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Stability Analysis of FitzHugh-Nagumo with Smooth Periodic Forcing

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Alan Lloyd Hodgkin and Andrew Huxley received the 1963 Nobel Prize in Physiology for their work describing the propagation of action potentials in the squid giant axon. Major analysis of their system of differential equations was performed by Richard FitzHugh, and later by Jin-Ichi Nagumo who created a tunnel diode circuit based upon FitzHugh's work. The resulting differential model, known as the FitzHugh-Nagumo (FH-N) oscillator, represents a simplification of the Hodgkin-Huxley (H-H) model, but still replicates the original neuronal dynamics (Izhikevich, 2010). We begin by providing a thorough grounding in the physiology behind the equations, then continue by introducing some of the results established by Kostova et al. for FH-N without forcing (Kostova et al., 2004). Finally, this sets up our own exploration into stimulating the system with smooth periodic forcing. Subsequent quantification of the chaotic phase portraits using a Lyapunov exponent are discussed, as well as the relevance of these results to electrocardiography.

Keywords: stability analysis, FitzHugh-Nagumo, chaos, Lyapunov exponent, electrocardiography

1. Introduction

As computational neuroscientist Eugene Izhikevich so aptly put it, "If somebody were to put a gun to the head of the author of this book and ask him to name the single most important concept in brain science, he would say it is the concept of a neuron (Izhikevich, 2010)." By no means are the concepts forwarded in his book restricted to brain science. Indeed, one may use the same techniques when studying most any physiological system of the human body in which neurons play an active role. Certainly this is the case for studying cardiac dynamics.

On a larger scale, neurons form an incredibly complex network that branches to innervate the entire body of an organism; it is estimated that a typical neuron communicates directly with over 10,000 other neurons (Izhikevich, 2010). This communication between neurons takes the form of the delivery and subsequent reception of a traveling electric wave, called an action potential (Alberts, 2010). These action potentials became the subject of Hodgkin and Huxley's groundbreaking research.

At any given time, the neuron possesses a certain voltage difference across its membrane, known as its potential. To keep the membrane potential regulated, the neuron is constantly adjusting the flow of ions into and out of the cell. The movement of any ion across the membrane is detectable as an electric current. Hence, it follows that any accumulation of ions on one side of the membrane or the other will result in a change in the membrane potential. When the membrane potential is 0 mV, there is a balance of charges inside and outside of the membrane.

Before we begin looking at Hodgkin and Huxley's model, we must first understand how the membrane adjusts the flow of ions into and out of the cell. Within the cell, there is a predominance of potassium, K+, ions. To keep K+ ions inside of the cell, there are pumps located on the membrane that use energy to actively transport K+ in but not out. Leaving the cell is actually a much easier task for K+: there are leak channels that "randomly flicker between open and closed states no matter what the conditions are inside or outside the cell...when they are open, they allow K+ to move freely (Alberts, 2010)."

Since the concentration of K+ ions is so much higher inside the cell than outside, there is a tendency for K+ to flow out of these leak channels along its concentration gradient. When this happens, there is a negative charge left behind by the K+ ions immediately leaving the cell. This build-up of negative charge is actually enough to, in a sense, catch the K+ ions in the act of leaving and momentarily halt the flow of charge across the membrane. At this precise moment, "the electrochemical gradient of K+ is zero, even though there is still a much higher concentration of K+ inside of the cell than out (Alberts, 2010)." For any cell, the resting membrane potential is achieved whenever the total flow of ions across the cell membrane is balanced by the charge existing inside of the cell. We may use an adapted version of the Nernst Equation to determine the resting membrane potential with respect to a particular ion (Alberts, 2010):

$$V = \log_{10} \frac{C_o}{C_i},$$

where V is the membrane potential (in mV), C_o is the ion concentration outside of the cell, and C_i is the ion concentration inside of the cell. A typical resting membrane potential is about -60mV.

Before we continue, it is important to revisit the concept of action potentials. Neurons communicate with each other through the use of electric signals that alter the membrane potential on the recipient neuron. To continue propagating this message, the change in membrane potential must travel the length of the entire cell to the next recipient. Across short distances, this is not a problem. However, longer distances prove to be a bit more of a challenge, since they require amplification of the electrical signal. This amplified signal, which can travel at speeds of up to 100 meters per second, is the action potential (Alberts, 2010).

Physiologically speaking, there are some key events taking place whenever an action potential is discharged. Once

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the cell receives a sufficient electrical stimulus, the membrane is rapidly depolarized; that is to say, the membrane potential becomes less negative. The membrane depolarization causes voltage-gated Na^+ channels to open. (At this point, we have not yet discussed the role of sodium in the cell. The important thing to understand is that the concentration of sodium is higher outside of the cell than on the inside.) When these Na^+ channels open up, they allow sodium ions to travel along their concentration gradient into the cell. This in turn causes more depolarization, which causes more channels to open. The end result, occurring in less than 1 millisecond, is a shift in membrane potential from its resting value of -60mV to approximately +40mV (Alberts, 2010). The value of +40mV represents the resting potential for sodium, and so at this point no more sodium ions are entering the cell.

Before the cell is ready to respond to another signal, it must first return to its resting membrane potential. This is accomplished in a couple of different ways. First, once all of the sodium channels have opened to allow a sufficient amount of Na^+ to flood the cell, they switch to an inactive conformation that prevents any more Na^+ ions from entering (imagine putting up a wall in front of an open door). Since the membrane is still depolarized at this point, the gates will stay open. This inactive conformation will persist as long as the membrane is sufficiently depolarized. Once the membrane potential goes back down, the sodium channels switch from inactive to closed (remove the wall and close the door) (Alberts, 2010).

At the same time that all of this is occurring, there are also potassium channels that have been opened due to the membrane depolarization. There is a time lag that prevents the potassium gates from responding as quickly as those for sodium. However, as soon as these channels are opened, the K^+ ions are able to travel along their concentration gradient out of the cell, carrying positive charges out with them. The result is a sudden *re*-polarization of the cell. This causes it to return to its resting membrane potential, and we start the process all over again (Alberts, 2010).

As a special note of interest, cardiac cells are slightly different from nerve cells in that there are actually two repolarization steps taking place once the influx of sodium has sufficiently depolarized the cell: *fast* repolarization from the exit of K^+ ions, and *slow* repolarization that takes place due to an increase in Ca^{2+} conductance (Rocsoreanu et al., 2000). For now, we will continue dealing solely with Na^+ and K^+.

At this point, it is time to take a look at the models these physiological processes inspired. Arguably the most important of these was created by Alan Lloyd Hodgkin and Andrew Huxley, two men who forever changed the landscape of mathematical biology, when, in 1952, they modeled the neuronal dynamics of the squid giant axon. Refer to Izhikevich (2010) or FitzHugh (1961) for the complete set of space-clamped Hodgkin-Huxley equations.

Shortly after Hodgkin and Huxley published their model, biophysicist Richard FitzHugh began an in-depth analysis of their work. He discovered that, while their model accurately captures the excitable behavior exhibited by neurons, it is difficult to fully understand *why* the math is in fact correct. This is due not to any oversight on the part of Hodgkin and Huxley, but rather because their model exists in four dimensions. To alleviate this problem, FitzHugh proposed his own two-dimensional differential equation model. It combines a model from Bonhoeffer explaining the "behavior of passivated iron wires," as well as a generalized version of the van der Pol relaxation oscillator (FitzHugh, 1961). His equations, which he originally titled the Bonhoeffer-van der Pol (BVDP) oscillator, are shown below (FitzHugh, 1961; Rocsoreanu et al., 2000):

$$\begin{cases} \dot{x} = c(y + x - x^3/3 + z), \\ \dot{y} = -(x - a + by)/c, \end{cases}$$

where, $1 - 2b/3 < a < 1, 0 < b < 1, b < c^2$.

In his model, for which applied mathematician Jin-Ichi Nagumo constructed the equivalent circuit the following year in 1962, *x* "mimics the membrane voltage," while *y* represents a recovery variable, or "activation of the outward current (Izhikevich, 2010)." Both *a* and *b* are constants he supplied (in his 1961 paper, FitzHugh fixes $a = 0.7$ and $b = 0.8$). The third constant, *c*, is left over from the derivation of the BVDP oscillator (he fixes $c = 3$). The last variable, *z*, represents the injected current. It is important to note that in the case of $a = b = z = 0$, the model becomes the original van der Pol oscillator (FitzHugh, 1961).

Many different versions of this model exist (Izhikevich, 2010; Kostova et al., 2004; Rocsoreanu et al., 2000), all of them differing by some kind of transform of variables. We will consider the model used by Kostova et al. in their paper (2004), which presents the FitzHugh-Nagumo model without diffusion:

$$\begin{cases} \dfrac{du}{dt} = \varepsilon g(u) - w + I, \\ \dfrac{dw}{dt} = u - aw, \end{cases}$$

Equation 1

where $g(u) = u(u - \lambda)(1 - u), 0 < \lambda < 1$ and $a, \varepsilon > 0$ (17). Here the state variable *u* is the voltage, *w* is the recovery variable, and *I* is the injected current.

2. Stability Analysis via a Linear Approximation

2.1 Examining the Nullclines

When studying dynamical systems, it is important to be familiar with the concept of nullclines. In a broader sense, a nullcline is simply an isocline, or a curve in the phase space along which the value of a derivative is constant. In particular, the nullcline is the curve along which the value of the derivative is zero. Taking another look at FH-N (Equation 1), we see that there are two potential nullclines, one where the derivative of *u* will be zero, and the other where the derivative of *w* will be zero:

$$\begin{cases} \dfrac{du}{dt} = \varepsilon g(u) - w + I = 0, \\ \dfrac{dw}{dt} = u - aw = 0. \end{cases}$$

One of these nullclines is cubic, and the other is linear (observe the red graphs in Figure 1). Consider an intersection of those two graphs. At that particular point, we know that $du/dt = dw/dt = 0$. Hence, at this point, neither of our state variables is changing. This point where our nullclines intersect is called an equilibrium or fixed point. Since our nullclines are a cubic and a line, geometrically we see that there could be as many as three possible intersections, and no fewer than one. Let us consider the case where $I = 0$. Our system then becomes:

$$\begin{cases} \dfrac{du}{dt} = \varepsilon g(u) - w = 0, \\ \dfrac{dw}{dt} = u - aw = 0. \end{cases}$$

Evaluating the system at the origin, where $u = w = 0$, we see that this is always an equilibrium when $I = 0$.

2.2 Linearizing FitzHugh-Nagumo

Unless otherwise stated, we will assume $I = 0$ for the next few sections. Similarly, (u_e, w_e) will always refer to an equilibrium of FH-N (not necessarily the origin). Let us define the functions f_1 and f_2 as the following:

$$f_1 := \varepsilon g(u) - w + I,$$
$$f_2 := u - aw.$$

Finally, we also set $b_1 = g'(u_e)$, a notation we get from Kostova et al. (2004).

2.2.1 Creating a Jacobian

We may linearize FH-N by constructing a Jacobian matrix as follows:

$$J(u,w) := \begin{bmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial w} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial w} \end{bmatrix}.$$

In terms of FH-N, we have:

$$J(u_e, w_e) := \begin{bmatrix} \varepsilon b_1 & -1 \\ 1 & -a \end{bmatrix}.$$

We see that for any equilibrium, $J(u_e, w_e)$ has the same form, since we have the substitution in place for b_1. Thus, we may generalize the eigenvalues of the above Jacobian to be the eigenvalues of any equilibrium. Solving the characteristic polynomial for our Jacobian, we get the following eigenvalues:

$$\mu_{1,2} = \frac{1}{2}(\varepsilon b_1 - a) \pm \frac{1}{2}\sqrt{(a - \varepsilon b_1)^2 + 4(a\varepsilon b_1 - 1)}.$$

Equation 2

As long as it is never the case that $\mathrm{Re}(\mu_1) = \mathrm{Re}(\mu_2) = 0$, the eigenvalues will always have a real part, and then our equilibrium is hyperbolic (see definition below). By the *Hartman – Grobman Theorem*, we know that we may use the Jacobian to analyze the stability of any fixed point of FH-N.

> *Hyperbolic Fixed Points (2-D)*:
> If $\mathrm{Re}(\mu) \neq 0$ for both eigenvalues, the fixed point is *hyperbolic* (Strogatz, 1994).
>
> *The Hartman-Grobman Theorem*:
> The local phase portrait near a hyperbolic fixed point is "topologically equivalent" to the phase portrait of the linearization; in particular, the stability type of the fixed point is faithfully captured by the linearization. Here topologically equivalent means that there is a homeomorphism that maps one local phase portrait onto the other, such that trajectories map onto trajectories and the sense of time is preserved (Strogatz, 1994).

2.2.2 Trace, Determinant, and Eigenvalues

From Poole (2011), we find two well-known results which tie together the trace, τ, and determinant, Δ, of a matrix with its eigenvalues. For any $n \times n$, A, with a complete set of eigenvalues, $(\mu_1, \mu_2, \ldots, \mu_n)$, we know:

$$\Delta_A = \mu_1 \mu_2 \cdots \mu_n, \text{ and}$$
$$\tau_A = \mu_1 + \mu_2 + \ldots + \mu_n.$$

Hence, for our Jacobian (J) evaluated at an equilibrium, we have:

$$\Delta_J = 1 - \varepsilon b_1 a,$$
$$\tau_J = \varepsilon b_1 - a.$$

For 2-dimensional systems especially, there are many flowcharts available to assist with classifying the stability of an equilibrium based upon the trace and determinant. One such flowchart may be found in Nagle et al. (2008). We will now proceed by exploring the different stability cases for a given set of real eigenvalues.

Case 1

Let $\varepsilon a b_1 < 1$. Then $\Delta_J > 0$. Evaluating the trace, we see that for $\varepsilon b_1 > a$, we get $\tau_J > 0$, which therefore means that we have a dominant positive eigenvalue. Since $\Delta_J > 0$, we know that both of our eigenvalues must then be positive. This gives us an unstable source. For $\varepsilon b_1 < a$, we

get $\tau_J < 0$. This time however, since $\Delta_J > 0$, both of our eigenvalues are negative, and so the system is a stable sink.

Case 2

Let $\varepsilon a b_1 > 1$. Then $\Delta_J < 0$. Hence, our eigenvalues are different signs. In this case, the equilibrium is an unstable saddle.

2.3 Bifurcation Analysis

An important area to study in the field of dynamics is *bifurcation theory*. A bifurcation occurs whenever a certain parameter in a system of equations is changed in a way that results in the creation or destruction of an equilibrium. Although there are many different classifications of bifurcations, we will focus only on one.

2.3.1 Hopf Bifurcation

Consider the complex plane. In a 2-D system, such as FH-N, a stable equilibrium will have eigenvalues that lie in the left half of the plane, that is, the $\mathrm{Re}(\mu) < 0$ half of the plane. Since these eigenvalues in general are the solutions to a particular quadratic equation, we need them both to be either real and negative, or complex conjugates in the same $\mathrm{Re}(\mu) < 0$ part of the plane. Given a stable equilibrium, we may de-stabilize it by moving one or both of the eigenvalues to the $\mathrm{Re}(\mu) > 0$ part of the complex plane. Once an equilibrium has been de-stabilized in this manner, a *Hopf bifurcation* has occurred (Strogatz, 1994).

2.3.2 Proposition 3.1 from Kostova, et al. (2004)

As the eigenvalues μ_1, μ_2 of any equilibrium (u_e, w_e) are of the form

$$\mu_{1,2} = \frac{1}{2}R \pm \frac{1}{2}\sqrt{R^2 + 4Q},$$

where $Q(\varepsilon, a, b_1) \equiv \varepsilon a b_1 - 1$ and $R(\varepsilon, a, b_1) \equiv \varepsilon b_1 - a$, a Hopf bifurcation occurs in cases when $R = 0$ and $Q < 0$ (Kostova et al., 2004).

Proof

Recall from earlier that we defined the Jacobian for FH-N as follows:

$$J(u,w) := \begin{bmatrix} \varepsilon g'(u) & -1 \\ 1 & a \end{bmatrix}.$$

Now we solve for the eigenvalues of this matrix evaluated at an equilibrium. From equation 2, we know our eigenvalues have the following form:

$$\mu_{1,2} = \frac{1}{2}(\varepsilon b_1 - a) \pm \frac{1}{2}\sqrt{(a - \varepsilon b_1)^2 + 4(a\varepsilon b_1 - 1)}.$$

Substituting in now for R and Q, we clearly have

$$\mu_{1,2} = \frac{1}{2}R \pm \frac{1}{2}\sqrt{R^2 + 4Q}.$$

If we allow $Q < 0$ and $R = 0$, our eigenvalues become:

$$\mu_{1,2} = \pm\frac{1}{2}\sqrt{4Q} = \pm i\sqrt{|Q|}.$$

Both of these eigenvalues are along the imaginary axis. This is the exact point at which a Hopf bifurcation occurs.

■

3. Chaos

3.1 Butterflies

We have really only focused on determining the stability of our fixed points, however there are many other interesting questions we can ask of a dynamical system. Two of these questions, which concern sensitivity dependence, we can lump together: how sensitive is our system to the initial conditions that we give it, and how sensitive is our system to a certain parameter that it calls?

The relevance of this first question was explored by meteorologist Edward Lorenz in 1961 (Gleick, 1987). At the time, he was studying weather forecasting models. He found that by slightly changing his initial input to the system, he could wildly, and quite unexpectedly, change the prediction given by his model. Consider the following question, which was actually the title of a talk given by Lorenz back in 1972 (Lorenz, 1993):

> *Does the Flap of a Butterfly's Wings in Brazil Set off a Tornado in Texas?*

This may at first seem frivolous, but the concept that drove him to ask in the first place digs a little bit deeper. Given some system that you use to make predictions (in essence, any mathematical model), do you expect that using roughly equivalent initial conditions will give you roughly the same prediction? Surprisingly, and this is what Lorenz discovered, the answer is not always yes.

Granted, this question depends on a lot of things, for instance how far apart your initial conditions are, how far into the future you wish to make predictions, and how different predictions need to be before you are willing to actually deem them "different." However, once we define explicitly what we are asking, we can learn a great deal about our system. When we start thinking about this in mathematical terms, the butterfly effect means that two solutions, initialized ever so slightly apart, will diverge exponentially as time progresses (assuming of course that our system in question possesses this property).

3.2 Modified BVDP with Smooth Periodic Forcing

With regards to the FitzHugh-Nagumo model, asking such a question as to whether it is sensitive to initial conditions is in most cases trivial. If we take a look at the vector field in the phase plane (see below, Figure 1), we see that none of our solutions will run away on some different path, since they are all restricted ($\varepsilon = 14, a = 1, \lambda = 0.1$).

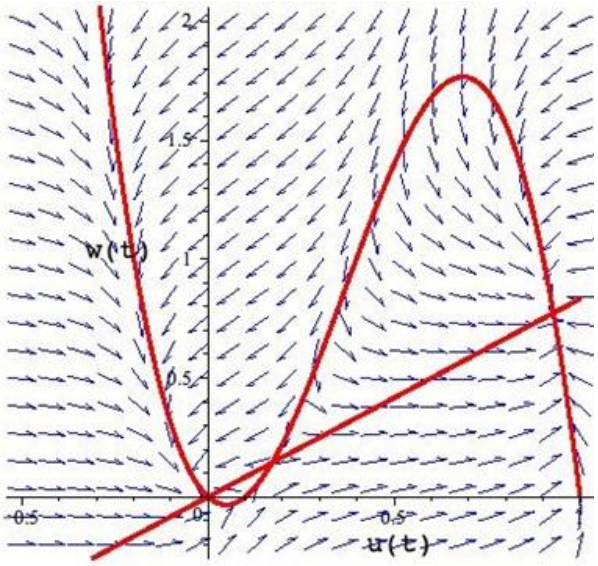

Figure 1: Direction Field for FitzHugh-Nagumo

Even more specifically however, we know that each solution starting in a certain neighborhood of the equilibrium will either converge asymptotically to the equilibrium, or periodically trace an orbit that is held within the neighborhood. There are no surprises here: as long as you initialize a solution in the neighborhood, you will get asymptotic convergence or an orbit.

But what happens when you start changing the parameters inside of the equations themselves? We will begin to examine this question by considering a modified version of the Bonhoeffer - van der Pol equation (Braaksma, 1993), which is a distant cousin of the FitzHugh-Nagumo model (remove the forcing function and do a change of variables to get FH-N):

$$\begin{cases} \varepsilon \dfrac{dx}{dt} = y - \left(\dfrac{1}{2}x^2 + \dfrac{1}{3}x^3 \right), 0 < \varepsilon << 1, \\ \dfrac{dy}{dt} = -(x + \alpha) + s(t), \alpha \in \Re. \end{cases}$$

Braaksma defines *s(t)* to be a Dirac δ-function calling *t* modulo some constant, *T*. While the Dirac function is especially useful for modeling neuronal dynamics, we decided to look at smooth forcing, an idea that we had not seen considered in any literary source. The function we ultimately ended up choosing is rather simple: we consider a smooth, periodic force, generated by $s(t) = \kappa \cos(t)$.

Consider the modified BVDP oscillator that fixes $\varepsilon = \alpha = 0.01$, and $\kappa = 0$. The phase diagram for a solution starting near the origin is shown in Figure 2. We will take some liberties by assuming that the physiological analog for this solution is similar to that of our original FH-N oscillator.

Refer to FitzHugh (1961) for a diagram of these analogs. As an overview, consider Figure 2, ignoring the phase diagram. Start near the origin (not necessarily tangent), and then trace an arc over to the bottom of the left branch of the cubic. Once there, follow the cubic up to the top of its knee. At the top (again, not necessarily tangent), trace another horizontal arc over to the other branch, and then follow the cubic back down to the origin. The resulting rhomboidal path roughly simulates a full oscillation, or physiologically, one neuron successfully reaching an active state.

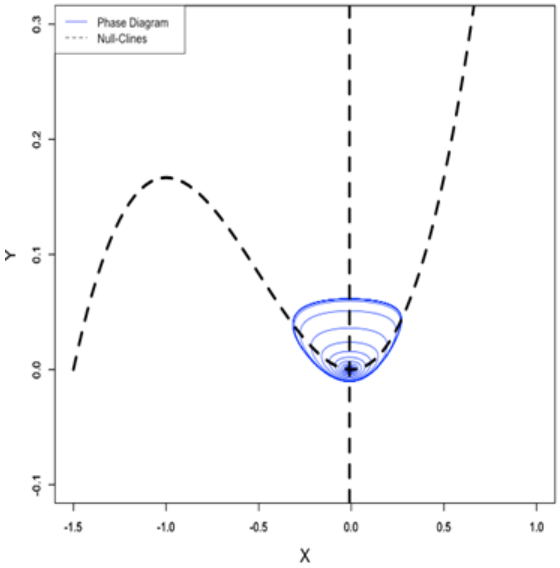

Figure 2: Modified BVDP Phase Portrait, kappa = 0.

Keeping ε and α fixed at their value of 0.01, we now set κ = 0.5 (Figure 3). In essence, we are delivering a continuously oscillating current of electricity, the magnitude of which does not exceed 0.5. We see now that a solution with the exact same starting conditions now sweeps all the way to the left side of the space before travelling up the left knee. From FitzHugh (1961), we know that this solution simulates a neuron experiencing four different active states.

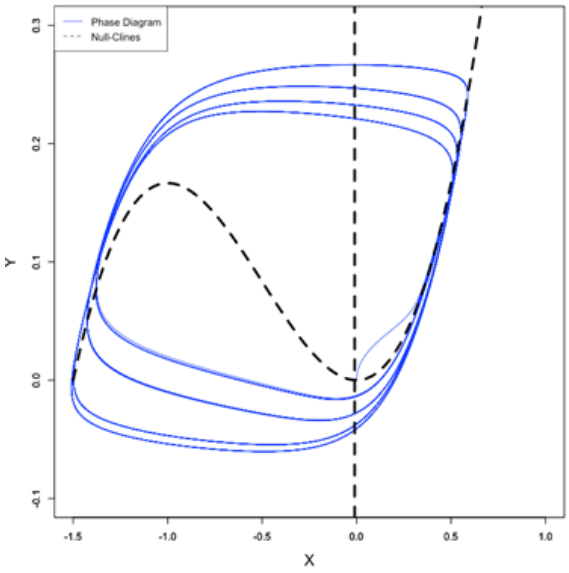

Figure 3: Modified BVDP Phase Portrait, kappa = 0.5.

Another important aspect of this portrait worth noting is the existence of what appear to be four periodic limit cycles through which our solution travels. Shown in Figure 4 is the *bifurcation diagram* for our bifurcating parameter, κ. We see that as the value of κ changes from 0.1 to 1, solutions exist possessing 2, 3, and 4 distinct limit cycles (we see that it is consistent with the phase portrait for κ = 0.5). For κ between 0 and 0.1 however, it is unclear what is happening. It appears as though dozens of limit cycles may potentially exist. Our system seems to be highly sensitive to the value of κ. The question now becomes whether or not this parameter sensitivity means that chaos is actually present.

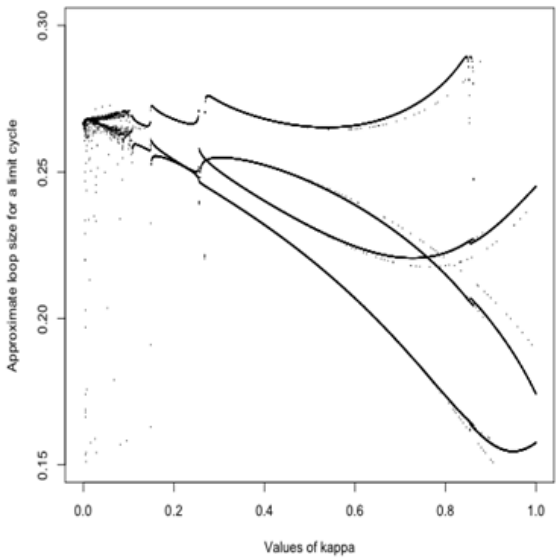

Figure 4: Bifurcation Diagram for kappa.

3.3 Lyapunov Exponents

Arguably the most popular way to quantify the existence of chaos is by calculating a *Lyapunov exponent*. An *n*-dimensional system will have *n* Lyapunov exponents, each corresponding to the rate of exponential divergence (or convergence) of two nearby solutions in a particular direction of the *n*-space. A positive value for a Lyapunov exponent indicates exponential divergence; thus, the presence of any one positive Lyapunov exponent means that the system is chaotic (Wolf, 1985).

3.3.1 Lyapunov Spectrum Generation

There have been numerous algorithms published outlining different ways for generating what are known as *Lyapunov spectra*. As previously mentioned, an *n*-dimensional system will have *n* Lyapunov exponents. Each Lyapunov exponent is defined as the limit of the corresponding Lyapunov spectrum calculated using one of these aforementioned algorithms. For our calculations, we consider the following method from Rangarajan that eliminates the need for reorthogonalization and rescaling (Rangarajan, 1998).

Suppose we have a two dimensional system of nonlinear differential equations, like the one below:

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1, x_2), \\ \dfrac{dx_2}{dt} = f_2(x_1, x_2). \end{cases}$$

We may describe a Jacobian for this system in the same way as we did back in Section 2:

$$J(x_1, x_2) := \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}.$$

Given our two dimensional system and its corresponding linearization, Rangarajan introduces three more differential equations to be coupled with the original system. The state variables λ_1 and λ_2 are the Lyapunov exponents, and θ is a third variable describing angular evolution of the solutions. The heart of the algorithm, equations for setting up the three new variables, is shown below (Rangarajan, 1998):

$$\frac{d\lambda_1}{dt} = J_{11}\cos^2(\theta) + J_{22}\sin^2(\theta) - \frac{1}{2}(J_{12} + J_{21})\sin(2\theta),$$
$$\frac{d\lambda_2}{dt} = J_{11}\sin^2(\theta) + J_{22}\cos^2(\theta) - \frac{1}{2}(J_{12} + J_{21})\sin(2\theta),$$
$$\frac{d\theta}{dt} = -\frac{1}{2}(J_{11} - J_{22})\sin(2\theta) + J_{12}\sin^2(\theta) - J_{21}\cos^2(\theta).$$

Coupling these three equations with our original system, we get a five dimensional system of differential equations. We now simultaneously solve all of these as we would any other system of differential equations, and the output corresponding to the values of λ_1 and λ_2 over time is the Lyapunov spectrum we seek.

3.3.2 The Lyapunov Spectra

Running the algorithm for our modified BVDP model with κ = 0.5 will produce the spectrum shown in Figure 5. Recall how we saw four stable limit cycles existing for the solution to this system. Hence, we would not expect either of our Lyapunov exponents to be greater than zero. Upon generating each of the Lyapunov spectra, we see that this is indeed the case. Both of the Lyapunov exponents for this particular system seem to settle down right away at two negative values, a result which is consistent with our expectations. In general, for roughly any system constructed with a κ value between 0.1 and 1, we can predict, at the very least, that both of our Lyapunov exponents will be less than zero.

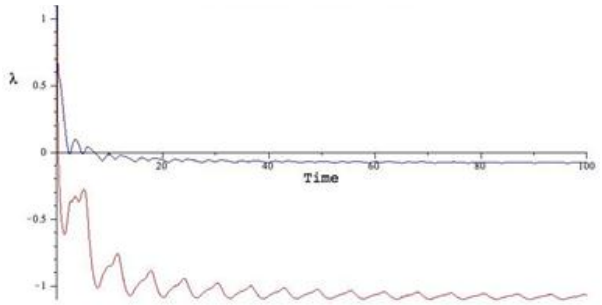

Figure 5: Lyapunov Spectrum for Modified BVDP, kappa = 0.5.

However, the same cannot be said for systems calling a value of κ between 0 and 0.1. Setting κ = 0.01, we may generate the phase portrait seen in Figure 6. Notice there are now numerous orbits, none of which are generating an active state, and none of which seem to have been traced more than once. Said another way, this solution, upon first glance at least, appears to be aperiodic. Aperiodicity is our first clue that chaos might be present in the model.

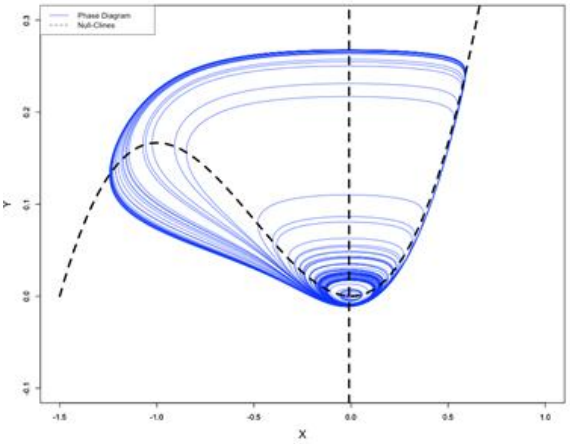

Figure 6: Modified BVDP Phase Portrait, kappa = 0.01.

Changing nothing except for the value of κ, we may now generate the Lyapunov spectrum corresponding to this new system (Figures 7 and 8). We see that one of these lines eventually makes its way underneath the horizontal axis, but the other hovers enticingly close to the axis. At first glance, it is difficult to tell whether or not it ever actually reaches the horizontal axis and/or goes negative. Figure 8 gives us a better look, as it zooms in on values between t = 80 and t =100; from this we see that the spectrum never actually crosses the axis between these values of t, but rather stays over it.

In terms of chaos, it is difficult to judge what is happening. While one of these lines ventures below the horizontal axis, the other is clearly oscillating strictly *above* the axis. We would be remiss to immediately conclude that chaos is in fact present. And we have two reasons for offering this conjecture:

1. We aren't sure how exactly the oscillations are being damped, and
2. There appears to be a decreasing trend to these oscillations, suggesting they may eventually pass beneath the horizontal axis.

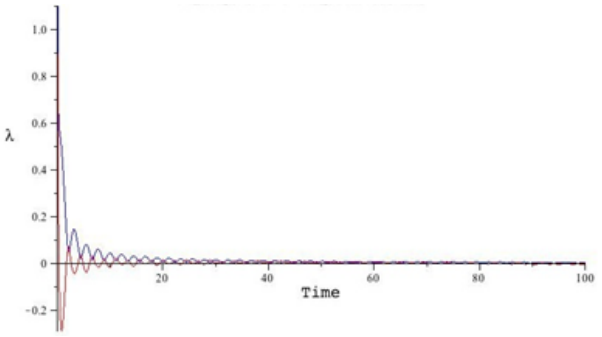

Figure 7: Lyapunov Spectrum for Modified BVDP, kappa = 0.01.

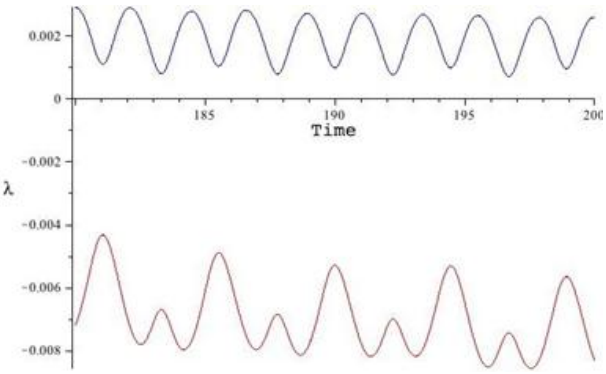

Figure 8: Lyapunov Spectrum for Modified BVDP, kappa = 0.01, $180 \leq t \leq 200$.

The first reason listed above presents issues for us since we need this output to approach some kind of limit. If it continues to behave like it is currently, we cannot say definitively whether it will asymptotically reach a limit or not (recall how the limit of cos(t) is undefined as t approaches infinity). Should it *not* asymptotically approach a limit, the only real conclusion we could offer is that we need to use a more robust algorithm. The second reason is not so much a problem as it is an observation that this output could be asymptotically approaching a positive, negative, or zero valued limit. For now, all we know is that one of our Lyapunov exponents appears to be negative, and the other is positive as far as our solver can tell us.

4. Discussion

"The healthy heart dances, while the dying organ can merely march (Browne, 1989)."
- Dr. Ary Goldberger, Harvard Medical School

The very nature of cardiac muscle stimulation fosters an environment for the propagation of chaos as we have previously described it. This may at first seem slightly counterintuitive. The word "chaos" itself connotes disorder. Certainly it would not immediately come to mind to describe a process as efficient as cardiac muscle contraction. And yet, what we find physiologically with heart rhythms is that a "...perfectly regular heart rhythm is actually a sign of potentially serious pathologies (Cain, 2011)." In particular,

many periodic processes manifest themselves as arrythmia, such as ventricular fibrillation or asystole (the absence of any heartbeat whatsoever) (Chen, 2000). Neither of these particular heart rhythms is conducive for sustaining life: automated external defibrillators (AEDs) were developed to counteract the presence of ventricular fibrillation in a patient; and asystole is the exact opposite of what is conducive for keeping a human alive.

At this point, it would appear as if chaos, at least in humans, is required for survival. Indeed, Harvard researcher Dr. Ary Goldberger was so moved by this idea that he made the above comment before a conference of his peers back in 1989. As the next few years unfold, it will be interesting to see what role, if any, chaos plays in assisting engineers with the development of new equipment to alter life-threatening cardiac arrhythmia in patients. The past twenty years especially have seen a tremendous increase in the demand for AEDs in public fora. Unfortunately, through an interview with a medical engineer at an AED manufacturer, we learned commercially available AEDs only treat ventricular fibrillation and ventricular tachycardia.

AEDs operate by applying a burst of electricity along the natural circuitry in the heart. This electrical stimulus causes a massive depolarization event to take place, triggering simultaneous contraction of a vast majority of cardiac cells. The hope is that this sufficiently resets the heart enough for the pacemaker to regain control. In terms of a forcing function, this is almost similar to stimulation via a Dirac δ-function. Hence, we find the underlying motivation for our exploration into alternative forcing functions.

If we consider our modified BVDP model to be a sufficient analog to cardiac action potential generation, then the solution in Figure 2 roughly represents a heart experiencing ventricular fibrillation. Application of our forcing function $s(t)=\kappa\cos(t)$ for amplitudes between 0.1 and 1 seems to positively impact this model by inducing active states. However, it is unknown whether or not this is a realistic or even adequate portrayal of positively intervening on an arrhythmic event.

In light of the quote from Dr. Goldberger, is it possible that we should be discounting periodic solutions? If a healthy heart rhythm is in fact chaotic, would this necessitate the generation of a chaotic solution? Thus far, the closest we have come to the aforementioned chaotic solution is one that indiscriminately oscillates along subthreshold or superthreshold orbits (see Figure 6), most of which do not even come close to simulating an active event in the cell. In essence, this would imply that the heart is "skipping a beat" each time it fails to generate an action potential. This is no closer to offering a viable heart rhythm, and is actually further off the mark, than our periodic solutions. Unfortunately, our search continues for an induced current that can generate both chaos and muscle contraction.

Another issue needing to be considered is the fact that we cannot, in our modified BVDP model with smooth periodic forcing, remove the forcing lest the neuron quit generating action potentials. Shown below in Figure 10 is the phase portrait for the modified BVDP model with a damped periodic forcing function, $s(t)=\frac{1}{t+1}\kappa\cos(t)$. We see maybe one action potential generated, and then the rest are all subthreshold excitations.

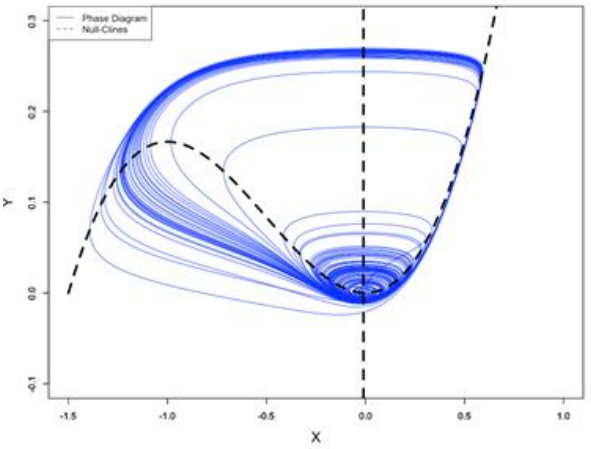

Figure 9: Modified BVDP Phase Portrait, Damped Forcing (kappa = 0.5).

At first glance, it would appear as though we would have to continuously induce our current. This imposes an entirely impractical, even dangerous, requirement on emergency service providers in the field. However, if our forcing function behaves at all like an AED, this result is not surprising. Once you strip away the forcing function, or in our case, once you evaluate solutions after *t* has grown sufficiently large, the underlying model describes a v-fib-like-event taking place. It would then only make sense that action potentials are no longer generated.

The question now is whether or not our forcing function could effectively take the place of a strong induced electrical spike, similar to that delivered by an AED. And if the answer is no, are there scenarios in which continuous application of our periodic current would be practical? Certainly no such scenario is imaginable for AEDs in an out-of-hospital environment, however the possibility remains that it could be useful within a highly controlled setting, such as inside of an operating room during surgery or built into an implantable pacemaker. Ultimately, this a question best left to the engineers and surgeons.

The reason why this is all so important is because sudden cardiac arrest (SCA) causes the deaths of more than 250,000 Americans each year (Heart Rhythm Foundation, 2012). Contrary to popular belief, SCA is first and foremost an electrical problem, triggered by faulty heart rhythms. It should not be confused with a heart attack, which is actually a blockage in one of the major blood vessels of the circulatory system. Certainly a heart attack could eventually become cardiac arrest if left untreated, but qualitatively they are entirely different events.

Whereas heart blockages and similar "plumbing problems" can be remedied by angioplasty or bypass surgery, SCA requires immediate intervention. Typically the window for successful interruption of a cardiac arrest episode will close within approximately eight to ten minutes of onset. Even with the proper training, like a CPR or First Aid course that incorporates the use of an AED, SCA results in death for most out-of-hospital patients. This is certainly not for lack of trying; there are just two big problems victims currently face: CPR is an inefficient substitute for the natural blood delivery of the heart, and AEDs are only effective against two

arrhythmia, v-fib and v-tach. Ideally, technology will be made widely available so that *any* arrhythmia could be treated in an out-of-hospital environment by the layperson.

5. Conclusion

The Hodgkin-Huxley system represents a landmark achievement in the field of biomathematics, however it is difficult to analyze and largely inaccessible due to the fact that it is a four-dimensional system of equations. Richard FitzHugh and Jin-Ichi Nagumo successfully captured the important qualities of the H-H equations, in a system with only two dimensions. Using a modified version of the FH-N equation from Kostova (2004) (Eq. 1), we were able to determine regions in the parameter space where equilibria would be stable or unstable, and, in one particular case, where we could create a Hopf bifurcation.

This set up our own exploration of a modified version of FH-N from Braaksma (1993), which we manipulated by introducing a smooth periodic forcing term ($\kappa\cos(t)$). Using charts from FitzHugh's 1961 paper as a basis for comparison, we saw that we could replicate phase portraits consistent with various instances of neuronal firing. In the realm of electrocardiography, our phase portraits were consistent with a successful contraction of the heart when $\kappa = 0.5$.

However, recent results indicate that healthy heartbeats will be mathematically chaotic. Quantification of our results via a bifurcation diagram of our bifurcating parameter, κ, showed us a region where we could have a chaotic system. And in fact, as far as our algorithm from Rangarajan (1998) can tell us, we were able to create chaotic system when $\kappa = 0.01$. Unfortunately, that chaotic system generated solutions consistent with an irregular heart rhythm.

If we assume that we can use the FH-N equation (or any slightly modified versions) to capture neuronal firing, then it is worth noting that "healthy" solutions to the system do not agree with recent results pointing towards the presence of chaos in healthy neurons. It will be interesting to see if in fact a chaotic solution can be generated to this or any similar system that also solves the problem of successfully firing.

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