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Assessing the Economic Tradeoffs Between Prevention and Suppression of Forest Fires

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To the Graduate Council:

I am submitting herewith a dissertation written by Elizabeth Trulia Heines entitled "Assessing the Economic Tradeoffs Between Prevention and Suppression of Forest Fires." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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Assessing the Economic Tradeoffs Between Prevention and Suppression of Forest Fires

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To my family and Chris Mesic.

Your love, support, and encouragement will never be forgotten.

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Abstract

The number of large-scale, high-severity forest fires occurring in the United States is increasing, as is the cost to suppress these fires. These trends have prompted investigations into alternative fuels methods to help prevent these large wildfires. One of the key challenges in studying the costs and benefits of forest fire prevention management is the incorporation of risk and uncertainty surrounding management decisions. We use a technique developed by William Reed to incorporate the stochasticity of the time of a forest fire into our optimal control problems. The goal of these problems is to determine the optimal fire prevention management spending rate and the optimal fire suppression spending which maximizes the expected value of a forest. Using these optimal control problems we explore the potential tradeoffs between prevention management spending and suppression spending, along with the overall economic viability of prevention management spending. The first optimal control problem we develop assumes that the effects of prevention management spending are instantaneous. We develop two parameter sets reflecting the 2011 Las Conchas Fire in New Mexico and 2014 Happy Camp Fire Complex in California and numerically solve our optimal control problem. For this problem, we perform a parameter sensitivity analysis to rank our parameters based on their impact on the value of a forest and the mean optimal prevention management spending rate. Furthermore, we adapt our optimal control problem so that it may be applied successively to simulate a sequence of fires. We perform a simulation study to determine how, on average, prevention management spending affects the value of a forest given an unknown number of fires over a fixed management horizon. The second optimal control problem we develop allows for the effects of prevention management spending to accumulate over time. We consider the numerical results and compare them to our first optimal control problem. Overall, our results support the conclusion that the

prevention management efforts offset rising suppression costs and increase the value of a forest overall. This work showcases a valuable tool which can guide forest managers and policymakers in their development of forest fire management plans.

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Chapter 1

Introduction

1.1 Background

The number of large-scale, high-severity forest fires occurring in the United States is increasing. From 2010 through 2016 there have been 180 fires in the U.S. that each burned more than 40,000 acres and from 1997 through 2016 there have been 177 fires in the U.S. that each burned more than 100,000 acres [33]. Despite a decreasing trend for the total number of fires occurring each year, the total number of acres being burned each year is increasing; see Figure 1.1 [33]. This suggests that fires are larger and more severe, on average. In 2015 alone, more than 65,000 wildfires burned over 10 million acres in total, this being 145% of the national 10-year average for total acres burned. Of these wildfires, approximately 1,050 were considered large or significant; that is, 1,050 fires burned either more than 300 acres of grassland and/or shrubland or more than 100 acres of timberland. In 2015, there were 44 fires that burned over 40,000 acres each [33]. Calkin et al. conclude that only 1% of wildfires account for 97.5% of the total number of acres burned [8].

In addition to increasingly large fires, the cost to suppress, contain, and extinguish these fires is increasing; see Figure 1.1. Twelve years between 2000 and 2015 saw total federal fire suppression costs over $1 billion, compared with no such years between 1985 and 1999. In particular, in 2015 federal fire suppression costs were over $2 billion [33]. In constant 2009 dollars, the ten year average for federal suppression expenditures has more than doubled from 1990-1999 to 2000-2009 [14]. One explanation for the increase in fire suppression

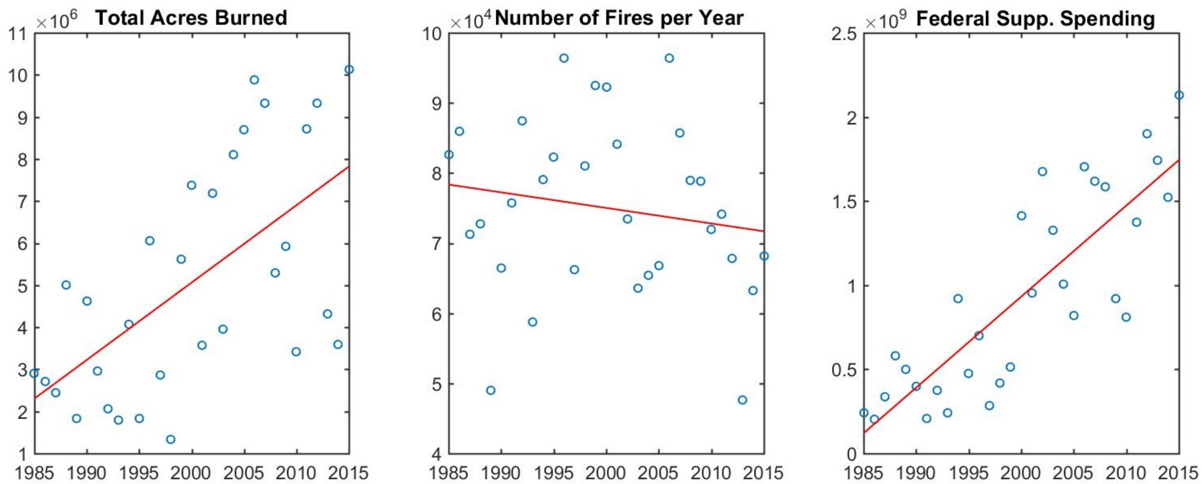


Figure 1.1: The plots above contain data from 1985 to 2015 on the total number of acres burned in wildfires, the total number of fires per year, and the total amount spent on federal fire suppression efforts. The data were taken from the National Interagency Fire Center [33].

costs includes decades of fire suppression and exclusion policies which have resulted in uncharacteristically continuous and dense forests with more ladder fuels [8]. In the past century there has been an active fire exclusion effort in the United States,; this means that wildland fires have not been allowed to burn despite the history and relationship of fire to a given ecosystem or region. As a result, some ecosystems have been significantly altered, leading to more continuous landscapes which support devastating severe fire events [16, 2]. In particular, fire-adapted ecosystems, where low-intensity surface fires were a common occurrence and were regenerative, now experience high-severity, stand-replacing fires where most of the trees are killed [16]. Other explanations for the increase in fire suppression costs include an expanding wildland-urban interface (WUI), prolonged drought (climate change), and the lack of financial accountability for fire managers [8].

These trends of increasing wildfire size and increasing federal suppression costs have prompted investigations into alternative methods to help prevent and manage these large wildfires. One such alternative is fuels management, defined in the USDA Forest Service Manual as the "practice of controlling flammability and reducing resistance to control of wildland fuels through mechanical, chemical, biological or manual means, or by fire, in support of land management objectives," [29]. In particular, this may include fire prevention management actions such as mechanical thinning, prescribed fire, etc. The Wildland Fire

Strategic Plan: 2015-2019 put forth by the National Park Service emphasizes the importance of determining areas where fuels management treatments are needed and the importance of developing appropriate programs to address these treatments [45]. According to Rummer *et al.* roughly 67 millions acres of forest have left their natural fire regime and are in need of some form of fuels management [43].

It is not feasible to experimentally test the impact of fuels treatments on suppression costs on a large scale[2, 50]. However, empirical evidence for the efficacy of fuels treatments to reduce fire hazard and the size of a fire has been observed following several past large fire events [2]. Even though the body of evidence in favor of fuels management is growing, fire suppression spending still outweighs expenditures on hazardous fuels reduction [18]. In fact, in extreme fire years emergency funding for fire suppression has been appropriated from funds designated for fuels management programs [48].

Despite the need for fuels management alternatives and despite the growing body of evidence in their favor, they have seen limited adoption. Constraints surrounding smoke, endangered species, regulatory review, and lack of societal acceptance inhibit timely implementation of fuel management strategies [48, 44]. Furthermore, there has been limited economic analysis concerning the viability of such fire prevention management strategies [17, 25, 29]. In particular, Mercer et al. stress that, "Two of the most important unanswered economic questions are whether the resources expended to reduce wildfire risk result in net economic gains and how to quantify the trade-offs between increasing expenditures on suppression and fuels management," [29]. The work within this dissertation aims to investigate and increase understanding of the trade-offs between fire prevention management spending and fire suppression spending on the overall net value of the forest.

1.2 Economic Prevention Management Models

Due to the dramatically increasing costs of fire suppression and the need for fire prevention management, researchers are turning to economics to inform wildland fire prevention management plans [31]. Here, we briefly summarize some methods and results found within the literature.

Mercer *et al.* use dynamic stochastic programming and a Monte Carlo simulation model to test the impact of alternative prescribed burning applications on the overall welfare of a forest [30]. They conclude that net economic gains may result from extensive prescribed burning in a specified area. However, this model is only applied to one specific area, does not account for ecosystems differences, is scale dependent, and thus, is not easily broadly applied [30]. In a different study, a standard-response model is modified and linear-integer optimization is used to examine the trade-offs between fuels management alternatives and initial wildfire suppression attack resource deployment. They find that a combination of the two would produce a global optimum for overall societal welfare [29]. Minas *et al.* (2015) present an integer programming model which fully integrates fuel treatment and fire suppression planning. They find that this integrated performance model outperforms models concerning initial attack coverage outcomes that consider fuel management and fire suppression in isolation [32].

This is a very complex subject as fire itself is very dynamic and has many spatial and temporal interdependencies. There have been many studies focusing on modeling fire behavior and spread. While we acknowledge the work done in these areas, it is not the focus of the work presented here. In a literature review of economic studies exploring the cost and benefits of wildland fires and their management, Milne *et al.* found that one of the key challenges in these studies is the incorporation of risk and uncertainty surrounding management decisions [31]. This is one of the key challenges we address in our work. In particular, we determine fire prevention management spending and suppression spending for a forest under threat of fire, given that the time of fire is unknown. We use techniques developed by William J. Reed in the late 1980s and early 1990s which convert an initially stochastic problem, due to the random time of an event, into a deterministic optimal control problem [42]. This enables us to address the issue of the inclusion of risk and uncertainty in economic fire models. In the subsection below, we summarize the method used by Reed to convert a stochastic control problem into a deterministic optimal control problem and we provide a brief overview of some other papers that utilize this method.

1.2.1 Optimal Control Applied to Resource Management Models with Risk of Catastrophic Collapse

In the late 1980s and early 1990s Reed wrote a series of papers exploring resource management modeling where the resource in question was vulnerable to random catastrophic collapse [38, 39, 40, 41, 9]. In these papers, a particular biological resource is under threat of collapse and the probability of collapse is taken to be a function of the state of the resource and the control measures being applied to the resource. Given that the time of the collapse of the biological resource is random, the problem being considered is stochastic. Reed developed a method to convert a problem with a single stochastic element, the time of collapse, into a deterministic optimal control problem where Pontryagin's Maximum Principle may be utilized [42]. We refer to this as Reed's method. A summary of this method, along with its different applications, are found in [42]. Next, we briefly outline Reed's Method, tailored slightly to more closely represent our particular application of it.

Consider the finite time horizon $[0, T]$ and suppose that we wish to maximize the net present value of a particular biological resource under threat of catastrophic collapse. Let $x(t)$ represent the stock of the biological resource under threat of collapse and suppose that the dynamics of the stock are governed by the differential equation

$$x'(t) = f\big(t, x(t), u(t)\big) \text{ with } x(0) = x_0, \tag{1.1}$$

where $u(t)$ is the function representing some control measure being taken on the resource stock x. Suppose that the collapse of the resource happens at time $\tau \in [0, T]$ and that the net present value of the resource up until the time of collapse is given by

$$\int_0^\tau g\big(t, x(t), u(t)\big)e^{-rt}dt. \tag{1.2}$$

Furthermore, suppose that the net present value of the resource from the time of collapse τ up to the end of the time horizon T are given by the function

$$e^{-r\tau}G\big(\tau, x(\tau), u(\tau)\big). \tag{1.3}$$

Let the time of collapse τ be a random variable taking values in $[0, T]$ where $\tau = T$ indicates that the time of the collapse of the resource occurs outside of the time horizon. We characterize the time of collapse random variable using a hazard function ψ defined by

$$\psi(t) = \lim_{\Delta t \to 0} \left\{ \frac{P(\text{resource collapses in } (t, t+\Delta t) | \text{alive at } t)}{\Delta t} \right\}. \tag{1.4}$$

The hazard function represents the conditional probability that the resource will collapse at time t given that it has not collapsed up to time t. While in this overview we simply write the hazard as a function of time, it can be chosen to be a function of the other state and control variables being used in the optimal control problem. The hazard function is related to the survivor function, which gives the probability that the resource survives until time t, in the following way:

$$S(t) = e^{-\int_0^t \psi(z) dz}. \tag{1.5}$$

Note that $S(0) = 1$. Moreover, by treating the time of collapse as a mixed-type random variable with a discrete component at the end of our time horizon T, we can use the survivor function to build its cumulative distribution function:

$$F(\tau) = \begin{cases} 1 - S(\tau) & \text{if } \tau < T \\ 1 & \text{if } \tau = T. \end{cases} \tag{1.6}$$

Let the net present value of the biological resource be given by

$$V(\tau, x, u) = \begin{cases} \int_0^\tau g(t, x, u) e^{-rt} dt + e^{-r\tau} G\big(\tau, x(\tau), u(\tau)\big) & \text{if } \tau < T \\ \int_0^\tau g(t, x, u) e^{-rt} dt & \text{if } \tau = T. \end{cases} \tag{1.7}$$

The expectation with respect the time of collapse random variable of $V(\tau, x, u)$ gives the expected net present value of the resource over the time horizon $[0, T]$ for a control u:

$$J(u) = E_{\mathcal{T}}\big\{V(\tau, x, u)\big\}, \tag{1.8}$$

subject to (1.1). Following a bit of calculus, we are able to write the expected net present value of the biological resource as

$$J(u) = \int_0^T \Big[g(t,x,u) + \psi(t)G(t,x,u)\Big]S(t)e^{-rt}dt, \tag{1.9}$$

subject to the state differential equation (1.1) and survivor function (1.5). Full details for going from (1.8) to (1.9) are included in the derivation of our own optimal control problem. Next, to complete the conversion of this stochastic problem to a deterministic optimal control problem, we introduce a new state variable y to represent the cumulative hazard over time:

$$y(t) = \int_0^t \psi(z)dz. \tag{1.10}$$

Thus, we can rewrite the survivor function as

$$S(t) = e^{-y(t)}. \tag{1.11}$$

The dynamics of the state variable y are thus given by

$$y'(t) = \psi(t) \text{ with } y(0) = 0, \tag{1.12}$$

where the initial condition follows from the fact that $S(0) = 1$. With the introduction of the state variable y we can now write down our deterministic optimal control problem maximizing the expected net present value of the biological resource:

$$\max_{u\in U} \int_0^T \Big[g(t,x,u) + \psi(t)G(t,x,u)\Big]e^{-rt-y(t)}dt \tag{1.13}$$

$$\text{subject to } x'(t) = f(t,x,u) \text{ with } x(0) = x_0 \tag{1.14}$$

$$y'(t) = \psi(t) \text{ with } y(0) = 0 \tag{1.15}$$

$$\text{where } U = \{u : [0,T] \to \mathbb{R} | u \text{ is Lebesgue measurable}\}. \tag{1.16}$$

As previously mentioned, Reed wrote a series of papers utilizing variations of this general technique, as have other researchers. The strength of this technique is its ability to incorporate the risk and uncertainty of the timing of a significant catastrophic event into resource management models, especially when there is a distinction between control management strategies employed before, during, or after the event. Investment in preventative management before a catastrophic event is often uncertain and risky because its benefits are not immediately realized and its effects may be difficult to quantify. Because of this, managers are often risk-averse and biased towards implementing control measures only after the event has taken place because the cost and benefits of these measures occur at approximately the same time [12]. Thus, the Reed method allows us to investigate and quantify how preventative management before a resource collapses may influence the timing of the event and the level of suppression management needed once the event has occurred. Next, we briefly describe some the different applications of this method.

Reed wrote a couple of papers investigating the optimal level of spending to protect an "on-going" commercial tree stand from fire and the optimal rotation age of the forest for harvest [39, 38]. The protection spending schedule before a fire is determined for a single fixed growth cycle of the stand. It is assumed that the stand will survive to some specified rotation age and will be cut and sold for profit, or the stand will be destroyed in its entirety by fire before the harvest age and no profits will be gained. An attempt was made to determine the protection schedule given an infinite number of cycles. However, due to the simplifications made, the solution can only be thought of as the protection spending over a single cycle with a payoff at the end of the cycle.

Finoff *et al.* applied Reed's method to an invasive species problem where the time of collapse of a biological resource represents the time of invasion [11]. They worked to determine the appropriate levels of prevention management before the invasion and levels of adaptive management and control once the invasive population had been established. They do so by considering an optimal control problem within an optimal control problem. That is, they considered the *ex post* problem, or the problem of maximizing net revenue from the forest after the invasion has occurred, to be its own optimal control problem. The solution to this problem was then fed into the *ex ante* problem, or the problem of maximizing the net revenue before the invasion, which upon application of the Reed method becomes its own optimal control problem. However, only some qualitative results concerning the levels of prevention and control are provided since explicit functional forms and parameters are not chosen for the problem.

Berry *et al.* and Horan *et al.* both applied Reed's method to study the management of emerging infectious diseases [7, 19]. Berry *et al.* considered the spread of influenza among human populations and allowed for the effects of prevention management to accumulate in a prevention stock over time and for the background hazard of an outbreak to increase over time. They concluded that prevention expenditures used to decrease the likelihood of an outbreak, which has an increasing background risk, are small in comparison to the potential losses that could occur in the face of a severe outbreak [7]. Horan's application of Reed's method to emerging infectious diseases was in the context of wildlife and livestock interactions. They concluded that while it is often optimal to manage pathogen risk before an outbreak, it is often too costly to try to fully eradicate the disease [19].

There are, of course, common elements throughout all of the applications of Reed's method listed above. However, due to the flexibility of Reed's method, there are many different variations that can be explored based on the context of the problem being considered, the initial setup, and assumptions being made. One of the main differences of our application of Reed's method to the rest is our successive application of the optimal control problem to study sequences of fires. This enables us to perform a simulation study allowing for a more comprehensive understanding of the effects of prevention management spending on the overall value of a forest. Many of the applications in the literature consider only one occurrence of the catastrophic event, despite the use of an infinite time horizon. We formulate our optimal control problem in a way so that we can consider the economic effects of prevention management spending on the value of the forest over a fixed amount of time for an unknown number of fires. Additionally, our method differs from others in that we explicitly determine a function describing the optimal value of a forest following a fire using scalar optimization. Moreover, this allows us to quantitatively examine the effects of prevention management spending and suppression spending on the overall economic value of a forest. By choosing functional forms and parameter ranges explicitly, we are also able to perform a parameter sensitivity analysis on one of our optimal control problems to determine which parameters have the most impact on the value of the forest and the mean optimal prevention management spending rate. To our knowledge this type of global sensitivity analysis has not been performed for other problems applying Reed's method.

1.3 Optimal Control Theory for Ordinary Differential Equations

Our goal is to examine the economic trade-offs between total prevention management spending before a fire and fire suppression spending at the time of fire. The Reed method of incorporating stochasticity into a problem in such a way that it can be converted into a deterministic optimal control problem allows us to incorporate the uncertainty surrounding the timing of forest fires directly into our problem. In this section we briefly highlight the fundamentals of optimal control theory for a single ordinary differential equation and one control.

In optimal control, we seek to maximize some goal through the control of a particular state variable. Let the variable $x(t)$ represent the state of the system we wish to influence by the control variable $u(t)$. The dynamics of the state x are governed by the ordinary differential equation

$$x'(t) = g\big(t, x(t), u(t)\big) \text{ with } x(0) = x_0. \tag{1.17}$$

Furthermore, suppose that the goal we are trying to maximize is affected by the time t, the state variable x, and the control variable u. This goal is typically represented by the integral of a function of these variables and is known as the objective functional:

$$\int_0^T f\big(t, x(t), u(t)\big)dt. \tag{1.18}$$

Thus, we aim to solve the optimal control problem

$$\max_{u \in U} \int_0^T f\big(t, x(t), u(t)\big)dt$$
$$\text{subject to } x'(t) = g\big(t, x(t), u(t)\big) \text{ with } x(0) = x_0, x(T) \text{ free.} \tag{1.19}$$

We can assume that the control set U contains either piecewise continuous or Lebesgue measurable functions. Our goal is to determine a control function in U which maximizes our objective functional subject to our state differential equation. We will denote this "optimal" control by u^* and the state corresponding to the optimal control by x^*.

In the 1950s, Pontryagin and his collaborators developed the first-order necessary conditions for solving optimal control problems, assuming the existence of an optimal control u^* and corresponding state variable x^*. In their most basic form, under appropriate differentiability conditions on f and g, these necessary conditions are found in Pontryagin's Maximum Principle [26]:

Theorem 1.1 (Pontryagin's Maximum Principle)**.** *If $u^*(t)$ and $x^*(t)$ are optimal for problem (1.19), then there exists an adjoint variable $\lambda(t)$ such that*

$$H\big(t, x^*(t), u(t), \lambda(t)\big) \leq H\big(t, x^*(t), u^*(t), \lambda(t)\big) \tag{1.20}$$

at each time t, where the Hamiltonian H is defined by

$$H\big(t, x(t), u(t), \lambda(t)\big) = f\big(t, x(t), u(t)\big) + \lambda(t)g\big(t, x(t), u(t)\big), \tag{1.21}$$

and the adjoint equation is determined by the differential equation

$$\lambda'(t) = -\frac{\partial H}{\partial x}, \tag{1.22}$$

with

$$\lambda(T) = 0. \tag{1.23}$$

The final time condition on the adjoint equation is known as the transversality condition. Thus, solving an optimal control problem, i.e. finding an optimal control u^* which maximizes the objective functional subject to the state variable x, is now a problem of maximizing the Hamiltonian function pointwise. The first-order necessary conditions to solve this problem include the optimality condition

$$\frac{\partial H}{\partial u} = 0 \text{ at } u^*, \tag{1.24}$$

and the adjoint equation

$$\lambda'(t) = -\frac{\partial H}{\partial x}, \tag{1.25}$$

with transversality condition

$$\lambda(T) = 0. \tag{1.26}$$

It is not always the case that the optimality condition (1.24) can be solved explicitly for a characterization of the optimal control u^*; in fact, we will that see this is the case with one of our optimal control problems. However, by Pontryagin's Maximum Principle it is still possible to numerically determine the optimal control by maximizing the Hamiltonian pointwise. The second-order condition

$$\frac{\partial^2 H}{\partial u^2} \leq 0, \tag{1.27}$$

should also be checked to confirm that the concavity of the Hamiltonian lends itself to a maximization problem. We quickly note that for a minimization problem the direction of the inequality is reversed. Additionally, Pontryagin's Maximum Principle also holds when there are bounds on the control, i.e.

$$a \leq u(t) \leq b. \tag{1.28}$$

In the case of a maximization problem, the optimal control is then characterized by

$$\begin{cases} u^* = a & \text{if } \frac{\partial H}{\partial u} < 0, \\ a \leq u^* \leq b & \text{if } \frac{\partial H}{\partial u} = 0, \\ u^* = b & \text{if } \frac{\partial H}{\partial u} > 0. \end{cases} \tag{1.29}$$

For a more complete introduction to optimal control with applications to biology, see [26, 5]; for applications in bioeconomics, see [22, 46, 10]. In the next section we outline the methods we use to numerically solve our optimal control problems.

1.4 Numerical Methods

Often, in application, analytical solutions for the state, control, and adjoint variables cannot be determined for an optimal control problem, and thus we must turn to numerical methods to approximate the solution. In this section, we briefly outline the steps needed to numerically approximate the solution to, or "solve", an optimal control problem [26]. Although our optimal control problem is built with continuous functions, in order to determine the numerical solution we must use vector approximations for our state, control, and adjoint variables. In the steps below, when referring to the state, control, and adjoint variables, we are referring to their vector approximations. The steps to solve our optimal control problem numerically are as follows:

1. Determine an initial guess for the optimal control u^*.
2. Solve the state differential equation $x'(t) = g(t, x, u)$ forward in time using the initial condition $x(0) = x_0$ and the current vector approximation for the optimal control.
3. Solve the adjoint differential equation backward in time using the transversality condition $\lambda(T) = 0$ and the current vector approximations for the state variable and optimal control.
4. Update the optimal control u^* using the new approximations for the state and adjoint variables. If the optimality condition could be explicitly solved for a characterization of u^*, then a convex combination of the approximation of the control from this current

iteration and the approximation from the previous iteration can be used to update u^*. If the optimality condition could not be used to determine a characterization for u^*, then the Hamiltonian can be maximized pointwise in order to create a new approximation for the control. Then a convex combination of this new control approximation and the approximation from the previous iteration are used to update u^*.

5. Check for the convergence of the state, adjoint, and control variables for a given tolerance. If the previous approximations for the variables and the new approximations for the variables are close enough to one another, then we output the current values of the variables as the solution. If convergence is not achieved, return to step 2.

Here, when we speak of achieving convergence of our variables, we are requiring the numerical approximations for a variable to be sufficiently close to the approximation of the variable from the previous iteration. Let v_{new} represent the current vector approximation for a single variable and let v_{old} represent the vector approximation for the same variable from the previous iteration. Thus, for convergence we require

$$\frac{||v_{new} - v_{old}||}{||v_{new}||} \leq \epsilon, \tag{1.30}$$

where ϵ is the accepted tolerance and $||\cdot||$ refers to the sum of the absolute values of the elements of the vector. Once this condition (1.30) is met by all variables, we output their current values as the numerical solution to our optimal control problem.

1.5 Overview

With the number of large, high-severity forest fires and the cost to suppress them on the rise in the United States, the economic viability of alternative fuels management programs needs to be evaluated. In particular, studies which jointly consider fire prevention management spending and fire suppression spending and incorporate the uncertainty surrounding the time of fire are needed. To address this, we develop two optimal control problems using Reed's method to investigate the trade-offs in prevention management spending and suppression spending and their effect on the overall expected net present value of the forest over a finite time horizon.

In Chapter 2, Sections 2 and 3, the initial formulation of our first optimal control problem is given in full detail and we derive the corresponding necessary conditions using Pontryagin's Maximum Principle. Once this is done, in Section 4 we build two parameter sets to investigate based on two recent fire events. We build parameter sets for the Las Conchas Fire which occurred in New Mexico in 2011 and for the Happy Camp Fire Complex which occurred in Northern California in 2014. We numerically approximate the solution to our optimal control problem in both cases and interpret the results. In Section 5 we perform a global parameter sensitivity analysis, using Latin Hypercube Sampling and partial rank correlation coefficients, to determine the parameters in our problem which have a significant impact on the value of our objective functional and the mean optimal prevention management spending rate. Once these parameters are determined, we take additional steps to rank them according to their level of impact on the output in question. In Section 6 we modify our optimal control problem so that we may consider the impact of prevention management spending on the value of a forest for an unknown sequence of fires over a fixed management horizon. We apply a slightly modified version of our optimal control problem multiple times in succession and sample for the times of fires using a cumulative distribution function built with the solution to our optimal control problem. Because the times of the fires are random, each trial is different. We perform a simulation study in order to determine the average value of the forest over our management horizon given that an unknown number of fires may occur, and look at the trade-offs between total prevention management spending and suppression spending.

In Chapter 3 we incorporate a new state variable representing cumulative prevention management stock into our optimal control problem. The reformulation of our problem is found in Section 2. In Section 3, we derive the necessary conditions and prove the existence of an optimal control. Following this proof, we consider some numerical results from our optimal control problem which now incorporates cumulative stock. Finally, we consider the limitations of our work and discuss some future extensions to be explored.

Chapter 2

Optimal Prevention Spending Strategies with Instantaneous Prevention Effects

2.1 Introduction

We wish to assess the economic trade-offs between fire prevention management spending and fire suppression efforts and their overall impact on the value of a forest under threat of fire. In particular, we want to incorporate the uncertainty surrounding the time of fire into our study as this is one of the key challenges in addressing and developing fire management strategies [31]. In order to do this, we turn to the method created by Reed to formulate a deterministic optimal control problem which includes the time of fire as a stochastic element. Our goal is to determine the optimal prevention management spending rate which will maximize the expected net present value of the forest over a finite time horizon.

In this chapter, we assume the effects of prevention management spending are instantaneous. We assume that prevention management spending at the time of fire will decrease the number of acres burned in the fire and that it will decrease the hazard of fire. Additionally, the fire event itself is taken to be instantaneous and therefore, only prevention management spending that occurs exactly at the time of fire will decrease the number of acres burned in the fire. Any prevention management spending before the time of fire does nothing to decrease the number of acres destroyed in the fire. We relax this assumption in Chapter 3.

2.2 Optimal Control Problem Formulation

Consider a forest with $\bar{A}$ acres over the finite time horizon $[0, T]$. Let $A(t)$ represent the number of unburned acres in the forest at any given time $0 \leq t \leq T$. Note $A(t) \leq \bar{A}$ for all $0 \leq t \leq T$. Suppose that the forest generates a flow of non-timber benefits B per unit time as a function of the number of unburned acres in the forest; that is, $B = B\big(A(t)\big)$. For now, suppose that the next large fire in the forest occurs at time τ with $0 < \tau < T$. We take the fire to be an instantaneous event. Prior to the fire, $t < \tau$, preventative management actions, such as prescribed fire or mechanical thinning, lower the likelihood of a large stand-replacing fire and decrease the severity of the fire. Let the cost of these prevention management actions over time be represented by the spending rate $h(t)$. Before the fire at time τ, the present value of the net revenue from the forest is given by

$$\int_0^{\tau} \big[B(\bar{A}) - h(t)\big] e^{-rt} dt, \tag{2.1}$$

where r is the discount rate. We assume that before the fire the forest is entirely unburned; i.e. for $t < \tau, A(t) = \bar{A}$.

The number of unburned acres destroyed in the fire, K, is assumed to be dependent on the *ex ante* prevention management spending rate at the time of the fire, $h(\tau)$, and the *ex post* fire suppression expenditures at the time of the fire, $x(\tau)$. That is,

$$K = K\big(h(\tau), x(\tau)\big). \tag{2.2}$$

It is reasonable to expect that prevention management spending in the time leading up to the fire will impact the number of the acres destroyed in the fire, K. However, at this point we are not considering any cumulative effects of prevention management spending. In our current problem, spending on prevention management up to the time of fire, $t < \tau$, does not have any effect on the number of acres destroyed in the fire. Only prevention management spending at the time of fire, $h(\tau)$, and suppression spending at the time of fire $x(\tau)$, affects the number of acres destroyed in the fire. In Chapter 3 we relax this restriction and explore cumulative effects of prevention management spending.

Furthermore, we assume that the number of forested acres burned in the fire, K, is decreasing with respect to increases in prevention management h and suppression spending x; i.e. $\frac{\partial K}{\partial h} < 0$ and $\frac{\partial K}{\partial x} < 0$. At the time of fire the number of unburned acres in the forest is instantaneously reduced by $K\big(h(\tau), x(\tau)\big)$ so that

$$A(\tau) = \bar{A} - K\big(h(\tau), x(\tau)\big). \tag{2.3}$$

Therefore, there will be a jump discontinuity in the number of the unburned acres A at the time of fire τ. We assume that another fire does not occur in our finite time horizon $[0, T]$ and that starting from the time of fire τ the number of unburned acres in the forest will increase according to the differential equation

$$A'(t) = \delta\big(\bar{A} - A(t)\big) \text{ with } A(\tau) = \bar{A} - K\big(h(\tau), x(\tau)\big), \tag{2.4}$$

so that $A(t)$ approaches $\bar{A}$ as time increases. Note that in application we are only concerned with determining the prevention management spending rate $h(t)$ up to the time of the first fire. For this reason we accept the assumption of at most one fire in $[0, T]$. We recognize the limitations of this assumption and in a later section in this chapter we consider sequences of fires in a fixed time interval.

As the fire event is taken to be instantaneous at time τ, so are the associated costs. The associated costs include the cost of suppressing the fire $x(\tau)$ and the cost of non-timber damages D. Non-timber damages account for the destruction of non-timber structures by the fire. These structures may include surrounding infrastructure such as buildings, roads, etc. The cost of non-timber damages D is assumed to be a function of the number of acres burned in the fire. That is,

$$D = D\Big(K\big(h(\tau), x(\tau)\big)\Big), \tag{2.5}$$

where the cost of damages is increasing with increasing K; that is, $\frac{\partial D}{\partial K} > 0$. In addition, we assume that the cost of damages is decreasing with respect to increases in prevention management spending and suppression spending, $\frac{\partial D}{\partial h} < 0$ and $\frac{\partial D}{\partial x} < 0$.

The function describing the flow of benefits B before and after the fire is the same. However, as the number of unburned acres before the fire is assumed constant, the flow of benefits before the fire will also be constant. After the fire, the number of unburned acres $A(t)$ is not constant, and thus the flow of benefits will vary accordingly. The net present value of the forest following a fire is given by the difference between the benefits accrued from the time of fire to the end of our time horizon and the instantaneous suppression and damage costs at the time of fire:

$$\int_{\tau}^{T} B\big(A(t)\big)e^{-rt}dt - \Big[D\Big(K\big(h(\tau), x(\tau)\big)\Big) + x(\tau)\Big]e^{-r\tau}, \tag{2.6}$$

subject to (2.4) and $x(\tau) \geq 0$. It is assumed that no prevention management actions are taken following a fire.

Let the *ex post* net value of the forest after the fire, with $e^{-r\tau}$ factored out, be defined by

$$JW\big(\tau, h(\tau), x(\tau)\big) = \int_{\tau}^{T} B\big(A(t)\big)e^{-r(t-\tau)}dt - \Big[D\Big(K\big(h(\tau), x(\tau)\big)\Big) + x(\tau)\Big]. \tag{2.7}$$

Given a time of fire τ and the corresponding prevention management spending rate $h(\tau)$ at that time, consider the following optimization problem

$$\begin{aligned} &\max_{x(\tau)} JW\big(\tau, h(\tau), x(\tau)\big) \\ &\quad \text{subject to} \\ &\quad A'(t) = \delta\big(\bar{A} - A(t)\big) \text{ with } A(\tau) = \bar{A} - K\big(h(\tau), x(\tau)\big) \\ &\quad x(\tau) \geq 0, \end{aligned} \tag{2.8}$$

with $x(\tau)$ being a real-valued scalar representing suppression spending for a given time of fire τ and the prevention management spending rate at the time of fire $h(\tau)$. Let $x^*(\tau)$ be the real-value scalar representing optimal suppression spending, for a given time of fire τ and prevention management spending rate $h(\tau)$ at the time of fire, which maximizes the value of the forest after the fire JW. The maximized value of the forest after the fire for a given τ and $h(\tau)$ is thus denoted by

$$JW^*\big(\tau, h(\tau)\big) = JW\big(\tau, h(\tau), x^*(\tau)\big). \tag{2.9}$$

The value of the forest following a fire JW is maximized when evaluated at $x^*(\tau)$. We note that JW^* can be positive or negative. However, we assume that increased prevention management expenditures h increases the *ex post* value of the forest; that is

$$\frac{\partial JW^*\big(\tau, h(\tau)\big)}{\partial h} > 0. \tag{2.10}$$

Once functional forms are chosen, we will explicitly determine $x^*(\tau)$ and $JW^*\big(\tau, h(\tau)\big)$ using scalar optimization techniques. The details surrounding this process are discussed later in this section.

Up to this point, for the purposes of problem formulation, we have assumed that a fire occurs within our finite time horizon $[0, T]$. However, this is not a necessary assumption as the time of the next fire τ could take place at any point in $(0, \infty)$. Now take $\tau \in (0, \infty)$ to be the time of fire. If the time of fire τ is less than T, then the total value of the forest over the time horizon $[0, T]$ is given by the sum of the net present value of the forest before the fire and the net present value of the forest after the fire up to time T,

$$\int_0^\tau \big[B(\bar{A}) - h(t)\big]e^{-rt}dt + \int_\tau^T B\big(A(t)\big)e^{-rt}dt - \Big[D\Big(K\big(h(\tau), x(\tau)\big)\Big) + x(\tau)\Big]e^{-r\tau}. \tag{2.11}$$

Note that this is the sum of (2.1) and (2.6) and it gives the value of the forest over the time horizon $[0, T]$.

If the time of the first fire τ is greater than or equal to T then we represent the value of the forest over the time horizon $[0, T]$ by

$$\int_0^T \left[B(\bar{A}) - h(t)\right] e^{-rt} dt. \tag{2.12}$$

In this case, we recognize that a fire will eventually occur, but because it does not occur within the interior of the time horizon $[0, T]$ we do not subtract the instantaneous suppression costs or costs from non-timber damages. Additionally, the number of unburned acres remains unchanged at $\bar{A}$ over $[0, T]$ in the case where the time of fire τ is not in our finite time horizon. We only consider the benefits B from the forest less the cost of prevention management spending h. In summary, the net present value of the forest over $[0, T]$ depends on the time of fire τ, the prevention management spending rate h, and the suppression spending $x(\tau)$ at the time of fire. It can be represented by the following piecewise function:

$$V\big(\tau, h, x(\tau)\big) = \begin{cases} \int_0^\tau \left[B(\bar{A}) - h(t)\right] e^{-rt} dt + \int_\tau^T B\big(A(t)\big) e^{-rt} dt \\ \quad - \Big[D\Big(K\big(h(\tau), x(\tau)\big)\Big) + x(\tau)\Big] e^{-r\tau} & \text{if } \tau < T, \\ \\ \int_0^T \left[B(\bar{A}) - h(t)\right] e^{-rt} dt & \text{if } \tau \geq T. \end{cases} \tag{2.13}$$

We can update this representation to incorporate the optimal value of the forest following the fire JW^*. Assuming an optimal value of the forest for $\tau \in [0, T]$ with optimal suppression $x^*(\tau)$ we rewrite (2.13). Recall that $e^{-r\tau}$ was factored out of (2.7) and hence, observe the discount factor in front of JW^*:

$$\mathcal{V}(\tau, h) = \begin{cases} \int_0^\tau \left[B(\bar{A}) - h(t)\right] e^{-rt} dt + e^{-r\tau} JW^*\big(\tau, h(\tau)\big) & \text{if } \tau < T \\ \\ \int_0^T \left[B(\bar{A}) - h(t)\right] e^{-rt} dt & \text{if } \tau \geq T. \end{cases} \tag{2.14}$$

The function $\mathcal{V}(\tau, h)$ represents the net present value of the forest over the time interval $[0, T]$ for a given time of fire τ and prevention management spending h. In the case that a fire happens within the time horizon, $\mathcal{V}$ incorporates the optimal net value of the forest following a fire $JW^*\big(\tau, h(\tau)\big)$.

When the large fire event will occur is unknown. Thus, the time of fire $\tau \in (0, \infty)$ represents an uncertainty in our system. To capture this uncertainty in our problem, we take the time of fire τ to be a realization of the random variable (RV) $\mathcal{T}$. The random variable is characterized by the hazard function ψ, defined as

$$\psi = \lim_{\Delta t \to 0} \left\{ \frac{Pr(\text{fire in } [t, t + \Delta t) | \text{no fire up to } t)}{\Delta t} \right\}. \tag{2.15}$$

The hazard function represents the conditional probability that a fire will occur at a time t given that no fire has occurred up to that time. For our problem, the hazard function is assumed to be a function of the *ex ante* prevention management spending rate,

$$\psi = \psi\big(h(t)\big). \tag{2.16}$$

Furthermore, we assume that the hazard is decreasing with respect to an increased prevention management spending rate, i.e. $\frac{\partial \psi}{\partial h} < 0$. A constant background hazard is assumed in the absence of *ex ante* prevention management spending.

The survivor function $S(t)$, which gives the probability of the forest surviving to time t with no fire, is related to the hazard function ψ in the following way:

$$S(t) = e^{-\int_0^t \psi\left(h(z)\right) dz}. \tag{2.17}$$

It follows that $S(0) = 1$, meaning the probability of surviving to time $t = 0$ is 1. While we assume that prevention spending can reduce hazard, we do not assume that prevention spending will indefinitely delay the occurrence of a large, stand-replacing fire. Therefore, we assume that the integral representing the cumulative hazard, $\int_0^t \psi\big(h(z)\big) dz$, will diverge to positive ∞ as $t \to \infty$, and thus $S(\infty) = 0$. The corresponding cumulative distribution function for $\mathcal{T}$ is related to the survivor function and is given by

$$F(\tau) = P(\mathcal{T} \le \tau) = 1 - S(\tau). \tag{2.18}$$

It follows that the probability density function for the time of fire RV $\mathcal{T}$ is given by

$$\begin{aligned} f(\tau) &= \frac{d}{d\tau}[1 - S(\tau)] \\ &= \psi\big(h(\tau)\big)S(\tau). \end{aligned} \tag{2.19}$$

Note that the random variable $\mathcal{T}$ can take values in $(0, \infty)$. However, as our problem assumes a finite time horizon, for the optimal control problem formulation the following mixed-type random variable, truncated at time T, is used:

$$\mathcal{T}_M = \begin{cases} \mathcal{T} & \text{if } \tau < T \\ T & \text{if } \tau \geq T. \end{cases} \tag{2.20}$$

As seen in (2.20) the mixed type random variable $\mathcal{T}_M$ takes values in $[0, T]$. It has the following distribution:

$$F_{\mathcal{T}_M}(\tau_M) = \begin{cases} 1 - S(\tau_M) & \text{if } \tau_M < T \\ 1 & \text{if } \tau_M = T. \end{cases} \tag{2.21}$$

Notice the potential for discontinuity at time T. Hence, we observe that the probability density function for $\mathcal{T}_M \in [0, T)$ is

$$f_{\mathcal{T}_M}(t) = \psi\big(h(t)\big)S(t). \tag{2.22}$$

Hence,

$$\begin{aligned} P(\mathcal{T}_M < T) &= \int_0^T \psi\big(h(t)\big)S(t)dt \\ &= \int_0^T \psi\big(h(t)\big)e^{-\int_0^t \psi(h(z))dz}dt \qquad (2.23) \\ &= 1 - S(T). \qquad (2.24) \end{aligned}$$

To move from (2.23) to (2.24), we use substitution along with the fact that $S(0) = 1$. The mixed type RV $\mathcal{T}_M$ has a discrete component. For $\mathcal{T}_M = T$, the probability mass is

$$\begin{aligned} P(\mathcal{T}_M = T) &= F_{\mathcal{T}_M}(T) - F_{\mathcal{T}_M}(T^-) \\ &= 1 - \big(1 - S(T)\big) \\ &= S(T). \end{aligned} \tag{2.25}$$

We can further confirm this by observing that

$$\begin{aligned} P(\mathcal{T}_M = T) &= P(\mathcal{T} \geq T) \\ &= \int_T^\infty \psi\big(h(t)\big) S(t) dt \\ &= \int_T^\infty \psi\big(h(t)\big) e^{-\int_0^t \psi\left(h(z)\right) dz} dt \\ &= S(T), \end{aligned} \tag{2.26}$$

as we assumed that $S(\infty) = 0$.

Earlier we saw that the present value of the net revenue from the forest, given a fire at time $\tau \in (0, \infty)$, is equal to (2.14). Treating the time of fire as a truncated random variable we can write (2.14) as

$$\mathcal{V}(\tau_M, h) = \begin{cases} \int_0^{\tau_M} \big[B(\bar{A}) - h(t)\big] e^{-rt} dt + JW^*\big(\tau_M, h(\tau_M)\big) e^{-r\tau_M} & \text{if } \tau_M < T \\ \int_0^T \big[B(\bar{A}) - h(t)\big] e^{-rt} dt & \text{if } \tau_M = T, \end{cases} \tag{2.27}$$

where τ_M is a realization of the mixed-type RV $\mathcal{T}_M$. Again, if $\tau_M = T$, no costs other than prevention management spending h are considered. Our goal is to determine the prevention management spending rate $h(t) \geq 0$ which maximizes the net present value of the forest over $[0, T]$ using deterministic optimal control. As written, our problem is currently stochastic. However, using techniques developed by Reed, we can convert this stochastic problem to deterministic by taking the expectation of (2.27) with respect to the RV $\mathcal{T}_M$ and introducing a state variable to represent cumulative hazard [42]. From here forward we drop the subscript M when discussing the mixed-type random variable for the time of fire for simplicity.

The expectation of (2.27), i.e. the expected net present value of the forest over $[0, T]$, is given by

$$\begin{aligned} J(h) &= E_{\mathcal{T}}\big\{\mathcal{V}(\tau, h)\big\} \\ &= \int_0^T \left[\int_0^\tau \big[B(\bar{A}) - h(t)\big] e^{-rt} dt + JW^*\big(\tau, h(\tau)\big) e^{-r\tau} \right] \psi\big(h(\tau)\big) S(\tau) d\tau \\ &\quad + S(T) \int_0^T \big[B(\bar{A}) - h(t)\big] e^{-rt} dt. \end{aligned} \tag{2.28}$$

To simplify this equation, first we distribute the integration in the first term across the sum,

$$\begin{aligned} J(h) = \int_0^T \int_0^\tau & \big[B(\bar{A}) - h(t)\big] \psi\big(h(\tau)\big) S(\tau) e^{-rt} dt d\tau \\ &+ \int_0^T JW^*\big(\tau, h(\tau)\big) \psi\big(h(\tau)\big) S(\tau) e^{-r\tau} d\tau \\ &+ S(T) \int_0^T \big[B(\bar{A}) - h(t)\big] e^{-rt} dt. \end{aligned} \tag{2.29}$$

Next, we switch the order of integration in the double integral contained in the first term of (2.29) to obtain:

$$\begin{aligned} J(h) = \int_0^T & \big[B(\bar{A}) - h(t)\big] e^{-rt} \left(\int_t^T \psi\big(h(\tau)\big) S(\tau) d\tau \right) dt \\ &+ \int_0^T JW^*\big(\tau, h(\tau)\big) \psi\big(h(\tau)\big) S(\tau) e^{-r\tau} d\tau \\ &+ S(T) \int_0^T \big[B(\bar{A}) - h(t)\big] e^{-rt} dt. \end{aligned} \tag{2.30}$$

Due to the relationship between the hazard function ψ and the survivor function S, see equation (2.17), the inner integral in the first term of (2.30) evaluates to $S(t) - S(T)$. Therefore,

$$\begin{aligned} J(h) = \int_0^T & \big[B(\bar{A}) - h(t)\big]\big(S(t) - S(T)\big)e^{-rt}dt \\ & + \int_0^T JW^*\big(\tau, h(\tau)\big)\psi\big(h(\tau)\big)S(\tau)e^{-r\tau}d\tau \\ & + S(T)\int_0^T \big[B(\bar{A}) - h(t)\big]e^{-rt}dt. \end{aligned} \tag{2.31}$$

After distributing in the first term in (2.31), we see that the terms containing $S(T)$ cancel one another and we arrive at

$$\begin{aligned} J(h) &= \int_0^T \big[B(\bar{A}) - h(t)\big]S(t)e^{-rt}dt + \int_0^T JW^*\big(t, h(t)\big)\psi\big(h(t)\big)S(t)e^{-rt}dt \\ &= \int_0^T \Big[B(\bar{A}) - h(t) + \psi\big(h(t)\big)JW^*\big(t, h(t)\big)\Big]S(t)e^{-rt}dt. \end{aligned} \tag{2.32}$$

This function, $J(h)$, represents the expected net present value of the forest over an interval $[0, T]$ subject to the survivor function $S(t)$. By introducing a new state variable y to represent cumulative hazard we complete the conversion our stochastic problem to deterministic. Let y represent cumulative hazard and be governed by the differential equation

$$y'(t) = \psi\big(h(t)\big) \text{ with } y(0) = 0. \tag{2.33}$$

The initial condition $y(0) = 0$ follows from the fact that $S(0) = 1$. Note that

$$y(t) = \int_0^t \psi\big(h(z)\big)dz. \tag{2.34}$$

Hence, the survivor function can be rewritten as

$$S(t) = e^{-y(t)}, \tag{2.35}$$

and this allows us to rewrite (2.32) with our new state variable y.

Our goal is to find a control h in our control set which maximizes the objective functional $J(h)$ with respect to the state variable y governed by differential equation (2.33). Therefore, our deterministic optimal control problem can be written as

$$\max_{h \in U} \int_0^T \Big[B(\bar{A}) - h(t) + \psi\big(h(t)\big) JW^*\big(t, h(t)\big)\Big] e^{-rt-y(t)} dt \tag{2.36}$$

$$\text{subject to } y'(t) = \psi\big(h(t)\big) \text{ with } y(0) = 0, \tag{2.37}$$

where

$$U = \big\{h : [0,T] \to [0,\infty) | h \text{ is piecewise continuous}\big\}. \tag{2.38}$$

Thus, our control problem with stochastic time of fire has been converted to a deterministic optimal control problem. Next, we choose our functional forms. Our choices allow us to explicitly determine the the optimal *ex post* value of the forest following a fire $JW^*\big(\tau, h(\tau)\big)$ and ultimately solve our optimal control problem by determining the optimal management spending $h(t)$ rate over $[0, T]$. Keep in mind that in application $h(t)$ is applied up to the time of the first fire or until the end of the time horizon, whichever comes first.

The benefits function B represents the flow of benefits from the forest and is a function of the number of unburned acres in the forest, $A(t)$. We assume that the flow of benefits is directly proportional to the number of unburned acres in the forest:

$$B\big(A(t)\big) = B_1 A(t), \tag{2.39}$$

where $B_1 \geq 0$ is a parameter representing the proportionality constant determining the flow of benefits from the forest. When the forest is completely unburned, i.e. all t such that $A(t) = \bar{A}$, the net flow of benefits from the forest is given by $B_1 \bar{A}$.

The function K represents the number of acres destroyed, or completely burned, by the fire at time τ. We use "destroyed" because the fire events we are interested in are high severity, stand-replacing fires. We are not considering low-severity fires because they are often regenerative for the ecosystem and are thus beneficial. This function is dependent on both prevention management and suppression expenditures:

$$K(h,x) = \frac{k}{(k_1 + h)(k_2 + x)}, \tag{2.40}$$

with parameters $k, k_1,$ and k_2 where $k > 0$ and $k_1, k_2 \geq 1$. The parameter k is related to the size of a fire. The parameter k_1 controls the magnitude of the effect of prevention management spending h on decreasing the number of acres killed. Similarly, the parameter k_2 controls the magnitude of the effect of suppression spending x on decreasing the number of acres burned. Furthermore, K is assumed to be decreasing with respect to prevention management expenditures h and suppression expenditures x, and we see that equation (2.40) satisfies these assumptions:

$$\frac{\partial K}{\partial x} = \frac{-k}{(k_1 + h)} \frac{1}{(k_2 + x)^2} < 0, \tag{2.41}$$

$$\frac{\partial K}{\partial h} = \frac{-k}{(k_2 + x)} \frac{1}{(k_1 + h)^2} < 0. \tag{2.42}$$

The non-timber damage function D accounts for the damages to non-timber structures caused by the fire. This may include surrounding buildings, roads, etc. It is assumed that the cost of non-timber damages is directly proportional to the number of acres destroyed in the fire:

$$\begin{aligned} D\big(K(h,x)\big) &= cK(h,x) \\ &= \frac{ck}{(k_1 + h)(k_2 + x)}. \end{aligned} \tag{2.43}$$

The parameter $c \geq 0$ represents the cost of damages in millions of dollars per thousand acres burned. Similar to the function representing the number of acres destroyed in the fire, K, increases in prevention management spending or in suppression spending will decrease the cost of non-timber damages, i.e. $\frac{\partial D}{\partial h} < 0$ and $\frac{\partial D}{\partial x} < 0$. See (2.41) and (2.42).

As described earlier, the hazard function ψ represents the conditional probability that a fire will occur at time t given that a fire has not occurred up to that time. The function chosen to represent the hazard,

$$\psi\big(h(t)\big) = be^{-vh(t)}, \tag{2.44}$$

is consistent with the literature [42, 38, 7, 12]. The parameter $0 < b < 1$ represents the background hazard, or the constant hazard rate when there is no prevention management spending. The constant $v > 0$ is used to control the effectiveness of preventative management spending $h(t)$ on reducing hazard. The condition that $\frac{\partial \psi}{\partial h} < 0$ is met by this function, as is the assumption that in the absence of prevention management spending the hazard is constant.

Within the *ex post* problem, following a fire at time τ, the number of unburned acres $A(t)$ in the forest will increase as time increases according to differential equation (2.4). Note that the regeneration of the forest is not dependent on *ex ante* prevention management expenditures h or *ex post* suppression expenditures x. This allows us to solve differential equation (2.4) using basic techniques. Thus,

$$A(t) = \bar{A} - K\big(h(\tau), x(\tau)\big)e^{-\delta(t-\tau)} \text{ for all } t \geq \tau. \tag{2.45}$$

Prior to the fire at time $t = \tau$, $A(t) = \bar{A}$.

Now that the functional forms have been defined and differential equation (2.4) solved, we optimize the value of the forest after the fire JW. Recall the *ex post* problem is to maximize

$$\begin{aligned} JW\big(\tau, h(\tau), x(\tau)\big) = \int_{\tau}^{T} B\big(A(t)\big)e^{-r(t-\tau)}dt - \Big[D\Big(K\big(h(\tau), x(\tau)\big)\Big) + x(\tau)\Big] \\ \text{subject to } x(\tau) \geq 0, \\ \text{where } A(t) = \bar{A} - K\big(h(\tau), x(\tau)\big)e^{-\delta(t-\tau)}, \end{aligned} \tag{2.46}$$

with respect to suppression spending at the time of fire, $x(\tau)$, for a given τ and $h(\tau)$. Using the functions defined above we can calculate an explicit form for $JW^*\big(\tau, h(\tau)\big)$. First, using the function for A (2.45) we integrate the flow of benefits with discounting over $[\tau, T]$, treating τ as a constant:

$$
\begin{aligned}
&\int_{\tau}^{T} B\big(A(t)\big)e^{-r(t-\tau)}dt \\
&\quad = e^{r\tau}\int_{\tau}^{T} B_1\big[\bar{A} - K\big(h(\tau), x(\tau)\big)e^{-\delta(t-\tau)}\big]e^{-rt}dt \\
&\quad = e^{r\tau}\left[B_1\bar{A}\int_{\tau}^{T} e^{-rt}dt - B_1K\big(h(\tau), x(\tau)\big)e^{\delta\tau}\int_{\tau}^{T} e^{-(\delta+r)t}dt\right] \\
&\quad = e^{r\tau}\left[B_1\bar{A}\left[\frac{-1}{r}e^{-rt}\right]_{t=\tau}^{t=T} - B_1K\big(h(\tau), x(\tau)\big)e^{\delta\tau}\left[\frac{-1}{\delta+r}e^{-(\delta+r)t}\right]_{t=\tau}^{t=T}\right] \\
&\quad = e^{r\tau}\left[\frac{B_1\bar{A}}{r}\Big(e^{-r\tau} - e^{-rT}\Big) - \frac{B_1K\big(h(\tau), x(\tau)\big)e^{\delta\tau}}{\delta+r}\Big(e^{-(\delta+r)\tau} - e^{-(\delta+r)T}\Big)\right] \\
&\quad = \frac{B_1\bar{A}}{r}\Big(1 - e^{-r(T-\tau)}\Big) - \frac{B_1K\big(h(\tau), x(\tau)\big)}{\delta+r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big).
\end{aligned} \tag{2.47}
$$

Hence, it follows that

$$
\begin{aligned}
JW\big(\tau, h(\tau), x(\tau)\big) &= \frac{B_1\bar{A}}{r}\Big(1 - e^{-r(T-\tau)}\Big) - \frac{B_1K\big(h(\tau), x(\tau)\big)}{\delta+r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big) \\
&\qquad - D\Big(K\big(h(\tau), x(\tau)\big)\Big) - x(\tau) \\
&= \frac{B_1\bar{A}}{r}\Big(1 - e^{-r(T-\tau)}\Big) \\
&\qquad - K\big(h(\tau), x(\tau)\big)\left[\frac{B_1}{\delta+r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big) + c\right] - x(\tau)
\end{aligned} \tag{2.48}
$$

The goal is to maximize $JW\big(\tau, h(\tau), x(\tau)\big)$ with respect to the instantaneous suppression costs $x(\tau)$ at the time of fire. We do this using scalar optimization and therefore we consider the partial derivative of (2.48) with respect to $x(\tau)$. It follows that

$$\begin{cases} x^*(\tau) = 0 & \text{if } \frac{\partial JW}{\partial x(\tau)} < 0, \\ x^*(\tau) \geq 0 & \text{if } \frac{\partial JW}{\partial x(\tau)} = 0. \end{cases} \tag{2.49}$$

As K is a function of x, the partial derivative of $JW\big(\tau, h(\tau), x(\tau)\big)$ with respect to $x(\tau)$ is,

$$\begin{aligned} \frac{\partial JW}{\partial x} &= -\left[\frac{B_1}{\delta + r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big) + c\right]\frac{\partial K}{\partial x} - 1 \\ &= \left[\frac{B_1}{\delta + r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big) + c\right]\frac{k}{(k_1 + h)(k_2 + x)^2} - 1. \end{aligned} \tag{2.50}$$

If $\frac{\partial JW}{\partial x(\tau)} = 0$, then $x^*(\tau) \geq 0$. To determine $x^*(\tau)$ in this case we take the partial derivative of JW with respect to $x(\tau)$, see (2.50) above, set the partial derivative equal to zero, and solve for $x(\tau)$:

$$0 = \left[\frac{B_1}{\delta + r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big) + c\right]\frac{k}{\big(k_1 + h(\tau)\big)\big(k_2 + x(\tau)\big)^2} - 1 \tag{2.51}$$

$$\big(k_2 + x(\tau)\big)^2 = \frac{k}{\big(k_1 + h(\tau)\big)}\left[\frac{B_1}{\delta + r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big) + c\right] \tag{2.52}$$

Taking the positive square root, as $x(\tau) \geq 0$, gives

$$0 \leq x^*(\tau) = x^*\big(\tau, h(\tau)\big) = \sqrt{\frac{k}{\big(k_1 + h(\tau)\big)}\left[\frac{B_1}{\delta + r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big) + c\right]} - k_2, \tag{2.53}$$

in the case that $\frac{\partial JW}{\partial x(\tau)} = 0$.

If $\frac{\partial JW}{\partial x(\tau)} < 0$ and $x^*(\tau) = 0$, then

$$\sqrt{\frac{k}{\big(k_1 + h(\tau)\big)}\left[\frac{B_1}{\delta + r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big) + c\right]} - k_2 < 0 = x^*(\tau). \tag{2.54}$$

Based on our choices for our functional forms, we see that optimal suppression spending x^* is not only a function of the time of fire, but also prevention management spending $h(\tau)$. Note that maximum of JW could occur at the endpoint. Thus, it follows that optimal suppression spending is given by

$$x^*\big(\tau, h(\tau)\big) = \max\left\{0, \sqrt{\frac{k}{\big(k_1 + h(\tau)\big)}\left[\frac{B_1}{\delta + r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big) + c\right]} - k_2\right\}. \tag{2.55}$$

Therefore, we substitute (2.55) into our JW representation (2.48) so that the optimal value of the forest following a fire is

$$JW^*\big(\tau, h(\tau)\big) = JW\Big(\tau, h(\tau), x^*\big(\tau, h(\tau)\big)\Big). \tag{2.56}$$

We note that a quick calculation shows $\frac{\partial^2 JW}{\partial x^2} \leq 0$ and so (2.53) is indeed a maximum of (2.48). Now that we have determined an explicit form for JW^* we can proceed with solving our optimal control problem. In the next section, we will derive the optimality system.

2.3 The Optimality System

Pontryagin's Maximum Principle (PMP) assumes the existence of an optimal control and the corresponding optimal state variables. Because of this, often the first step after formulating an optimal control problem is to prove the existence of an optimal control. However, due to nonlinearities in our problem this is not easily verified. Specifically, the hazard function $\psi\big(h(t)\big)$ and the optimal *ex post* value of the forest function $JW^*\big(t, h(t)\big)$ are highly nonlinear. We apply PMP despite this, given that the concavity of the Hamiltonian can be numerically verified for our choices of functions and parameters. First, we work through the derivation of the conditional current-value optimality system.

2.3.1 Derivation

The standard Hamiltonian for our optimal control problem (2.36) is given by

$$H = e^{-rt-y(t)}\Big[B(\bar{A}) - h(t) + \psi\big(h(t)\big)JW^*\big(t, h(t)\big)\Big] + \lambda(t)\psi\big(h(t)\big), \tag{2.57}$$

where $\lambda(t)$ is the adjoint function associated with the state variable y. The associated optimality condition uses the partial derivative of the Hamiltonian with respect to the control h,

$$\frac{\partial H}{\partial h} = e^{-rt-y(t)}\left[-1 + JW^*\big(t, h(t)\big)\frac{\partial \psi}{\partial h} + \psi\big(h(t)\big)\frac{\partial JW^*}{\partial h}\right] + \lambda(t)\frac{\partial \psi}{\partial h} = 0. \tag{2.58}$$

The adjoint differential equation and the transversality condition are

$$\begin{aligned} \lambda'(t) &= -\frac{\partial H}{\partial y} \\ &= e^{-rt-y(t)}\Big[B(\bar{A}) - h(t) + \psi\big(h(t)\big)JW^*\big(t, h(t)\big)\Big], \end{aligned} \tag{2.59}$$

with

$$\lambda(T) = 0. \tag{2.60}$$

In our objective functional we include the discount term e^{-rt} so that the value of the forest at each time is discounted back to time 0. In order to interpret our results in terms of current values (values at time t, not time $t = 0$) we use a current-value adjoint and a current-value Hamiltonian [23]. This also serves to simplify our problem.

Define $\mu(t) = \lambda(t)e^{rt}$ as the current-value adjoint and define the current value Hamiltonian $\mathbf{H}$ as

$$\begin{aligned} \mathbf{H} &= e^{rt}H \\ &= e^{-y(t)}\Big[B(\bar{A}) - h(t) + \psi\big(h(t)\big)JW^*\big(t, h(t)\big)\Big] + \mu(t)\psi\big(h(t)\big). \end{aligned} \tag{2.61}$$

The partial derivative of the current-value Hamiltonian with respect to the control is:

$$\frac{\partial \mathbf{H}}{\partial h} = e^{-y(t)}\big[-1 + JW^*\big(t, h(t)\big)\frac{\partial \psi}{\partial h} + \psi\big(h(t)\big)\frac{\partial JW^*}{\partial h}\big] + \mu(t)\frac{\partial \psi}{\partial h}. \tag{2.62}$$

When deriving the current-value adjoint differential equation and transversality condition, extra care must be taken. Observe that as $\mu(t) = e^{rt}\lambda(t)$,

$$\begin{aligned} \mu'(t) &= re^{rt}\lambda + e^{rt}\lambda'(t) \\ &= r\mu - e^{rt}\frac{\partial H}{\partial y} \\ &= r\mu - \frac{\partial(e^{rt}H)}{\partial y} \\ &= r\mu - \frac{\partial \mathbf{H}}{\partial y} \end{aligned} \tag{2.63}$$

Thus, the differential equation for the current-value adjoint equation is

$$\mu'(t) = r\mu + e^{-y(t)}\Big[B(\bar{A}) - h(t) + \psi\big(h(t)\big)JW^*\big(t, h(t)\big)\Big]. \tag{2.64}$$

It quickly follows that the current-value transversality condition is

$$\mu(T) = e^{rT}\lambda(T) = 0. \tag{2.65}$$

We can take this simplification technique one step further. The survivor function, written as $e^{-y(t)}$ in the objective functional, serves as a type of premium added to the discount factor [42]. In a similar fashion, we construct the *conditional* current-value Hamiltonian and adjoint equation to once again simplify our problem. Let the conditional current-value adjoint function be given by

$$\rho(t) = e^{y(t)}\mu(t). \tag{2.66}$$

Note that this new adjoint function is simply the current-value adjoint function divided by the survivor function. The conditional current-value Hamiltonian $\mathcal{H}$ is

$$\mathcal{H} = e^{y(t)}\mathbf{H} \tag{2.67}$$

$$= B(\bar{A}) - h(t) + \psi\big(h(t)\big)JW^*\big(t, h(t)\big) + \rho(t)\psi\big(h(t)\big). \tag{2.68}$$

The partial derivative of the conditional current value Hamiltonian $\mathcal{H}$ with respect to the control is

$$\frac{\partial \mathcal{H}}{\partial h} = -1 + JW^*\big(t, h(t)\big)\frac{\partial \psi}{\partial h} + \frac{\partial JW^*}{\partial h}\psi\big(h(t)\big) + \rho(t)\frac{\partial \psi}{\partial h}. \tag{2.69}$$

Once again, special care must be taken when deriving the conditional current-value adjoint differential equation. Since $\rho(t) = e^{y(t)}\mu(t)$, the differential equation for the conditional current value adjoint is given by

$$\rho'(t) = y'(t)\mu e^{y(t)} + \mu'(t)e^{y(t)} \tag{2.70}$$

$$= \psi\big(h(t)\big)\rho(t) + \mu'(t)e^{y(t)} \tag{2.71}$$

$$= \psi\big(h(t)\big)\rho(t) + \left(r\mu - \frac{\partial \mathbf{H}}{\partial y}\right)e^{y(t)} \tag{2.72}$$

$$= \Big(r + \psi\big(h(t)\big)\Big)\rho(t) - e^{y(t)}\frac{\partial \mathbf{H}}{\partial y} \tag{2.73}$$

Hence, the conditional current-value adjoint differential equation is

$$\rho'(t) = \Big(r + \psi\big(h(t)\big)\Big)\rho(t) + B(\bar{A}) - h(t) + \psi\big(h(t)\big)JW^*\big(t, h(t)\big), \tag{2.74}$$

with transversality condition

$$\rho(T) = e^{y(T)}\mu(T) = 0. \tag{2.75}$$

The hazard function $\psi\big(h(t)\big)$ is nonlinear in h, as is the function $JW^*\big(t, h(t)\big)$, which represents the optimal value of the forest following a forest fire for a given time of fire and a given amount of prevention management spending at the time of fire. Because of these nonlinearities we cannot calculate a closed form for the characterization of the optimal control from the partial derivative of $\mathcal{H}$ with respect to h (2.69). Instead, in order to determine the optimal control h^* we must turn to numerical methods. In particular, we utilize the fact that the PMP states that the optimal control maximizes the Hamiltonian with respect to the control h pointwise at each t to numerically determine the optimal control. Before moving to numerical results, we investigate the concavity of the conditional current-value Hamiltonian.

2.3.2 Concavity of Hamiltonian

Here, we justify the use of PMP for our maximization problem by examining the concavity of the conditional current-value Hamiltonian $\mathcal{H}$, given by equation (2.67), with respect to the control h. We are interested in showing that

$$\frac{\partial^2 \mathcal{H}}{\partial h^2} \leq 0, \tag{2.76}$$

as we are seeking to maximize our objective functional (2.36).

The first partial derivative of the conditional current-value Hamiltonian $\mathcal{H}$ with respect to the control is simply the equation given by (2.69). We differentiate (2.69) with respect to h to obtain the second partial derivative of $\mathcal{H}$ with respect to h:

$$\begin{aligned} \frac{\partial^2 \mathcal{H}}{\partial h^2} &= JW^*\big(t, h(t)\big)\frac{\partial^2 \psi}{\partial h^2} + \frac{\partial \psi}{\partial h}\frac{\partial JW^*}{\partial h} + \psi\big(h(t)\big)\frac{\partial^2 JW^*}{\partial h^2} + \frac{\partial JW^*}{\partial h}\frac{\partial \psi}{\partial h} + \rho\frac{\partial^2 \psi}{\partial h^2} \\ &= \frac{\partial^2 \psi}{\partial h^2}\big(JW^*\big(t, h(t)\big) + \rho(t)\big) + 2\frac{\partial \psi}{\partial h}\frac{\partial JW^*}{\partial h} + \psi\big(h(t)\big)\frac{\partial^2 JW^*}{\partial h^2}. \end{aligned} \tag{2.77}$$

Next, we examine the signs of the three terms in (2.77) to determine if $\frac{\partial^2 \mathcal{H}}{\partial h^2} \leq 0$. The signs of the partial derivatives of the functions $\psi\big(h(t)\big)$ and $JW^*\big(t, h(t)\big)$ with respect to the control h are calculated below.

Recall that the hazard function $\psi\big(h(t)\big)$ is given by (2.44). Hence, we see that $\psi\big(h(t)\big) > 0$. Furthermore, it is easily shown that its first and second partial derivatives with respect to the control $h(t)$ are

$$\frac{\partial \psi}{\partial h} = -vbe^{-vh(t)} = -v\psi < 0, \tag{2.78}$$

and

$$\frac{\partial^2 \psi}{\partial h^2} = v^2 be^{-vh(t)} = v^2 \psi > 0. \tag{2.79}$$

Now, we calculate the partial derivatives of $JW^*\big(t, h(t)\big)$. In our earlier work, we showed that optimal suppression spending x^* is a function of the time of fire and of the prevention spending at the time of fire. Hence,

$$\begin{aligned} JW^*\big(t, h(t)\big) &= JW\Big(t, h(t), x^*\big(t, h(t)\big)\Big) \\ &= \frac{B_1 \bar{A}}{r}\Big(1 - e^{-r(T-t)}\Big) - \frac{B_1 K\Big(h(t), x^*\big(t, h(t)\big)\Big)}{\delta + r}\Big(1 - e^{-(\delta+r)(T-t)}\Big) \\ &\quad - D\bigg(K\Big(h(t), x^*\big(t, h(t)\big)\Big)\bigg) - x^*\big(t, h(t)\big) \\ &= \frac{B_1 \bar{A}}{r}\Big(1 - e^{-r(T-t)}\Big) \\ &\quad - K\Big(h(t), x^*\big(t, h(t)\big)\Big)\left[\frac{B_1}{\delta + r}\Big(1 - e^{-(\delta+r)(T-t)}\Big) + c\right] - x^*\big(t, h(t)\big) \end{aligned} \tag{2.80}$$

To make our calculations a little simpler, define

$$\alpha(t) = \frac{B_1}{\delta + r}\Big(1 - e^{-(\delta+r)(T-t)}\Big) + c. \tag{2.81}$$

It easily follows that

$$\alpha(t) \geq 0, \tag{2.82}$$

since the parameters involved are all chosen to be be greater than or equal to 0. Now we can write $JW^*\big(t, h(t)\big)$ as

$$JW^*\big(t,h(t)\big) = \frac{B_1\bar{A}}{r}\Big(1-e^{-r(T-t)}\Big) - \alpha(t)K\Big(h(t),x^*\big(t,h(t)\big)\Big) - x^*\big(t,h(t)\big). \tag{2.83}$$

We can also simplify $x^*\big(t,h(t)\big)$ and $K\Big(h(t),x^*\big(t,h(t)\big)\Big)$ using $\alpha(t)$. Therefore, when $x^*>0$ we have

$$\begin{aligned} x^*\big(t,h(t)\big) &= \sqrt{\frac{k}{\big(k_1+h(t)\big)}\left[\frac{B_1}{\delta+r}\Big(1-e^{-(\delta+r)(T-t)}\Big)+c\right]}-1 \\ &= \sqrt{\frac{k\alpha(t)}{\big(k_1+h(t)\big)}}-1 \end{aligned} \tag{2.84}$$

Additionally, when $x^*>0$

$$\begin{aligned} K\Big(h(t),x^*\big(t,h(t)\big)\Big) &= \frac{k}{\big(k_1+h(t)\big)\Big(k_2+x^*\big(t,h(t)\big)\Big)} \\ &= \sqrt{\frac{k}{\alpha(t)\big(k_1+h(t)\big)}}. \end{aligned} \tag{2.85}$$

By examining the signs of the partial derivatives of $K\Big(h(t),x^*\big(t,h(t)\big)\Big)$ and x^* we can learn more about the signs of the partial derivatives of $JW^*\big(t,h(t)\big)$. The first and second partial derivatives of $x^*\big(t,h(t)\big)$, when $x^*>0$, are given by

$$\frac{\partial x^*}{\partial h} = -\frac{1}{2}\sqrt{\frac{k\alpha(t)}{\big(k_1+h(t)\big)^3}} < 0, \tag{2.86}$$

and

$$\frac{\partial^2 x^*}{\partial h^2} = \frac{3}{4}\sqrt{\frac{k\alpha(t)}{\big(k_1+h(t)\big)^5}} > 0. \tag{2.87}$$

The partial derivatives of the function K evaluated at $x^*\big(t,h(t)\big)$, when $x^*>0$, with respect to the control $h(t)$ are

$$\frac{\partial K\Big(h(t), x^*\big(t, h(t)\big)\Big)}{\partial h} = -\frac{1}{2}\sqrt{\frac{k}{\alpha(t)\big(k_1 + h(t)\big)^3}} < 0, \tag{2.88}$$

and

$$\frac{\partial^2 K\Big(h(t), x^*\big(t, h(t)\big)\Big)}{\partial h^2} = \frac{3}{4}\sqrt{\frac{k}{\alpha(t)\big(k_1 + h(t)\big)^5}} > 0. \tag{2.89}$$

With this information we can now quickly calculate the first and second partial derivatives of $JW^*\big(t, h(t)\big)$ with respect to h and determine their signs,

$$\frac{\partial JW^*}{\partial h} = -\alpha(t)\frac{\partial K\Big(h(t), x^*\big(t, h(t)\big)\Big)}{\partial h} - \frac{\partial x^*}{\partial h} \tag{2.90}$$

$$\frac{\partial^2 JW^*}{\partial h^2} = -\alpha(t)\frac{\partial^2 K\Big(h(t), x^*\big(t, h(t)\big)\Big)}{\partial^2 h} - \frac{\partial^2 x^*}{\partial h^2}. \tag{2.91}$$

Hence, it follows from (2.88) - (2.91) that

$$\frac{\partial JW^*}{\partial h} \geq 0 \tag{2.92}$$

and

$$\frac{\partial^2 JW^*}{\partial h^2} \leq 0. \tag{2.93}$$

Now, when considering $\frac{\partial^2 \mathcal{H}}{\partial h^2}$ given by (2.77), it is easily seen that the second term is negative as $\frac{\partial \psi}{\partial h} < 0$ and $\frac{\partial JW^*}{\partial h} \geq 0$. Similarly, the third term is negative as $\psi > 0$ and $\frac{\partial^2 JW^*}{\partial h^2} \leq 0$. The first term in (2.77) makes determining the sign of $\frac{\partial^2 \mathcal{H}}{\partial h^2}$ difficult. Using that $\frac{\partial^2 \psi}{\partial h^2} > 0$, if

$$JW^*\big(t, h(t)\big) + \rho(t) < 0 \tag{2.94}$$

then we automatically satisfy the concavity condition. That is, having

$$JW^* + \rho < 0 \tag{2.95}$$

is sufficient, but not necessary, to verify the concavity of the conditional current-value Hamiltonian $\mathcal{H}$. Unfortunately, we have not been able to determine the sign of this sum analytically, but we have had success in verifying (2.94) numerically for our given functions and parameter choices.

We also note that the concavity result also holds in the case that $x^*\big(t, h(t)\big) = 0$. Observe that when $x^* = 0$,

$$K(h(t), x^*(t, h(t))) = \frac{k}{\big(k_1 + h(t)\big)} > 0. \tag{2.96}$$

The first and second partial derivatives of K when $x^* = 0$ are

$$\frac{\partial K\Big(h(t), x^*\big(t, h(t)\big)\Big)}{\partial h} = -\frac{k}{\big(k_1 + h(t)\big)^2} < 0, \tag{2.97}$$

and

$$\frac{\partial^2 K\Big(h(t), x^*\big(t, h(t)\big)\Big)}{\partial h^2} = \frac{2k}{\big(k_1 + h(t)\big)^3} > 0. \tag{2.98}$$

The signs of the first and second partial derivatives of K with respect to h when $x^* = 0$ are the same as the signs of the first and second partial derivatives of K when $x^* > 0$. Thus, the conclusions reached concerning the concavity of the conditional current value Hamiltonian $\mathcal{H}$ for the case when $x^* > 0$ also hold for the case when $x^* = 0$. We offer the numerical verification of (2.76) as justification for our use of the PMP. Next, we examine some numerical results for our optimal control problem.

2.4 Numerical Results

Now that we have formulated our optimal control problem and the associated optimality system, we solve it numerically and interpret the results. However, we must first choose values to assign to the parameters in our problem. To do this, we examine information from two recent large fire events and use it to help us build a more realistic model. We consider only large fire events to guide our parameter choices for our optimal control problem because large fires contribute the most to destroyed wildland and suppression spending overall [8]. Moreover, detailed information on large fire events is more readily available. In particular, we use the 2011 Las Conchas Fire, which burned near Los Alamos, New Mexico, and the 2014 Happy Camp Fire Complex, which burned in Northern California, to build two parameter sets to examine and compare. These fires were chosen, not only because they were large and expensive, but also because they happened in different geographic locations in the United States. The forest compositions and recent fire history differ across these locations and thus illustrate the potential versatility of this model. We reiterate that we are using the data from these fires to build a more realistic problem, not to draw any retrospective conclusions concerning prevention management or suppression spending decisions made at the time of these fires. We begin with the Las Conchas Fire and follow with the Happy Camp Complex.

2.4.1 2011 Las Conchas Fire

A fallen power line ignited the Las Conchas Fire on June 26, 2011. Within 14 hours the fire had burned across 43,000 acres and destroyed dozens of homes [47]. The Las Conchas Fire continued to burn over the course of the summer through sections of Santa Fe National Forest, Bandelier National Monument, and Valles Caldera National Preserve near Los Alamos, New Mexico. The fire was finally contained at the beginning of August 2011 [35, 47]. Over 150,000 acres burned in the fire and over \$40 million were spent on fire suppression efforts [35, 47, 54]. In addition to suppression costs, over 110 structures were destroyed or damaged during the fire [35, 47]. Using this information, along with information concerning the forest type and fire history of the area, we build our first parameter set to examine.

The parameter $\bar{A}$ represents the “size of the forest” in units of thousands of acres. Interpreting this parameter can be ambiguous because the area through which a fire burns is not constrained by arbitrary intangible man-made boundaries. The Las Conchas Fire mainly burned through the Santa Fe National Forest, Bandelier National Monument, and Valles Caldera National Preserve and thus we take the value for $\bar{A}$ to be roughly the combined

Table 2.1: The table below includes the parameter values chosen to reflect the 2011 Las Conchas Fire.

Parameter	Units	Value	Justification
$\bar{A}$	acres(1000)	1700	size of SFNF, BNM, VCNP
r	/time	0.04	standard discount rate
k	acres(1000)×\$^{2}/time	7000	$k \approx$ size of fire × suppression \$
k_1	\$ (mil.)/time	1	assumed
k_2	\$ (mil.)/time	1	assumed
δ	/time	0.05	Pipo: 70-250 years to mature
b	———	0.2	high frequency of fires in region
c	\$ (mil.)/ acres(1000)	0.1	114 buildings destroyed, 156,000 acres burned
B_1	\$ (mil.)/time	0.02	calculated from x^* formula
v	———	1	assumed

size of these three regions[47]. The size of the Santa Fe National Forest is 1.6 million acres; the size of Bandelier National Monument is over 33,000 acres; the size of Valles Caldera National Preserve is over 89,000 acres [34, 53, 57]. Thus, the combined size of these three areas is approximately 1,722 thousand acres. Rounding down, we take $\bar{A} = 1,700$. In the remainder of this subsection, when we refer to "the forest", we are referring to the combined region of the Santa Fe National Forest, Bandelier National Monument, and Valles Caldera National Preserve.

The parameter δ represents the regeneration rate of the forest following a fire. When we refer to "burned" acres we are referring to acres in the forest which have been completely devastated by fire. When we refer to "unburned" acres we are referring to acres of forest which have not been devastated by fire. Thus, it is possible that some acres have experienced fire, but one with a lower intensity that did not destroy the stand. These low intensity fires are not considered in our problem. We assume that before the fire the forest is entirely unburned and remains so until the time of fire. Thus, there is no regeneration of the forest during this period. At the time of fire, the number of unburned acres in the forest is reduced instantaneously by $K\big(h(\tau), x(\tau)\big)$. From the time of fire τ until the end of the time horizon T, the number of unburned acres in the forest increases from $\bar{A} - K\big(h(\tau), x(\tau)\big)$ toward $\bar{A}$

according to the differential equation (2.4). In this equation, δ controls the rate at which the number of unburned acres increases. We choose δ based on the dominant tree type in the forest, which in the Santa Fe National Forest is Ponderosa Pine (Pipo). The age of Ponderosa Pine at maturity is 70-250 years [52]. Thus, if an acre of the forest is destroyed by the fire, we are assuming it will take 70-250 years until that acre will once again be considered unburned. As the regeneration of the forest after a fire is governed by equation (2.45) we can estimate an appropriate value for δ. Assuming that at the time of fire the number of unburned acres is reduced by half, we choose a value for δ so that the number of unburned acres after 100 years has approximately returned to $\bar{A}$. As such, we choose $\delta = 0.05$.

A common choice for a continuous discount rate is 0.04, and as such we also choose this for our problem: $r = 0.04$.

The parameter b, found in the hazard function $\psi\big(h(t)\big)$, represents the background fire hazard. Recall that

$$\psi\big(h(t)\big) = be^{-vh(t)}. \tag{2.99}$$

Thus, in the absence of prevention management spending the hazard function is constant at the value b; i.e. $\psi(h = 0) = b$. Often, for a region, the mean fire interval has been estimated. The mean fire interval is based on the long-term fire history for a given region and is estimated using dendrochronology techniques. [20, 13]. For the southwest region, the mean fire interval for the a Ponderosa Pine ecosystem is between 2-40 years [13, 3, 15]. However, historically, fires in these ecosystems were frequent, low-intensity fires which stayed on the ground. These ecosystems had evolved and adapted to this fire regime and so these fires were beneficial and regenerative for the ecosystems. The exclusion of fire in these regions, resulted in an uncharacteristic accumulation of ladder fuels, and has contributed to more high-severity, stand-replacing fires which ultimately devastate the forest. These are the types of fires we are interested in studying. As such, because the mean fire interval is based on the long-term historical fire regime of a region, we do not use it to determine the value for the parameter b. Instead, we examine the more recent fire history of the Los Alamos region to determine a value for b.

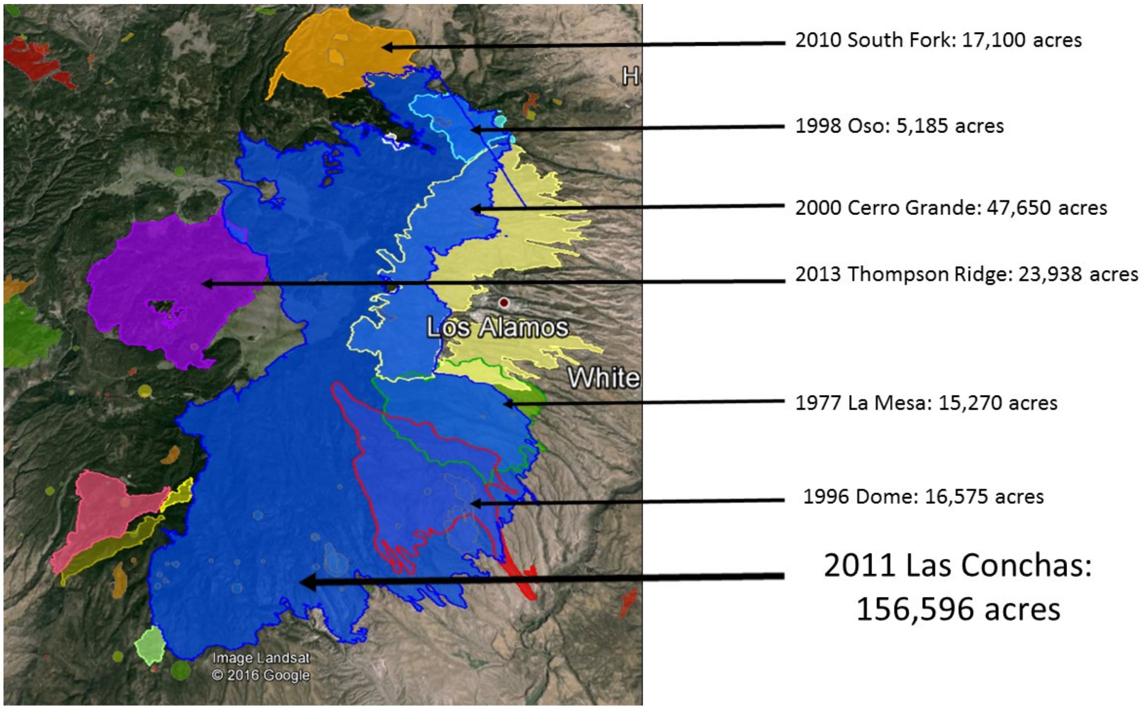


Figure 2.1: The figure above details some of the recent fire history around the area where the Las Conchas Fire took place. Selected fires are listed with the year in which they occurred and the number of acres burned. The map indicates the geographic area burned by each fire.

In Figure 2.1, we see that there have been several large fire events over the past 20 years near the area where the Las Conchas Fire burned. This figure does not represent a complete history of fire events in the Santa Fe National Forest. For a complete history, see [58]. In particular, we focus on the fact that in between 2000 and 2013, there were 4 large fire events. In 2000, the Cerro Grande Fire burned 47,650 acres. In 2010, the South Fork Fire burned 17,100 acres. In 2011, the Las Conchas Fire burned 156,593 acres and in 2013, the Thompson Ridge Fire burned 23,938 acres. So, on average, we say that a large fire might occur every 3 to 4 years. Thus, we want the value for the parameter b to reflect a high probability that a large fire will occur in a relatively short amount of time.

Recall that the survivor function $S(t)$ represents the probability that the forest will not have a large, stand-replacing fire (i.e. "survive") up to time t and is given by equation (2.17). In order to determine b, for simplicity we assume no prevention management spending, $h = 0$. Hence, the survivor function (2.17) in the case without prevention management spending is given by

$$S(t) = e^{-bt}. \tag{2.100}$$

To capture the probability that large, high-severity fires happen frequently in the region, we choose $b = 0.2$. For $b = 0.2$ the probability of the forest surviving to 3.5 years with no fire is approximately 0.5.

The parameters $k, k_1,$ and k_2 are all included in the function K which represents the number of acres burned in the instantaneous fire. Recall that this function is given by

$$K(h, x) = \frac{k}{(k_1 + h)(k_2 + x)}. \tag{2.101}$$

The parameter k_1 controls the strength of the impact that prevention management spending $h(\tau)$ at the time of fire will have on the reduction of acres destroyed in the fire. The parameter k_2 controls the strength of the impact that suppression spending $x(\tau)$ at the time of fire will have on the reduction of acres destroyed in the fire. Given that values for these parameters could not be estimated from the literature, for simplicity we choose $k_1 = k_2 = 1$. This choice also lends itself to a more straightforward interpretation for the the parameter k.

The parameter k, given $k_1 = k_2 = 1$, represents the number of acres that will be completely burned in the fire given that there is no prevention management spending or suppression spending at the time of the fire. When thought about this way, one might expect that the fire severity parameter k should be less than or equal to the number of unburned acres in the forest, $k \leq \bar{A}$. However, in the case of these large fire events it is very reasonable to expect that there will always be fire suppression efforts and thus fire suppression spending. Because we expect $x(\tau) > 0$, it is reasonable for the fire severity parameter k, which represents the number of acres destroyed when $h(\tau) = 0$ and $x(\tau) = 0$, to be greater than the number of acres in the forest. That is, it is reasonable to have $k \geq \bar{A}$.

In particular, given that we know the number of acres burned in the Las Conchas Fire and the amount spent on suppression, we can use the function K to estimate a value for the fire severity parameter k. Assuming that there is no prevention management spending at the time of fire, we have

$$K(x) = \frac{k}{k_2 + x}. \tag{2.102}$$

We have already chosen $k_2 = 1$. Thus $k \approx K \times (1 + x)$, where K represents the number of acres burned (in thousands) in the fire and x represents the amount spent on suppression (in millions). For the Las Conchas Fire, approximately 157 thousand acres were burned and over \$40M were spent on suppression. Thus, $k \approx 6,500$ and we round up and choose $k = 7,000$ given that estimates for suppression costs range between \$40M and \$50M.

The parameter v is found in the hazard function ψ. It represents the the effectiveness of prevention management spending on reducing hazard, with larger values of v indicating that prevention management spending is more effective at reducing hazard. In the literature, v is often taken to be 1 [39, 7]. Because of this, and lack of other criteria on which to base the value of this parameter, we also choose $v = 1$.

The parameter c represents the non-timber damages cost in millions of dollars per thousand acres burned. To determine the value of this parameter we use knowledge about the number of structures destroyed in the fire, the number of acres burned in the fire, and the median cost of homes in the area. In the Las Conchas Fire, 114 buildings were destroyed or damaged in the fire [47]. The median value of homes in the region ranges between \$100,000 and \$450,000 [51]. Recall that the function determining the cost of non-timber damages is given by

$$D = cK(h, x). \tag{2.103}$$

First, we estimate the cost of non-timber damages D for the Las Conchas Fire by multiplying the number of building destroyed by the median value of homes in the area. For this estimate, we take the median value of the buildings to be \$150,000 and the number of buildings destroyed or damaged to be 114. For the median value of homes, we choose near the bottom end of the range because we do not have specific information on the types of building damaged and destroyed or the extent of the damage inflicted. Our estimate for D is thus \$17.2 million. We then divide this estimate for D by the number of acres destroyed in the fire, $K = 157$, to estimate an appropriate value for c; $c \approx \frac{D}{K}$. For the Las Conchas Fire in particular, we round and take $c = 0.1$.

Lastly, we look to determine an appropriate value for the parameter B_1, which represents the flow of benefits from the forest given in units millions of dollars per thousand acres. The challenge of valuing a forest is very complex and is its own problem in and of itself [37]. Thus, as to not become too bogged down in this problem, we use our equation for optimal suppression spending (2.53) to determine a value for B_1 based on our other parameter choices and the amount of money spent on suppression for the Las Conchas Fire. Recall that in the case that $x^* > 0$:

$$x^*\big(\tau, h(\tau)\big) = \sqrt{\frac{k}{\big(k_1 + h(\tau)\big)}\left[\frac{B_1}{\delta + r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big) + c\right]} - k_2.$$

In order to determine a reasonable estimate for B_1, we assume that the amount of suppression spending was optimal and we allow this amount to stand in for x^*. We then solve equation (2.53) for B_1 and approximate its value using our previous parameter choices. Furthermore, we assume $h(\tau) = 0, \tau = 0,$ and $T = 500$. This leads to the choice of $B_1 = 0.02$ for the Las Conchas Fire. Choosing B_1 in this manner also insures that the amount spent on suppression x^*, in the case of no prevention management spending, is comparable to realistic large fire suppression costs.

We recognize that the selection of some of these parameter values is not literature driven and that we are making several assumptions and choices in their selection. Because of this, we perform a sensitivity analysis in a later section to determine which parameters have the most impact on the overall expected net present value of the forest and the mean optimal prevention management spending rate.

Now that we have chosen values for all 10 parameters within our optimal control problem, we can work toward determining a numerical solution. As was seen in our derivation of the optimality system, an analytical solution for the optimal management prevention spending h^* cannot be determined due to the nonlinearities in our problem. As such, we turn to numerical methods in order to solve our optimal control problem. In particular, because an explicit characterization of the optimal control h^* cannot be obtained, we will determine h^* by maximizing the conditional current-value Hamiltonian pointwise.

We use code developed in MATLAB to solve our optimal control problem. In the code, we first define our parameter values and then we create MATLAB functions for the hazard function ψ, the optimal suppression spending function x^*, and the function for the optimal value of the forest after a fire JW^*. We initialize two vectors for optimal suppression x^*, cumulative hazard y, management prevention spending h, and the adjoint equation ρ. One vector will store the "old" values for the variable and one vector will store the "updated" or current values for the variable. We update x^* using the equation (2.53). Keep in mind that x^* is a function of management spending h. Then, we update the cumulative hazard y using a simple MATLAB integration function to give the cumulative hazard at different time points in our time horizon. The adjoint differential equation for ρ is solved for using the MATLAB ordinary differential equation solver `ode45`. Given an updated x, y, and ρ, we numerically determine a new h by maximizing the Hamiltonian pointwise at each time step with respect to h using the MATLAB function `fminbnd`. Here, we choose the upper bound for this function to be very large so that there is effectively no bound on h, since we do not require this in our problem. For the lower bound required in the MATLAB function `fminbnd`, we use -0.01. After applying this function, we use a convex combination of our old and new h for our updated h. Next, we check for convergence of our variables by comparing the magnitude of the differences between the old and current variable vectors. If the difference between old and updated vectors across all variables is sufficiently small, convergence is achieved, and we are done. If not, we continue the updating process until convergence is achieved. Now that we have chosen parameter values and outlined the process used to numerically solve our optimal control problem, we will go through the results for this particular set of parameters related to the 2011 Las Conchas Fire.

Figure 2.2 contains some key results concerning the solution to our optimal control problem. In particular, the optimal prevention management spending h^*, the optimal suppression spending x^*, and the survivor function $S(t)$ are included. For comparison, we also include in the plots the case where zero dollars are spent on prevention management spending, $h = 0$. Before interpreting our results, we make a note about the time horizon T.

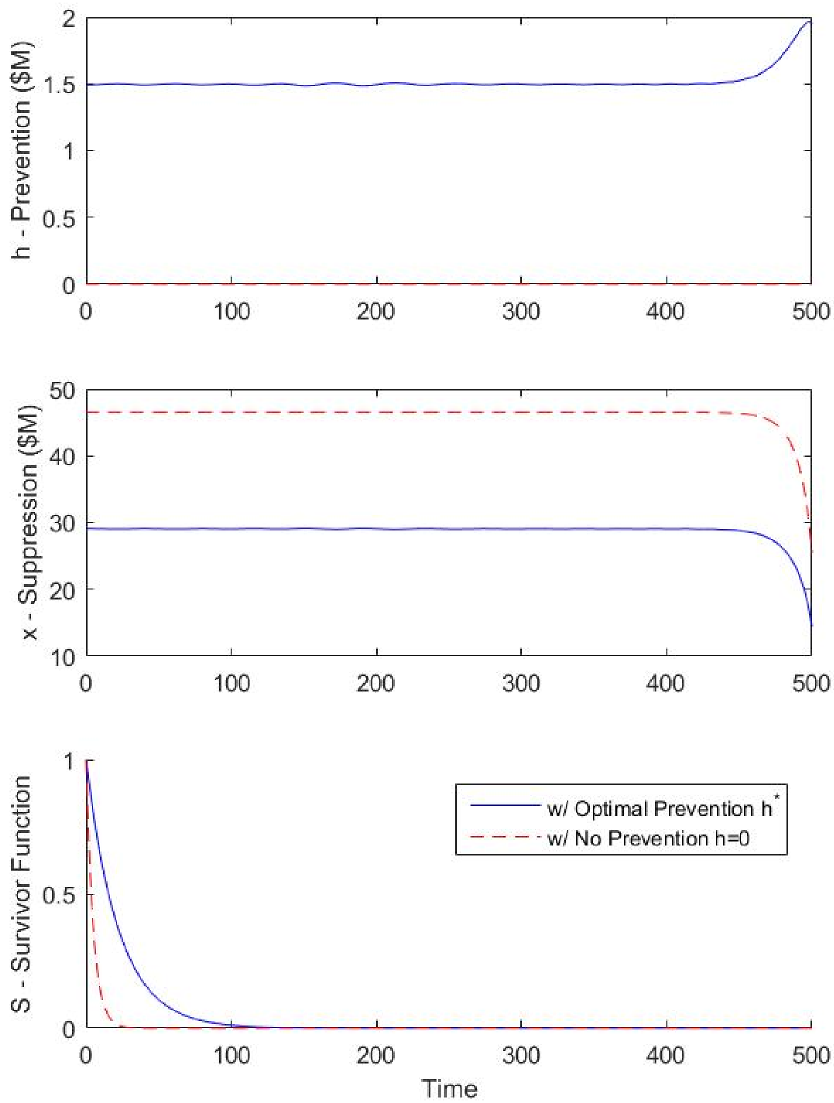


Figure 2.2: The plots above contain the results of our optimal control problem using the Las Conchas Fire parameter set. For comparison, in each plot we include the case with optimal prevention management spending h^* and the case with no prevention spending $h = 0$. The top plot gives the optimal prevention management spending h^* and no prevention spending $h = 0$. The middle plot contains the optimal suppression spending x^* and suppression spending when $h = 0$. The bottom plot gives the survivor function in the case with optimal prevention management spending h^* and in the case with no prevention spending $h = 0$.

We consider a time horizon of $T = 500$ years. We realize that in terms of application to management, 500 years may seem unrealistic, especially given that only one fire is allowed for in the entire time horizon. However, there are a couple of reasons that such a long time horizon is considered. First, we choose a long time horizon so that any tail effects from the finite time horizon can be reasonably ignored. The second reason for the long time horizon centers around how the results should be interpreted and applied. Our goal is to determine how money should be spent on prevention management given that the time of the next fire to occur is unknown. Thus, in a real-life scenario, the optimal management prevention spending given by h^* would only be applied up to the time of the first fire τ. By choosing T very large, we essentially "guarantee" that the time of the next fire will fall within our time horizon. This is validated by the survivor function which is essentially zero after 100 years in both the optimal prevention management spending case and the no prevention management spending case.

As can be seen in Figure 2.2, the optimal prevention management spending rate h^* is approximately constant at 1.5 million dollars per year over the course of the time horizon, with an increase to 2 million near the end of the time horizon. We believe the increase near the final time to be due to end effects and that it can be inconsequentially ignored. First, in application it is not reasonable to expect that no large fire would happen for 450 years. Second, the impact of these end effects on the value of the objective functional is very small due to the discount factor and additional premium from the survivor function in the objective functional. Thus, the increase in h^* near the end of the time horizon contributes a negligible amount to the overall value of the forest. Hence, we interpret the graph of h^* as saying that approximately \$1.5 million should be spent on prevention management per year, up to the time of the first fire, which in practice is unknown.

Recall that the fire event in our problem is taken to be instantaneous, along with its associated costs. As seen in Figure 2.2, the function representing optimal suppression spending in the optimal prevention management spending case is approximately constant at \$29M over the time horizon, except for effects at the end. It is important to recall that suppression spending is a one-time instantaneous cost at the time of the fire and therefore, the suppression cost of \$29M only occurs once in application at the time of fire.

This is in contrast to the optimal prevention management spending h^*, discussed in the previous paragraph, which is ongoing up to the time of fire. In the case without prevention management spending, instantaneous suppression spending is roughly \$46.5M at the time of fire, given that the function for x^* in the case without prevention management spending is approximately constant over the course of the time horizon. As expected, instantaneous suppression spending in the case without prevention management spending is greater than instantaneous suppression spending in the optimal prevention case. This is expected as $\frac{\partial x^*}{\partial h} < 0$. Moreover, instantaneous suppression spending decreases approximately 38% when optimal prevention management spending is applied, in comparison to the corresponding amount without prevention management spending.

The survivor functions for the optimal prevention management spending case and the no prevention management spending case are given in Figure 2.2. As can be seen, in the case without prevention management, the probability that the forest survives to time t drops to near zero ($S(t) < 0.01$) around 23.5 years. In the case of optimal prevention management spending, the probability that the forest survives to time t drops to near zero around 103 years. The survival probability of the forest increases with increased prevention management spending. This is due to slower accumulation of hazard y over time. Recall that cumulative hazard y is a strictly increasing function given by

$$y(t) = \int_0^t \psi\big(h(z)\big)dz, \tag{2.104}$$

and relates to the survivor function in the following way:

$$S(t) = e^{-y(t)}. \tag{2.105}$$

Hence, optimal prevention management spending increases the survivability of the forest.

Given that the time of fire is treated as a random variable in our problem, one measure that we would also like to compare between the two cases of optimal prevention management and no prevention management is the expected, or mean, time of the next fire. In order to determine this value, we calculate the expected value of the time of fire random variable. Recall that for the formulation of the problem, we considered the time of fire RV as a mixed type RV that has a discrete component at the end of our time horizon T. As a result, the expected value of the time of fire (mixed-type) random variable is given by

$$E[\mathcal{T}] = \int_0^T t\psi\big(h(t)\big)e^{-y(t)}dt + TS(T). \tag{2.106}$$

However, because our time horizon was taken to be large, $S(T)$ is very small (see Figure 2.2), and we can calculate the expected value of the time of fire simply by using the probability density function (2.19) for $\mathcal{T}$ over $[0, T]$. Thus, the expected value of the time of fire random variable is justifiably approximated by

$$E[\mathcal{T}] = \int_0^T t\psi\big(h(t)\big)e^{-y(t)}dt. \tag{2.107}$$

In the case of no prevention management spending, this is reduced to

$$\int_0^T bte^{-bt}dt. \tag{2.108}$$

Thus, the mean time of fire in the case without prevention management spending is 5 years. In the case of optimal prevention management spending, the mean time of fire is approximately 22.3 years. Hence, on average, the time of the fire in the optimal case is approximately 17 years later than the no prevention case. Furthermore, in the case of multiple fires, we might expect that over a fixed amount of time there will be fewer fires when optimal prevention management spending is employed compared to when there is no prevention management spending.

Another measure we wish to consider is the expected net present value of the forest over $[0, T]$. This is given by the value of the objective functional in our optimal control problem, either evaluated at the optimal control h^* or evaluated when $h = 0$. In the no prevention management spending case, the expected value of the forest over $[0, T]$ is approximately \$772.6M. In the optimal prevention spending case, the expected value of the forest over $[0, T]$ is approximately \$801.2M. Thus, the value of forest is larger when optimal prevention management spending is applied.

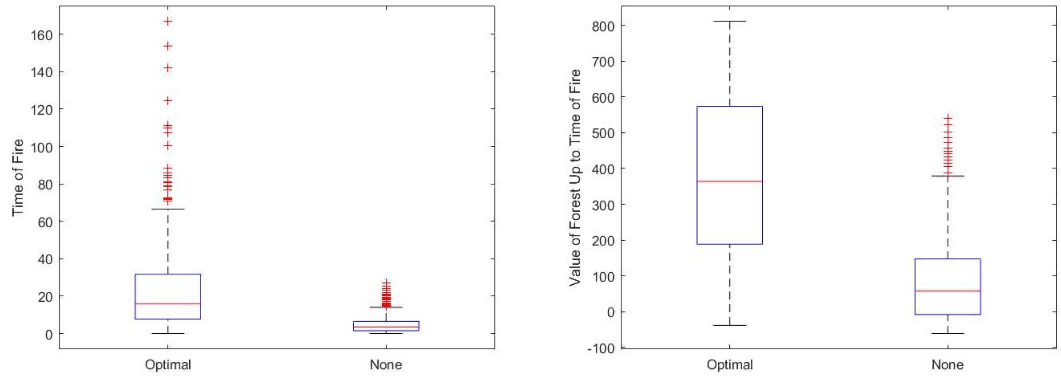
We recognize that a 500 year time horizon is unrealistic in terms of management plans, especially given that it is extremely unlikely that only one fire would occur in that time period. Furthermore, in application the optimal control h^* is only applied up to the time of the first fire and after the fire there is no more information concerning prevention management spending. We want to examine how using the results from this optimal control problem might look in practice. To do this we perform a quick simulation study. Using the cumulative distribution function for the time of fire random variable, we can sample for different times of fire. Using the sampled time of fire and the solution to our optimal control problem, we can calculate the value of the forest up to the time of fire, including suppression and non-timber damage costs at the time of fire. We can also investigate the total amount of prevention management spending up to the time of fire and suppression spending at the time of fire.

We solve our optimal control problem one time for our given set of Las Conchas Fire parameters. This solution provides us with h^*, y^*, and x^* over the full time horizon $[0, T]$. We use y^* to build the cumulative distribution function for the time of fire random variable and sample from it 500 times. This provides us with 500 fire times. For each sampled time of fire we calculate the value of the forest up to the time of fire (including suppression and non-timber damages at the time of fire), the total amount of prevention management spending up to the sampled time of fire, and the amount of suppression spending at the time of fire. Each time we are using the same functions from the solution of the optimal control problem; what changes is where we truncate the functions based on the sampled time of fire. Additionally, we consider the problem in the case without prevention management spending and sample 500 fire times representing this case. We calculate the value of the forest up to the time of fire and suppression costs in this case. This allows us to compare the cases with and without prevention management spending. The results are contained in Figure 2.3 and Tables 2.2 and 2.3.

The mean time of fire is later in the case with optimal prevention management spending than in the case without prevention management spending; this is expected from our work earlier. In particular, here the mean of the sampled times of fire in the case with optimal prevention management spending is 23.3 years and the mean time of fire in the case without prevention management spending is 4.93 years. There are subtle differences between these numbers and numbers reported earlier; we attribute these differences to sampling. The difference in the mean time of fire suggests that the total number of fires that might occur over a fixed amount of time is smaller in the case where optimal prevention management spending is applied.

The value of the forest up to the time of fire in the case with optimal prevention management is larger than in the case without prevention management spending. In particular, the average value of the forest up to the time of the first fire in the case without prevention management spending is \$85.92M and in the case with optimal prevention management spending is \$376M. Notice that these values are less than previously reported because we are not considering the value of the forest over the full time horizon $[0, T]$, but are only calculating the value of the forest up to the time of fire, including the costs from the fire, and because we do not allow for the accumulation of benefits following the fire. Thus the difference in the value of the forest between the two cases is large because in the optimal prevention case the time of fire is, on average, later than in the case without prevention management spending. Thus, in the optimal prevention management spending case benefits have more time to accrue. Additionally, suppression and non-timber costs are less in this case because of the application of optimal prevention management spending h^*. However, making a direct and meaningful comparison of the value of the forest between these two cases is difficult because the amount of time over which the value of the forest is considered in each trial is different.

Recall that for the solution to our optimal control problem, the function representing optimal suppression spending x^* is approximately constant (see Figure 2.2). Thus, it is not surprising that in our simulation study the average amount of suppression spending across 500 different sampled times of fire is approximately \$29M in the optimal prevention management spending case. Furthermore, the average amount of suppression spending across 500 different sampled times of fire is approximately \$46.5M in the case without prevention management spending. In the case with optimal prevention management spending, on average \$34.9M is spent in total on prevention management up to the time of fire. However, given that the sampled times of fire vary substantially, so does this value as spending is accumulating over different amounts of time. Thus, similar to the value of the forest, it



(a) **(b)**

Figure 2.3: The optimal control problem is solved once. Then a time of fire is sampled 500 times and the value of the forest up to the time of fire (including suppression and non-timber damage costs) is calculated. Figure (a) is a boxplot illustrating the distribution of the sampled times of fire and Figure (b) is a boxplot illustrating the distribution of the value of the forest up to the time of fire.

Table 2.2: This table contains basic descriptive statistics concerning the sampled time of fire for the Las Conchas parameter set for 500 samples.

Time of Fire		
Prev.	Optimal	None
Mean	23.3	4.93
Median	16	3.5
Std.	23.16	4.78

can be difficult to interpret this measurement because of the fire time differences. Better comparisons and interpretations could be drawn if a fixed time horizon was considered.

Overall, we see that in the case of optimal prevention management spending h^* the value of the forest and the mean time of fire are larger than in the case without prevention management spending $h = 0$. However, we recognize that it is unrealistic to assume that only one fire will occur in 500 years, especially since we chose the background hazard b to reflect a high frequency of fires in the region. We consider the value of the forest at the mean time of fire in each case (optimal prevention vs. no prevention), but this does not provide results which are directly comparable to one another. This is because in the optimal prevention case we determine the value of the forest at approximately 23 years. In the no prevention

Table 2.3: This table contains basic descriptive statistics concerning the value of the forest up to the sampled time of fire, including non-timber damage and suppression costs at the sampled time of fire, for 500 simulations.

Value of Forest UP TO Time of Fire		
Prev.	Optimal	None
Mean	376.8	89.52
Median	363.9	57.8
Std.	235.6	125.7

case, we determine the value of the forest at 5 years. It could be reasonably expected that up to 3 or more fires could happen in the no prevention management case before the first fire occurs in the optimal prevention management case and so it is misleading to directly compare these values. Thus, in order to make better comparisons concerning the value of the forest and the trade-offs between prevention management and suppression spending, we would like to apply our optimal control problem to a sequence of fires over a fixed amount of time. Using similar sampling techniques used above and with some modification to our optimal control problem we are able to do so. This work is included in Section 2.6. Next, we construct our parameter set reflecting the 2014 Happy Camp Complex and determine the numerical results.

2.4.2 2014 Happy Camp Complex

Lightning strikes ignited 18 small fires in the Klamath National Forest in Northern California in August 2014. Three of these fires were not quickly contained and eventually spread and merged into one large fire, now known as the Happy Camp Fire Complex [24]. This fire complex was contained by the end of October. A total of 134,056 acres were burned in the fire, with 98% of the acres burned being within the Klamath National Forest [24, 36]. The suppression costs for the fire were over $88 million, which on average came to over $1 million spent on fire suppression per day [24, 55]. In addition to suppression costs, there were 8 buildings destroyed or damaged during the fire [55]. Using this information, along with information concerning the forest type and recent fire history, we determine parameter values to best represent this fire.

Table 2.4: The table below includes the parameter values chosen to reflect the 2014 Happy Camp Complex.

Parameter	Units	Value	Justification
$\bar{A}$	acres(1000)	1700	size of Klamath NF
r	/time	0.04	standard discount rate
k	acres(1000)$\times$\$^{2}/time	12000	$k \approx$ size of fire $\times$ suppression \$
k_1	\$ (mil.)/time	1	assumed
k_2	\$ (mil.)/time	1	assumed
δ	/time	0.05	Douglas-fir: 60-180 yrs to mature
b	———	0.05	frequency of fires in region
c	\$ (mil.)/ acres(1000)	0.006	8 buildings destroyed/damaged, 134,000 acres burned.
B_1	\$ (mil.)/time	0.05	calculated from x^* formula.
v	———	1	assumed

The Klamath National Forest covers 1,737,774 acres and, as such, we take the size of the forest parameter to be $\bar{A} = 1,700$. Conveniently, as the interpretation of $\bar{A}$ can be ambiguous, this is the same value chosen for $\bar{A}$ in the Las Conchas example. This allows for a more direct comparison of the two fire examples. The forest in the Klamath National Forest is a mixed conifer/hardwood forest, with some Ponderosa Pine stands found at low elevations and Douglas-Fir at higher elevations. We saw earlier that Ponderosa Pine takes between 70 and 250 years to mature. Depending on forest conditions, a Douglas Fir can grow to be very large in a matter of decades or it may grow hundreds of years and remain relatively small [59]. Douglas Fir trees can be classified into 4 age classes; the first two age classes documented for Douglas Fir range between 60 and 180 years [59]. We will use this to guide our choice for δ. When choosing δ for the Ponderosa Pine we chose $\delta = 0.05$ because after 100 years the number of unburned acres has approximately returned to $\bar{A}$ in a situation where the number of unburned acres is reduced by half. It is reasonable to make this same choice for Douglas-Fir. As Douglas-Fir and Ponderosa Pine both make up this forest, for consistency with the Ponderosa Pine, we once again choose $\delta = 0.05$. We continue to choose the discount rate to be $r = 0.04$ and the management effectiveness parameter in the hazard function to be $v = 1$.

To choose the value for the parameter b we look at the recent fire history in the Klamath National Forest. Since 1970, fires burning over 15 thousand acres have occurred in 1977, 1987, 2008, and 2014. So, we reasonably say that a large fire event might occur every 10-20 years. Therefore, we choose $b = 0.05$. For $b = 0.05$, the probability of the forest surviving to 15 years with no large fire occurring is approximately 0.5.

For simplicity, we still choose $k_1 = k_2 = 1$. We use $k \approx K \times (1 + x)$ to approximate the fire severity parameter k. Given the cost of suppression at \$88 million and the number of acres burned at 134 thousand acres, we round up, and set $k = 12,000$.

To approximate the cost parameter c, we need an approximation for the cost of non-timber damages D. The median cost of homes in the area ranges between \$50,000 and \$200,000 [51]. For the estimate, we take the median value to be \$100,000. We then multiply this by the number of buildings destroyed or damaged in the fire, 8, and divide by the number of acres burned in the fire, 134 thousand. This gives $c = 0.006$.

To determine an appropriate value for the flow of benefits parameter B_1 we use the equation for optimal suppression spending x^*, solved for B_1. As before, we take x^* to be the amount spent on suppression and we assume $h(\tau) = 0$, $\tau = 0$, and $T = 500$. Using the parameter values already determined, this leads to $B_1 = 0.0589$. However, because it is recognized that suppression spending was suboptimal for this fire, we will round down to $B_1 = 0.05$ [24].

With parameters values chosen, we determine the numerical solution. In Figure 2.4 the optimal prevention management spending h^*, the optimal suppression spending x^*, and the survivor function $S(t)$ are given. For comparison, we also include in the plots the case where zero dollars are spent on prevention management spending, $h = 0$. Once again, for consistency, the time horizon T is taken to be 500 years.

As can be seen in Figure 2.4, the optimal prevention management spending rate h^* is approximately constant at 1.35 million dollars per year over the majority of the time horizon. Hence, we interpret the graph of h^* as saying that approximately \$1.35 million should be spent on prevention management per year, up to the time of the first fire.

In this example, optimal suppression spending in the optimal prevention management spending case is approximately 52 million dollars over the time horizon, except for effects

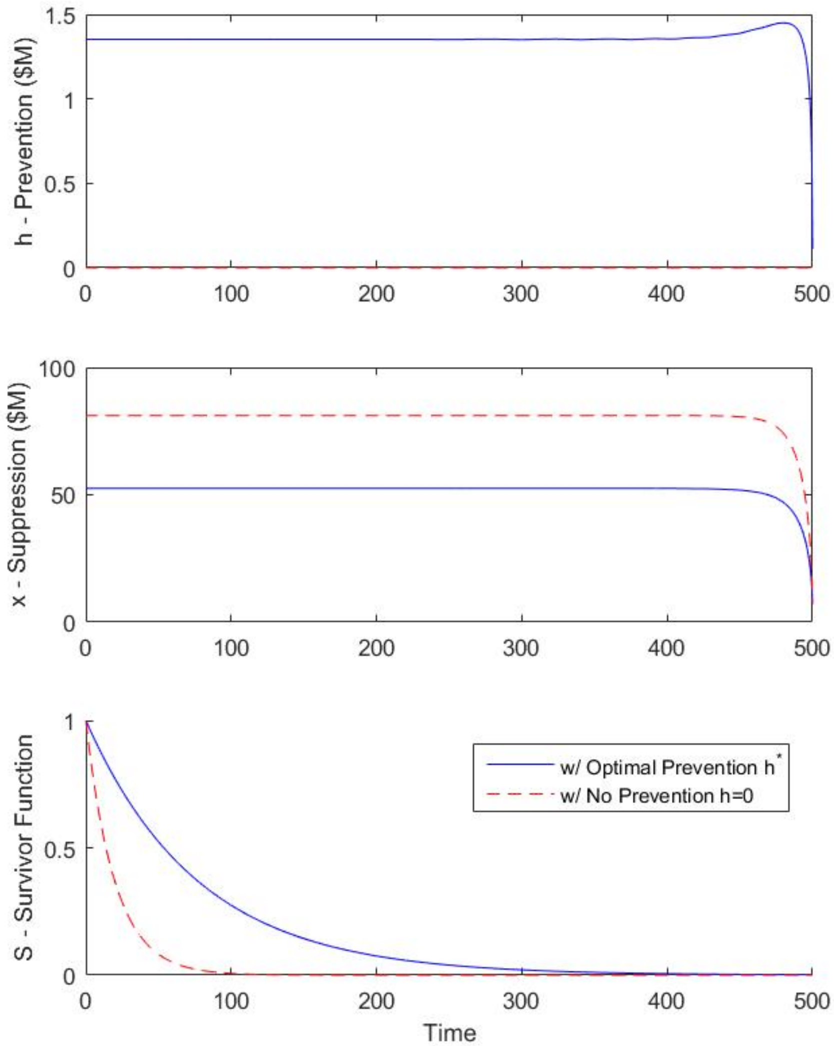


Figure 2.4: The plots above contain the results of our optimal control problem using the Happy Camp Complex parameter set. For comparison, in each plot we include the case with optimal prevention management spending h^* and the case with no prevention spending $h = 0$. The top plot gives the optimal prevention management spending h^* and no prevention spending $h = 0$. The middle plot contains the optimal suppression spending x^* and suppression spending when $h = 0$. The bottom plot gives the survivor function in the case with optimal prevention management spending h^* and in the case with no prevention spending $h = 0$.

at the end. Suppression spending is a one-time instantaneous cost at the time of the fire and therefore, the suppression cost of \$52M would only occur once in application. In the no prevention management spending case, instantaneous suppression spending is roughly \$81M. Instantaneous suppression spending decreases by 36% in the case of optimal prevention management spending, in comparison to the case without prevention management spending.

The survivor functions for both the optimal prevention management spending case and the no prevention management spending case are given in Figure 2.2. In the case without prevention management, the probability that the forest survives to time t drops to near zero ($S(t) < 0.01$) around 92.5 years. In the case of optimal prevention management spending, the probability that the forest survives to time t drops to near zero around 356 years.

Again, we calculate and compare the mean of the time of fire RV. The mean time of fire in the case of no prevention management spending is 20 years. In the case of optimal prevention management spending, the mean time of fire is approximately 76.4 years. Hence, on average, the time of the fire in the optimal case occurs over fifty-five years later than the no prevention management spending case.

The expected net present value of the forest over $[0, T]$ is given by the value of the objective functional in our optimal control problem evaluated at the optimal control h^*. In the no prevention management spending case, the value of the forest over $[0, T]$ is approximately \$2,035 million dollars. In the optimal prevention spending case, the value of the forest over $[0, T]$ is approximately \$2,073 million dollars. Again, for this example, the value of forest is still larger in this case where optimal prevention management spending is applied.

As in the previous example, we again see that in the case of optimal prevention management spending the value of the forest is greater and the mean time of fire is also larger than in the case where there is no prevention management spending. We do not perform the same small simulation study here as we did for the Las Conchas Fire example.

Table 2.5: The table below contains the lower and upper bounds of the parameter values to be used in our LHS/PRCC analysis. The baseline value for a given parameter is simply the average of the lower and upper bounds.

Parameter	Lower Bound	Upper Bound	Baseline Value
$\bar{A}$	1250	1900	1575
δ	0.025	0.075	0.05
B_1	0.01	0.05	0.03
r	0.03	0.05	0.04
b	0.1	0.2	0.15
c	0.01	0.75	0.38
v	0.5	1.5	1
k	5000	20000	12500
k_1	1	5	3
k_2	1	25	13

2.5 Parameter Sensitivity Analysis

As seen in the previous subsection, the values chosen for our parameters are not all strictly data driven or from literature sources. Thus, we perform a global sensitivity analysis to determine which parameters have the most significant impact on the expected value of the forest, $J(h^*)$ and the average prevention management spending rate h^*[21, 27]. We use Latin Hypercube Sampling (LHS) and Partial Rank Correlation Coefficient (PRCC) analysis to determine the parameters to which the value of the objective functional evaluated at the optimal control h^* and the average prevention management spending rate h^* are most sensitive.

LHS was introduced in 1979 by M.D. McKay as an improved alternative to simple random sampling in Monte Carlo studies[28]. In particular, the advantage of using LHS over simple random sampling is its efficiency. The LHS method provides similar accuracy as simple random sampling methods, but with fewer iterations, making it particularly useful for computationally expensive models [21, 27, 28].

There are 10 parameters in our optimal control problem that we will investigate. They are listed in the first column of Table 2.5. First, we must determine an appropriate range over which to investigate each of the parameters, or “inputs”. For each parameter we must choose an appropriate lower and upper bound for the parameter range. These choices for the upper and lower bounds are also found in Table 2.5. Next, we will discuss how these ranges were chosen.

Our range for the size of forest parameter $\bar{A}$ is based on the sizes of national forests in the United States and is given in thousands of acres [56]. The forest regeneration rate δ, found in the differential equation (2.4), parameter range is centered around our original choice of $\delta = 0.05$ for both Douglas-fir and Ponderosa Pine in the previous fire examples. Smaller choices for δ reflect forest types which take longer to recover following a fire and larger choices of δ reflect forest types which regenerate more quickly. Recall that for a particular fire event we determine a value for the flow of benefits parameter B_1 by solving the equation representing optimal fire suppression spending (2.55) for B_1 in terms of x^*. Then using our previously chosen parameter values and the cost of a particular fire event we obtain an estimate for the value of B_1. To determine a parameter range of B_1 to be used in the LHS/PRCC analysis, several fire events were considered and the range chosen is a reflection of the value of B_1 across these scenarios. Note that this range contains our choice of B_1 for both the Las Conchas and Happy Camp fire examples. The non-timber damage cost parameter c was chosen to capture the possibility of fire events in both isolated forest areas (smaller c) and well-developed forest areas (larger c). The range for the fire severity parameter k is chosen to capture a variety of high severity fires. The upper bound for the range is much larger than either of the choices in our examples because in our sensitivity analysis we are allowing for a range of values for k_1 and k_2, and not simply setting $k_1 = k_2 = 1$. The choice for the background hazard parameter b was chosen to reflect the frequency with which large fire events may occur in a given area.

The range for the discount rate parameter r is centered and varied around our original choice of $r = 0.04$. The range for the prevention management effectiveness parameter v, found in the hazard function, is centered and varied around our original choice of $v = 1$. The parameter range for k_1, associated with prevention management spending h, is chosen to be smaller than the range for parameter k_2 because, at a given point in time, less is spent on prevention management than on suppression. We first perform our sensitivity analysis for the output corresponding to the expected net present value of the forest $J(h^*)$.

In order to properly use LHS, we must first verify that the output in question, the value of the objective functional evaluated at the optimal control h^*, is monotonic with respect to each parameter[28]. That is, we solve our optimal control problem multiple times across the range of a given parameter, with all other parameters held at their baseline values. The baseline value for a parameter is simply the average of the lower and upper bound given for that parameter. Baseline values for the parameters being investigated in the sensitivity analysis are found in Table 2.5. We then verify that the value of the objective functional evaluated at the optimal control h^* is monotonic with respect to changes in the parameter. We repeat this process for every parameter that we are investigating in our sensitivity analysis. We quickly see in Figure 2.5 that, for all 10 parameters we are investigating with their given parameter ranges, the value of the objective functional evaluated at the optimal control h^* is monotonic with respect to variations in the parameters. Thus, we may move forward with our LHS/PRCC sensitivity analysis.

As monotonicity between the individual parameters and the output has been established, the LHS parameter matrix can be generated. The LHS matrix is an $N \times 10$ matrix where N is the number of trials to run and 10 is the number of parameters to investigate. A trial here simply refers to a single application and solution of our optimal control problem. We assume a uniform distribution for all 10 parameters across their parameter ranges because the parameter ranges are not strictly data driven or from literature sources. Choosing $N = 50$, each parameter range is then partitioned equally into 50 intervals and from each interval a sample is taken. Thus, each parameter is strategically sampled 50 times across its range. These 50 samples are stored in a column vector. The 10 column vectors, one for each parameter, make up the LHS matrix. For each individual parameter column vector in the LHS matrix, the sampled values are permuted so that they are not necessarily ordered. Thus, one row of the LHS matrix contains the parameter values to be used in a single trial of our optimal control problem.

Once the LHS matrix has been generated we solve our optimal control problem 50 times, once for each row vector of parameter values from the LHS matrix. For each trial, the value

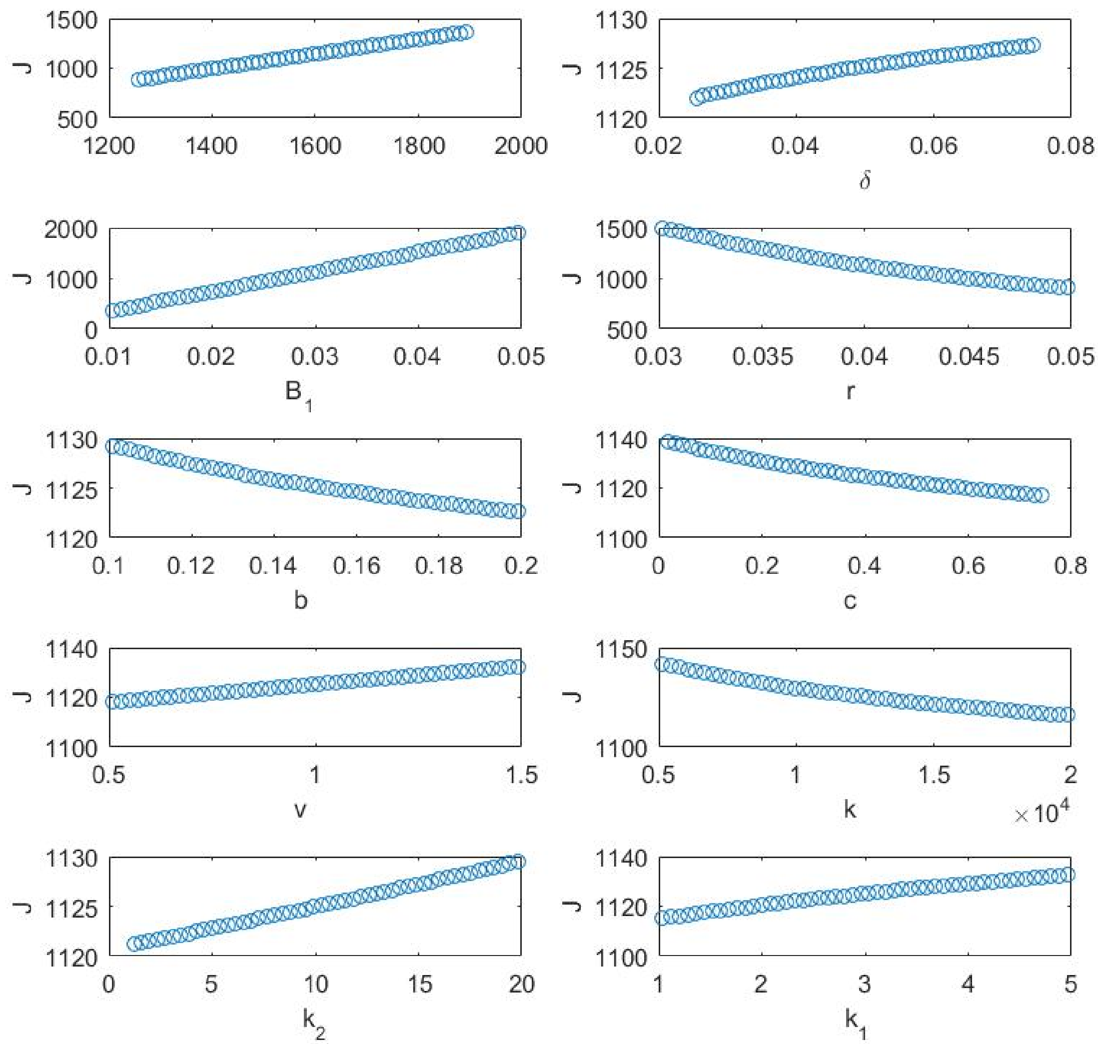


Figure 2.5: Above are the monotonicity plots of $J(h^*)$ with the 10 parameters we are investigating for our LHS/PRCC analysis varied across their chosen ranges. It is easily seen that the value of the objective functional evaluated at the optimal h^* is monotonic with respect to each parameter. For a single plot, one parameter is varied across its full range while all other parameters are held at their baseline values.

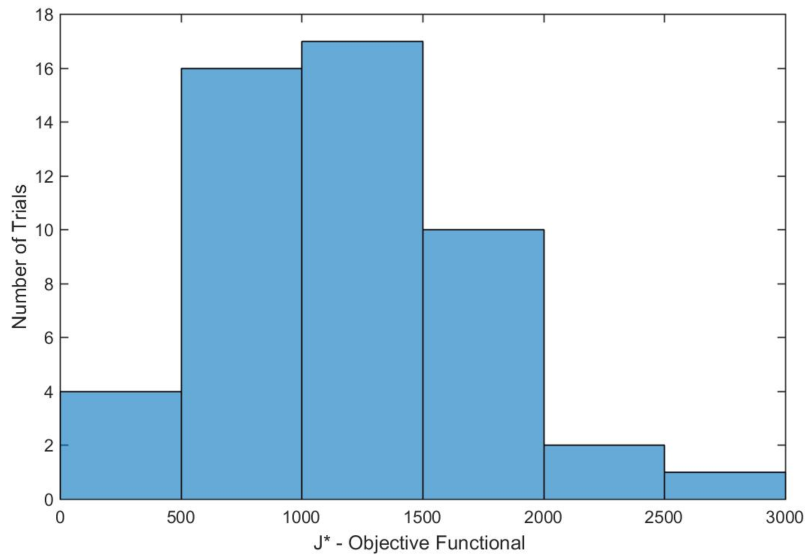


Figure 2.6: This histogram shows how the value of the objective functional $J(h^*)$ evaluated at the optimal control is distributed across the $N = 50$ parameter combinations in the LHS matrix.

Table 2.6: The table below gives some basic descriptive statistics for the distribution of the value of the objective functional evaluated at the optimal control across the $N = 50$ parameter combinations in the LHS matrix.

Distribution Stats - $J(h^*)$	
Mean	\$1,153M
Median	\$1,103M
Std.	\$525M

of the objective functional evaluated at the optimal control h^* is obtained. The histogram in Figure 2.6 allows us to see the variation in the expected value of the forest $J(h^*)$ across the different parameter sets.

Some descriptive statistics concerning the distribution of $J(h^*)$ over 50 trials are found in Table 2.6. The mean of $J(h^*)$ for the 50 trials is $\mu = \$1,153M$. Given that the standard deviation for the output, $\sigma = \$525M$, is large in comparison to the mean, it is clear that the uncertainty present in the value of $J(h^*)$ is substantial. That is, variation in our choice of parameter values has a significant impact on the value of the objective functional evaluated at the optimal control. Hence, we follow this work with a PRCC sensitivity analysis to

Table 2.7: In this table, the partial rank correlation coefficients for each parameter associated with the output $J(h^*)$, along with the corresponding p-values, are listed. Using a significance level of $\alpha = 0.05$ we see that 5 of the 10 parameters investigated are significantly different from zero. They are highlighted in yellow.

Parameter	PRCC	p-value
$\bar{A}$	0.88	$<< 0.05$
δ	-0.04	0.82
B_1	0.99	$<< 0.05$
r	-0.86	$<< 0.05$
b	-0.09	0.57
c	-0.37	0.02
v	0.08	0.61
k	-0.30	0.052
k_1	-0.10	0.54
k_2	0.37	0.02

determine which parameters are the most significant contributors to this uncertainty, i.e. which parameters have the largest impact on the value of $J(h^*)$.

We want to examine the relationship between our individual inputs, the parameters in our optimal control problem, and the output, the value of the objective functional evaluated at the optimal control. Partial correlation coefficients assess the degree of linear association between one input and the output, while controlling for the effects of the other inputs. Partial rank correlation coefficients (PRCCs) assess the degree of monotonicity between one input and the output, while controlling for the effects of the other inputs. That is, a PRCC is a sensitivity measure which allows us to assess nonlinear, but monotonic relationships between inputs and an output[21, 27]. A PRCC is calculated for each parameter investigated. For a given input, a PRCC which is close to 1 in magnitude indicates that the input has a large impact on the value of the output.

The PRCC for each parameter and associated p-values in Table 2.7 are calculated using the MATLAB function `partialcorr`(). The p-values are used to assess whether or not the PRCCs are significantly different from zero. Using a significance level of $\alpha = 0.05$, we see that 5 of our 10 parameters have PRCCs significantly different from zero. These

Table 2.8: This table lists the PRCCs and their corresponding Fisher transforms for the parameters which were shown to have the most impact on the value of the objective functional evaluated at the optimal control h^*.

Parameter	PRCC γ	Fisher Transform γ'
$\bar{A}$	0.88	1.38
B	0.99	2.53
r	-0.86	-1.31
c	-0.37	-0.39
k_2	0.37	0.39

parameters include the size of the forest parameter $\bar{A}$, the flow of benefits parameter B_1, the discount rate parameter r, the nontimber damage cost parameter c, and the suppression spending effectiveness parameter k_2. We interpret this to mean that the parameters which have PRCCs significantly different from zero have a significant impact on the value of the objective functional evaluated at the optimal control h^*.

Now that we know which parameters have a significant impact on the value of the output, we would like to make a comparison of these significant parameters to see which ones have the strongest impact, in magnitude, on $J(h^*)$. That is, we would like to rank the significant parameters from most impact to least impact. A simple comparison of the absolute values of the PRCCs does not suffice. To determine if a given parameter has a greater impact on the output than another, we must determine if there are significant statistical differences in their corresponding PRCCs.

In order to perform statistical comparison tests for PRCCs, we must first apply the following log transformation to each PRCC:

$$\gamma' = \frac{1}{2} \ln \left| \frac{1+\gamma}{1-\gamma} \right|, \tag{2.109}$$

where γ is the original PRCC and γ' is the transformed PRCC[27, 4]. The log transformed PRCC γ' is known as the Fisher tranform and is approximately Gaussian $\mathcal{N}(\mu, \sigma^2)$ with

$$\mu = \frac{1}{2} \ln \left| \frac{1+\gamma}{1-\gamma} \right| \text{ and } \sigma^2 = \frac{1}{N-3-p}, \tag{2.110}$$

where N is the number of trials and p is the number of parameters controlled for when the PRCC is calculated. Table 2.8 gives the Fisher transformed PRCCs for the parameters whose PRCCs are significantly different from zero.

We can compare the values of two PRCCs by examining the z-statistic

$$z = \frac{\gamma_1' - \gamma_2'}{\sqrt{\frac{1}{N_1-3-p_1} + \frac{1}{2-3-p_2}}}, \tag{2.111}$$

which follows a $\mathcal{N}(0,1)$ distribution. Here, $N_1 = N_2 = 50$ is the number of trials and the value p_i, $i = 1,2$, represents the number of parameters controlled for when the PRCC γ_i is calculated [27]. For our problem, $p_1 = p_2 = 9$ since we are investigating 10 parameters. We note that increases in some parameters increase the value of the output, while increases in some parameters will decrease the value of the output $\big($see Figure 2.5$\big)$. We are most interested in determining which parameters have the largest impact on the output in magnitude, regardless of whether that impact is positive or negative. This guides the development of the family of hypotheses we wish test to determine the ranking of the significant parameters.

To properly rank the PRCCs according to their impact on the output $J(h^*)$ in magnitude, we must perform multiple pairwise comparison tests. In particular, we test the null hypothesis that all PRCCs are equal

$$H_0 : |\gamma_{\bar{A}}'| = |\gamma_{B_1}'| = |\gamma_c'| = |\gamma_{k_2}'| = |\gamma_r'| \tag{2.112}$$

against the alternative hypotheses

$$H_A : |\gamma_i'| \neq |\gamma_j'|, \tag{2.113}$$

for every pair $(i,j) \in \{\bar{A}, B_1, c, k_2, r\}$ where $i \neq j$. Thus, we have a family of $\binom{5}{2} = 10$ pairwise hypothesis tests to perform in order to effectively rank our 5 significant parameters.

When performing multiple comparison tests we must be careful to consider the increased likelihood of a rare event; that is, when considering multiple tests, we are more likely to reject the null hypothesis when it is true, a type I error. Given that we are performing 10 hypothesis tests and have chosen a significance level of $\alpha = 0.05$, the probability that we reject at least one of the null hypotheses (i.e. the probability of at least one rare event, the probability of at least one type I error) is

$$\begin{aligned} P(\geq 1 \text{ significant event}) &= 1 - P(0 \text{ significant events}) \\ &= 1 - (1 - 0.05)^{10} \\ &= 0.40126. \end{aligned} \tag{2.114}$$

In other words, using a significance level of $\alpha = 0.05$ for each of the 10 tests, the probability of at least one significant event (at least one rejection of the null hypothesis) is approximately 40%. This is known as the familywise error rate (FWER). We would like to control the FWER on our family of hypothesis tests in order to control the number of false positives. To do so, we need to differentiate between a per test significance level $\alpha[PT]$, read "alpha per test," and a per family significance level $\alpha[PF]$, read "alpha per family." Given a family of hypothesis tests we would like to control the familywise error rate at the level of $\alpha[PF] = 0.05$. The FWER for a given $\alpha[PT]$ is given by

$$\alpha[PF] = 1 - (1 - \alpha[PT])^C, \tag{2.115}$$

where C is the number of hypothesis tests [1]. This is known as the Sidak equation which can be rewritten to give the $\alpha[PT]$ for a given $\alpha[PF]$:

$$\alpha[PT] = 1 - (1 - \alpha[PF])^{\frac{1}{C}}. \tag{2.116}$$

Thus, given that we want $\alpha[PF] = 0.05$ and we are performing $C = 10$ tests, we can solve for $\alpha[PT]$. However, to determine $\alpha[PT]$ we use the simpler Bonferroni approximation:

$$\alpha[PT] \approx \frac{\alpha[PF]}{C}. \tag{2.117}$$

The Bonferroni approximation is the linear approximation of the Sidak equation and its use is well-established in the literature as a procedure to control FWER [1].

Thus, to control the type I error for the family of 10 hypotheses we are testing at a significance level of $\alpha[PF] = 0.05$, we use a per test significance level of

$$\alpha[PT] \approx \frac{\alpha[PF]}{C} = \frac{0.05}{10} = 0.005. \tag{2.118}$$

Using the Bonferroni corrected per test significance level of $\alpha[PT] = 0.005$ we control the FWER at approximately a significance level of $\alpha[PF] = 0.05$. Observe,

$$\begin{aligned} P(\geq 1 \text{ significant event}) &= 1 - P(0 \text{ significant events}) \\ &= 1 - (1 - 0.005)^{10} \\ &= 0.049. \end{aligned} \tag{2.119}$$

Next, we perform our family of hypothesis tests, with $\alpha[PT] = 0.005$. Recall that the null hypothesis for each test in the family is that all PRCCs are equal (2.112). We are testing the null hypothesis against the alternative hypotheses that a pair of PRCCs are not equal. There are 10 pairs of PRCCs as we are ranking 5 parameters. The z-statistic that we are using to test the hypotheses is given by equation (2.111).

The results of the 10 pairwise comparison tests are included in Table 2.9. The z-score associated with $\alpha[PT] = 0.005$ is 2.807. Therefore, if the z-statistic for a given hypothesis test is is greater than 2.807, then the null hypothesis is rejected and if the z-statistic is less than 2.807, then we fail to reject the null hypothesis. To reject the null hypothesis means that the two PRCCs considered in the alternative hypothesis are significantly different from one another. Thus, the parameter with the larger PRCC in absolute value has a greater impact on the output $J(h^*)$. To fail to reject the null hypothesis means that there is not enough evidence to conclude that the two PRCCs being compared are significantly different and hence no conclusions about which parameter has a greater impact on the output can be drawn. The full results of our hypothesis tests are summarized in Table 2.9, and we now explain specific comparisons in detail.

Table 2.9: The table below contains the results of the hypothesis tests to determine the ranking of our significant parameters according to their impact on the output. To control the FWER, using the Bonferroni approximation, we use a per test significance level of 0.005 to determine whether or not to reject the null hypothesis.

Hypothesis Test Results - $J(h^*)$		
Alternative Hypothesis	z-statistic	Conclusion
$\lvert\gamma'_{\bar{A}}\rvert \neq \lvert\gamma'_{B_1}\rvert$	5.540	reject null
$\lvert\gamma'_{\bar{A}}\rvert \neq \lvert\gamma'_{c}\rvert$	4.304	reject null
$\lvert\gamma'_{\bar{A}}\rvert \neq \lvert\gamma'_{k_2}\rvert$	4.304	reject null
$\lvert\gamma'_{\bar{A}}\rvert \neq \lvert\gamma'_{r}\rvert$	0.359	FAIL TO REJECT
$\lvert\gamma'_{B_1}\rvert \neq \lvert\gamma'_{c}\rvert$	9.843	reject null
$\lvert\gamma'_{B_1}\rvert \neq \lvert\gamma'_{k_2}\rvert$	9.843	reject null
$\lvert\gamma'_{B_1}\rvert \neq \lvert\gamma'_{r}\rvert$	5.899	reject null
$\lvert\gamma'_{c}\rvert \neq \lvert\gamma'_{k_2}\rvert$	0	FAIL TO REJECT
$\lvert\gamma'_{c}\rvert \neq \lvert\gamma'_{r}\rvert$	3.944	reject null
$\lvert\gamma'_{k_2}\rvert \neq \lvert\gamma'_{r}\rvert$	3.944	reject null

First, we test the null hypothesis (2.112) against the alternative hypothesis $|\gamma'_{\bar{A}}| \neq |\gamma'_{B_1}|$. With a z-statistic of $5.540 > 2.807$, we reject the null hypothesis and conclude that the parameter B_1 has a greater impact in magnitude on the output than the parameter $\bar{A}$. Here, we note that the parameter B_1 has the PRCC closest to 1 in magnitude and $\bar{A}$ has the second largest PRCC. Thus, it is not surprising, and is expected, that when considering the alternative hypotheses $|\gamma'_{B_1}| \neq |\gamma'_{c}|$, $|\gamma'_{B_1}| \neq |\gamma'_{k_2}|$, and $|\gamma'_{B_1}| \neq |\gamma'_{r}|$, we also reject the null hypothesis. Hence, we conclude that the impact of B_1 on the value of the output is also greater than the impact of the parameters c, k_2 and r on the output. Therefore, B_1, the flow of benefits parameter, is the parameter which has the greatest impact on the output in magnitude.

Next, we compare the PRCCs of the parameters $\bar{A}$ and r. With a z-statistic of $0.359 < 2.807$ we fail to reject the null hypothesis. That is, we cannot make any conclusions about whether $\bar{A}$ or r has a greater impact on $J(h^*)$ in magnitude. Thus, the parameters $\bar{A}$ and r, which were shown to have significantly less impact on the output than the parameter B_1, were not shown to be significantly different from one another when considering their impact on the output in absolute value. The z-statistics for comparing the impact of $\bar{A}$ against c and $\bar{A}$ against k_2 are equal because, in terms of absolute value, c and k_2 have the same PRCC. Hence, with a z-statistic of $4.304 > 2.807$ we reject the null hypothesis and conclude that the impact of $\bar{A}$ on the output is greater than the impact of c and greater than the impact of k_1 on the output.

Once again, as the PRCCs for c and k_2 are equal in absolute value, comparing the impact of r against c on the output is equivalent to comparing the impact of r against k_2 on the output. With a z-statistic of $z = 3.944 > 2.807$, we reject the null hypothesis and conclude that r has a greater impact on the output than parameters c and k_2. Finally, since the PRCCs of c and k_2 are equal in magnitude, the z-statistic is 0, and clearly we fail to reject the null hypothesis. We conclude that in terms of magnitude, the parameters c and k_2 have a similar impact on the output $J(h^*)$. In summary, we use the Bonferroni approximation to determine a per test significance level for the family of hypothesis tests used in order to rank the parameters according to their impact on the value of the objective functional. We conclude that the parameter B_1 has the strongest impact on the expected net present value of the forest, followed by parameters $\bar{A}$ and r, followed by parameters c and k_2.

The per test significance level used for family of hypothesis tests above was determined using the Bonferroni approximation to the Sidak equation. This allowed us to control the FWER at a level of approximately 0.05. However, the Bonferroni correction is very conservative and can often miss important features of data because of this [6, 49]. Another method often used when considering a family of hypothesis tests controls the false discovery rate, or the expected ratio of false discoveries (type I errors) to the total number of discoveries (rejections of the null hypothesis), instead of controlling FWER.

In 1995, Yoav Benjamini and Yosef Hochberg developed a procedure, referred to as the BH-procedure, to tackle the multiplicity problem by controlling the false discovery rate (FDR). FDR-controlling procedures provide less stringent control of type I errors, but tend to have more statistical power than FWER-controlling procedures [6]. Recall that a type I error is made when the null hypothesis is rejected even though it is true and that statistical power refers to the probability that the hypothesis test rejects the null hypothesis when the null hypothesis is false. For the BH-procedure, we take the 10 hypothesis tests $H_1, \ldots, H_{10}$

Table 2.10: The table below presents the results of the BH-procedure on our family of hypothesis tests. The hypothesis tests are ranked according to their p-values from smallest to largest. We fail to reject the null hypothesis twice. These results are the same as when using the Bonferroni correction.

i	Alt. Hypothesis	p-values	$\frac{i}{10}\alpha$	conclusion
1	$\lvert\gamma'_{B_1}\rvert \neq \lvert\gamma'_c\rvert$	3.639E-22	0.005	reject null
2	$\lvert\gamma'_{B_1}\rvert \neq \lvert\gamma'_{k_2}\rvert$	3.639E-22	0.01	reject null
3	$\lvert\gamma'_{B_1}\rvert \neq \lvert\gamma'_r\rvert$	1.109E-08	0.015	reject null
4	$\lvert\gamma'_{\bar{A}}\rvert \neq \lvert\gamma'_{B_1}\rvert$	8.652E-08	0.02	reject null
5	$\lvert\gamma'_{\bar{A}}\rvert \neq \lvert\gamma'_c\rvert$	3.792E-05	0.025	reject null
6	$\lvert\gamma'_{\bar{A}}\rvert \neq \lvert\gamma'_{k_2}\rvert$	3.792E-05	0.03	reject null
7	$\lvert\gamma'_r\rvert \neq \lvert\gamma'_c\rvert$	0.000167	0.035	reject null
8	$\lvert\gamma'_r\rvert \neq \lvert\gamma'_{k_2}\rvert$	0.000167	0.04	reject null
9	$\lvert\gamma'_{\bar{A}}\rvert \neq \lvert\gamma'_r\rvert$	0.374	0.045	FAIL TO REJECT
10	$\lvert\gamma'_{\bar{c}}\rvert \neq \lvert\gamma'_{k_2}\rvert$	0.399	0.05	FAIL TO REJECT

and their corresponding z-statistics and calculate the associated p-values, $p_1, \ldots, p_{10}$. Then, we rank the hypotheses according to their p-values from smallest to largest. Let the ordered hypotheses and p-values be denoted by $H_{(1)}, \ldots, H_{(10)}$ and $p_{(1)}, \ldots, p_{(10)}$. Next, for $i \in \{1, \ldots, 10\}$, we determine the largest i such that $p_{(i)} \leq \frac{i}{10}\alpha$ where $\alpha = 0.05$. Then, we reject all null hypotheses for $j = 1, \ldots, i$ [6].

The results of the BH-procedure for our family of 10 hypothesis tests are summarized in Table 2.10. As can be seen, we fail to reject the null hypothesis for the tests in which the alternative hypotheses are $|\gamma'_{\bar{A}}| \neq |\gamma'_r|$ and $|\gamma'_{\bar{c}}| \neq |\gamma'_{k_2}|$. That is, the parameters $\bar{A}$ and r have a similar impact on the output in magnitude and the parameters c and k_1 have a similar impact on the output in magnitude. Thus, the results for the ranking of the parameters $\bar{A}, B_1, c, k_2$, and r are the same using the FWER-controlling Bonferroni correction and the FDR-controlling BH-procedure.

Next, we perform this global sensitivity analysis on the mean optimal prevention management spending rate h^*. In the two fire examples considered earlier in this chapter, we saw that the optimal prevention management spending rate h^* was approximately constant, except for some tail effects near the end of the time horizon. Thus, the output we consider

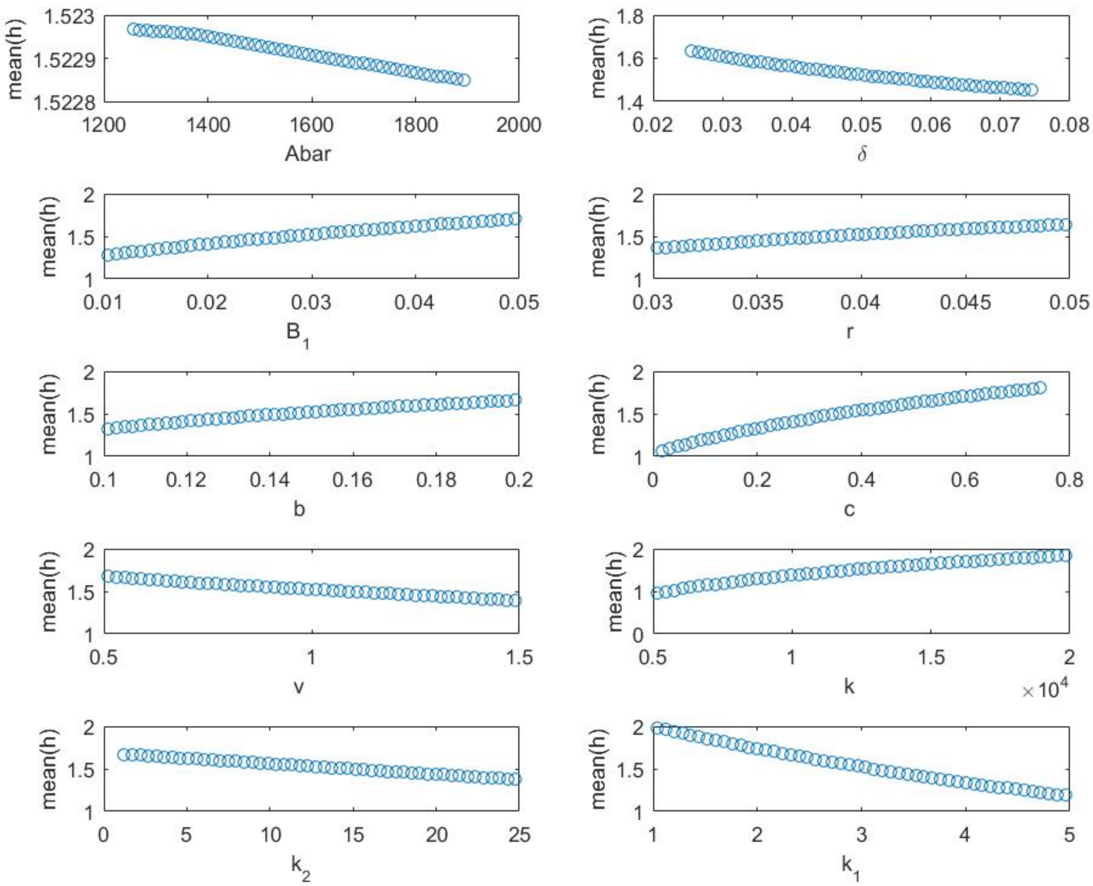


Figure 2.7: The ten plots above establish the monotonicity of the mean optimal prevention management spending rate h^* with respect to each parameter. These plots also demonstrate the how increasing values for the parameters affects the output.

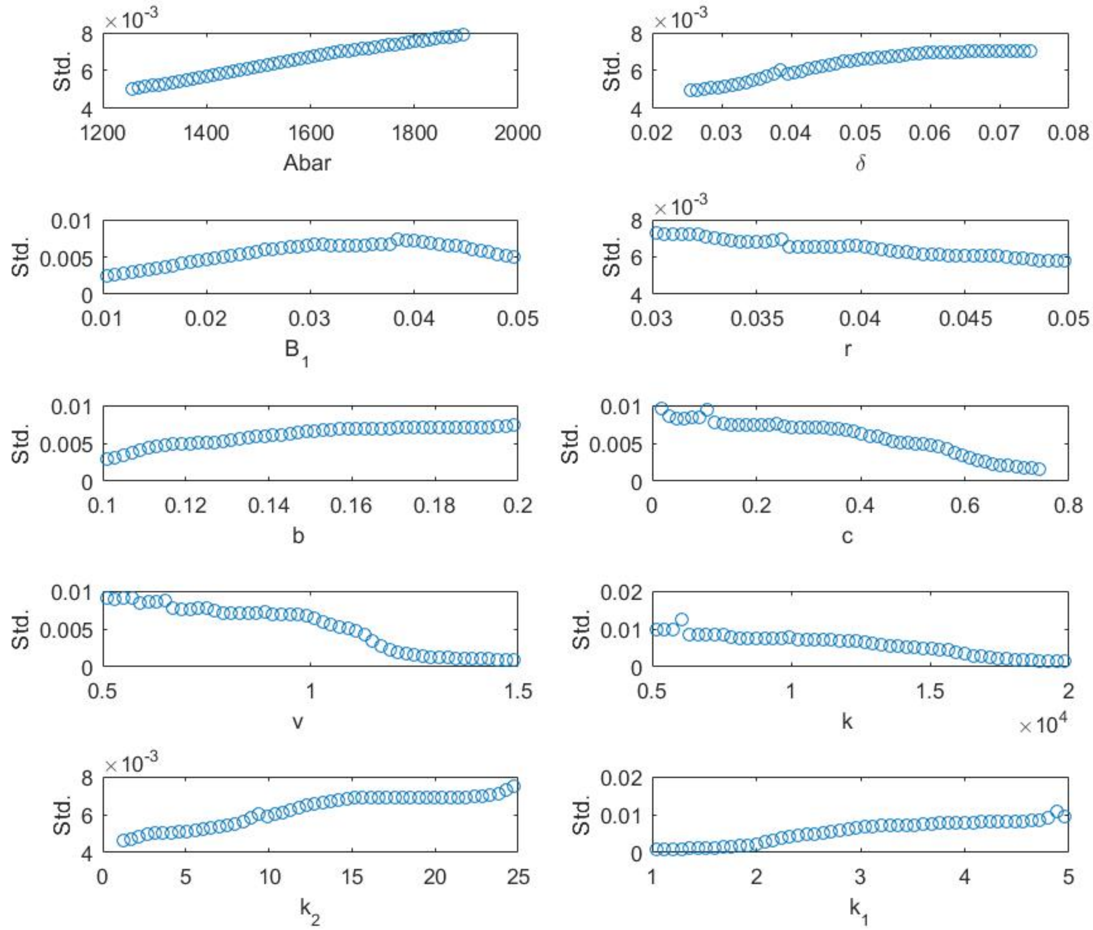


Figure 2.8: The 10 plots show that the standard deviation of h^* is very close to zero across the ranges of all parameters. This illustrates that optimal prevention management spending is approximately constant over the first 400 years of the time horizon.

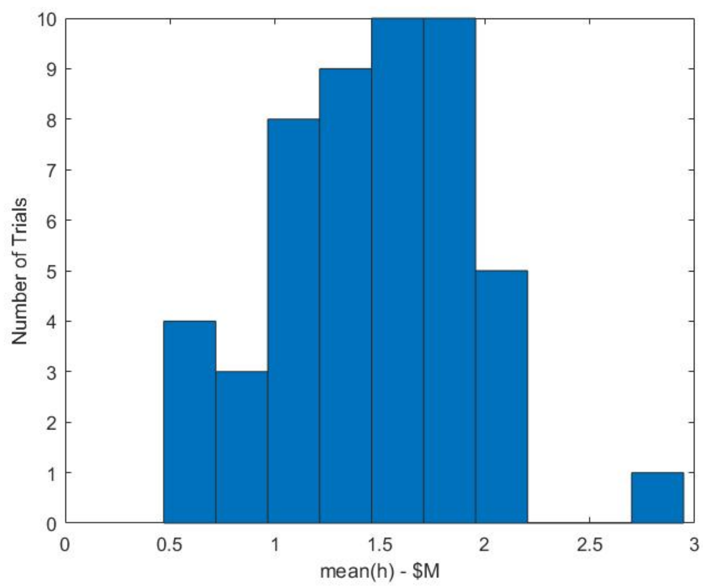


Figure 2.9: The histogram above describes the distribution of the mean optimal prevention management spending rate h^* over 400 years across the $N = 50$ parameter combinations in the LHS matrix.

in the sensitivity analysis is the mean optimal prevention spending rate h^* over the first 400 years of the time horizon. In this way, we ignore te tail effects present in h^*. We test the sensitivity of this output to the same 10 parameters considered earlier. Additionally, we use the same parameter ranges as before.

First, we verify that the output in question, the mean optimal prevention management spending rate h^* over 400 years, is monotonic with respect to each parameter. The results are contained in Figure 2.7. It is easily seen that the monotonicity requirement is met. Furthermore, these plots reveal to us how each parameter affects the optimal prevention management spending h^*. This is useful because we were not able to determine an analytical representation of h^*. For instance, we see that the mean optimal prevention management spending rate h^* is increasing with respect to the parameters $B_1, r, b, c,$ and k and decreasing with respect to the parameters $\bar{A}, \delta, v, k_1,$ and k_2. Beyond demonstrating monotonicity, we also want to demonstrate that the mean optimal prevention management spending rate h^* over 400 years is a reasonable output to consider. While calculating the mean of h^* in order to test the monotonicity of each parameter, we also calculated the standard deviation of h^*. The results are contained in Figure 2.8. The 10 plots show that the standard deviation of

Table 2.11: The table below provides some basic descriptive statistics for the distribution of mean optimal prevention management spending h^* over 400 years across the $N = 50$ parameter combinations in the LHS matrix.

Distribution Statistics - mean(h^*)	
Mean	\$1.46M/yr.
Median	\$1.51M/yr.
Std.	\$0.48M/yr.

h^* over 400 years is very close to zero across the ranges of all 10 parameters. This illustrates that the optimal prevention management spending rate h^* is approximately constant over the first 400 years of the time horizon, and thus, it is reasonable to consider the mean of h^* as output for parameter sensitivity analysis.

Monotonicity has been established and the use of the mean of h^* over 400 years as an output to consider has been justified. Thus, we continue with the sensitivity analysis. The LHS matrix is generated in the same manner previously described. For each trial from the LHS matrix, the mean optimal prevention management spending h^* over 400 years is obtained. In Figure 2.9, the histogram shows the variation in the mean of h^* across the different parameter sets. Some descriptive statistics concerning the distribution of the mean of h^* over 400 years for $N = 50$ trials are contained in Table 2.11. The mean of the mean of h^* for 50 trials is $\mu = 1.46$. The standard deviation for the output is $\sigma = 0.48$. We follow this with a PRCC sensitivity analysis to determine which parameters are the most significant contributors to the uncertainty present in the mean optimal prevention management spending rate h^*.

The PRCC for each parameter and their associated p-values are found in Table 2.12. Using a significance level of $\alpha = 0.05$ we see that 9 of the 10 parameters investigated are significantly different from zero; the only parameter not included is $\bar{A}$. Now that we know which parameters have a significant impact on the output, we rank the significant parameters from most impact to least impact. We calculate the Fisher transform of the PRCCs for the significant parameters using equation (2.109). The results are contained in Table 2.13. We can compare the values of two PRCCs by examining the z-statistic given by equation (2.111). In order to properly rank the PRCCs of our parameters on the output in magnitude, we must

Table 2.12: In this table, the partial rank correlation coefficients for each parameter corresponding to the output mean h^*, along with the associated p-values, are listed. Using a significance level of $\alpha = 0.05$ we see that 9 of the 10 parameters investigated are significantly different from zero. They are highlighted in yellow.

Parameter	PRCC	p-value
$\bar{A}$	0.11	0.504
δ	-0.33	0.034
B_1	0.73	$<< 0.05$
r	0.38	0.014
b	0.45	0.003
c	0.85	$<< 0.05$
v	-0.48	0.002
k	0.885	$<< 0.05$
k_1	-0.87	$<< 0.05$
k_2	-0.58	$<< 0.05$

Table 2.13: This table lists the PRCCs and their corresponding Fisher transforms for the parameters which were shown to have the most impact on the mean optimal prevention management spending rate h^* over 400 years.

Parameter	PRCC γ	Fisher Transform γ'
δ	-0.33	-0.346
B_1	0.73	0.929
r	0.38	0.402
b	0.45	0.480
c	0.85	1.253
v	-0.48	-0.517
k	0.885	1.400
k_1	-0.87	-0.663
k_2	-0.58	-1.331

perform of $\binom{9}{2} = 36$ pairwise comparison tests. In particular, we test the null hypothesis that all transformed PRCCs are equal

$$H_0 : |\gamma'_\delta| = |\gamma'_{B_1}| = |\gamma'_c| = |\gamma'_{k_2}| = |\gamma'_r| = |\gamma'_v| = |\gamma'_k| = |\gamma'_{k_1}| = |\gamma'_b|, \tag{2.120}$$

against the alternative hypotheses

$$H_A : |\gamma'_i| \neq |\gamma'_j|, \tag{2.121}$$

for every pair $(i, j) \in \{B_1, b, \delta, c, k, k_1, k_2, r, v\}$ where $i \neq j$. Given our family of hypothesis tests we control the familywise error rate at the level of $\alpha[PF] = 0.05$. To determine the $\alpha[PT]$ we use the Bonferroni approximation:

$$\alpha[PT] \approx \frac{\alpha[PF]}{C} = \frac{0.05}{36} \approx 0.0014. \tag{2.122}$$

This $\alpha[PT]$ corresponds to the z-score 2.99. Thus, if the z-score is less than 2.99 we fail to reject the null hypothesis and thus there is not sufficient evidence to conclude that the two PRCCs being compared are significantly different from one another. If the z-score for a particular test is greater than 2.99 we reject the null hypothesis and conclude that the two PRCCs being compared are significantly different. The full results of our hypothesis tests are contained in Table 2.14.

We summarize the results here. The parameter k has the largest PRCC. The PRCC for k is significantly larger than the PRCCs for the parameters δ, r, b, v, and k_2. The PRCC of the parameter k is not significantly different from the PRCCs for parameters k_1, c, and B_1. The PRCCs parameters k_1 and c are the two next largest PRCCs and they are not significantly different from one another. Additionally, they are not significantly different from the parameters k_2 and B_1. They are both significantly larger than the parameters v, b, r, and δ. The parameter k_2 has the fourth largest PRCC and is not significantly different from the parameters k_1, c, B_1, v, b, r, and δ. It is significantly smaller than the parameter k. The PRCC for the parameter B_1 is not significantly different from the PRCCs for any of the other parameters tested. The parameters δ, r, b, and v are the parameters with the lowest PRCCs and they are not significantly different from one another. Additionally, they are

Table 2.14: The table below contains the results of the hypothesis tests to determine the ranking of our significant parameters according to their impact on the mean optimal prevention management spending rate. To control the FWER, using the Bonferroni approximation, we use a per test significance level of $\alpha[PT] = 0.0014$, corresponding to a z-score of 2.99, to determine whether or not to reject the null hypothesis.

Alternative Hypothesis	z-statistic	Conclusion
$\lvert\gamma'_\delta\rvert \neq \lvert\gamma'_{B_1}\rvert$	2.54	FAIL TO REJECT
$\lvert\gamma'_\delta\rvert \neq \lvert\gamma'_r\rvert$	0.25	FAIL TO REJECT
$\lvert\gamma'_\delta\rvert \neq \lvert\gamma'_b\rvert$	0.59	FAIL TO REJECT
$\lvert\gamma'_\delta\rvert \neq \lvert\gamma'_c\rvert$	5.46	reject null
$\lvert\gamma'_\delta\rvert \neq \lvert\gamma'_v\rvert$	0.74	FAIL TO REJECT
$\lvert\gamma'_\delta\rvert \neq \lvert\gamma'_k\rvert$	4.59	reject null
$\lvert\gamma'_\delta\rvert \neq \lvert\gamma'_{k_2}\rvert$	1.38	FAIL TO REJECT
$\lvert\gamma'_\delta\rvert \neq \lvert\gamma'_{k_1}\rvert$	4.29	reject null
$\lvert\gamma'_{B_1}\rvert \neq \lvert\gamma'_r\rvert$	2.29	FAIL TO REJECT
$\lvert\gamma'_{B_1}\rvert \neq \lvert\gamma'_b\rvert$	1.95	FAIL TO REJECT
$\lvert\gamma'_{B_1}\rvert \neq \lvert\gamma'_c\rvert$	1.42	FAIL TO REJECT
$\lvert\gamma'_{B_1}\rvert \neq \lvert\gamma'_v\rvert$	1.80	FAIL TO REJECT
$\lvert\gamma'_{B_1}\rvert \neq \lvert\gamma'_k\rvert$	2.05	FAIL TO REJECT
$\lvert\gamma'_{B_1}\rvert \neq \lvert\gamma'_{k_2}\rvert$	1.16	FAIL TO REJECT
$\lvert\gamma'_{B_1}\rvert \neq \lvert\gamma'_{k_1}\rvert$	1.75	FAIL TO REJECT
$\lvert\gamma'_r\rvert \neq \lvert\gamma'_b\rvert$	0.34	FAIL TO REJECT
$\lvert\gamma'_r\rvert \neq \lvert\gamma'_c\rvert$	3.71	reject null
$\lvert\gamma'_r\rvert \neq \lvert\gamma'_v\rvert$	0.50	FAIL TO REJECT
$\lvert\gamma'_r\rvert \neq \lvert\gamma'_k\rvert$	4.35	reject null
$\lvert\gamma'_r\rvert \neq \lvert\gamma'_{k_2}\rvert$	1.13	FAIL TO REJECT
$\lvert\gamma'_r\rvert \neq \lvert\gamma'_{k_1}\rvert$	4.05	reject null
$\lvert\gamma'_b\rvert \neq \lvert\gamma'_c\rvert$	3.37	reject null
$\lvert\gamma'_b\rvert \neq \lvert\gamma'_v\rvert$	0.16	FAIL TO REJECT
$\lvert\gamma'_b\rvert \neq \lvert\gamma'_k\rvert$	4.01	reject null
$\lvert\gamma'_b\rvert \neq \lvert\gamma'_{k_2}\rvert$	0.79	FAIL TO REJECT
$\lvert\gamma'_b\rvert \neq \lvert\gamma'_{k_1}\rvert$	3.71	reject null
$\lvert\gamma'_c\rvert \neq \lvert\gamma'_v\rvert$	3.21	reject null
$\lvert\gamma'_c\rvert \neq \lvert\gamma'_k\rvert$	0.64	FAIL TO REJECT
$\lvert\gamma'_c\rvert \neq \lvert\gamma'_{k_2}\rvert$	2.57	FAIL TO REJECT
$\lvert\gamma'_c\rvert \neq \lvert\gamma'_{k_1}\rvert$	0.34	FAIL TO REJECT
$\lvert\gamma'_v\rvert \neq \lvert\gamma'_k\rvert$	3.85	reject null
$\lvert\gamma'_v\rvert \neq \lvert\gamma'_{k_2}\rvert$	0.64	FAIL TO REJECT
$\lvert\gamma'_v\rvert \neq \lvert\gamma'_{k_1}\rvert$	3.55	reject null
$\lvert\gamma'_k\rvert \neq \lvert\gamma'_{k_2}\rvert$	3.21	reject null
$\lvert\gamma'_k\rvert \neq \lvert\gamma'_{k_1}\rvert$	0.30	FAIL TO REJECT
$\lvert\gamma'_{k_2}\rvert \neq \lvert\gamma'_{k_1}\rvert$	2.91	FAIL TO REJECT

not significantly different from the parameters k_2 and B_1 and they are significantly smaller than the parameters k, c, and k_1. Thus, there is not necessarily a straightforward way to rank these parameters according to their impact on mean optimal prevention management spending. Broadly, we conclude that the parameters k, k_1, c, and B_1 have the largest impact on the output. The parameter k_2 has a moderate impact on the output and the parameters δ, r, b, and v have the least impact on the mean optimal prevention management spending rate.

In conclusion, we performed a global sensitivity analysis on ten parameters to determine their impact on the expect net present value of the forest $J(h^*)$ and the mean optimal prevention management spending rate h^* over 400 years. We found that the flow of benefits parameter B_1 has the largest impact on the value of the objective functional $J(h^*)$ evaluated at the optimal control. The discount rate r and size of the forest $\bar{A}$ are the parameters which have the second largest impact on this output. The parameters c and k_2 also have a significant impact on the output, but the least out of the 5 parameters ranked. The 5 remaining parameters were shown to not have a significant impact on the value of the output. We found that 9 of 10 parameters had a significant impact on the mean optimal prevention management spending rate. As a result, 36 pairwise comparison tests were needed to fully examine the relationships between the different PRCCs. A straightforward ranking could not be made. Instead, we broadly conclude that the fire severity parameter k, the fire severity management effectiveness parameter k_1, the non-timber damages cost parameter c, and the benefits parameter B_1 have the largest impact on the mean of h^* over 400 years. Furthermore, we conclude that fire severity suppression effectiveness parameter k_2 has a moderate impact on the output and that the forest regeneration rate δ, the discount rate r, the background hazard b, and the hazard management effectiveness parameter v have the least impact on the mean of h^*.

It is important to note that the results of this sensitivity analysis are dependent on the ranges chosen for the parameters and of course the significance levels used. For instance, this same analysis for the output $J(h^*)$ was run with a larger range for the parameter B_1: 0.005 to 0.1. In this case the only parameters with PRCCs significantly different from zero were $B_1, \bar{A},$ and r. Moreover, when the range for k_2 was decreased (keeping the range for B_1

used in the analysis presented here) to $[1, 20]$, once again the only parameters with PRCCs significantly different from zero were $B_1, \bar{A},$ and r. Also, by examining Table 2.7, we see that by choosing a significance level of $\alpha = 0.01$ or $\alpha = 0.1$ would have an impact on the results of our analysis. However, given the choices made for the present analysis, using LHS/PRCC sensitivity analysis we determined that the value of the objective functional evaluated at the optimal control is most sensitive to changes in the parameter B_1, followed by parameters $\bar{A}$ and r, and finally parameters c and k_2.

2.6 Applying Optimal Prevention Strategies to a Sequence of Fires

Our goal is to explore the effects of prevention management spending on the value of a forest over a fixed number of years given that a sequence of an unknown number of large fire events may occur within this time. Let this fixed management horizon that we wish to consider a sequence of fires over be Y years long, where $Y < T$. In short, the choice of our optimal control problem time horizon T is very large to accommodate sampling concerns, while the choice of Y is reflective of a more realistic management time horizon.

To achieve our goal we apply our optimal control problem successively to an unknown sequence of fires that happens within a fixed amount of time Y. Here, we give a quick description of the overall method. First, we solve our optimal control problem for a given parameter set. Then, we need to sample for the time of fire. Recall that solving our optimal control problem gives us the optimal prevention management spending h^* over $[0, T]$, which determines cumulative hazard y^* which is in turn used to construct the cumulative distribution function for the time of fire random variable, $\mathcal{T}$. We sample from this distribution to determine the time of the first fire, $\tau_1 \in [0, T]$. Thus, if $\tau_1 \leq Y$, then the restriction of the optimal control $h^*(t)$ over $[0, \tau_1] \subset [0, T]$ gives the prevention management spending for t such that $0 \leq t \leq \tau_1$. Now, we wish to apply our optimal control problem again to determine the optimal prevention management spending from the time of the first fire until the time of the second fire, or until the time Y, whichever comes first. However, the number of unburned acres directly following a fire at time τ_1 is no longer $\bar{A}$, as assumed in our original optimal control problem. At the time of fire τ_1, a specific number of acres are destroyed in the fire, K. In order to apply our optimal control problem again following this first fire, we must relax the assumption that before a fire the forest is entirely unburned. In the following subsection, we modify our original optimal control problem to allow for a time-varying number of unburned acres before the time of fire.

This small modification allows us to successively apply our optimal control problem. Following the fire at τ_1, a new initial condition is set for the number of unburned acres in a fire and our modified optimal control problem over $[0, T]$ is solved. We use the prevention management spending from the second application of our problem to construct the cumulative distribution function for the time of fire RV $\mathcal{T}$ and sample the time of fire τ_2. Note that τ_2 is chosen in the context of the time horizon $[0, T]$. In terms of our management horizon $[0, Y]$, the second fire occurs at time $\tau_1 + \tau_2$. If $\tau_1 + \tau_2 \geq Y$, we are done. If not, we continue to sample times of fires and solve our optimal control problem until a fire occurs at a time greater than Y. At this point we will have determined the prevention management over Y years, which we will denote h_Y, for a previously unknown number of fire.

Prevention management spending h_Y over Y years is a piecewise function, created by concatenating the truncated optimal prevention management spending rate functions determined from the successive application of our modified optimal control problem. It is important to note that while, later on, we refer to h_Y as "optimal" prevention management spending, we are not optimizing prevention management spending over $[0, Y]$. We are optimizing the prevention management spending between each fire event using our reformulated optimal control problem. We also determine J_Y, the value of the forest over Y years evaluated at h_Y, and consider the trade-offs in total prevention management spending and suppression spending. Because we are sampling the times of the fires, each time we determine h_Y and J_Y over Y years will be different. Thus, we perform a simulation study and perform multiple trials. We then examine some basic descriptive statistics for the value of the forest over Y years, the number of fires over Y years, and the total amount of prevention management spending and suppression spending over Y years. For comparison, we also consider the case when $h_Y = 0$. Before the simulation study, we will go through the details of our optimal control problem reformulation.

2.6.1 Optimal Control Problem Reformulation

We reformulate our original optimal control problem to allow for time-varying unburned acres before a fire. It is important to recognize that for this optimal control problem formulation we still maintain that only one fire will occur in $[0, T]$. Once we make this modification we will be able to truncate and apply the optimal control problem successively in order to consider a sequence of fires.

Let $A(t)$ be the number of unburned acres in a forest before a fire at time $\tau \in [0, T]$. In our original optimal control problem formulation, we assume that before the fire the forest is completely unburned, that is $A(t) = \bar{A}$ for $t < \tau$. We wish to relax this assumption. Now, before a fire at time τ the present value of the net revenue from the forest is given by

$$\int_0^\tau \Big[B\big(A(t)\big) - h(t)\Big] e^{-rt} dt, \tag{2.123}$$

where the number of unburned acres $A(t)$ before the fire is governed by the differential equation

$$A'(t) = \delta\big(\bar{A} - A(t)\big) \text{ with } A(0) = A_0 \leq \bar{A}. \tag{2.124}$$

We assume that the regeneration of the forest is only dependent on the number of unburned acres in the forest and the initial condition A_0; it is not dependent on any control variables being solved for in the current application of the problem. When considering a sequence of fires, we choose the first A_0. We will see that, when the optimal control problem is applied successively, the initial condition will be determined by the number of acres burned in the previous fire, which is a function of previous optimal suppression spending and optimal prevention management spending at the time of fire. However, the initial condition in the current application will be dependent on these functions from the previous application of the problem and thus are known. Therefore, we can simply treat the initial condition as a constant and not as a function of h and x. This becomes clearer as we work through a particular sequence of fires. The differential equation for the number of unburned acres given by (2.124) can be solved, giving

$$A(t) = \bar{A} - (\bar{A} - A_0)e^{-\delta t}. \tag{2.125}$$

As before, the number of acres destroyed in the fire, K, is dependent on the *ex ante* prevention management expenditures at the time of the fire, $h(\tau)$, and the *ex post* fire suppression expenditures at the time of the fire, $x(\tau)$. That is,

$$K = K\big(h(\tau), x(\tau)\big). \tag{2.126}$$

We continue to assume that the number of acres burned in the fire K is decreasing with respect to increases in prevention management and suppression spending; i.e. $\frac{\partial K}{\partial h} < 0$ and $\frac{\partial K}{\partial x} < 0$.

We define the *ex post* problem following a fire event at time τ similarly to our original problem. However, we must take into consideration that the number of unburned acres A in the forest before a fire is varying over time.

Let $\hat{A}(t)$ represent the number of unburned acres in the forest following a fire at time τ. The fire event at time τ is taken to be instantaneous and so the number of unburned acres destroyed in the fire K is taken into account at the time of fire τ. Thus, the number of unburned acres at time τ, $\hat{A}(\tau)$, represents the number of acres that remain in the forest after the number of acres destroyed K in the fire have been accounted for:

$$\hat{A}(\tau) = A(\tau) - K\big(h(\tau), x(\tau)\big). \tag{2.127}$$

That is, $A(\tau)$ is the number of unburned acres in the forest before we account for the number of acres burned in the fire and $\hat{A}(\tau)$ is the number of unburned acres remaining in the forest after we account for the number of acres burned in the fire.

Thus, at the time of fire there is a jump discontinuity at the time of fire between $A(\tau)$ and $\hat{A}(\tau)$. For this new problem formulation, where the number of unburned acres A before a fire is determined by function (2.125) and is not assumed constant, we still assume that another fire does not occur in our finite time horizon $[0, T]$, even though ultimately we will be considering a sequence of fires. We assume that starting from the time of fire τ the number of unburned acres in the forest will increase according to the differential equation

$$\hat{A}'(t) = \delta\big(\bar{A} - \hat{A}(t)\big) \text{ with } \hat{A}(\tau) = A(\tau) - K\big(h(\tau), x(\tau)\big), \tag{2.128}$$

so that $\hat{A}(t)$ increases toward $\bar{A}$ as time increases. Keep in mind that for this problem reformulation we are still only considering one fire and thus do not consider how the number of unburned acres will change based on a subsequent fire. Note that $A(\tau)$ is known according to (2.125). The solution to this differential equation is

$$\hat{A}(t) = \bar{A} - \Bigg(\bar{A} - \Big(A(\tau) - K\big(h(\tau), x(\tau)\big)\Big)\Bigg)e^{-\delta(t-\tau)}. \tag{2.129}$$

As before, the fire event is taken to be instantaneous and so are the associated costs. The cost of suppressing the fire $x(\tau)$ and the cost of repairing non-timber damages D are subtracted from the non-timber benefits that accrue after the fire. The cost of non-timber damages is a function of the number of acres destroyed in the fire:

$$D = D\Big(K\big(h(\tau), x(\tau)\big)\Big). \tag{2.130}$$

Again, we assume that the cost of damages is increasing with increasing K; that is, $\frac{\partial D}{\partial K} > 0$. Additionally, we assume that the cost of damages is decreasing with respect to increases in prevention management spending and suppression spending, $\frac{\partial D}{\partial h} < 0$ and $\frac{\partial D}{\partial x} < 0$.

The function describing the flow of benefits before and after the fire is the same, even though we distinguish between unburned acres before the fire and unburned acres after the fire, A and $\hat{A}$, respectively. The net present value of the forest following a fire is given by the difference between the benefits accrued from the time of fire to the end of our time horizon and the instantaneous suppression and non-timber damage costs:

$$\int_{\tau}^{T} B\big(\hat{A}(t)\big)e^{-rt}dt - \Big[D\Big(K\big(h(\tau), x(\tau)\big)\Big) + x(\tau)\Big]e^{-r\tau}, \tag{2.131}$$

subject to (2.129) and $x(\tau) \geq 0$. Again, it is assumed that there is no prevention management spending following a fire. When we move to consider sequences of fires we will have prevention management following each fire event, but this is because we are essentially "resetting" our optimal control problem after every fire.

Let the value of the forest after the fire, with $e^{-r\tau}$ factored out, be defined by

$$JW\big(\tau, A(\tau), h(\tau), x(\tau)\big) = \int_{\tau}^{T} B\big(\hat{A}(t)\big)e^{-r(t-\tau)}dt - \Big[D\Big(K\big(h(\tau), x(\tau)\big)\Big) + x(\tau)\Big]. \tag{2.132}$$

Note that the *ex post* value of the forest is now a function of the time of fire τ, the prevention management spending $h(\tau)$, suppression spending $x(\tau)$, and of the number of unburned acres $A(\tau)$ at the time of fire, before the effects of the fire have been considered. We say that JW is a function of $A(\tau)$ and not $\hat{A}(\tau)$ because $\hat{A}$ is determined by the boundary condition containing $A(\tau)$ and the differential equation (2.128). Hence, given a time of fire τ, the corresponding prevention management spending at that time $h(\tau)$, and the number of unburned acres $A(\tau)$ before the effects of the fire have been considered, the optimal *ex post* value of the forest is the solution to

$$\begin{aligned} &\max_{x(\tau)} \int_{\tau}^{T} B\big(\hat{A}(t)\big)e^{-r(t-\tau)}dt - \Big[D\Big(K\big(h(\tau), x(\tau)\big)\Big) + x(\tau)\Big] \\ &\quad \text{subject to } x(\tau) \geq 0, && (2.133) \\ &\quad \text{where } \hat{A}(t) = \bar{A} - \bigg(\bar{A} - \Big(A(\tau) - K\big(h(\tau), x(\tau)\big)\Big)\bigg)e^{-\delta(t-\tau)}, && (2.134) \end{aligned}$$

with $x(\tau)$ being a real-valued scalar representing suppression spending. Let $x^*(\tau)$ be the real-value scalar representing optimal suppression spending for a given τ, $h(\tau)$, and $A(\tau)$, which maximizes the value of the forest after the fire. The maximized *ex post* value of the forest for a given τ, $h(\tau)$, and $A(\tau)$ is henceforth denoted by

$$JW^*\big(\tau, A(\tau), h(\tau)\big) = JW\big(\tau, A(\tau), h(\tau), x^*(\tau)\big). \tag{2.135}$$

The value of the forest following a fire JW is maximized when evaluated at $x^*(\tau)$. Once again, JW^* can be positive or negative and we assume that

$$\frac{\partial JW^*\big(\tau, A(\tau), h(\tau)\big)}{\partial h} > 0. \tag{2.136}$$

Once functional forms are chosen we explicitly determine $x^*(\tau)$, and thus $JW^*\big(\tau, A(\tau), h(\tau)\big)$, using scalar optimization techniques. The details surrounding this process are discussed later in this section.

What follows is very similar to the work done for the original problem formulation and, therefore, some details have been omitted for brevity. Up to this point we have assumed that a fire at a time τ occurs within our finite time horizon $[0, T]$. However, this is not a necessary assumption as the time of the next fire could take place at any time in $(0, \infty)$. Let $\tau \in (0, \infty)$ be the time of fire. If the time of fire τ is less than T, then the total value of the forest over the time horizon $[0, T]$ is given by the sum of the net value of the forest before the fire and the net value of the forest after the fire up to time T,

$$\int_0^\tau \big[B\big(A(t)\big) - h(t)\big]e^{-rt}dt + \int_\tau^T B\big(\hat{A}(t)\big)e^{-rt}dt - \Big[D\Big(K\big(h(\tau), x(\tau)\big)\Big) + x(\tau)\Big]e^{-r\tau}, \tag{2.137}$$

where $A(t)$ is given by (2.125) and $\hat{A}(t)$ is given by (2.129). Note that this total value is the sum of (2.123) and (2.132) and gives the value of the forest over the full time horizon $[0, T]$.

If the time of the first fire τ is greater than or equal to T, then we represent the value of the forest over the time horizon $[0, T]$ by

$$\int_0^T \Big[B\big(A(t)\big) - h(t)\Big]e^{-rt}dt, \tag{2.138}$$

where $A(t)$ is given by (2.125).

In this case, we recognize that a fire will eventually occur, but because it does not occur within the time horizon $[0, T]$ we do not subtract the instantaneous suppression costs or non-timber damage costs. In summary, the value of the forest over $[0, T]$ depends on the time of fire τ, the prevention management spending h, and the initial condition $A_0 = A(0)$

for the number of unburned acres in the forest before a fire. The value of the forest can thus be represented by the piecewise function

$$V(A_0,\tau,h) = \begin{cases} \int_0^{\tau} \Big[B\big(A(t)\big) - h(t)\Big]e^{-rt}dt + \int_{\tau}^{T} B\big(\hat{A}(t)\big)e^{-rt}dt \\ \quad -\Big[D\Big(K\big(h(\tau),x(\tau)\big)\Big) + x(\tau)\Big]e^{-r\tau} & \text{if } \tau < T, \\ \\ \int_0^{T} \Big[B\big(A(t)\big) - h(t)\Big]e^{-rt}dt & \text{if } \tau \geq T, \end{cases} \tag{2.139}$$

where

$$A(t) = \bar{A} - (\bar{A} - A_0)e^{-\delta t}, \tag{2.140}$$

and

$$\hat{A}(t) = \bar{A} - \Bigg(\bar{A} - \Big(A(\tau) - K\big(h(\tau),x(\tau)\big)\Big)\Bigg)e^{-\delta(t-\tau)}. \tag{2.141}$$

We write that V is a function of A_0 because $A(t)$ is determined by our choice for A_0 and thus $A(\tau)$, which in part determines $\hat{A}(t)$, is determined by A_0.

We can update equation (2.139) to incorporate the optimal value of the forest following the fire JW^*. Assuming an optimal value of the forest after a fire in $[0,T]$ with optimal suppression $x^*(\tau)$, we rewrite the value of the forest function (2.139). Recall that $e^{-r\tau}$ was factored out of (2.132) and observe the discount factor in front of JW^*:

$$\mathcal{V}(A_0,\tau,h) = \begin{cases} \int_0^{\tau} \Big[B\big(A(t)\big) - h(t)\Big]e^{-rt}dt + e^{-r\tau}JW^*\big(\tau,A(\tau),h(\tau)\big) & \text{if } \tau < T \\ \\ \int_0^{T} \Big[B\big(A(t)\big) - h(t)\Big]e^{-rt}dt & \text{if } \tau \geq T, \end{cases} \tag{2.142}$$

where $A(t)$ is given by (2.125). Note that $\hat{A}$ is completely contained within JW^*. The equation $\mathcal{V}(A_0,\tau,h)$ represents the net present value of the forest of over the whole time interval $[0,T]$ for a given time of fire τ, prevention management spending h, and initial number of unburned acres in the forest A_0. In the case that a fire happens within the time horizon, $\mathcal{V}$ now incorporates the optimal value of the forest following a fire $JW^*\big(\tau, A(\tau), h(\tau)\big)$.

From here we follow the procedure detailed in the derivation of our original optimal control problem. The time of fire τ is now treated as a random variable characterized by the hazard function $\psi\big(h(t)\big)$. Recall that the hazard function is used to build the cumulative distribution function for the time of fire random variable. In order to convert the problem from stochastic to deterministic, the expectation of the value function (2.142) is taken with respect to the time of fire random variable and a new state variable for cumulative hazard y is introduced. The steps required for taking this expectation are not changed by our problem modifications and thus the details are omitted. In short, the reformulated optimal control problem is given by

$$\max_{h\in U} \int_0^T \Big[B\big(A(t)\big) - h(t) + \psi\big(h(t)\big) JW^*\big(t, A(t), h(t)\big)\Big] e^{-rt-y(t)} dt$$
$$\text{subject to } y'(t) = \psi\big(h(t)\big) \text{ with } y(0) = 0,$$

where

$$A(t) = \bar{A} - (\bar{A} - A_0)e^{-\delta t}, \tag{2.143}$$

and

$$U = \big\{h : [0,T] \to [0,\infty) | h \text{ is piecewise continuous}\big\}. \tag{2.144}$$

Next, we determine the closed form for $JW^*\big(t, A(t), h(t)\big)$. We use the same functional forms chosen for the original problem formulation. Recall the *ex post* problem:

$$\max_{x(\tau)} \int_\tau^T B\big(\hat{A}(t)\big) e^{-r(t-\tau)} dt - \Big[D\Big(K\big(h(\tau), x(\tau)\big)\Big) + x(\tau)\Big] \tag{2.145}$$
$$\text{subject to } x(\tau) \geq 0,$$
$$\text{where } \hat{A}(t) = \bar{A} - \Big(\bar{A} - \Big(A(\tau) - K\big(h(\tau), x(\tau)\big)\Big)\Big) e^{-\delta(t-\tau)}. \tag{2.146}$$

Using the solution to the state differential equation for $\hat{A}(t)$ above, we integrate the flow of benefits from the time of fire τ to the end of our time horizon T:

$$\begin{aligned}
&\int_{\tau}^{T} B\big(\hat{A}(t)\big)e^{-r(t-\tau)}dt \\
&\quad = \int_{\tau}^{T} B_1\Bigg(\bar{A} - \Big(\bar{A} - \big(A(\tau) - K\big(h(\tau), x(\tau)\big)\big)\Big)e^{-\delta(t-\tau)}\Bigg)e^{-r(t-\tau)}dt \\
&\quad = B_1\bar{A}\int_{\tau}^{T} e^{-r(t-\tau)}dt \\
&\qquad - B_1\Big(\bar{A} - \big(A(\tau) - K\big(h(\tau), x(\tau)\big)\big)\Big)\int_{\tau}^{T} e^{-(\delta+r)(t-\tau)}dt \\
&\quad = \frac{B_1\bar{A}}{r}\Big(1 - e^{-r(T-\tau)}\Big) \\
&\qquad - \frac{B_1\Big(\bar{A} - \big(A(\tau) - K\big(h(\tau), x(\tau)\big)\big)\Big)}{\delta + r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big).
\end{aligned} \tag{2.147}$$

Hence, the *ex post* value of the forest is given by

$$
\begin{aligned}
&JW\big(\tau, A(\tau), h(\tau), x(\tau)\big) \\
&\quad = \frac{B_1\bar{A}}{r}\Big(1 - e^{-r(T-\tau)}\Big) \\
&\qquad - \frac{B_1\Big(\bar{A} - \Big(A(\tau) - K\big(h(\tau), x(\tau)\big)\Big)\Big)}{\delta + r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big) \\
&\qquad - \Big[D\Big(K\big(h(\tau), x(\tau)\big)\Big) + x(\tau)\Big] \\
&\quad = \frac{B_1\bar{A}}{r}\Big(1 - e^{-r(T-\tau)}\Big) \\
&\qquad - \frac{B_1\big(\bar{A} - A(\tau)\big)}{\delta + r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big) \\
&\qquad - \frac{B_1\Big(K\big(h(\tau), x(\tau)\big)\Big)}{\delta + r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big) \\
&\qquad - \Big[D\Big(K\big(h(\tau), x(\tau)\big)\Big) + x(\tau)\Big] \\
&\quad = \frac{B_1\bar{A}}{r}\Big(1 - e^{-r(T-\tau)}\Big) - \frac{B_1\big(\bar{A} - A(\tau)\big)}{\delta + r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big) \\
&\qquad - K\big(h(\tau), x(\tau)\big)\left[\frac{B_1}{\delta + r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big) + c\right] - x(\tau). \qquad (2.148)
\end{aligned}
$$

Our goal is to maximize $JW\big(\tau, A(\tau), h(\tau), x(\tau)\big)$ with respect to the one-time suppression costs $x(\tau)$. We do this using scalar optimization and thus consider the partial derivative of JW (2.148) with respect to $x(\tau)$. It follows that

$$
\begin{cases}
x^*(\tau) = 0 & \text{if } \frac{\partial JW}{\partial x(\tau)} < 0, \\
x^*(\tau) \geq 0 & \text{if } \frac{\partial JW}{\partial x(\tau)} = 0.
\end{cases} \qquad (2.149)
$$

Thus, we take the partial derivative of JW (2.148) with respect to $x(\tau)$:

$$\frac{\partial JW}{\partial x} = \frac{-B_1}{\delta + r}\left(1 - e^{-(\delta+r)(T-\tau)}\right)\frac{\partial K}{\partial x} - c\frac{\partial K}{\partial x} - 1$$
$$= -\left[\frac{B_1}{\delta + r}\left(1 - e^{-(\delta+r)(T-\tau)}\right) + c\right]\frac{\partial K}{\partial x} - 1$$
$$= \left[\frac{B_1}{\delta + r}\left(1 - e^{-(\delta+r)(T-\tau)}\right) + c\right]\frac{k}{(k_1 + h)(k_2 + x)^2} - 1. \tag{2.150}$$

If $\frac{\partial JW}{\partial x(\tau)} = 0$, then $x^*(\tau) \geq 0$. To determine $x^*(\tau)$ in this case we take the partial derivative of JW with respect to $x(\tau)$ above, set the partial derivative equal to zero, and solve for $x(\tau)$:

$$0 = \left[\frac{B_1}{\delta + r}\left(1 - e^{-(\delta+r)(T-\tau)}\right) + c\right]\frac{k}{(k_1 + h)(k_2 + x)^2} - 1 \tag{2.151}$$
$$1 = \frac{1}{(k_1 + h)}\frac{k}{(k_2 + x)^2}\left[\frac{B_1}{\delta + r}\left(1 - e^{-(\delta+r)(T-\tau)}\right) + c\right] \tag{2.152}$$
$$(k_2 + x)^2 = \frac{k}{(k_1 + h)}\left[\frac{B_1}{\delta + r}\left(1 - e^{-(\delta+r)(T-\tau)}\right) + c\right]. \tag{2.153}$$

Taking the positive square root, as $x(\tau) \geq 0$, gives

$$0 \leq x^* = x^*\big(\tau, h(\tau)\big) = \sqrt{\frac{k}{\big(k_1 + h(\tau)\big)}\left[\frac{B_1}{\delta + r}\left(1 - e^{-(\delta+r)(T-\tau)}\right) + c\right]} - k_2, \tag{2.154}$$

in the case that $\frac{\partial JW}{\partial x(\tau)} = 0$.

If $\frac{\partial JW}{\partial x(\tau)} < 0$ and $x^*(\tau) = 0$, then

$$\sqrt{\frac{k}{\big(k_1 + h(\tau)\big)}\left[\frac{B_1}{\delta + r}\left(1 - e^{-(\delta+r)(T-\tau)}\right) + c\right]} - k_2 < 0 = x^*(\tau). \tag{2.155}$$

The maximum of JW could occur at the endpoint. This it follows that optimal suppression is given by

$$x^*\big(\tau, h(\tau)\big) = \max\left\{0, \sqrt{\frac{k}{\big(k_1 + h(\tau)\big)}\left[\frac{B_1}{\delta + r}\left(1 - e^{-(\delta+r)(T-\tau)}\right) + c\right]} - k_2\right\}. \tag{2.156}$$

Notice that x^* in our original problem formulation (2.55) and x^* above (2.156) are identical. Thus, optimal suppression spending at the time of fire is not dependent on the number of unburned acres in the forest at the time of fire, even though the optimal value of the forest following a fire JW^* is. The optimal suppression spending is related to the severity of the fire through the function K, which is only dependent on prevention management spending h and suppression spending x. The simplifying assumption that K is not a function of unburned acres is leftover from our original formulation. This was a proper assumption to make since the number of unburned acres before a fire was originally assumed to be constant. Perhaps in a future version of this problem, the function K could be taken to also be a function of unburned acres before the fire A. Based on our choices for our functional forms, we see that optimal suppression spending x^* is a function of the time of fire τ, and the prevention management spending $h(\tau)$.

From here, we substitute x^* into JW to obtain the optimal value of the forest after a fire. Therefore,

$$JW^*\big(\tau, A(\tau), h(\tau)\big) = JW^*\Big(\tau, A(\tau), h(\tau), x^*\big(\tau, h(\tau)\big)\Big). \tag{2.157}$$

Hence, our reformulated optimal control problem is:

$$\max_h \int_0^T \Big[B\big(A(t)\big) - h(t) + \psi\big(h(t)\big)JW^*\big(t, A(t), h(t)\big)\Big]e^{-rt-y(t)}dt$$
$$\text{subject to } y'(t) = \psi(h(t)) \text{ with } y(0) = 0$$

where

$$A(t) = \bar{A} - (\bar{A} - A_0)e^{-\delta t}, \tag{2.158}$$

and

$$U = \big\{h : [0, T] \to [0, \infty) | h \text{ is piecewise continuous}\big\}. \tag{2.159}$$

Notice that assuming $A_0 = \bar{A}$ reduces this new formulation of our optimal control problem to the previous formulation of the problem. So the modified optimal control problem is simply a more generalized version of our original problem. Also, because we are not introducing any new control or state variables to our problem, the optimality system mirrors that of the original problem, with only some minor modifications. Instead of going through the entire derivation, we simply write down the conditional current-value Hamiltonian and conditional current-value adjoint differential equation associated with state variable y. The conditional current-value Hamiltonian is given by

$$\mathcal{H} = B\big(A(t)\big) - h(t) + \psi\big(h(t)\big)JW^*\big(t, A(t), h(t)\big) + \rho(t)\psi\big(h(t)\big), \tag{2.160}$$

and the associated conditional current-value adjoint equation corresponding to the state variable y is given by

$$\rho'(t) = \Big(r + \psi\big(h(t)\big)\Big)\rho(t) + B\big(A(t)\big) - h(t) + \psi\big(h(t)\big)JW^*\big(t, A(t), h(t)\big), \tag{2.161}$$

with transversality condition

$$\rho(T) = 0. \tag{2.162}$$

We numerically solve the reformulated optimal control using the same methods as the original problem. Now, we solve our reformulated optimal control problem, which allows for non-constant unburned acres A before a fire, with several different initial conditions A_0 to illustrate the effects of this generalization on our problem. We use the parameter values determined for the 2011 Las Conchas Fire for continuity. These parameter values are listed in Table 2.1.

Here, we compare the effects of different initial conditions $A(0) = A_0$ on the solution to the optimal control problem. We solve the reformulated optimal control problem using the initial conditions $A_0 = 0.25\bar{A}$, $0.5\bar{A}$, $0.75\bar{A}$, and $\bar{A}$. Note that the case where $A_0 = \bar{A}$ is identical to our earlier results for the Las Conchas Fire. The plots in Figure 2.10 show the number of unburned acres A, the optimal prevention management spending $h^*(t)$, the

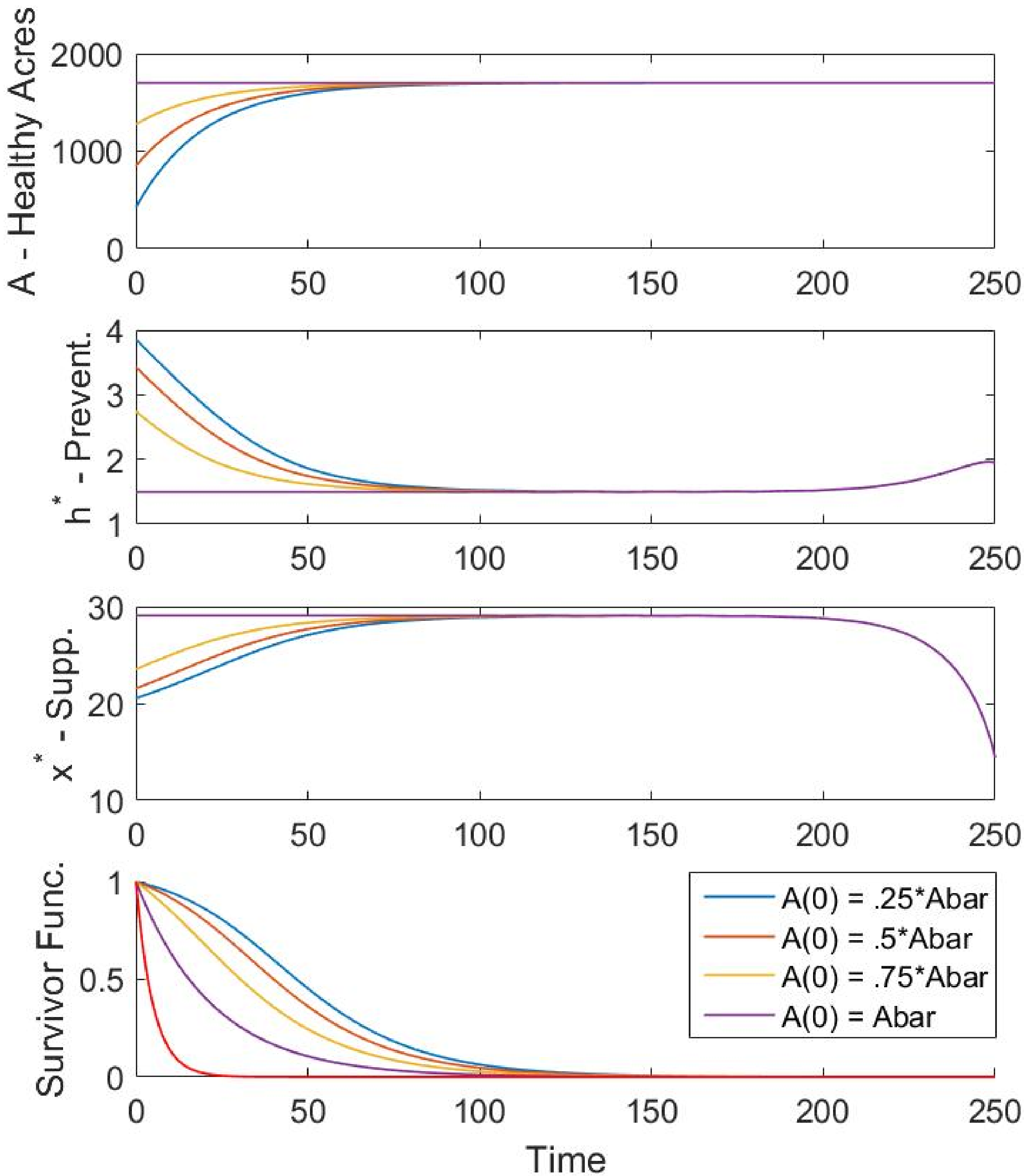


Figure 2.10: The plots above, in order from top to bottom, give the number of unburned acres A in the forest, the optimal prevention management spending h^*, the optimal suppression spending x^*, and the survivor functions $S(t)$, for different initial condition scenarios over $[0, T]$ where $T = 250$ years. In particular, we consider the cases when $A_0 = 0.25\bar{A}$, $A_0 = 0.5\bar{A}$, $A_0 = 0.75\bar{A}$, and $A_0 = \bar{A}$. The extra red line on the survivor functions plot gives the survivor function in the case when $A_0 = \bar{A}$ and $h(t) = 0$ over $[0, T]$ as a means of comparison.

Table 2.15: This table gives the value of the objective functional (the value of the forest over $[0,T]$) evaluated at the optimal control h^* for different initial conditions A_0. It also gives the value of the objective functional evaluated when $h=0$ for comparison. Additionally, the mean of the time of fire random variable is given in each case.

$A(0)$	$0.25\bar{A}$	$0.5\bar{A}$	$.75\bar{A}$	$\bar{A}$
Value of Forest ($h=0$)	488.41	582.91	677.39	771.87
Value of Forest (h^*)	496.22	597.51	700.25	801.10
Mean Time of Fire	50.7	44.63	35.94	22.3

optimal instantaneous suppression spending x^*, and the survivor functions over $[0,T]$ with $T=250$.

The topmost plot in Figure 2.10 shows the how the number of unburned acres A evolves over time according to (2.125) given different initial conditions A_0. As expected, the number of unburned acres approaches $\bar{A}$ over time. Directly below that, the plot for the optimal prevention management spending $h^*(t)$ is given. Notice that prevention management spending is greater with fewer unburned acres. In all cases where the initial condition A_0 is less than $\bar{A}$, as the number of unburned acres $A(t)$ decreases toward $\bar{A}$, the optimal prevention spending $h^*(A_0<\bar{A})$ approaches $h^*(A_0=\bar{A})$. The third plot from the top gives the optimal instantaneous suppression spending x^*. Recall that x^* is not dependent on $A(t)$; however, it is dependent on prevention management spending h, with $\frac{\partial x^*(\tau)}{\partial h(\tau)} \leq 0$. Thus, as seen in Figure 2.10, with fewer unburned acres, prevention management spending h is increased and suppression spending is decreased. Once again, in all cases where $A_0 \leq \bar{A}$, as the number of unburned acres approaches $\bar{A}$, the optimal suppression spending $x^*(A_0<\bar{A})$ increases toward $x^*(A_0=\bar{A})$. The bottom plot gives the survivor functions for the different initial condition cases. The dark red line gives the survivor function in the case where prevention spending management is taken to be zero and $A(t)=\bar{A}$ for all $t\in[0,T]$. As can be seen, fewer unburned acres A leads to greater survivability of the forest. This is because fewer unburned acres A leads to increased prevention management spending h^* and in turn, because $\frac{\partial\psi}{\partial h}<0$, there is less accumulation of hazard. With less accumulation of hazard, the probability of surviving to time t without a fire occurring increases.

Table 2.15 summarizes some results concerning our reformulated optimal control problem using different initial conditions A_0. The value of the objective functional evaluated at the optimal control h^*, the value of the objective functional evaluated with $h = 0$, and the mean of the time of fire random variable given the optimal control h^* are given. Recall that the mean time of fire is given by the equation (2.107). The value of the objective functional evaluated at the optimal control h^* is greater than the value of the objective functional evaluated at $h = 0$ for each tested initial condition. As expected, when as the value of the initial condition A_0 increases toward $\bar{A}$, the value of the forest increases. This is because more benefits are accruing and less is being spent on prevention management. We also see that the expected value of the time of fire random variable decreases as the initial condition increases toward $\bar{A}$. When the number of unburned acres A is low, more is spent on prevention management h^*. This is largely because for fewer unburned acres, the *ex post* value of the forest is less. Thus, increased spending on prevention management spending will increase the *ex post* value of the forest JW^*. The time of fire random variable is characterized by the hazard function $\psi\big(h^*(t)\big)$ and $\frac{\partial \psi}{\partial h} < 0$. Thus, in cases with fewer unburned acres, more is spent on prevention management, and thus cumulative hazard y is decreased. Since the cumulative distribution function of the time of fire random variable is given by equation (2.18) and the survivor function is given by (2.35), the mean of the time of fire random variable is reduced when initial conditions A_0 are closer to $\bar{A}$.

Now that the number of unburned acres in the forest is not assumed to be constant, we are able to more realistically consider sequences of fires. This is addressed in the following subsection.

2.6.2 Fire Sequence Simulation

Reformulating our optimal control problem to allow for non-constant unburned acres before a fire makes it possible to consider sequences of fires. In essence, we solve our optimal control problem, use y^* to build the CDF of RV $\mathcal{T}$, sample for a time of fire, truncates the solution appropriately, and then solve our optimal control problem again with an updated initial condition A_0 for the number of unburned acres in the forest. This new initial condition takes into account the number of acres destroyed in the fire according to the previous solution of the optimal control problem. We continue to do this until the time of the n^{th} fire, n unknown, is beyond a specified amount of time, Y. It is important to realize that in this process we are applying and solving our reformulated optimal control problem multiple times in succession. We are not optimizing over a given or a random sequence of fires. We are optimizing prevention management spending between each fire event using our reformulated optimal control problem. From this point, unless specified, when we refer to our optimal control problem, we are referring to the reformulated optimal control problem that allows for non-constant burned acres before a fire.

Now we discuss some important differences between the quantities Y and T. The parameter T represents the length of the time horizon considered for our optimal control problem. The parameter Y represents the the length of the management horizon over which we want to consider a sequence of fires. Over the course of the management horizon $[0, Y]$ our optimal control problem will be solved (and the solution truncated) several times. Recall that we require that $Y < T$. First, we must determine the number of years, Y, we want to consider a sequence of fires over. We will refer to it as the management horizon. The choice for Y will be reflective of a management scenario being considered. As such, we do not consider Y to be too large. For example, we would not choose $Y = 500$ as it is unrealistic to consider a management plan over that length of time. Additionally, choosing $Y = 500$ would be very computationally expensive as many fires could occur in that time. Furthermore, we require $Y < T$. In particular, T should be chosen so that $S(T)$ is very small (close to zero). This is because the CDF for the mixed type time of fire random variable is

$$\begin{cases} 1 - S(t) & \text{if } t < T \\ 1 & \text{if } t > T. \end{cases} \tag{2.163}$$

For sampling purposes, we want to be able to approximate the cumulative distribution function by

$$1 - S(t). \tag{2.164}$$

That is, in effect, we want to "guarantee" that the sampled time of fire falls within $[0, T]$. Thus, we choose T to be very large, similar to how we have chosen it previously. If T is too small, such that the survival probability $S(T)$ is not close enough to zero, then (2.164) will not serve as an adequate approximation of the CDF for $\mathcal{T}$. Thus, when we sample for the time of fire, many sample times could occur at the endpoint T and we could be forcing more fires into our management horizon than would otherwise occur.

Here, we explain the process used for a single simulation of a sequence of fires. Within a single simulation, or trial, we solve our optimal control problem multiple times and as such we will need to distinguish between the different state and control variables corresponding to the different solutions for the optimal control problem. To do this, we use numerical subscripts to indicate which solution to which the variables correspond. Next, we go through a single simulation in detail.

First, we solve our optimal control problem for a given set of parameters and initial condition $A_1(0) = \bar{A}$. As a result, we know the optimal prevention management spending $h_1^*(t)$, the optimal instantaneous suppression spending $x_1^*(t)$, the optimal cumulative hazard $y_1^*(t)$, and the number of unburned acres $A_1(t)$ over the time horizon $[0, T]$. Note the subscript 1 on the variables denotes that these functions correspond to the solution of our optimal control problem the first time we solve it. The number of unburned acres $A_1(t)$ is unaffected by either control variables x and h and as such, we do not use the star notation with it; it is completely determined by (2.125). After numerically determining the solution, we build the CDF for the time of fire RV $\mathcal{T}$ and sample for the time of the first fire. Recall that we are approximating the cumulative distribution for the time of fire random variable $\mathcal{T}$ by

$$F(\tau) = P(\mathcal{T} \leq \tau) = 1 - S(\tau). \tag{2.165}$$

For the problem formulation and derivation we take further steps and consider the time of fire as a mixed type random variable. For the sampling of the time if fire, we do not need to take these extra steps because we choose T to be very large and thus the probability of the forest not experiencing a fire near time T is essentially zero. Furthermore, recall that the survivor function is given by

$$S(t) = e^{-y(t)}, \tag{2.166}$$

where y represents cumulative hazard. Thus, given $y_1^*(t)$, we can numerically sample for the first time of fire using the cumulative distribution function for the time of fire random variable. Let $\tau_1 \in [0, T]$ be the sampled time of the first fire. If $\tau_1 > Y$, then the value of the forest, denoted by J_Y, over Y years is given by

$$J_Y = \int_0^Y \Big[B\big(A_1(t)\big) - h_1^*(t)\Big]e^{-rt}dt, \tag{2.167}$$

and we are done. We do not consider costs of suppression or non-timber damages because the time of the first fire is outside of our management horizon Y.

If $\tau_1 = Y$, then the value of the forest up to time $\tau_1 = Y$ is given by

$$J_Y = \int_0^Y \Big[B\big(A_1(t)\big) - h_1^*(t)\Big]e^{-rt}dt - \Big[D\Big(K\big(h_1^*(\tau_1), x_1^*(\tau_1)\big)\Big) + x_1^*(\tau_1)\Big]e^{-r\tau_1}, \tag{2.168}$$

and we are done. Here, we consider the costs of suppression and non-timber damages because the time of the first fire occurs at the end of the management horizon Y.

If $\tau_1 < Y$, then value of the forest up to time τ_1 is given by

$$\int_0^{\tau_1} \Big[B\big(A_1(t)\big) - h_1^*(t)\Big]e^{-rt}dt - \Big[D\Big(K\big(h_1^*(\tau_1), x_1^*(\tau_1)\big)\Big) + x_1^*(\tau_1)\Big]e^{-r\tau_1}, \tag{2.169}$$

and we need to solve our optimal control problem again and sample for the time of the next fire since $\tau_1 < Y$. The expression directly above is not labeled as J_Y because we have not yet accounted for the whole management horizon. Note that $A_1(t)$ here is given by the solution to the differential equation (2.125).

Now that a fire has occurred, and $\tau_1 < Y$, the number of unburned acres is less than $\bar{A}$ and we need to set the initial condition $A_2(0)$ to prepare for the next application of our optimal control problem. We solve our optimal control problem again, but now instead of the initial condition for A_2 being $\bar{A}$, we set it so that it takes into account the number of acres destroyed in the fire at time τ_1. In particular, we set our new initial condition to be

$$A_2(0) = A_1(\tau) - K\big(h_1^*(\tau_1), x_1^*(\tau_1)\big), \tag{2.170}$$

where $A_1(\tau) = \bar{A}$. Note that $A_1(\tau) = \bar{A}$ because for our first solution of our optimal control problem we chose the initial condition for A to be at an equilibrium point. We point out again that while this initial condition is dependent on prevention management spending and suppression spending, it is from the previous optimal control solution, and thus completely known. Thus, the initial condition can be treated as a constant.

Note that while we are considering these fires in sequence, the time horizon $[0, T]$ of our optimal control problem remains the same. At each fire event, we are in essence "resetting" our problem. With our new initial condition, we solve our optimal control problem using the same set of parameters over $[0, T]$ and once again, as a result, we will know the optimal prevention management spending $h_2^*(t)$, the optimal instantaneous suppression spending $x_2^*(t)$, the optimal cumulative hazard $y_2^*(t)$, and the number of unburned acres $A_2(t)$ over the time horizon $[0, T]$. Thus, as before, we sample for the time of the second fire, τ_2, using the CDF constructed using y_2^*. The sampled time τ_2 is associated with the time horizon $[0, T]$. We have to be careful when translating this to our management horizon. Thus, the time of the second fire in the context of our management horizon $[0, Y]$ is $\tau_1 + \tau_2$, the sum of the first sampled time of fire and the second sampled time of fire.

If $\tau_1 + \tau_2 > Y$, then the value of the forest over Y years is given by

$$J_Y = \int_0^{\tau_1} \Big[B\big(A_1(t)\big) - h_1^*(t)\Big] e^{-rt} dt - \Big[D\Big(K\big(h_1^*(\tau_1), x_1^*(\tau_1)\big)\Big) + x_1^*(\tau_1)\Big] e^{-r\tau_1} \tag{2.171}$$

$$+ \int_0^{Y-\tau_1} \Big[B\big(A_2(t)\big) - h_2^*(t)\Big] e^{-rt} dt. \tag{2.172}$$

Here, we take into account the cost associated with the first fire because it falls within $[0, Y]$. We do not take into account the costs associated with the second fire because because $\tau_1 + \tau_2 > Y$. Also notice that the limits of integration for the second integral are from 0 to $Y - \tau_1$. We begin at the time $t = 0$ because the optimal control problem is solved over $[0, T]$. We only integrate up to $Y - \tau_1$ because there are only $Y - \tau_1$ years from the time of the first fire τ_1 to the end of the management horizon Y.

If $\tau_1 + \tau_2 \leq Y$, then the value of the forest up to $\tau_1 + \tau_2$ years is given by

$$\begin{aligned} &\int_0^{\tau_1} \Big[B\big(A_1(t)\big) - h_1^*(t)\Big]e^{-rt}dt - \Big[D\Big(K\big(h_1^*(\tau_1), x_1^*(\tau_1)\big)\Big) + x_1^*(\tau_1)\Big]e^{-r\tau_1} \\ &+ \int_0^{\tau_2} \Big[B\big(A_2(t)\big) - h_2^*(t)\Big]e^{-rt}dt - \Big[D\Big(K\big(h_2^*(\tau_2), x_2^*(\tau_2)\big)\Big) + x_2^*(\tau_2)\Big]e^{-r\tau_2}. \end{aligned} \tag{2.173}$$

If $\tau_1 + \tau_2 = Y$, we are done since $\tau_2 = Y - \tau_1$. However, if $\tau_1 + \tau_2 < Y$, once again, we must sample for another time of fire and solve our problem again. We set our new initial condition for unburned acres,

$$A_3(0) = A_2(\tau_2) - K\big(h_2^*(\tau_2), x_2^*(\tau_2)\big), \tag{2.174}$$

solve our optimal control problem, and sample the next time of fire. We continue to do this until the sum of the sampled fire times is greater than or equal to Y.

Suppose that the n^{th} time of fire τ_n sampled gives $\tau_1 + \tau_2 + \cdots + \tau_n > Y$. Then, the value of the forest over Y years is given by

$$\begin{aligned} J_Y = &\int_0^{\tau_1} \Big[B\big(A_1(t)\big) - h_1^*(t)\Big]e^{-rt}dt - \Big[D\Big(K\big(h_1^*(\tau_1), x_1^*(\tau_1)\big)\Big) + x_1^*(\tau_1)\Big]e^{-r\tau_1} \\ &+ \int_0^{\tau_2} \Big[B\big(A_2(t)\big) - h_2^*(t)\Big]e^{-rt}dt - \Big[D\Big(K\big(h_2^*(\tau_2), x_2^*(\tau_2)\big)\Big) + x_2^*(\tau_2)\Big]e^{-r\tau_2} \\ &+ \cdots + \int_0^{Y-(\tau_1+\cdots+\tau_{n-1})} \Big[B\big(A_n(t)\big) - h_n^*(t)\Big]e^{-rt}dt, \end{aligned} \tag{2.175}$$

where $A_i(t)$ is governed by

$$A_i'(t) = \delta\big(\bar{A} - A_i(t)\big) \text{ with } A_i(0) = A_{i-1}(\tau_{i-1}) - K\big(h_{i-1}^*(\tau_{i-1}), x_{i-1}^*(\tau_{i-1})\big), \tag{2.176}$$

for $i = 2, \ldots, n$. Notice that the expenses from the final fire are not deducted from the value of the forest. This is because the final fire occurs outside of $[0, Y]$. If $\tau_1 + \tau_2 + \cdots + \tau_n = Y$ then

$$J_Y = \sum_{i=1}^{n} \Bigg[\int_0^{\tau_i} \Big[B\big(A_i(t)\big) - h_i^*(t)\Big] e^{-rt} dt \\ - \Big[D\Big(K\big(h_i^*(\tau_i), x_i^*(\tau_i)\big)\Big) + x_i^*(\tau_i)\Big] e^{-r\tau_i} \Bigg]. \tag{2.177}$$

Let $h_Y(t)$ be the prevention management spending over Y years where the time of the n^{th} fire is greater than or equal to Y. It is given by the following:

$$h_Y(t) = \begin{cases} h_1^*|_{[0,\tau_1]}(t) & \text{for } 0 \le t < \tau_1 \\ h_2^*|_{[0,\tau_2]}(t - \tau_1) & \text{for } \tau_1 \le t < \tau_1 + \tau_2 \\ \cdots & \\ h_{n-1}^*|_{[0,\tau_{n-1}]}\big(t - (\tau_1 + \cdots + \tau_{n-2})\big) & \text{for } \tau_1 + \cdots + \tau_{n-2} \le t \\ & \quad < \tau_1 + \cdots + \tau_{n-1} \\ h_n^*|_{\left[0, Y-(\tau_1+\tau_2+\cdots+\tau_{n-1})\right]}\big(t - (\tau_1 + \cdots + \tau_{n-1})\big) & \text{for } \tau_1 + \cdots + \tau_{n-1} \le t \le Y \end{cases}. \tag{2.178}$$

Simply put, we are concatenating the truncated optimal prevention management spending functions that were solved for between fire events. We can define the number of unburned acres over Y years, A_Y, similarly:

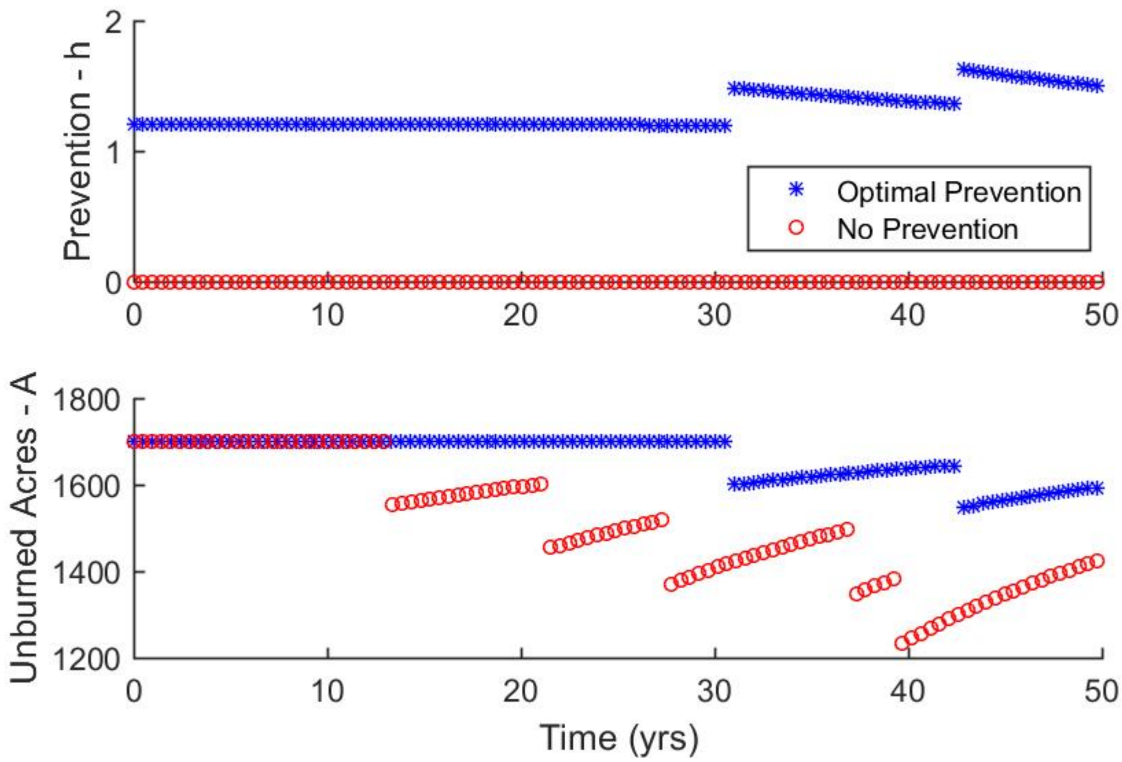


Figure 2.11: The top plot gives the management prevention spending. We compare the cases where no there is no management prevention spending and optimal prevention spending as determined by our optimal control problem which has been solved and truncated several times in succession. Recall that the time horizon for the optimal control problem is $T = 500$ years. The bottom plot gives the number of unburned acres over $[0, Y]$ where $Y = 50$. Every jump discontinuity represents a fire event. Notice that in the optimal prevention management spending case there are 2 fire events in 50 years and in the no prevention management spending case there are 5 fire events in 50 years.

$$A_Y(t) = \begin{cases} A_1|_{[0,\tau_1]}(t) & \text{for } 0 \le t < \tau_1 \\ A_2|_{[0,\tau_2]}(t-\tau_1) & \text{for } \tau_1 \le t < \tau_1+\tau_2 \\ \cdots & \\ A_{n-1}|_{[0,\tau_{n-1}]}\big(t-(\tau_1+\cdots+\tau_{n-2})\big) & \text{for } \tau_1+\cdots+\tau_{n-2} \le t \\ & \quad < \tau_1+\cdots+\tau_{n-1} \\ A_n|_{\left[0,Y-(\tau_1+\tau_2+\cdots+\tau_{n-1})\right]}\big(t-(\tau_1+\cdots+\tau_{n-1})\big) & \text{for } \tau_1+\cdots+\tau_{n-1} \le t \le Y \end{cases} . \tag{2.179}$$

For suppression spending x_Y over the management horizon $[0, Y]$, recognize that we only need amount spent at the times of the fires. However, we still define x_Y in the same manner we define h_Y:

$$x_Y(t) = \begin{cases} x_1^*|_{[0,\tau_1]}(t) & \text{for } 0 \leq t < \tau_1 \\ x_2^*|_{[0,\tau_2]}(t-\tau_1) & \text{for } \tau_1 \leq t < \tau_1 + \tau_2 \\ \cdots & \\ x_{n-1}^*|_{[0,\tau_{n-1}]}\big(t - (\tau_1 + \cdots + \tau_{n-2})\big) & \text{for } \tau_1 + \cdots + \tau_{n-2} \leq t \\ & \quad < \tau_1 + \cdots + \tau_{n-1} \\ x_n^*|_{\left[0, Y-(\tau_1+\tau_2+\cdots+\tau_{n-1})\right]}\big(t - (\tau_1 + \cdots + \tau_{n-1})\big) & \text{for } \tau_1 + \cdots + \tau_{n-1} \leq t \leq Y. \end{cases} \tag{2.180}$$

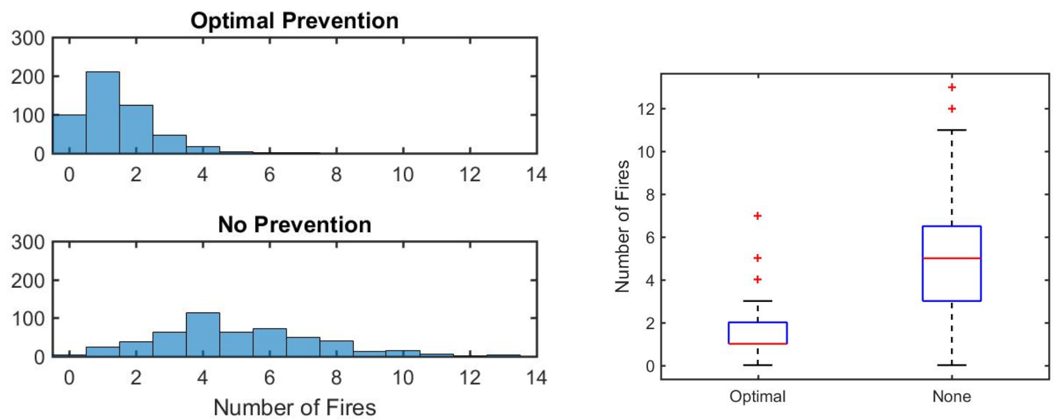
This allows us to concisely write J_Y once we define the times of fire as they are in the management horizon $[0, Y]$. Recall that the sampled times of fire τ_1, τ_2, $\ldots \tau_n$ correspond with the optimal control problem time horizon $[0, T]$. Let $\tau_1' = \tau_1$, $\tau_2' = \tau_1 + \tau_2$, $\ldots$, and $\tau_n' = \tau_1 + \tau_2 + \cdots + \tau_n$. Observe that we can now write J_Y as:

$$J_Y = \begin{cases} \int_0^T \Big(B\big(A_Y(t)\big) - h_Y(t)\Big)e^{-rt}dt & \\ \qquad - \sum_{i=1}^{n} \Big[D\Big(K\big(h_Y(\tau_i'), x_Y(\tau_i')\big)\Big) + x_Y(\tau_i')\Big]e^{-r\tau_i'} & \text{if } \tau_n' = Y \\ \int_0^T \Big(B\big(A_Y(t)\big) - h_Y(t)\Big)e^{-rt}dt & \\ \qquad - \sum_{i=1}^{n-1} \Big[D\Big(K\big(h_Y(\tau_i'), x_Y(\tau_i')\big)\Big) + x_Y(\tau_i')\Big]e^{-r\tau_i'} & \text{if } \tau_n' > Y. \end{cases} \tag{2.181}$$

Figure 2.11 provides one example of what the management prevention schedule $h_Y(t)$ and number of unburned acres $A_Y(t)$ might be in one simulation where a sequence of fires is considered over $Y = 50$ years. Because we sample for the times of the fires, every fire sequence simulation will be different. The set of parameter values used are based on the values determined for the Las Conchas Fire and are found in Table 2.1. The only difference is that we choose a smaller value for the background hazard b. Here, we let $b = 0.1$ instead of choosing $b = 0.2$. These plots also show what the number of unburned acres $A_Y(t)$ might look like given no prevention management spending $\big(h_Y(t) = 0$ over $[0, Y]\big)$. This simulation is determined separately from the optimal case. This is because h determines y which is used to build the CDF used for sampling a time of fire. The jump discontinuities in the plots correspond to the different fire events. In the no prevention management spending case 5 fires occur in 50 years and in the optimal prevention case 2 fires occur. We are careful again to distinguish that, here, the use of "optimal prevention" refers to the fact that the prevention management spending between each fire is optimized according to our optimal control problem. We are not optimizing over the full sequence of fires in the management horizon.

The simulation depicted in Figure 2.11 is just one example, or trial, of how a sequence of fires might result. Because we are sampling the times of fires, every trial is different. Thus, in order to create a more comprehensive picture concerning the effect of prevention management spending over a fixed management horizon for sequences of fires, we conduct many simulations and calculate statistics concerning the results. For our simulation study, a management horizon of $Y = 50$ years is considered and 500 trials are run. The set of parameter values, based on the values determined for the Las Conchas fire, used for the simulation study are found in Table 2.1, with the exception that now $b = 0.1$. Note in particular the large value for T and the initial condition $A_1(0)$. The initial condition $A_1(0) = \bar{A}$ only holds for the first solution of the optimal control problem in a single trial. Following that, the initial condition for the number of unburned acres following the first fire in a single trial are determined based on the sampled time of fire.

For the simulation study, 500 trials are conducted to determine the value of the forest J_Y over 50 years, given that an unknown number fires may occur in this time period for each trial. In addition to calculating value of the forest J_Y using the prevention management schedule h_Y found according to the optimal control problems, for comparison, we also calculate the value of the forest given that no money is spent on prevention management, $J_Y(h_Y = 0)$. It is important to note that these two cases are determined completely independently from one another as the cumulative distribution function used to sample the times of the fires is a function of prevention management spending. We also consider total prevention management



(a) (b)

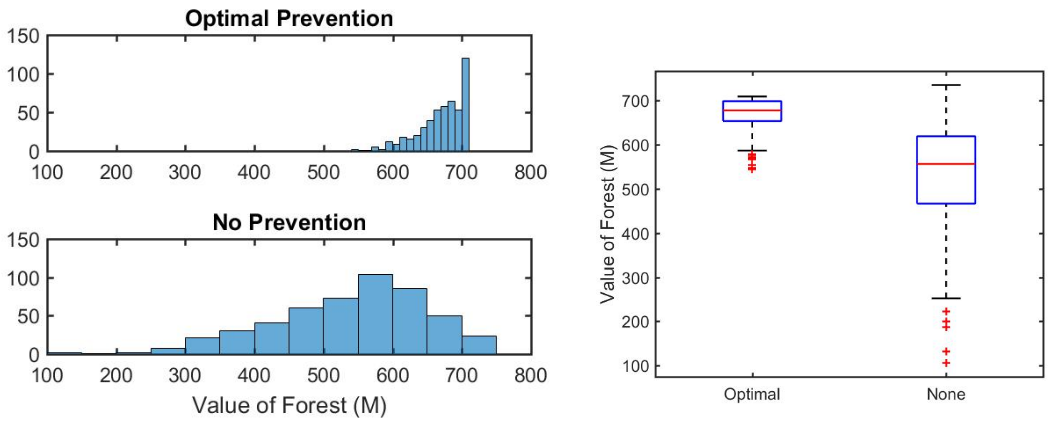
Figure 2.12: Above are two histograms and a boxplot describing the distribution of the number of fires in a 50 year period in two different cases for the 500 trials run in our simulation study. For each trial, in addition to counting the number of fires when the optimal prevention management spending is used, for comparison, we count the number of fires in a 50 year period for the case when prevention management spending is 0 over. The exact values for these statistics are found in Table 2.16.

Table 2.16: This table provides some basic statistics concerning the distribution of the number of fires in 50 years across 500 trials for the case with prevention management spending determined using optimal control and the case with no prevention management spending.

Number of Fires		
	Optimal	None
Mean	1.4	5.0
Median	1	5
Std.	1.1	2.4

spending and suppression spending in each case, in addition to the number of fires that occur in the management horizon. The results from the simulation study are discussed below.

Figure 2.12 and corresponding Table 2.16 provide details concerning the distribution of the number of fires in 50 years across 500 simulations. In the optimal prevention management case there are fewer large forest fires than in the no prevention case. In particular, the mean number of fires in the case with optimal prevention management is 1.4 and in the case with no prevention management spending is 5.0. We would like to show that this difference of means is statistically significant. It can be shown using a two-sample t-test (assuming non-equal variances) that the difference between the means for the number of fires in 50 years is



(a) (b)

Figure 2.13: Above are two histograms and a boxplot describing the distribution of the value of the forest in two different cases over 50 years for the 500 trials run in our simulation study. For each trial, in addition to determining the value of the forest with the prevention management spending determined using our optimal control problem, for comparison, the value of the forest was determined for the case when prevention management spending is 0 over the 50 years. The exact values for these statistics are found in Table 2.17.

significant, using a significance level of $\alpha = 0.05$. Similarly, the median number of fires in the optimal case is 1 and the median number of fires in the no spending case is 5. Furthermore, the standard deviation of the number of fires is approximately 1 in the optimal prevention management case and greater than 2 in the no prevention management case. Considering a variation of one standard deviation around the mean, this could be interpreted to mean that in the no prevention management case, 3-7 large fire events could be reasonably expected in a 50 year period, whereas in the optimal prevention case, only 0-2 fires. Moving from a case where there is no spending on prevention management to the case with optimal prevention management shown here, there is on average a 72% reduction in the number of fires that occur within 50 years. Hence, applying optimal prevention management spending h_Y reduces the risk of fire for a forest.

Figure 2.13 and corresponding Table 2.17 provide details concerning the distribution of the value of the forest J_Y over 50 years across 500 simulations. In the optimal prevention case the mean value of the forest over 50 years is \$671 million dollars and in the case without prevention management the mean value of the forest over 50 years is \$536 million dollars. We would like to show that this difference of means is statistically significant. It can be

Table 2.17: This table provides some basic statistics concerning the distribution of the value of the forest across 500 trials for the case with prevention management spending determined using optimal control and the case with no prevention management spending. In particular, we note that the mean and median value of the forest is larger in the optimal case, in addition to having a smaller standard deviation.

Value of Forest		
	Optimal	None
Mean	671	536
Median	677	556
Std.	34.0	111.7

shown using a two-sample t-test (assuming non-equal variances) that the difference between the means for the value of the forest is significant, using a significance level of $\alpha = 0.05$. We also see that the median value of the forest over 50 years is \$677 million dollars in the optimal prevention management case and \$556 million dollars in the no prevention management case. Additionally, the standard deviation of the distribution for the value of the forest in the optimal prevention management case is 34.0, compared to 111.7 in the no prevention management case. That is, the standard deviation is three times larger in the case without prevention management spending compared to the case with optimal prevention management. Hence, the value of the forest over 50 years with multiple fires, is less variable and likely greater in the case of optimal prevention management compared to the case without prevention management.

We also calculate the total prevention management spending and suppression spending over 50 years. In the case without prevention management spending, on average \$236M is spent on forest fire suppression over 50 years. In the case with optimal prevention management spending, on average \$42M is spent on suppression over 50 years and \$65M is spent on prevention management spending over 50 years. That is, in the case applying the optimal prevention management spending, on average only \$107M is spent on prevention management and suppression combined. Thus, we can conclude that prevention management has the potential to offset high suppression costs and decrease spending overall.

The results of our sensitivity analysis in the previous section revealed that the benefits parameter B_1 has the greatest impact on the expected net present value of the forest over

Table 2.18: This table compares total prevention management spending and suppression spending for the case where $B_1 = 0.02$ and $B_1 = 0.04$.

	$B_1 = 0.02$	$B_1 = 0.04$
Avg. Optimal Prevention Spending Total	\$65M	\$78M
Avg. Suppression Spending Total (w/ Prev.)	\$42M	\$44M
Avg. Suppression Spending Total (w/o Prev.)	\$236M	\$315M

our 500 year time horizon. We perform the above simulation study again with a larger value for B_1 to examine how the value for this parameter might affect the results for the simulation study. We are interested in doing this because we know that our choice of B_1 is not literature or data-based. We rerun our simulation study with $B_1 = 0.04$; all other parameters remain the same. The results are contained in Tables 2.18 - 2.22.

In Table 2.18 we see that, in the optimal prevention management spending case, the average optimal prevention management spending total over 50 years and the average suppression spending total are larger in the case where $B_1 = 0.04$. In particular, we see that in the case where $B_1 = 0.02$, \$65M is spent on prevention management over 50 years and in the case where $B_1 = 0.04$, \$78M is spent on prevention management. Thus, over 50 years there is only a \$13M difference between the two cases. The difference between cases concerning the average total suppression spending is even smaller, with total suppression spending at \$42M in the case where $B_1 = 0.02$ and and \$44M in the case where $B_1 = 0.04$.

Tables 2.20 and 2.19 contain the statistics concerning the number of fires that occur in 50 years for the cases $B_1 = 0.02$ and $B_1 = 0.04$. As is seen, the values in these tables are nearly identical. This tells us that the variation in the value for B_1 does not have a large effect on the number of fires that occur in 50 years. This is expected as the prevention management spending rate is also similar in the two cases.

The statistics used to compare the value of the forest over 50 years for the two cases are found in Tables 2.22 and 2.21. Here, we see a much more substantial difference between the two cases. In particular, in the case where $B_1 = 0.02$, when optimal prevention management spending is applied the mean value of the forest over 50 years is \$671M and in the case where $B_1 = 0.04$ the mean value of the forest over 50 years is \$1,400M. That is, the value of the forest over 50 years for the case where $B_1 = 0.04$ is over double the value for the case where

Table 2.19: This table contains statistics for mean, median, and std. for the number of fires occurring in 50 years for 500 simulations. We note that these results are very similar to the case for $B_1 = 0.02$.

Number of Fires - $B_1 = 0.04$		
Prev.	Optimal	None
Mean	1.2	5.1
Median	1	5
Std.	1.0	2.3

Table 2.20: This table contains the statistics for the mean, median, and standard deviation for the number of fires occurring in 50 years for 500 simulations in the case that $B_1 = 0.02$. We note that these results are very similar to the case when $B_1 = 0.04$.

Number of Fires - $B_1 = 0.02$		
Prev.	Optimal	None
Mean	1.4	5.0
Median	1	5
Std.	1.1	2.4

$B_1 = 0.02$. However, this is not too surprising because we know from our sensitivity analysis that the parameter B_1 has the most impact on the value of the objective functional $J(h^*)$ evaluated at the optimal control. So, since 0.04 is double 0.02, it is not surprising that this difference is reflected in the value of the forest over 50 years.

From this comparison we see that while the benefits parameters B_1 does have a significant impact on the value of the forest, slight variation in its value does not have a large effect on the optimal prevention management spending rate or the number of fires that occur in 50 years. Thus, even though determining B_1 may be difficult we have shown that it its largest impact is on the value of the forest, not the prevention management spending rate.

Our results reveal that, on average, in the case of optimal prevention management spending, h_Y, there are fewer fires and an increased value of the forest in comparison to the case with no prevention management spending. Furthermore, the standard deviation around the average number of fires and value of the forest is much smaller in the optimal prevention management case in comparison to the no prevention management case. This suggests that using optimal prevention management spending is a less risky management

Table 2.21: This table contains the statistics for the mean, median, and standard deviation for the value of the forest over 50 years for 500 simulations in the case that $B_1 = 0.04$.

Value of Forest - $B_1 = 0.04$		
Prev.	Optimal	None
Mean	1400	1207
Median	1410	1221
Std.	39.9	137.7

Table 2.22: This table contains the statistics for the mean, median, and standard deviation for the value of the forest over 50 years for 500 simulations in the case that $B_1 = 0.02$.

Value of Forest - $B_1 = 0.02$		
Prev.	Optimal	None
Mean	671	536
Median	677	556
Std.	34.0	111.7

option when compared to the case without prevention management spending. Additionally, we see that prevention management spending can offset high suppression costs and decrease the total amount of spending overall.

2.7 Conclusions

Our goal is to examine the economic trade-offs between prevention management spending and suppression spending as applied to large forest fire events when the time of fire is unknown. To do this, we formulated a problem with stochastic time of fire and converted it to a deterministic optimal control problem using Reed's method. Within this chapter, we present the numerical results from our optimal control problem applied to two parameter sets representing two recent fire events, a global parameter sensitivity analysis evaluating the impact of our parameters on the expected net value of the forest and the mean optimal prevention management spending rate, and a simulation study concerning the effects of prevention management spending over a finite management horizon given an unknown sequences of fires. Here, we summarize these results and draw some further conclusions.

For our original optimal control problem our goal is to maximize the expected net present value of the forest J over a finite time horizon $[0, T]$ with respect to the control variable h, representing the prevention management spending rate. Moreover, we assume that before the fire the forest was entirely unburned. The formulation of our original optimal control problem begins by considering the time of fire τ as given. We construct the *ex ante* problem to give the net present value of the forest before a fire. Here, the prevention management spending rate is subtracted from the constant flow of benefits rate from the forest over time. The fire at time τ and all associated costs are taken to be instantaneous. Thus, we construct the *ex post* problem to give the net value of the forest from the time of fire up to the end of time horizon T. Suppression spending x and non-timber damage costs D are taken into account in the *ex post* problem. Furthermore, we assume that there is no prevention management spending following a fire. Next, we solve the *ex post* problem by maximizing the value of the forest following a fire with respect to suppression spending at the time of fire using scalar optimization. Once the optimal suppression spending x^*, and thus the optimal value of the forest JW^* following a fire, is determined, we treat the time of fire as a random variable and apply Reed's method. The random variable for the time of fire is characterized using a hazard function $\psi(h)$. With ψ we build the cumulative distribution function for the random variable and thus we can calculate the expectation of the piecewise function representing the value of the forest for a given time of fire with respect to the time of fire random variable. This, along with the addition of a state variable y capturing cumulative hazard, allows us to convert what was a stochastic problem into a deterministic optimal control problem.

To our solve our optimal control problem, we use numerical methods in MATLAB to determine an approximation to the optimal prevention management spending rate h^* by maximizing the conditional current-value Hamiltonian pointwise. We use two large, recent fire events to guide the development of two parameters sets to examine. We compare the results for the 2011 Las Conchas Fire and the 2014 Happy Camp Complex. In both cases we see an increased value of the forest and an increased mean time of fire in the case of optimal prevention management spending in comparison to the case without prevention management spending. For both fires, the parameters $\bar{A}, \delta, k_1, k_2, r$, and v were chosen to be the same. Thus, the parameters that differed between the two fire parameter sets are B_1, b, c, and k.

The parameter b represents the background hazard of fire found in the hazard function ψ. It is an important parameter when calculating the expectation of the time of fire random variable (2.107). Using the recent fire history of the region each selected fire burned, we choose $b = 0.2$ for the Las Conchas Fire and $b = 0.05$ for the Happy Camp Complex. The mean time of fire for the Las Conchas Fire example in the case of optimal prevention management spending h^* is 22.3 years; in the case of no prevention management spending $h = 0$ the mean time of fire is 5 years. This is nearly a 350% increase in the mean time of fire from the case without prevention management spending to the case with optimal prevention management spending. The mean time of fire for the Happy Camp Complex in the case of optimal prevention management h^* is 76.4 years; in the case without prevention management spending the mean time of fire is 20 years. Thus, by applying optimal prevention management spending we see a nearly 280% increase in the mean time of fire from the case without prevention management spending. In both fire examples, optimal prevention management spending increases the mean time of fire in a substantial way. The percent increase from the no prevention management case to the optimal prevention management case is larger in the Las Conchas Fire example than in the Happy Camp Complex example because on average more was spent on prevention management for the Las Conchas Fire example solution. In the Las Conchas example, approximately \$1.5M is spent per year on prevention management while in the Happy Camp Complex example on average only \$1.35M is spent per year.

From our parameter sensitivity analysis, we determine which parameters have a significant impact on the value of the objective functional $J(h^*)$ and rank the significant parameters according to their level of impact. We use Latin Hypercube Sampling and calculate partial rank correlation coefficients for each parameter in our optimal control problem. Using a significance level of $\alpha = 0.05$, we determine that parameters $\bar{A}, B_1, c, k_2$, and r have a partial rank correlation coefficient significantly different from zero. Next, we rank these parameters according to the strength of their impact on $J(h^*)$. In order to do this we perform a family of 10 pairwise hypothesis tests at an adjusted per test significance level to control for the familywise error rate. After making the appropriate adjustments to the

PRCCs in order to compare them with a z-test, we find that of the parameters found to have the most significant impact on the value of the objective functional, the parameter B_1 has the most impact on $J(h^*)$, followed by the parameters $\bar{A}$ and r, and finally the parameters c and k_2. Given that $\bar{A}$ and r are the same for both fire examples, it is reasonable that the difference in the $J(h^*)$ for the two fire examples is largely accounted for by the difference in the choice of the B_1 parameter.

From our parameter sensitivity analysis, we see that benefits parameter B_1 has the greatest impact on the value of the objective functional $J(h^*)$, which is the expected net present value of the forest when the optimal prevention management spending rate h^* is applied. For the Las Conchas Fire we choose $B_1 = 0.02$ and for the Happy Camp Complex we choose $B_1 = 0.05$; there is roughly an 86% difference between these parameters. Thus, it is unsurprising that the percent difference in $J(h^*)$ for the two fires is 88.5% as $J(h^*) = \$801.2$M for the Las Conchas Fire and $J(h^*) = \$2,073$M for the Happy Camp Complex. As the parameters $\bar{A}$ and r, which are the parameters which have the second-greatest impact on $J(h^*)$, are the same across the two fire examples, it is unsurprising that the difference in the expected net present value of the forest can be accounted for by the difference in the choice for B_1.

We also test the sensitivity of the mean of the optimal prevention management spending rate h^* over 400 years to changes across 10 parameters. Nine of the ten parameters considered were found to have a significant impact on the output. Because of this, 36 pairwise comparison tests had to be performed in order to properly compare the impact of the parameters on the output. While we could not determine a strict ranking of the parameters as we did with the output $J(h^*)$, we were able to broadly categorize the parameters in terms of impact. We conclude that the parameters k, k_1, c, and B_1 have the most impact on the output, k_2 has a moderate impact on the output, and the parameters b, v, r, and δ have the least impact on the output.

Following our parameter sensitivity analysis we modify our optimal control problem so that we can examine the effects of prevention management spending for an unknown sequence of fires in a given management horizon $[0, Y]$. By allowing for time-varying unburned acres A before a fire, we are able to apply our reformulated optimal control problem in succession and simulate a sequence of fires. In short, we solve our optimal control problem for a given set of parameters and use the resulting solution for y^* to build the cumulative distribution function for the time of fire random variable. This enables us to sample for a time of fire. If the sampled time of fire is greater than the length of our management horizon Y, then we are done and we calculate the value of the forest over Y years given the solution from our optimal control problem. If the time of fire sampled is less than the length of the management horizon, then we substitute the appropriate initial condition for the number of unburned acres into our optimal control problem and we solve it again, build the CDF for the time of fire RV, and sample for a time of fire. If the sum of the sampled times of fire is greater than or equal to the management horizon we are done and we calculate the value of the forest over the management horizon. If not, we once again reset the initial condition for the number of unburned acres and run our optimal control problem again. This process repeats until the sum of sampled fire times is greater than or equal to the length of the management horizon Y. This is just one example of how a sequence of fires might play out. Thus, to form a more comprehensive picture concerning the value of the forest for an unknown sequence of fires, we run this simulation 500 times, each time recording the value of the forest J_Y over management horizon, the number of fires that occur within each simulation, and total prevention and management spending. We run 500 simulations using our optimal control problem to determine prevention management spending over 50 years and for comparison we run 500 simulations with no prevention management spending.

For 500 simulations and a management horizon of $Y = 50$ years, we find that the mean value of the forest J_Y in the no prevention management case is \$536M with a standard deviation of \$111.7M. In the case using prevention management h_Y determined by the successive application of our optimal control problem, we find the mean value of the forest J_Y to be \$671M with a standard deviation of \$34.0M. Therefore, in the case of applying prevention management, not only is the mean value of the forest increased over our management horizon, the standard deviation around that mean is much smaller than in the case where there is no prevention management spending. Thus, with the application of h_Y, the value of the forest is greater and is less variable than in the case where prevention management spending is not applied to the forest.

We also examine how prevention management spending might affect the number of large fires that occur within the management horizon $[0, Y]$. In 500 simulations, we find that in the no prevention management case the mean number of fires was 5.0 with a standard deviation of 2.4. In the case with prevention management h_Y the mean number of fires in Y years is 1.4 fires with a standard deviation of 1.1. Thus, by applying prevention management h_Y the forest experiences fewer fires and is at less risk of experiencing devastating large fire events.

Lastly, we determine the trade-offs between prevention management spending and suppression spending. In the case without prevention management spending, on average \$236M was spent on fire suppression over the course of 50 years. In the case with applying optimal prevention management spending, only \$42M was spent on average on suppression over 50 years and \$65M was spent on prevention management. That is, when optimal prevention management is employed, not only are high suppression costs drastically reduced, total spending on prevention management and suppression is less than the case without prevention management.

Through our work with fire sequences we saw with prevention management on average a 25% increase in the value of the forest over a 50 year management horizon when moving from a case without prevention management. Furthermore, the value of the forest is significantly less variable when prevention management is applied. We also saw a 72% reduction in the number of forest fires occurring over 50 years when considering optimal prevention management spending compared to no prevention management spending. In terms of trade-offs between prevention management spending and suppression spending, we observed an 88% reduction in suppression spending on average with prevention management, and a 55% reduction in spending overall.

Overall, this optimal control formulation with the use of Reed's method demonstrates that prevention management spending, while an upfront cost, can ultimately increase the economic value of a forest given that prevention management spending works to decrease the hazard of fire in the forest and to decrease the number of acres devastated by large and severe fire events. Additionally, prevention management spending can help offset large fire suppression costs. By reducing the number of large, severe fire events in a forest there is less risk to the forest and communities nearby or integrated within the forested region. Moreover, these conclusions hold and are strengthened further by the results from our simulation study concerning sequences of fires. This simulation study reveals that the application of prevention management spending can increase the value of the forest and reduce the number of fires within a fixed period of time. In particular, the relatively small standard deviations for the mean of the number of fires in and the value of the forest over 50 years indicates that prevention management overall reduces risk and uncertainty for forest managers. Thus, overall, this work has shown that prevention management spending can offset high fire suppression costs and can be economically beneficial overall. In particular, we believe that this optimal control problem formulation and corresponding numerical results illustrate a valuable tool for understanding the trade-offs between prevention management and suppression spending.

Chapter 3

Optimal Prevention Strategies with Cumulative Prevention Effects

3.1 Background

In our original optimal control problem introduced in Section 2.2, we assume that the effects of prevention management spending h are instantaneous. That is, we assume that prevention management spending decreases the number of acres burned in the fire K, and therefore suppression costs x^*, only if the spending occurs exactly at the time of fire. Due to this assumption, prevention management spending before the time of fire does nothing to decrease the number of acres destroyed in the fire or reduce suppression costs. In this chapter, we relax this assumption and allow for cumulative effects of prevention management spending. This way prevention management spending accumulates stock over time and thus prevention management spending before the time of fire might contribute to the reduction of fire severity and fire hazard at the time of fire.

Let $z(t)$ be a state variable representing the cumulative prevention management stock and let it be governed by the differential equation

$$z'(t) = h(t) - \gamma z(t) \text{ with } z(0) = z_0. \tag{3.1}$$

As can be seen in equation (3.1), prevention stock z increases with prevention management spending h and decays at of rate of γ proportional to the current level of cumulative prevention management stock. Thus, $z(t)$ reflects cumulative benefits of prevention management spending while also capturing the impermanence of prevention management treatments.

We incorporate cumulative prevention management stock z into our optimal control problem through the *ex post* problem JW^*, more specifically through the function K, and the hazard function ψ. We now assume that the level of cumulative prevention management $z(\tau)$ stock at the time of fire will reduce the number of acres burned in the fire, i.e. $\frac{\partial K}{\partial z} < 0$. Additionally, we now assume that increases in prevention stock will reduce hazard, i.e. $\frac{\partial \psi}{\partial z} < 0$. The details of how this affects the formulation of our optimal control problem are included in the following section.

3.2 Optimal Control Problem Formulation

The formulation of our optimal control problem with a state variable representing cumulative prevention management stock z is very similar to our previous optimal control problem formulations. Therefore, we focus on the components of this new formulation that are affected by the inclusion of the new state variable z. First, we formulate the *ex post* problem. Suppose that a fire instantaneously occurs at time $\tau \in [0, T]$. The benefits accrued following the fire are given by

$$\int_{\tau}^{T} B\big(\hat{A}(t)\big) e^{-rt} dt, \tag{3.2}$$

where the number of unburned acres $\hat{A}(t)$ following the fire is determined by the solution to

$$\hat{A}'(t) = \delta\big(\bar{A} - \hat{A}(t)\big) \text{ with } \hat{A}(\tau) = A(\tau) - K\big(z(\tau), x(\tau)\big), \tag{3.3}$$

where $A(\tau)$ represents the number of unburned acres in the forest at the time of fire before we account for the effects of the fire. Note that in this problem formulation, we are using

$A(t)$ and $\hat{A}(t)$ in the same way that we did for the problem formulation in Section 2.6. The solution to this differential equation and boundary condition (3.3) is given by

$$\hat{A}(t) = \bar{A} - \Big(\bar{A} - \big(A(\tau) - K\big(z(\tau), x(\tau)\big)\big)\Big)e^{-\delta(t-\tau)}. \tag{3.4}$$

Furthermore, we emphasize that the number of acres burned in the fire K is given by

$$K = K(z, x) = \frac{k}{(k_1 + z)(k_2 + x)}. \tag{3.5}$$

We see that the number of acres destroyed in the fire $K\big(z(\tau), x(\tau)\big)$ is a function of cumulative prevention management stock $z(\tau)$ at the time of the fire, instead of prevention management spending $h(\tau)$ at the time of fire, and suppression spending $x(\tau)$ at the time of fire. Thus, the net present value of the forest following a fire is given by the difference between the benefits accrued from the time of fire to the end of our time horizon and the instantaneous suppression and non-timber damage costs:

$$\int_{\tau}^{T} B\big(\hat{A}(t)\big)e^{-rt}dt - \Big[D\Big(K\big(z(\tau), x(\tau)\big)\Big) + x(\tau)\Big]e^{-r\tau}, \tag{3.6}$$

subject to equation (3.4) and $x(\tau) \geq 0$. It is still assumed that there is no prevention management spending h following the fire.

Let the *ex post* value of the forest, with $e^{-r\tau}$ factored out, be defined by

$$JW\big(\tau, A(\tau), z(\tau), x(\tau)\big) = \int_{\tau}^{T} B\big(\hat{A}(t)\big)e^{-r(t-\tau)}dt - \Big[D\Big(K\big(z(\tau), x(\tau)\big)\Big) + x(\tau)\Big]. \tag{3.7}$$

The *ex post* value of the forest JW is a function of the time of fire τ, cumulative prevention management stock $z(\tau)$ at the time of fire, suppression spending $x(\tau)$ at the time of fire, and the number of unburned acres $A(\tau)$ at the time of fire before the effects of the fire have been considered. In particular, we point out that JW is no longer explicitly a function of h. Hence, given a time of fire τ, the corresponding cumulative prevention management stock $z(\tau)$ at that time, and the number of unburned acres $A(\tau)$, the optimal *ex post* value of the forest is the solution to the following optimal control problem:

$$\max_{x(\tau)} \int_{\tau}^{T} B\big(\hat{A}(t)\big)e^{-r(t-\tau)}dt - \Big[D\Big(K\big(z(\tau), x(\tau)\big)\Big) + x(\tau)\Big] \tag{3.8}$$

$$\text{subject to } x(\tau) \geq 0, \tag{3.9}$$

$$\text{where } \hat{A}(t) = \bar{A} - \Bigg(\bar{A} - \Big(A(\tau) - K\big(z(\tau), x(\tau)\big)\Big)\Bigg)e^{-\delta(t-\tau)}, \tag{3.10}$$

with $x(\tau)$ being a real-valued scalar representing suppression spending at the time of fire. Let $x^*(\tau)$ be the real-value scalar representing optimal suppression spending for a given time of fire τ, cumulative prevention management stock $z(\tau)$, and the number of unburned acres $A(\tau)$, which maximizes the value of the forest after the fire. The maximized value of the forest after the fire for a given τ, $h(\tau)$, and $A(\tau)$ is henceforth denoted by

$$JW^*\big(\tau, A(\tau), z(\tau)\big) = JW\big(\tau, A(\tau), z(\tau), x^*(\tau)\big). \tag{3.11}$$

We want to emphasize that JW^* is a function of $A(\tau)$ because $\hat{A}$ is a function of $A(\tau)$, see equation (3.4). Thus, JW^* is not written as a function of $\hat{A}$, but rather as a function of $A(\tau)$.

The value of the forest following a fire JW is maximized when evaluated at $x^*(\tau)$. We assume that

$$\frac{\partial JW^*\big(\tau, A(\tau), z(\tau)\big)}{\partial z} > 0. \tag{3.12}$$

That is, we assume that increases in cumulative prevention management stock z increase the optimal *ex post* value of the forest JW^*. Later in this chapter, we explicitly determine $x^*(\tau)$, and thus $JW^*\big(\tau, A(\tau), z(\tau)\big)$, using scalar optimization techniques.

Now, we consider the *ex ante* problem, which gives the net present value of the forest before a fire. Let $A(t)$ be the number of unburned acres in the forest before a fire. Before a fire at time $\tau \in [0, T]$, the present value of the net revenue from the forest is given by

$$\int_0^{\tau} \Big[B\big(A(t)\big) - h(t) - \frac{\epsilon}{2}h^2(t)\Big]e^{-rt}dt, \tag{3.13}$$

where $A(t)$ is given by the solution to the differential equation

$$A'(t) = \delta(\bar{A} - A(t)) \text{ with } A(0) = A_0 \leq \bar{A}. \tag{3.14}$$

The solution to this differential equation is

$$A(t) = \bar{A} - (\bar{A} - A_0)e^{-\delta t}. \tag{3.15}$$

Notice that in the integrand of the objective functional (3.13) we include a quadratic cost term. This term is not present in our original optimal control problem. Without this term, the final formulation of this new optimal control problem is linear in the control h. This is because JW^* is no longer explicitly a function of h and the differential equation governing z is also linear in h. Additionally, we are choosing our hazard function ψ to now be a function of cumulative prevention management stock z, instead of a function of prevention management h:

$$\psi = \psi\big(z(t)\big) = be^{-vz(t)}. \tag{3.16}$$

Due to the difficulty of determining the possible singular control and of obtaining convergence of the numerical algorithm in the "linear in the control" case, we decided to incorporate a quadratic control term into the objective functional.

What follows is very similar to the work done for the original problem formulation and therefore, the details have been omitted for brevity. Up to this point we have assumed that a fire occurs within our finite time horizon $[0, T]$. Let $\tau \in (0, \infty)$ be the time of fire. If the time of fire τ is less than T, then the net present value of the forest over the time horizon $[0, T]$ is given by the sum of the net value of the forest before the fire and the net value of the forest after the fire up to time T,

$$\int_0^\tau \big[B\big(A(t)\big) - h(t) - \frac{\epsilon}{2}h^2(t)\big]e^{-rt}dt + \int_\tau^T B\big(\hat{A}(t)\big)e^{-rt}dt - \Big[D\Big(K\big(z(\tau), x(\tau)\big)\Big) + x(\tau)\Big]e^{-r\tau}, \tag{3.17}$$

where $A(t)$ is given by equation (3.15) and $\hat{A}(t)$ is given by equation (3.4). Note that this is the sum of (3.13) and (3.6) and gives the net present value of the forest over the full time horizon $[0, T]$.

If the time of fire τ is greater than or equal to T, then we represent the net present value of the forest over the time horizon $[0, T]$ by

$$\int_0^T \Big[B\big(A(t)\big) - h(t) - \frac{\epsilon}{2}h^2(t)\Big]e^{-rt}dt, \tag{3.18}$$

where $A(t)$ is given by (3.15).

The value of the forest can thus be represented by the piecewise function

$$V(A_0, \tau, h, z) = \begin{cases} \int_0^\tau \big[B\big(A(t)\big) - h(t) - \frac{\epsilon}{2}h^2(t)\big]e^{-rt}dt \\ \quad + \int_\tau^T B\big(\hat{A}(t)\big)e^{-rt}dt \\ \quad -\Big[D\Big(K\big(z(\tau), x(\tau)\big)\Big) + x(\tau)\Big]e^{-r\tau} & \text{if } \tau < T, \\ \\ \int_0^T \big[B\big(A(t)\big) - h(t) - \frac{\epsilon}{2}h^2(t)\big]e^{-rt}dt & \text{if } \tau \geq T, \end{cases} \tag{3.19}$$

where

$$A(t) = \bar{A} - (\bar{A} - A_0)e^{-\delta t}, \tag{3.20}$$

and

$$\hat{A}(t) = \bar{A} - \Bigg(\bar{A} - \Big(A(\tau) - K\big(z(\tau), x(\tau)\big)\Big)\Bigg)e^{-\delta(t-\tau)}. \tag{3.21}$$

We can update equation (3.19) to incorporate the optimal value of the forest JW^* following the fire. Assuming an optimal value of the forest after a fire in $[0, T]$ with optimal suppression $x^*(\tau)$, we rewrite the value of the forest piecewise function (3.19):

$$\mathcal{V}(A_0,\tau,h,z)=\begin{cases}\int_0^{\tau}\big[B\big(A(t)\big)-h(t)-\frac{\epsilon}{2}h^2(t)\big]e^{-rt}dt & \\ \quad +e^{-r\tau}JW^*\big(\tau,A(\tau),z(\tau)\big) & \text{if } \tau<T \\ & \\ \int_0^{T}\big[B\big(A(t)\big)-h(t)-\frac{\epsilon}{2}h^2(t)\big]e^{-rt}dt & \text{if } \tau\geq T.\end{cases} \tag{3.22}$$

Recall that the function $\hat{A}$ is completely incorporated into the function JW^* and because $\hat{A}$ is a function of A at a particular time, we write JW^* as a function of $A(t)$, not $\hat{A}$. The function $\mathcal{V}(A_0,\tau,h,z)$ represents the value of the forest of over the whole time interval $[0,T]$ for a given time of fire τ, prevention management spending rate h, cumulative prevention management stock z, and initial number of unburned acres in the forest A_0. In the case that a fire happens within the time horizon, $\mathcal{V}$ now incorporates the optimal value of the forest $JW^*\big(\tau,A(\tau),z(\tau)\big)$ following a fire.

From here we follow the procedure detailed in the derivation of our original optimal control problem to convert a stochastic problem to a deterministic optimal control problem. The time of fire τ is now treated as a random variable characterized by the hazard function ψ. As mentioned above, the hazard function is now a function of cumulative prevention management stock z: $\psi=\psi\big(z(t)\big)$. This change does not alter the mechanics of the procedure used to convert our problem from stochastic to deterministic. The survivor function and cumulative distribution function are constructed in the same way. In order to convert the problem from stochastic to deterministic, the expectation of the value function is taken with respect to the time of fire random variable and a new state variable for cumulative hazard y is introduced. In short, the optimal control problem, now with state variable z representing cumulative prevention management stock, is given by

$$\max_{h\in U}\int_0^T\Big[B\big(A(t)\big)-h(t)-\frac{\epsilon}{2}h^2(t)+\psi\big(z(t)\big)JW^*\big(t,A(t),z(t)\big)\Big]e^{-rt-y(t)}dt \tag{3.23}$$

$$\text{subject to } y'(t)=\psi\big(z(t)\big) \text{ with } y(0)=0 \tag{3.24}$$

$$z'(t)=h(t)-\gamma z(t) \text{ with } z(0)=z_0, \tag{3.25}$$

where

$$A(t) = \bar{A} - (\bar{A} - A_0)e^{-\delta t}, \tag{3.26}$$

and

$$U = \big\{h : [0,T] \to [0,M] | h \text{ is piecewise continuous}\big\}. \tag{3.27}$$

Notice that for this problem formulation we have set an upper bound M on the control variable h. This bound is necessary in the proofs concerning the existence of an optimal control. The proof of the existence of the optimal control is found in a later section.

Now that we have formulated the optimal control problem, we determine the closed form for $JW^*\big(t, A(t), z(t)\big)$. We use the same functional forms chosen for the original problem formulation. However, we reiterate that the function which determines the number of acres destroyed in the fire K is now a function of cumulative prevention management stock z, instead of h, and suppression spending x. Beyond this change, nothing is else is altered and so all of the steps taken to determine x^* are the same as in the previous chapter. Thus, we will simply write that optimal suppression is given by

$$x^*\big(\tau, z(\tau)\big) = \max\left\{0, \sqrt{\frac{k}{\big(k_1 + z(\tau)\big)}\left[\frac{B_1}{\delta + r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big) + c\right]} - k_2\right\}. \tag{3.28}$$

Optimal suppression spending x^* is a function of the time of fire τ and the cumulative prevention management stock $z(\tau)$ at the time of fire. For the remainder of the problem, we will continue to write $x^* = x^*(\tau)$, instead of $x^*\big(\tau, z(\tau)\big)$, for simplicity. We substitute x^* into JW to obtain the optimal value of the forest after a fire. Hence,

$$JW^*\big(\tau, A(\tau), z(\tau)\big) = JW^*\big(\tau, A(\tau), z(\tau), x^*(\tau)\big). \tag{3.29}$$

That is, the optimal *ex post* value of the forest is dependent on the time of fire τ, the number of unburned acres in the forest $A(\tau)$ before we account for the number of acres burned in the fire, and cumulative prevention management stock $z(\tau)$ at the time of fire. Next, we derive the optimality system and prove the existence of an optimal control.

3.3 Necessary Conditions

In this section, we derive the optimality system for our new optimal control problem which includes a new state variable z for cumulative prevention management stock. We leave out the details leading to the derivation of the conditional current-value Hamiltonian because the steps are the same as encountered in Section 2.3. We simply present the results. Let H represent the standard Hamiltonian with $\lambda_1(t)$ representing the adjoint equation corresponding to the state variable y and with $\lambda_2(t)$ representing the adjoint equation corresponding to the state variable z. The standard Hamiltonian is given by

$$\begin{aligned} H = &\Big[B\big(A(t)\big) - h(t) - \frac{\epsilon}{2}h^2(t) + \psi\big(z(t)\big)JW^*\big(t, A(t), z(t)\big)\Big]e^{-rt-y(t)} \\ &+ \lambda_1(t)\psi\big(z(t)\big) + \lambda_2(t)\big(h(t) - \gamma z(t)\big). \end{aligned} \tag{3.30}$$

Define the conditional current-value Hamiltonian by

$$\mathcal{H} = e^{rt+y(t)}H, \tag{3.31}$$

and the corresponding conditional current-value adjoint equations by

$$\rho_i(t) = e^{rt+y(t)}\lambda_i(t), \tag{3.32}$$

for $i = 1, 2$.

Hence, the conditional current-value Hamiltonian is given by

$$\begin{aligned} \mathcal{H} = &B\big(A(t)\big) - h(t) - \frac{\epsilon}{2}h^2(t) + \psi\big(z(t)\big)JW^*\big(t, A(t), z(t)\big) \\ &+ \rho_1(t)\psi\big(z(t)\big) + \rho_2(t)\big(h(t) - \gamma z(t)\big), \end{aligned} \tag{3.33}$$

The conditional current-value adjoint differential equation corresponding to the state variable y is given by

$$\rho_1'(t) = \Big(r + \psi\big(z(t)\big)\Big)\rho_1(t) + B\big(A(t)\big) - h(t) - \frac{\epsilon}{2}h^2(t) + \psi\big(z(t)\big)JW^*\big(t, A(t), z(t)\big), \quad (3.34)$$

with transversality condition

$$\rho_1(T) = 0. \quad (3.35)$$

The conditional current-value adjoint equation corresponding to the state variable z is given by

$$\rho_2'(t) = \Big(r + \gamma + \psi\big(z(t)\big)\Big)\rho_2(t) - \psi\big(z(t)\big)\frac{\partial JW^*}{\partial z} - JW^*\big(t, A(t), z(t)\big)\frac{\partial \psi}{\partial z} - \rho_1(t)\frac{\partial \psi}{\partial z}, \quad (3.36)$$

with transversality condition

$$\rho_2(T) = 0. \quad (3.37)$$

The partial derivative of the conditional current-value Hamiltonian with respect to the control variable h, is given by

$$\frac{\partial \mathcal{H}}{\partial h} = -1 - \epsilon h(t) + \rho_2(t). \quad (3.38)$$

Given the bounds on the control $0 \leq h(t) \leq M$, it follows that

$$\begin{cases} h^* = 0 & \text{if } \frac{\partial \mathcal{H}}{\partial h} < 0, \\ h^* = \frac{\rho_2(t) - 1}{\epsilon} & \text{if } \frac{\partial \mathcal{H}}{\partial h} = 0, \\ h^* = M & \text{if } \frac{\partial \mathcal{H}}{\partial h} > 0. \end{cases} \quad (3.39)$$

Combining the above, we have that

$$h^*(t) = \min\left\{M, \max\left\{0, \frac{\rho_2(t) - 1}{\epsilon}\right\}\right\}. \tag{3.40}$$

We quickly point out that

$$\frac{\partial^2 \mathcal{H}}{\partial h^2} = -\epsilon < 0, \tag{3.41}$$

and thus, our optimal control problem satisfies the concavity requirement for a maximization problem.

Ideally, when solving our optimal control problem numerically, we would use this characterization of h^* to update and determine h^*. However, using a forward-backward sweep method, it was difficult to achieve convergence of all of our variables. Thus, once again, we determined h^* by maximizing the conditional current-value Hamiltonian with respect to the control h pointwise at each time t. Our numerical results are discussed in a later section. In the subsection below, we prove the existence of an optimal control.

3.3.1 Existence of an Optimal Control

Our optimal control problem including a state variable for cumulative prevention management stock z is given by

$$\begin{aligned}
&\sup_h \int_0^T \Big[B\big(A(t)\big) - h(t) - \frac{\epsilon}{2}h^2(t) + \psi\big(z(t)\big)JW^*\big(t, A(t), z(t)\big)\Big]e^{-rt-y(t)}dt \\
&\text{subject to } y'(t) = \psi\big(z(t)\big) \text{ with } y(0) = 0 \\
&\qquad z'(t) = h(t) - \gamma z(t) \text{ with } 0 \le z(0) = z_0 < \infty \\
&\qquad 0 \le h(t) \le M,
\end{aligned} \tag{3.42}$$

where $A(t)$ is given by equation (3.15). Note the objective functional as

$$J(h) = \int_0^T \Big[B\big(A(t)\big) - h(t) - \frac{\epsilon}{2}h^2(t) + \psi\big(z(t)\big)JW^*\big(t, A(t), z(t)\big)\Big]e^{-rt-y(t)}dt, \tag{3.43}$$

and the set of admissible controls as

$$U = \{h : [0, T] \to [0, M] | h \text{ is Lebesgue measurable}\}. \tag{3.44}$$

Our goal is to prove the existence of $h^* \in U$ such that

$$J(h^*) = \max_{h \in U} J(h). \tag{3.45}$$

First, we prove two lemmas concerning the boundedness of our state variables and the boundedness of the objective functional $J(h)$. Once these facts are established we move on to the proof of the existence of an optimal control.

Lemma 3.1 (Boundedness of State Variables)**.** *Given the control set U and the state differential equations for y and z, there exist constants $C_1, C_2, C_3,$ and C_4 such that $|y(t)| \leq C_1, |y'(t)| \leq C_2, |z(t)| \leq C_3,$ and $|z'(t)| \leq C_4$ for all $h \in U$ and for all $t \in [0, T]$.*

Proof: Let $t \in [0, T]$ and $h \in U$. First, we show that the derivative $y'(t)$ of the cumulative hazard function is bounded. By our functional choice for ψ, observe:

$$|y'(t)| = \left|\psi\big(z(t)\big)\right| \tag{3.46}$$

$$= \left|be^{-vz(t)}\right| \tag{3.47}$$

$$\leq b = C_1. \tag{3.48}$$

It immediately follows that $y(t)$ is bounded as its derivative is bounded and we are considering a finite time horizon. Note that $y(0) = 0$. By the fundamental theorem of calculus we have

$$
\begin{aligned}
|y(t)| &= \left| \int_0^t \psi\big(z(s)\big) ds \right| \\
&\leq \int_0^t \left| \psi\big(z(s)\big) \right| ds \\
&\leq \int_0^T b ds && (3.49) \\
&= bT = C_2. && (3.50)
\end{aligned}
$$

Next, we show that $z(t)$ is bounded. Using an integrating factor we solve the differential equation for z and obtain

$$
z(t) = z_0 e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)} h(s) ds. \tag{3.51}
$$

Hence,

$$
\begin{aligned}
|z(t)| &= \left| z_0 e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)} h(s) ds \right| \\
&\leq z_0 + \int_0^T \left| e^{-\gamma(t-s)} h(s) \right| ds \\
&\leq z_0 + \int_0^T |M| \, ds \\
&= z_0 + MT = C_3. && (3.52)
\end{aligned}
$$

Finally, using the result above, we show that $z'(t)$ is bounded:

$$
\begin{aligned}
|z'(t)| &= |h(t) - \gamma z(t)| \\
&\leq |h(t)| + |\gamma z(t)| \\
&\leq M + \gamma(z_0 + MT) = C_4. && (3.53)
\end{aligned}
$$

□

Lemma 3.2.

$$\sup_{h\in U} J(h) < \infty \tag{3.54}$$

Proof: First, we show that that exists a constant C_5 such that $|JW^*| \leq C_5$. Observe:

$$\begin{aligned}
&\left|JW^*\big(\tau, A(\tau), z(\tau)\big)\right| && (3.55)\\
&= \left|JW\big(\tau, A(\tau), z(\tau), x^*(\tau)\big)\right| && (3.56)\\
&= \Bigg|\frac{B_1\bar{A}}{r}\left(1-e^{-r(T-\tau)}\right) - \frac{B_1\big(\bar{A}-A(\tau)\big)}{\delta+r}\left(1-e^{-(\delta+r)(T-\tau)}\right)\\
&\quad - K\big(z(\tau), x^*(\tau)\big)\left[\frac{B_1}{\delta+r}\left(1-e^{-(\delta+r)(T-\tau)}\right)+c\right] - x^*(\tau)\Bigg| && (3.57)\\
&\leq \left|\frac{B_1\bar{A}}{r}\left(1-e^{-r(T-\tau)}\right)\right| + \left|\frac{B_1\big(\bar{A}-A(\tau)\big)}{\delta+r}\left(1-e^{-(\delta+r)(T-\tau)}\right)\right|\\
&\quad + \left|K\big(z(\tau), x^*(\tau)\big)\left[\frac{B_1}{\delta+r}\left(1-e^{-(\delta+r)(T-\tau)}\right)+c\right]\right| + |x^*(\tau)| && (3.58)\\
&\leq \frac{B_1\bar{A}}{r} + \frac{B_1\bar{A}}{\delta+r} + \frac{B_1 k}{\delta+r} + kc + |x^*(\tau)| && (3.59)\\
&\leq C_5, && (3.60)
\end{aligned}$$

as

$$\begin{aligned}
|x^*(\tau)| &= \left|\sqrt{\frac{k}{\big(k_1+z(\tau)\big)}\left[\frac{B_1}{\delta+r}\left(1-e^{-(\delta+r)(T-\tau)}\right)+c\right]} - k_2\right| && (3.61)\\
&\leq \left|\sqrt{\left[\frac{B_1 k}{\delta+r}\left(1-e^{-(\delta+r)(T-\tau)}\right)+c\right]}\right| && (3.62)\\
&\leq \left|\sqrt{\left[\frac{B_1 k}{\delta+r}+c\right]}\right|. && (3.63)
\end{aligned}$$

For $h \in U$, it follows from our choices for functional forms that

$$
\begin{aligned}
|J(h)| &= \left| \int_0^T \Big[B\big(A(t)\big) - h(t) - \frac{\epsilon}{2} h^2(t) + \psi\big(z(t)\big) JW^*\big(t, A(t), z(t)\big) \Big] e^{-rt-y(t)} dt \right| && (3.64) \\
&\le \int_0^T \Big| \Big[B\big(A(t)\big) - h(t) - \frac{\epsilon}{2} h^2(t) + \psi\big(z(t)\big) JW^*\big(t, A(t), z(t)\big) \Big] \Big| \left| e^{-rt-y(t)} \right| dt && (3.65) \\
&\le \int_0^T |B_1 A(t)| + |h(t)| + \left| \frac{\epsilon}{2} h^2(t) \right| + \big|\psi\big(z(t)\big)\big| \, \big| JW^*\big(t, A(t), z(t)\big) \big| \, dt && (3.66) \\
&\le \int_0^T \big(B_1 \bar{A} + M + \frac{\epsilon}{2} M^2 + bC_5 \big) dt && (3.67) \\
&= T(B_1 \bar{A} + M + \frac{\epsilon}{2} M^2 + bC_5). && (3.68)
\end{aligned}
$$

Hence, we conclude that $\sup_{h \in U} J(h) < \infty$.

□

We now present our result for the existence of an optimal control.

Theorem 3.3. *There exists an optimal control $h^* \in U$ which maximizes the objective functional $J(h)$.*

Proof: By Lemma 3.2 we have shown that

$$\sup_{h \in U} J(h) < \infty. \tag{3.69}$$

Therefore, let $\{h_n(\cdot)\}_{n \ge 1} \subset U$ be a maximizing sequence, i.e.

$$\lim_{n \to \infty} J(h_n) = \max_{h \in U} J(h), \tag{3.70}$$

where $y_n(\cdot)$ and $z_n(\cdot)$ are the state trajectories corresponding to $h_n(\cdot)$. Note that $U \subset L^\infty([0,T])$. In particular, $\{h_n(t)\}_{n \ge 1}$ is uniformly bounded in $L^\infty([0,T])$ by M. As the time interval $[0,T]$ is finite, it follows that $\{h_n(t)\}_{n \ge 1}$ is also uniformly bounded in $L^2([0,T])$. Bounded sequences in L^2 have weakly convergent subsequences. Thus, there exists a subsequence $\{h_n(\cdot)\}$ and $h^*(\cdot)$ such that

$$h_n \rightharpoonup h^* \text{ weakly in } L^2([0,T]). \tag{3.71}$$

By Lemma 3.1, we know there exist constants $C_1, C_2, C_3,$ and C_4 such that $|y(t)| \leq C_1, |y'(t)| \leq C_2, |z(t)| \leq C_3,$ and $|z'(t)| \leq C_4$ for all $h \in U$ and for all $t \in [0,T]$. Let $C = \max\{C_2, C_4\}$. For all $t_a, t_b \in [0,T]$ it follows that

$$|y_n(t_b) - y_n(t_a)| \leq C\,|t_b - t_a| \tag{3.72}$$

$$|z_n(t_b) - z_n(t_a)| \leq C\,|t_b - t_a|\,. \tag{3.73}$$

Thus, y_n and z_n are uniformly Lipschitz continuous. Hence the family of functions $\{y_n(\cdot), z_n(\cdot)\}$ is equicontinuous. Thus, by the Arzela-Ascoli theorem there exists a subsequence (y_n, z_n) and (y^*, z^*) such that

$$(y_n, z_n) \to (y^*, z^*) \text{ uniformly on } [0,T]. \tag{3.74}$$

Now, we show that (y^*, z^*) satisfies the DE system corresponding to h^*.

We have that $y_n'(t) = \psi\big(z_n(t)\big)$, and thus it follows by the fundamental theorem of calculus that $y_n(t) = \int_0^t \psi\big(z_n(s)\big)ds$ since $y_n(0) = 0$. Hence, by the uniform convergence of z_n to z^*, and the continuity of ψ, for $t \in [0,T]$ we have

$$y^*(t) = \lim_{n\to\infty} y_n(t) \tag{3.75}$$

$$= \lim_{n\to\infty} \int_0^t \psi\big(z_n(s)\big)ds \tag{3.76}$$

$$= \int_0^t \lim_{n\to\infty} \psi\big(z_n(s)\big)ds \tag{3.77}$$

$$= \int_0^t \psi\big(z^*(s)\big)ds. \tag{3.78}$$

Furthermore, $z_n'(t) = h(t) - \gamma z_n(t)$ with $z_n(0) = z_0$ gives that

$$z_n(t) = z_0 e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)} h_n(s)ds. \tag{3.79}$$

Note that $h_n \rightharpoonup h^*$ in $L^2([0,T])$ and that $e^{-\gamma(t-s)} \in L^2([0,T])$. Thus, by definition of weak convergence we have that

$$
\begin{aligned}
z^*(t) &= \lim_{n\to\infty} z_n(t) && (3.80)\\
&= \lim_{n\to\infty}\left[z_0 e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)} h_n(s)ds\right] && (3.81)\\
&= z_0 e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)} h^*(s)ds. && (3.82)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\max_{h\in U} J(h) &= \lim_{n\to\infty} J(h_n) && (3.83)\\
&= \lim_{n\to\infty}\int_0^T \left[B\big(A(t)\big) - h_n(t) - \frac{\epsilon}{2}h_n^2(t)\right.\\
&\qquad \left. + \psi\big(z_n(t)\big)JW^*\big(t, A(t), z_n(t)\big)\right] e^{-rt-y_n(t)}dt && (3.84)\\
&= \lim_{n\to\infty}\left[\int_0^T B\big(A(t)\big)e^{-rt-y_n(t)}dt\right.\\
&\qquad - \int_0^T h_n(t)e^{-rt-y_n(t)}dt\\
&\qquad + \int_0^T \frac{\epsilon}{2}h_n^2(t)e^{-rt-y_n(t)}dt\\
&\qquad \left. + \int_0^T \psi\big(z_n(t)\big)JW^*\big(t, A(t)z_n(t)\big)e^{-rt-y_n(t)}dt\right]. && (3.85)
\end{aligned}
$$

For the first term in equation (3.85), recall that $A(t)$ is given by equation (3.15). Furthermore, note that $y_n \to y^*$ uniformly and $e^{-rt-y_n(t)}$ is continuous on a compact set and thus uniformly continuous. Hence, $e^{-rt-y_n(t)} \to e^{-rt-y^*(t)}$ uniformly. Therefore,

$$
\lim_{j\to\infty}\int_0^T B\big(A(t)\big)e^{-rt-y_n(t)}dt = \int_0^T B\big(A(t)\big)e^{-rt-y^*(t)}dt. \tag{3.86}
$$

For the second term in equation (3.85), observe:

$$\left|\int_0^T h_n(t)e^{-rt-y_n(t)}dt - \int_0^T h^*(t)e^{-rt-y^*(t)}dt\right| \tag{3.87}$$

$$= \left|\int_0^T \left(h_n(t)e^{-rt-y_n(t)} - h^*(t)e^{-rt-y^*(t)}\right)dt\right| \tag{3.88}$$

$$= \left|\int_0^T \left(h_n(t)e^{-rt-y_n(t)} - h_n(t)e^{-rt-y^*(t)} + h_n(t)e^{-rt-y^*(t)} - h^*(t)e^{-rt-y^*(t)}\right)dt\right| \tag{3.89}$$

$$\leq \left|\int_0^T \left(h_n(t)e^{-rt-y_n(t)} - h_n(t)e^{-rt-y^*(t)}\right)dt\right| + \left|\int_0^T \left(h_n(t)e^{-rt-y^*(t)} - h^*(t)e^{-rt-y^*(t)}\right)dt\right| \tag{3.90}$$

$$\leq \left|\int_0^T h_n(t)\left(e^{-rt-y_n(t)} - e^{-rt-y^*(t)}\right)dt\right| + \left|\int_0^T e^{-rt-y^*(t)}\big(h_n(t) - h^*(t)\big)dt\right|. \tag{3.91}$$

In the first term of equation (3.91), by the boundedness of all $h \in U$ by M and the uniform convergence of $e^{-rt-y_n(t)} \to e^{-rt-y^*(t)}$, we have that

$$\left|\int_0^T h_n(t)\left(e^{-rt-y_n(t)} - e^{-rt-y^*(t)}\right)dt\right| \tag{3.92}$$

$$\leq \int_0^T |h_n(t)| \left|e^{-rt-y_n(t)} - e^{-rt-y^*(t)}\right| dt \tag{3.93}$$

$$\leq \int_0^T |M| \left|e^{-rt-y_n(t)} - e^{-rt-y^*(t)}\right| dt \tag{3.94}$$

$$\to 0 \text{ as } n \to \infty. \tag{3.95}$$

In the second term of equation (3.91), note that $e^{-rt-y^*(t)} \in L^2([0,T])$, so by the weak convergence of $h_n \rightharpoonup h^*$:

$$\left|\int_0^T e^{-rt-y^*(t)}(h_n(t) - h^*(t))dt\right| \to 0 \text{ as } n \to \infty. \tag{3.96}$$

Therefore, combining the work above, we have for the second term in equation (3.91) that

$$\lim_{n\to\infty}\int_0^T h_n(t)e^{-rt-y_n(t)}dt = \int_0^T h^*(t)e^{-rt-y^*(t)}dt. \tag{3.97}$$

For the third term in equation (3.85), as $y_n \to y^*$ uniformly and $e^{-\frac{1}{2}\left(rt+y_n(t)\right)}$ is uniformly continuous on a compact set, we have that $e^{-\frac{1}{2}\left(rt+y_n(t)\right)} \to e^{-\frac{1}{2}\left(rt+y^*(t)\right)}$ uniformly. Furthermore, this implies convergence in $L^2([0,T])$. As $h_n(t) \rightharpoonup h^*(t)$ in $L^2([0,T])$, it follows that their product

$$e^{-\frac{1}{2}\left(rt+y_n(t)\right)}h_n(t) \rightharpoonup e^{-\frac{1}{2}\left(rt+y^*(t)\right)}h^*(t), \tag{3.98}$$

converges weakly in $L^1([0,T])$. Since $F(a) = a^2$ is convex we have by weak lower semi-continuity of convex functions that

$$\int_0^T e^{-rt-y^*(t)}\big(h^*(t)\big)^2 dt \le \liminf_{n\to\infty}\int_0^T e^{-rt-y_n(t)}h_n^2(t)dt. \tag{3.99}$$

The fourth term in equation (3.85) has products of uniformly continuous functions with uniform convergence $(y_{n_{k_j}}, z_{n_{k_j}}) \to (y^*, z^*)$ on $[0,T]$. Hence, we have

$$\lim_{n\to\infty}\int_0^T \psi\big(z_n(t)\big)JW^*\big(t, A(t), z_n(t)\big)e^{-rt-y_n(t)}dt \tag{3.100}$$

$$= \int_0^T \psi(z^*(t))JW^*(t, A(t), z^*(t))e^{-rt-y^*(t)}dt. \tag{3.101}$$

Thus, combining all of our work above, we have that

$$\max_{h\in U} J(h) = \lim_{n\to\infty}\int_0^T \Big[B\big(A(t)\big) - h_n(t) - \frac{\epsilon}{2}h_n^2(t) + \psi\big(z_n(t)\big)JW^*\big(t, A(t), z_n(t)\big)\Big]e^{-rt-y_n(t)}dt \tag{3.102}$$

$$\geq \liminf_{n\to\infty}\int_0^T \Big[B\big(A(t)\big) - h_n(t) - \frac{\epsilon}{2}h_n^2(t) + \psi\big(z_n(t)\big)JW^*\big(t, A(t), z_n(t)\big)\Big]e^{-rt-y_n(t)}dt \tag{3.103}$$

$$\geq \int_0^T \Big[B\big(A(t)\big) - h^*(t) - \frac{\epsilon}{2}\big(h^*(t)\big)^2 + \psi\big(z^*(t)\big)JW^*\big(t, A(t), z^*(t)\big)\Big]e^{-rt-y^*(t)}dt \tag{3.104}$$

$$= J(h^*). \tag{3.105}$$

□

3.4 Numerical Results

Now that we have formulated our optimal control problem to include cumulative prevention management stock and have derived the associated optimality system, we solve it numerically. First, we discuss parameter choices. For the parameters common to our original problem formulation, we use the values chosen for the Las Conchas Fire. These parameter values are found in Table 2.1. Beyond this, we have to choose values for the new parameters $A_0, \epsilon, \gamma, z_0$, and M. Recall that A_0 is the initial condition for the number of unburned acres in the forest at the beginning of our time horizon, ϵ is the coefficient associated with the quadratic cost term in the objective functional, γ determines the rate at which cumulative prevention management stock decays, z_0 is the initial condition for cumulative prevention management stock, and M is the upper bound on the prevention management spending rate h.

For the quadratic cost parameter ϵ we choose $\epsilon = 2$ and we do not vary it. Our initial hope was to choose ϵ small so that the quadratic cost term did not play a large role in the objective functional and so that our results for this new formulation might be more directly comparable to our previous results for the Las Conchas Fire without cumulative prevention management stock. However, when attempting to solve the problem numerically, we found that achieving convergence of our variables to be very sensitive to changes in ϵ. Even decreasing ϵ from 2 to 1 gives rise to a case where we cannot achieve convergence.

The parameter γ gives the rate of decay for cumulative prevention management stock. We choose a few values to examine that represent a few different scenarios. Choosing values close to zero indicates a slow decline of stock, while larger values for γ indicate a quick decline of the benefits of prevention management efforts. Thus, we solve our optimal control problem using $\gamma = 0.5, 1, 5$ and let our baseline value for the parameter be $\gamma = 1$ when we vary other parameters.

The parameter z_0 is the initial condition for the cumulative prevention management stock. Its value can be used to reflect whether or not, or to what extent, there have been prevention management efforts in an area prior to the application of our optimal control problem. We compare a few values for z_0, choosing $z_0 = 0, 1, 5$. Setting the initial condition to zero implies that no prevention management efforts have been recently made in the forest. We let $z_0 = 0$ be our baseline value for the parameter as this choice is consistent with our prior assumptions when determining parameter values for the Las Conchas Fire.

The parameter M is the upper bound on prevention management spending h. We choose $M = B_1\bar{A} = 34$. That is, we stipulate that prevention management spending rate h is never greater than the flow of benefits when the forest is entirely unburned. We do not vary this parameter because in all cases tested, this upper bound is not reached by h^* and does not even come close to it.

We recognize that the selection of these parameters is not literature driven, but more scenario driven. Furthermore, because small changes in some of these parameters (particularly ϵ) leads to difficulties in achieving convergence, we do not perform a global sensitivity analysis on these parameters as we did in the previous chapter. Instead we perform a local sensitivity analysis where we consider several different parameter scenarios by varying one parameter and holding the others constant at their stated baseline values.

Once again, we use code developed in MATLAB to solve our optimal control problem. The process used is similar to what we used in our original problem and thus we do not go over it in detail here. The main differences are the inclusion of a new state variable z and its associated adjoint equation ρ_2. Furthermore, even though we have an explicit characterization for h^*, given by (3.40), we still use the function `fminbnd` to determine h^* by optimizing the Hamiltonian pointwise.

For this problem, we consider a time horizon of $T = 5$. While this time horizon may be considered a more realistic time scale to consider for application to management, it is very different from the time horizon we chose in the previous chapter. Ideally, we would like to be able to choose $T = 500$ as we did for our previous problem formulations. However, for very large values of T the numerical approximations for the solution would not converge, except under extreme cases for parameter values. Unfortunately, because of this, it is difficult to directly compare these new results to our previous results from the previous chapter. However, we are able to determine numerical solutions for our previous optimal control problem without cumulative prevention management stock for the shorter time horizon and thus are able to make some comparisons to the work in the previous chapter.

Comparison to our original optimal control problem is also made difficult by the inclusion of a quadratic cost term in the objective functional. However, we are able to easily formulate an "intermediate" optimal control problem where we insert a quadratic cost term in our original optimal control problem so that some comparisons of our problem with and without stock can be made. These comparisons are included later in this section. We also point out that because of this short time horizon we are not able to meaningfully calculate the mean of the time of fire random variable because we are not able to closely approximate the mixed-type random variable by its continuous counterpart. Thus we are not able to examine the impact of prevention management spending on the mean time of fire or consider sequences of fires.

First, we consider the results when our optimal control problem is solved at the baseline values for γ and z_0 and we vary the initial condition for the number of unburned acres with $A_0 = 0.5\bar{A}, \bar{A}$. The results can be found in Table 3.1 and Figure 3.1. In the case when $A_0 = 0.5\bar{A}$, the expected net present value of the forest for 5 years is \$60.85 M and in the case when $A_0 = \bar{A}$, the expected net present value of the forest is \$130.5 M. Of course, because there are fewer unburned acres A in the forest in the case when $A_0 = 0.5\bar{A}$ and

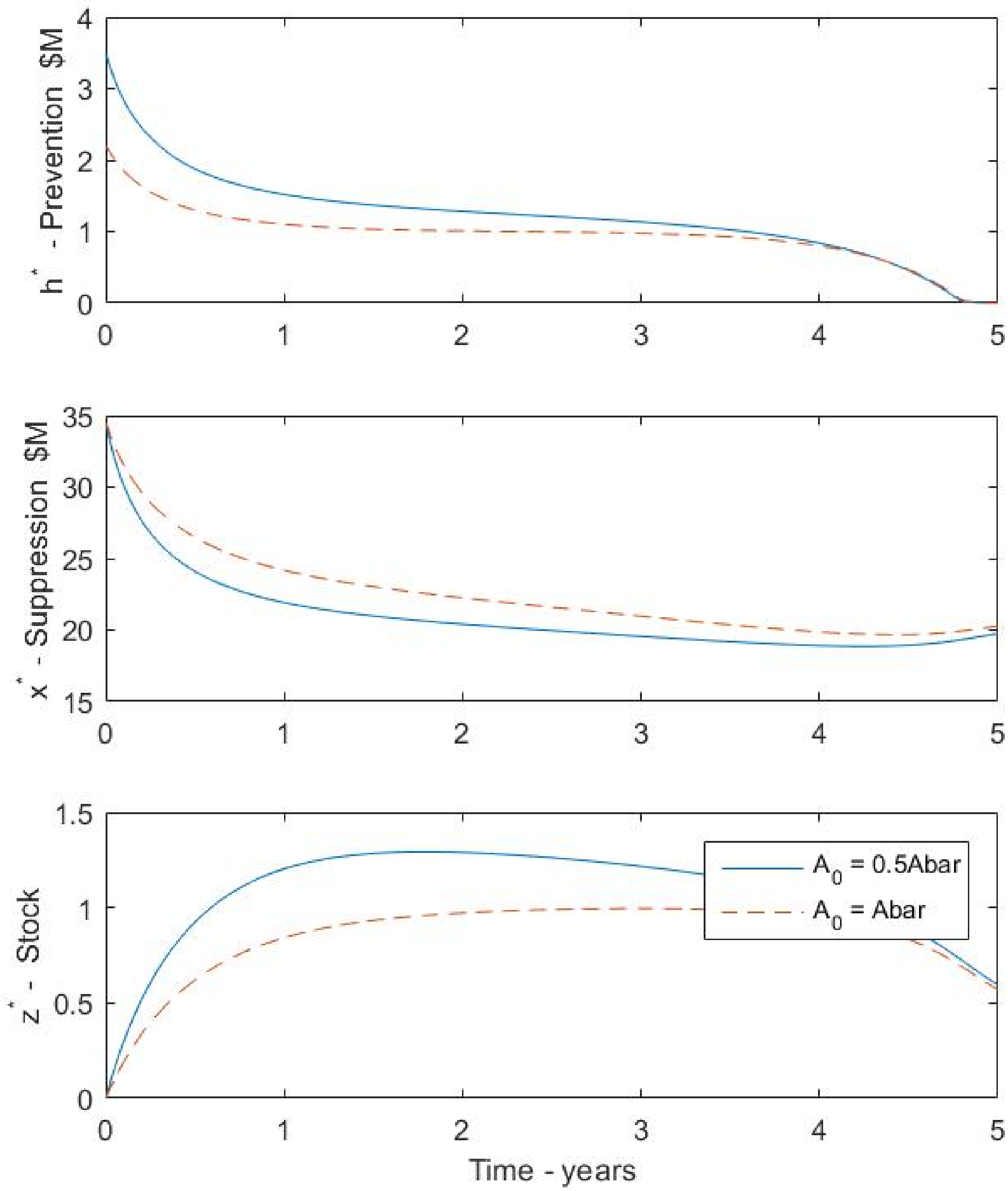


Figure 3.1: The plots above show the results of our optimal control problem with two different initial conditions for the number of healthy acres in the forest A_0. We use $A_0 = 0.5\bar{A}$ and $A_0 = \bar{A}$ and compare prevention management spending h^*, suppression spending x^*, and cumulative prevention management stock z^*.

Table 3.1: The table above gives the value of the objective functional $J(h^*)$ evaluated at the optimal control for the different parameter scenarios tested.

Parameter	$J(h^*)$ ($M)
$A_0 = 0.5\bar{A}$	60.85
$A_0 = \bar{A}$	130.50
$\gamma = 0.5$	135.42
$\gamma = 1$	130.50
$\gamma = 5$	118.04
$z_0 = 0$	130.50
$z_0 = 1$	134.69
$z_0 = 5$	140.52

benefits are a function of unburned acres, it is not surprising that the expected net present value of the forest $J(h^*)$ is less in the case where $A_0 = 0.5\bar{A}$. This is consistent with our results from the previous chapter. Also contributing to this difference in values for $J(h^*)$ is that, in the case of $A_0 = 0.5\bar{A}$, there is more spending on prevention management than in the case where $A_0 = \bar{A}$. Moreover, the higher level of prevention management spending in the case where $A_0 = 0.5\bar{A}$ leads to a greater accumulation of prevention management stock z^*, which explains the lower optimal suppression spending x^* as $\frac{\partial x^*}{\partial z} < 0$.

Next, we consider the results when the decay rate γ is varied, with $A_0 = \bar{A}$ and $z_0 = 0$. See Figure 3.2 and the corresponding rows in Table 3.1. As the rate of decay γ of cumulative prevention management stock increases the expected net present value of the forest $J(h^*)$ decreases. In the case where γ is largest, prevention spending h^* is lowest, along with the value of the forest $J(h^*)$.

Let's examine the solution to the state differential equation for z to try and gain a better understanding of this state variable:

$$z(t) = z_0 e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)} h(s) ds. \tag{3.106}$$

Let's suppose, for instance, that prevention management spending is constant with $h = C$, and let's suppose that $z_0 = 0$. Then with a simple calculation we see that

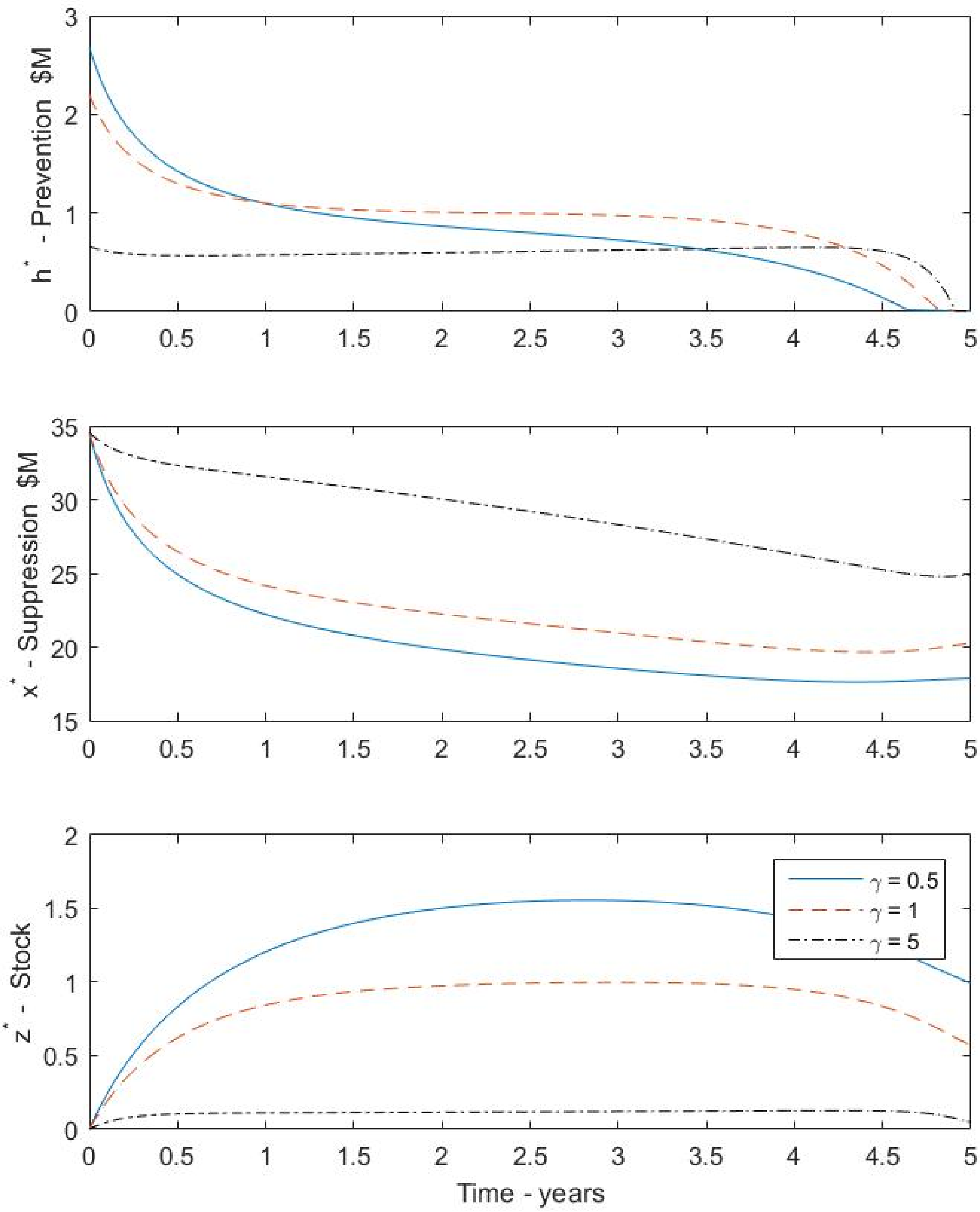


Figure 3.2: The plots show the results of our optimal control problem with three different values for the parameter γ, which controls the rate of decay of cumulative prevention management stock z . We use $\gamma = 0.5$, $\gamma = 1$, and $\gamma = 5$ and compare prevention management spending h^*, suppression spending x^*, and cumulative prevention management stock z^*.

$$z(t) = \frac{C}{\gamma}\Big(1 - e^{-\gamma t}\Big). \tag{3.107}$$

This suggests that prevention management spending needs to be relatively high, in comparison to the levels of h^* in our previous formulation, in order for stock to accumulate in a meaningful way. For instance, in the case where $\gamma = 5$, even if $h = 2$, the cumulative prevention management stock would never rise above 0.4 and hence would not be very effective at reducing suppression costs or hazard. In this case a very high decay rate γ for cumulative stock is "worse" than when the the effects of prevention management spending are instantaneous. Thus, in cases where the rate of decay is very high, the utility of prevention management spending is greatly decreased. As we can see in our quick example with h constant, $\gamma >> 1$ has a substantial effect on how prevention spending contributes to the stock.

In the large γ case when $\gamma = 5$ (see Figure 3.2), optimal prevention management spending h^* is approximately constant at \$0.6M for the first 4.5 years of the 5 year time horizon. The cumulative prevention management stock z^* is very low, around \$0.1 M, which is expected due to our previous analysis. In the case when $\gamma = 1$, optimal prevention spending begins near \$2 million and reduces to roughly \$1 million after one year. During this first year prevention management spending is less than in the case when $\gamma = 0.5$. After one year, h^* is larger in the case when $\gamma = 1$. As can be seen for the prevention stock z^*, a smaller rate of decay $\gamma = 0.5$ allows for a quick accumulation of stock, which stays relatively high, even with decreasing levels of optimal prevention management spending. As is seen, different values of γ have a varying effect on prevention management spending, meaning that there is not a strict monotonic relationship between the value of γ and the level of prevention management spending h^*.

Finally, we vary the parameter for the initial condition for cumulative prevention management stock z_0. We use values $z_0 = 0, 1, 5$ and let $A_0 = \bar{A}$ and $\gamma = 1$. As seen in Figure 3.3, initially prevention management spending h^* is higher for lower values of initial cumulative prevention management stock z_0. However, near $t = 3$ these three trajectories begin to coincide. Unsurprisingly, the three different x^* and z^* trajectories all come together

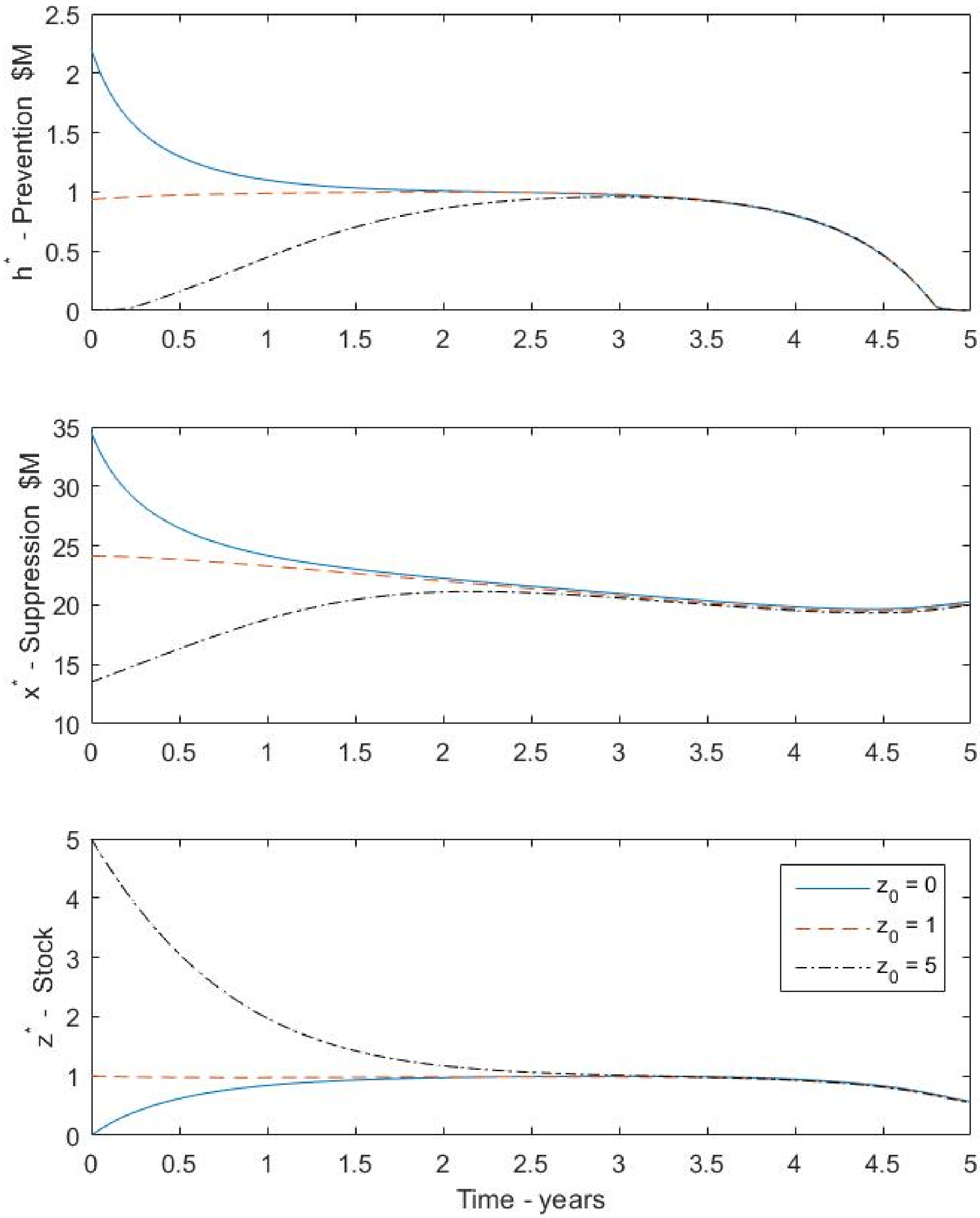


Figure 3.3: The plots show the results of our optimal control problem with three different initial conditions for cumulative prevention management stock z_0. We use $z_0 = 0$, $z_0 = 1$, and $z_0 = 5$ and compare prevention management spending h^*, suppression spending x^*, and cumulative prevention management stock z^*.

Table 3.2: In this table we list the value of the objective functional evaluated at the optimal control for three different cases.

Case	$J(h^*)$ \$M
w/ stock, w/ quad. cost: $z_0 = 1, \gamma = 1$	134.69
no stock, w/ quad. cost	134.05
no stock, no quad. cost	141.47

near the same time. Thus, it appears that for this given set of parameters, there is an optimal stock level, and despite the value chosen for the initial stock level z_0, prevention management h^* is chosen so that the optimal stock level is eventually reached. As we see in Table 3.1 the expected net present value of the forest $J(h^*)$ increases with increasing values for z_0. This is likely due to a lower prevention management spending rate h^* in the cases for larger values of z_0.

We wish to compare our optimal control problem with cumulative prevention management stock to our original optimal control problem where the effects of prevention management spending are taken to be instantaneous. Making a direct comparison of these two cases is difficult because in the optimal control problem with cumulative stock we include a quadratic cost term in the objective functional. This quadratic cost term is not found in our original problem. However, it is straightforward to make the appropriate modifications and construct an intermediate optimal control problem that includes the quadratic cost term in the objective functional, but does not include the state variable for cumulative prevention management stock. This allows better comparisons to be made. This intermediate optimal control problem is given by:

$$\max_{h \in U} \int_0^T \Big[B(\bar{A}) - h(t) - \frac{\epsilon}{2} h^2(t) + \psi\big(h(t)\big) JW^*\big(t, h(t)\big)\Big] e^{-rt - y(t)} dt \tag{3.108}$$

$$\text{subject to } y'(t) = \psi\big(h(t)\big) \text{ with } y(0) = 0,$$

$$h(t) \geq 0, \tag{3.109}$$

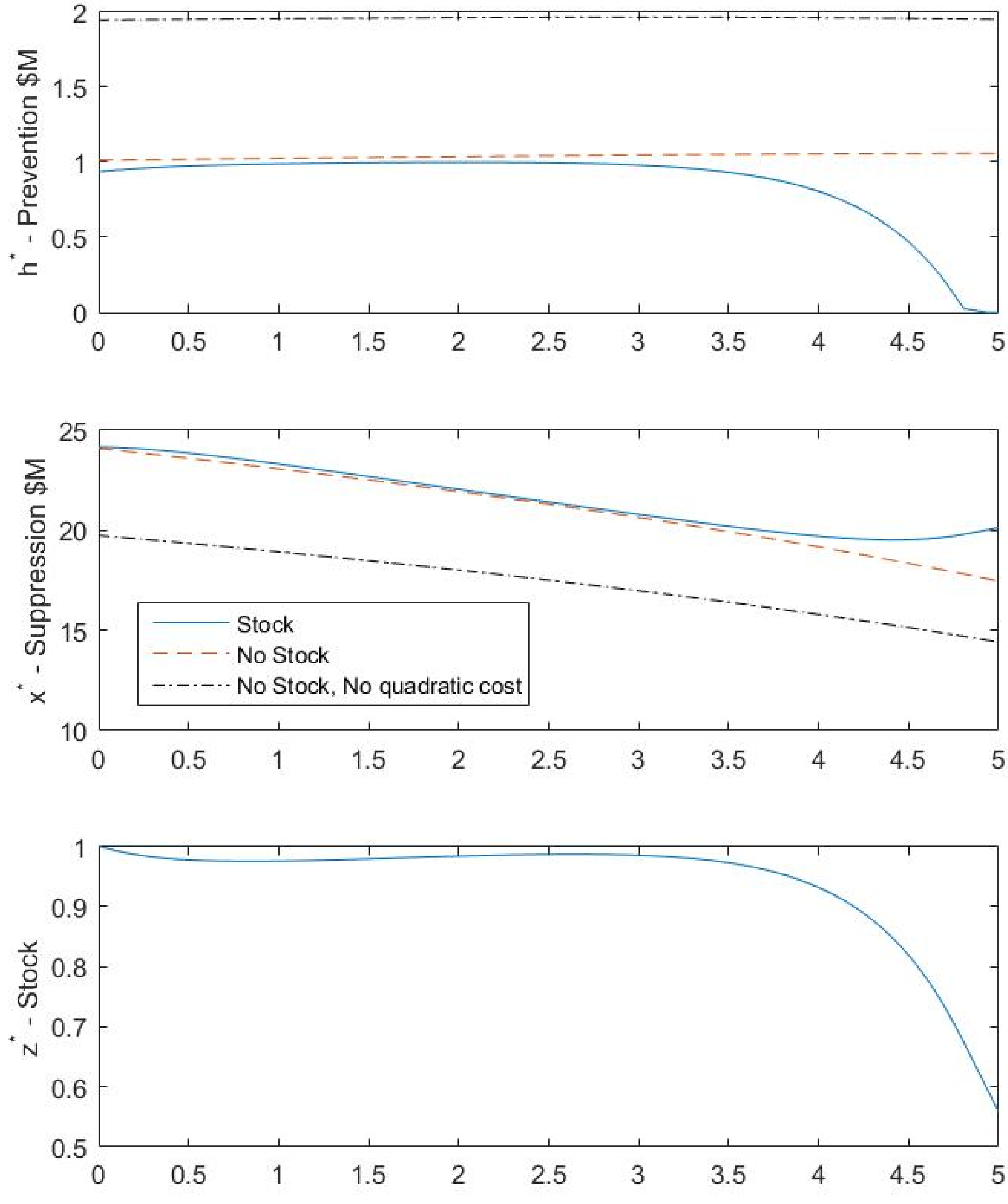


Figure 3.4: Results from three different optimal control problems are displayed: with cumulative prevention management stock and quadratic cost term, no cumulative prevention management stock and quadratic cost term, and no cumulative prevention management stock and no quadratic cost term. We use $z_0 = 1, \gamma = 1$, and $A_0 = \bar{A}$.

where

$$U = \{h : [0, T] \to [0, \infty) | h \text{ is piecewise continuous}\}. \tag{3.110}$$

Notice that this intermediate optimal control problem is the same as our original optimal control problem, except for the inclusion of the quadratic cost term in the objective functional. The changes to the adjoint equations and optimality system are small and thus we do not include them in detail here.

The results for the case comparisons are contained in Table 3.2 and Figure 3.4. Keep in mind that for our original optimal control problem (without stock), there is no upper bound on the prevention management spending rate h and in the case with stock the upper bound is $M = 34$. This difference is not important because the optimal prevention management spending rate does not approach or come close to the upper bound M. For the case with cumulative stock, we choose $z_0 = 1$ and $\gamma = 1$. We choose $z_0 = 1$ because we want to closely match the situation in the optimal control problem without cumulative stock so that direct comparisons can be made. We choose a nonzero initial condition for cumulative stock z because in the instantaneous prevention effects optimal control problem, prevention management spending is effective immediately. In contrast, stock takes time to accumulate. Thus, if we choose $z_0 = 0$ it will take time for stock to accumulate and be effective; this is not reflective of the no stock situation. By choosing $z_0 = 1$, we allow the cumulative stock z to affect the hazard and number of acres burned in the fire early in the time horizon in a way similar to the problem without cumulative prevention stock. As is seen in Table 3.2, the values of the objective functional evaluated at the optimal control are nearly equal in the cases where the quadratic cost is considered. One significant difference between these cases is that in the cumulative stock case, optimal prevention management spending h^* decreases to zero as we approach the end of the time horizon. This is because h^*, given by (3.40), is determined by adjoint equation $\rho_2(t)$ which has transversality condition $\rho_2(T) = 0$. Hence, unless we were to include a salvage term to change this, h^* will always be pulled to zero at $t = T$ in the case including cumulative stock.

We also can compare the quadratic cost case without stock to the case without quadratic cost (also without stock). As seen in Figure 3.4, in the case when there is not a quadratic cost term, optimal prevention management spending is nearly double the case when there is a quadratic cost term. Moreover, the expected value of the forest $J(h^*)$ is greater in the case when the quadratic cost term is not incorporated. Thus, the inclusion of a quadratic cost term in the objective functional has a substantial impact on the solution to our optimal control problem with cumulative prevention management stock.

Furthermore, because we chose $\gamma = 1$, stock is not accumulating over time because prevention management spending h^* is slightly less than one. Thus, over the first three years of the time horizon, the stock z^* stays approximately constant near one. Let's compare this case to the case where $\gamma = 0.5$. In Figure 3.5 we can see that with a lower rate of decay γ, cumulative prevention management stock is able to accumulate over time rather than remaining approximately constant even as prevention management spending h^* decreases. This in turn leads to a greater expected value of the forest. In particular, we see that $J(h^*) = \$140.5\text{M}$ in the case when $\gamma = 0.5$. Therefore, given that the rate of decay γ of cumulative prevention management stock is small enough to allow for meaningful accumulation of prevention management stock, it is possible to realize lower prevention management spending levels and an increased value of the forest.

3.5 Conclusions

Our goal is to examine the economic trade-offs between prevention management spending and suppression spending applied to large forest fires when the time of fire is unknown. In this chapter we add a cumulative prevention management stock state variable z to our optimal control problem. The inclusion of this state variable allows for the effects of prevention management spending to accumulate over time or decay very rapidly, depending on the choice for γ. In this formulation of the optimal control problem, the fire hazard for the forest is a function of cumulative prevention management stock z instead of prevention management spending h as in the previous chapter. Additionally, in this chapter the function determining the number of acres burned in the fire K is a function of cumulative prevention management stock z and suppression spending x. Thus, the function for the optimal *ex post* value of the forest JW^* becomes a function of z instead of h. Initially, the addition of cumulative stock in

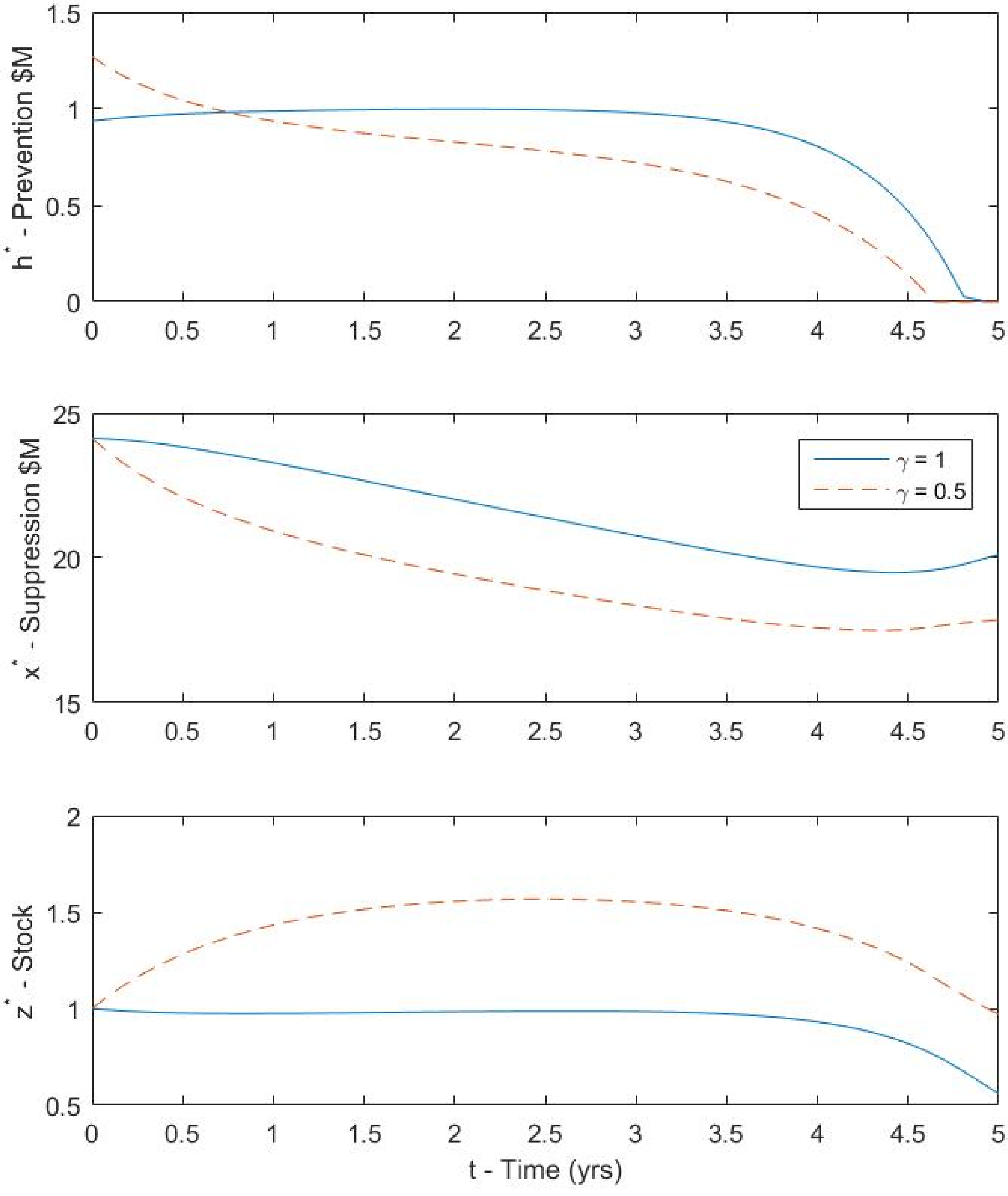


Figure 3.5: In the plots above, we compare optimal prevention management stock h^*, optimal suppression spending x^*, and optimal cumulative prevention management stock z^* in the cumulative prevention management stock optimal control problem for two different values for the stock decay parameter: $\gamma = 0.5, 1$. Here, we take $z_0 = 1$ in both cases.

our problem removed all of the nonlinearities in h, making the problem linear in the control. This led to numerical difficulties when trying to solve the optimal control problem. As a result we added a quadratic cost term to our objective functional. Furthermore, because the only nonlinearity of h comes from the quadratic cost term in the objective functional, we are able to prove the existence of an optimal control for this particular problem.

Numerically, we solve this optimal control problem by maximizing the conditional current-value Hamiltonian point-wise to determine h^*. Issues with achieving convergence limited the range of the parameters we could consider. Thus, we did not perform a full global parameter sensitivity analysis as we did in Chapter 2. In particular, choosing values for $\epsilon < 2$ and choosing large values for T led to issues with convergence. Thus, for all scenarios we chose $\epsilon = 2$ and $T = 5$. We varied the parameters A_0, z_0, and γ to examine their local effect on the expected value of the forest $J(h^*)$ and corresponding controls and stock z. Larger values for the initial condition A_0 for the number of unburned acres led to increased values for $J(h^*)$ and a decreased prevention management spending rate h^*. Larger values of initial cumulative prevention management stock z_0 also led to larger values of $J(h^*)$ as less prevention management spending h^* was required. Increased values for the rate of stock decay γ led to decreased values for $J(h^*)$ with mixed results for its effects on h^*. Also, when varying the initial cumulative prevention management stock z_0, we see that all trajectories eventually merge to one trajectory over time.

In order to make comparisons between our optimal control problem with instantaneous prevention management effects and our optimal control problem with cumulative prevention management effects, we construct an intermediate optimal control problem assuming instantaneous prevention management effects with an additional quadratic cost term in the objective functional. We conclude from our work that given a small enough rate of decay γ for cumulative prevention management stock, less prevention management spending h^* is required and the expected net present value of the forest $J(h^*)$ is increased from the cases when γ is large and when stock is not considered. Thus, in cases when the cumulative prevention management stock decays too quickly, less is spent on prevention management as it is not worth the investment. Additionally, we see that the quadratic cost term in the objective functional has a substantial effect on the prevention management spending rate h^*.

There are limitations to this work. The necessity of the quadratic cost term in the optimal control problem with cumulative prevention effects is unfortunate, especially given that its coefficient cannot be brought below one without causing convergence issues. As we see in our comparisons for our problems with and without cumulative prevention stock in Figure 3.4, the difference in the prevention management spending rate h^* in the cases without stock is substantial. In particular we see that in the case without the quadratic cost term (without stock) h^* is nearly double the control in the case with the quadratic cost term (also without stock). However, there is only about a 5% difference in the overall value of the forest $J(h^*)$. Hence, its effects are not too extreme.

Overall, in comparison to our original problem, the convergence of the cumulative optimal control problem was much more sensitive to parameter variation. This sensitivity prevented us from performing a full-scale global parameter sensitivity analysis as we did for the optimal control problem in Chapter 2. Additionally, large values of the time horizon T also caused convergence issues and thus we were forced to use a short time horizon, $T = 5$. Recall that for Reed's method, where the time of fire is treated as a random variable, in application the prevention management spending h^* is valid only up to the time of the first fire. Thus, if the time of the first fire occurs after $T = 5$ years, then past 5 years there is no information on how prevention management spending should be applied. Moreover, because of this tight restriction on T, we could not investigate sequences of fires as we did in the previous chapter or consider how the mean time of fire was affected by prevention management spending. If better numerical techniques could be found to solve this optimal control problem, some of these restrictions and limitations may be lessened.

Furthermore, we recognize that there are limitations to this body of work overall. Reed's method allowed us to construct a deterministic optimal control problem which incorporates the stochasticity of the time of fire. However, we make several assumptions and simplifications in the formulation of our problem. Firstly, and most significantly, we largely simplify our treatment of the actual forest fire event. Forest fires are dynamic, spread and evolve over time and space, and can be largely influenced by the weather. There is an extensive body of work in the literature that attempts to understand, model, and simulate this behavior. We do not attempt to incorporate this work into our problem. Within our optimal control problem, a forest fire is treated as a static, instantaneous event; we do not attempt to capture any of its spatial or temporal complexity within our optimal control problems. Instead of trying to capture the complexity and stochasticity of the physical forest fire within our optimal control problems, we focus on capturing the stochasticity of the timing of the forest fire. By treating the forest fire as an instantaneous event we are able to explicitly determine optimal suppression spending x^* and thus, the optimal value of the forest following a fire, JW^*. This in turn enables us to numerically examine the effects of prevention management spending on overall suppression costs and the value of the forest.

Additionally, in our treatment of prevention management spending and suppression spending, we make no distinctions for the different types of management actions that may be taken. For example, there are many varied types of prevention management actions, including prescribed fire, mechanical thinning, chemical treatments, etc. Depending on the location of the forest, its accessibility, the dominant tree type/ecosystem, and many other factors, certain types of prevention actions may be more beneficial or applicable than others. We make no conclusions concerning what kind of prevention management actions should be taken for a particular forest. We only conclude, given that spending on prevention management actions reduce the number of acres burned in a fire and the hazard of fire for the forest, that prevention management spending is beneficial to the forest economically. Similarly, the different strategies and methods that could be used to suppress a large forest fire are not taken into account. Furthermore, we recognize that in practice there is not an upper bound on how much will be spent to suppress a fire, especially if a fire is threatening the safety of humans or the communities in which they live. However, the Las Conchas Fire and the Happy Camp Complex were some of the largest and most expensive fires in their respective regions in recent history. Thus, for most large fires that might occur, our optimal control problems are likely overestimating the number of acres burned in a fire, along with the cost of suppression. Note that we do not allow for any kind of variability in the characteristics of the fire event beyond how prevention management spending/cumulative stock and suppression spending affect it.

Determining values for our many parameters, while guided by previous fire events, was at times challenging, in particular for the the flow of benefits parameter B_1. According to our sensitivity analysis, the parameter B_1 has the largest impact on the expected value of the forest $J(h^*)$, and hence we recognize the importance of its role in our problem. However, economic valuation of a forest is a very complex problem in and of itself. Thus, the method we utilize to choose B_1 may seem simplistic, but it is consistent with our other parameter choices and allows the cost of suppression spending to mirror that of an actual fire event.

Another simplification we make comes in our treatment of the forest and how fire affects it. In our problem we consider the number of unburned acres of a forest A and the number of acres burned in the fire K. We describe an acre of forest as "burned" if it has been completely destroyed by a stand-replacing fire, as these are the types of fires in which we are most interested. It is a simplification to assume that the entire fire will be stand-replacing as many fires are of mixed-severity over the full landscape they burn. Furthermore, we do not consider the "health" of the forest in our problem. We recognize that fire in certain ecosystems, especially low-severity surface fires, can be regenerative. Because of fire exclusion practices, many forests today are considered overgrown and unhealthy. Prevention management efforts could work to improve the health of the forest overall. In an extension of this problem, it may be possible to consider the health of the forest as a state variable which we attempt to maximize using a salvage term in our objective functional. The flow of benefits could also be considered at the final time as a function of the health of the forest, in addition to being a function of the number of unburned acres in the forest A.

There are many other possibilities for extensions and modifications of this optimal control problem, including any changes or improvements that may alleviate the limitations of our problem mentioned above. For instance, the function determining the number of acres burned in the forest K could be taken as a function of unburned acres A, in addition to prevention management spending/cumulative prevention management stock and suppression spending. The hazard function ψ could be taken to have increasing background hazard in order to account for changes in risk due to climate change; it could also be taken as a function of the health of the forest or the number of unburned acres in the forest. Additionally, we could adapt Reed's method for a discrete problem.

In conclusion, we were able to adapt and apply Reed's method in order to investigate the economic trade-offs between prevention management spending and suppression spending and their overall effects on the economic value of a forest under threat of fire. We considered an optimal control problem where the effects of prevention management spending were instantaneous and found that, when considering one fire, the mean time of fire and expected value of the forest were increased given optimal prevention management spending. We extended this optimal control problem so that we could consider a sequence of fires within a set management horizon. The results of our simulation study indicate that by applying prevention management the overall value of the forest is increased and more stable than when no prevention management efforts are made. Furthermore, the risk of fire for the forest is greatly reduced given that the number of fires within the management horizon is decreased and has smaller standard deviation in the case where prevention management is applied. Another variation of our optimal control problem allows for the effects from prevention management spending to accumulate in a stock over time. Given that the stock does not decay too rapidly, the results indicate that less prevention management spending may be needed than in the case without prevention management stock, leading to an increased expected value of the forest.

Overall, our results support the use of preventative management efforts for forests under threat of fire to offset rising suppression costs and increase the value of the forest overall. This work showcases a valuable tool which could guide forest managers and policymakers in their development of forest fire management plans.

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