathbb{R} \tag{1.3}$$

where Γ is a finite measure on J_0 , and $J_0 = \{t: r^{\frac{1}{\alpha}} < |t| \leq 1\}$. Hereafter, M will always represent a strictly r-SS(α) random measure. Note that if Γ is symmetric in (1.3) and k_α is replaced by $\overline{k}_\alpha$, then the corresponding random measure is a symmetric r-SS(α) random measure. Hereafter M_0 will represent a symmetric r-SS(α) random measure. For the existence and properties of r-SS(α) random measures, see [19] .

The following theorem on the comparison of tails of distributions of $M_0(A)$ and $M_{\alpha,0}(A)$, for $A \in \mathcal{R}$, is from Rosinski [23, p. 100] and will be used in Chapter II.

1.2.3 Theorem [23, p. 100] . There exist positive constants C_1 and C_2 , which depend only on r, α, and Γ , such that

$$C_1 P(C_1 |M_{\alpha,0}(A)| > t) \leq P(|M_0(A)| > t) \leq C_2 P(C_2 |M_{\alpha,0}(A)| > t) \qquad (1.5)$$

for every $A \in R$ and $t > 0$.

1.3. Fourier Integral Theorem

In this section, we state a direct corollary of a theorem generally known as the Fourier Integral Theorem. This corollary will be used in Chapter 2. Details on this theorem and its proof can be found in Bochner's monograph [3] .

1.3.1 Proposition [3, p. 51]. Let $c_1, c_2 \in \mathbb{R}$, and let f_1, f_2 be monotonic functions on $[0, \infty)$. Let $f = c_1 f_1 + c_2 f_2$. Then

$$\frac{1}{2} f(0+) = \frac{1}{\pi} \int_0^\infty \int_0^\infty f(t) \cos \alpha t \, dt \, d\alpha \ , \qquad (1.6)$$

if one of the following two conditions holds:

(i) $\int_0^\infty |f_j(t)| \, dt < \infty$ for $j = 1, 2$.

(ii) $\lim_{t\to\infty} f_j(t) = 0$ for $j = 1, 2$, and there exists $N \in \mathbb{N}$ such that $\int_N^\infty \left|\frac{f_j(t)}{t}\right| dt < \infty$ for $j = 1, 2$.

We note here that under condition (ii) the integrals appearing on the right hand side of the formula (1.6) are improper Riemann integrals.

1.4. Borel Structure on the k-Dimensional Tetrahedron

We begin with the following notations: For $k \in \mathbb{N}$, let $\Delta_k = \{(t_1, \ldots, t_k) \in [0, 1]^k : 0 \leq t_1 < t_2 < \ldots < t_k \leq 1\}$, the k-dimensional tetrahedron; for $k, n \in \mathbb{N}$, let $\Lambda_k^n = \{(i_1, i_2, \ldots, i_k) \in \mathbb{N}^k : 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}$; for $k \in \mathbb{N}$, let $A_k = \{A_1 \times \ldots \times A_k \subset \Delta_k : A_1, A_2, \ldots, A_k \in I\}$, where I is the class of all finite disjoint unions of (all) subintervals of $[0, 1]$. Further, let C_k and $\overline{C}_k$ be, respectively, the ring and the algebra generated by A_k. The main facts about the ring C_k and the algebra $\overline{C}_k$ that are important to us are included in the following propositions. These are standard results and are stated, for instance, in [7, p. 31; 10, p. 10] without proof. We include short proofs of these here for completeness. To prove the first proposition we need the following lemma whose proof is deferred until the end of this section.

1.4.1. Lemma. If $A, B \in A_k$, then

(i) $A \backslash B$, and

(ii) $A \cup B$ are finite disjoint unions of elements of A_k.

1.4.2. Proposition. C_k is the class of all finite disjoint unions of elements of A_k.

Proof. Let C be the class of all finite disjoint unions of elements of A_k. Since C_k is a ring containing A_k, we have $C \subset C_k$. To prove that $C \supset C_k$, it is sufficient to show that C is a ring containing A_k. Clearly, $\phi \in C$, $A_k \in C$, and $A \cup B \in C$ whenever $A, B \in C$.

It remains only to be shown that $A\setminus B \in C$, if $A, B \in C$. Let $B \in A_k$ and $A = A_1 \cup \ldots \cup A_\ell \in C$, where $A_1, \ldots, A_\ell \in A_k$. Because $A_1, \ldots, A_\ell$, and B are elements of A_k, we see by Lemma 1.4.1 that $A_1\setminus B, \ldots, A_\ell\setminus B \in C$, and since $A\setminus B = (A_1 \cup \ldots \cup A_\ell)\setminus B = (A_1\setminus B) \cup \ldots \cup (A_\ell \setminus B)$, it follows that $A\setminus B \in C$. Now let $A \in C$ and $B = B_1 \cup \ldots \cup B_n$, where $B_1, \ldots, B_n$ are disjoint elements of A_k. Now we show that $A\setminus B \in C$, by induction on n. Since $A\setminus(B_1 \cup B_2) = (A\setminus B_1)\setminus B_2$, $A \in C$, and $B_1, B_2 \in A_k$, we have, by what we have shown above, that $A\setminus B \in C$. In a similar manner, $(A\setminus(B_1 \cup \ldots \cup B_{n-1}))\setminus B_n \in C$ if we assume that $A\setminus(B_1 \cup \ldots \cup B_{n-1}) \in C$ and $B_n \in A_k$. Since $A\setminus(B_1 \cup \ldots \cup B_n) = (A\setminus(B_1 \cup \ldots \cup B_{n-1}))\setminus B_n$, we see by induction that if $A, B \in C$ and if $B = B_1 \cup \ldots \cup B_n$ for some $B_1, \ldots, B_n \in A_k$, then $A\setminus B \in C$. Therefore, C is a ring containing A_k, and hence $C \supset C_k$. ■

1.4.3. Proposition [7, p. 31; 10, p. 11]. If $B \in C_k$, then there exist $n \in \mathbb{N}$, $\nu \subset \Lambda_k^n$, and subintervals $I_1, I_2, \ldots, I_n$ of $[0, 1]$ such that $I_1 < I_2 < \ldots < I_n$ and

$$B = \bigcup_{(s_1,\ldots,s_k)\in\nu} I_{s_1} \times \ldots \times I_{s_k},$$

where for any two subsets A and B of $[0, 1]$, we write $A < B$ if $x < y$ for all $x \in A$ and $y \in B$.

Proof. Let $B \in C_k$. Then B is a finite disjoint union of elements of A_k. Thus, since every element of A_k can be written as a finite disjoint union of sets of the form $A_1 \times \ldots \times A_k$, where

$A_1, \ldots, A_k$ are subintervals of $[0, 1]$, we can write

$$B = \bigcup_{j=1}^{\ell} B_{j1} \times \ldots \times B_{jk}$$

for some $\ell \in \mathbb{N}$, where for $j = 1, 2, \ldots, \ell$, the sets $B_{j1}, B_{j2}, \ldots, B_{jk}$ are disjoint subintervals of $[0, 1]$. Now we can find intervals $I_1, I_2, \ldots, I_n$ of $[0, 1]$ such that $I_1 < I_2 < \ldots < I_n$ and such that for $j = 1, 2, \ldots, \ell$, each set $B_{j1}, B_{j2}, \ldots, B_{jk}$ can be expressed as a finite (disjoint) union of $I_1, I_2, \ldots, I_n$. Hence,

$$B = \bigcup_{(s_1, s_2, \ldots, s_k) \in \nu} I_{s_1} \times I_{s_2} \times \ldots \times I_{s_k}$$

for some $\nu \subset \Lambda_k^n$. ■

<u>1.4.4. Proposition</u> [10, p. 13]. If $A \in \overline{C}_k$, then there exists an increasing sequence of sets $\{A_j\}_{j=1}^{\infty} \subset C_k$ such that $A = \bigcup_{j=1}^{\infty} A_j$.

<u>Proof.</u> Let $U = \{B \subset \Delta_k : B \cap A \in C_k$, for all $A \in C_k\}$. Using the fact that C_k is a ring, we see that U is an algebra containing C_k and hence $\overline{C}_k \subset U$. Now we show that there exists an increasing sequence $\{C_j\}_{j=1}^{\infty} \subset C_k$ such that $\Delta_k = \bigcup_{j=1}^{\infty} C_j$. For any $(t_1, t_2, \ldots, t_k) \in \Delta_k$, there exist rational numbers $s_1, s_2, \ldots, s_{k-1}$ such that $0 < t_1 \leq s_1 < t_2 \leq s_2 < t_3 \leq s_4 < \ldots \leq s_{k-1} < t_k < 1$. So,

$$\Delta_k = \bigcup_{(s_1, s_2, \ldots, s_{k-1}) \in \mathbb{Q}^{k-1} \cap \Delta_{k-1}} B_{(s_1, s_2, \ldots, s_{k-1})} ,$$

where $\mathbb{Q}^{k-1} = \underbrace{\mathbb{Q} \times \ldots \times \mathbb{Q}}_{(k-1)\text{ times}}$, and

$B_{(s_1, s_2, \ldots, s_{k-1})} = [0, s_1] \times (s_1, s_2] \times \ldots \times (s_{k-1}, 1]$ for every $(s_1, s_2, \ldots, s_{k-1}) \in \mathbb{Q}^{k-1} \cap \Delta_{k-1}$. Let $C_j = \bigcup_{\ell=1}^{j} B_{\psi(\ell)}$, where ψ is a bijection of $\mathbb{N}$ onto $\mathbb{Q}^{k-1} \cap \Delta_{k-1}$. Thus for the increasing sequence of sets $\{C_j\}_{j=1}^{\infty}$, we have $\Delta_k = \bigcup_{j=1}^{\infty} C_j$; also, for every j , the set $C_j \in \mathcal{C}_k$ since $B_{\psi(\ell)} \in \mathcal{C}_k$ for every ℓ .

To conclude the proof, let $A \in \overline{\mathcal{C}}_k$. Since $\overline{\mathcal{C}}_k \subset \mathcal{U}$, we have $A \in \mathcal{U}$ and hence $A \cap C_j \in \mathcal{C}_k$ for $j = 1, 2, \ldots$. Setting $A_j = A \cap C_j$, we have an increasing sequence of sets $\{A_j\}_{j=1}^{\infty} \subset \mathcal{C}_k$ with $A = A \cap \Delta_k = A \cap (\bigcup_{j=1}^{\infty} C_j) = \bigcup_{j=1}^{\infty} (A \cap C_j) = \bigcup_{j=1}^{\infty} A_j$. ■

Finally, we have the following proposition about the Borel σ-algebra on Δ_k .

<u>1.4.5. Proposition.</u> $\mathcal{B}(\Delta_k) = \sigma(\overline{\mathcal{C}}_k)$, where $\sigma(\overline{\mathcal{C}}_k)$ is the σ-algebra generated by $\overline{\mathcal{C}}_k$.

<u>Proof.</u> Since $\mathcal{C}_k \subset \mathcal{B}(\Delta_k)$, we have $\sigma(\overline{\mathcal{C}}_k) \subset \mathcal{B}(\Delta_k)$. Now we show that $\mathcal{B}(\Delta_k) \subset \sigma(\overline{\mathcal{C}}_k)$. We note that $\mathcal{B}(\Delta_k) = \mathcal{B}([0, 1]^k) \cap \Delta_k = \sigma(\mathcal{A}) \cap \Delta_k = \sigma(\mathcal{A} \cap \Delta_k)$ (see Ash [15, p. 5]) , where

$\mathcal{A} = \{I_1 \times I_2 \times \ldots \times I_k : I_1, I_2, \ldots, I_k$ are subintervals of $[0, 1]\}$.

As mentioned in the proof of Proposition 1.4.5, we have

$\Delta_k = \bigcup_{\ell=1}^{\infty} B_{\psi(\ell)} = \bigcup_{\ell=1}^{\infty} E_1^{(\ell)} \times \ldots \times E_k^{(\ell)}$, where $E_1^{(\ell)}, \ldots, E_k^{(\ell)}$ are subintervals of $[0, 1]$ such that $E_1^{(\ell)} < \ldots < E_k^{(\ell)}$ for $\ell = 1, 2, \ldots$. Thus

$$(I_1 \times \ldots \times I_k) \cap \Delta_k = \bigcup_{\ell=1}^{\infty} (E_1^{(\ell)} \cap I_1) \times \ldots \times (E_k^{(\ell)} \cap I_k) . \qquad (1.7)$$

Since $E_1^{(\ell)} \cap I_1$, $E_2^{(\ell)} \cap I_2, \ldots, E_k^{(\ell)} \cap I_k$ are subintervals of $[0, 1]$

and $E_1^{(\ell)} \cap I_1 < \dots < E_k^{(\ell)} \cap I_k$, we have that $(E_1^{(\ell)} \cap I_1) \times \dots \times (E_k^{(\ell)} \cap I_k) \in C_k$ for $\ell = 1, 2, \dots$. Hence, by (1.9), the set $(I_1 \times \dots \times I_k) \cap \Delta_k \in \sigma(\overline{C}_k)$. Therefore $A \cap \Delta_k \subset \sigma(\overline{C}_k)$, and it follows that $\mathcal{B}(\Delta_k) = \sigma(A \cap \Delta_k) \subset \sigma(\overline{C}_k)$. ■

Proof of Lemma 1.4.1. (i) Let $A = A_1 \times \dots \times A_k \in A_k$, and let $B = B_1 \times \dots \times B_k \in A_k$. By induction on k, we can show that

$$(A_1 \times \dots \times A_k) \setminus (B_1 \times \dots \times B_k)$$

$$= \bigcup_{j=1}^{k} (A_1 \cap B_1 \times \dots \times A_{j-1} \cap B_{j-1} \times A_j \cap B_j^c \times A_{j+1} \times \dots \times A_k) . \qquad (1.8)$$

For $i \neq j$, we have $(A_1 \cap B_1 \times \dots \times A_i \cap B_i \times A_i \cap B_i^c \times A_{j+1} \times \dots \times A_k) \cap (A \cap B_1 \times \dots \times A_j \cap B_j \times A_j \cap B_j^c \times A_{j+1} \times \dots \times A_k) = \emptyset$. Also, for $j = 1, 2, \dots, k$, the sets $A_j \cap B_j^c$ and $A_j \cap B_j \in \mathcal{I}$, since $A_j, B_j \in \mathcal{I}$. Thus, the right-hand side of (1.8) is a finite disjoint union of elements of A_k.

(ii) Let $A, B \in A_k$. Since, by (i), $A \setminus B$ is a finite disjoint union of elements of A_k and since $A \cup B = (A \setminus B) \cup B$, we have that $A \cup B$ is a finite disjoint union of elements of A_k. ■

1.5. Caratheodory-Hahn-Kluvanek Extension Theorem

In this section, we introduce vector measures and state a part of the Caratheodory-Hahn-Kluvanek extension theorem. This material is adopted from the book Vector Measures [5] by Diestel, and Uhl. Jr.

Recall that an F-space is a complete topological vector space whose topology is induced by an invariant metric. Throughout this section

A will denote an algebra of subsets of a set S, and $\sigma(A)$ will denote the σ-algebra generated by A.

1.5.1. Definitions. Let X be an F-space. A function $m: A \to X$ is called a finitely additive vector measure, or simply a vector measure, if $m(A_1 \cup A_2) = m(A_1) + m(A_2)$ for any two disjoint sets $A_1, A_2 \in A$.

A vector measure m is said to be countably additive, if in the topology of X, $m(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n)$ for every sequence $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint elements of A such that $\bigcup_{n=1}^{\infty} A_n \in A$.

Let λ be a finite, non-negative, countably additive measure on A. A vector measure m is said to be λ-continuous if $\lim_{\lambda(A)\to 0} m(A) = 0$.

The extension of a finitely additive vector measure on A to a countably additive vector measure on $\sigma(A)$, for Banach valued vector measures, is given by a part of the Caratheodory-Hahn-Kluvanek extension theorem [5, p.27]. The same proof can be adopted for the extension of F-space valued vector measures.

1.5.2. Theorem [5, p. 27]. Let X be an F-space, and let $m: A \to X$ be a λ-continuous vector measure. Then there exists a unique extension $\overline{m}$ of m to $\sigma(A)$ such that $\overline{m}$ is a λ-continuous countably additive vector measure on $\sigma(A)$.

Finally, we close this section with a definition.

1.5.3 Definition. Let X be an F-space with an invariant metric d, and let $m: A \to X$ be a vector measure. For each $x \in X$, let $||x||$ denote the distance $d(x, 0)$. We call the extended nonnegative function $||m||: A \to [0, \infty]$ defined by

$$\|m\|(A) = \sup \|\sum_{A_j \in \Pi} s_j\, m(A_j)\|$$

for every $A \in A$, the semivariation of m , where the supremum is taken over all partitions Π of A into finitely many disjoint elements of A and over all finite sequences (S_j) such that $|s_j| \leq 1$ for all j .

The vector measure m is said to be of bounded semivariation if $\|m\|(S) < \infty$.

1.6. Random Multilinear Forms

In this section, we present the definition of random multilinear forms, some notations, and the 'multilinear contraction principle' which is obeyed by certain Banach spaces and is related to the topic of random multilinear forms. We adopt this material from [12, 13] which contain more information on random multilinear forms.

1.6.1 Notations and Definitions. (i) For a Banach space X , let $L_p(X)$ denote the set of all X-valued random variables ξ such that $\|\xi\|_p < \infty$, where

$$\|\xi\|_p = \begin{cases} (E\,\|\xi\|^p)^{\frac{1}{p}} & \text{if } 0 < p < \infty , \\ E(\frac{\|\xi\|}{1 + \|\xi\|}) & \text{if } p = 0 . \end{cases}$$

(ii) Let $F_{k,X}$ denote the set of all maps $F\colon \mathbb{N}^k \to X$ such that F is zero for all but finitely many elements of $\mathbb{N}^k$, and $F((i_1,\ldots, i_k)) = 0$ whenever $i_j = i_\ell$ for some j and ℓ such that

$1 \leq j, \ell \leq k$. A map $F \in \mathcal{F}_{k,X}$ is called *tetrahedronal* if $F((i_1,\ldots, i_k)) = 0$ whenever $i_j > i_\ell$ for some j and ℓ such that $1 \leq j < \ell \leq k$; a map $F \in \mathcal{F}_{k,X}$ is called *symmetric* if $F((i_1,\ldots, i_k)) = F((i_{\pi(1)},\ldots, i_{\pi(k)}))$ for all permutations π of $\{1, 2,\ldots, k\}$. Let $\mathcal{F}^{\tau}_{k,X}$ and $\mathcal{F}^{s}_{k,X}$, respectively, denote the set of all tetrahedronal $F \in \mathcal{F}_{k,X}$ and the set of all symmetric $F \in \mathcal{F}_{k,X}$. Let $\mathbb{R}^{\mathbb{N}}$ and $L_0(\mathbb{R})^{\mathbb{N}}$, respectively, denote the set of all sequences of real numbers and the set of all sequences of real random variables. For each $F \in \mathcal{F}_{k,X}$, let $\Psi_F\colon \underbrace{\mathbb{R}^{\mathbb{N}} \times \ldots \times \mathbb{R}^{\mathbb{N}}}_{k \text{ times}} \to X$ be the map defined by

$$\Psi_F((\underline{t}^{(1)},\ldots, \underline{t}^{(k)})) = \sum_{(i_1,\ldots,i_k)\in\mathbb{N}^k} F((i_1,\ldots, i_k))t^{(1)}_{i_1}\ldots t^{(k)}_{i_k}$$

for all $(\underline{t}^{(1)},\ldots, \underline{t}^{(k)}) \in \underbrace{\mathbb{R}^{\mathbb{N}} \times \ldots \times \mathbb{R}^{\mathbb{N}}}_{k \text{ times}}$, where $\underline{t}^{(j)} = (t^{(j)}_1, t^{(j)}_2,\ldots) \in \mathbb{R}^{\mathbb{N}}$ for $j = 1, 2,\ldots, k$; let $\Phi_F\colon \underbrace{L_0(\mathbb{R})^{\mathbb{N}} \times \ldots \times L_0(\mathbb{R})^{\mathbb{N}}}_{k \text{ times}} \to X$ be the map defined by

$$\Phi_F((\underline{\xi}^{(1)},\ldots, \underline{\xi}^{(k)})) = \sum_{(i_1,\ldots,i_k)\in\mathbb{N}^k} F((i_1,\ldots,i_k))\xi^{(1)}_{i_1}\ldots\xi^{(k)}_{i_k}$$

for all $(\underline{\xi}^{(1)},\ldots, \underline{\xi}^{(k)}) \in \underbrace{L_0(\mathbb{R})^{\mathbb{N}} \times \ldots \times L_0(\mathbb{R})^{\mathbb{N}}}_{k \text{ times}}$, where $\underline{\xi}^{(j)} = (\xi^{(j)}_1, \xi^{(j)}_2,\ldots)$ for $j = 1,2,\ldots, k$. For each $F \in \mathcal{F}_{k,X}$, the map Ψ_F (respectively, Φ_F) is called a k-linear form (respectively, a random k-linear form). Let $\langle F; \underline{t}^{(1)},\ldots, \underline{t}^{(k)}\rangle$ (respectively, $\langle F; \underline{\xi}^{(1)},\ldots, \underline{\xi}^{(k)}\rangle$) denote $\Psi_F((\underline{t}^{(1)},\ldots, \underline{t}^{(k)}))$ (respectively,

$\Phi_F((\underline{\xi}^{(1)},\dots,\underline{\xi}^{(k)})))$, and let $\langle F; (\underline{t})^k\rangle$ (respectively, $\langle F; (\underline{\xi})^k\rangle$) denote $\langle F; \underbrace{\underline{t},\dots,\underline{t}}_{k \text{ times}}\rangle$ (respectively, $\langle F; \underbrace{\underline{\xi},\dots,\underline{\xi}}_{k \text{ times}}\rangle$) .

1.6.2. Remark. It follows from the definition of $F^{\tau}_{k,X}$ that, if $F \in F^{\tau}_{k,X}$ then there exists an $n \in \mathbb{N}$ such that $\langle F; \underline{\xi}^{(1)},\dots,\underline{\xi}^{(k)}\rangle$

$= \sum_{1\le i_1<\dots<i_k\le n} F((i_1,\dots,i_k))\, \xi^{(1)}_{i_1} \dots \xi^{(k)}_{i_k}$ for all

$(\underline{\xi}^{(1)},\dots,\underline{\xi}^{(k)}) \in \underbrace{L_0(\mathbb{R})^{\mathbb{N}} \times \dots \times L_0(\mathbb{R})^{\mathbb{N}}}_{k \text{ times}}$.

1.6.3. Definition [13]. A Banach space X is said to satisfy the multilinear contraction principle (in short, M.C.P.) if there exists a $p \in (0, \infty)$ and a constant $C > 0$ (depending only on p) such that, for all $n \in \mathbb{N}$, for all finite subsets $\{x_{ij}: i,j = 1,2,\dots, n\}$ of X , and for all $\{s_{ij}: i,j = 1,2,\dots, n\} \subset \{-1, 1\}$, the inequality

$$\|\sum_{i,j=1}^{n} x_{ij}\, s_{ij}\, \varepsilon^{(1)}_i\, \varepsilon^{(2)}_j\|_p \le C\|\sum_{i,j=1}^{n} x_{ij}\, \varepsilon^{(1)}_i\, \varepsilon^{(2)}_j\|_p$$

holds, where $(\varepsilon^{(j)}_1, \varepsilon^{(j)}_2,\dots)$ for $j = 1, 2$ are independent copies of the sequence of independent identically distributed Rademacher random variables. We recall here that a random variable ε with $P(\varepsilon = 1) = P(\varepsilon = -1) = \frac{1}{2}$ is called a Rademacher random variable.

Pisier has shown that every Banach lattice satisfies the M.C.P.. Thus, in particular, $\mathbb{R}$ satisfies the M.C.P..

1.7. Marcinkiewicz-Paley-Zygmund Condition

It follows easily that if $\{\xi_n\}_{n=1}^{\infty} \subset L_p(X)$ converges in the pth norm, then it converges in the qth norm for any $0 \leq q \leq p$. A condition is stated in this section, under which the convergence of any sequence $\{\xi_n\}_{n=1}^{\infty} \subset L_p(X)$ in all the $L_q(X)$ norms are equivalent for $0 \leq q \leq p$. This condition, originated from the papers of Paley-Zygmund and Marcinkiewicz-Zygmund, was formulated by Krakowiak and Szulga [12]. The following definition and the two propositions are adopted from [13].

1.7.1. Definition. A family $C \subset L_p(X)$ is said to satisfy the Marcinkiewicz-Paley-Zugmund condition with exponent $0 < p < \infty$, if there exists $\delta > 0$ such that

$$P\{\|\xi\| > \delta \|\xi\|_p\} > \delta \quad \text{for all} \quad \xi \in C .$$

If $C \subset L_p(X)$ satisfies the above condition, then it is written as $C \in MPZ(p)$.

The following proposition is very useful.

1.7.2. Proposition [10, 12]. Let $C \subset L_p(X)$. Then

(i) The following three conditions are equivalent.

(a) $C \in MPZ(p)$.

(b) For any $q \in (0, p)$, $\sup_{\xi \in C} \frac{\|\xi\|_p}{\|\xi\|_q} < \infty$.

(c) There exists a $q \in (0, p)$ such that $\sup_{\xi \in C} \frac{\|\xi\|_p}{\|\xi\|_q} < \infty$.

(ii) If $C \in MPZ(p)$ then $C^0 \in MPC(p)$ where C^0 is the $L_0(X)$-closure of C. Moreover, for all $q \in [0, p]$, the topologies induced by all the $L_q(X)$ norms are equivalent.

1.7.3. Proposition [13, p. 769]. Let $\underline{\theta} = (\theta_1, \theta_2, \dots)$ be a sequence of independent identically distributed symmetric α-stable random variables (i.e. the ch. function $\hat{L}_{\theta_1}(\cdot)$ of θ_1 is given by

$$\hat{L}_{\theta_1}(t) = \exp\{-c|t|^{\alpha}\} , \quad t \in \mathbb{R} ,$$

where c is some real number). Then the class $\{\langle F; (\underline{\theta})^k \rangle : F \in F^{\tau}_{k,X}\} \in MPZ(p)$ for every $0 < p < \alpha$.

CHAPTER II

COMPARISON THEOREMS

2.1. Introduction

In this chapter, we develop the results needed to compare the multiple stochastic integrals with respect to a strictly r-SS(α) random measure M and the standard $S(\alpha)$ random measure $M_{\alpha,0}$, when $\alpha \in (0, 1) \cup (1, 2)$. Recall that Theorem 1.2.3 compares the tail probabilities of $M_0(A)$ and $M_{\alpha,0}(A)$, uniformly over $A \in \mathcal{R}$. First we extend this result for an arbitrary (not necessarily symmetric) strictly r-SS(α) random measure M when $1 < \alpha < 2$, and for a strictly r-SS(α) random measure M when $0 < \alpha < 1$ under the additional condition that the distribution of $M(A)$ is not one-sided for at least one $A \in \mathcal{R}$. Then we define multiple stochastic integrals with respect to M and $M_{\alpha,0}$ on the space of all (Banach valued) C_k-measurable simple functions. Finally, we use a result of Kwapień [14] and establish a theorem that compares the moments of the multiple integral relative to M with the corresponding moments of the multiple integral relative to $M_{\alpha,0}$.

2.2. Comparison of the tail probabilities of $M(A)$ and $M_{\alpha,0}(A)$

The following theorem yields the comparison between the tail probabilities of $M(A)$ and $M_{\alpha,0}(A)$.

2.2.1. Theorem. Let M be a strictly r-SS(α) random measure given by (1.3) and let $M_{\alpha,\beta}$ be a strictly stable random measure given by (1.2).

(i) If $1 < \alpha < 2$, or if $0 < \alpha < 1$ and the distribution of $M(A)$ is not one-sided for some $A \in R$, then there exist positive constants C_1, C_2, and C_3 which depend only on r, α, and Γ, and do not depend on A, such that

$$C_1 P(C_1 |M_{\alpha,0}(A)| > t) \leq P(|M(A)| > t) \leq C_2 P(C_3 |M_{\alpha,0}(A)| > t) \qquad (2.1)$$

for all $t > 0$, and for all $A \in R$.

(ii) If $1 < \alpha < 2$, then

$$\frac{1}{2} P(2^{\frac{1-\alpha}{\alpha}} |M_{\alpha,0}(A)| > t) \leq P(|M_{\alpha,\beta}(A)| > t)$$

$$\leq (\frac{\alpha}{\alpha - 1}) P(2^{\frac{1}{\alpha}} |M_{\alpha,0}(A)| > t) \qquad (2.2)$$

for all $t > 0$, and for all $A \in R$.

In order to establish the above theorem, we need a preliminary result (Proposition 2.2.3) concerning a distributional property of $M(A)$. The proof of this proposition uses a formula that is proved first in the following lemma. This lemma is a direct consequence of an inversion formula noted without proof by Pitman [18, p. 394]. We supply a proof of this formula in the case of strictly $S(\alpha)$ and strictly r-SS(α) random variables.

<u>2.2.2. Lemma.</u> For $u > 0$, let ξ_u be a random variable whose ch. function $\hat{L}_{\xi_u}(\cdot)$ is given by either

$$\hat{L}_{\xi_u}(t) = \exp\{-u|t|^{\alpha} \int_{J_0} |s|^{\alpha} k_{\alpha}(ts)\, \Gamma(ds)\}, \quad t \in \mathbb{R}, \qquad (2.3)$$

where Γ, J_0, and k_α are as given in (1.3), or

$$\hat{L}_{\xi_u}(t) = \exp\{-u|t|^\alpha (1 - i\ \beta_u \tan\frac{\pi\alpha}{2} \cdot \operatorname{sgn}(t))\}\ ,\ t \in \mathbb{R}\ , \tag{2.4}$$

where $|\beta_u| \leq 1$. Then

$$1 - 2\ F_{\xi_u}(0) = \frac{2}{\pi}\int_0^\infty \frac{1}{t}\ Im(\hat{L}_{\xi_u}(t))\ dt\ , \tag{2.5}$$

where F_{ξ_u} is the distribution function of ξ_u , $Im(\hat{L}_{\xi_u}(t))$ is the imaginary part of $\hat{L}_{\xi_u}(t)$, and the integral in (2.5) is a Lebesgue integral.

Proof. Pitman [18, p. 394] has shown that

$$\frac{1}{t}\ Im(\hat{L}_{\xi_u}(t)) = \int_0^\infty K_u(x) \cos tx\ dx\ , \tag{2.6}$$

for every $t > 0$, where $K_u(x) = 1 - F_{\xi_u}(x) - F_{\xi_u}(-x)$ for $x \geq 0$. We note that the integral in (2.6) is an improper Riemann integral. Now we show that K_u satisfies the hypotheses of Proposition 1.3.1. We observe that for $x \in [0, \infty)$ we have $K_u(x) = f_1(x) - f_2(x)$, where $f_1(x) = 1 - F_u(x)$ and $f_2(x) = F_u(-x)$. For $j = 1, 2$, we have

$$\int_1^\infty |\frac{f_j(x)}{x}|\ dx \leq \int_1^\infty \frac{P(|\xi_u| > x)}{x}\ dx$$

$$\leq (E|\xi_u|^{\frac{\alpha}{2}}) \int_1^\infty x^{-\frac{\alpha}{2} - 1}\ dx < \infty\ , \tag{2.7}$$

where (2.7) holds by Chebychev's inequality and the fact that $E|\xi_u|^{\frac{\alpha}{2}} < \infty$ (see [19]) . Thus, since f_j is monotonic on $[0, \infty)$ and $\lim_{x\to\infty} f_j(x) = 0$ for $j = 1, 2$, it now follows that the function K_u satisfies the hypotheses of Proposition 1.3.1. Therefore, by Proposition 1.3.1, we have

$$\frac{1}{2} K_u(0+) = \frac{1}{\pi}\int_0^\infty \left(\int_0^\infty K_u(x) \cos tx\, dx\right)dt \ , \tag{2.8}$$

where the integrals in (2.8) are improper Riemann integrals. Since $\hat{L}_{\xi_u}(t)$ is absolutely integrable over $\mathbb{R}$, the distribution function F_{ξ_u} is absolutely continuous. Hence the function K_u is continuous at zero, and

$$K_u(0) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty K_u(x) \cos tx\, dx\right) dt \ ; \tag{2.9}$$

hence, by (2.6) and (2.9),

$$1 - 2\, F_{\xi_u}(0) = K_u(0) = \frac{2}{\pi}\int_0^\infty \left(\frac{1}{t}\, \mathit{Im}(\hat{L}_{\xi_u}(t))\right) dt \ .$$

Finally, we show that the integral in (2.5) is a Lebesgue integral. We know from [19, p. 142] that $C_0 = \inf_{t\neq 0} \mathrm{Re}\, k_\alpha(t) > 0$, $C_1 = \sup_{t\neq 0} |\mathit{Im}\, k_\alpha(t)| < \infty$, and $\int_{J_0} |s|^\alpha\, \Gamma(ds) < \infty$. Thus, if $\hat{L}_{\xi_u}(\cdot)$ is given by (2.3), then recalling the inequality $|\sin x| \leq x$ for $x \geq 0$, we have

$$\int_0^\infty \left|\frac{1}{t}\, \mathcal{I}m(\hat{\mathcal{L}}_{\xi_u}(t))\right| dt \le \int_0^\infty C_1' \, e^{-ut^\alpha C_0'} \, ut^{\alpha-1} \, dt \tag{2.10}$$

$$= \frac{C_1'}{C_0' \, \alpha} < \infty \quad ,$$

where $C_0' = C_0(\int_{J_0} |s|^\alpha \, \Gamma(ds))$ and $C_1' = C_1(\int_{J_0} |s|^\alpha \, \Gamma(ds))$. If ξ_u is given by (2.4), then using again the inequality $|\sin x| \le x$ for $x \ge 0$, we have that

$$\int_0^\infty \left|\frac{1}{t}\, \mathcal{I}m(\hat{\mathcal{L}}_{\xi_u}(t))\right| dt$$

$$= \int_0^\infty \left|\frac{1}{t}\, e^{-ut^\alpha} \sin(u\beta_u \, t^\alpha \tan \tfrac{\pi\alpha}{2})\right| dt$$

$$\le \int_0^\infty e^{-ut^\alpha} \, u \cdot \tan \tfrac{\pi\alpha}{2} \cdot t^{\alpha-1} \, dt < \infty$$

for all $u > 0$ because $\alpha \in (0, 1) \cup (1, 2)$. ■

2.2.3. Proposition. (i) If $\alpha \in (0, 1)$ and the distribution of $M(A)$ is not one-sided for some $A \in \mathcal{R}$, or if $\alpha \in (1, 2)$, then there exist constants c_1 and c_2 , depending only on r, α, and Γ and not on A , such that

$$0 < c_1 \le F_{M(A)}(0) \le c_2 < 1 \tag{2.11}$$

for all $A \in \mathcal{R}$ with $\mu(A) \neq 0$. Here $F_{M(A)}$ is the distribution function of the random variable $M(A)$.

(ii) Let $\alpha \in (0, 1) \cup (1, 2)$. If ξ is a random variable with the ch. function $\hat{L}_\xi(\cdot)$ given by

$$\hat{L}_\xi(t) = \exp\{-|t|^\alpha(1 - i\beta_0 \cdot \tan \frac{\pi\alpha}{2} \cdot \mathrm{sgn}(t))\} ,$$

where $|\beta_0| \leq 1$, then

$$F_\xi(0) = \frac{1}{2} - \frac{1}{\pi\alpha} \tan^{-1}(\beta_0 \tan \frac{\pi\alpha}{2}) . \tag{2.12}$$

If $0 < \alpha < 1$, then $M_{\alpha,1}(A)$ and $M_{\alpha,-1}(A)$ are one-sided for all $A \in R$. If $0 < \alpha < 1$ and if $\beta(A) = \beta_0$ for all $A \in R$ with $|\beta_0| < 1$, then there exists a constant $c \in (0, 1)$ which depends only on α and β_0 such that

$$F_{M_{\alpha,\beta_0}(A)}(0) = \frac{1}{2} - \frac{1}{\pi\alpha} \tan^{-1}(\beta_0 \tan \frac{\pi\alpha}{2}) = c \tag{2.13}$$

for all $A \in R$ with $\mu(A) \neq 0$. If $1 < \alpha < 2$, and β is arbitrary as in (1.2), then

$$0 < 1 - \frac{1}{\alpha} \leq F_{M_{\alpha,\beta}(A)}(0) \leq \frac{1}{\alpha} < 1 \tag{2.14}$$

for all $A \in R$ with $\mu(A) \neq 0$.

Proof of (i). Let ξ_u be given by (2.3) . We define $g\colon (0, \infty) \to [0, 1]$ by $g(u) = F_{\xi_u}(0)$ for every $u > 0$. We first show that $\inf_{u>0} g(u) = g(u_0)$ and $\sup_{u>0} g(u) = g(u_1)$ for some

$u_0, u_1 \in [r, 1]$. For this, we first observe that for every $u \in (0, \infty)$, there exists an integer ℓ (which depends on u) such that $ur^{\ell} \in [r, 1]$ and $g(u) = g(ur^{\ell})$. In fact, by the substitution $\omega = r^{\frac{\ell}{\alpha}} t$, we have

$$\frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} e^{-ur^{\ell} t^{\alpha}\int_{J_0} |s|^{\alpha}\, Re\, k_{\alpha}(ts)\Gamma(ds)}$$

$$\cdot \sin(-ur^{\ell} t^{\alpha} \int_{J_0} |s|^{\alpha}\, Im\, k_{\alpha}(ts)\, \Gamma(ds))\, \frac{1}{t}\, dt$$

$$= \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} e^{-u\omega^{\alpha}\int_{J_0} |s|^{\alpha}\, Re\, k_{\alpha}(\omega s)\, \Gamma(ds)}$$

$$\cdot \sin(-u\omega^{\alpha}\int_{J_0} |s|^{\alpha}\, Im\, k_{\alpha}(\omega s)\, \Gamma(ds))\frac{1}{\omega}\, d\omega\ , \qquad (2.15)$$

since

$$ur^{\ell}\, t^{\alpha} \int_{J_0} |s|^{\alpha}\, Re\, k_{\alpha}(ts)\, \Gamma(ds)$$

$$\equiv u\int_{J_0} \sum_n r^{-(n-\ell)}\, \{1 - \cos\, (r^{\frac{n}{\alpha}}\, t|s|)\}\, \Gamma(ds)$$

$$= u \int_{J_0} \sum_n r^{-(n-\ell)}\, \{1 - \cos(r^{\frac{n-\ell}{\alpha}}\, \omega|s|)\}\, \Gamma(ds)$$

$$= u\omega^{\alpha} \int_{J_0} |s|^{\alpha}\, Re\, k_{\alpha}(\omega s)\, \Gamma(ds)\ ,$$

and

$$ur^{\ell}\, t^{\alpha} \int_{J_0} |s|^{\alpha}\, Im\; k_{\alpha}(ts)\; \Gamma(ds)$$

$$\equiv \begin{cases} u \displaystyle\int_{J_0} \operatorname{sgn}(s) \sum_n r^{-(n-\ell)} \{r^{\frac{n}{\alpha}}|s|t - \sin(r^{\frac{n}{\alpha}}|s|t)\}\; \Gamma(ds) & \text{if } \alpha \in (1, 2) \\ u \displaystyle\int_{J_0} \operatorname{sgn}(s) \sum_n r^{-(n-\ell)} \{-\sin(r^{\frac{n}{\alpha}}|s|t)\}\; \Gamma(ds) & \text{if } \alpha \in (0, 1) \end{cases}$$

$$\equiv \begin{cases} u \displaystyle\int_{J_0} \operatorname{sgn}(s) \sum_n r^{-(n-\ell)} \{r^{\frac{n-\ell}{\alpha}}|s|\omega - \sin(r^{\frac{n-\ell}{\alpha}}|s|\omega)\}\Gamma(ds) & \text{if } \alpha \in (1,2) \\ u \displaystyle\int_{J_0} \operatorname{sgn}(s) \sum_n r^{-(n-\ell)} \{-\sin(r^{\frac{n-\ell}{\alpha}}|s|\omega)\}\Gamma(ds) & \text{if } \alpha \in (0, 1) \end{cases}$$

$$= u\omega^{\alpha} \int_{J_0} |s|^{\alpha}\, Im\; k_{\alpha}(\omega s)\; \Gamma(ds)\ .$$

Hence, by Lemma 2.2.2 and (2.14), we have $g(u) = g(ur^{\ell})$. Now by (2.5) and (2.10) we see that $|1 - 2\, g(u)| \leq \frac{2}{\pi} \int_0^{\infty} C_1' \, e^{-rt^{\alpha}C_0'}\, t^{\alpha-1}\, dt < \infty$ for all $u \in [r, 1]$. Hence, by (2.5) and the Lebesgue Dominated Convergence Theorem, g is continuous on $[r, 1]$. Therefore $\inf_{u>0} g(u) = g(u_0)$ and $\sup_{u>0} g(u) = g(u_1)$ for some $u_0, u_1 \in [r, 1]$. Finally, we note that when $1 < \alpha < 2$, F_{ξ_u} is not one-sided for all $u > 0$ (see [27, pp. 293-298]) ; when $0 < \alpha < 1$, assuming that F_{ξ_u} is not one-sided for some $u > 0$, F_{ξ_u} is not one sided for all $u > 0$ (see [20; 17, pp. 179-195]). Therefore,

$$0 < g(u_0) \leq g(u) \leq g(u_1) < 1 \quad \text{for all} \quad u > 0 \ .$$

Hence $0 < g(u_0) \leq F_{M(A)}(0) \leq g(u_1) < 1$ for all $A \in \mathcal{R}$ with $\mu(A) \neq 0$. Here we recall that the Γ appearing in (1.3) and (2.3) are identical.

Proof of (ii). For $u > 0$, let ξ_u be a random variable given by (2.4). Then by (2.5), we have

$$1 - 2\, F_{\xi_u}(0) = \frac{2}{\pi} \int_0^\infty \frac{1}{t}\, e^{-ut^\alpha} \sin(ut^\alpha \beta_u \tan \frac{\pi\alpha}{2})\, dt \ . \tag{2.16}$$

Let $\psi\colon (0, \infty) \times [-1, 1] \to [-1, 1]$ be the function defined by $\psi(u, v) = \frac{2}{\pi} \int_0^\infty \frac{1}{t}\, e^{-ut^\alpha} \sin(ut^\alpha v \tan \frac{\pi\alpha}{2})\, dt$ for every $(u, v) \in (0, \infty) \times [-1, 1]$. By the substitution $\omega = ut^\alpha$, we see that

$$\psi(u, v) = \frac{2}{\pi\alpha} \int_0^\infty e^{-\omega} \frac{1}{\omega} \sin(\omega v \tan \frac{\pi\alpha}{2})\, d\omega$$

$$= \frac{2}{\pi\alpha} \tan^{-1}(v \cdot \tan \frac{\pi\alpha}{2}) \ , \tag{2.17}$$

using methods of Laplace Transforms (see [4, p. A-197]). We note here that (2.17) holds for any $\alpha \in (0, 1) \cup (1, 2)$. From (2.17) it follows that if $\beta(A) = \beta_0$ for all $A \in \mathcal{R}$, then (2.12) holds. Also, from (2.17) it follows that if $0 < \alpha < 1$ and $\beta_0 = \pm 1$, then $F_{M_{\alpha,1}(A)}(0) = 0$ and $F_{M_{\alpha,-1}(A)}(0) = 1$; if $0 < \alpha < 1$ and $|\beta_0| < 1$, then for all $A \in \mathcal{R}$ with $\mu(A) \neq 0$,

$$F_{M_{\alpha,\beta_0}(A)}(0) = \frac{1}{2} - \frac{1}{\pi\alpha} \tan^{-1}(\beta_0 \tan \frac{\pi\alpha}{2})$$

which is in $(0, 1)$. If $1 < \alpha < 2$ (and β is arbitrary as in (2.4)), then $\sup \psi(u, v) = \psi(u, -1) = \frac{2}{\pi\alpha} \tan^{-1}(-\tan \frac{\pi\alpha}{2}) = \frac{2}{\pi\alpha} (\pi - \frac{\pi\alpha}{2}) = \frac{2}{\alpha} - 1$, and $\inf \psi(u, v) = \psi(u, 1) = \frac{2}{\pi\alpha} \tan^{-1}(\tan \frac{\pi\alpha}{2}) = \frac{2}{\pi\alpha} (\frac{\pi\alpha}{2} - \pi) = 1 - \frac{2}{\alpha}$. Thus, from (2.16), we have

$$0 < 1 - \frac{1}{\alpha} \leq F_{\xi_u}(0) \leq \frac{1}{\alpha} < 1$$

for all $u > 0$. In particular (2.14) holds for all $A \in \mathcal{R}$ with $\mu(A) \neq 0$. ■

2.2.4. Remark. Let $\alpha \in (0, 1) \cup (1, 2)$. Let ξ be a random variable with the ch. function $\hat{L}_\xi(\cdot)$ given by

$$\hat{L}_\xi(t) = \exp\{\{-|t|^\alpha(1 - i\beta \cdot \tan \frac{\pi\alpha}{2} \cdot \mathrm{sgn}(t))\}\}, \ t \in \mathbb{R},$$

where $|\beta| \leq 1$. Zolotarev [28, p. 79] has calculated, in a way different from that shown in the proof of Proposition 2.2.3, the value of the distribution function of ξ at zero using the integral representation of its density function.

Finally, to prove Theorem 2.2.1, we need the following lemma which is a slight modification of the weak symmetrization inequalities of Loève [16, p. 257].

2.2.5 Lemma [16]. Let ξ and ξ_1 be two independent identically distributed random variables. Then

$$[\min\{P(\xi > 0), P(\xi < 0)\}]\, P(|\xi| > t)$$

$$\leq P(|\xi - \xi_1| > t) \leq 2\, P(|\xi| > \tfrac{t}{2}) \qquad (2.18)$$

for any $t > 0$.

Proof. For any $t > 0$, we have

$$P(\xi - \xi_1 > t) \geq P(\xi > t, \xi_1 < 0) = P(\xi > t)\, P(\xi < 0) \qquad (2.19)$$

and

$$P(\xi - \xi_1 < -t) \geq P(\xi_1 > 0, \xi < -t) = P(\xi > 0)\, P(\xi < -t) \qquad (2.20)$$

The equalities in (2.19) and (2.20) follow from the fact that ξ and ξ_1 are independent and identically distributed. Thus we have

$$\begin{aligned} P(|\xi - \xi_1| > t) &= P(\xi - \xi_1 > t) + P(\xi - \xi_1 < -t) \\ &\geq P(\xi > t)\, P(\xi < 0) + P(\xi > 0)\, P(\xi < -t) \\ &\geq [\min\{P(\xi > 0), P(\xi < 0)\}]\, P(|\xi| > t) \,. \end{aligned}$$

The other inequality of Lemma 2.2.5 follows as in [16, p. 257] . ∎

Now we prove Theorem 2.2.1.

Proof of Theorem 2.2.1.

Proof of (i). Let M' be an independent copy of M and let $\tilde{M}$ be the symmetrization $M - M'$ of M . Then for any $A \in \mathcal{R}$ with

$\mu(A) \neq 0$, we have the ch. function $\hat{L}_{\tilde{M}(A)}(\cdot)$ of $\tilde{M}(A)$ is given by

$$\hat{L}_{\tilde{M}(A)}(t) = \exp\{-\mu(A) \int_{J_0} |ts|^{\alpha} \overline{k}_{\alpha}(ts)\, \tilde{\Gamma}(ds)\} , \quad t \in \mathbb{R}$$

where $\tilde{\Gamma}$ is the symmetrization of Γ . By Theorem 1.2.3, there exist constants C_1' and C_2' which depend only on r, α, and Γ , and do not depend on A , such that

$$C_1'P(C_1'|M_{\alpha,0}(A)| > t) \leq P(|\tilde{M}(A)| > t) \leq C_2'P(C_2'|M_{\alpha,0}(A)| > t) \qquad (2.21)$$

for all $t > 0$, and for all $A \in \mathcal{R}$ with $\mu(A) \neq 0$. By Proposition 2.2.3(i) , there exist constants c_1 and c_2 which depend only on r, α, and Γ , such that

$$0 < c_1 \leq P(M(A) < 0) \leq c_2 < 1 \qquad (2.22)$$

for all $A \in \mathcal{R}$ with $\mu(A) \neq 0$. Applying Lemma 2.2.5 to $M(A)$ and $M'(A)$ and using (2.22), we obtain

$$cP(|M(A)| > t) \leq P(|\tilde{M}(A)| > t) \leq 2P(|M(A)| > \tfrac{t}{2}) \qquad (2.23)$$

for all $t > 0$, where $c = \min(c_1, 1 - c_2)$. From (2.23) and (2.21) we get (2.1), where $C_1 = \frac{C_1'}{2}$, $C_2 = \frac{C_2'}{c}$, and $C_3 = C_2'$. We note that these constants depend only on r, α, and Γ , and they do not depend on A .

Proof of (ii). By Proposition 2.2.4(ii), we have that

$$0 < 1 - \frac{1}{\alpha} \le P(M_{\alpha,\beta}(A) < 0) \le \frac{1}{\alpha} < 1 \tag{2.24}$$

for all $A \in R$ with $\mu(A) \neq 0$. Let $\tilde{M}_{\alpha,\beta} = M_{\alpha,\beta} - M'_{\alpha,\beta}$, where $M'_{\alpha,\beta}$ is an independent copy of $M_{\alpha,\beta}$. By (2.24) and Lemma 2.2.5 we have

$$(1 - \frac{1}{\alpha})\, P(|M_{\alpha,\beta}(A)| > t) \le P(|\tilde{M}_{\alpha,\beta}(A)| > t) \le 2P(|M_{\alpha,\beta}(A)| > \frac{t}{2}) \tag{2.25}$$

for all $t > 0$, and for all $A \in R$ with $\mu(A) \neq 0$. Since $\tilde{M}_{\alpha,\beta}(A)$ is distributed as $2^{\frac{1}{\alpha}} M_{\alpha,0}(A)$, we have

$$P(|\tilde{M}_{\alpha,\beta}(A)| > t) = P(2^{\frac{1}{\alpha}} |M_{\alpha,0}(A)| > t) \tag{2.26}$$

for all $t > 0$, and for all $A \in R$ with $\mu(A) \neq 0$. Now (2.2) follows from (2.25) and (2.26). ∎

2.3. Definition of Multiple Stochastic Integral

Let $S_{k,X}$ denote the space of all X-valued C_k-measurable simple functions on Δ_k; i.e., if $f \in S_{k,X}$, then there exist some elements $x_1, x_2, \ldots, x_n$ of X and disjoint elements $C_1, C_2, \ldots, C_n$ of C_k such that $f = \sum_{j=1}^{n} x_j \chi_{C_j}$. Now we proceed to define on $S_{k,X}$ the multiple integral for functions in $S_{k,X}$ with respect to an r-SS(α) random measure.

2.3.1. Definitions. For any $C \in C_k$, we define

$$M^k(C) = \sum_{j=1}^{n} M(A_{j_1}) \ldots M(A_{j_k}) , \tag{2.27}$$

where, by Proposition 1.4.2, $C = \bigcup_{j=1}^{n} A_{j_1} \times \dots \times A_{j_k}$, a finite disjoint union of elements of A_k. Similarly, replacing M by M_0 (respectively, $M_{\alpha,\beta}$) in (2.27), we define M_0^k (respectively, $M_{\alpha,\beta}^k$).

2.3.2. Note. It is standard to show that M^k is well defined (see Halmos [8, p. 149]). Indeed, if $C \in C_k$ has two representations, $\bigcup_{j=1}^{n} A_{j_1} \times \dots \times A_{j_k}$ and $\bigcup_{i=1}^{\ell} B_{i_1} \times \dots \times B_{i_k}$, each of which is a finite disjoint union of elements of A_k, then

$$\begin{aligned}
&\sum_{j=1}^{n} M(A_{j_1}) \dots M(A_{j_k}) \\
&= \sum_{j=1}^{n} M^k(A_{j_1} \times \dots \times A_{j_k}) \\
&= \sum_{j=1}^{n} M^k[(A_{j_1} \times \dots \times A_{j_k}) \cap (\bigcup_{i=1}^{\ell} B_{i_1} \times \dots \times B_{i_k})] \\
&= \sum_{j=1}^{n} M^k[\bigcup_{i=1}^{\ell} (A_{j_1} \cap B_{i_1} \times \dots \times A_{j_k} \cap B_{i_k})] \\
&= \sum_{j=1}^{n} \sum_{i=1}^{\ell} M(A_{j_1} \cap B_{i_1}) \dots M(A_{j_k} \cap B_{i_k}) .
\end{aligned}$$

Similarly we can show that

$$\sum_{i=1}^{\ell} M(B_{i_1}) \dots M(B_{i_k}) = \sum_{i=1}^{\ell} \sum_{j=1}^{n} M(A_{j_1} \cap B_{i_1}) \dots M(A_{j_k} \cap B_{i_k}) .$$

Thus, $M^k(C)$ does not depend on the representation of C.

2.3.3. Definitions. For any $f \in S_{k,X}$, we define the k-tuple stochastic integral $I_k(f)$ with respect to semistable random measure M by

$$I_k(f) = \sum_{j=1}^{n} x_j \, M^k(C_j) \ , \tag{2.28}$$

where $f = \sum_{j=1}^{n} x_j \chi_{C_j}$ for some $x_1, \ldots, x_n \in X$ and disjoint elements $C_1, C_2, \ldots, C_n$ of C_k. For $f \in S_{k,X}$, the k-tuple stochastic integral of f with respect to a strictly stable random measure $M_{\alpha,\beta}$ is defined by replacing M by $M_{\alpha,\beta}$ in (2.28). For $f \in S_{kX}$, we will denote by $I_k^0(f)$ the k-tuple stochastic integral of f with respect to symmetric r-SS(α) random measure M. Again by a standard procedure we see that for $f \in S_{k,X}$, $I_k(f)$ does not depend on the representation of f (see Ash [1, p. 36]).

2.4. Comparison of Moments of $I_k(f)$ and $I_k^{\alpha,0}(f)$.

The key to the development of the multiple stochastic integrals with respect to a strictly r-SS(α) random measure M (respectively, M_0) for a larger class of $\mathcal{B}_k$-measurable functions is the comparison of the moments of $I_k(f)$ (respectively, $I_k^0(f)$) with the corresponding moments of $I_k^{\alpha,0}(f)$ for $f \in S_{k,X}$. We present the comparison in the following theorem.

2.4.1. Theorem. (i) If $1 < p < \alpha < 2$, then there exist positive constants C_1 and C_2 which depend only on k, r, p, α, and Γ such that

$$C_1 \| I_k^{\alpha,0}(f) \|_p \le \| I_k(f) \|_p \le C_2 \| I_k^{\alpha,0}(f) \|_p \tag{2.29}$$

for all $f \in S_{k,X}$. Analogously, there exist positive constants C_1 and C_2 which depend only on k, p, and α such that

$$C_1 \| I_k^{\alpha,0}(f) \|_p \leq \| I_k^{\alpha,\beta}(f) \|_p \leq C_2 \| I_k^{\alpha,0}(f) \|_p \qquad (2.30)$$

for all $f \in S_{k,X}$.

(ii) If $0 < p < \alpha < 1$, then, replacing I_k by I_k^0 in (2.29), an analogue of (2.29) holds for I_k^0 .

For the proof of this theorem we need the result which follows.

2.4.2. Proposition (Kwapień [14, Theorem 1]). Let $(\eta_1, \eta_2, \ldots, \eta_n)$ and $(\xi_1, \xi_2, \ldots, \xi_n)$ be two finite sequences of independent symmetric random variables such that $P(|\eta_i| \geq t) \leq KP(L|\xi_i| \geq t)$ for some constants K and L , for $i = 1, \ldots, n$, and for all $t > 0$. Let X be a vector space and let $Q\colon \mathbb{R}^n \to X$ be a polynomial defined by

$$Q(t_1, \ldots, t_n) = \sum_{k=1}^{n} \sum_{1 \leq i_1 < \ldots < i_k \leq n} c_{i_1 \ldots i_k} t_{i_1} \ldots t_{i_k} ,$$

where the coefficients $c_{i_1 \ldots i_k}$ are elements of X . Then for any measurable convex function $\Phi\colon X \to \mathbb{R}$, the inequality

$$E(\Phi(Q(\eta_1, \eta_2, \ldots, \eta_n))) \leq E(\Phi(Q(KL\xi_1, KL\xi_2, \ldots, KL\xi_n))) \qquad (2.31)$$

holds.

The above result yields the following proposition.

2.4.3. Proposition. Let $(\eta_1, \eta_2, \ldots, \eta_n)$ and $(\xi_1, \xi_2, \ldots, \xi_n)$ be two sequences of independent symmetric random variables such that for some K and L , the inequality

$$P(|\eta_i| \geq t) \leq KP(L|\xi_i| \geq t)$$

holds for all $t > 0$ and for $i = 1, 2, \ldots, n$. If $p \in (1, 2)$ (respectively, if $p \in (0, 1)$) , then

$$E\|\langle F; (\underline{\eta})^k\rangle\|_p \leq (KL)^k \, E\|\langle F: (\underline{\xi})^k\rangle\|_p \tag{2.32}$$

(respectively,

$$E\|\langle F; (\underline{\eta})^k\rangle\|_p \geq (KL)^k \, E\|\langle F; (\underline{\xi})^k\rangle\|_p) \tag{2.33}$$

for all $F \in F^{\tau}_{k,X}$, where $\underline{\eta} = (\eta_1, \ldots, \eta_n, 0, \ldots)$ and $\underline{\xi} = (\xi_1, \ldots, \xi_n, 0, \ldots)$.

Proof. For any $F \in F^{\tau}_{k,X}$, we consider the polynomial $Q_F: \mathbb{R}^n \to X$ given by $Q_F(t_1, \ldots, t_n) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} F((i_1, \ldots, i_k))t_{i_1} \ldots t_{i_k}$ for all $(t_1, \ldots, t_n) \in \mathbb{R}^n$, and the measurable convex function $\Phi: X \to \mathbb{R}$ given by $\Phi(x) = \|x\|^p$ for all $x \in X$ if $p \in (1, 2)$ (respectively, $\Phi(x) = -\|x\|^p$ for all $x \in X$ if $p \in (0, 1)$) . We have by Proposition 2.4.2 that

$$E(\Phi(Q_F(\eta_1, \ldots, \eta_n))) \leq E(\Phi(Q_F(KL\xi_1, \ldots, KL\xi_n))) .$$

Thus, using the facts that $E\|\langle F; (\underline{\eta})^k\rangle\|^p = E\|Q_F(\eta_1, \ldots, \eta_n)\|^p$, $E\|\langle F; (\underline{\xi})^k\rangle\|^p = E\|Q_F(\xi_1, \ldots, \xi_n)\|^p$, and $Q_F(KL\xi_1, \ldots, KL\xi_n) = (KL)^k \, Q_F(\xi_1, \ldots, \xi_n)$, we have (2.32) and (2.33) . ∎

2.4.4 Proposition (Krakowiak and Szulga [11, Cor. 2.2]) . Let $F \in F^{\tau}_{k,X}$, and let $(\xi_1,\dots, \xi_n)$ be a sequence of independent real random variables such that for $p > 1$, $E|\xi_j|^p < \infty$ and $E\xi_j = 0$ for $j = 1, 2,\dots, n$. Let $(\varepsilon_1, \varepsilon_2,\dots, \varepsilon_n)$ be a sequence of independent Rademacher random variables. Then there exists a positive constant C which depends only on k and p such that

$$C^{-1} E\|\langle F; (\underline{\xi})^k\rangle\|^p \le E\|\langle F: (\underline{\varepsilon\xi})^k\rangle\|^p$$

$$\le C\, E\|\langle F; (\underline{\xi})^k\rangle\|^p \tag{2.34}$$

for all $F \in F^{\tau}_{k,X}$, where $\underline{\xi} = (\xi_j,\dots, \xi_n, 0,\dots)$ and $\underline{\varepsilon\xi} = (\varepsilon_1\xi_1,\dots, \varepsilon_n\xi_n, 0,\dots)$.

Combining 2.4.3 and 2.4.4 we get Proposition 2.4.5 which is an analogue of Proposition 2.4.3 for two finite sequences of independent, not necessarily symmetric, real random variables $(\xi_1,\dots, \xi_n)$ and $(\eta_n,\dots, \eta_n)$.

2.4.5 Proposition. Let $(\eta_1,\dots, \eta_n)$ and $(\xi_1,\dots, \xi_n)$ be two sequences of independent, not necessarily symmetric, real random variables such that for some positive real numbers K and L the inequality $P(|\eta_i| \ge t) \le KP(L|\xi_i| \ge t)$ holds for all $t > 0$ and for $i = 1, 2,\dots, n$. Let $p \in (1, 2)$, and let $E|\xi_i|^p < \infty$ and $E\eta_i = E\xi_i = 0$ for $i = 1, 2,\dots, n$. Then there is a positive constant $C_1 = C(KL)^k$, depending only on k, K, L, and p , such that

$$E\|\langle F; (\underline{\eta})^k\rangle\|^p \le C_1 E\|\langle F: (\underline{\xi})^k\rangle\|^p \tag{2.35}$$

for all $F \in F^{\tau}_{k,X}$, where $\underline{\eta} = (\eta_1, \ldots, \eta_n, 0, \ldots)$, and $\underline{\xi} = (\xi_1, \ldots, \xi_n, 0, \ldots)$ and C is the constant appearing in (2.34).

Proof. Let $\varepsilon_1, \ldots, \varepsilon_n$ be independent Rademacher random variables; let $(\varepsilon'_1, \ldots, \varepsilon'_n)$ and $(\eta'_1, \ldots, \eta'_n)$ be copies of $(\varepsilon_1, \ldots, \varepsilon_n)$ and $(\eta_1, \ldots, \eta_n)$ respectively such that $(\varepsilon_1, \ldots, \varepsilon_n)$, $(\xi_1, \ldots, \xi_n)$, $(\eta'_1, \ldots, \eta'_n)$ and $(\varepsilon'_1, \ldots, \varepsilon'_n)$ are independent. Since $(\varepsilon'_1\eta'_1, \ldots, \varepsilon'_n\eta'_n)$ and $(\varepsilon_1\xi_1, \ldots, \varepsilon_n\xi_n)$ are two finite sequences of independent symmetric real random variables such that

$$\begin{aligned} P(|\varepsilon'_i\eta'_i| \geq t) &= P(|\eta'_i| \geq t) \\ &\leq KP(L|\xi_i| \geq t) = KP(L|\varepsilon_i\xi_i| \geq t) \end{aligned}$$

for all $t > 0$ and for $i = 1,2,\ldots, n$, we have by Proposition 2.4.3,

$$E\|\langle F; (\underline{\varepsilon'\eta'})^k\rangle\|^p \leq (KL)^k \, E\|\langle F; (\underline{\varepsilon\xi})^k\rangle\|^p \tag{2.36}$$

for all $F \in F^{\tau}_{k,X}$, where $\underline{\varepsilon'\eta'} = (\varepsilon'_1\eta'_1, \ldots, \varepsilon'_n\eta'_n, 0, \ldots)$ and $\underline{\varepsilon\xi} = (\varepsilon_1\xi_1, \ldots, \varepsilon_n\xi_n, 0, \ldots)$. Applying Proposition 2.4.4 to the finite sequences $(\eta'_1, \ldots, \eta'_n)$ and $(\xi_1, \ldots, \xi_n)$, we see that there exist positive constants C_1 and C_2 depending only on k and p, such that

$$C_1 E\|\langle F; (\underline{\eta}')^k\rangle\|^p \leq E\|\langle F; (\underline{\varepsilon'\eta'})^k\rangle\|^p \tag{2.37}$$

and

$$E\|\langle F; (\underline{\varepsilon\xi})^k\rangle\|^p \leq C_2 \, E\|\langle F: (\underline{\xi})^k\rangle\|^p . \tag{2.38}$$

Since $E\|\langle F;\ (\underline{\eta})^k\rangle\|^p = E\|\langle F;\ (\underline{\eta}')^k\rangle\|^p$, we have, from (2.36), (2.37), and (2.38) that

$$\begin{aligned} C_1 E\|\langle F;\ (\underline{\eta})^k\rangle\|^p &= C_1 E\|\langle F;\ (\underline{\eta}')^k\rangle\|^p \\ &\leq E\|\langle F;\ (\underline{\varepsilon}'\underline{\eta}')^k\rangle\|^p \\ &\leq (KL)^k\ E\|\langle F;\ (\underline{\varepsilon\xi})^k\rangle\|^p \\ &\leq (KL)^k\ C_2 E\ \|\langle F;\ (\underline{\xi})^k\rangle\|^p\ . \qquad \blacksquare \end{aligned}$$

2.4.6. Remark. Since Proposition 2.4.4 was available only in the case of random variables $\xi_1,\ldots,\ \xi_n$ such that $E|\xi_j|^p < \infty$ for some $p \in (1,\ 2)$ and $E\xi_j = 0$ for $j = 1,2,\ldots,\ n$, we have Proposition 2.4.5 when $1 < p < \alpha < 2$. However, when $0 < \alpha < 1$, we conjecture that (2.34) holds when

$$(\xi_1,\ldots,\ \xi_n) = (M(A_1),\ldots,\ M(A_n))$$

(or, $(M_{\alpha,\beta}(A_1),\ldots,\ M_{\alpha,\beta}(A_n)))$, a finite sequence of independent strictly r-SS(α) (or strictly α-stable) random variables whose distributions are not one-sided. Hereafter, when $0 < \alpha < 1$, we will consider only symmetric r-SS(α) random measures.

2.4.7. Lemma. For each $f \in \mathcal{S}_{k,X}$, there exist $F \in \mathcal{F}^{\tau}_{k,X}$ and some subintervals $A_1,\ldots,\ A_\ell$ of $[0,\ 1]$ with $A_1 < \ldots < A_\ell$ such that

$$f = \sum_{1\leq i_1<\ldots<i_k\leq \ell} F((i_1,\ldots,\ i_k))\ \chi_{A_{i_1}\times\cdots\times A_{i_k}} \ .$$

Proof. Let $f \in S_{k,X}$. Then there exist $x_1, \dots, x_n \in X$ and disjoint elements $C_1, \dots, C_n$ of C_k such that $f = \sum_{j=1}^{n} x_j \chi_{C_j}$.

Since $C_j \in C_k$ for $j = 1, \dots, n$, by Proposition 1.4.4 we can find subintervals $I_1^{(j)}, \dots, I_{\ell_j}^{(j)}$ of $[0, 1]$ and $\alpha_j \subset \Lambda_k^{\ell_j}$ such that $C_j = \bigcup_{(i_1, \dots, i_k)\ \alpha_j} I_{i_1}^{(j)} \times \dots \times I_{i_k}^{(j)}$; we can find subintervals $A_1, \dots, A_\ell$ of $[0, 1]$ such that $A_1 < \dots < A_\ell$ and such that each element of $\{I_{i_s}^{(j)} : 1 \le j \le n,\ 1 \le s \le k, \text{ and } (i_1, \dots, i_k) \in \alpha_j\}$ can be expressed as a finite (disjoint) union of elements of $\{A_1, \dots, A_\ell\}$. Thus, there exist subsets $\lambda_1, \dots, \lambda_n$ of Λ_k^ℓ such that $C_j = \bigcup_{(s_1, \dots, s_k) \in \lambda_j} A_{s_1} \times \dots \times A_{s_k}$ for $j = 1, 2, \dots, n$. Let $\lambda = \bigcup_{j=1}^{n} \lambda_j$. We define $F: \mathbb{N}^k \to X$ such that for each $(i_1, \dots, i_k) \in \mathbb{N}^k$,

$$F((i_1, \dots, i_k)) = \begin{cases} x_j & \text{if } (i_1, \dots, i_k) \in \lambda_j,\ 1 \le j \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $F \in F_{k,X}^{\tau}$; since $f = \sum_{j=1}^{n} x_j \chi_{C_j}$, we have

$$f = \sum_{1 \le i_1 < \dots < i_k \le \ell} F((i_1, \dots, i_k)) \chi_{A_{i_1} \times \dots \times A_{i_k}}. \qquad ■$$

Finally, we prove Theorem 2.4.1.

Proof of Theorem 2.4.1.

Proof of (i). Let $f \in S_{k,X}$. Then by Lemma 2.4.7, there exist $F \in F_{k,X}^{\tau}$ and some subintervals $A_1, \dots, A_\ell$ of $[0, 1]$ with

$A_1 < \dots < A_\ell$ such that $f = \sum_{1 \le i_1 < \dots < i_k \le \ell} F((i_1, \dots, i_k)) \chi_{A_{i_1} \times \dots \times A_{i_k}}$.

Therefore,

$$I_k(f) = \sum_{1 \le i_1 < \dots < i_k \le \ell} F((i_1, \dots, i_k)) M(A_{i_1}) \dots M(A_{i_k})$$

$$= \langle F; (M(A_1), \dots, M(A_\ell), 0, \dots)^k \rangle . \tag{2.39}$$

Similarly

$$I_k^{\alpha,0}(f) = \langle F; (M_{\alpha,0}(A_1), \dots, M_{\alpha,0}(A_\ell), 0, \dots)^k \rangle . \tag{2.40}$$

By (2.1) there exist positive constants C_1', C_2', and C_3' , depending only on r, α, and Γ , such that

$$C_1' P(C_1' |M_{\alpha,\beta}(A_j)| > t) \le P(|M(A_j)| > t) \le C_2' P(C_3' |M_{\alpha,0}(A_j)| > t) \tag{2.41}$$

for all $t > 0$ and for $j = 1, 2, \dots, \ell$. Since (2.41) holds for the sequences of independent random variables $(M(A_1), \dots, M(A_\ell))$ and $(M_{\alpha,0}(A_1), \dots, M_{\alpha,0}(A_\ell))$, by Proposition 2.4.5 there exist constants C_1 and C_2 , depending only on r, α, k, p, and Γ such that

$$C_1 \| \langle F; (M_{\alpha,0}(A_1), \dots, M_{\alpha,0}(A_\ell), 0, \dots)^k \rangle \|_p$$

$$\le \| \langle F; (M(A_1), \dots, M(A_\ell), 0, \dots)^k \rangle \|_p$$

$$\le C_2 \| \langle F; (M_{\alpha,0}(A_1), \dots, M_{\alpha,0}(A_\ell), 0, \dots)^k \rangle \|_p \tag{2.42}$$

for all $F \in F_{k,X}$. We note that these constants do not depend on the sets $A_1, \ldots, A_\ell$. Hence, by (2.39) and (2.40), we have (2.29) .

In the case of strictly stable random measure, we have by (2.2)

$$\frac{1}{2} P(2^{\frac{1-\alpha}{\alpha}} |M_{\alpha,0}(A_j)| > t) \leq P(|M_{\alpha,\beta}(A_j)| > t)$$
$$\leq (\frac{\alpha}{\alpha - 1}) P(2^{\frac{1}{\alpha}} |M_{\alpha,\beta}(A_j)| > t) \qquad (2.43)$$

for $j = 1,2,\ldots, \ell$. Replacing (2.41) by (2.43) in the above argument, we get (2.30) .

<u>Proof of (ii).</u> If $0 < p < \alpha < 1$, then by Theorem 1.2.3 there exist positive constants C_1' and C_2' , depending only on r, α, and Γ , such that

$$C_1'P(C_1'|M_{\alpha,0}(A_j)| > t) \leq P(|M_0(A_j)| > t) \leq C_2'P(C_2'|M_{\alpha,0}(A_j)| > t) \quad (2.44)$$

for all $t > 0$ and for $j = 1, 2, \ldots, \ell$. Thus, by (2.44) $(M_0(A_1), \ldots, M_0(A_\ell))$ and $(M_{\alpha,0}(A_1), \ldots, M_{\alpha,0}(A_\ell))$ are two finite sequences of independent real random variables that satisfy the hypotheses of Proposition 2.4.3(ii). Hence there exist positive constants C_1 and C_2 , depending only on r, α , and Γ , such that

$$C_1\|\langle F; (M_{\alpha,0}(A_1), \ldots, M_{\alpha,0}(A_\ell), 0, \ldots)^k\rangle\|_p$$
$$\geq \|\langle F; (M_0(A_1), \ldots, M_0(A_\ell), 0, \ldots)^k\rangle\|_p$$
$$\geq C_2\|\langle F; (M_{\alpha,0}(A_1), \ldots, M_{\alpha,0}(A_\ell), 0, \ldots)^k\rangle\|_p \qquad (2.45)$$

for every $F \in \mathcal{F}_{k,X}$. We note that these constants do not depend on $A_1, \ldots, A_\ell$ or F. Thus it follows from (2.42), (2.43) and (2.45) that

$$C_1 \| I_k^{\alpha,0}(f) \|_p \geq \| I_k^0(f) \|_p \geq C_2 \| I_k^{\alpha,0}(f) \|_p$$

for all $f \in \mathcal{S}_{k,X}$ whenever $0 < p < \alpha < 1$. ■

CHAPTER III

MULTIPLE STOCHASTIC INTEGRALS

Recall that in section 2.3 of Chapter II, we defined the multiple stochastic integrals I_k and I_k^0 of Banach valued C_k-measurable simple functions with respect to r-SS(α) random measures. In this chapter, we extend the definitions of these multiple stochastic integrals I_k and I_k^0 to a larger class of Banach valued $\mathcal{B}_k$-measurable functions on Δ_k. For the integral I_k we shall restrict ourselves to the case $1 < \alpha < 2$. We do this because of the unavailability of analogs of the crucial inequalities (2.34) and of Proposition 5.1 of [13] for the case $0 < \alpha \leq 1$. Thus, throughout this chapter, M and $M_{\alpha,\beta}$ will represent, respectively, a strictly r-SS(α) random measure and a strictly S(α) random measure with the restriction that $1 < \alpha < 2$. Our approach in extending the definitions of the integrals I_k and I_k^0 is similar to that of Krakowiak and Szulga [10] for the symmetric stable case.

3.1. Extension of M^k to $\mathcal{B}_k$

In section 2.3, we defined the finitely additive vector measures M^k and M_0^k on C_k. Now we extend from C_k to $\mathcal{B}_k$, the vector measure M^k for $1 < \alpha < 2$ and the vector measure M_0^k for all α. The proof is similar to the one given by Krakowiak and Szulga [13, Theorem 5.4] in the case of the vector measure $M_{\alpha,0}^k$. Before we state the extension theorem, we state two propositions which are consequences of Theorem 2.4.1. The first proposition and the last part of the second proposition give a relationship between M^k and the 'control measure' μ^k on C_k, where μ^k is the restriction of the measure $\underbrace{\mu \times \dots \times \mu}_{k \text{ times}}$ to Δ_k. This relationship is crucial to the proof of the extension theorem.

3.1.1 Proposition. (i) Let $1 < p < \alpha < q$. Then there exist positive constants C_1' and C_2' which depend only on α, r, p, q, $\mu([0, 1])$, k, and Γ such that

$$C_1'\|f\|_{L_\alpha(\Delta_k;\mathbb{R})} \leq \|I_k(f)\|_p \leq C_2'\|f\|_{L_q(\Delta_k;\mathbb{R})} \tag{3.2}$$

for all $f \in \mathcal{S}_{k,\mathbb{R}}$. Analogously,

$$C_1'\|f\|_{L_\alpha(\Delta_k;\mathbb{R})} \leq \|I_k^{\alpha,\beta}(f)\|_p \leq C_2'\|f\|_{L_q(\Delta_k;\mathbb{R})} \tag{3.3}$$

holds for all $f \in \mathcal{S}_{k,\mathbb{R}}$, where C_1' and C_2' depend only on α, r, p, q, k, and $\mu([0, 1])$.

(ii) If $0 < p < \alpha < q$ and $\alpha \neq 1$, then there exist positive constants C_1' and C_2' which depend only on α, r, p, q, $\mu([0, 1])$, k, and Γ such that

$$C_1'\|f\|_{L_\alpha(\Delta_k;\mathbb{R})} \leq \|I_k^0(f)\|_p \leq C_2'\|f\|_{L_q(\Delta_k;\mathbb{R})} \tag{3.4}$$

for all $f \in \mathcal{S}_{k,\mathbb{R}}$.

Proof. By Proposition 5.1 and Corollary 5.2 of [13] (see also [10, p. 12]) it follows that there exist positive constants C' and C''

which depend only on α , $\mu([0, 1])$, k, p, and q such that

$$C'\|f\|_{L_\alpha(\Delta_k;\mathbb{R})} \leq \|I_k^{\alpha,0}(f)\|_p \leq C''\|f\|_{L_q(\Delta_k;\mathbb{R})} \tag{3.5}$$

for all $f \in S_{k,\mathbb{R}}$. Recall from 2.4.1 that if $1 < p < \alpha$, then

$$C_1\|I_k^{\alpha,0}(f)\|_p \leq \|I_k(f)\|_p \leq C_2\|I_k^{\alpha,0}(f)\|_p , \tag{3.6}$$

$$(C_1\|I_k^{\alpha,0}(f)\|_p \leq \|I_k^{\alpha,\beta}(f)\|_p \leq C_2\|I_k^{\alpha,0}(f)\|_p) , \tag{3.7}$$

and if $0 < p < \alpha$ and $\alpha \neq 1$, then

$$C_1\|I_k^{\alpha,0}(f)\|_p \leq \|I_k^0(f)\|_p \leq C_2\|I_k^{\alpha,0}(f)\|_p \tag{3.8}$$

for all $f \in S_{k,\mathbb{R}}$. Thus (3.5) together with (3.6), (3.7), and (3.8) yields (3.2), (3.3), and (3.4) .

3.1.2 Proposition. (i) [Krakowiak and Szulga, 10, 13]. The class $\{I_k^{\alpha,0}(f): f \in S_{k,\ddot{X}}\} \in MPZ(p)$ for every $p \in [0, \alpha)$.

(ii) Let $C = \{I_k(f): f \in S_{k,X}\}$ $(C = \{I_k^0(f): f \in S_{k,X}\}$, respectively). Then $C^0 \in MPZ(p)$ for every $0 < p < \alpha$. Recall that C^0 is the $L_0(X)$-closure of C . (Hence, for any $0 < p' < p < \alpha$, the $L_{p'}$ and L_p norms of elements of C are comparable; see Proposition 1.7.2).

(iii) If $1 < \alpha < 2$, then (2.29), (2.30), (3.2), and (3.3) hold for every $p \in (0, \alpha)$.

Proof of (i). Let $f \in S_{k,X}$. Then, by Proposition 2.4.3, there exist $F \in F^{\tau}_{k,X}$ and subintervals $A_1, \ldots, A_n$ of $[0, 1]$ such that $A_1 < \ldots < A_n$ and

$$I_k^{\alpha,0}(f) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} F(i_1, \ldots, i_k))\, M_{\alpha,0}(A_{i_1}) \ldots M_{\alpha,0}(A_{i_k}) \;.$$

Since for any $A \in R$, $M_{\alpha,0}(A)$ has the same distribution as $\mu(A)^{\frac{1}{\alpha}}\theta$, where θ is a symmetric α-stable random variable, we have that $I_k^{\alpha,0}(f)$ has the same distribution (hence the same moments) as $\langle G; (\theta)^k \rangle$, where $\underline{\theta} = (\theta_1, \theta_2, \ldots)$ is a sequence of independent symmetric α-stable random variables and $G\colon \mathbb{N}^k \to X$ is the map given by

$$G((i_1, \ldots, i_k)) = \begin{cases} F((i_1, \ldots, i_k))(\mu(A_{i_1}) \ldots \mu(A_{i_k}))^{\frac{1}{\alpha}} \\ \qquad\qquad \text{if } 1 \leq i_1 < \ldots < i_k \leq n \;, \\ \\ 0 \qquad \text{otherwise.} \end{cases}$$

Thus, since $\{\langle G; (\underline{\theta})^k \rangle : G \in F^{\tau}_{k,X}\} \in \mathrm{MPZ}(p)$ for $0 < p < \alpha$, we have by Proposition 1.7.3, that $\{I_k^{\alpha,0}(f) : f \in S_{k,X}\} \in \mathrm{MPZ}(p)$, for $0 < p < \alpha$.

Proof of (ii). We first show that $C \in \mathrm{MPZ}(p)$ for every $p \in [0, \alpha)$ and $1 < \alpha < 2$. Now, for $1 < p' < p < \alpha$, Theorem 2.4.1 yields positive constants C_1 and C_2 which depend only on r, α,

k, p', p, and Γ such that

$$C_1 \| I_k^{\alpha,0}(f) \|_{p'} \leq \| I_k(f) \|_{p'} \leq C_2 \| I_k^{\alpha,0}(f) \|_{p'}$$

for all $f \in S_{k,X}$. From (i) above, we have $\{ I_k^{\alpha,0}(f) : f \in S_{k,X} \} \in \mathrm{MPZ}(p)$ for every $p \in [0, \alpha)$. Hence it follows from Proposition 1.7.2 that $C \in \mathrm{MPZ}(p)$ for every $p \in [0, \alpha)$. Thus, by Proposition 1.7.2(ii), $C^0 \in \mathrm{MPZ}(p)$ for every $p \in [0, \alpha)$. The other case follows similarly.

<u>Proof of (iii).</u> We show that an analogue of (2.29) holds for $0 \leq p \leq 1 < \alpha < 2$; the proofs of the other cases are similar. Let $1 < \alpha' < \alpha$. Since $0 < p \leq 1 < \alpha' < \alpha$, we have, by part (ii) above and (2.29),

$$\| I_k(f) \|_p \leq \gamma^{-1}_{,p} \| I_k(f) \|_{\alpha'} \geq \gamma^{-1}_{\alpha,p} \, C_1 \| I_k^{\alpha,0}(f) \|_{\alpha'}$$

$$\geq \gamma^{-1}_{\alpha,p} \, C_1 \, \| I_k^{\alpha,0}(f) \|_p \; ;$$

similarly,

$$\| I_k(f) \|_p \leq \| I_k(f) \|_{\alpha'} \leq C_2 \| I_k^{\alpha,0}(f) \|_{\alpha'}$$

$$\leq C_2 \, \gamma'_{\alpha,p} \| I_k^{\alpha,0}(f) \|_p \, ,$$

where

$$\gamma_{\alpha,p} = \sup_{f \in S_{k,X}} \frac{\|I_k(f)\|_{\alpha'}}{\|I_k(f)\|_p}$$

and

$$\gamma'_{\alpha,p} = \sup_{f \in S_{k,X}} \frac{\|I_k^{\alpha,0}(f)\|_{\alpha'}}{\|I_k^{\alpha,0}(f)\|_p}$$

are finite by part (ii) and C_1 and C_2 are the constants appearing in (2.29). ■

Now we state and prove our extension theorem.

3.1.3 Theorem. Let $0 < p < \alpha < q$. If $1 < \alpha < 2$, then M^k extends uniquely to a countably additive $L_p(\mathbb{R})$ valued, μ^k-continuous vector measure on $\mathcal{B}_k$. Also, there exist constants C_1 and C_2 which depend only on $r, k, \alpha, \mu([0, 1])$, and Γ such that

$$C_1(\mu^k(A))^{\frac{1}{\alpha}\min(1,p)} \le \|M^k\|(A) \le C_2(\mu^k(A))^{\frac{1}{q}\min(1,p)} \tag{3.9}$$

for all $A \in \mathcal{B}_k$, where $\|M^k\|$ is the semivariation of M^k on $\mathcal{B}_k$. Analogously, the above holds for M_0^k where $\alpha \in (0, 1) \cup (1, 2)$ and for $M_{\alpha,\beta}^k$ when $\alpha \in (1, 2)$.

Proof. By Proposition 3.1.1 and 3.1.2(iii), there exist positive constants C_1 and C_2 which depend only on $\alpha, r, \mu([0, 1]), k, p, q$, and Γ such that

$$C_1(\mu^k(A))^{\frac{1}{\alpha}} \le \|M^k(A)\|_p \le C_2(\mu^k(A))^{\frac{1}{q}} \tag{3.10}$$

for all $A \in C_k$. Using the inequalities (3.10), we now extend M^k, from C_k to $\overline{C}_k$ first, and then to $\mathcal{B}_k$. Let $A \in \overline{C}_k$. By Proposition 1.4.4, there exists an increasing sequence of sets $\{A_j\}_{j=1}^{\infty} \subset C_k$ such that $\bigcup_{j=1}^{\infty} A_j = A$. By the finite additivity of M^k on C_k, for $j > \ell$, we have

$$\|M^k(A_j) - M^k(A_\ell)\|_p = \| {}^k(A_j \setminus A_\ell)\|_p .$$

By (3.10), we have

$$\|M^k(A_j) - M^k(A_\ell)\|_p \le C_2(\mu^k(A_j \setminus A_\ell))^{\frac{1}{q}} = C_2(\mu^k(A_j) - \mu^k(A_\ell))^{\frac{1}{q}} .$$

Thus, since $\{\mu^k(A_j)\}_{j=1}^{\infty}$ is Cauchy, we have that $\{M^k(A_j)\}_{j=1}^{\infty}$ is Cauchy in $L_p(\mathbb{R})$ and hence is convergent in $L_p(\mathbb{R})$. We define $M^k(A)$ as the $L_p(\mathbb{R})$ limit of $\{M(A_j)\}_{j=1}^{\infty}$. Now we show that M^k is well defined and finitely additive on $\overline{C}_k$. Let $A \in \overline{C}_k$ and let $\{A_j\}_{j=1}^{\infty}$ and $\{B_j\}_{j=1}^{\infty}$ be two increasing sequences of sets in C_k such that $\bigcup_{j=1}^{\infty} A_j = A = \bigcup_{j=1}^{\infty} B_j$. Taking $f = \chi_{A_j} - \chi_{B_j}$ in (3.2), we get that

$$\begin{aligned} \|M^k(A_j) - M^k(B_j)\|_p &\le C_2' \|\chi_{A_j} - \chi_{B_j}\|_{L_q(\Delta_k;\mathbb{R})} \\ &= C_2'(\mu^k(A_j \,\Delta\, B_j))^{\frac{1}{q}} \end{aligned}$$

which tends to zero as $j \to \infty$. Therefore, $\lim_{j\to\infty} M^k(A_j) = \lim_{j\to\infty} M^k(B_j)$

and hence M^k is well defined on $\overline{C}_k$. For disjoint sets $A, B \in \overline{C}_k$, let $\{A_j\}_{j=1}^{\infty}$ and $\{B_j\}_{j=1}^{\infty}$ be two increasing sequences of sets in C_k such that $A = \bigcup_{j=1}^{\infty} A_j$ and $B = \bigcup_{j=1}^{\infty} B_j$. Then $M^k(A \cup B) = \lim_{j\to\infty} M^k(A_j \cup B_j)$ $= \lim_{j\to\infty} (M^k(A_j) + M^k(B_j)) = M^k(A) + M^k(B)$ by the definition of M^k on $\overline{C}_k$ and the finite additivity of M^k on C_k. Now we show that (3.10) holds for every $A \in \overline{C}_k$. In fact, if we let $A \in \overline{C}_k$ and let the increasing sequence $\{A_j\}_{j=1}^{\infty} \subset C_k$ be such that $\bigcup_{j=1}^{\infty} A_j = A$, then (3.10) holds for each A_j and hence, by the definition of $M^k(A)$, taking the limit as $j \to \infty$, we obtain that (3.10) holds for all $A \in \overline{C}_k$. Thus $M^k << \mu^k$ on $\overline{C}_k$ and, by Theorem 1.5.2, M^k can be extended to a countably additive μ^k-continuous vector measure on B_k.

Finally, we show that the semivariation of the vector measure M^k on B_k satisfies (3.9). For this, first we show that if $B_1,\ldots, B_n$ are disjoint elements of $\overline{C}_k$, and if $s_1,\ldots, s_n \in [-1, 1]$, then

$$C_1\left(\sum_{i=1}^{n} |s_i|^{\alpha}\, \mu^k(B_i)\right)^{\frac{1}{\alpha}} \leq \left\| \sum_{i=1}^{n} s_i\, M^k(B_i)\right\|_p$$

$$\leq C_2\left(\sum_{i=1}^{n} |s_i|^{q}\, \mu^k(B_i)\right)^{\frac{1}{q}} \tag{3.11}$$

By Proposition 1.4.4, there exist increasing sequences $\{B_1^{(j)}\}_{j=1}^{\infty},\ldots, \{B_n^{(j)}\}_{j=1}^{\infty} \subset C_k$ such that $B_i = \bigcup_{j=1}^{\infty} B_i^{(j)}$ for $j = 1, 2,\ldots, n$; since for each j, the sets $B_1^{(j)},\ldots, B_n^{(j)}$ are disjoint, we have by (3.2) that

$$C_1\left(\sum_{i=1}^{n} |s_i|^{\alpha}\, \mu^k(B_i^{(j)})\right)^{\frac{1}{\alpha}} \leq \left\| \sum_{i=1}^{n} s_i\, M^k(B_i^{(j)})\right\|_p$$

$$\leq C_2\Big(\sum_{i=1}^{n} |s_i|^q \, \mu^k(B_i^{(j)})\Big)^{\frac{1}{q}} \quad . \tag{3.12}$$

Thus, by the definition of M^k on $\overline{C}_k$, letting $j \to \infty$ in (3.12) we obtain (3.11). Now let $s_1, \ldots, s_n \in [-1, 1]$, and let $B_1, \ldots, B_n$ be elements of $\mathcal{B}_k$ which form a partition of $B \in \mathcal{B}_k$. By the nature of the extension (in Theorem 1.5.2; see [5, p. 29]) of M^k from $\overline{C}_k$ to $\mathcal{B}_k$, we can find sequences $\{B_1^{(j)}\}_{j=1}^{\infty}, \ldots, \{B_n^{(j)}\}_{j=1}^{\infty} \subset \overline{C}_k$ such that for each j, the sets $B_1^{(j)}, \ldots, B_m^{(j)}$ are disjoint and such that, for $i = 1, 2, \ldots, n$, the limit $\lim_{j\to\infty} \mu^k\,(B_i^{(j)} \,\Delta\, B_i) = 0$ and $M^k(B_i)$ is the $L_p(\mathbb{R})$ limit of $M^k(B_i^{(j)})$ as $j \to \infty$. From (3.11), for each $j = 1, 2, \ldots,$ we have that

$$C_1\Big(\sum_{i=1}^{n} |s_i|^{\alpha} \mu^k(B_i^{(j)})\Big)^{\frac{1}{\alpha}} \leq \Big\|\sum_{i=1}^{n} s_i \, M^k(B_i^{(j)})\Big\|_p$$

$$\leq C_2\Big(\sum_{i=1}^{n} |s_i|^q \, \mu^k(B_i^{(j)})\Big)^{\frac{1}{q}} . \tag{3.13}$$

Thus, as $j \to \infty$, we have

$$C_1\Big(\sum_{i=1}^{n} |s_i|^{\alpha} \, \mu^k(B_i)\Big)^{\frac{1}{\alpha}} \leq \Big\|\sum_{i=1}^{n} s_i \, M^k(B_i)\Big\|_p$$

$$\leq C_2\Big(\sum_{i=1}^{n} |s_i|^q \, \mu^k(B_i)\Big)^{\frac{1}{q}} .$$

Since $|s_1|, \ldots, |s_n| \leq 1$, we get

$$C_1\left(\sum_{i=1}^{n} |s_i|^{\alpha}\, \mu^k(B_i)\right)^{\frac{1}{\alpha}} \leq \left\| \sum_{i=1}^{n} s_i\, M^k(B_i)\right\|_p \leq C_2\left(\sum_{i=1}^{n} \mu^k(B_i)\right)^{\frac{1}{q}} .$$

Now,

$$\left[C_1\left(\sum_{i=1}^{n} \mu^k(B_i)\right)^{\frac{1}{\alpha}}\right]^{\min(1,p)} \leq \sup_{s_1,\ldots,s_n \in [-1,1]} \left[C_1\left(\sum_{i=1}^{n} |s_i|^{\alpha}\, \mu^k(B_i)\right)^{\frac{1}{\alpha}}\right]^{\min(1,p)}$$

$$\leq \sup_{s_1,\ldots,s_n \in [-1,1]} \left\| \sum_{i=1}^{n} s_i M^k(B_i)\right\|_p^{\min(1,p)}$$

$$\leq \left[C_2\left(\sum_{i=1}^{n} \mu^k(B_i)\right)^{\frac{1}{q}}\right]^{\min(1,p)} . \qquad (3.14)$$

Thus, taking the supremum over all finite sequences (s_k) with $|s_k| \leq 1$ for all k, and over all finite partitions of $B \in \mathcal{B}_k$, we obtain, from inequalities (3.14),

$$C_1(\mu^k(B))^{\frac{1}{\alpha}\min(1,p)} \leq \|M^k\|(B) \leq C_2(\mu^k(B))^{\frac{1}{q}\min(1,p)} ,$$

for some constants C_1 and C_2 which depend only on r, α, k, p, q, $\mu[0, 1]$, and Γ. ■

3.1.4 Definition. Let X be a Banach space.

(i) For any $B \in \mathcal{B}_k$ and any $\mathcal{B}_k$-measurable X-valued simple function f on Δ_k, we define

$$\int_B f\, dM^k = \sum_{j=1}^{n} x_j\, M^k(B_j \cap B) \qquad (3.15)$$

where $B_1,\dots,B_n$ are disjoint elements of $\mathcal{B}_k$, $x_1,\dots,x_n \in X$, and $f = \sum_{j=1}^{n} x_j \chi_{B_j}$. The integrals $\int_B f\, dM^k_{\alpha,\beta}$ and $\int_B f\, dM^k_0$ are defined similarly.

(ii) Let f be an X-valued $\mathcal{B}_k$-measurable function on Δ_k. We say that f is M^k-integrable if there exists a sequence $\{f_i\}_{i=1}^{\infty}$ of X-valued $\mathcal{B}_k$-measurable simple functions on Δ_k such that

$$f_j \to f \quad \text{in measure } \mu^k \text{ , as } j \to \infty ,$$

and such that, for any $B \in \mathcal{B}_k$, the sequence

$$\{\int_B f_j \, dM^k\}_{j=1}^{\infty} \quad \text{converges in } L_0(X) ;$$

we define the integral $\int_B f\, dM^k$ as the $L_0(X)$-limit of the sequence $\{\int_B f_j\, dM^k\}_{j=1}^{\infty}$. Similarly, we define the M^k_0-integrability and $M^k_{\alpha,\beta}$-integrability of an X-valued $\mathcal{B}_k$-measurable function on Δ_k and, in the case of an integrable function, the corresponding integral.

Before showing that the above integrals are well defined, we show that for each $B \in \mathcal{B}_k$ and for any sequence $\{f_j\}_{j=1}^{\infty}$ of $\mathcal{B}_k$-measurable X-valued simple functions, the convergence of $\{\int_B f_j\, dM^k\}_{j=1}^{\infty}$ in $L_0(X)$ is equivalent to its convergence in $L_p(X)$, $0 < p < \alpha$. We recall from Proposition 3.1.2 that $C^0 \in MPZ(p)$ for $0 \le p < \alpha$, where $C = \{I_k(f) : f \in S_{k,X}\}$. Hence it is sufficient to show that the integral $\int_B f_j \, dM^k \in C^0$ for any $B \in \mathcal{B}_k$ and for any $\mathcal{B}_k$-measurable simple function f_j. This will be established in Proposition 3.1.5.

<u>3.1.5 Proposition.</u> Let X be a Banach space and let $C = \{I_k(f) \colon f \in \mathcal{S}_{k,X}\}$. Then $xM^k(B) \in C^0$ for any $x \in X$ and $B \in \mathcal{B}_k$, where C^0 is the $L_0(X)$ closure of C.

<u>Proof</u>. Let $x \in X$ and let $\mathcal{G} = \{B \in \mathcal{B}_k \colon xM^k(B) \in C^0\}$. We prove that $\mathcal{G} = \mathcal{B}_k$ by showing that $\mathcal{G}$ is a monotone class (see Ash [1, p. 19]) and that $\mathcal{G} \supseteq \bar{C}_k$. Clearly,

$$\{xM^k(A) \colon A \in C_k\} = \{I_k(x\chi_A) \colon A \in C_k\} \subset C \subset C^0$$

and hence, $C_k \subset \mathcal{G}$. Let $\{A_j\}_{j=1}^{\infty} \subset \mathcal{G}$ be an increasing sequence of sets such that $A = \bigcup_{j=1}^{\infty} A_j$. Since M^k is a countably additive $L_p(\mathbb{R})$-valued vector measure on $\mathcal{B}_k$, we have that the sequence $\{M^k(A_j)\}_{j=1}^{\infty}$ converges to $M^k(A)$ in $L_p(\mathbb{R})$, $0 < p < \alpha$. Hence the sequence $\{xM^k(A_j)\}_{j=1}^{\infty}$ converges to $xM^k(A)$ in probability; consequently $x\,M^k(A) \in C^0$ and $A \in \mathcal{G}$. Since $C_k \subset \mathcal{G}$, using Proposition 1.4.4 to find, for any $A \in \bar{C}_k$, an increasing sequence $\{A_j\}_{j=1}^{\infty} \subset C_k$ such that $A = \bigcup_{j=1}^{\infty} A_j$, we now have that $A \in \mathcal{G}$. Finally, we show that $\mathcal{G}$ is closed under complementation.

Let $A \in \mathcal{G}$. Since $A \in \mathcal{G}$ and $\Delta_k \in \bar{C}_k \subset \mathcal{G}$, there exist sequences $\{I_k(f_j)\}_{j=1}^{\infty}$ and $\{I_k(g_j)\}_{j=1}^{\infty} \subset C$, which converge in probability to $xM^k(\Delta_k)$ and to $xM^k(A)$, respectively. Now the sequence $\{I_k(f_j - g_j)\}_{j=1}^{\infty} \subset C$ and

$$I_k(f_j - g_j) = I_k(f_j) - I_k(g_j) \xrightarrow{P} xM^k(\Delta_k) - xM^k(A)$$

$$= xM^k(\Delta_k \setminus A)$$

as $j \to \infty$. Thus $\Delta_k \setminus A \in G$ if $A \in G$. Thus, we have shown that G is a monotone class containing $\bar{C}_k$ and hence, by the Monotone Class Theorem (see Ash [1, p. 19]), we conclude that $G = B_k$. ■

3.1.6. Proposition. For each M^k-integrable function f, and each $B \in B_k$, the integral $\int_B f\, dM^k$ is well defined.

We first prove this proposition when f is a real valued B_k-measurable function on Δ_k, and then we prove this when f is a Banach valued B_k-measurable function on Δ_k. The proof of this proposition when f is real valued is taken from Dunford and Schwartz [6, p. 324].

Proof of Proposition 3.1.6. If f is real valued, let $\{f_j\}_{j=1}^{\infty}$ and $\{g_j\}_{j=1}^{\infty}$ be two sequences of real valued B_k-measurable simple functions such that $\{f_j\}_{j=1}^{\infty}$ and $\{g_j\}_{j=1}^{\infty}$ converge to f in measure μ^k and such that for each $B \in B_k$, the sequences $\{\int_B f_j\, dM^k\}_{j=1}^{\infty}$ and $\{\int_B g_j\, dM^k\}_{j=1}^{\infty}$ converge in $L_0(\mathbb{R})$ (and equivalently, by Propositions 3.1.2 and 3.1.5 in $L_p(\mathbb{R})$ for $0 < p < \alpha$). For each j, we define $\nu_j(B) = \int_B h_j\, dM^k$ for every $B \in B_k$, where $h_j = f_j - g_j$. We show that for each $B \in B_k$, $\{\nu_j(B)\}_{j=1}^{\infty}$ converges to zero in $L_p(\mathbb{R})$, $0 < p < \alpha$. Since M^k is a μ^k-continuous $L_p(\mathbb{R})$-valued vector measure on B_k and h_j is a simple function, ν_j is an $L_p(\mathbb{R})$-valued μ^k-continuous vector measure on B_k. Also, we know that the

$L_p(X)$-limit $\lim_{j\to\infty} \nu_j(B)$ exists for every $B \in \mathcal{B}_k$. Hence, by the Vitali-Hahn-Saks Theorem (see Dunford and Schwartz [6, p. 158]), we have $\lim_{\mu^k(B)\to 0} \nu_j(B) = 0$, uniformly for $j = 1, 2, \ldots$. Thus, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, if

$$\mu^k(A) < \delta \text{ , then } \left\|\int_A h_j \, dM^k\right\|_p^{\min(1,p)} < \varepsilon \tag{3.16}$$

for all j. Since $h_j \xrightarrow{\mu^k} 0$ as $j \to \infty$, there exists $N \in \mathbb{N}$ such that $\mu^k\{s: |h_j(s)| > \varepsilon\} < \delta$ for all $j \geq N$. Now

$$\begin{aligned} &\left\|\int_B h_j \, dM^k\right\|_p^{\min(1,p)} \\ &\quad \leq \left\|\int_{B\setminus\{s:|h_j(s)|>\varepsilon\}} h_j \, dM^k\right\|_p^{\min(1,p)} \\ &\qquad + \left\|\int_{B\cap\{s:|h_j(s)|>\varepsilon\}} h_j \, dM^k\right\|_p^{\min(1,p)} \end{aligned} \tag{3.17}$$

Since h_j is a real valued simple function and $|h_j(s)| \leq \varepsilon$ for all $s \in B\setminus\{s: |h_j(s)| > \varepsilon\}$, we know that

$$\begin{aligned} &\left\|\int_{B\setminus\{s:|h_j(s)|>\varepsilon\}} h_j \, dM^k\right\|_p^{\min(1,p)} \\ &\qquad \leq \varepsilon^{\min(1,p)} \|M^k\| (B\setminus\{s: |h_j(s)| > \varepsilon\}) \end{aligned} \tag{3.18}$$

Therefore, from (3.16), (3.17), and (3.18), we have

$$\left\|\int_B h_j \, dM^k\right\|_p^{\min(1,p)} \leq \varepsilon^{\min(1,p)} \|M^k\| \, (B \setminus \{s: |h_j(s)| > \varepsilon\}) + \varepsilon$$
$$\leq C_2(\varepsilon^q \mu^k(B))^{\frac{\min(1,p)}{q}} + \varepsilon \, .$$

Hence, for each $B \in \mathcal{B}_k$, $\lim_{j\to\infty} \nu_j(B) = 0$ in $L_p(\mathbb{R})$.

When f is Banach valued it suffices to show that if $\{f_j\}_{j=1}^{\infty}$ is a sequence of X-valued $\mathcal{B}_k$-measurable simple functions such that $f_j \to 0$ in measure μ^k and, for a fixed $B \in \mathcal{B}_k$, $\int_B f_j \, dM^k \to Z_B$ in $L_0(X)$ as $j \to \infty$, then $Z_B = 0$ a.s. Since each f_j is a simple function, we have

$$x^*\left(\int_B f_j \, dM^k\right) = \int_B (x^* \cdot f_j) \, dM^k \in L_0(\mathbb{R})$$

for any $x^* \in X^*$, the dual of X. Since $\{\int_B f_j \, dM^k\}_{j=1}^{\infty}$ converges to Z_B in $L_0(X)$, there exists a subsequence $\{\int_B f_{j_\ell} \, dM^k\}_{\ell=1}^{\infty}$ converging to Z_B a.s., and hence for each $x^* \in X^*$, we have that $x^*(\int_B f_{j_\ell} \, dM^k)(\omega) \to (x^* \cdot Z_B)(\omega)$ for almost all ω as $\ell \to \infty$. Therefore the sequence $\{\int_B (x^* \cdot f_{j_\ell}) \, dM^k\}_{\ell=1}^{\infty}$ converges to $x^* \cdot Z_B$ in $L_0(\mathbb{R})$. Since $f_{j_\ell} \xrightarrow{\mu_k} 0$ as $\ell \to \infty$, we note that the real valued sequence $\{x^* \cdot f_{j_\ell}\}_{\ell=1}^{\infty}$ converges to zero in measure μ^k for each $x^* \in X^*$. Thus by what we showed in the real-valued case, we have $x^* \cdot Z_B = 0$ off N_{x^*} with $P(N_{x^*}) = 0$. Now we show that $Z_B = 0$ a.s. Since the sequence $\{\int_B f_{j_\ell} \, dM^k\}_{\ell=1}^{\infty}$ converges to Z_B a.s., there exists a measurable set N_0 with $P(N_0) = 0$ such that

$$\{Z(\omega):\ \omega \in \Omega \setminus N_0\} \subset \overline{\bigcup_{\ell=1}^{\infty} \{(\int_B f_{j_\ell}\ dM^k)(\omega):\ \ \omega \in \Omega \setminus N_0\}}\ .$$

Since f_{j_ℓ} is a simple function for each ℓ , we have $\{(\int_B f_{j_\ell}\ dM^k)(\omega):\ \omega \in \Omega \setminus N_0\}$ is finite dimensional and hence $\bigcup_{\ell=1}^{\infty} \{(\int_B f_{j_\ell}\ dM^k)(\omega):\ \ \omega \in \Omega \setminus N_0\}$ is separable. Therefore, $\{Z(\omega):\ \ \omega \in \Omega \setminus N_0\}$ is separable.

Let $D = \{y_1, y_2, \ldots\}$ be a countable dense subset of $\{Z(\omega):\ \omega \in \Omega \setminus N_0\}$. By the Hahn-Banach Theorem, for each (nonzero) $y_j \in D$, there exists $y_j^* \in X^*$ with $\|y_j^*\| = 1$ and $y_j^*(y_j) = \|y_j\|$. If $\omega \notin N_0 \cup (\bigcup_j N_{y_j^*})$, then $y_j^*(Z_B(\omega)) = 0$ for $j = 1, 2, \ldots$. Suppose that $Z_B(\omega) \neq 0$. Then there exists a nonzero $y_j \in D$ with $\|Z_B(\omega) - y_j\| < \frac{\|Z_B(\omega)\|}{3}$. Therefore $y_j^*(y_j - Z_B(\omega)) = \|y_j\|$ and hence

$$\|y_j - Z_B(\omega)\| = \sup_{\|y^*\|=1} |y^*(y_j - Z_B(\omega))| \geq \|y_j\|\ .$$

Thus,

$$\frac{\|Z_B(\omega)\|}{3} > \|y_j - Z_B(\omega)\| \geq \|y_j\|\ .$$

This implies that

$$\frac{\|Z_B(\omega)\|}{3} > \|y_j - Z_B(\omega)\| \geq |\ \|y_j\| - \|Z_B(\omega)\|\ |$$

$$= \|Z_B(\omega)\| - \|y_j\|$$

$$> \|Z_B(\omega)\| - \frac{\|Z_B(\omega)\|}{3}$$

$$= \frac{2}{3}\|Z_B(\omega)\| ,$$

which is absurd. Hence, $Z_B = 0$ off $N_0 \cup [\bigcup_{j=1}^{\infty} N_{y_j^*}]$ with $P(N_0 \cup [\bigcup_{j=1}^{\infty} N_{y_j^*}]) = 0$. ■

Similarly, the integrals with respect to M_0^k and $M_{\alpha,\beta}^k$ are well defined for their respective integrable functions.

3.1.7 Remark. In Definition 3.1.4(ii), we can replace the sequence $\{f_j\}_{j=1}^{\infty}$ of $\mathcal{B}_k$-measurable simple functions by a sequence $\{g_j\}_{j=1}^{\infty}$ of C_k-measurable simple functions such that $g_j \xrightarrow{\mu^k} f$ as $j \to \infty$, and such that $\{\int_B g_j \, dM^k\}_{j=1}^{\infty}$ converges in $L_0(X)$.

Proof. Let $\{f_j\}$ be as in Definition 3.1.4(ii). Suppose that for each j , $f_j = \sum_{i=1}^{\ell_j} x_i^{(j)} \chi_{B_i^{(j)}}$, where $B_1^{(j)},\ldots, B_{\ell_j}^{(j)}$ are disjoint elements of $\mathcal{B}_k$ and $x_1^{(j)},\ldots, x_{\ell_j}^{(j)} \in X$. Let $q > \max(1, \alpha)$ and $0 < p < \alpha$. Now for each j , we will find $g_j \in \mathcal{S}_{k,X}$ such that

$$\|f_j - g_j\|_q^{\min(1,p)} < \frac{1}{j} \text{ and } \left\|\int_B f_j \, dM^k - \int_B g_j \, dM^k\right\|_p^{\min(1,p)} < C_2 \left(\frac{1}{j}\right)$$

for every $B \in \mathcal{B}_k$, where C_2 is the constant appearing in (3.9). By the Caratheodory Theorem and Proposition 1.4.4 there exists $A_i^{(j)} \in C_k$ such that

$$\mu^k(B_i^{(j)} \, \Delta \, A_i^{(j)})^{\frac{1}{q}\min(1,p)} < \frac{1}{2j \, \ell_j \max\limits_{1 \le i \le \ell_j} (\|x_i^{(j)}\|^p , \|x_i\|)}$$

for $i = 1,\dots,\ell_j$. For each j, we define $g_j = \sum_{i=1}^{\ell_j} x_i^{(j)} \chi_{A_i^{(j)}}$. Clearly $g_j \in \mathcal{S}_{k,X}$ and, by the triangle inequality, the property of the semivariation, and (3.9), it follows that for each $B \in \mathcal{B}_k$, we have

$$
\begin{aligned}
&\left\|\int_B f_j \, dM^k - \int_B g_j \, dM^k\right\|_p^{\min(1,p)} \\
&\quad = \left\|\sum_{i=1}^{\ell_j} x_i^{(j)} M^k(B \cap B_i^{(j)}) - \sum_{i=1}^{\ell_j} x_i^{(j)} M^k(B \cap A_i^{(j)})\right\|_p^{\min(1,p)} \\
&\quad = \left\|\sum_{i=1}^{\ell_j} x_i^{(j)} (M^k(B \cap B_i^{(j)}) - M^k(B \cap A_i^{(j)}))\right\|_p^{\min(1,p)} \\
&\quad = \left\|\sum_{i=1}^{\ell_j} x_i^{(j)} (M^k(B \cap B_i^{(j)} \setminus B_i \cap A_i^{(j)}) \right. \\
&\qquad\qquad \left. - M^k(B \cap A_i^{(j)} \setminus B \cap B_i^{(j)}))\right\|_p^{\min(1,p)} \\
&\quad \le \sum_{i=1}^{\ell_j} \|x_i^{(j)}\|^{\min(1,p)} \, \|M^k(B \cap (B_i^{(j)} \setminus A_i^{(j)}))\|_p^{\min(1,p)} \\
&\qquad\qquad + \sum_{i=1}^{\ell_j} \|x_i^{(j)}\|^{\min(1,p)} \, \|M^k(B \cap (A_i^{(j)} \setminus B_i^{(j)}))\|_p^{\min(1,p)} \\
&\quad \le \sum_{i=1}^{\ell_j} \|x_i^{(j)}\|^{\min(1,p)} \, \|M^k\| \, (B \cap (B_i^{(j)} \setminus A_i^{(j)})) \\
&\qquad\qquad + \sum_{i=1}^{\ell_j} \|x_i^{(j)}\|^{\min(1,p)} \, \|M^k\| \, (B \cap (A_i^{(j)} \setminus B_i^{(j)}))
\end{aligned}
$$

$$\leq C_2 \sum_{i=1}^{\ell_j} \|x_i^{(j)}\|^{\min(1,p)} \quad \mu^k(B \cap (B_i^{(j)} \setminus A_i^{(j)}))^{\frac{1}{q}\min(1,p)}$$

$$+ C_2 \sum_{i=1}^{\ell_j} \|x_i^{(j)}\|^{\min(1,p)} \quad \mu^k(B \cap (A_i^{(j)} \setminus B_i^{(j)}))^{\frac{1}{q}\min(1,p)}$$

$$\leq 2C_2 \sum_{i=1}^{\ell_j} \|x_i^{(j)}\|^{\min(1,p)} \quad \mu^k(B_i^{(j)} \,\Delta\, A_i^{(j)})^{\frac{1}{q}\min(1,p)}$$

$$\leq C_2 \left(\frac{1}{j}\right) .$$

Also,

$$\|f_j - g_j\|_q^{\min(1,p)}$$

$$= \left(\int_{\Delta_k} \left\| \sum_{i=1}^{\ell_j} x_i^{(j)} \chi_{B_i^{(j)}} - \sum_{i=1}^{\ell_j} x_i^{(j)} \chi_{A_i^{(j)}} \right\|^q d\mu^k\right)^{\frac{1}{q}\min(1,p)}$$

$$= \left(\int_{\Delta_k} \left\| \sum_{i=1}^{\ell_j} x_i^{(j)} \chi_{B_i^{(j)} \setminus A_i^{(j)}} - \sum_{i=1}^{\ell_j} x_i^{(j)} \chi_{A_i^{(j)} \setminus B_i^{(j)}} \right\|^q d\mu^k\right)^{\frac{1}{q}\min(1,p)}$$

$$\leq \Big[\sum_{i=1}^{\ell_j} \Big\{ \Big(\int_{\Delta_k} \|x_i^{(j)}\|^q \chi_{B_i^{(j)} \setminus A_i^{(j)}} \, d\mu^k\Big)^{\frac{1}{q}}$$

$$+ \Big(\int_{\Delta_k} \|x_i^{(j)}\|^q \chi_{A_i^{(j)} \setminus B_i^{(j)}} \, d\mu^k\Big)^{\frac{1}{q}} \Big\} \Big]^{\min(1,p)}$$

$$\leq \Big[2\Big(\sum_{i=1}^{\ell_j} \|x_i^{(j)}\|^q \mu^k(B_i^{(j)} \,\Delta\, A_i^{(j)})\Big)^{\frac{1}{q}} \Big]^{\min(1,p)}$$

$$\leq 2 \sum_{i=1}^{\ell_j} \|x_i^{(j)}\|^{\min(1,p)} \left(\mu^k(B_i^{(j)} \,\Delta\, A_i^{(j)})\right)^{\frac{1}{q}\min(1,p)} < \frac{1}{j} .$$

Now $g_j \xrightarrow{\mu^k} f$ as $j \to \infty$ since $f_j \xrightarrow{\mu^k} f$ as $j \to \infty$. In fact, for any $\delta > 0$, we have

$$\mu^k\{s: \|g_j(s) - f_j(s)\| > \tfrac{\delta}{2}\}$$

$$\leq \mu^k\{s: \|g_j(s) - f_j(s)\| > \tfrac{\delta}{2}\} + \mu^k\{s: \|f_j(s) - f(s)\| > \tfrac{\delta}{2}\} .$$

$$\leq \frac{\|g_j - f_j\|_q^q}{(\frac{\delta}{2})^q} + \mu^k\{s: \|f_j(s) - f(s)\| > \frac{\delta}{2} \}$$

$$\leq \left(\frac{1}{j}\right)^{\frac{q}{\min(1,p)}} \frac{1}{(\frac{\delta}{2})^q} + \mu^k\{s: \|f_j(s) - f(s)\| > \tfrac{\delta}{2}\} ,$$

and, since $f_j \xrightarrow{\mu^k} f$ as $j \to \infty$, we have that $g_j \xrightarrow{\mu^k} f$ as $j \to \infty$. Finally, we show that for any $B \in \mathcal{B}_k$, the convergence of $\{\int_B f_j \, dM^k\}_{j=1}^{\infty}$ in $L_0(X)$ (equivalently in $L_p(X)$), implies the convergence of $\{\int_B g_j \, dM^k\}_{j=1}^{\infty}$ in $L_0(X)$ (equivalently, in $L_p(X)$). Let $\varepsilon > 0$ be given. Since

$$\left\|\int_B g_j \, dM^k - \int_B g_\ell \, dM^k\right\|_p^{\min(1,p)}$$

$$\leq \left\|\int_B g_j \, dM^k - \int_B f_j \, dM^k\right\|_p^{\min(1,p)}$$

$$+ \left\|\int_B f_j \, dM^k - \int_B f_\ell \, dM^k\right\|_p^{\min(1,p)}$$

$$+ \left\| \int_B f_\ell \, dM^k - \int_B g_\ell \, dM^k \right\|_p^{\min(1,p)}$$

$$\leq \frac{1}{j} C_2 + \left\| \int_B f_j \, dM^k - \int_B f_\ell \, dM^k \right\|_p^{\min(1,p)} + \frac{1}{\ell} C_2$$

and $\{\int_B f_j \, dM^k\}_{j=1}^\infty$ converges in $L_p(X)$, there exists $N \in \mathbb{N}$ such that for any $j, \ell \geq N$ we have $\left\| \int_B g_j \, dM^k - \int_B g_\ell \, dM^k \right\|_p^{\min(1,p)} < \varepsilon$. Thus the sequence $\{\int_B g_j \, dM^k\}_{j=1}^\infty$ converges in $L_p(X)$ and hence in $L_0(X)$.

3.2. M^k-Integrability

In this section we prove two results. First, in Theorem 3.2.1 we show that the class of all M^k-integrable functions (or M_0^k-integrable functions) is the same as the class of $M_{\alpha,0}^k$-integrable functions. Second, using Theorem 3.2.1, we prove Theorem 3.2.2 which is an analogue of Theorem 5.5 of [13] in the case of M^k (and M_0^k) . This theorem states an equivalent but simpler condition for the M^k-integrability of a Banach valued $\mathcal{B}_k$-measurable function when the Banach space satisfies the Multilinear Contraction Principle.

3.2.1 Theorem. For any Banach space X , a $\mathcal{B}_k$-measurable function $f: \Delta_k \to X$ is M^k-integrable (or M_0^k-integrable) iff it is $M_{\alpha,0}^k$-integrable.

Proof. Let f be $M_{\alpha,0}^k$-integrable. By Remark 3.1.7, there exists a sequence $\{f_j\}_{j=1}^\infty \subset \mathcal{S}_{k,X}$ such that $f_j \xrightarrow{\mu^k} f$,

$\int_B f_j \, dM^k_{\alpha,0} \to \int_B f \, dM^k_{\alpha,0}$ in $L_0(X)$ and equivalently in $L_p(X)$, $0 < p < \alpha$, for every $B \in \mathcal{B}_k$. Suppose that f is not M^k-integrable. Then there exists a $B \in \mathcal{B}_k$ and an $\varepsilon > 0$ such that for any $N \in \mathbb{N}$ there exist $j_n, \ell_n > n$ such that

$$\left\| \int_B (f_{j_n} - f_{\ell_n}) \, dM^k \right\|_p \geq \varepsilon . \tag{3.19}$$

By the Caratheodory Theorem and Proposition 1.4.4 there exists a sequence $\{A_m\}_{m=1}^{\infty} \subset \mathcal{C}_k$ such that $\mu^k(B \,\Delta\, A_m) < \frac{1}{m}$ for $m = 1,2,\ldots$. For each fixed n , let $f_{j_n} - f_{\ell_n} = \sum_{i=1}^{q_n} x_i^{(n)} \chi_{A_i^{(n)}}$ where $A_1^{(n)},\ldots, A_{q_n}^{(n)} \in \mathcal{C}_k$ and $x_1^{(n)},\ldots, x_{q_n}^{(n)} \in X$. Therefore,

$$\left\| \int_B (f_{j_n} - f_{\ell_n}) \, dM^k \right\|_p = \left\| \sum_{i=1}^{q_n} x_i^{(n)} M^i(B \cap A_i^{(n)} \right\|_p$$

$$= \lim_{m\to\infty} \left\| \sum_{i=1}^{q_n} x_i^{(n)} M^k(A_m \cap A_\ell^{(n)}) \right\|_p$$

$$= \lim_{m\to\infty} \left\| \int_{A_m} (f_{j_n} - f_{\ell_n}) \, dM^k \right\|_p$$

and

$$\left\| \int_B (f_{j_n} - f_{\ell_n}) \, dM^k_{\alpha,0} \right\|_p = \lim_{m\to\infty} \left\| \int_{A_m} (f_{j_n} - f_{\ell_n}) \, dM^k_{\alpha,0} \right\|_p .$$

Since $f_{j_n} - f_{\ell_n} \in \mathcal{S}_{k,X}$ and $A_m \in \mathcal{C}_k$, we have, by the right hand side of (2.29),

$$C_2 \left\| \int_{A_m} (f_{j_n} - f_{\ell_n})\, dM^k_{\alpha,0} \right\|_p \geq \left\| \int_{A_m} (f_{j_n} - f_{\ell_n})\, dM^k \right\|_p .$$

Taking the limit as $m \to \infty$, we have

$$C_2 \left\| \int_B (f_{j_n} - f_{\ell_n})\, dM^k_{\alpha,0} \right\|_p \geq \left\| \int_B (f_{j_n} - f_{\ell_n})\, dM^k \right\|_p \geq \varepsilon .$$

Thus, there exists an $\varepsilon > 0$ such that for any n , there exist $j_n, \ell_n > n$ with

$$\left\| \int_B (f_{j_n} - f_{\ell_n})\, dM^k_{\alpha,0} \right\|_p \geq \frac{\varepsilon}{C_2}$$

which contradicts the fact that the sequence $\{\int_B f_j \, dM^k_{\alpha,0}\}_{j=1}^{\infty}$ converges in $L_p(X)$. Therefore f is M^k-integrable. Conversely, if f is M^k-integrable, using a similar argument as above and the left hand side of (2.29) we get that f is $M^k_{\alpha,0}$-integrable.

Similarly, we note that for $\alpha \in (0, 1) \cup (1, 2)$ then the class of M^k_0-integrable functions and the class of all $M^k_{\alpha,0}$-integrable functions coincide. We recall that M_0 is a symmetric r-SS(α) random measure. ■

3.2.2 Theorem. Let X be a Banach space satisfying the M.C.P. A $\mathcal{B}_k$-measurable function $f: \Delta_k \to X$ is M^k-integrable (respectively, M^k_0-integrable) iff there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{S}_{k,X}$ such that

(i) $f_n \to f$ in measure μ^k as $n \to \infty$,

and

(ii) $\{I_k(f_n)\}_{n=1}^{\infty}$ (respectively, $\{I^0_k(f_n\}_{n=1}^{\infty}$) is Cauchy in $L_0(X)$.

Proof. By Remark 3.1.7, it is enough to show the "only if" part of this theorem. So, let $\{f_n\}_{n=1}^{\infty} \subset S_{k,X}$ be such that $f_n \xrightarrow{\mu^k} f$ as $n \to \infty$ and such that the sequence $\{I_k(f_n)\}_{n=1}^{\infty}$ is Cauchy in $L_0(X)$. Equivalently, by Proposition 3.1.2(ii), $\{I_k(f_n)\}_{n=1}^{\infty}$ is Cauchy in $L_p(X)$, $0 < p < \alpha$. By Theorem 2.4.1 and Proposition 3.1.2(ii), it follows that $\{I_k(f_n)\}_{n=1}^{\infty}$ is Cauchy in $L_0(X)$ iff $\{I_k^{\alpha,0}(f_n)\}_{n=1}^{\infty}$ is Cauchy in $L_0(X)$. Thus, by Theorem 5.5 of [13], f is $M_{\alpha,0}^k$-integrable. Hence by Theorem 3.2.1, f is M^k-integrable. ■

Finally, we state from Krakowiak and Szulga [13] some facts about $M_{\alpha,0}^k$-integrable functions. Let $L_{M_{\alpha,0}^k}(X)$ denote the class of all $M_{\alpha,0}^k$-integrable functions. Then

$$L_{M_{\alpha,0}^k}(X) \subseteq L_\alpha(X)$$

when $0 < \alpha < 2$ and

$$\bigcup_{q>\alpha} L_q(X) \subseteq L_{M_{\alpha,0}^k}(X)$$

when X is of stable type α. We recall that a Banach space is X is of stable type α if for some constant $c > 0$ and some $p \in (0, \alpha)$,

$$\|\sum_j x_j \theta_j\|_p \le c(\sum_j \|x_j\|^\alpha)^{\frac{1}{\alpha}},$$

for every finite sequence $(x_j) \subset X$, where $(\theta_1, \theta_2, \ldots)$ is a sequence of independent α-stable random variables.

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